Global stability and positive recurrence of a stochastic SIS model with Lévy noise perturbation

Tomás Caraballo[*,a], Adel Settati[b], Mohamed El Fatini[c], Aadil Lahrouz[b], Abdelouahid Imlahi[b]

[a]Dpto. Ecuaciones Diferenciales y Análisis Numérico, Facultad de Matemáticas, Universidad de Sevilla, Tarfia s/n, 41012-Sevilla, Spain
[b]Laboratory of Mathematics and Applications, Department of Mathematics, Faculty of Sciences and Techniques, B.P. 416-Tanger Principale, Tanger, Morocco.
[c]Ibn Tofail University, FS, Department of Mathematics, BP 133, Kénitra, Morocco.

Abstract

Focusing on epidemic model in random environments, this paper uses white noise and Lévy noise to model the dynamics of the SIS epidemic model subject to the random changes of the external environment. We show that the jump encourages the extinction of the disease in the population. We first, give a rigorous proof of the global stability of the disease-free equilibrium state. We also establish sufficient conditions for the persistence of the disease. The presented results are demonstrated by numerical simulations.

Key words: White noise, Lévy jumps, Extinction, Persistence, Positive recurrence.

1. Introduction

In recent decades, many mathematical models have been proposed for the transmission dynamics of infectious diseases [9, 5, 24, 12, 10, 15] such as, SIS (susceptible-infective-susceptible), SIRS (susceptible-infective-removed-susceptible), SEIR (susceptible-exposure-infective-recovered). The development of such models is aimed at both understanding the observed mechanisms of infectious diseases and predicting the consequences of the introduction of public health interventions to control the spread of diseases, and thus helps us in devising effective strategies to minimize the destruction caused by infectious diseases. The SIS models are appropriate representations of

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[*]Corresponding author
Email address: `caraball@us.es` (Tomás Caraballo)

the population dynamics for some bacterial agent diseases such as meningitis, and for protozoan agent diseases such as malaria, and many sexually transmitted one such as gonorrhea where exposed individuals typically become infective within 24 hours and do not gain immunity to the disease once they are infected [13]. The literature contains several forms to model SIS diseases in a deterministic and stochastic continuous framework (see, e.g., [24, 12, 11, 20, 16]). The classical SIS model is described by the following system:

$$\begin{cases} dS = (\mu - \mu S - \beta SI + \gamma I)\, dt, \\ dI = (-(\mu + \gamma)I + \beta SI)\, dt, \end{cases} \tag{1.1}$$

where $S(t)$ and $I(t)$ denote respectively the frequency of susceptible individuals and the frequency of infective individuals at time t of the disease. This model considers vital dynamics with death rate μ coinciding with birth rate, which implies that $S + I = 1$. In addition, β is the infection coefficient, and γ is the recovery rate. The asymptotic behavior of the deterministic version of (1.1) is determined by the epidemic threshold $\mathcal{R}_0 = \frac{\beta}{\mu+\gamma}$ [24]. That is, if $\mathcal{R}_0 \leq 1$, then the free-disease equilibrium state $E_0\,(1,0)$ is globally asymptotically stable. While if $\mathcal{R}_0 > 1$, E_0 becomes unstable and there exists an endemic equilibrium state $E_*\left(\frac{1}{\mathcal{R}_0}, \frac{\mathcal{R}_0 - 1}{\mathcal{R}_0}\right)$ which is globally asymptotically stable.

Because epidemiological models are inevitably affected by environmental noises, a stochastic version of the deterministic model (1.1)

$$\begin{cases} dS = (\mu - \mu S - \beta SI + \gamma I)\, dt - \sigma SI dB, \\ dI = (-(\mu + \gamma)I + \beta SI)\, dt + \sigma SI dB, \end{cases} \tag{1.2}$$

was considered by Gray et al. [11], where B is standard Brownian motion. The authors proved the uniqueness and positivity of the solution, they also established conditions for extinction and persistence of $I(t)$. In the case of persistence, they showed the existence of a stationary distribution and derived expressions for its mean and variance. However, the white noise perturbation is not the only way to introduce stochasticity into a deterministic model. There is another type of environmental noises. For instance, telegraph noise which can be illustrated as a switching between two or more regimes of environment with different characteristic factors [12, 16, 17]. Another type of environmental noise is Lévy jumps [6, 7, 8, 3, 4, 22]. This is one of the useful stochastic models which appears frequently in many applications. Its introducing into the underlying population model when the later may suffer sudden environmental shocks such a massive diseases like avian influenza and SARS, earthquakes, hurricanes, etc. The authors in [22]

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have studied the asymptotic behavior of a stochastic SIR model with jumps around the equilibria of the deterministic counterpart system. In this paper, we will discuss the effect of white noise and jump perturbation on the simple SIS epidemic model (1.1). First, we prove that there is a unique positive solution to the system with a positive initial value. Then, we establish sufficient conditions for extinction and persistence of the disease. Finally, we introduce some numerical simulations to support the main results.

2. Model formulation and well-posedness of solutions

Suppose that there is a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\{\mathcal{F}_t\}_{t \geq 0}$ contains all $\mathbb{P}$-null sets). Let $B(t)$ be a standard Brownian motion defined on this probability space. To introduce the Lévy noise, let us define $N(dt, dz)$ as a Poisson counting measure independent of $B(t)$ and $\nu(.)$ as the intensity measure defined on a measurable subset $\mathbb{Y}$ of $[0, \infty)$. We assume that ν is a Lévy measure and $\nu(\mathbb{Y}) < \infty$. Let $\widetilde{N}(dt, du) = N(dt, du) - \nu(du)dt$ be the compensated Poisson measure (where dt is the Lebesgue measure). Lastly, take $H(.)$ as the affects of random jumps on the transmission rate β, where $-1 \leq H(z) \leq 1$ for every $z \in \mathbb{Y}$, and $H(.)$ is continuously differentiable. Considering stochastic perturbations on β

$$\beta dt \rightsquigarrow \beta dt + \sigma dB(t) + \int_{\mathbb{Y}} H(z) \widetilde{N}(dt, dz),$$

the SIS model (1.2) becomes the following stochastic differential equation with jumps:

$$
\begin{cases}
dS(t) = \left(\mu - \mu S(t) - \beta S(t)I(t) + \gamma I(t)\right) dt \\
\qquad\quad - \sigma S(t)I(t)dB(t) - \int_{\mathbb{Y}} H(z)S(t^-)I(t^-)\widetilde{N}(dt, dz), \\
\\
dI(t) = \left(-(\mu + \gamma)I(t) + \beta S(t)I(t)\right) dt \\
\qquad\quad + \sigma S(t)I(t)dB(t) + \int_{\mathbb{Y}} H(z)S(t^-)I(t^-)\widetilde{N}(dt, dz),
\end{cases}
\tag{2.1}
$$

where $\mu, \beta, \gamma > 0$ and $\sigma \geq 0$. Here $S(t^-)$ and $I(t^-)$ are the left limits of $S(t)$ and $I(t)$ respectively. Thereafter, for convenience and simplicity, these two left limits are denoted simply by $S(t)$ and $I(t)$. To begin the analysis of the model, define the subsets

$$\mathbb{R}_+^2 = \left\{(x_1, x_2) | x_i > 0, \ i = 1, 2\right\}, \qquad \Delta = \left\{x \in \mathbb{R}_+^2; \ x_1 + x_2 = 1\right\},$$

and introduce the following theorem ensuring that the solution remains in Δ, that is the model is well posed and thus biologically meaningful.

