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Numerical Methods for Infinite Decision-making Processes

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The new computational methodology due to Yaroslav Sergeyev (see [25–27]) makes it possible to evaluate numerically the terminal features of complete, sequential decision-making processes. By standard numerical methods, these processes have indeterminate features or seem to support paradoxical conclusions. We show that they are better regarded as a class of problems for which the numerical methods based on Sergeyev's methodology provide a uniform technique of resolution.

Keywords: Infinite decision, supertask, fair lottery, paradox, grossone, complete sequence, infinite payoff, infinitesimal probability.

1 INTRODUCTION

Infinite decision-making processes have given rise to a rich literature over the last twenty years (e.g. [2,6,9,10,14,15,17,22]). Several decision-making models have been deployed to highlight problems for classical decision theory in an environment with infinite features. A recurring pattern, on which I shall focus in this paper, displays a net loss as the result of the infinite iteration of individually advantageous actions. Such a pattern arises from a sequential process whose individual stages are standardly marked by numerals in a finite base. This selection of numeral specifications mandates a distinctive way of handling infinite processes, namely reasoning that focusses on finite initial segments only. The result is a systematic failure numerically to capture

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certain features of infinite decision-making, most notably the fact that it should reach a point of completion at which overall payoffs or overall features are uniquely determined. These features are, in particular, uniformly indeterminate by standard numerical means. It is because they cannot be computed that paradoxes arise. As a consequence, it is to be expected that more powerful numerical methods should resolve paradoxes by replacing indeterminate values with numerically determinate ones. A toy example can usefully illustrate the kind of paradoxical issue discussed in this paper.

Suppose that an infinite stack of dollar bills, labelled by the numerals $1, 2, 3, \ldots$, which denote the finite, positive integers, is available. An agent A is asked repeatedly to choose one of the following methods of payment:

(a) At stage n, dollar bill n is paid out;
(b) At stage n, dollar bill n is paid out if and only if n is even.

It is presupposed that, for each method of payment, there are as many stages as there are dollar bills and that the stages about which information can be computed are identified by a numeral label. In particular, given a numeral n, it is possible to compute the payment made at stage n, denoted by $p_a(n)$ in case the first method of payment is chosen, and by $p_b(n)$ otherwise. The overall payment at stage n under a fixed (i.e. constantly chosen) method is:

$$\sum_{i=1}^{n} p_j(i), \text{ with } j \in \{a, b\}.$$

Trivially, at an arbitrary stage $n \geq 1$, it is more lucrative for agent A to choose the first method of payment over the second. An intriguing question is whether this remains true once an actually infinite, completed sequence of payments is carried out. This question may be attacked by evaluating:

$$\sum_{i=1}^{\infty} p_a(i) \text{ and } \sum_{i=1}^{\infty} p_b(i).$$

A problem arises from the fact that both series are divergent. Divergent behaviour reduces to the behaviour of finite partial sums at the limit and it is not obvious that limits can accurately describe actually infinite sequences of transactions. Suppose, for the sake of argument, that limits are accurate enough. It is then tempting to conclude that A should treat a uniform choice of method a and a uniform choice of method b as equivalent at infinity. A puzzling scenario appears, in which the persistent advantage of choosing method a over method b seems to break down at infinity. But, if limiting behaviour is a faithful indicator of 'outcomes at infinity', there is no reason to take note of the sums of series only. An asymptotic comparison between the series involved can also be carried out. The comparison yields:

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} p_a(i)}{\sum_{i=1}^{n} p_b(i)} = 2 \neq 1,$$

and thus points to the fact that the series are not asymptotically equivalent. The last conclusion may well suggest, against the previous surmise, that agent A should not treat the two methods of payment as equivalent even at infinity. Thus, two equally plausible appeals to limiting processes lead to inconsistent conjectures as to whether or not payment a is superior to payment b when either can be uniformly chosen infinitely many times.

The last remarks throw into relief the fact that standard sequential reasoning on the finite stages 1, 2, 3, . . . of an actually infinite decision-making process does not provide a uniquely determined picture of its overall features. Two mathematical responses to this state of affairs recommend themselves.

One is to allow as meaningful models of decision-making processes only feasible sequential processes: this can be done by e.g. exploiting the approach of [23], which works with a version of Peano Arithmetic relativised to a finite row of natural numbers of the form $0, \ldots, \Box$, where $\Box$ is an absolute resource bound. In this case, it may be argued that paradoxes arise when decision-making process are forced beyond feasible length (this is sure to happen if infinitely long ones are taken into account). As soon as their features are pushed beyond the reach of computation, there is no hope to recover them other than by hinting at the lost numerical information by a symbol like ∞. On the contrary, under suitable feasibility constraints, paradoxes are replaced by actual numerical computations. Analogues of the results discussed in sections 1 to 3 may be obtained along these lines, when finite bounds to the length of sequential decision-making are determined. Sazonov's alternative approach to feasibility (see [24]) which does not work with a bound $\Box$ but with a 'vague' horizon between feasible and unfeasible numbers, may also prove helpful in a similar but less straightforward manner. If carried out in the context of the axiomatic theory FEAS (see section 4 of [24]), however, this latter approach severely restricts the resources that can be invoked to build sequential decision-making models. For example, FEAS is inconsistent with the existence of the doubling-function $f(x) = x + x$.

