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The ice response to an oscillating load moving along a frozen channel

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Abstract. Unsteady response of an ice cover to an oscillating load moving along a frozen rectangular channel is studied for large times. The channel is filled with ideal incompressible fluid. The ice cover is modelled by a thin elastic plate. The flow caused by the deflection of the ice cover is potential. The problem is formulated within the linear theory of hydroelasticity. External load is modelled by a smooth localized pressure distribution. The load has periodic magnitude and moves along the channel with constant speed. Joint system of equations for the ice plate and the flow potential is closed by initial and boundary conditions: the ice plate is frozen to the walls of the channel, the flow velocity potential satisfies the impermeability condition at the rigid walls of the channel and linearized kinematic and dynamic conditions at the ice-liquid interface; at the initial time the load is stationary, the fluid in the channel is at rest and the stationary ice deflection is determined from the plate equation for the initial magnitude of the load. The problem is solved with the help of the Fourier transform along the channel. The ice deflection profile across the channel is sought in the form of the series of the eigenmodes of the ice cover oscillations in a channel. The solution of the problem is obtained in quadratures and consists of three parts: (1) symmetric with respect to the load deflection corresponding to the stationary load; (2) deflection corresponding to steady waves propagating at the load speed; (3) deflection corresponding to waves propagating from the load and caused by the oscillations of the load. The number of the last waves, depending on the parameters, can not exceed four for each eigenmode. In this article the results of the analytical and numerical analysis of the considered problem is presented.

1. Introduction

The problems of loads moving along an unbounded ice cover without vertical walls are well studied (see, for example, an excellent review in [1]). The problems are usually solved within the theory of linear hydroelasticity. It is known that the ice deflection is strongly dependent on the speed of the load. If the load speed is below a certain critical value, the ice deflection is localized near the load and quickly decays with the distance from the load. For higher speeds of the load, outgoing waves are formed in the far field if viscous damping in ice is not included in the mathematical model. At the critical speed, the linear theory without damping predicts unbounded ice response. To obtain the estimates of the ice response for the critical speed of the load either nonlinear effects [2] or viscous damping [3] or both are included in the ice model. The presence of the channel walls complicates the problem. There are an infinite number of



critical speeds for a frozen channel, in contrast to the ice sheet of infinite extent, for which there are two simple critical values of the load speed. Each critical speed for an ice cover in a channel corresponds to a mode of propagating along the channel and sloshing across the channel waves studied by Korobkin et al. [4]. For a speed of the load different from the critical values, the linear theory of hydroelasticity can be used to estimate the strains in the ice cover. We restrict ourselves to a constant and different from the critical values speed of the load and the strain/deflection distributions which are stationary in the coordinate system moving together with the load. These stationary distributions can be obtained directly within ice models with viscous damping, where ice deflection decays exponentially with distance from the moving load, see [5]. Decreasing viscous damping, we obtain higher strains in the ice cover and longer region of significant deflections along the channel. Both deflections and strains are given by infinite series and integrals. Numerical integration becomes challenging for small damping. To estimate the strains, we need them for zero damping, where the approach from [5] does not work.

Another approach to stationary strain/deflection distributions caused by a load moving along the channel for zero damping in the ice cover [6] is employed in the present study. In this approach, the stationary distributions are approached in time t starting from the initial rest state. At $t = 0$ the load, which is modelled by an external pressure acting on the ice plate, is at rest. The initial ice deflection satisfies the stationary equation of thin elastic plate with proper boundary conditions on the walls of the channel. The pressure distribution generated by the load is symmetric with respect to the central line of the channel. Then the load starts to move at a constant speed U along the channel and the load magnitude starts to oscillate with the given constant frequency ω' . The ice deflection decays far ahead and far behind the moving load as $|x - Ut| \rightarrow \infty$ at any finite time t . The unsteady problem of hydroelasticity is solved with the help of the Fourier transform along the channel and the normal-mode method [7] for the ice deflection. A second-order differential equations in time for the principal coordinates of the normal modes are derived and solved analytically. As a result, the ice deflection is presented by an infinite series of regular Fourier integrals. The integrals are evaluated for large times by asymptotic methods, see [6]. The asymptotic behaviours of the integrals depend on the speed of the load with respect to the critical speeds of the propagating-sloshing wave modes. It is shown that for large times the ice deflection consists of symmetric deflection localized near the load and a system of waves in front and behind the load. The number of these waves is obtained and the wave amplitudes are evaluated numerically.