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Theorem 2.1. *Define* $C(z, x) = (1 - H(z)x)(1 + H(z)(1 - x))$, *for* $(z, x) \in \mathbb{Y} \times (0, 1)$. *If*

$$\sup_{0 < x < 1} \int_{\mathbb{Y}} \left(\log \frac{1}{C(z, x)} \right) \nu(dz) < \infty. \tag{2.2}$$

Then, the set Δ *is almost surely positively invariant by the system* (2.1), *that is, if* $(S_0, I_0) \in \Delta$, *then* $\mathbb{P}\left((S(t), I(t)) \in \Delta \right) = 1$ *for all* $t \geq 0$.

Proof. Let $(S_0, I_0) \in \Delta$. Firstly, it should be noted that the total population is constant, that is,

$$N = S + I = 1 \qquad \text{almost surely (briefly a.s.).} \tag{2.3}$$

This, since we have $N(t) - 1 = (N(0) - 1)e^{-\mu t}$. Then, if $(S(s), I(s)) \in \mathbb{R}_+^2$ for all $s \in [0, t]$ a.s., we get

$$S(s), \, I(s) \in (0, 1) \qquad \text{for all} \quad s \in [0, t] \quad \text{a.s..} \tag{2.4}$$

Secondly, the coefficients of the equation are locally Lipschitz. Then, for any given initial value $(S_0, I_0) \in \Delta$, there is a unique maximal local solution $(S(t), I(t))$ for $t \in [0, \tau_e)$, where τ_e is the explosion time. Let $\varepsilon_0, \, \varepsilon > 0$ such that $S_0, \, I_0 > \varepsilon_0$ and $\varepsilon \leq \varepsilon_0$. Considering the stoping time

$$\tau_\varepsilon = \inf\{t \in [0, \tau_e), \quad S(t) \leq \varepsilon \quad \text{or} \quad I(t) \leq \varepsilon\},$$

and defining the function for $X(S, I) \in \mathbb{R}_+^2$, $\mathbf{V}(X) = \log\left(\frac{1}{S}\right) + \log\left(\frac{1}{I}\right)$. We have by Itô's formula, for all $t \geq 0$, $s \in [0, t \wedge \tau_\varepsilon]$

$$d\mathbf{V}(X(s)) = \left(2\mu + \gamma - \frac{\mu}{S(s)} + \beta I(s) - \frac{\gamma I(s)}{S(s)} + \frac{\sigma^2 I^2(s)}{2} - \beta S(s) + \frac{\sigma^2 S^2(s)}{2} \right) ds$$

$$+ \int_{\mathbb{Y}} \left(-\left(\log\left(1 - H(z)I(s) \right) + H(z)I(s) \right) - \left(\log\left(1 + H(z)S(s) \right) - H(z)S(s) \right) \right) \nu(dz) ds$$

$$+ \sigma(I(s) - S(s))dB - \int_{\mathbb{Y}} \left(\log\left(1 - H(z)I(s) \right) + \log\left(1 + H(z)S(s) \right) \right) \widetilde{N}(ds, dz).$$

Using (2.2) and (2.4), we obtain

$$d\mathbf{V}(X(s)) \leq \left(2\mu + \gamma + \beta + \sigma^2 + 2\nu(\mathbb{Y}) + \sup_{0 < x < 1} \int_{\mathbb{Y}} \log \frac{1}{C(z, x)} \nu(dz) \right) ds$$

$$+ \sigma(I(s) - S(s))dB - \int_{\mathbb{Y}} \left(\log\left(1 - H(z)I(s) \right) + \log\left(1 + H(z)S(s) \right) \right) \widetilde{N}(ds, dz).$$

$$\tag{2.5}$$

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Integrating (2.5) and taking the expectation, yields that for all $t \geq 0$

$$\mathbb{E}\mathbf{V}(X(t \wedge \tau_\varepsilon)) \leq \mathbf{V}(X(0)) + Kt \leq 2\log\left(\frac{1}{\varepsilon_0}\right) + Kt, \qquad (2.6)$$

where

$$K = 2\mu + \gamma + \beta + \sigma^2 + 2\nu(\mathbb{Y}) + \sup_{0 < x < 1} \int_{\mathbb{Y}} \log \frac{1}{C(z,x)} \nu(dz).$$

Suppose that $\tau_e < \infty$, then there is $t > 0$ such that $\mathbb{P}(\tau_e < t) > 0$. Hence, for any $\varepsilon < \epsilon_0$ we have $\mathbb{P}(\tau_\varepsilon < t) > 0$. Let $\omega \in (\tau_\varepsilon < t)$. By the definition of τ_ε, we have $S(\tau_\varepsilon)(\omega) \leq \varepsilon$ or $I(\tau_\varepsilon)(\omega) \leq \varepsilon$ (here we note that we do not have equality because S and I are not necessarily left-continuous). Therefore

$$\log\left(\frac{1}{\varepsilon}\right) \leq \mathbf{V}(X(t \wedge \tau_\varepsilon)(\omega)).$$

This, together with (2.6), gives that

$$\log\left(\frac{1}{\varepsilon}\right) \mathbb{P}(\tau_\varepsilon \leq t) \leq E\left(\mathbf{V}(X(\tau_\varepsilon)) I_{\{\tau_\varepsilon < t\}}\right) \leq E\left(\mathbf{V}(X(t \wedge \tau_\varepsilon))\right)$$

$$\leq 2\log\left(\frac{1}{\varepsilon_0}\right) + Kt. \,(2.7)$$

From (2.7) and the fact that $\tau_\varepsilon \leq \tau_e$, we obtain

$$\mathbb{P}(\tau_e \leq t) \leq \mathbb{P}(\tau_\varepsilon \leq t) \leq \frac{2\log\left(\frac{1}{\varepsilon_0}\right) + Kt}{\log\left(\frac{1}{\varepsilon}\right)}.$$

Letting $\varepsilon \to 0$, leads to the contradiction $\mathbb{P}(\tau_e \leq t) = 0$. Hence $\tau_e = \infty$ a.s.. According to (2.3), the proof of the theorem is completed. $\square$