An alternative approach, which will be pursued in this paper and can be extended beyond the arithmetic of the natural numbers, starts from the observation that certain numerical features of infinite decision-making processes elude numerical computation by traditional means. The answer to this problem is not now to restrict attention to feasible features in an existing representation by numeral terms, but to expand computation by introducing numeral terms that can describe the characteristics of completed, infinite sequential processes. The key point is that a mathematically serviceable notion of a completed, infinite process, which is necessary if a notion of ‘overall payoff’ is to be numerically handled, cannot rely, in particular, on familiar sequential arguments but must make use of a new numerical standpoint. Yaroslav Sergeyev’s new computational methodology involving infinitely large and infinitely small numbers provides the needed standpoint, while it does not call for any special restriction on the resources appealed to in order to build infinite decision-making models.

2 A NOVEL COMPUTATIONAL PARADIGM

Sergeyev’s computational methodology (see [25–27]) is a fairly recent but already very successful approach that has proved its fruitfulness in several areas of pure and applied mathematics, including linear and nonlinear programming ([13,19–21]), numerical integration ([5,18,30]), Turing machines ([28, 29]) and cellular automata ([11, 12]). The purpose of this paper is to apply this methodology to infinite decision-making. Sergeyev’s key ideas have been presented in a number of distinct ways. In this section, we follow the axiomatic approach presented in [7]. Lolli works with a suitable conservative extension of Peano Arithmetic: it follows that, if Peano Arithmetic is consistent, then Lolli’s theory is (a proof of conservativeness is outlined in [7], p.13.). The theory is based on second-order predicative arithmetic, which may be thought of as a multi-sorted version of first-order Peano arithmetic* . Besides the familiar axioms of Peano arithmetic (with induction formulated as a single, second-order axiom, but such that only instantiations to first-order definable subsets are allowed), Lolli introduces an infinite list of axioms of the form $n < ①$, where n is a numeral denoting a finite number and ① (read: gross-one) is a new constant symbol intended to denote an infinitely large number. Call any total function from $\{1, 2, 3, \ldots\}$ to some nonempty set A a complete sequence, also symbolised as $a_1, a_2, a_3, \ldots$. Lolli’s axiomatic approach aims at capturing the idea that a model of arithmetic is an environment in which a complete sequence can be singled out as a designated, infinitely long initial segment. More precisely, in Lolli’s framework complete sequences are total functions from $\{0, 1, \ldots, ① - 1\}$ to A (equivalence with the above definition can however be proved. For a more detailed discussion of complete sequences, see [28, 29]). This is in line with Sergeyev’s idea of using the numeral ① as a symbol identifying a unit of measure for the actually infinite collection of natural numbers. The next step in this context is to use initial segments of the numeral system specifiable by means

* See the remarks in [7], p.9

of terms in the language of Lolli's theory as abstract 'rulers'. To this end, a suitable function symbol μ can be introduced. The function μ takes second-order terms (denoting sets of numbers) as arguments and first-order terms (numerals in the arithmetical language with ①) as values. Under the assumptions that $\mu(\{x\}) = 1$, for any singleton $\{x\}$, and that μ is additive on disjoint bounded sets[†], it can be proved that $\mu(\emptyset) = 0$ and that $X \subseteq Y$ implies the inequality $\mu(x) \leq \mu(y)$. Moreover, strict inclusion implies a strict inequality. If $S_x = \{y | y < x\}$ is the initial segment of a model determined by x, then $\mu(S_x) = x$. Finally, Lolli shows that every bounded set X is in one-to-one correspondence with an initial segment S_z and, thus, has measure $\mu(X) = z$. This guarantees that μ behaves like the familiar cardinality function on finite sets and that it measures every subset of an initial segment, in particular every subset of a complete sequence (note that universal quantifiers in the last statements are instantiated by first-order definable sets). It is also guaranteed that strict subsequences have strictly smaller measures: in short, μ satisfies the principle that the whole is greater than the part.

However significant, the existence of measures is of relatively little value if there are not enough computable ones. Lolli's theory takes care of this *desideratum* by introducing an infinite list of axioms that guarantee the existence of objects denoted by $①/2, ①/3, \ldots, ①/n, \ldots$, for each finite numeral n. In presence of these divisibility axioms it becomes possible to prove, for instance, that the set of even numbers in a complete sequence, defined by the condition $\exists y(x < ① \wedge x = 2y)$, has measure $①/2$, the same as the measure of the set of odd numbers, defined by the condition $\exists y(x < ① \wedge x = 1 + 2y)$ (note that there is a bijection between the last two sets, but not between any of them and a complete sequence). In general, the condition $\exists y(x < ① \wedge x = k + yn)$, where k, n are finite numerals with $k < n$, defines a part of a complete sequence of size $①/n$. It can be proved that two parts of a complete sequence have the same numerical measure under μ if and only if a bijection links them.

Before returning to the toy example from the previous section, it is convenient to establish one inequality that be used in subsection 3.2. To this end, define $\llcorner log_2 x \lrcorner$ as the greatest n such that $2^n \leq x$. Then:

Lemma 2.1. $\llcorner \log_2 ① \lrcorner < ① - 2$.

