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The Tolerance Approach in Multiobjective Linear Fractional Programming

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Abstract

When solving a multiobjective programming problem by the weighted sum approach, weights represent the relative importance associated to the objectives. As these values are usually imprecise, it is important to analyze the sensitivity of the solution under possible deviations on the estimated values. In this sense, the tolerance approach provides a direct measure of how weights may vary simultaneously and independently from their estimated values while still retaining the same efficient solution.

This paper provides an explicit expression to the maximum tolerance on weights in a multiobjective linear fractional programming problem when all the denominators are equal. An application is also presented to illustrate how the results may help the decision maker to choose a most satisfactory solution in a production problem.

Key Words: managerial decision making; multiobjective; sensitivity; fractional programming.
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1 Introduction

Fractional programming is an important tool widely used in the complex field of managerial decision making. It arises in a variety of practical areas such as agricultural planning (Bereanu,1953), models for minimizing wastage (Gilmore and Gomory,1961), optimal policy for a Markov chain (Derman,1962; Fox,1966), information theory (Meister and Oetlli,1967), shipping schedule (Bitran and Novaes,1974), investment portfolios (Ziemba, Parkan and Brooks-Hill,1974), econometric models (Bradley and Frey,1974), inventory models (Stancu-Minasian, 1980), etc. Sensitivity analysis is a fundamental issue when dealing with these problems because it allows to control the permanence of the solution when the data involved in the model are imprecise.

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Recent work on this field has presented a new perspective on sensitivity analysis in mathematical programming called the tolerance approach. It permits to deal with simultaneous and independent perturbations of the coefficients in a variety of problems, that arise in management science and business administration. It provides a direct but simple sensitivity measure of the variable values that the decision maker controls with respect to simultaneous and independent changes in the problem parameters. This approach avoids the difficulty of the classic multiparametric analysis, that yields to a region difficult for the decision maker to understand. Besides it does not require the specification of increase or decrease directions that is needed in other methods such as the 100 percent rule.

Wendell (1985,1984) and Ravi and Wendell (1989) presented the tolerance approach for dealing with variations in a standard linear programming problem. The approach incorporates the possibility of using a priori information about the variability of the coefficients in order to obtain larger tolerance percentages. Hansen, Labbé and Wendell (1989) proposed the tolerance approach to address sensitivity analysis of the multiobjective linear problem. They showed how to calculate the maximum tolerance with respect to the weights that generate an efficient solution. Recently, Mármol and Puerto (1997) have studied how specially structured information on the importance of the objectives can be exploited to yield a larger maximum tolerance percentage in a multiobjective linear problem. The sensitivity in the scalar fractional case has also been approached from this perspective. Dutta, Rao and Tiwari (1992) considered the tolerance approach in a linear fractional programming problem, obtaining expressions of the tolerance that depend on the equivalent linear problem.

The purpose of this paper is to study the sensitivity of the solution in multiobjective linear fractional programming problems provided that all denominators are equal. Specifically, we apply the tolerance approach with respect to the weights used to generate efficient solutions.

The paper proceeds as follows. In section 2 the mathematical statement of the problem is provided and the expression of the maximum tolerance is obtained. An application of a production problem illustrating the results of section 2 is presented in section 3. Finally, section 4 is devoted to the conclusions and to outline possible extensions.

2 Tolerance on weights in a multiobjective linear fractional programming problem.

Consider the multiobjective linear fractional programming problem (MOLFP):

$$\max\left[q_1(x) = \frac{c^{1t}x + \alpha_1}{d^t x + \beta}, \dots, q_p(x) = \frac{c^{pt}x + \alpha_p}{d^t x + \beta}\right]$$
$$\text{s.t. } x \in X = \{x \in \mathbb{R}^n / Ax \leq b, x \geq 0, A \in M_{m.n}, b \in \mathbb{R}^m\}$$

where X is a nonempty bounded set, and such that $d^t x + \beta \neq 0$, $\forall x \in X$; $c^r, d \in \mathbb{R}^n$ and $\alpha_r, \beta \in \mathbb{R}$, $r = 1, ..., p$, and assume that A has full rank.

As optimal solution in the traditional sense is impossible if multiple criteria are involved, we adopt the usual concept of efficient or Pareto optimal solution.