Given that $(S_0, I_0) \in \Delta$, we have $(S(t), I(t)) \in \Delta$ and then $S(t) + I(t) = 1$, for all $t \geq 0$. Hence, it is sufficient to study SDE (2.1) for $I(t)$

$$\begin{aligned} dI(t) &= I(t)\left(\beta(1 - I(t)) - (\mu + \gamma)\right)dt + \sigma I(t)(1 - I(t))dB(t) \\ &\quad + \int_{\mathbb{Y}} H(z)I(t)(1 - I(t))\widetilde{N}(dt, dz). \qquad (2.8) \\ &\triangleq f_1(I(t))dt + f_2(I(t))dB(t) + \int_{\mathbb{Y}} f_3(I(t), z)\widetilde{N}(dt, dz). \end{aligned}$$

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The Itô formula associated with (2.8) is given, for any twice continuously differentiable $V(.)$, by

$$dV(I(t)) = \quad \mathcal{L}V(I(t))dt + f_2(I(t))\frac{\partial V(I(t))}{\partial x}dB(t)$$
$$+ \int_{\mathbb{Y}} [V(I(t) + f_3(I(t),z)) - V(I(t))]\,\widetilde{N}(dt,dz). \quad (2.9)$$

Where $\mathcal{L}$ is the generator of the process $I(t)$ defined, for each $x \in (0,1)$, by

$$\mathcal{L}V(x) = \quad f_1(x)\frac{\partial V(x)}{\partial x} + \frac{1}{2}f_2^2(x)\frac{\partial^2 V(x)}{\partial x^2}$$
$$+ \int_{\mathbb{Y}} \left[V(x + f_3(x,z)) - V(x) - f_3(x,z) \times \frac{\partial V(x)}{\partial x} \right]\nu(dz). (2.10)$$

In what follows we will concentrate on the stochastic differential equation (2.8) in order to study their asymptotic behaviors and investigate the influence of the white and Lévy noises on this behaviors. Thereafter we suppose that

$$\sup_{0<x<1} \int_{\mathbb{Y}} (\log(1 + H(z)x))^2\,\nu(dz) < \infty. \quad (2.11)$$

Remark 2.2. *It is easy to show that $|H| < \eta$, where $\eta < 1$, is a sufficient condition for (2.2) and (2.11).*

3. Global stability

Refering to Khasminskii et al. [14] and Mao [19], the trivial solution $I \equiv 0$ of system (2.8) is said to be:
 i) stochastically stable (or stable in probability) if for every $\varepsilon > 0$,

$$\lim_{i_0 \to 0} \mathbb{P}(\sup_{t \geqslant 0} I(t,i_0) > \varepsilon) = 0, \quad (3.1)$$

where $I(t,i_0)$ denotes the solution of system (2.8) starting from $I_0 = i_0$,
 ii) globally asymptotically stable (or stochastically asymptotically stable in the large) if it is stochastically stable and for all $i_0 \in \mathbb{R}$,

$$\mathbb{P}\left(\lim_{t \to +\infty} I(t,i_0) = 0 \right) = 1.$$

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Theorem 3.1. *For any initial value $(S_0, I_0) \in \Delta$. If (2.2) and (2.11) hold, and*

$$\Lambda < 0, \tag{3.2}$$

where

$$\Lambda = \begin{cases} C, & \text{if } B \leq 0, \\ \frac{B^2}{4A} + C, & \text{if } B > 0, \end{cases} \tag{3.3}$$

and

$$A = \frac{1}{2}\left(\sigma^2 + \frac{1}{2}\int_{\mathbb{Y}} H^2(z)\nu(dz)\right),$$

$$B = -\beta + \sigma^2 + \frac{1}{2}\int_{\mathbb{Y}} H^2(z)\nu(dz), \tag{3.4}$$

$$C = \beta - \left(\mu + \gamma + \frac{1}{2}\left(\sigma^2 + \frac{1}{2}\int_{\mathbb{Y}} H^2(z)\nu(dz)\right)\right),$$

then the disease-free $I = 0$ of system (2.8) is stochastically stable.

Proof. Let $(S_0, I_0) \in \Delta$ and define the Lyapunov function

$$V(I) = \kappa I^{\frac{1}{\kappa}}, \tag{3.5}$$

where κ is a real positive constant to be chosen in the following. First, let us show that there exists a sufficiently large $\kappa > 0$ and $C(\kappa) > 0$ such that

$$\mathcal{L}V(I) \leq -C(\kappa)I^{\frac{1}{\kappa}}. \tag{3.6}$$

From (2.10), we have

$$\begin{aligned}
\mathcal{L}V = {}& -(\mu+\gamma)I^{\frac{1}{\kappa}} + \beta(1-I)I^{\frac{1}{\kappa}} - \frac{1}{2}\left(\frac{1}{\kappa}-1\right)\sigma^2(1-I)^2 I^{\frac{1}{\kappa}} \\
& + \kappa\int_{\mathbb{Y}}\left((I + H(z)(1-I)I)^{\frac{1}{\kappa}} - 1 - \frac{1}{\kappa}H(z)(1-I)I^{\frac{1}{\kappa}}\right)\nu(dz) \\
= {}& I^{\frac{1}{\kappa}}\left(-(\mu+\gamma) + \beta(1-I) - \frac{1}{2}\sigma^2(1-I)^2\right. \\
& \left. + \kappa\int_{\mathbb{Y}}\left((1 + H(z)(1-I))^{\frac{1}{\kappa}} - 1 - \frac{1}{\kappa}H(z)(1-I)\right)\nu(dz) + \frac{1}{2\kappa}\sigma^2(1-I)^2\right).
\end{aligned} \tag{3.7}$$

On the other hand, using the Taylor-Lagrange formula and the fact that $|H(z)(1-I)| < 1$, one can easily show that there exists $c = c(\kappa, H(z)(1-I))$ such that $0 < c < \frac{1}{\kappa}$ and

$$
\begin{aligned}
(1 + &H(z)(1-I))^{\frac{1}{\kappa}} - 1 \\
&= \frac{1}{\kappa} \log(1 + H(z)(1-I)) + \frac{1}{2\kappa^2}(\log(1 + H(z)(1-I)))^2 e^{c\log(1+H(z)(1-I))} \\
&\leq \frac{1}{\kappa} \log(1 + H(z)(1-I)) + \frac{1}{2\kappa^2}(\log(1 + H(z)(1-I)))^2 e^{\frac{\log(2)}{\kappa}}. \qquad (3.8)
\end{aligned}
$$

Hence, from (3.7) and (3.8), we get

$$
\begin{aligned}
\mathcal{L}V \leq I^{\frac{1}{\kappa}} \bigg(&-(\mu + \gamma) + \beta(1-I) - \frac{1}{2}\sigma^2(1-I)^2 \\
&+ \int_{\mathbb{Y}} (\log(1 + H(z)(1-I)) - H(z)(1-I))\, \nu(dz) \\
&+ \frac{2^{\frac{1}{\kappa}-1}}{\kappa} \sup_{0<x<1} \int_{\mathbb{Y}} (\log(1 + H(z)x))^2\, \nu(dz) + \frac{1}{2\kappa}\sigma^2 \bigg). \qquad (3.9)
\end{aligned}
$$