Proof. If $X = \{g(n) : 0 < g(n) \leq k\}$ is a bounded set determined by a strictly increasing function g such that $g(0) > 0$, then $g^{-1}(X)$ is an initial segment S_x of measure x. Because g^{-1} is a bijection between S_x and

[†] Here a set is bounded if it is included in some initial segment of a model, which is a linearly ordered set.

X, it follows that $\mu(g(X)) = x$. Consider the special case of this situation in which $g(n) = 2^n$, $k = ①$ and $X = \{2^n : 0 < 2^n \leq ①\}$. Then $g^{-1}(X)$ is the initial segment $S_{\llcorner log_2 ① \lrcorner}$ and $\mu(X) = \llcorner log_2 ① \lrcorner$. Note that X is a set of even numbers, strictly included in the set E of even numbers in $\{1, \ldots, ①\}$. Now $\mu(E) = ①/2$ and, since μ is strictly monotonic relative to inclusion, $\mu(X) < \mu(E) = ①/2 < ① - 2$ (because $① = ①/2 + ①/2 > ①/2 + 2$). The result follows. □

It is now possible to turn to a simple treatment of the toy example from section 1.1.

3 EXCHANGES AND BETS

The formal apparatus from section 2 affords a way of introducing numerical models for a process involving a complete sequence of consecutive stages. Since the stages are in one-to-one correspondence with $\{0, 1, \ldots, ① - 1\}$, there are ① of them. As a consequence, the stages may be conveniently named by the numerals $1, 2, 3, \ldots, ① - 1, ①$. The toy example from section 1.1. is now amenable to a straightforward numerical treatment. Method a, when uniformly chosen, allocates exactly one bill per stage: its overall payoff, allocated after a complete sequence of ① stages, amounts to ① dollar bills. Because only the even stages of $S_①$ correspond to a payment under method b, the overall payoff in this case is $①/2 < ①$. Agent A is better off choosing method a, irrespective of whether only finitely many transactions or a completed sequence of transactions are proposed to her.

This conclusion may well look trivial, but this is because the toy example to which it applies is trivial: this example looks like a perplexing puzzle only when it is studied by numerical methods that leave its outcome indeterminate because computationally out of reach. In other words, its troublesome appearance is not due to the type of infinite process at hand but rather of the limitations of the numerical resources adopted to tackle it. This is true in general. The decision-making puzzles found in the literature are all constructed on a numeral treatment of complete sequences that focusses exclusively on their ‘early’, i.e. finite, stages, leaving aside the stages closer to their outcome, let alone the outcome itself.

3.1 Infinitely many bets

The toy example from section 1.1. is a simplified version of a much more sophisticated setting devised in [9]. This, in turn, is a nonlinear variant of a simpler paradox presented in [6]. To handle the simpler decision-making paradox first will help clarify how the formal approach in [7] can be used and it will make clearer how similar paradoxes can be tackled.

In the case at hand, an agent is faced with three alternatives a, b, c, where a is highly undesirable, b highly desirable, and c indifferent to her. A complete sequence of offers[‡] is deployed, according to the following template: the n-th offer proposes a regime of a for n days, followed by a regime of b for the next $n+1$ days, and a regime of c for all remaining days. Crucially, some actually infinite sequence of days is the time-span over which offers are made. Now it is sensible to reject the first bet (1 day of a and 2 days of b) in favour of the second (2 days of a and 3 days of b), which provides a longer period of highly attractive enjoyment. By the same clue, it is sensible to reject the second bet in favour of the third, and so on. A problematic conclusion seems to loom: an agent who accepts the complete sequence of bets should end up choosing a constant regime of a.

If this setting is described by a standard numerical model, individual bets with finite features are numerically specifiable, but no numerical specification of the overall number of days involved the offers or of the number of offers can be given. Without these specifications, one is confronted with an endless sequence of decisions labelled by finite numerals, whose completion is not defined: talk of a conclusive decision on the entire system of infinitely many offers becomes hazy. If, on the other hand, the numerical determination ① is introduced to identify the stages of a complete sequence of bets, then its overall features can be treated in a numerical way. The ①-th offer, modelled on the pattern of every other offer, requires ① days to to be spent under the regime of a and $① + 1$ to be spent under the regime of b. Thus, if ① is also the set whose number of elements determines the supply of available days, then there cannot be ① bets, but only $①/2 - 1$, a strictly smaller, infinitely large number.

Now, a completed, infinite sequence of days has the numerical size ①. Given this size, it is easy to realise that the last offer guarantees $①/2 - 1$ days of a, followed by $①/2 + 1$ days of b and no days of c. The agent who discarded earlier offers for later ones, should be satisfied with offer $①/2 - 1$, which is the last available one, if her preferences convince her that the largest administration b she can avail herself to is worth suffering a strictly shorter, but still rather long, administration of a. It is of the essence to remark that there is no difference for the agent in question between going through the offers sequentially, by discarding earlier ones in favour of later ones, or

[‡] This is what [6], p.260 means, even if the less accurate reference to a countably infinite system is used. This reference is not inappropriate, but occurs in a theoretical framework in which a complete sequence is indistinguishable from any of its infinite subsequences. It is in fact the lack of numerical determinations that tell such sequences apart that engenders paradoxes.

considering them as a whole. The impression of a difference arises in the absence of a computationally amenable notion of complete sequence. With these observations in mind, it is possible better to appreciate the sophisticated form of the toy example from section 1 devised by Barrett and Arntzenius.