Definition 2.1. $x^* \in X$ is an efficient solution of the MOLFP if

$$\nexists\, x \in X / q_r(x) \geq q_r(x^*),\ r = 1, ..., p;\ \exists i / q_i(x) > q_i(x^*)$$

Since MOLFP has identic denominators, efficient solutions are solutions of weighted sum problems. Therefore, for each $w^0 \in W = \{w \in \mathbb{R}^p / w_r > 0,$ $r = 1, ..., p\}$, we get an efficient solution of the MOLFP as the optimal solution of the weighted problem (WFP)

$$\max \sum_{r=1}^{p} w_r^0 \frac{c^{rt}x + \alpha_r}{d^t x + \beta}$$
$$\text{s.t. } Ax \leq b,$$
$$x \geq 0$$

When solving this problem by the weighted sum approach, each objective $(c^{rt}x + \alpha_r)/(d^t x + \beta)$ is associated with a positive weight w_r^0 and all are combined into a composite criterion function. Each component of this vector w^0 represents the relative importance that the decision maker gives to the objective. A great difficulty with weighting problems is that, in many situations, the decision maker may be unable to specify a weighting vector. Thus, as the efficient solution obtained depends on the values assigned to the weights, it is important to analyze the effect that possible variations in the estimated values of weights produce on the efficient solution obtained.

In order to deal with simultaneous and independent perturbations from their estimated values w^0, we focus on the following perturbed fractional problem (PFP):

$$\begin{aligned} \max & \sum_{r=1}^{p} (w_r^0 + \gamma_r w_r') \frac{c^{rt}x + \alpha_r}{d^t x + \beta} \\ \text{s.t. } & Ax \leq b, \\ & x \geq 0 \end{aligned}$$

The w_r' $(r = 1, \dots, p)$ are given values and γ_r are real parameters. This general perturbation scheme allows to handle a wide range of cases. If $w_r' = w_r^0$, $r = 1, \dots, p$, γ_r represents the percentage deviation from the estimated value w_r^0. If $w_r' = 1$, γ_r represents an additive perturbation from w_r'. Moreover, if the value w_r^0 is precisely known, we can suppress variation in the value of this weight setting $w_r' = 0$. In particular, when all the weights are expressed in terms of the first one, they can be normalized setting $w_1^0 = 1$, $w_1' = 0$.

To address the problem, let x^* be the basic efficient solution of MOLFP, obtained from the vector of weights w^0. Let B denote the optimal basis to WFP, and we use the notation $A_{.j}$ to denote the jth column of matrix A.

Definition 2.2. A finite nonnegative number is called an allowable tolerance τ for PFP if and only if the same basis B is optimal in PFP as long as the absolute value of each perturbation γ_r does not exceed τ, $r = 1, \dots, p$. We define τ^* as a maximum tolerance for PFP if τ^* is the least upper bound of the set $\{\tau : \tau \text{ is a tolerance for PFP }\}$.

Hansen, Labbé and Wendell (1989) developed the expression for the maximum tolerance to simultaneous and independent variations of weights, for the multiobjective linear weighted problem

$$\begin{aligned} \max & \sum_{r=1}^{p} w_r^0 c^{rt} x \\ \text{s.t. } & Ax \leq b, \\ & x \geq 0 \end{aligned}$$

If weights are perturbed as $w_r^0 + \gamma_r w_r'$, the maximum tolerance τ^* is computed as

$$\tau^* = \min_{j\in J}\left\{\tau_j = \frac{-\sum_{r=1}^{p} w_r^0(c_j^r - c_B^{rt}B^{-1}A_{.j})}{\sum_{r=1}^{p} w_r'|c_j^r - c_B^{rt}B^{-1}A_{.j}|}\right\}$$

where J is the index set of the nonbasic variables.

Based on this result, the following theorem establishes the expression of the maximum tolerance within which weights associated to the objectives may deviate simultaneously and independently from their estimated values w^0 in the weighted fractional problem WFP.