Using the following inequality

$$
\log(1 + x) - x \leq -\frac{x^2}{4}, \quad -1 < x \leq 1, \qquad (3.10)
$$

we have from (3.9)

$$
\begin{aligned}
\mathcal{L}V \leq I^{\frac{1}{\kappa}} \bigg(&-(\mu + \gamma) + \beta(1-I) - \frac{1}{2}\bigg(\sigma^2 + \frac{1}{2}\int_{\mathbb{Y}} H^2(z)\nu(dz)\bigg)(1-I)^2 \\
&+ \frac{2^{\frac{1}{\kappa}-1}}{\kappa} \sup_{0<x<1} \int_{\mathbb{Y}} (\log(1 + H(z)x))^2\, \nu(dz) + \frac{1}{2\kappa}\sigma^2 \bigg) \qquad (3.11)
\end{aligned}
$$

$$
= I^{\frac{1}{\kappa}} \bigg(\Gamma(1-I) + \frac{2^{\frac{1}{\kappa}-1}}{\kappa} \sup_{0<x<1} \int_{\mathbb{Y}} (\log(1 + H(z)x))^2\, \nu(dz) + \frac{1}{2\kappa}\sigma^2 \bigg), (3.12)
$$

where the function Γ is defined by

$$
\Gamma(x) = -\frac{1}{2}\bigg(\sigma^2 + \frac{1}{2}\int_{\mathbb{Y}} H^2(z)\nu(dz)\bigg)x^2 + \beta x - (\mu + \gamma). \qquad (3.13)
$$

By studying the quadratic form Γ in $(0,1)$, it is easy to see that

$$
\Gamma(1-I) \leq \Lambda, \qquad (3.14)
$$

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where Λ is defined in (3.3). This implies

$$\mathcal{L}V \leq I^{\frac{1}{\kappa}} \left(\Lambda + \frac{2^{\frac{1}{\kappa}-1}}{\kappa} \sup_{0<x<1} \int_{\mathbb{Y}} (\log(1 + H(z)x))^2 \, \nu(dz) + \frac{1}{2\kappa}\sigma^2 \right) \quad (3.15)$$

By (3.2), we can choose a sufficiently large κ such that (3.6) holds with

$$-C(k) \triangleq \Lambda + \frac{2^{\frac{1}{\kappa}-1}}{\kappa} \sup_{0<x<1} \int_{\mathbb{Y}} (\log(1 + H(z)x))^2 \, \nu(dz) + \frac{1}{2\kappa}\sigma^2 < 0.$$

Now, for every $\varepsilon > 0$, let ϱ_ε be the first time that I jumps ε, that is

$$\varrho_\varepsilon = \inf\{t \geq 0 : I(t) \geq \varepsilon\}.$$

By (2.9) and (3.6), we have

$$\begin{aligned} d\left((I(t))^{\frac{1}{\kappa}}\right) &\leq \frac{\sigma}{\kappa}(I(t))^{\frac{1}{\kappa}}(1 - I(t))dB_t \\ &+ \int_{\mathbb{Y}} \left[(I(t) + H(z)I(t)(1 - I(t)))^{\frac{1}{\kappa}} - (I(t))^{\frac{1}{\kappa}} \right] \widetilde{N}(dt, dz). \end{aligned}$$

Integrating the above inequality between 0 and $T \wedge \varrho_\varepsilon$, and taking expectation in both sides, one can easily have for $T > 0$

$$E\left((I(\varrho_\varepsilon))^{\frac{1}{\kappa}} I_{\{\varrho_\varepsilon < T\}}\right) \leq E\left(I(T \wedge \varrho_\varepsilon)\right)^{\frac{1}{\kappa}} \leq I_0^{\frac{1}{\kappa}}. \quad (3.16)$$

Note that the equality $I(\varrho_\varepsilon) = \varepsilon$ do not hold because I is not necessarily left-continuous, but

$$I(\varrho_\varepsilon) \geq \varepsilon. \quad (3.17)$$

Indeed, for all $h > 0$ there exists $t_h \in \{t \geq 0 : I(t) \geq \varepsilon\}$ such that $\varrho_\varepsilon \leq t_h \leq \varrho_\varepsilon + h$, so we can easily construct a sequence $(t'_n)_{n \geq 1}$ such that for all $n \geq 1$,

$$I(t'_n) \geq \varepsilon, \quad \varrho_\varepsilon \leq t'_n \leq \varrho_\varepsilon + 2^{-n}. \quad (3.18)$$

Since I is right-continuous, we get from (3.18)

$$\varepsilon \leq \lim_{n \to \infty} I(t'_n) = \lim_{t \searrow \varrho_\varepsilon} I(t) = I(\varrho_\varepsilon),$$

and then (3.17) holds. Combining (3.16) and (3.17), we obtain for all $T > 0$

$$\mathbb{P}\left(\varrho_\varepsilon < t\right) \leq I_0^{\frac{1}{\kappa}} \varepsilon^{-\frac{1}{\kappa}}.$$

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Letting $t \longrightarrow \infty$ we get $\mathbb{P}\left(\varrho_\varepsilon < \infty\right) < I_0^{\frac{1}{\kappa}} \varepsilon^{-\frac{1}{\kappa}}$, that is

$$\mathbb{P}\left(\sup_{t \geqslant 0} I(t) < \varepsilon\right) \geq 1 - I_0^{\frac{1}{\kappa}} \varepsilon^{-\frac{1}{\kappa}}.$$

Hence, $\lim_{I_0 \to 0} \mathbb{P}\left(\sup_{t \geqslant 0} I(t) \geq \varepsilon\right) = 0$. This makes end to the proof of Theorem. 3.1. $\qquad\square$

Here, we will discuss the extinction of the stochastic differential equation (2.8). Let us begin with the following strong law of large numbers for local martingales (see, e.g., Lipster [18]).

Lemma 3.2. *Let M_t, $t \geq t_0$, be a local martingale vanishing at time 0. Define $\varphi_{M_t} := \int_{t_0}^{t} \frac{d<M_s, M_s>}{(1+s)^2}$, where $< M_t, M_t >$ is Meyer's angle bracket process. Then*

$$\mathbb{P}\left(\lim_{t \to \infty} \frac{M_t}{t} = 0\right) = 1, \quad \textit{provided that} \quad \mathbb{P}\left(\lim_{t \to \infty} \varphi_{M_t} < \infty\right) = 1.$$

Theorem 3.3. *Let Λ be as in (3.3) and let assumptions (2.2) and (2.11) hold. Then*

$$\limsup_{t \to \infty} \frac{1}{t} \log(I(t)) \leq \Lambda \quad a.s..$$

Moreover, if $\Lambda < 0$, then the solution $I(t)$ of the stochastic differential equation (2.8) tends to zero exponentially almost surely.

