A FORCING NOTION COLLAPSING $\aleph_3$ AND PRESERVING ALL OTHER CARDINALS

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ABSTRACT. I construct, in ZFC, a forcing notion that collapses $\aleph_3$ and preserves all other cardinals. The existence of such a forcing answers a question of Uri Abraham from 1983.

1. INTRODUCTION

It is a trivial fact that, under CH, the collapse of ω_2 to ω_1 with countable conditions, $\mathrm{Coll}(\omega_1, \omega_2)$, is a forcing notion collapsing $\aleph_2$ and preserving all other cardinals. In [1], Abraham addresses, and answers affirmatively, the question whether the existence of a forcing notion with the above property can be proved in ZFC alone.[1] It should be pointed out that Todorčević provides a different proof of the same conclusion in [13], Theorem 6. In fact he shows, in ZFC, that for every nonzero $n < \omega$ there is a partial order $\mathcal{Q}$ collapsing ω_n to ω_1, preserving ω_1, and preserving all cardinals above ω_n. Given a stationary subset S of $[\omega_n]^{\aleph_0}$ of size $\aleph_n$, which always exists by a standard covering argument (cf. Lemma 3.1), and a bijection $i : \omega_n \longrightarrow S$, $\mathcal{Q}$ is the set, ordered by reverse inclusion, of all finite subsets of S linearly ordered by the relation $x < y$ iff $i(x) \in y$.

For a set X of ordinals, $\mathrm{Add}(\omega, X)$ denotes the finite–support product of copies of Cohen forcing indexed by the ordinals in X. Also, given cardinals κ, λ, the set of functions $p \subseteq \kappa \times \lambda$ such that $|p| < \kappa$, ordered by reverse inclusion, is denoted by $\mathrm{Coll}(\kappa, \lambda)$. Abraham's construction in ZFC of a forcing $\mathcal{P}$ collapsing exactly $\aleph_2$ proceeds in the following way. First he fixes a subset $A \subseteq \omega_2$ such that $\omega_2^{L[A]} = \omega_2$. His forcing $\mathcal{P}$ is the iteration $\mathrm{Add}(\omega, \omega_1) * \mathrm{Coll}(\omega_1, \omega_2)^{L[A][\dot{G}]}$, where of course $\mathrm{Coll}(\omega_1, \omega_2)^{L[A][\dot{G}]}$ denotes $\mathrm{Coll}(\omega_1, \omega_2)$ as computed in the inner model $L[A][\dot{G}]$. This makes sense as $\mathrm{Add}(\omega, \omega_1)$ is in fact in $L[A]$ and every $\mathrm{Add}(\omega, \omega_1)$–generic filter over $\mathbf{V}$ is of course generic over $L[A]$. It is immediate to check that $\mathcal{P}$ collapses ω_2 and preserves all higher cardinals. What needs some argument is to show that $\mathcal{P}$

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[1]Note that $\mathrm{Coll}(\omega_1, \omega_2)$ collapses $2^{\aleph_0}$ to ω_1 if CH fails.

preserves ω_1, and this is where the Cohen reals come into play. For this one proves that $\mathrm{Coll}(\omega_1,\omega_2)^{L[A][\dot{G}]}$ is σ–distributive in $\mathbf{V}^{\mathrm{Add}(\omega,\omega_1)}$. ($\mathrm{Coll}(\omega_1,\omega_2)^{L[A][\dot{G}]}$ is certainly σ–directed closed in $L[A]^{\mathrm{Add}(\omega,\omega_1)}$, but no longer so in $\mathbf{V}^{\mathrm{Add}(\omega,\omega_1)}$ in general.) Given an $\mathrm{Add}(\omega,\omega_1)$–generic G, a condition $p\in \mathrm{Coll}(\omega_1,\omega_2)^{L[A][G]}$, and a $\mathrm{Coll}(\omega_1,\omega_2)^{L[A][G]}$–name $\dot{f}$ in $\mathbf{V}[G]$ for an ω–sequence of ordinals, one finds a certain descending sequence $(p_n)_{n<\omega}\in L[A][G]$ of conditions in $\mathrm{Coll}(\omega_1,\omega_2)^{L[A][G]}$ in such a way that $\bigcup_n p_n$ happens to decide $\dot{f}(m)$ for every $m<\omega$. The Cohen reals are used to guide the construction of $(p_n)_n$. The way $(p_n)_n$ is constructed ensures, by a density argument, that for every $m<\omega$ there is some p_n deciding $\dot{f}(m)$. Of course $L[A][G]$ does not know about the fact that $\bigcup_n p_n$ decides $\dot{f}(m)$ for all m as $\dot{f}$ is not even in $L[A][G]$, but $\mathbf{V}[G]$ does.

In [1], Abraham asks if this result can be extended to higher cardinals; in particular, he asks whether it is true, in ZFC, that there is a forcing notion collapsing $\aleph_3$ and preserving all other cardinals. In this paper I answer this question, affirmatively, by proving the following theorem.

Theorem 1.1. *(*ZFC*) There is a partial order $\mathcal{P}$ with the following properties.*

(1) *$\mathcal{P}$ collapses $\aleph_3$.*
(2) *$\mathcal{P}$ preserves all cardinals above $\aleph_3$.*
(3) *$\mathcal{P}$ is $<\omega_2$–distributive.*

I will actually give two proofs of Therem 1.1 for the reason I will describe next.

My original proof of the above result was in two steps, as follows:

(1) In a first step, one proves that there is a cardinal–preserving forcing adding a set $A\subseteq\omega_3$ such that ω_3 computes ω_3 correctly and such that the collection of internally club[2] elementary submodels of $H(\omega_3)^{L[A]}$ in $L[A]$ is stationary in $\mathbf{V}$.
(2) Working in a forcing extension as given by (1), one can extend Abraham's construction, using a natural forcing $\mathrm{Add}_{\mathbb{B}}(\omega_1)$ for adding $\aleph_1$–many mutually generic Baumgartner clubs – rather than $\mathrm{Add}(\omega,\omega_1)$ – in order to predict the relevant objects.

Subsequently, Veličković observed that part (2) can be replaced by a significantly simpler argument using Neeman's forcing with chains of models of two types (countable and internally club of size $\aleph_1$) over an extension as given by (1). Thus, if my original proof of Theorem 1.1

[2] I will properly introduce all undefined notions soon.

was a natural extension of Abraham's proof of the main result from [1], Veličković's modification extends Todorčević's proof of that result.

Even if Veličković's proof is simpler, I think the proof strategy in (2) is itself of sufficient interest and may found further applications. For this reason, I have opted for giving both proofs (first the original proof and then Veličković's). The reader who is interested only in the result may of course want to skip the first proof of Theorem 1.1, after Section 3, and read only the second proof.

In the next section I will lay the ground towards the first proof. After introducing the forcing $\mathrm{Add}_{\mathbb{B}}(X)$, for a set X of ordinals, and giving its main properties (in Subsection 2.1), I give a sketch of how to lift Abraham's construction using $\mathrm{Add}_{\mathbb{B}}(\omega_1)$ (in Subsection 2.2). In Section 3, I briefly discuss covering properties for inner models, and show how to always find a cardinal–preserving forcing adding a suitable partial square–sequence on ω_2; it will then follow that in the generic extension there is a subset A of ω_3 such that $\omega_3^{L[A]} = \omega_3$ and such the suitable form of covering holds for $H(\omega_3)^{L[A]}$. This is part (1) of both proofs of Theorem 1.1 referred to above. In Section 4, and building on the construction in Section 3, I give the two proofs of Theorem 1.1. Section 5 is a short section addressing a question of a similar flavour as Abraham's question, regarding the possibility that CH fails in a suitably absolute way, and I conclude the paper with some natural questions in Section 6.

Much of the notation in this paper follows the set–theoretic standards set forth in [5] or [7]. In particular, given an isomorphism $\varphi : \mathbb{P} \longrightarrow \mathbb{Q}$ between forcing notions and a $\mathbb{P}$–name $\dot{x}$, $\hat{\varphi}(\dot{x})$ denotes the $\mathbb{Q}$–name given by $\{(\varphi(p), \hat{\varphi}(\dot{y})) : (p, \dot{y}) \in \dot{x}\}$. If κ is a cardinal and α is an ordinal, $\kappa^{+\alpha}$ denotes the α–th cardinal λ such that $\lambda \geq \kappa$; in other words, if $\kappa = \aleph_\gamma$, then $\kappa^{+\alpha} = \aleph_{\gamma+\alpha}$. If $\kappa < \lambda$ are infinite regular cardinals, $S^\lambda_\kappa = \{\alpha < \lambda : \mathrm{cf}(\alpha) = \kappa\}$.

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2. Approaching Abraham's question

One first naive approach towards answering Abraham's question affirmatively would be to try to push his construction one cardinal up, i.e., replacing everywhere ω_1 and ω_2 by ω_2 and ω_3, respectively. This will not work in general, though; for one thing, even replacing Cohen forcing by the version, in a suitable $L[A]$, of the forcing for adding a Cohen subset of ω_1 might well collapse cardinals. Thus, it seems that one will need a gentler way to add (many) subsets of ω_1 if this approach is to work.

2.1. **Adding many Baumgartner clubs.** Let $\mathbb{B}$ denote Baumgartner's forcing from [3] for adding a club by finite approximations: $\mathbb{B}$ is the set, ordered by reverse inclusion, of all finite functions $p \subseteq \omega_1 \times \omega_1$ which can be extended to a strictly increasing and continuous function $F : \omega_1 \longrightarrow \omega_1$. $\mathbb{B}$ canonically adds a new club of ω_1, which I will call a Baumgartner club. The following definition is from [2].