Theorem 2.1. *Let x^* be an efficient basic solution of the MOLFP obtained from a vector of weights w^0, B the basis associated to x^*, c_B, d_B the basic parts of c, d and J the index set of the nonbasic variables. The maximum tolerance τ^* is given by:*

$$\tau^* = \min_{j\in J}\tau_j$$

where

$$\tau_j = \frac{-\sum_{r=1}^{p} w_r^0(\overline{c}_j^r - q_r(x^*)\overline{d}_j)}{\sum_{r=1}^{p} w_r'|\overline{c}_j^r - q_r(x^*)\overline{d}_j|}, \qquad \begin{aligned}\overline{c}_j^r &= c_j^r - c_B^{rt}B^{-1}A_{.j}\\ \overline{d}_j &= d_j - d_B^tB^{-1}A_{.j}\end{aligned}$$

Proof: As a consequence of the one to one transformation of Charnes and Cooper (1962), that consists of $t = 1/(d^tx+\beta)$, $y = tx$, WFP is equivalent to the following linear weighted problem (PLP)

$$\begin{aligned}\max &\sum_{r=1}^{p} w_r^0(c^{rt}y + \alpha_r t)\\ \text{s.t. } & Ay - bt \le 0\\ & d^ty + \beta t = 1\\ & y \ge 0, t > 0\end{aligned}$$

Given an optimal basic solution for this problem (y^*, t^*), we can obtain the expression of the maximum tolerance from the corresponding τ_j computed as

$$\tau_j = \frac{-\sum_{r=1}^{p} w_r^0 (c_j^r - c_{\underline{B}}^{rt} \underline{B}^{-1} \underline{A}_{.j})}{\sum_{r=1}^{p} w_r' |c_j^r - c_{\underline{B}}^{rt} \underline{B}^{-1} \underline{A}_{.j}|}$$

where matrix $\underline{A} = \begin{pmatrix} A & -b \\ d^t & \beta \end{pmatrix}$, $\underline{B}$ is the optimal basis of PLP. As $t^* \neq 0$, we can assume that the basic variables of the optimal solution (y^*, t^*) of PLP are $y_1^*, \ldots, y_m^*, t^*$. It follows that $x_i^* = y_i^*/t^*$, $i = 1, \ldots, m$ are the basic variables of the optimal solution of WFP. Hence matrix $\underline{B}$ can be written as $\underline{B} = \begin{pmatrix} B & -b \\ d_B^t & \beta \end{pmatrix}$, where B is the basis associated to x^*. In order to compute $\underline{B}^{-1}$, we partition matrix $\underline{B}$ and we obtain

$$\underline{B}^{-1} = \begin{pmatrix} B^{-1}(I - \frac{1}{d_B^t B^{-1} b + \beta} b d_B^t B^{-1}) & \frac{B^{-1} b}{d_B^t B^{-1} b + \beta} \\ \frac{-d_B^t B^{-1}}{d_B^t B^{-1} b + \beta} & \frac{1}{d_B^t B^{-1} b + \beta} \end{pmatrix}$$

also, we can write

$$c_{\underline{B}}^{rt} \underline{B}^{-1} \underline{A}_{.j} = (c_B^{rt}, \alpha_r) \underline{B}^{-1} \begin{pmatrix} A_{.j} \\ d_j \end{pmatrix}$$

Then we have:

$$c_j^r - c_{\underline{B}}^{rt} \underline{B}^{-1} \underline{A}_j = (c_j^r - c_B^{rt} B^{-1} A_j) - \frac{c_B^{rt} B^{-1} b + \alpha_r}{d_B^t B^{-1} b + \beta} (d_j - d_B^t B^{-1} A_{.j}) =$$
$$= \overline{c}_j^r - q_r(x^*) \overline{d}_j$$

and the result follows.

Notice that the value of τ^* can even be infinite, meaning that any vector of perturbations will lead to the same basic efficient solution. Particularly, when $w' = w^0$, if $\overline{c}_j^r - q_r(x^*) \overline{d}_j = 0$, $\forall j \in J$, $\forall r = 1, \ldots, p$, then $\tau^* = +\infty$ and any percentage deviation from w^0 will preserve the basis.