Proof. By the Itô's formula, we have

$$
\begin{aligned}
\log(I(t)) &= \log(I_0) + \int_0^t \left(-\frac{1}{2}\sigma^2(1 - I(s))^2 + \beta(1 - I(s)) - (\mu + \gamma)\right) ds \\
&\quad + \int_0^t \int_{\mathbb{Y}} \left(\log\left(1 + H(z)(1 - I(s))\right) - H(z)(1 - I(s))\right) \nu(dz)ds \\
&\quad + \int_0^t \sigma(1 - I(s))dB(s) + \int_0^t \int_{\mathbb{Y}} \log\left(1 + H(z)(1 - I(s))\right) \widetilde{N}(ds, dz). \quad (3.19)
\end{aligned}
$$

Using the inequality (3.10), we get

$$
\begin{aligned}
\log(I(t)) &\leq \int_0^t \left(-\frac{1}{2}\left(\sigma^2 + \frac{1}{2}\int_{\mathbb{Y}} H^2(z)\nu(dz)\right)(1 - I(s))^2 - \beta I(s) + \beta - (\mu + \gamma)\right) ds \\
&\quad + \log(I_0) + \int_0^t \sigma(1 - I(s))dB(s) + \int_0^t \int_{\mathbb{Y}} \log\left(1 + H(z)(1 - I(s))\right) \widetilde{N}(ds, dz). \\
&\triangleq \log(I_0) + \int_0^t \Gamma(1 - I(s))ds + M_t^0 + M_t^1, \quad (3.20)
\end{aligned}
$$

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where the function Γ is defined in (3.13) and

$$M_t^0 = \int_0^t \sigma(1-I(s))dB(s), \; M_t^1 = \int_0^t \int_{\mathbb{Y}} \log\left(1 + H(z)(1 - I(s))\right) \widetilde{N}(ds, dz).$$
(3.21)

M_t^0 and M_t^1 are real-valued local martingales having the following Meyer's angle bracket process

$$< M_t^0, M_t^0 > = \int_0^t \sigma^2(1 - I(s))^2 ds \le \sigma^2 t,$$

and

$$\begin{aligned} < M_t^1, M_t^1 > &= \int_0^t \int_{\mathbb{Y}} \left(\log\left(1 + H(z)(1 - I(s))\right)\right)^2 \nu(dz)ds \\ &\le \left[\sup_{0<x<1} \int_{\mathbb{Y}} (\log(1 + H(z)x))^2 \, \nu(dz)\right] t. \end{aligned}$$

Then, by (2.11) and the large number low cited in Lemma 3.2, we have

$$\limsup_{t\to\infty} \frac{M_t^0}{t} = 0, \quad \limsup_{t\to\infty} \frac{M_t^1}{t} = 0 \quad a.s..$$
(3.22)

It follows from (3.14) and (3.20) that

$$\limsup_{t\to\infty} \frac{1}{t} \log(I(t)) \le \Lambda + \limsup_{t\to\infty} \frac{M_t^0}{t} + \limsup_{t\to\infty} \frac{M_t^1}{t}.$$
(3.23)

We therefore obtain the desired assertion from (3.22) and (3.23). $\qquad\square$

Clearly, Theorems 3.1 and 3.3 lead to the following corollary which shows that the disease-free $I = 0$ of system (2.8) is globally asymptotically stable.

Corollary 3.4. *Let assumptions (2.2) and (2.11) hold. If $\Lambda < 0$, then the disease-free $I = 0$ of system (2.8) is globally asymptotically stable.*

4. Persistence of the disease

In the remainder of the paper, in addition to the constants A, B, C and the function Γ defined in (3.4) and (3.13) respectively, we will also use the function $\Gamma' : \mathbb{R} \to \mathbb{R}$ defined by

$$\Gamma'(x) = -A'x^2 + B'x + C',$$
(4.1)

where

$$A' = \frac{1}{2}\left(\sigma^2 + \int_{\mathbb{Y}} H^2(z)\nu(dz)\right) \geq A,$$

$$B' = -\beta + \sigma^2 + \int_{\mathbb{Y}} H^2(z)\nu(dz) \geq B,$$

$$C' = \beta - \left(\mu + \gamma + \frac{1}{2}\left(\sigma^2 + \int_{\mathbb{Y}} H^2(z)\nu(dz)\right)\right) \leq C.$$

Theorem 4.1. *Let assumptions (2.2) and (2.11) hold.*
(i) If $0 \leq H(z) < 1$, $z \in \mathbb{Y}$ and $C' > 0$ then,

$$\limsup_{t \to \infty} I(t) \geq \xi', \quad a.s., \tag{4.2}$$

where, ξ' is the positive root of $\Gamma'(x) = 0$, that is

$$\xi' = \frac{B' + \sqrt{B'^2 + 4A'C'}}{2A'}. \tag{4.3}$$

(ii) If $C > 0$ then,

$$\liminf_{t \to \infty} I(t) \leq \xi, \quad a.s., \tag{4.4}$$

where, ξ is the positive root of $\Gamma(x) = 0$, that is

$$\xi = \frac{B + \sqrt{B^2 + 4AC}}{2A}. \tag{4.5}$$

(iii) If $0 \leq H(z) < 1$, $z \in \mathbb{Y}$ and $C' > 0$ then, the epidemic is persistent in the sense that $I(t)$ reaches the set $[\xi', \xi]$ infinitely often with probability one.

Proof. (i) From (3.19) and the inequality

$$-\frac{x^2}{2} \leq \log(1 + x) - x, \quad x \geq 0, \tag{4.6}$$

we have

$$\log(I(t)) \geq \log(I_0) + \int_0^t \Gamma'(I(s))ds + M_t^0 + M_t^1, \tag{4.7}$$

where M_t^0 and M_t^1 are defined by (3.21) respectively. By the condition $C' > 0$, we have

$$\Gamma'(0) = C' > 0 \text{ and } \Gamma'(1) = -A' + B' + C' = -(\mu + \gamma) < 0.$$

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Then $\Gamma'(x)$ admits a unique root $\xi' \in (0,1)$. Moreover $\Gamma'(x)$ is decreasing around ξ'. So, we can easily show for all sufficiently small $\varepsilon > 0$ and all x such that $0 < x \le \xi' - \varepsilon$ that

$$\Gamma'(x) \ge \Gamma'(\xi' - \varepsilon), \quad \text{and} \quad \Gamma'(\xi' + \varepsilon) < 0 < \Gamma'(\xi' - \varepsilon). \tag{4.8}$$

We now begin to prove assertion (4.2). If it were not true, then there would be a sufficiently small $\varepsilon > 0$ such that

$$\mathbb{P}\left(\limsup_{t \to \infty} I(t) \le \xi' - 2\varepsilon\right) > 0.$$

Let us put

$$\Omega_1 = \left\{\limsup_{t \to \infty} I(t) \le \xi' - 2\varepsilon\right\}.$$