Definition 2.1. Let X be a set of ordinals. $\mathrm{Add}_{\mathbb{B}}(X)$ is the following forcing notion: Conditions in $\mathrm{Add}_{\mathbb{B}}(X)$ are pairs of the form $p = (f, \mathcal{F})$ with the following properties.

(1) f is a finite function with $\mathrm{dom}(f) \subseteq X$ and such that $f(\alpha) \in \mathbb{B}$ for every $\alpha \in \mathrm{dom}(f)$.
(2) $\mathcal{F}$ is a finite function with $\mathrm{dom}(\mathcal{F}) \subseteq \omega_1$ such that for every $\delta \in \mathrm{dom}(\mathcal{F})$,
  (a) δ is an indecomposable ordinal,
  (b) $\mathcal{F}(\delta)$ is a countable subset of X,
  (c) $\delta \in \mathrm{dom}(f(\alpha))$ and $f(\alpha)(\delta) = \delta$ for all $\alpha \in \mathrm{dom}(f) \cap \mathcal{F}(\delta)$, and
  (d) $\mathrm{ot}(\mathcal{F}(\delta')) < \delta$ for every $\delta' \in \mathrm{dom}(\mathcal{F} \restriction \delta)$.

Given $\mathrm{Add}_{\mathbb{B}}(X)$ conditions $(f_0, \mathcal{F}_0)$, $(f_1, \mathcal{F}_1)$, $(f_1, \mathcal{F}_1)$ extends $(f_0, \mathcal{F}_0)$ iff

- $\mathrm{dom}(f_0) \subseteq \mathrm{dom}(f_1)$ and $f_0(\alpha) \subseteq f_1(\alpha)$ for every $\alpha \in \mathrm{dom}(f_0)$, and
- $\mathrm{dom}(\mathcal{F}_0) \subseteq \mathrm{dom}(\mathcal{F}_1)$ and $\mathcal{F}_0(\delta) \subseteq \mathcal{F}_1(\delta)$ for every $\delta \in \mathrm{dom}(\mathcal{F}_0)$.

Given a set X of ordinals, $\mathrm{Add}_{\mathbb{B}}(X)$ can be seen as a particularly simple finite–support product of copies of $\mathbb{B}$ incorporating side conditions – these are the $\mathcal{F}$'s in the conditions. $\mathrm{Add}_{\mathbb{B}}(X)$ is designed to add mutually generic Baumgartner clubs indexed by the ordinals in X while preserving all cardinals.[3] In fact, one obtains Proposition 2.2 by arguments which are either trivial or essentially contained in [2].

Given two functions $\mathcal{F}$ and $\mathcal{G}$, let $\mathcal{F} \oplus \mathcal{G}$ denote the function $\mathcal{H}$ with domain $\mathrm{dom}(\mathcal{F}) \cup \mathrm{dom}(\mathcal{G})$ such that

[3] It is easy to see that both the finite–support product and the full–support product of countably many copies of $\mathbb{B}$ collapse ω_1.

- $\mathcal{H}(x) = \mathcal{F}(x)$ for every $x \in \mathrm{dom}(\mathcal{F}) \setminus \mathrm{dom}(\mathcal{G})$,
- $\mathcal{H}(x) = \mathcal{G}(x)$ for every $x \in \mathrm{dom}(\mathcal{G}) \setminus \mathrm{dom}(\mathcal{F})$, and
- $\mathcal{H}(x) = \mathcal{F}(x) \cup \mathcal{G}(x)$ for every $x \in \mathrm{dom}(\mathcal{F}) \cap \mathrm{dom}(\mathcal{G})$.

Also, given a set X, let δ_X denote $X \cap \delta$ in case $X \cap \delta \in \omega_1$.

The following proposition enumerates the main properties of $\mathrm{Add}_{\mathbb{B}}(X)$ relevant to us here.

Proposition 2.2. *Let $\mathbf{W}$ be an inner model such that $\omega_1^{\mathbf{W}} = \omega_1$ and let $X \in \mathbf{W}$ be a set of ordinals. Then the following holds.*

(1) *For every $\mathrm{Add}_{\mathbb{B}}(X)^{\mathbf{W}}$–generic G over $\mathbf{V}$ and every $\alpha \in X$,*
$$\bigcup \{f(\alpha) \,:\, (f, \mathcal{F}) \in G \text{ for some } \mathcal{F},\ \alpha \in \mathrm{dom}(f)\}$$
is the strictly increasing enumerating function of a Baumgartner club over $\mathbf{V}$.

(2) *If every countable subset of X in $\mathbf{V}$ is included in a countable set in $\mathbf{W}$, then $\mathrm{Add}_{\mathbb{B}}(X)^{\mathbf{W}}$ is proper in $\mathbf{V}$. In fact, in that case, given any cardinal θ, any countable elementary substructure N of $H(\theta)$ such that $\mathrm{Add}_{\mathbb{B}}(X)^{\mathbf{W}} \in N$, and any $(f, \mathcal{F}) \in \mathrm{Add}_{\mathbb{B}}(X)^{\mathbf{W}} \cap N$, if f' is the function with domain $\mathrm{dom}(f)$ such that*
$$f'(\alpha) = f(\alpha) \cup \{\langle \delta_N, \delta_N \rangle\}$$
for all $\alpha \in \mathrm{dom}(f)$ and $Y \in [X]^{\aleph_0} \cap \mathbf{W}$ is such that $N \cap X \subseteq Y$, then the pair $(f', \mathcal{F} \cup \{\langle \delta_N, Y \rangle\})$ is, in $\mathbf{V}$, an $(N, \mathrm{Add}_{\mathbb{B}}(X)^{\mathbf{W}})$–generic extension of $(f, \mathcal{F})$.

(3) *In $\mathbf{V}$, $\mathrm{Add}_{\mathbb{B}}(X)^{\mathbf{W}}$ has the $\aleph_2$–c.c.*

(4) *Let $X' \in \mathbf{W}$ be a set of ordinals and let $g : X \longrightarrow X'$ be an order–preserving bijection in $\mathbf{W}$. Then the function sending $(f, \mathcal{F}) \in \mathrm{Add}_{\mathbb{B}}(X)^{\mathbf{W}}$ to*
$$(\{\langle g(\alpha), f(\alpha) \rangle \,:\, \alpha \in \mathrm{dom}(f)\}, \{\langle \delta, g``\mathcal{F}(\delta) \rangle \,:\, \delta \in \mathrm{dom}(\mathcal{F})\})$$
is an isomorphism between $\mathrm{Add}_{\mathbb{B}}(X)^{\mathbf{W}}$ and $\mathrm{Add}_{\mathbb{B}}(X')^{\mathbf{W}}$.

(5) *For every partition (X_0, X_1) of X in $\mathbf{W}$, $\mathrm{Add}_{\mathbb{B}}(X)^{\mathbf{W}}$ can be naturally represented as the product $\mathrm{Add}_{\mathbb{B}}(X_0)^{\mathbf{W}} \times \mathrm{Add}_{\mathbb{B}}(X_1)^{\mathbf{W}}$; in fact, the function sending a condition $((f_0, \mathcal{F}_0), (f_1, \mathcal{F}_1)) \in \mathrm{Add}_{\mathbb{B}}(X_0)^{\mathbf{W}} \times \mathrm{Add}_{\mathbb{B}}(X_1)^{\mathbf{W}}$ to $(f_0 \cup f_1, \mathcal{F}_0 \oplus \mathcal{F}_1)$ is an isomorphism between $\mathrm{Add}_{\mathbb{B}}(X_0)^{\mathbf{W}} \times \mathrm{Add}_{\mathbb{B}}(X_1)^{\mathbf{W}}$ and $\mathrm{Add}_{\mathbb{B}}(X)^{\mathbf{W}}$.*

Proof. Only the proof of conclusion (2) is not completely straightforward. For the reader's convenience I am giving a proof of this conclusion suggested by the referee and somewhat simpler than the original proof from [2].

Let $\bar{p} = (\bar{f}, \bar{\mathcal{F}}_1)$ be an extension of $(f', \mathcal{F} \cup \{\langle \delta_N, Y \rangle\})$ in $\mathrm{Add}_{\mathbb{B}}(X)$ and let $D \in N$ be a dense subset of $\mathrm{Add}_{\mathbb{B}}(X)$ in N. We want to find a condition in $D \cap N$ compatible with $\bar{p}$.

We define a function g with domain $\mathrm{dom}(\bar{f}) \cap N$, by letting $g(\alpha) = \bar{f}(\alpha) \restriction \delta_N$ for all $\alpha \in \mathrm{dom}(\bar{f}) \cap N$. We define also $p = (g, \bar{\mathcal{F}} \cap N)$. Then p is clearly a condition in $\mathrm{Add}_{\mathbb{B}}(X)$ weaker than p_1. If we take an arbitrary extension $p^\dagger = (f^\dagger, \mathcal{F}^\dagger)$ of p in $D \cap N$ and ask if $p^\dagger$ is compatible with $\bar{p}$ (as we would like it to be), there could be two problems with compatibility. The first possible problem is that for some $\delta \in E = (\mathrm{dom}(\bar{\mathcal{F}}) \cap N) \setminus \mathrm{dom}(\mathcal{F}^\dagger)$, for some $\delta' < \delta$ in the domain of $\mathcal{F}^\dagger$ it is not the case that $\mathrm{ot}(\mathcal{F}^\dagger(\delta')) < \delta$. The second possible problem is that for some $\delta \in \mathrm{dom}(\bar{\mathcal{F}}) \cap N$ there is some $\alpha \in \mathrm{dom}(f^\dagger) \cap \bar{\mathcal{F}}(\delta)$ such that $f^\dagger(\alpha)(\delta) \neq \delta$. (Note that this may only happen for $\alpha \in \mathrm{dom}(f^\dagger) \setminus \mathrm{dom}(g)$.)