3.2 Barrett and Arntzenius' paradox

In [9, 10], Barrett and Arntzenius present and discuss a setup that involves a choice between the following actions:

(a) receive the dollar bill with the least index from the available reserve of bills;

(b) receive 2^{n+1} dollar bills and return the dollar bill with the least index from the amount owned.

Barrett and Arntzenius further assume that, whenever the second action is chosen, the dollar bill to be returned is eventually destroyed. A paradox now arises from the fact that, after finitely many stages, a constant selection of the second action yields a greater payoff, whilst, at infinity, a constant choice of the same action seems to require that a decision maker should return every dollar bill, ending up with none. At the same time, a less greedy decision maker, who chooses the first action all the time, is bound to find herself ultimately in possession of the whole infinite stack of dollar bills.

The problem with this account is that, once again, complete sequences of actions are dealt with by numerical methods biassed in favour of their finite beginnings. This critical point is already perceived in [16], but it may be fully vindicated by the numerical approach developed so far. The central issue is that it is not *a priori* clear whether there are enough stages for an agent choosing action b all the time to return as many dollar bills as there are in the given stack. In particular, if, as in subsection 3.1, a complete sequence of ① dollar bills is available and many of them are mobilised at each transaction regulated by action b, while only one of them is returned and destroyed, there may well arise a discrepancy between the number of possible transactions and the overall number of returned dollar bills. The possibility of such a discrepancy is concealed when ∞ must be used to denote some overall features of the decision-making process as a whole.

By contrast, the number of transactions compatible with a constant choice of action b on a stack of ① dollar bills is a computable value in presence of the richer numeral resources used so far. To find this value, let the quantity M_k denote the number of bills disposed of at step k (note that the term k may refer to a finite or infinitely large number). Once this quantity is known, the quantity N_k, denoting the number of bills earned at step k, can be computed. Now, for any first-order term k in the language of Lolli's theory, we have:

$$M_k = \sum_{n=1}^{k} 2^{n+1} = 2^{k+2} - 4$$

$$N_k = \sum_{n=1}^{k} (2^{n+1} - 1) = 2^{k+2} - (k+4)$$

When $k = \llcorner log_2 ① \lrcorner - 2$, $M_k \leq ① - 4$. At $k = \llcorner log_2 ① \lrcorner$, the overall payoff becomes:

$$2^{\llcorner log_2 ① \lrcorner + 2} - 4.$$

Since, by definition, $2^{\llcorner log_2 ① \lrcorner + 1} > ①$, we obtain:

$$\begin{aligned} 2^{\llcorner log_2 ① \lrcorner + 2} - 4 &= 2^{\llcorner log_2 ① \lrcorner + 1} + (2^{\llcorner log_2 ① \lrcorner + 1} - 4) \\ &> 2^{\llcorner log_2 ① \lrcorner + 1} \\ &> ①, \end{aligned}$$

which guarantees that action b, if it is constantly chosen, can be chosen at most $\llcorner log_2 ① \lrcorner$ times. A decision maker who chooses action a for $\llcorner log_2 ① \lrcorner$ times earns $\llcorner log_2 ① \lrcorner$ dollar bills, so it remains to verify that $N_{\llcorner log_2 ① \lrcorner} > \llcorner log_2 ① \lrcorner$, i.e.:

$$2^{\llcorner log_2 ① \lrcorner + 2} > 2 \llcorner log_2 ① \lrcorner + 4,$$

which is equivalent to:

$$2^{\llcorner log_2 ① \lrcorner + 1} - 2 > \llcorner log_2 ① \lrcorner,$$

Since the left-hand side is greater than $① - 2$, it suffices to show that:

$$① - 2 > \llcorner log_2 ① \lrcorner,$$

which is the last inequality from section 2. The paradox proposed by Barrett and Arntzenius can now be accounted for in a nuanced way. An agent who constantly chooses action a over action b receives a smaller payoff at each decision, but she dramatically increases the number of decisions available to her. An agent constantly choosing action b, on the other hand, can help herself to single greater payoffs but curtails the sequence of transactions, as well as incurring a loss of one dollar bill per stage. In other words, the constant selection of action a guarantees a way of securing all available dollar bills, whereas action b undermines that possibility. Nonetheless, the constant selection of action b yields an infinitely large payoff, as opposed to no payoff. This constant selection is also superior to the constant selection of action a if the latter can only be repeated $\llcorner log_2 ① \lrcorner$ times (the former cannot be repeated more than $\llcorner log_2 ① \lrcorner$ times). Such an intricate picture is largely overshadowed by classical sequential reasoning.

Before considering further instances of infinite decision-making processes, it is worth pausing to note that, if the processes described so far were all so timed that the decisions involved had to take place at a geometrically increasing pace, they would look like what the philosophical literature since [8] has called super-tasks, i.e. tasks completed within a finite amount of time and constituted by infinitely many sub-tasks. In effect, the literature on decision-making paradoxes is an outgrowth of the super-task literature. Super-tasks that are unrelated to decision-making can be tackled using the computational paradigm applied in this paper: this has been shown in [26] and [4] in connection with the well-known Thomson's Lamp super-task and various physical super-tasks respectively. It is worth noting that, in presence of Sergeyev's computational methodology, the total amount of time taken by completing an infinite, numerically specified, number of tasks is computable even if the tasks do not succeed one another at a sufficiently fast pace (e.g. if each task took 1 second then ①/2 would take the infinitely long time of ①/2 seconds).