Hence, for every $\omega \in \Omega_1$, there is a $T(\omega) > 0$ such that

$$I(s) \le \xi' - \varepsilon \qquad \forall \ s \ge T(\omega), \tag{4.9}$$

From (4.8) and (4.9) we get, for any $s \ge T(\omega)$

$$\Gamma'(I(s)) \ge \Gamma'(\xi' - \varepsilon) > 0. \tag{4.10}$$

Moreover, by (3.22) there is a set $\Omega_2 \subset \Omega$ with $\mathbb{P}(\Omega_2) = 1$ such that for every $\omega \in \Omega_2$,

$$\limsup_{t \to \infty} \frac{M_t^0}{t} = 0, \quad \limsup_{t \to \infty} \frac{M_t^1}{t} = 0. \tag{4.11}$$

Noting $\mathbb{P}(\Omega_1 \cap \Omega_2) > 0$ and fixing $\omega \in \Omega_1 \cap \Omega_2$. It then follows from (4.7) and (4.10), that, for $t \ge T(\omega)$

$$\log(I(t)) \ge \log(I_0) + \int_0^{T(\omega)} \Gamma'(I(s))ds + \Gamma'(\xi' - \varepsilon)(t - T(\omega)) + M_t^0 + M_t^1. \tag{4.12}$$

From (4.10), (4.11), (4.12) we get

$$\liminf_{t \to \infty} \frac{1}{t} \log(I(t)) \ge \Gamma'(\xi' - \varepsilon) > 0.$$

Whence $\lim_{t \to \infty} I(t) = \infty$. But this contradicts (4.9). The required assertion (4.2) must therefore hold.

(ii) Similarly, from (3.20) we have

$$\log(I(t)) \le \log(I_0) + \int_0^t \Gamma(I(s))ds + M_t^0 + M_t^1. \tag{4.13}$$

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If (4.4) were not true, we could then find an $\eta > 0$ sufficiently small such that $\mathbb{P}(\Omega_3) > 0$ where,

$$\Omega_3 = \left\{ \liminf_{t \to \infty} I(t) \geq \xi + 2\eta \right\}.$$

Hence, for every $\omega \in \Omega_3$, there is a $T'(\omega) > 0$ such that

$$I(s) \geq \xi + \eta \qquad \forall \ s \geq T'(\omega), \tag{4.14}$$

which implies that

$$\Gamma(I(s)) \leq \Gamma(\xi + \eta) < 0 \qquad \forall \ s \geq T'(\omega). \tag{4.15}$$

It then follows from (4.13) and (4.15) that, for $t \geq T'(\omega)$

$$\log(I(t)) \leq \log(I_0) + \int_0^{T'(\omega)} \Gamma(I(s))ds + \Gamma(\xi + \eta)(t - T'(\omega)) + M_t^0 + M_t^1. \tag{4.16}$$

Now, fix any $\omega \in \Omega_2 \cap \Omega_3$. Similarly to the proof of (i) we get, from (4.11), (4.15) and (4.16) that

$$\limsup_{t \to \infty} \frac{1}{t} \log(I(t)) \leq \Gamma(\xi + \eta) < 0.$$

Whence $\lim_{t \to \infty} I(t) = 0$. But this contradicts (4.14). This completes the proof of assertion (4.4).

(iii) It suffices to show that $\xi' \leq \xi$. We have for $x \in (0, 1)$

$$\Gamma(x) - \Gamma'(x) = \frac{1}{4}(x - 1)^2 \int_{\mathbb{Y}} H^2(z)\nu(dz) \geq 0. \tag{4.17}$$

On the other hand, the functions Γ and Γ' are decreasing around ξ and ξ' respectively, which implies with (4.17) that $\xi \geq \xi'$. □

5. Persistence in mean of the disease

Theorem 5.1. *Let assumptions (2.2) and $H > 0$ hold. If $B < 0$ and $C' > 0$ then*

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t I(s)ds \geq \frac{C'}{\beta - \frac{1}{2}\sigma^2}, \quad \limsup_{t \to \infty} \frac{1}{t} \int_0^t I(s)ds \leq \frac{C}{-B}, \quad a.s.,$$

that is, the solutions of stochastic model system (2.8) starting from any point $(S_0, I_0) \in \Delta$ are strongly persistent in mean.

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Proof. From (4.7) and by $I^2 \leq I$, we have

$$\log(I(t)) \geq \log I_0 + C't - \left(\beta - \frac{1}{2}\sigma^2\right) \int_0^t I(s)ds + M_t^0 + M_t^1. \qquad (5.1)$$

By (4.11), for every $\omega \in \Omega_2$ and $\varepsilon > 0$ sufficiently small there exists $T(\omega, \varepsilon)$ such that for all $t \geq T$, we have

$$\log I_0 + C't + M_t^0 + M_t^1 \geq (C' - \varepsilon)t,$$

which implies with (5.1) that for all $t \geq T$

$$\frac{1}{\left(\beta - \frac{1}{2}\sigma^2\right)} \frac{d}{dt}\left[e^{\left(\beta - \frac{1}{2}\sigma^2\right)\int_0^t I(s)ds}\right] = I(t)e^{\left(\beta - \frac{1}{2}\sigma^2\right)\int_0^t I(s)ds} \geq e^{(C'-\varepsilon)t}. \qquad (5.2)$$

Noting that $-\beta + \frac{1}{2}\sigma^2 \leq B < 0$. Hence, integrating (5.2) from T to t and dividing both sides by t yields that

$$\frac{1}{\left(\beta - \frac{1}{2}\sigma^2\right)t}\log\left[e^{\left(\beta - \frac{1}{2}\sigma^2\right)\int_0^T I(s)ds} + \frac{\beta - \frac{1}{2}\sigma^2}{C' - \varepsilon}\left(e^{(C'-\varepsilon)t} - e^{(C'-\varepsilon)T}\right)\right].$$
$$\leq \frac{1}{t}\int_0^t I(s)ds$$

Using l'Hospital's rule, one can easily have

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t I(s)ds \geq \frac{C' - \varepsilon}{\beta - \frac{1}{2}\sigma^2}.$$

Letting $\varepsilon \to 0$, we get the desired estimation.