In order to handle the first problem, in the above definition of p we replace $\bar{\mathcal{F}} \cap N$ by $\{\langle \delta, \emptyset \rangle : \delta \in \mathrm{dom}(\bar{\mathcal{F}} \restriction \delta_N)\}$, i.e., we let $p = (g, \{\langle \delta, \emptyset \rangle : \delta \in \mathrm{dom}(\bar{\mathcal{F}} \restriction \delta_N)\})$. For the second problem we work as follows.

Claim 2.3. *For any countable set E there is a condition $p_E = (f_E, \mathcal{F}_E)$, $p_E \in D$, such that p_E extends p and $E \cap (\mathrm{dom}(\mathcal{F}_E) \setminus \mathrm{dom}(g)) = \emptyset$.*

Proof. Otherwise, by elementarity of N we can take a counterexample $E \in N$. Then $E \subseteq N$ since E is countable. But then $\bar{p}$ brings about a contradiction since $N \cap (\mathrm{dom}(\bar{f}) \setminus \mathrm{dom}(g)) = \emptyset$. □

Using the claim enables one to construct a $\subseteq$–increasing sequence $(E_\zeta)_{\zeta<\omega_1} \in N$ of countable subsets of X such that for all ζ there is some extension $p_\zeta = (f_\zeta, \mathcal{F}_\zeta)$ of p, $p_\zeta \in D$, such that $E_\zeta \cap (\mathrm{dom}(f_\zeta) \setminus \mathrm{dom}(g)) = \emptyset$ and $\mathrm{dom}(f_\zeta) \subseteq E_{\zeta+1}$. We may of course assume that $p_\zeta \in N$ for each $\zeta < \delta_N$.

Define $Z = \bigcup\{\bar{\mathcal{F}}(\delta) : \delta \in \mathrm{dom}(\bar{\mathcal{F}}), \delta < \delta_N\}$. Then $\mathrm{ot}(Z) < \delta$ and so there is some $\zeta < \omega_1$ such that $Z \cap (E_{\zeta+1} \setminus E_\zeta) = \emptyset$. But then $p_\zeta \in D \cap N$ is readily seen to be compatible with $\bar{p}$ since no member of $\mathrm{dom}(f_\zeta) \setminus \mathrm{dom}(g)$ can be trapped by Z.

□

It is worth pointing out that the above proof can be slightly simplified in the case $X \subseteq \omega_1$ (which in fact is the only case we will need here).

Given an ordinal $\alpha < \omega_1$ and a generic filter G for $\mathrm{Add}_{\mathbb{B}}(\omega_1)$, I will write $F^G(\alpha)$ to denote the enumerating function of the α-th Baumgartner club adjoined by G, i.e., the function $\bigcup\{f(\alpha) : (f, \mathcal{F}) \in G \text{ for some } \mathcal{F}, \alpha \in \mathrm{dom}(f)\}$.

2.2. **Lifting Abraham's construction using** $\mathrm{Add}_{\mathbb{B}}(\omega_1)$**.** There is a clear analogy between the properties of $\mathrm{Add}_{\mathbb{B}}(X)$ listed in Proposition 2.2 and corresponding properties of $\mathrm{Add}(\omega, X)$, and it turns out that this analogy suffices to make $\mathrm{Add}_{\mathbb{B}}(\omega_1)$ a suitable candidate to be used in place of $\mathrm{Add}(\omega, \omega_1)$ in the approach to Abraham's problem sketched before,[4] *provided* one can find an inner model $L[A]$, for $A \subseteq \omega_3$, such that $\omega_3^{L[A]} = \omega_3$ and such that a suitable form of covering with respect to internally club models holds for $L[A]$. Indeed, given the existence of such an inner model $L[A]$, one can construct with the help of $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$ a forcing notion which collapses $\aleph_3$ and preserves all other cardinals. The forcing is $\mathcal{P} = \mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]} * \mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}]}$. It is straightforward to show that $\mathcal{P}$ collapses $\aleph_3$ and preserves all higher cardinals. Similarly as in Abraham's proof, working in an extension $\mathbf{V}[G]$ of $\mathbf{V}$ by $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$ one proves that $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][G]}$ preserves ω_1 and ω_2, and in fact that it is $<\omega_2$–distributive, and for this one uses the Baumgartner clubs coming from the generic G in order to guide, in $\mathbf{V}[G]$, the construction, taking place in $L[A][G]$, of a certain ω_1–sequence of conditions in $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][G]}$ which will end up deciding all values $\dot{F}(i)$ of a fixed $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}]}$–name $\dot{F}$ in $\mathbf{V}[G]$ for an ω_1–sequence of ordinals.

In order to answer Abraham's question, the missing ingredient here is of course the inner model $L[A]$. However, one can show that the existence of such an $L[A]$ can always be forced while preserving cardinals. In fact, the existence of such an $L[A]$ follows from the existence of a partial square sequence on ω_2, and one can prove that there is always a forcing preserving cardinals and adding such an object. This forcing is a natural "small" version of Neeman's forcing $\mathbb{S}$quare from [10]. Combining these two constructions, it follows that ZFC proves the existence of a partial order collapsing ω_3 and preserving all other cardinals.

It is worth pointing out a significant dissimilarity between the construction in [1] and the (second part of the) present construction. In Abraham's proof, the verification of the σ–directedness in $\mathbf{V}^{\mathrm{Add}(\omega,\omega_1)}$ of his $\mathrm{Coll}(\omega_1, \omega_2)^{L[A][\dot{G}]}$ makes use of a certain descending sequence $(p_n)_n$ of conditions in $\mathrm{Coll}(\omega_1, \omega_2)^{L[A][\dot{G}]}$, as mentioned before, where the conditions p_n are picked from within a suitable countable structure N (and where the construction is guided by any Cohen real r which is generic over some intermediate extension $\mathbf{V}'$ of $\mathbf{V}$ such that $N \in \mathbf{V}'$). In the present construction, on the other hand, one cannot just pick an

[4]It in fact suffices to take $\mathrm{Add}_{\mathbb{B}}(\omega_1)$ rather than $\mathrm{Add}_{\mathbb{B}}(\omega_2)$.

$\aleph_1$–sized structure N and do a similar thing. The problem is that we would like to pick the members of our descending ω_1–sequence of conditions from within N, but if we want to do that we will typically get stuck at stages of countable cofinality since N cannot be closed under ω–sequences if CH fails. Instead, one has to consider a certain $\in$–chain $(M_\nu)_{\nu<\omega_1}$ of countable structures in $L[A]$ – which we will be able to find thanks to the suitable form of covering for $L[A]$ – and argue in a somewhat subtler way. The disagreement at this point between Abraham's construction and the present construction is the reason why here one can do with $\mathrm{Add}_{\mathbb{B}}(\omega_1)$ rather than $\mathrm{Add}_{\mathbb{B}}(\omega_2)$. Also, even if we are working here with these countable structures M_ν, it would not suffice to add reals in the first step of the construction rather than subsets of ω_1, and the reason is that we will need our construction to be able to point, at any given stage, to M_ν for some arbitrarily high $\nu < \omega_1$.

3. Covering lemmas for suitable inner models and forcing partial square while preserving cardinals

Let X be a set and λ a cardinal. Recall that $D \subseteq [X]^\lambda$ is said to be a club of $[X]^\lambda$ iff D is cofinal in $([X]^\lambda, \subseteq)$ and the union of any $\subseteq$–increasing λ–sequence of members of D is in D, and that a set $S \subseteq [X]^\lambda$ is stationary if and only if $S \cap D \neq \emptyset$ for every club D of $[X]^\lambda$. A proof of the following standard covering lemma appears in [1].

Lemma 3.1. *Let $\mathbf{W} \subseteq \mathbf{V}$ be a transitive inner model of $\mathbf{V}$. Let κ be a cardinal in $\mathbf{V}$, let $n < \omega$, and suppose $(\kappa^{+n})^{\mathbf{W}} = (\kappa^{+n})^{\mathbf{V}}$. Then $(\kappa^{+i})^{\mathbf{W}} = \kappa^{+i}$ for every $i \leq n$, and for every $X \in \mathbf{W}$ such that $|X|^{\mathbf{W}} = \kappa^{+n}$ and every $i < n$, every subset of X of size κ^{+i} is included in a subset of size κ^{+i} in $\mathbf{W}$ and every club D of $[X]^{\kappa^{+i}}$ in $\mathbf{W}$ is a stationary subset of $[X]^{\kappa^{+i}}$ in $\mathbf{V}$.*

A simple observation that will be used repeatedly in this paper is that if κ is an infinite cardinal and $A \subseteq \kappa$, then $H(\kappa)^{L[A]}$ has size κ in $L[A]$ and in fact $H(\kappa)^{L[A]} = L_\kappa[A]$ since every bounded subset of κ in $L[A]$ is in $L_\alpha[A]$ for some $\alpha < \kappa$.

The above covering lemma suffices for the purposes of [1] but does not seem to be enough for us here. Recall that an infinite set X is internally club (IC, for short) iff $X = \bigcup_{i<\mathrm{cf}(|X|)} X_i$ where, for each i, $|X_i| < |X|$ and $(X_j)_{j<i} \in X$. What we seem to need in our situation is the existence of an inner model of the form $L[A]$, for $A \subseteq \omega_3$, satisfying the following enhanced form of Lemma 3.1: $L[A]$ computes ω_3 correctly (and therefore it computes also ω_1 and ω_2 correctly) and the set of $X \in [H(\omega_3)^{L[A]}]^{\aleph_1}$ in $L[A]$ which are, in $L[A]$, internally approachable

is stationary in $\mathbf{V}$. I will work now towards showing that it is always possible to force, without collapsing cardinals, in such a way that in the extension there is such an $A \subseteq \omega_3$.