4 PROBABILITY MODELS

If Peano Arithmetic is consistent, then its conservative extension in [7] is too and, by the completeness theorem (which applies to multi-sorted first-order logic), has a model. Let Nat be a model of Lolli's theory. By a well-known set-theoretic construction it is possible to use it to build an ordered ring Int of integers in which the classical, complete sequences $\{\ldots, -2, -1, \}$ and $\{1, 2, \ldots\}$, together with zero, are represented by the infinitely long segment $\{-①, \ldots, -1, 0, 1, 2, \ldots ①\}$. Moreover, the environment Int can, by familiar means, be used to build an ordered field Frac, in which, in particular, the elements of the complete sequence $\{1, 2, \ldots, ①\}$ have multiplicative inverses. Although the measure μ could easily be extended in a unique way to Int or Frac, extensions will not be needed for present purposes. What matters is to have a sufficiently rich collection of numerical specifications arising from a model of Lolli's theory. Thanks to it, field arithmetic can be made to interact with the numerical methods exemplified so far. Such an outcome of is of special interest when decision-making processes involving probabilities are concerned. The next subsection discusses a widely debated example.

4.1 A fair lottery

A fair lottery is one in which draws are taken from the complete sequence $\{1, 2, 3, \ldots\}$ and it is assumed that the event of drawing n, for any n in the sequence, has a fixed probability. To describe a fair lottery numerically is impossible by classical means, because the desired uniform, discrete distribution is not defined. On the other hand, by the methods employed so far, it is easy to deploy the distribution $U[1, ①]$, which assigns to the extraction of any number from a complete sequence the infinitely small, nonzero probability $1/①$.

Because fair lotteries do not admit of any numerical description by classical means, they have to be handled by means of intuitive considerations, as opposed to numerical computations. Such considerations usually lead to problems, because they support the formulation of hypotheses that cannot be checked by a computation.

As an illustration of this phenomenon, consider the discussion of a system of infinitely many bets attached to a fair lottery, as it appears in [6], pp. 256–257. Bet n between agents A and B requires that A pay B \$2 if n is drawn and that she receive $\$1/2^n$ if n is not drawn. It is then argued that A:

> counts each of these bets as favourable, since each offers a greater-than-infinitesimal chance of a finite gain, and threatens a merely infinitesimal chance of a finite loss ([6], p.257).

It should be observed that, because only standard mathematical resources are presupposed, the adjective ‘infinitesimal’ from the above quotation does not correspond to any numerical specification. It should be understood as providing an intuitive qualification, since the only standard infinitesimal is 0, which, if deemed meaningful, does not produce a paradox[¶] . If some nonzero, infinitesimal probability is involved, then the result of any draw appears to lead to a net loss for A: paradox resurfaces.

The question is that not ‘any’ draw in a sequence of bets may be specified, if there are not enough numerical specifications at hand. As a matter of fact, agent A is never in a position to evaluate other than early bets, occurring after only finitely many have been considered. For this reason alone, it is plausible to conjecture that she may misjudge the local features of the sequence of bets at later stages. If, as usual, we work with a complete sequence, agent A has to perform ① evaluations. Just before carrying out half of this task, she is confronted with bet $①/2 - 1$. The expected loss in this bet is $2/①$ dollars, an infinitely small amount. Her expected gain, however, is:

$$\frac{1}{2^{\frac{①}{2}-1}}\left(1-\frac{1}{①}\right) < \frac{1}{2^{\frac{①}{2}-1}}.$$

[¶] In classical terms, if the probability of drawing any finite n is zero, then the overall payoff is computed as the sum of a geometric series. Thus it is, with certainty, \$1, a net gain. No losses can occur but it is entirely unclear what kind of event a single draw is in this case.

Since, for finite n, $n < 2^n$ and $n < ①/2 - 1$, the last quantity is infinitely small and, thus, the expected gain is, too. Now the ratio:

$$\frac{\frac{2}{①}}{\frac{1}{2^{\frac{①}{2}-1}}} = \frac{2^{\frac{①}{2}}}{①}$$

can be evaluated by means of the following lemma, which is a theorem of Lolli's theory:

Lemma 4.1. $2^{\frac{①}{2}} > 2①$.