Turning now to the estimation required for $\limsup\limits_{t\to\infty} \dfrac{1}{t}\displaystyle\int_0^t I(s)ds$. From (4.13), we get

$$\log(I(t)) \leq \log I_0 + Ct + B\int_0^t I(s)ds + M_t^0 + M_t^1. \qquad (5.3)$$

As above, for every $\omega \in \Omega_2$ and $\varepsilon > 0$ sufficiently small there exists $T'(\omega, \varepsilon)$ such that for all $t \geq T'$, we have

$$\log I_0 + Ct + M_t^0 + M_t^1 \leq (C + \varepsilon)t,$$

which implies with (5.3) for all $t \geq T'$, that

$$\frac{1}{-B}\frac{d}{dt}\left[e^{-B\int_0^t I(s)ds}\right] = I(t)e^{-B\int_0^t I(s)ds} \leq e^{(C+\varepsilon)t}. \qquad (5.4)$$

Integrating (5.4) from T' to t, dividing both sides by t and letting $t \to \infty$ and $\varepsilon \to 0$, we obtain the desired results. $\qquad \square$

6. Positive recurrence

Recall that a $\mathbb{R}$-valued process $X(t, x_0)$ is recurrent with respect to the bounded set $\mathcal{U} \subset \mathbb{R}$ if $\mathbb{P}(\tau^{x_0} < \infty) = 1$, for any initial value $x_0 \notin \mathcal{U}$, where τ^{x_0} is the hitting time of $\mathcal{U}$ for $X(t, x_0)$, that is $\tau^{x_0} = \inf\{t > 0, X(t, x_0) \in \mathcal{U}\}$. The process $X(t, x_0)$ is said to be positive recurrent with respect to $\mathcal{U}$ if it is recurrent with respect to $\mathcal{U}$ and $\mathbb{E}(\tau^{x_0}) < \infty$, for any $x_0 \notin \mathcal{U}$.
From (4.2) if $H \geq 0$ and $C' > 0$, then $I(t)$ rises to or above the level ξ' infinitely often with probability one, which means that $I(t)$ is recurrent with respect to $[\xi', \infty)$. The following theorem shows that $I(t)$ is also positive recurrent with respect to $[\xi', \infty)$.

Theorem 6.1. *For any initial value $I_0 \in (0, 1)$. If (2.2) holds, $H \geq 0$, and*

$$\beta - \left(\mu + \gamma + \frac{1}{2}\left(\sigma^2 + \int_{\mathbb{Y}} H^2(z)\nu(dz)\right)\right) > 0, \tag{6.1}$$

then $I(t)$ is positive recurrent with respect to $[\xi'_-, \infty)$, where ξ'_- is any number strictly less than ξ'.

Proof. Let $\rho > 0$ be a real positive constant to be chosen in the following. From (2.10), we have

$$\mathcal{L}\left(I^{-\rho}\right) = \rho I^{-\rho}\left((\mu + \gamma) - \beta(1 - I) + \frac{1}{2}\sigma^2(1 - I)^2 + \frac{\rho}{2}\sigma^2(1 - I)^2 + \right.$$

$$\left. + \frac{1}{\rho}\int_{\mathbb{Y}}\left((1 + H(z)(1 - I))^{-\rho} - 1 + \rho H(z)(1 - I)\right)\nu(dz)\right) \tag{6.2}$$

Applying the mean value theorem, one can easily show that there exists $c' = c'(\rho, H(z)(1 - I)) > 0$ such that $-\rho < -c' < 0$ and

$$(1 + H(z)(1 - I))^{-\rho} - 1 + \rho H(z)(1 - I) \leq -\rho \log(1 + H(z)(1 - I))$$

$$+ \rho H(z)(1 - I) + \frac{\rho^2}{2}(\log 2)^2(1 + H(z)(1 - I))^{-c'}. \tag{6.3}$$

Combining (6.3) with inequality (4.6) and using $H \geq 0$ yields

$$(1 + H(z)(1 - I))^{-\rho} - 1 + \rho H(z)(1 - I)$$

$$\leq \frac{\rho}{2}H^2(z)(1 - I)^2 + \frac{\rho^2}{2}(\log 2)^2. \tag{6.4}$$

Hence, from (6.2) and (6.4), we get

$$\mathcal{L}\left((I)^{-\rho}\right)$$

$$\leq \rho I^{-\rho}\left((\mu+\gamma)-\beta+\frac{1}{2}\left(\sigma^2+H^2(z)\right)+\left(\beta-\left(\sigma^2+\int_{\mathbb{Y}}H^2(z)\nu(dz)\right)\right)\right)I$$

$$+\frac{1}{2}\left(\sigma^2+\int_{\mathbb{Y}}H^2(z)\nu(dz)\right)I^2+\frac{\rho}{2}((\log 2)^2+\sigma^2)\right)$$

$$=-\rho I^{-\rho}\left(\Gamma'(I)-\frac{\rho}{2}((\log 2)^2+\sigma^2)\right). \tag{6.5}$$

where $\Gamma'(.)$ is defined in (4.1).

Now, suppose $I_0 = i_0$, such that $\xi' > i_0$, let $\varrho_{\xi'-\varepsilon}$ be the first time that I jumps $\xi' - \varepsilon$, that is

$$\varrho_{\xi'-\varepsilon} = \inf\{t \geq 0 : I(t) \geq \xi' - \varepsilon\}.$$

From (4.2), we have $\mathbb{P}(\varrho_{\xi'-\varepsilon} < \infty) = 1$. So, by virtue of Dynkin's formula, we have

$$E\left((I(\varrho_{\xi'-\varepsilon}))^{-\rho}\right) - i_0^{-\rho} \;\; = \;\; E\left(\int_0^{\varrho_{\xi'-\varepsilon}}\mathcal{L}\left((I(s))^{-\rho}\right)ds\right). \tag{6.6}$$

By (6.5) and (6.6), one has

$$E\left(\int_0^{\varrho_{\xi'-\varepsilon}}I^{-\rho}(s)\left(\Gamma'(I(s))-\frac{\rho}{2}((\log 2)^2+\sigma^2)\right)ds\right) \leq \frac{i_0^{-\rho}}{\rho}. \tag{6.7}$$

Moreover, if (6.1) is satisfied, then for all sufficiently small $\varepsilon > 0$ such that $I \leq \xi' - \varepsilon$, we have

$$0 < \Gamma'(\xi'-\varepsilon) \leq \Gamma'(I). \tag{6.8}$$

Hence, from (6.7) and (6.8) we have

$$(\xi'-\varepsilon)^{-\rho}\left(\Gamma'(\xi'-\varepsilon)-\frac{\rho}{2}((\log 2)^2+\sigma^2)\right)E\left(\varrho_{\xi'-\varepsilon}\right) \leq \frac{i_0^{-\rho}}{\rho}.$$

Choosing ρ sufficiently small such that

$$0 < \Gamma'(\xi'-\varepsilon)-\frac{\rho}{2}((\log 2)^2+\sigma^2),$$

gives

$$E\left(\varrho_{\xi'-\varepsilon}\right) \leq \frac{i_0^{-\rho}}{\rho\left(\xi'-\varepsilon\right)^{-\rho}\left(\Gamma'(\xi'-\varepsilon)-\frac{\rho}{2}((\log 2)^2+\sigma^2)\right)} < \infty.$$

This completes the proof of Theorem 6.1 □

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7. Examples and computer simulations