This study is motivated by experiments in ice tanks, operations on ice in rivers and channels such as cargo transportation or ice breaking to avoid flooding, and ice-structure interaction. The strains calculated with zero damping are higher than the real ones. For safe transportation on ice, one needs to compare the computed strains with a strain critical value and determine safe speed of transportation. Knowing amplitudes of the waves generated by the moving load, we can find places and estimate the values of the highest strains far ahead and behind the load. It is expected that there are such speeds of the load that the maximum strains behind and or in front of the load are achieved at the walls of the channel. On the other hand, oscillations of the load can create extra waves in the ice cover. These waves can increase maximum stresses in the ice cover, that leads to the destruction of the ice cover in the non-resonant cases for a load with a constant magnitude. Linear hydroelastic problem for steady forced vibrations of a semi-infinite ice cover under the effect of localized external load was studied in [8]. The problem was solved within the shallow water theory. It was shown that, in the case of a free edge for the ice plate, there are edge waves. The edge waves are localized near the edge of the plate and exponentially attenuate as they move away from the edge. In this article, in addition to the movement of the load, periodic oscillation of its magnitude and the effects of the oscillations on the formation of the ice deflections are studied.

2. Formulation of the problem

The unsteady hydroelastic waves caused by an oscillating load moving along a frozen channel are considered. The channel has a rectangular cross section with finite depth H , $-H < z < 0$, and width $2L$, $-L < y < L$. The channel is infinitely long, $-\infty < x < \infty$. The channel is filled with ideal incompressible fluid of density ρ_l . The flow caused by the ice deflection is potential. The thickness of the ice cover h_i , the rigidity of the ice D and the ice density ρ_i are constant. The vertical displacement of the ice sheet, $w(x, y, t)$, satisfies the equation of thin elastic plate with clamped boundary conditions at $y = \pm L$

$$Mw_{tt} + D\nabla_2^4 w = p(x, y, 0, t) - P(x - Ut, y), \quad (-\infty < x < \infty, -L < y < L, z = 0), \quad (1)$$

$$w = 0, \quad w_y = 0 \quad (-\infty < x < \infty, y = \pm L), \quad (2)$$

where $\nabla_2^4 = \nabla_2^2 \cdot \nabla_2^2 = \partial^4/\partial x^4 + 2\partial^4/(\partial x^2 \partial y^2) + \partial^4/\partial y^4$, $M = h_i \rho_i$ is the mass of the ice plate per unit area, $p(x, y, 0, t)$ is the hydrodynamic pressure acting on the lower surface of the ice plate and $P(x, y, t)$ is the external pressure distribution on the upper surface of the ice plate caused by the moving load. The flow velocity potential $\varphi(x, y, z, t)$ satisfies the Laplace equation, impermeability conditions at the rigid walls, and the kinematic and dynamic boundary conditions at the ice-liquid interface

$$\nabla^2 \varphi(z, y, z, t) = 0 \quad (-\infty < x < \infty, -L < y < L, -H < z < 0), \quad (3)$$

$$\varphi_y = 0 \quad (y = \pm L), \quad \varphi_z = 0 \quad (z = -H), \quad \varphi_z = w_t \quad (z = 0). \quad (4)$$

$$p(x, y, 0, t) = -\rho_l \varphi_t(x, y, 0, t) - \rho_l g w(x, y, t) \quad (-\infty < x < \infty, -L < y < L, z = 0), \quad (5)$$

where g is the gravitational acceleration and $\varphi(x, y, z, t)$ is the velocity potential of the flow in the channel caused by the moving load.

The load is modelled by a localized smooth pressure distribution, $P(x, y, t)$, moving along a centre line of the channel at constant speed U with periodic magnitude, $P_0/2 \leq P \leq 3P_0/2$

$$P(x, y, t) = P(x - Ut, y, t) = P_0 P_1(X) P_2(y) (1 + \cos(\omega' t)/2) \quad (-\infty < x < \infty, -L < y < L), \quad (6)$$

$$P_1(X) = (\cos(\pi c_1 X) + 1)/2 \quad (c_1 |X| < L), \quad P_1(X) = 0 \quad (c_1 |X| \geq L),$$

$$P_2(y) = (\cos(\pi c_2 y) + 1)/2 \quad (c_2 |y| < L), \quad P_2(y) = 0 \quad (c_2 |y| \geq L),$$

where $X = x - Ut$, P_0 is the magnitude of the stationary load, c_1, c_2 – nondimensional parameters characterising area of the load and ω' is the oscillating frequency.