It is important to point out that the expression of the tolerance established in Theorem 2.1 is easy to calculate. All the elements needed to calculate the maximum tolerance can be determined from the simplex tableau associated to the basic efficient solution. Vectors $\underline{c}$, $\underline{d}$ can be interpreted as the reduced costs of vector c and d associated to the given basis, and can be easily computed in a simplex tableau. The value $\overline{c}_j - q(x^*)\overline{d}_j$ is the reduced cost that appears in some of the algorithms that look for optimal solutions of linear fractional programming problems (see e.g. Martos, 1964).

In addition, this approach permits to detect those weights that will lead to a different efficient solution. For instance, when determining the maximum tolerance percentage it is possible to obtain the 'nearest' adjacent solution in the sense of minimum percentual variation on the estimated weights. In this case, the solution is obtained solving the weighted problem with the following perturbations:

for the index j such that $\tau^* = \tau_j$

$$\begin{array}{l} \text{if } \overline{c}_j^r - q_r(x^*)\overline{d}_j > 0, \text{ then } \gamma_r = \tau^* \\ \text{if } \overline{c}_j^r - q_r(x^*)\overline{d}_j < 0, \text{ then } \gamma_r = -\tau^* \\ \text{if } \overline{c}_j^r - q_r(x^*)\overline{d}_j = 0, \text{ then } \gamma_r \text{ arbitrary} \end{array}$$

This solution could be of interest to the decision maker in the procedure to solve the multiobjective problem.

3 An application: a production problem.

		Demand per unit of product	
Capacity available		P1	P2
Machines (hours)	800	1	3
Owned capital	1000	4	2
Profit per unit		1	2
Employment		5	3
Pollution		3	2,5

Table 1

Let us consider a company that manufactures two products P_1 and P_2. Assume that the costs arising and the capital demands required are both proportional to individual activities. Regardless of the production program to be determined, there are fixed charges amounting to 200 dollars, a fixed capital demand amounting to 400 dollars and a fixed employment demand amounting to 100 hours. Furthermore, the production of P_2 is more than the production of P_1 at most in 200 units. Further production data appear in Table 1. Due to the fixed capital demand, there are 1000 dollars left for the variable capital demand.

One of the objectives of the company is the maximization of owned capital profitability. In order to receive state aids to business development, the company also wishes to maximize the employment in relation to the owned capital. In addition, as pollution generated by the production process is penalized depending on the relative size of the company, pollution by owned capital employed must be minimized. From these data, we establish the following linear fractional problem with three objectives: profitability, employment by owned capital and pollution by owned capital:

$$\max \left[\frac{x_1 + 2x_2 - 200}{4x_1 + 2x_2 + 400}, \frac{5x_1 + 3x_2 + 100}{4x_1 + 2x_2 + 400}, \frac{-3x_1 - 2.5x_2}{4x_1 + 2x_2 + 400} \right]$$

$$\begin{aligned} \text{s.t.} \quad & -x_1 + x_2 \leq 200 \\ & x_1 + 3x_2 \leq 800 \\ & 4x_1 + 2x_2 \leq 1000 \\ & x_1, x_2 \geq 0 \end{aligned}$$

where x_i are the units of the products P_i $(i = 1, 2)$ that the company may produce.

In order to find a solution to the problem, the company provides an estimation of the importance associated to each of the objectives: the first objective is three times more important than the third one, and the second objective is twice as important as the first one. It follows that the vector of weights is $w^0 = (3,6,1)$ and the associated efficient solution is to produce 140 units of product P_1 and 220 units of product P_2. In this case, the profitability is 19/70, the employment by owned capital is 73/70, and the pollution generated by owned capital is 97/140.

Suposse the company wants to analyze the sensitivity of the solution obtained with respect to the importance associated to the objectives. The maximum tolerance percentage will give a measure of how the weights may deviate from their estimated values while retaining the same solution. The efficient tableau associated to the solution is given in Table 2.

				d	4	2	0	0	0	
				c^3	-3	-2.5	0	0	0	
				c^2	5	3	0	0	0	
d_B	c_B^3	c_B^2	c_B^1	$B \setminus c^1$	1	2	0	0	0	b
4	-3	5	1	P_1	1	0	0	-1/5	3/10	140
2	-2.5	3	2	P_2	0	1	0	2/5	-1/10	220
0	0	0	0	P_3	0	0	1	-3/5	2/5	120
				$c_j^1 - z_j^{c1}$	0	0	0	-3/5	-1/10	$q^{1*} = \frac{19}{70}$
				$c_j^2 - z_j^{c2}$	0	0	0	-1/5	-12/10	$q^{2*} = \frac{73}{70}$
				$c_j^3 - z_j^{c3}$	0	0	0	2/5	13/20	$q^{3*} = \frac{-97}{140}$
				$d_j - z_j^d$	0	0	0	0	-1	

Table 2: Efficient tableau.