7.1. Extinction

Example 1. *Let us illustrate the first extinction condition of Corollary 3.4. Choosing*

$$\mu = 0.014, \quad \beta = 0.31, \quad \gamma = 0.32, \quad \sigma = 0.11, \quad H(z) = |z|, \quad z \in [-1, 1],$$

yields $A = 0.1171$, $B = -0.0657$, and $C = -0.1511 < 0$. Hence, the first extinction condition of Corollary 3.4 is satisfied. The computer simulations in Figure 1, using the Euler Maruyama method (see e.g. [2]), illustrates this result.
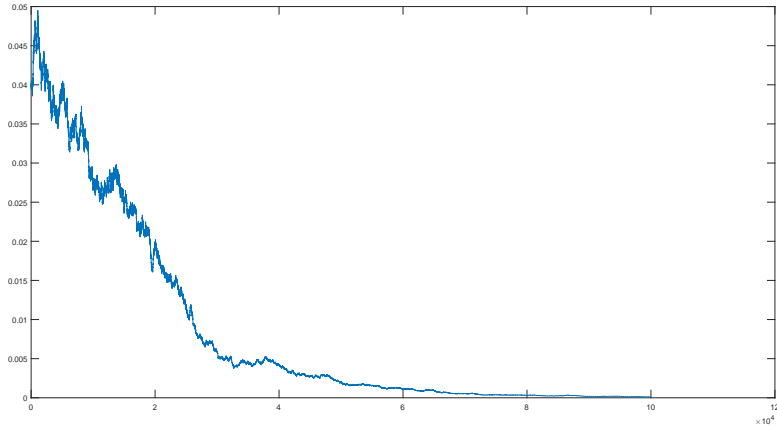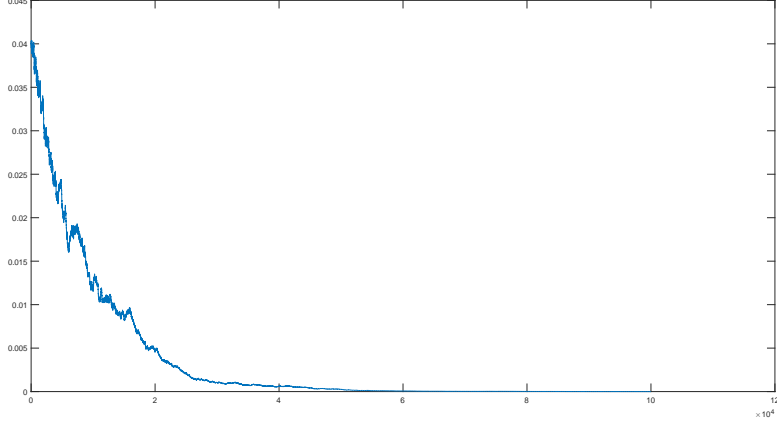

Figure 1: Computer simulation of a single path of $I(t)$ for the SDE model (2.8) with initial condition $I_0 = 0.4$ using the parameter values of Example 1.

Example 2. *To illustrate the second extinction condition of Corollary 3.4, we set*

$$\mu = 0.06, \quad \beta = 0.22, \quad \gamma = 0.3, \quad \sigma = 0.1, \quad H(z) = |z|, \quad z \in [-1, 1], .$$

This implies

$$A = 0.1171, \quad B = 0.0143, \quad C = -0.2111,$$

and

$$\frac{B^2}{4A} + C = -0.2111 < 0.$$

Hence, the second extinction condition of Corollary 3.4 is satisfied. The computer simulations in Figure 2 supports this result clearly.

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Figure 2: Computer simulation of a single path of $I(t)$ for the SDE model (2.8) with initial condition $I_0 = 0.4$ using the parameter values of Example 2.

7.2. Persistence

Example 3. *Let us choose*

$$\mu = 0.017, \quad \beta = 1.3, \quad \gamma = 0.5, \quad \sigma = 0.2, \quad H(z) = 1/(1 + |z|), \ z \in [-1, 1].$$

We compute

$$A = 0.27, \quad B = -0.76, \quad C = 0.513 > 0,$$

and

$$A' = 0.52, \quad B' = -0.26, \quad C' = 0.263 > 0.$$

Hence, the persistence conditions of Theorem 4.1 are satisfied and the corresponding estimates hold with $\xi = 0.5626$ and $\xi' = 0.5038$ that is, $I \in [0.5038, 0.5626]$ infinitely often with probability one, which is clearly illustrated by Figure 3.

8. Conclusion

This work studied a stochastic SIS epidemic model with constant population size driven by a white and Lévy noises. We first proved the existence and uniqueness of their global positive solutions. Then, the long-term behavior of the stochastic SIS epidemic model is investigated. We obtained sufficient criteria for global stability, extinction, persistence and positive recurrence of the epidemic in the population. The stochastic population model
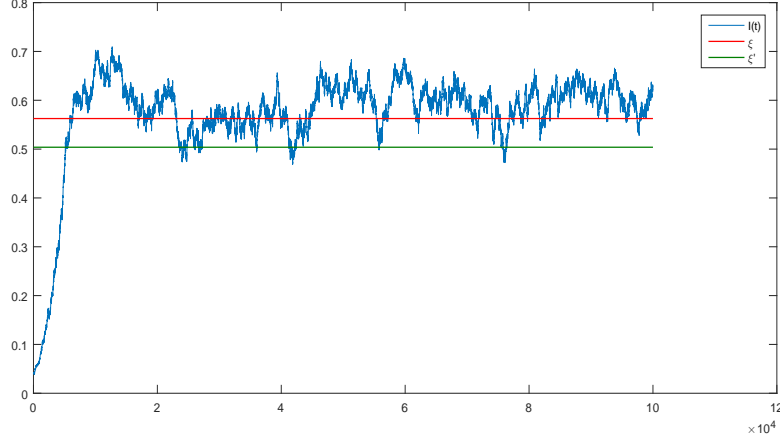
Figure 3: Results of one simulation run of SDE (2.8) with initial condition $I_0 = 0.04$ using the parameter values of Example 3. Here $I \in [0.5038, 0.5626]$ infinitely often with probability one.

(2.1) is one of several possible stochastic versions for the deterministic model (1.1). Further generalizations can be made for model system (2.1). For instance, adding switching component makes the formulation more realistic, but it adds some difficulties in the analysis of the constructed model. The motivation is that the population may suffer sudden changes int their parameters [23, 12]. For example, the transmission rate in winter will be much different to that in summer. Thus, we use a continuous-time Markov chain $r(t)$ to model these sudden changes. Hence, one can study the following stochastic hybrid SIS epidemic model with Poisson jumps.

$$
\begin{cases}
dS(t) = (\mu(r(t)) - \mu(r(t))S(t) - \beta(r(t))S(t)I(t) + \gamma(r(t))I(t))\,dt \\
\quad -\sigma(r(t))S(t)I(t)dB(t) + \int_{\mathbb{Y}} H(z, r(t))S(t^-)I(t^-)\widetilde{N}(dt, dz), \\
dI(t) = (-(\mu(r(t)) + \gamma(r(t)))I(t) + \beta(r(t))S(t)I(t))\,dt \\
\quad +\sigma(r(t))S(t)I(t)dB(t) + \int_{\mathbb{Y}} H(z, r(t))S(t^-)I(t^-)\widetilde{N}(dt, dz),
\end{cases}
\tag{8.1}
$$

We leave the model (8.1) for further study.

Acknowledgments

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