A club–sequence is a sequence $\vec{C} = (C_\delta : \delta \in S)$, for S a set of ordinals, such that each C_δ is a club of δ. A club–sequence $\vec{C}$ is coherent if and only if for every $\delta \in \mathrm{dom}(\vec{C})$ and every limit point ϵ of C_δ, $\epsilon \in \mathrm{dom}(\vec{C})$ and $C_\epsilon = C_\delta \cap \epsilon$.

The following result is due to Shelah (see [11], Lemma 4.4 (3)).

Proposition 3.2. *(Shelah) Suppose $\kappa < \lambda$ are infinite regular cardinals. Then there is decomposition $\lambda^+ = \bigcup_{i<\lambda} S_i$ such that for every $i < \lambda$ there is a coherent club–sequence $(C_\delta : \delta \in S_i)$ such that* $\mathrm{ot}(C_\delta) = \kappa$ *for every $\delta \in S_i$ of cofinality κ.*

Given an infinite cardinal λ and a regular cardinal $\kappa \leq \lambda$, I will call a coherent club–sequence $(C_\delta : \delta \in S)$ such that

- $S \subseteq \lambda^+$, and
- $\{\delta \in S \cap S^{\lambda^+}_\kappa : \mathrm{ot}(C_\delta) = \kappa\}$ is a stationary subset of λ^+

a *partial square sequence on λ^+ concentrating on cofinality κ*. I will also refer to such an object as a $\square^{\mathsf{p}}_\lambda$–sequence concentrating on cofinality κ. If $\kappa = \lambda$, I will just say *partial square sequence on λ^+* and $\square^{\mathsf{p}}_\lambda$–sequence.

The following lemma can now be established.

Lemma 3.3. *Suppose there is a $\square^{\mathsf{p}}_{\omega_1}$–sequence $\vec{C} = (C_\delta : \delta \in S)$. Then there is $A \subseteq \omega_3$ with the following properties.*

(1) $\omega_3^{L[A]} = \omega_3$
(2) *For every cardinal $\theta > \omega_3$, the set of $N \preccurlyeq H(\theta)$ such that*
  - $|N| = \aleph_1$,
  - $N \cap H(\omega_3)^{L[A]} \in L[A]$, *and*
  - $N \cap H(\omega_3)^{L[A]}$ *is IC in* $L[A]$

  is a stationary subset of $[H(\theta)]^{\aleph_1}$.

Proof. It follows from Proposition 3.2 for $\kappa = \omega_1$ and $\lambda = \omega_2$ that we may fix a $\square^{\mathsf{p}}_{\omega_2}$–sequence $\vec{D} = (D_\delta : \delta \in T)$ concentrating on cofinality ω_1. Let $e_\nu : \omega \longrightarrow \nu$ be a bijection for each $\nu \in [\omega, \omega_1)$, $f_\alpha : \omega_1 \longrightarrow \alpha$ a bijection for each $\alpha \in [\omega_1, \omega_2)$, and $g_\beta : \omega_2 \longrightarrow \beta$ a bijection for each $\beta \in [\omega_2, \omega_3)$. Let $A \subseteq \omega_3$ code $\vec{C}$, $\vec{D}$, $\vec{e} = (e_\nu : \nu \in [\omega, \omega_1))$, $\vec{f} = (f_\alpha : \alpha \in [\omega_1, \omega_2))$ and $\vec{g} = (g_\beta : \beta \in [\omega_2, \omega_3))$ in some canonical way.

Conclusion (1) clearly holds for $L[A]$. Let $i : \omega_3 \longrightarrow H(\omega_3)^{L[A]} = L_{\omega_3}[A]$ be a bijection in $L[A]$. We may assume that i is the initial segment of length ω_3 of the canonical well–order $<_{L[A]}$ of $L[A]$. In order to verify (2), let $F : [H(\theta)]^{<\omega} \longrightarrow H(\theta)$ be a function in $\mathbf{V}$. Our aim is to find some N that is closed under F for which the properties of (2) hold. Let $(Q_\sigma)_{\sigma<\omega_3}$ be a $\subseteq$–continuous $\subseteq$–increasing sequence of elementary submodels of $H(\theta)$ of cardinality $\aleph_2$ containing A, closed under F, and such that for all σ, $\beta_\sigma = Q_\sigma \cap \omega_3 \in \omega_3$.

Let σ be of cofinality ω_1 such that $\beta = \beta_\sigma \in T$ and $\mathrm{ot}(D_\beta) = \omega_1$. Let also $\alpha \in S$ be such that $\mathrm{ot}(C_\alpha) = \omega_1$ and such that there is an elementary substructure N of Q_σ of size $\aleph_1$ containing A, closed under F, and such that

- $N \cap \omega_3 = g_\beta``\alpha$ and
- $D_\beta \subseteq g_\beta``\alpha$

To see that α can be found, note that since Q_σ is closed under F and contains A, there is a club E_0 of $[Q_\sigma]^{\aleph_1}$ consisting of $N \preccurlyeq Q_\sigma$ closed under F and such that $A \in N$ and $D_\beta \subseteq N$. We have that $E_1 = \{N \cap \omega_3 : N \in E_1\}$ contains a club E_2 of $[\beta]^{\aleph_1}$ and that $E_3 = \{g_\beta``\alpha : \alpha < \omega_2\}$ is a club of $[\beta]^{\aleph_1}$, and of course club–many members of $E_2 \cap E_3$ are ordinals. Let $(\rho_\alpha)_{\alpha<\omega_2}$ be the strictly increasing enumeration of the club C of ordinals in $E_2 \cap E_3$. Since the set of $\alpha \in S$ such that $\mathrm{ot}(C_\alpha) = \omega_1$ is stationary, we may find such an α in C. But then α is as required.

Let $M = i``(g_\beta``\alpha)$. Since $M \in L[A]$, $F``[N]^{<\omega} \subseteq N$, and $M = N \cap H(\omega_3)^{L[A]}$, it suffices to show that M is IC in $L[A]$.

For this, let $(C_\alpha(\xi))_{\xi<\omega_1}$ and $(D_\beta(\xi))_{\xi<\omega_1}$ be the strictly increasing enumerations of C_α and D_β, respectively. Let also ZFC* be a suitable fragment of ZFC without the Power set Axiom (in the language for $(L_{\omega_3}[A], \in, A)$). We are going to define now a $\subseteq$–increasing sequence $(M_\nu)_{\nu<\omega_1}$ of countable sets such that $\bigcup_{\nu<\omega_1} M_\nu = M$, together with an increasing sequence $(\xi^\epsilon_{\nu+1})_{\nu+1<\omega_1}$ of countable ordinals for $\epsilon = 0, 1, 2$. For convenience we start with $M_0 = \emptyset$. Suppose $\nu < \omega_1$, $\nu > 0$, and suppose $(M_{\nu'})_{\nu'<\nu}$ and $(\xi^\epsilon_{\nu'+1})_{\nu'+1<\nu}$ have been defined for each $\epsilon < 3$. If ν is a limit ordinal we let $M_\nu = \bigcup_{\nu'<\nu} M_{\nu'}$. If $\nu = \bar\nu + 1$, then we let $(\xi^2_\nu, \xi^1_\nu, \xi^0_\nu)$ be the least triple (ξ^2, ξ^1, ξ^0), in any canonical well–order of $\omega \times \omega_1 \times \omega_1$ fixed throughout, such that

(i) $\xi^\epsilon > \xi^\epsilon_{\nu'+1}$ for all $\nu' < \bar\nu$ and $\epsilon < 3$,
(ii) $J^A_{D_\beta(\xi^2)} \models$ ZFC*, and
(iii) $\{\xi^0_{\bar\nu}, (C_\alpha(\xi))_{\xi\leq\xi^1_{\bar\nu}}, (D_\beta(\xi))_{\xi\leq\xi^2_{\bar\nu}}\} \in M'$, where M' is the canonical Skolem closure of $g_{D_\beta(\xi^2)}``(f_{C_\alpha(\xi^1)}``\xi^0))$ in $J^A_{D_\beta(\xi^2)}$.

By our choice of $(\beta_\sigma)_{\sigma<\omega_3}$, together with the fact that β is a limit of $\{\beta_\sigma : \sigma < \omega_3\}$ of cofinality ω_1 and D_β is a club of β and the fact that $\vec{D}$ is a coherent sequence, we can find some ξ^2 such that $\xi^2 > \xi^2_{\nu'+1}$

for all $\nu' < \bar{\nu}$, $J^A_{D_\beta(\xi^2)} \models \mathrm{ZFC}^*$, and $(D_\beta(\xi))_{\xi \leq \xi^2_{\bar{\nu}}} \in i``(g_{D_\beta(\xi^2)}``\alpha) \preccurlyeq J^A_{D_\beta(\xi^2)}$. Then ξ^1 and ξ^0 can be found easily using the fact that, by the coherence of $\vec{C}$, all objects in (iii) are in $i``(f_{D_\beta(\xi^2)}``\alpha)$. This shows that $(\xi^2_\nu, \xi^1_\nu, \xi^0_\nu)$ exists. We let M_ν be the canonical Skolem closure of $g_{D_\beta(\xi^2_\nu)}``(f_{C_\alpha(\xi^1_\nu)}``\xi^0_\nu))$ in $J^A_{D_\beta(\xi^2_\nu)}$.