Proof. Note that $2^{\frac{①}{2}}$ is the number of binary sequences of length $①/2$. It suffices to verify that, among these sequences, there are $2①$ distinct ones and that these do not produce an exhaustive list. In particular, there are $①/2$ distinct binary sequences with exactly one entry equal to 1 and $①/2 - 1$ further, distinct binary sequences can obtained from them by replacing the occurrence of 0 in their last entry with an occurrence of 1. Adding the constant sequence whose entries are all equal to 0, we obtain a total of ① distinct sequences. A further supply of $①/2 - 2$ sequences is provided by those with exactly three entries equal to 1, two of which are the first and last entry. Moreover, there are $①/2 - 4$ sequences with exactly five entries equal to 1 and such that their first and last two entries are equal to 1. The total count of distinct sequences is now $2① - 6$. Now the sequence with exactly the first n consecutive entries equal to 0 and $n = 1, 2, \ldots, 6$ provide six further distinct sequences. As a result, there are at least $2①$ binary sequences of length $①/2$. In fact, there must be infinitely many more, since, for instance, the sequences with exactly the first n consecutive entries equal to 0, for $n = 7, 8, \ldots ①/2$ have been omitted. □

The last lemma implies that the expected loss in the bet number $①/2 - 1$ is more than twice the expected gain. This situation persists in every subsequent bet. Agent A, if put in a position that enables her to carry out numerical computations of gains and losses for each bet, eventually realises that more than half (i.e. $①/2$) of the bets proposed by B lead to an expected loss greater than twice the expected gain. In other words, A must conclude that more than half of the bets offered her are unfavourable and reject the whole system of bets because too many individual bets are unfavourable. This conclusion contradicts what was suggested about A's behaviour by the last quotation: A's hypothesis was plausible only because she lacked numerical methods to evaluate the system of bets in a computationally accurate way. There is in fact no conflict between her view of the whole system of bets and her view of the individual bets, if she can intervene numerically on both.

4.2 Coin tosses

We have just considered infinitely long binary sequences. These may be regarded as outcomes of some numerically specifiable, infinite iteration of a coin toss. In particular, since a complete sequence of coin has length ①, there are $2^{①}$ outcomes. It is possible to use their powerset as sample space for the uniform, discrete distribution $U[1, 2^{①}]$. Under this distribution, the event of obtaining a sequence of ① heads (or tails) with a fair coin has infinitesimal probability $1/2^{①}$. Working with this probability model, a betting problem framed after the fashion of the Saint Petersburg paradox and presented in [6] is effectively dealt with. This problem is of special interest because it shows that, on adopting Sergeyev's computational paradigm, it becomes possible to find numerical values for expectations otherwise undefined.

The problem revolves around the question whether or not an agent should accept a complete sequence of bets based on the repeated tossing of a fair coin. What matters to each one of the bets is how soon tails comes up for the first time. For instance, if tails comes up at the first toss, Bet 1 yields a gain of $3 dollars, as well as a loss of $1 if tails never comes up. If the first toss does not correspond to an event of tails, then Bet 1 is void and Bet 2 is proposed, which involves a gain of $9 if tails comes up at the second toss for the first time, but a loss of $4 if it does not come up at the first toss. The table below describes the first three bets. The events of the first occurrence of tails at the first, second or third toss respectively are referred to as 1st T, 2nd T and 3rd T in the top row of the table. 'No T' refers to the event that corresponds to a complete sequence of heads. The second row of the table lists the probabilities of these events.

Occurrence of T	No T	1st T	2nd T	3rd T
Probability	?	1/2	1/4	1/8
Bet 1	Lose $1	Gain $3		
Bet 2		Lose $4	Gain $9	
Bet 3			Lose $10	Gain $21

In Bet 1, [6] assigns a probability 0 to a loss of $1, but this numerical specification can be improved upon by means of the probability model described at the beginning of this section, where this probability is $1/2^{①}$. The system of bets does not stop at 3, but it is clear from the table above that each column but the last involves a loss of $1. Given the way in which Gains and Losses are defined, by the recursive conditions:

$$Gain(n+1) = 2Gain(n) + 3 \text{ and } Loss(n+1) = 2Loss(n) - 2,$$

for any column labelled by a finite numeral n, the column next to it with leads to a loss of $1. The system of infinitely many bets thus appears to imply a sure loss, but it is also possible to consider each bet in isolation and note that it leads to a gain more than twice greater than its loss at half the probability. In other words, each single bet looks advantageous. The discrepancy that these remarks are meant to bring out is one between the system of bets as a whole and as a sequence of individual bets. Individual bets look advantageous but, when taken collectively, they seem to produce a net disadvantage.

This is by now the familiar situation that arises when certain features of the process cannot be numerically evaluated. In a complete sequence of ① bets, i.e. the model used by [6] with a numerical specification of its length, it is possible to compute the numerical value of the $(k+1)$-th gain G_{k+1} for k finite or infinitely large. Since $G_{k+1} = 3(2^{k+1} - 1)$ and the probability of G_{k+1} is $1/2^{k+1}$, the overall expected gain resulting from a complete sequence of bets is:

$$G = \sum_{k=0}^{①-1} \frac{1}{2^{k+1}} G_{k+1}.$$

The above summation can be computed fas follows:

$$3\sum_{k=0}^{①-1}\left(1 - \frac{1}{2^{k+1}}\right) = 3① - 3\sum_{k=0}^{①-1}\frac{1}{2^{k+1}} = 3(① - 1) + \frac{3}{2^{①}},$$

which, if one adds the loss in Bet 1 for no occurrence of tails, an event with probability $1/2^{①}$, leads to the value:

$$G' = 3(① - 1) + \frac{1}{2^{①-1}}.$$

Since Bet ① is the last bet, losses should be computed up to the last one relative to toss number $① - 1$. The k-th loss is given by:

$$L_k = 2^{k+1} + 2^{k-1} + 2^{k-2} + \ldots + 2^2 + 2 + 1 = 2^{k+1} + 2^k - 2.$$

Since $2^{-k}L_k = 3 - 2^{-(k+1)}$, it is again possible to compute the overall expected loss:

$$L = \sum_{k=1}^{①-1}\left(3 + \frac{1}{2^{k-1}}\right) = 3① - 5 + \frac{1}{2^{①-1}}.$$

It follows that $G - L > G' - L = 2$: the full system of bets entails a net gain. Note that, if losses had been computed up to $k = ①$, the last quantity would have been -1, a net loss, but this loss could have been included only if a system of $① + 1$ bets is taken into account, with the accompanying $① + 1$-th gain, which again leads to a net overall gain.