As we consider multiplicative perturbations, then $w_r' = w_r$, $j = 1, 2, 3$. It follows from Theorem 2.1 that $\tau_4 = 13/17$, $\tau_5 = 11/35$ and $\tau^* = 11/35$ which means that weights can vary simultaneously and independently within 31.42% of their estimated values yielding to the same solution. In this production problem an optimal policy would consist of the same quantity of products P_1 and P_2 as long as the weights remain within a 31.42% of their estimated values (3,6,1).

The tolerance region is:

$$\{w \in I\!R^3 / w_1 \in [72/35, 138/35], w_2 \in [144/35, 276/35], w_3 \in [24/35, 46/35]\}$$

as shown in Figure 1. If the company believes that the weights may vary beyond the maximum tolerance limit, another solution must be explored. One of the vertices of the tolerance region is a vector of weights that leads to obtain an adjacent efficient solution. As $\tau^* = \tau_5$, this vector is given by $w^* = (138/35, 144/35, 24/35)$. A nearby solution is obtained by substituting in the basis the slack variable associated to the third constraint by the one associated to the first constraint. Notice that vector w^* is proportional to (23,24,4), thus if the relative importances of the objectives are specified by this vector, the best solution consists of producing 50 units of product P_1 and 250 units of product P_2. This is the nearest solution in the indicated sense.

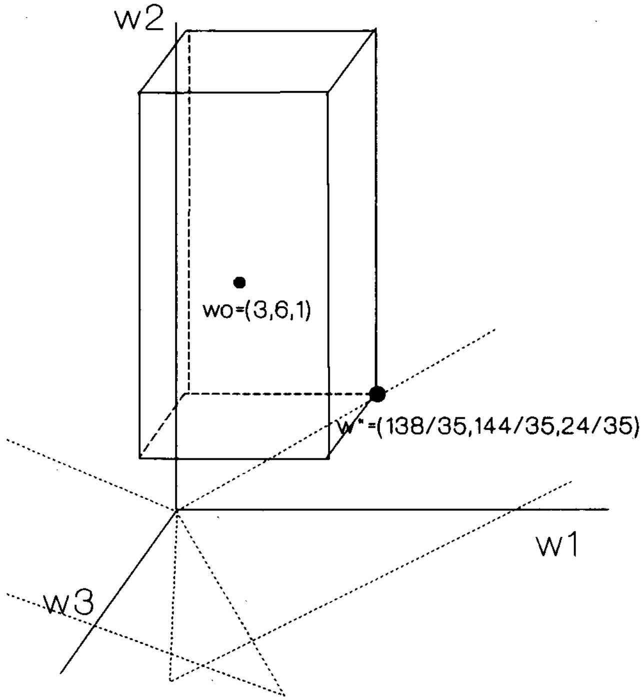

Figure 1: Tolerance region.

4 Concluding remarks and extensions.

We have considered a multiobjective linear fractional problem solved by the weighted sum approach. In this setting, the weights represent the relative importance that the decision maker assigns to the different objectives. Due to the difficulty to establish weights exactly, it is important to analyze the effect of possible deviations on the solution obtained. The paper provides an explicit expression to the maximum tolerance of weights that is a direct measure of the sensitivity of the solution.

The calculation does not depend on the specific method used to solve the weighted problem, because only the simplex tableau associated to the solution is needed. Furthermore, this procedure permits to obtain the vector of weights that provides an adjacent efficient solution. One of the main implications of this result is that can be easily incorporated to any interactive algorithm in order to assure that the solution obtained is satisfactory.

The approach can be extended easily to the cases where additional information about the importance of the objectives is available in the same way as Hansen et al. (1989) and Mármol and Puerto (1997) do for multiobjective linear programming.

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