Each M_ν is a countable subset of M, and of course $(M_\nu)_{\nu<\omega_1}$ is $\subseteq$–continuous by construction. If $\nu = \bar{\nu}+1$, then using the uniformity of the definition of $(M_{\nu'})_{\nu' \leq \bar{\nu}}$ and of $(\xi^\epsilon_{\nu'+1})_{\nu'+1<\nu}$ (for $\epsilon < 3$), together with the fact that $M_\nu \models \mathrm{ZFC}^*$ contains $\xi^0_{\bar{\nu}}$, $(C_\alpha(\xi))_{\xi \leq \xi^1_{\bar{\nu}}}$ and $(D_\beta(\xi))_{\xi \leq \xi^2_{\bar{\nu}}}$, it is easy to check that $(M_{\nu'})_{\nu' \leq \bar{\nu}} \in M_\nu$. Finally, it is easy to check that $\bigcup_{\nu<\omega_1} M_\nu = M$. $\square$

By a result of Magidor ([8]), it is consistent that there is no partial square sequence on ω_2. On the other hand, the existence of such an object can always be forced while preserving all cardinals:

Lemma 3.4. *There is a partial order $\mathcal{P}_0$ such that*

(1) $|\mathcal{P}_0| = \aleph_2$,
(2) $\mathcal{P}_0$ *preserves* ω_1 *and* ω_2, *and*
(3) $\mathcal{P}_0$ *forces the existence of a* $\square^p_{\omega_1}$*–sequence.*

Proof. This follows from, essentially, analysing the relevant proofs from [10], Section 3. Let $\vec{e} = (e_\alpha : \alpha \in [\omega_1, \omega_2))$ be such that $e_\alpha : \omega_1 \longrightarrow \alpha$ is a bijection for each α. Let $\mathcal{F}$ be a countable set of Skolem functions for $(H(\omega_2), \in, \vec{e})$, and for every $X \subseteq \omega_2$ let $\mathrm{Sk}(X)$ be the closure of X under all functions in $\mathcal{F}$. Let $\mathcal{S}$ consist of all sets of the form $e_\alpha``\nu$, for some uncountable $\alpha < \omega_2$ and some $\nu < \omega_1$, such that $\mathrm{Sk}(e_\alpha``\nu) \cap \omega_2 = e_\alpha``\nu$ and $\omega_1 \cap e_\alpha``\nu = \nu$. Let $\mathcal{T}$ consist of all $\alpha < \omega_2$ such that $\mathrm{Sk}(\alpha) \cap \omega_2 = \alpha$ and such that $e_\alpha``\nu \in \mathrm{Sk}(\alpha)$ for club–many $\nu \in \omega_1$. Borrowing terminology from Neeman's [9] and [10], let us call members of $\mathcal{S} \cup \mathcal{T}$ *nodes*. Strictly speaking, though, these sets are not nodes in Neeman's sense since they are not themselves models of a suitable fragment of ZFC.

Claim 3.5. *The set of countable $N \preccurlyeq H(\omega_2)$ such that $N \cap \omega_2$ is a node is a stationary subset of $[H(\omega_2)]^{\aleph_0}$ and the set of $N \preccurlyeq H(\omega_2)$ such that $|N| = \aleph_1$ and $N \cap \omega_2$ is a node is a stationary subset of $[H(\omega_2)]^{\aleph_1}$.*

Proof. This claim is quite standard but I include a proof for completeness. Let $F : [H(\omega_2)]^{<\omega} \longrightarrow H(\omega_2)$ be a function. For the first part we pick some $\alpha < \omega_2$ such that $F``[\alpha]^{<\omega} \subseteq \alpha$ and then, noting that $\{e_\alpha``\nu : \nu \in [\omega, \omega_1)\}$ is a club of $[\alpha]^{\aleph_0}$, pick some $\nu < \omega_1$ such that $F``[e_\alpha``\nu]^{<\omega} \subseteq e_\alpha``\nu$. For the second part we define a strictly increasing and continuous sequence $(\alpha_\nu)_{\nu<\omega_1}$ of ordinals in ω_2, together with a sequence $(N_\nu)_{\nu<\omega_1}$ of countable elementary submodels of $H(\theta)$, for θ large enough, containing F and such that for all ν, $(e_{\alpha_{\nu'}})_{\nu'\leq\nu} \in N_{\nu+1}$ and $N_{\nu+1}\cap\omega_2 = e_{\alpha_{\nu+1}}``\delta$ for some $\delta < \omega_1$. This is of course possible by the first part. Now it is easy to check that $\alpha = \sup_{\nu<\omega_1}\alpha_\nu$ is a node, as witnessed by the club of $\nu < \omega_1$ such that $e_\alpha``\nu = \bigcup_{\nu'<\nu} e_{\alpha_{\nu'}}``\nu$. □

$\mathcal{P}_0$ will be $\mathbb{S}$quare$^{\mathsf{p}}$, where $\mathbb{S}$quare$^{\mathsf{p}}$ = $\mathbb{S}$quare$^{\mathsf{p}}(\mathcal{S},\mathcal{T})$ is the following rendering of Neeman's forcing $\mathbb{S}$quare from [10], Section 3:

Conditions in $\mathbb{S}$quare$^{\mathsf{p}}$ are pairs (s, c) with the following properties.

1. $s \in \mathbb{P}_{\mathsf{side}}(\mathcal{S},\mathcal{T})$. Here, this means that $s \in [\mathcal{S}\cup\mathcal{T}]^{<\omega}$ is closed under intersections and that there is a (necessarily unique) enumeration $(Q_k)_{k<n}$ of s such that $Q_k \in \mathrm{Sk}(Q_{k+1})$ for all $k+1 < n$.
2. c is a function on s.
3. For each $Q \in s$, $c(Q)$ is an $\in$–linear set of countable nodes from $s \cap Q$.
4. For each $Q \in s$, if $\mathcal{T}$ is cofinal in $\sup(Q\cap\omega_2)$, then $\mathcal{T}$ is cofinal in $\sup(M\cap\omega_2)$ for every $M \in c(Q)$. If $\mathcal{T}$ is bounded in $\sup(Q\cap\omega_2)$, then $c(Q) = \emptyset$.
5. For each $Q \in s$, if $M \in c(Q)$, then $c(M) = c(Q) \cap M$.
6. If $\alpha \in \mathcal{T}\cap s$, $M \in \mathcal{S}\cap s$, $\alpha \in M$, and $\mathcal{T}$ is cofinal in α, then $M \cap \alpha \in c(\alpha)$.

Given $\mathbb{S}$quare$^{\mathsf{p}}$–conditions (s, c) and (s^*, c^*), we say that (s^*, c^*) extends (s, c) iff the following holds.

- (i) $s^* \leq s$ in $\mathbb{P}_{\mathsf{side}}(\mathcal{S},\mathcal{T})$. This means that $s \subseteq s^*$.
- (ii) $c(Q) \subseteq c^*(Q)$ for every $Q \in s$.
- (iii) For $Q \in s$, $R \in c(Q)\cup\{Q\}$, and $P \in c^*(Q)\cap\mathrm{Sk}(R)$, if $\mathrm{Sk}(P) \supseteq c(Q)\cap R \neq \emptyset$, then $\mathrm{Sk}(P) \supseteq s \cap R$.

By the proof of [10], Claim 3.3, the extension relation on $\mathbb{S}$quare$^{\mathsf{p}}$ is transitive. Clearly, $\mathbb{S}$quare$^{\mathsf{p}}$ has size $\aleph_2$. The proof of [10], Lemma 3.4 and Corollary 3.5 establishes the following claim.

Claim 3.6. *Let Q be a node and $(s, c) \in \mathrm{Sk}(Q) \cap \mathbb{S}quare^{\mathsf{p}}$. Then*

- *there is an extension (s', c') of (s, c) such that $Q \in s'$, and*
- *if $(s', c') \in \mathbb{S}quare^{\mathsf{p}}$ is such that $Q \in s'$ and $N \preccurlyeq H((2^{\aleph_1})^+)$ is such that $\vec{e} \in N$ and $N \cap \omega_2 = Q$, then (s', c') is $(N, \mathbb{S}quare^{\mathsf{p}})$–generic.*

By Claims 3.5 and 3.6 it then follows that $\mathbb{S}$quare$^{\mathsf{p}}$ preserves ω_1 and ω_2.

The corresponding form of Claim 3.6 from [10] does not necessarily hold for $\mathbb{S}$quare$^{\mathsf{p}}$, and this is ultimately the reason why $\mathbb{S}$quare$^{\mathsf{p}}$ cannot be used to add a full $\square_{\omega_1}$–sequence rather than just a partial square sequence. However, the following weak form of this claim can be proved easily.

Claim 3.7. *Let $(s, c) \in \mathbb{S}quare^{\mathsf{p}}$, $\alpha \in s \cap \mathcal{T}$, and $\beta \in \mathcal{T} \cap \alpha$ such that $Q \in \mathrm{Sk}(\beta)$ for every $Q \in s \cap \mathrm{Sk}(\alpha)$. Then $(s \cup \{\beta\}, c) \in \mathbb{S}quare^{\mathsf{p}}$.*

Finally, the corresponding version of the proof of [10], Lemma 3.8 using the above Claim 3.7 instead of Claim 3.6 from [10], combined with the present Claims 3.6 and 3.5, establishes that $\mathbb{S}\mathrm{quare}^{\mathsf{p}}$ adds a $\square^{\mathsf{p}}_{\omega_1}$–sequence.

□

4. Collapsing exactly $\aleph_3$

As mentioned in the introduction, I will give two proofs of Theorem 1.1. The first one is the proof that I initially found, using $\mathrm{Add}_{\mathbb{B}}(\omega_1)$. The second proof, due to Veličković, is considerably simpler.