This outcome is not transparent in a standard numerical model because the latter lacks the resources to discriminate between the expected values of infinitely long sequences of bets, in particular complete ones.

5 PERSPECTIVES ON INFINITE DECISIONS

The infinite decisions examined in subsections 3.1, 4.1 and 4.2 respectively illustrate each of three types of decision-making paradox discussed in [6]. These examples were originally intended to illustrate the possibility of decision-making scenarios in which a collection of individually attractive deals proves harmful as a whole (see [6], p. 280). It was shown that the inconsistency between local and global evaluation of deals is in fact a discrepancy between numerically accessible evaluations (local ones) and numerically inaccessible evaluations (global ones). In particular, the numerical methods provided by Sergeyev's computational methodology afford a treatment under which both the local and global features of a decision-making process are amenable to numerical computation. In fact, something subtler can be said. Specific mathematical properties, like the non-conglomerability of certain probability measures or the discontinuity of certain functions, arise in the context of paradoxes of infinite decision-making. Such properties do not highlight absolute features of the relevant processes but the absence of adequate numerical methods to handle them. The next two-subsections illustrate this phenomenon.

5.1 Non-conglomerability

In section 4.1, we introduced a probability model for a fair lottery on a complete sequence. Arntzenius et al (2004) seek to set one up but, in order to do so in the absence of a direct treatment of complete sequences and numerical infinities, they are forced to land with a non-conglomerable probability measure(see [6], pp.274–275). Non-conglomerability on a complete sequence amounts to the fact that there are an event $X \subseteq \{1, 2, 3, \ldots\}$ and a countable partition $\mathcal{S} = \{S_1, S_2, S_3, \ldots\}$ of $\{1, 2, 3, \ldots\}$ such that some $x \in (0, 1)$ satisfies the inequalities $P(X) < x < P(X|S_i)$ for each $i \in \mathbb{N}$. When this happens, a probability measure locally larger than x is globally smaller than the same quantity. It is clear that such measures are well-suited to setting up the kinds of conflicts between local and global decisions studied so far.

[6] get a non-conglomerable measure by exploiting a construction from [3] in order to guarantee the existence of a fair lottery on a complete sequence. In their lottery, the event e.g. of drawing an even number has probability $1/2$, as in the numerical model from section 4.1. Set $x = 7/12$: the probability value $1/2 < 7/12$ is assigned by the given lottery to the set X of even

numbers but, conditional on a restriction to $S_i \in \mathcal{S} = \{S_1 = \{1, 2, 4\}, S_2 = \{3, 6, 8\}, S_3 = \{5, 10, 12\}, \ldots\}$, the probability of picking an even number is $2/3 > 7/12$.

The same choices of X and $\mathcal{S}$ do not lead, in presence of Sergeyev's numerical methods, to non-conglomerability. Since there are ①$/2$ even numbers in a complete sequence, the probability of picking one under $U[1,$ ①$]$ is $1/2$ as above. Because there also are ①$/2$ odd numbers, the partition $\mathcal{S}$ has ①$/2$ distinct cells. Clearly, not all of them can contain two distinct even numbers. If this were the case, there would be ①$/2 +$ ①$/2 +$ ①$/2 =$ ① $+$ ①$/2 >$ ① distinct numbers in a complete sequence, whose length is only ①. In fact, only ①$/4$ cells from $\mathcal{S}$ contain even numbers and this means that the probability of picking an even number conditional on a restriction to $S_i \in \mathcal{S}$ is 0 for an infinitely large number of choices of S_i, with i infinitely large. This result can be achieved because the relatively inaccurate qualification 'countable', applied to the size of the partition $\mathcal{S}$, is replaced by more refined numerical distinctions like ①$/4$ and ①$/2$. In general, it is impossible to offer an instance of non-conglomerability while working with Sergeyev's numeral system, where the standard possibility of identifying distinct, infinitely large numbers is foreclosed.

5.2 Discontinuity

In [17], Bartha and his collaborators identify the general structure of decision-making paradoxes by an appeal to a notion of discontinuity. We shall show that this notion is a pointer to the employment of inadequate numerical methods or, more constructively, a pointer to the need for more effective numerical methods.

The notion of discontinuity at play is elicited from a particular abstract setting: given an item X, partitioned into a complete sequence $X_1, X_2, X_3, \ldots$ of pieces, an agent A is sequentially offered each piece but she is also told that she will incur a large loss if she takes them all. The decision-making problem A faces consists in selecting the most profitable moment to stop. Equivalently, A must decide how many pieces she will accept in sequential order. In [17] numerical specifications of utilities are employed, under which accepting X_1 increases A's utility by 5 units, while accepting X_n, with $n > 1$, produces an increase of $5/(n-1) - 5/n$ units. At the same time, taking all pieces in the sequence produces a net loss of 1000 utility units.