4.1. **First proof of Theorem 1.1.** Let $\mathcal{P}_0$ be a poset as in Lemma 3.4. By Lemma 3.3, in $\mathbf{V}_1 = \mathbf{V}^{\mathcal{P}_0}$ we may fix $A \subseteq \omega_3$ such that $\omega_3^{L[A]} = \omega_3$ and such that for every cardinal $\theta > \omega_3$, the set of $N \preccurlyeq H(\theta)$ such that

- $|N| = \aleph_1$,
- $N \cap H(\omega_3)^{L[A]} \in L[A]$, and
- $N \cap H(\omega_3)^{L[A]}$ is IC in $L[A]$

is a stationary subset of $[H(\theta)]^{\aleph_1}$. Still in $\mathbf{V}_1$, let $\mathcal{P}_1 = \mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]} * \dot{\mathcal{Q}}$, where $\dot{\mathcal{Q}}$ is, in $L[A]^{\mathrm{Add}_{\mathbb{B}}(\omega_1)}$, a name for $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}]}$. Our poset will be $\mathcal{P} = \mathcal{P}_0 * \dot{\mathcal{P}}_1$, where $\dot{\mathcal{P}}_1$ is a $\mathcal{P}_0$–name for $\mathcal{P}_1$.

By a standard density argument it is clear that $\mathcal{P}$ collapses ω_3. Since $\mathcal{P}_0$ preserves cardinals, it will be enough to show that in $\mathbf{V}^{\mathcal{P}_0}$, $\mathcal{P}_1$ has a dense set of size $\aleph_3$ and preserves both ω_1 and ω_2. Let us work from now on in $\mathbf{V}_1 = \mathbf{V}^{\mathcal{P}_0}$.

Claim 4.1. *$\mathcal{P}_1$ has a dense set of size $\aleph_3$.*

Proof. Since $\aleph_3^{\aleph_2} = \aleph_3$ holds in $L[A]$, $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$ is, in $L[A]$, a forcing notion of size at most $\aleph_3$ and, by Proposition 2.2 (3), with the $\aleph_2$–c.c., which in turn implies, by $(\aleph_3^{\aleph_1})^{L[A]} = \aleph_3$, that there are at most $\aleph_3$–many antichains of $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$ in $L[A]$. Working in an extension of $L[A]$ by $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$, every condition in $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}]}$ can be canonically coded by a subset of α, for some $\alpha < \omega_3$, and by the above there are at most $\aleph_3^{\aleph_2} = \aleph_3$ many nice $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$–names in $L[A]$ for subsets of any such given α. Finally, the set $\mathcal{P}_2$ of $(p, \dot{q}) \in \mathcal{P}_1$ such that $\dot{q}$ is a canonical name for a condition in $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}]}$ coded by a nice $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$–name for a bounded subset of ω_3 is dense in $\mathcal{P}_1$, and by what we have seen $\mathcal{P}_2$ has size $\aleph_3$. □

By Proposition 2.2 (2) together with Lemma 3.1, we have that $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$ is proper in $\mathbf{V}_1$. It thus remains to show, in an extension of $\mathbf{V}_1$ by $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$, that $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}]}$ is $<\omega_2$–distributive. For this, let G be $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$ generic over $\mathbf{V}_1$, let $\dot{F} \in \mathbf{V}_1$ be an $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$–name for a $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}]}$–name for a function from ω_1 into the ordinals, and let $q \in \mathrm{Coll}(\omega_2, \omega_3)^{L[A][G]}$. For the remainder of this proof, if $Y \subseteq \omega_1$, then let $\dot{G}_Y$ denote the canonical $\mathrm{Add}_{\mathbb{B}}(Y)$–name for the generic object added by $\mathrm{Add}_{\mathbb{B}}(Y)^{L[A]}$.

By our choice of A we may find, in $\mathbf{V}_1$, an elementary substructure N of some large enough $H(\theta)$ containing everything relevant – which includes $\dot{F}$ – of size $\aleph_1$ and such that $M = N \cap H(\omega_3)^{L[A]} \in L[A]$ is such that $M = \bigcup_{\nu<\omega_1} M_\nu$ for a $\subseteq$–continuous sequence $(M_\nu)_{\nu<\omega_1} \in L[A]$ of countable elementary substructures of $H(\omega_3)^{L[A]}$ such that $(M_{\nu'})_{\nu' \leq \nu} \in M_{\nu+1}$ for all ν. Let also $\pi : \omega_1 \to M$ be a surjection in $L[A]$ such that for every ν, $\pi \restriction \delta_{M_\nu}$ is a surjection from δ_{M_ν} onto M_ν. We may assume $\pi \restriction \delta_{M_\nu} \in M_{\nu+1}$ for all $\nu < \omega_1$.

Remember that for every $\delta \in \omega_1$, $F^G(\delta)$ denotes the enumerating function of the δ-th Baumgartner club adjoined by G, i.e., the function $\bigcup\{f(\delta) : (f, \mathcal{F}) \in G \text{ for some } \mathcal{F},\ \delta \in \mathrm{dom}(f)\}$. We build in $L[A][G]$ a decreasing sequence $(q^\nu)_{\nu<\omega_1}$ of conditions in $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][G]}$, together with an increasing sequence $(\gamma^\nu)_{\nu<\omega_1}$ of countable ordinals. We use the Baumgartner clubs of ω_1 added by G in order to guide the construction of these sequences. Specifically, we run the construction as follows.

(1) $q^0 = q$ and $\gamma_0 = 0$.
(2) If ν is a non-zero limit ordinal and both $(q^{\nu'})_{\nu'<\nu}$ and $(\gamma^{\nu'})_{\nu'<\nu}$ have been defined, then $q^\nu = \bigcup_{\nu'<\nu} q^{\nu'}$ and $\gamma^\nu = \sup_{\nu'<\nu} \gamma^{\nu'}$.
(3) Suppose $\nu < \omega_1$ is an ordinal and q^ν and γ^ν have been defined. Let $q^{\nu+1}$ be $\dot{q}_G$ if $\dot{q}$ is an $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$–name, $F^G(\delta_{M_{\gamma^\nu}})(1) = \gamma$ with $\gamma > \gamma^\nu$, $F^G(\delta_{M_{\gamma^\nu}})(0) = \xi$ with $\xi < \delta_{M_\gamma}$, $\pi(\xi) = \dot{q}$, and $\dot{q}_G$ is a condition in $\mathrm{Coll}(\omega_2, \omega_3)$ extending q^ν. In this case let also $\gamma^{\nu+1} = \gamma$. Otherwise let $q^{\nu+1} = q^\nu$ and $\gamma^{\nu+1} = \gamma^\nu + 1$.

In other words, $(q^\nu)_{\nu<\omega_1}$ and $(\gamma^\nu)_{\nu<\omega_1}$ are obtained from $q^0 = q$ and $\gamma^0 = 0$ by taking unions at nonzero limit stages. At successor stages $\nu + 1$, we use the $\delta_{M_{\gamma^\nu}}$–th Baumgartner club C naturally added by G in order to define $q^{\nu+1}$ and $\gamma^{\nu+1}$: We look at whether the second member γ of the strictly increasing enumerating function F of C is such that $\gamma > \gamma^\nu$ and whether the first member of F is some index

$\xi < \delta_{M_\gamma}$ such that $\pi(\xi)$ happens to be a $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$–name $\dot{q}$ whose interpretation by G is a condition in $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][G]}$ extending q^ν. If that is the case, then we set $q^{\nu+1} = \dot{q}_G$ and $\gamma^{\nu+1} = \gamma$, and otherwise we just set $q^{\nu+1} = q^\nu$ and $\gamma^{\nu+1} = \gamma^\nu + 1$. Note that the construction of $(q^{\nu'})_{\nu' \leq \nu+1}$ and $(\gamma^{\nu'})_{\nu' \leq \nu+1}$ takes place in $M_{\gamma^{\nu+1}+1}[G]$, and that in fact these sequences are definable in that model from $\pi \restriction \delta_{M_{\gamma^{\nu+1}}}$ and $(M_{\nu'})_{\nu' \leq \gamma^{\nu+1}}$. Note also that $\gamma^\nu \geq \nu$ for all ν.

In the end, we let q^* be $\bigcup_{\nu<\omega_1} q^\nu$. Note that $q^* \in L[A][G]$. For each $\nu < \omega_1$, let $\dot{q}^\nu$ and $\dot{\gamma}^\nu$ be $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$–names in $L[A]$ for, respectively, q^ν and γ^ν. Let also $\dot{q}^*$ be an $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$–name in $L[A]$ for q^*. Note that we may assume, for each ν, that both $\dot{q}^\nu$ and $\dot{\gamma}^\nu$ are in $M_{\gamma^\nu+1}$.

Next we will see that, thanks to the genericity of G, q^* is an extension of q which turns out to decide all of $\dot{F}_G$. This will conclude the proof of the theorem.

Claim 4.2. *q^* decides $\dot{F}_G(i)$ for every $i < \omega_1$.*

Proof. Let $\dot{q}' \in \mathbf{V}_1$ be some $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$–name for a condition in $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}_{\omega_1}]}$ extending $\dot{q}^*$ and deciding the value of $\dot{F}(i)$. Let $(f, \mathcal{F})$ be a condition in $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$. In $\mathbf{V}_1$, let Q be a countable elementary substructure of $H(\theta)$ containing $(\dot{q}^\nu)_{\nu<\omega_1}$, $(\dot{\gamma}^\nu)_{\nu<\omega_1}$, $\dot{q}'$, $\dot{F}$ and i.

Let f' be the function with domain $\mathrm{dom}(f)$ sending α to $f(\alpha) \cup \{\langle \delta_Q, \delta_Q \rangle\}$ and let $\mathcal{F}' = \mathcal{F} \cup \{\langle \delta_Q, \delta_Q \rangle\}$. Then $(f', \mathcal{F}')$ is an extension of $(f, \mathcal{F})$ which is $(Q, \mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]})$–generic by Proposition 2.2 (2). Hence, $(f', \mathcal{F}')$ forces the following.

- (a) There is a condition in $\dot{G}_{\omega_1} \cap Q$ forcing that $\dot{q}'$ is a condition in $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}_{\omega_1}]}$ deciding $\dot{F}(i)$.
- (b) For every $\nu < \delta_Q$ there is a condition in $\dot{G}_{\omega_1} \cap Q$ forcing that $\dot{\gamma}^\nu < \delta_Q$ and that $\dot{q}'$ extends $\dot{q}^\nu$ in $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}_{\omega_1}]}$.

Let now $R \in \mathbf{V}_1$ be a countable elementary substructure of N containing all relevant objects – which now includes $\dot{F}$, i, $(\dot{q}^\nu)_{\nu<\delta_Q}$, and $(\dot{\gamma}^\nu)_{\nu<\delta_Q}$ – and such that $\delta := \delta_R > \delta_Q$ is such that $M_\delta = R \cap H(\omega_3)^{L[A]}$. Of course $\dot{q}'$ is not in R (as it was defined only after defining all $\dot{q}^\nu$). However, we will be able to find a certain $\bar{q} \in R$ reflecting the relevant properties of $\dot{q}'$. Indeed, since $(f', \mathcal{F}')$ forces (a) and (b), by existential completeness there is an $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$–name $\bar{q} \in R$ such that $(f', \mathcal{F}')$ forces the following.

- (c) There is a condition in $\dot{G}_{\omega_1} \cap Q$ forcing that $\bar{q}$ is a condition in $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}_{\omega_1}]}$ deciding $\dot{F}(i)$.

(d) For every $\nu < \delta_Q$ there is a condition in $\dot{G}_{\omega_1} \cap Q$ forcing that $\dot{\gamma}^\nu < \delta_Q$ and that $\bar{q}$ extends $\dot{q}^\nu$ in $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}_{\omega_1}]}$.

Let $Y = \omega_1 \setminus \{\delta_Q\}$ and let $\varphi : \mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]} \longrightarrow \mathrm{Add}_{\mathbb{B}}(Y)^{L[A]}$, $\varphi \in R \cap L[A]$, be an isomorphism which is the identity on $\mathrm{Add}_{\mathbb{B}}(\delta_Q)^{L[A]}$. This φ exists by Proposition 2.2 (4).

Let f'' be the function with domain $\mathrm{dom}(f')$ sending $\alpha \in \mathrm{dom}(f')$ to $f'(\alpha) \cup \{\langle \delta, \delta \rangle\}$ and let $\mathcal{F}'' = \mathcal{F}' \cup \{\langle \delta, \delta \setminus \{\delta_Q\} \rangle\}$. Then $(f'', \mathcal{F}'')$ is a condition in $\mathrm{Add}_{\mathbb{B}}(Y)^{L[A]}$ and, since $(f'', \mathcal{F}'')$ extends $(f', \mathcal{F}') \in \mathrm{Add}_{\mathbb{B}}(\delta_Q)^{L[A]}$, viewed as an $\mathrm{Add}_{\mathbb{B}}(Y)^{L[A]}$–condition, φ is the identity on $\mathrm{Add}_{\mathbb{B}}(\delta_Q)^{L[A]}$, and $(f', \mathcal{F}')$ forces (c) and (d) in $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$, we have that $\hat{\varphi}(\bar{q}) \in R$ is an $\mathrm{Add}_{\mathbb{B}}(Y)^{L[A]}$–name such that $(f'', \mathcal{F}'')$ forces the following in $\mathrm{Add}_{\mathbb{B}}(Y)^{L[A]}$.

(e) There is some $(g, \mathcal{G}) \in \dot{G}_Y$ with $\mathrm{dom}(g) \cup \mathrm{dom}(\mathcal{G}) \cup \bigcup \mathrm{range}(\mathcal{G}) \subseteq \delta_Q$ forcing that $\hat{\varphi}(\bar{q})$ is a condition in $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}_Y]}$ deciding $\hat{\varphi}(\dot{F})(i)$.

(f) For every $\nu < \delta_Q$ there is some $(g, \mathcal{G}) \in \dot{G}_Y$ with $\mathrm{dom}(g) \cup \mathrm{dom}(\mathcal{G}) \cup \bigcup \mathrm{range}(\mathcal{G}) \subseteq \delta_Q$ forcing that $\hat{\varphi}(\bar{q})$ extends $\hat{\varphi}(\bar{q}^\nu)$ in $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}_Y]}$ and forcing $\hat{\varphi}(\dot{\gamma}^\nu) < \delta_Q$.

Since $(f'', \mathcal{F}'')$ is $(R, \mathrm{Add}_{\mathbb{B}}(Y)^{L[A]})$–generic by Proposition 2.2 (2), we can fix an extension $(\bar{f}, \bar{\mathcal{F}})$ of $(f'', \mathcal{F}'')$ and an $\mathrm{Add}_{\mathbb{B}}(Y)^{L[A]}$–name $q^\dagger \in R \cap H(\omega_3)^{L[[A]} = M_\delta$ for a $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}_Y]}$–condition such that $(\bar{f}, \bar{\mathcal{F}})$ forces the following in $\mathrm{Add}_{\mathbb{B}}(Y)^{L[A]}$.

(g) There is some $(g, \mathcal{G}) \in \dot{G}_Y$ with $\mathrm{dom}(g) \cup \mathrm{dom}(\mathcal{G}) \cup \bigcup \mathrm{range}(\mathcal{G}) \subseteq \delta_Q$ forcing that $q^\dagger$ decides $\hat{\varphi}(\dot{F})(i)$.

(h) For every $\nu < \delta_Q$ there is some $(g, \mathcal{G}) \in \dot{G}_Y$ with $\mathrm{dom}(g) \cup \mathrm{dom}(\mathcal{G}) \cup \bigcup \mathrm{range}(\mathcal{G}) \subseteq \delta_Q$ forcing that $q^\dagger$ extends $\hat{\varphi}(\dot{q}^\nu)$ in $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}_Y]}$ and forcing $\hat{\varphi}(\dot{\gamma}^\nu) < \delta_Q$.

Let now $(h, \bar{\mathcal{F}})$ be a condition in $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$ extending $(\bar{f}, \bar{\mathcal{F}}) \in \mathrm{Add}_{\mathbb{B}}(Y)^{L[A]} \subseteq \mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$ and such that $h(\delta_Q)(1) = \delta$ and $h(\delta_Q)(0) = \xi$ for some $\xi < \delta$ such that $\pi(\xi) = \hat{\psi}(q^\dagger)$, where $\psi = \varphi^{-1}$.

Now comes a subtle point, which uses the fact that $\mathrm{Add}_{\mathbb{B}}(Y)^{L[A]}$ is, not only isomorphic to, but also a complete suborder of $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$ (which is true by Proposition 2.2 (5)): Using once again that φ is the identity on $\mathrm{Add}_{\mathbb{B}}(\delta_Q)^{L[A]}$, together with the fact that $(\bar{f}, \bar{\mathcal{F}})$ forces (g) in $\mathrm{Add}_{\mathbb{B}}(Y)^{L[A]}$ and that, as I said, $\mathrm{Add}_{\mathbb{B}}(Y)^{L[A]}$ is a complete suborder of $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$, we have that $(h, \bar{\mathcal{F}})$ forces in $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$ that

(i) there is some $(g, \mathcal{G}) \in \dot{G}_{\omega_1}$ with $\mathrm{dom}(g) \cup \mathrm{dom}(\mathcal{G}) \cup \bigcup \mathrm{range}(\mathcal{G}) \subseteq \delta_Q$ forcing that $\pi(\xi)$ decides $\hat{\psi}(\hat{\varphi}(\dot{F}))(i)$ $(= \dot{F}(i))$.

Similarly, again by the above and since $(\bar{f}, \bar{\mathcal{F}})$ forces (h) in $\mathrm{Add}_{\mathbb{B}}(Y)^{L[A]}$, we have that $(h, \bar{\mathcal{F}})$ forces in $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$ that

(j) for every $\nu < \delta_Q$ there is some $(g, \mathcal{G}) \in \dot{G}_{\omega_1}$ with $\mathrm{dom}(g) \cup \mathrm{dom}(\mathcal{G}) \cup \bigcup \mathrm{range}(\mathcal{G}) \subseteq \delta_Q$ forcing that $\pi(\xi)$ extends $\hat{\psi}(\hat{\varphi}(\dot{q}^\nu))$ $(= \dot{q}^\nu)$ in $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}_{\omega_1}]}$ and forcing $\hat{\psi}(\hat{\varphi}(\dot{\gamma}^\nu)) = \dot{\gamma}^\nu < \delta_Q$.