At each finite stage, accepting one more piece is a rational strategy, thus [17] can represent each finite, partial sequence of favourable choices for all pieces up to the n-th by a binary sequence with a constant initial segment (a string of occurrences of 1, standing for acceptance). Each sequence corresponds to an action a_n and it can be immediately verified that these actions have a point-wise limit a, which is an infinite string of occurrences of 1. At the same time, utilities are strictly increasing on the sequence (a_n) but $u(a)$ is not an upper bound for $(u(a_n))$, since it falls below all of its elements. The utility function u is discontinuous at a (a rigorous version of this statement can be found in [17], pp. 641–642). Discontinuity, as it arises here, seems to mark the defeat of sequential strategic behaviour.

It is noteworthy that [17] calls a 'ALL' and assigns it a utility, as is done with any finite stage a_n. However plausible it may be to imagine A in possession of a complete sequence of pieces X_i, there are no standard means to render this situation numerically as the outcome of a complete sequential process. This shortcoming becomes becomes even more striking when [17] suggests an ordering of A's actions according to utility. After observing that a is not preferred to any a_i, where i can only be a finite label, the authors go on to consider the distinct action of taking all elements of the partition but X_1. The utility attached to this event is evaluated as -995 units, which can only make sense if one thinks that the penalty for taking the entire partition is applied before X_1 is returned. If X_1 had not been taken in the first stage of the decision-making process, no penalty could have been applied. But to take the surrender of X_1 as a discrete event to follow the completion of a complete sequence of decisions, is, again, to work with the qualitative notion of a completion event to which no numerical specification can be assigned.

In order to get around this difficulty, it may be possible to envisage a process in which X_1 is rejected and then relabelled to appear as X_2 with a lower utility. Then X_2 should be relabelled to appear as X_3 and so on. In this case, however, two problems arise. One concerns the opportunity of employing a formal approach to decisions that turns distinct qualitative descriptions into equivalent actions (returning X_1 or never taking it), thus making decision-making hopelessly ambiguous. The other problem is simply that the difficulty with rendering completion for an infinite sequence of decisions reappears with respect to an infinite sequence of label changes.

It is of special interest that, despite these setbacks, [17] engages as far as possible with numerical evaluations of utilities. For instance, the action of taking only the partition elements with an even index is evaluated as producing a utility lower than -995 (estimated at -997). In fact, the event a is positioned in the middle of a utility ordering in which an increasing sequence of finite acquisitions of X_i follows a, carrying greater utilities, and an increasing sequence of greater losses, produced by taking infinitely many pieces in various ways, precedes a. Willingness to even set up this ordering presupposes an interest in drawing numerical distinctions between infinite processes that lie beyond the scope of standard numerical specifications. In short, the analysis indicates a method to be desired, since handling finite numerals in computations cannot guarantee accurate estimates, if it produces any.

To fulfil the desire in [17], it suffices to note that a partition of X into a complete sequence of pieces can be represented, under Sergeyev's numerical approach, as $\{X_1, X_2, \ldots, X_{①-1}, X_①\}$. The utility of it last element $X_①$ is $5/(① - 1) - 5/①$ units and the overall utility of taking the full partition (minus the penalty) can be found by computing the infinitely long summation below:

$$\begin{aligned}
5 + \sum_{i=2}^{①} \frac{5}{i-1} - \frac{5}{i} &= 5 + 5\sum_{i=2}^{①} \frac{1}{i(i-1)} \\
&= 5 + 5\sum_{i=2}^{①} \left(\frac{1}{i-1} - \frac{1}{i}\right) \\
&= 5 + 5\left(1 - \frac{1}{2} + \frac{1}{2} - \ldots + \frac{1}{① - 1} - \frac{1}{①}\right) \\
&= 10 - \frac{5}{①},
\end{aligned}$$

where $5/①$ is infinitely small (if one took only the even pieces, the third equality above would involve the alternating harmonic series. The finite part of the sum of its first ① terms, which is independent of the order of summation, is $log 2$. For a thorough discussion with computations see [1], pp. 8066-8069. It is easy to see that, whenever a number of elements from the partition of X is taken, which is smaller than ①, only a utility bounded above by the last utility value is gained by agent A. Her best strategy is to reject the least valuable element $X_①$ and take every other element. This allows her to incur no penalty while attaining a utility that is still infinitely close to 10 units. It is also clear that taking all elements but X_1 leads to no penalty and a utility of $5 - 5/①$ units, as opposed to the negative utility conjectured by [17].

The discontinuity identified by an appeal to standard mathematics can now be handled by recognising that a large penalty is applied at an identifiable stage of the infinite decision A is confronted with, provided that a condition is satisfied (all pieces are taken). The latter condition can be specified by numerical means (all ① pieces are taken) and overall utilities can be computed both when the condition is satisfied and when it is not. The problem faced by [17] resides in the decision to opt for the adoption of standard mathematics where the formulation of the problem at hand calls for the introduction of alternative resources that allow its qualitative features to be translated into numerical specifications.

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