Thanks to (j), we know that $(h, \bar{\mathcal{F}})$ forces in $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$ that $\dot{\gamma}^{\delta_Q} = \sup_{\nu<\delta_Q} \dot{\gamma}^\nu = \delta_Q$ and that $\pi(\xi)$ extends $\dot{q}^{\delta_Q} = \bigcup_{\nu<\delta} \dot{q}^\nu$. Hence, by the definition of q^{δ_Q+1} in (3) and the fact that $\delta_{M_{\delta_Q}} = \delta_Q$, it follows that $(h, \bar{\mathcal{F}})$ forces that $\dot{q}^{\delta_Q+1} = \pi(\xi)$ and $\dot{q}^{\delta_Q+1}$ decides $\dot{F}(i)$. It follows that $(h, \bar{\mathcal{F}})$ forces that $\dot{q}^*$ decides $\dot{F}(i)$. This concludes the proof of the claim since $(h, \bar{\mathcal{F}})$ extends $(f, \mathcal{F})$. □

Claim 4.2 completes the proof of Theorem 1.1. □

4.2. **Second proof of Theorem 1.1 (due to Veličković).** As in the first proof, we start out by considering a poset $\mathcal{P}_0$ as in Lemma 3.4. By Lemma 3.3, in $\mathbf{V}_1 = \mathbf{V}^{\mathcal{P}_0}$ we may fix $A \subseteq \omega_3$ such that $\omega_3^{L[A]} = \omega_3$ and such that for every cardinal $\theta > \omega_3$, the set of $N \preccurlyeq H(\theta)$ such that

- $|N| = \aleph_1$,
- $N \cap H(\omega_3)^{L[A]} \in L[A]$, and
- $N \cap H(\omega_3)^{L[A]}$ is IC in $L[A]$

is a stationary subset of $[H(\theta)]^{\aleph_1}$.

In $\mathbf{V}^{\mathcal{P}_0}$, we let $\mathcal{P}_1$ be the forcing consisting of finite chains of models of two types (in Neeman's terminology), countable and IC of size $\aleph_1$, coming from $H(\omega_3)^{L[A]}$. Specifically, let $\mathcal{P}_1$ be the forcing, ordered by reverse inclusion, of finite sets p with the following properties.

(1) Every $Q \in p$ belongs to $L[A]$ and is either
  (a) a countable elementary substructure of $H(\omega_3)^{L[A]}$, or
  (b) an IC elementary substructure of $H(\omega_3)^{L[A]}$.
(2) There is an enumeration $(Q_i)_{i<n}$ of p such that for all i, if $i+1 < n$, then $Q_i \in Q_{i+1}$.
(3) For all M, $N \in p$, if M is countable, N is uncountable, and $N \in M$, then $N \cap M \in p$.

Let $\mathcal{N}$ be the set of IC $N \preccurlyeq H((2^{\aleph_2})^+)$ of size $\aleph_1$ such that $N \cap H(\omega_3)^{L[A]} \in L[A]$ and such that $N \cap H(\omega_3)^{L[A]}$ is IC in $L[A]$. Since $\mathcal{N}$ is, in $\mathbf{V}$, a stationary subset of $[H((2^{\aleph_2})^+)]^{\aleph_1}$, we also have that the set $\mathcal{M}$ of countable $M \preccurlyeq H((2^{\aleph_2})^+)$ such that $M \cap H(\omega_3)^{L[A]} \in L[A]$ is a stationary subset of $[H((2^{\aleph_2})^+))]^{\aleph_0}$. Since, by Neeman's analysis in [9] of the forcing of finite two–type chains of models, $\mathcal{P}_1$ is both $\mathcal{M}$–proper and $\mathcal{N}$–proper (i.e., for every $Q \in \mathcal{M} \cup \mathcal{N}$, every condition in $Q \cap \mathcal{P}_1$ can be extended to a condition which is $(Q, \mathcal{P}_1)$–generic), it follows from the stationarity of $\mathcal{M}$ and $\mathcal{N}$ that forcing with $\mathcal{P}_1$ over $\mathbf{V}^{\mathcal{P}_0}$ preserves both ω_1 and ω_2. Also by Neeman's analysis, we have that $\mathcal{P}_1$ adds an $\in$–increasing sequence of models $(N_i)_{i\in I}$ of size $\aleph_1$ such that $H(\omega_3)^{L[A]} = \bigcup_{i\in I} N_i$, and hence collapses ω_3. Finally, forcing with $\mathcal{P}_1$ over $\mathbf{V}^{\mathcal{P}_0}$ preserves all cardinals above ω_3 since $\mathcal{P}_1 \subseteq H(\omega_3)^{L[A]}$ and $|H(\omega_3)^{L[A]}| = \aleph_3$. □

5. CH may fail in an absolute way

The following is a question of a similar flavour as the main question addressed in this paper: Is there always a set $A \subseteq \omega_2$ such that $\omega_2^{L[A]} = \omega_2$ and $L[A] \models$ CH? In other words, is it always possible to code a sequence of injections $g_\alpha : \alpha \longrightarrow \omega_1$ (for $\alpha < \omega_2$) in such a way that not more than $\aleph_1$–many reals be also coded? Note that this is always possible if we replace ω_2 with ω_1. Indeed, if $A \subseteq \omega_1$, then a simple condensation argument yields that $L[A] \models 2^{\aleph_0} \leq \kappa$, for $\kappa = \omega_1^V$, and on the other hand one can obviously always find $A \subseteq \omega_1$ such that $\omega_1^{L[A]} = \omega_1$.

In this short section I observe that the answer to the above question is no: It is consistent that CH fails in an absolute way, in the sense that any inner model computing ω_2 will think that CH fails.

Recall that if κ is an infinite cardinal, a κ^+–Aronszajn tree T is said to be special if there is a function $f : T \longrightarrow \kappa$ such that for every $\xi < \kappa$, $f^{-1}(\xi)$ is a collection of pairwise incomparable nodes of T. We say that f specializes T. It is a standard fact that if $2^{<\kappa} = \kappa$, then a standard construction, in ZFC, of a special $\aleph_1$–Aronszajn tree can be lifted to κ^+ to produce a special κ^+–Aronszajn (see e.g. [5]).

Theorem 5.1. *Suppose there are no special $\aleph_2$–Aronszajn trees. If $A \subseteq \omega_2$ is such that $\omega_2^{L[A]} = \omega_2$, then $L[A] \models \neg$ CH.*

Proof. Suppose otherwise. Since CH holds in $L[A]$, there is a tree $T \in L[A]$ such that $L[A] \models T$ is a special $\aleph_2$–Aronszajn tree. Let $f : T \longrightarrow \omega_1$ be a specializing function in $L[A]$. Since $\omega_2^{L[A]} = \omega_2$, we immediately get that f witnesses, in V, that T is an $\aleph_2$–Aronszajn tree there. Contradiction. □

6. Some open questions

Many questions in this area remain open. The following seem to be some such questions.

Question 6.1. *Is there, in* ZFC*, a forcing notion collapsing* $\aleph_4$ *and preserving all other cardinals?*

We could also focus on ZF instead of ZFC. It is easy to see that, in ZF, $\mathrm{Coll}(\omega, \omega_1)$ is a forcing notion collapsing $\aleph_1$ and preserving all other alephs.

Question 6.2. *Is there any natural number* $n > 1$ *such that* ZF *proves the existence of a forcing notion collapsing* $\aleph_n$ *but preserving all other alephs?*

The problem of forcing $\Box_{\omega_1}$ by finite conditions while preserving cardinals has received some attention in recent years. Indeed, each of [4], [6] and [10] contains forcing notions adding a $\Box_{\omega_1}$–sequence.[5] All these papers need to use some form of GCH as additional hypothesis. As we have seen in Lemma 3.4, a partial square sequence on ω_2 can always be added while preserving cardinals. However, the following question seems to be open.

Question 6.3. *Is there, in* ZFC*, a forcing notion preserving all cardinals and adding a* $\Box_{\omega_1}$ *sequence?*

The following seems to be a natural question regarding the absolute failure of CH considered in Section 5.

Question 6.4. *Does the nonexistence of any* $A \subseteq \omega_2$ *such that* $\omega_2^{L[A]} = \omega_2$ *and* $L[A] \models \mathrm{CH}$ *have consistency strength beyond* ZFC*?*

The last question I want to mention concerns Club–Guessing on $S^{\omega_2}_{\omega_1}$: Let us say that a club–sequence $\vec{C} = (C_\delta : \delta \in S)$ is club–guessing iff for every club $D \subseteq \sup(S)$ there is some δ such that $C_\delta \subseteq D$.

Question 6.5. *Suppose* $\aleph_\omega$ *is a strong limit. Is there a partial order* $\mathcal{P}$ *with the following properties?*

(1) $|\mathcal{P}| < \aleph_\omega$
(2) $\mathcal{P}$ *preserves* ω_1, ω_2 *and* ω_3.
(3) $\mathcal{P}$ *forces the existence of a club–guessing sequence* $\vec{C} = (C_\delta : \delta \in S^{\omega_2}_{\omega_1})$

The main motivation for asking Question 6.5 comes from the following (probably folklore) observation.

Observation 6.6. Suppose ZFC proves that if $\aleph_\omega$ is a strong limit, then there is a partial order satisfying (1)–(3) in Question 6.5. Then ZFC proves that if $\aleph_\omega$ is a strong limit, then $2^{\aleph_\omega} < \aleph_{\omega_3}$.

[5] A version of the corresponding forcing in [10] was of course the forcing used in the proof of Lemma 3.4.

Proof. Suppose $\aleph_\omega$ is strong limit and yet $2^{\aleph_\omega} > \aleph_{\omega_3}$. If $\mathcal{P}$ satisfies (1)–(3) in Question 6.5, then after forcing with $\mathcal{P}$ it is still true that $2^{\aleph_\omega}$ is a strong limit and $2^{\aleph_\omega} > \aleph_{\omega_3}$. Now, working in this extension, we run Shelah's proof of $2^{\aleph_\omega} < \aleph_{\omega_4}$ if $\aleph_\omega$ is a strong limit ([12]), using the club–guessing sequence on $S^{\omega_2}_{\omega_1}$ we have added rather than a club–guessing sequence on $S^{\omega_3}_{\omega_1}$ as in Shelah's proof – which ZFC gives us for free – and derive a contradiction in the same way. □

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