At the initial moment the flow is at rest and the ice deflection $w(x, y, t)$ satisfies initial conditions

$$w(x, y, 0) = w_0(x, y), \quad w_t(x, y, 0) = 0. \quad (7)$$

The last condition is obtained from the kinematic condition (4) for $t = 0$. The RHS of the first condition in (7) is non zero. The equation of the plate (1) with $t = 0$ provides

$$D\nabla_2^4 w_0(x, y) = -\rho_l g w_0(x, y) - P(x, y, 0). \quad (8)$$

The solution of the problem (1) – (8) depends on the density of the liquid, ρ_l , parameters of the ice, ρ_i, h_i, D , parameters of the channel, H, L , parameters of the load, U, c_1, c_2, ω' and initial data. The problem and the solution are time-dependent. We shall find large time behaviour of the ice deflection, $w(x, y, t)$, and the effect of the load oscillations (term $\cos(\omega' t)$) on the ice deflections for some given values of the parameters of the problem. The solution will be given by the asymptotic formulas for amplitudes of the hydroelastic waves and consisted of three parts:

(1) symmetric ice response with respect to the load corresponding to the stationary load and (2) deflection corresponding to waves propagating at the load speed, both are established for $t \rightarrow \infty$; and (3) deflection corresponding to waves propagating from the load and caused by the oscillations of the load. The last waves propagate from the load with encounter frequency and their phase depend on time. It will be shown that number of these waves is limited. Obtained formulas provide largest possible strains in the ice cover for the given load. If one wants find the ice deflection for given value of time then other methods of the solution should be used. Asymptotic analysis is presented for the rectangular channel and the given form of the load, but can be extended to non-rectangular channels and different forms of the load.

Steady state solution of similar problem of the moving load on ice was studied in [5]. The problem is stationary and was formulated for a Kelvin-Voigt viscoelastic plate with non-zero retardation time, $\tau > 0$ s. The steady state solution of the problem, $w(x,y,t) = w(X,y) = w(x - Ut, y)$, was investigated in the coordinate system moving with the load. The solution did not depend on time and that approach did not require initial data. The problem was solved numerically. Due to the viscous effect of the plate all numerical results showed quick decay of the ice deflection away from the load. The effects of the channel walls, the plate thickness and speed of the load on the solution were studied. We should note that determination of the value of the retardation time is complicated and not established yet. In this article we shall find the asymptotic behaviour of the ice deflection for an elastic plate with no viscous effects in the nonstationary problem (1) – (8). Models of ice response without damping are less physical, however, they may provide helpful estimates of maximum strains and bearing capacity of the ice cover. Such estimates are known for loads moving on the ice cover of infinite extent.

3. Method of the solution

The considered initial-boundary value problem is solved with the help of the Fourier transform along the channel

$$w(X, y, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} w^F(\xi, y, t) e^{-i\xi x} d\xi, \quad w^F(\xi, y, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} w(x, y, t) e^{-i\xi x} dx \quad (9)$$

where ξ is the parameter of the transform. The equation (1) provides

$$Mw_{tt}^F + D \left(w_{yyyy}^F - 2\xi^2 w_{yy}^F + \xi^4 w^F \right) = -\rho_l \varphi_t^F - \rho_l g w^F - P^F(\xi, y) e^{-i\xi Ut} (1 + \cos(\omega' t)/2), \quad (10)$$

where $\varphi^F(\xi, y, z, t)$ and $P^F(\xi, y) e^{-i\xi Ut}$ are corresponding Fourier images of $\varphi(x, y, z, t)$ and $P_o P_1(X) P_2(y)$. Then the Fourier image of the ice deflection, $w^F(\xi, y, t)$, is sought in the form

$$w^F(\xi, y, t) = w^a + w^b = \sum_{n=1}^{\infty} [a_n(\xi, t) \psi_n(\xi, y) + b_n(\xi, t) \psi_n(\xi, y)]. \quad (11)$$

Here $\psi_n(\xi, y)$ are the eigenmodes of the ice cover oscillations in the channel, and $a_n(\xi, t)$ and $b_n(\xi, t)$ are the principle coordinates of the modes. The functions $\psi_n(k, y)$ were calculated in [4]. They describe the hydroelastic waves propagating along the channel

$$w(x, y, t) = \text{Re} \left[A^n \psi_n(k, y) e^{i(kx - \omega_n t)} \right], \quad (12)$$

where $\omega_n(k)$ is the wave frequency, $k > 0$ is the wave number and A^n is the wave amplitude. The function $w^a = \sum_{n=1}^{\infty} a_n(\xi, t) \psi_n(\xi, y)$ in (11) describes established at large times hydroelastic waves caused by the load with constant magnitude

$$Mw_{tt}^a + D \left(w_{yyyy}^a - 2\xi^2 w_{yy}^a + \xi^4 w^a \right) + \rho_l g w^a = -\rho_l \varphi_t^a - P^F(\xi, y) e^{-i\xi Ut}, \quad (13)$$

and function w^b describes unsteady hydroelastic waves corresponding to the effect of the load oscillations

$$Mw_{tt}^b + D \left(w_{yyyy}^b - 2\xi^2 w_{yy}^b + \xi^4 w^b \right) + \rho_l g w^b = -\rho_l \varphi_t^b - \frac{P^F(\xi, y) e^{-i\xi U t} \cos(\omega' t)}{2}, \quad (14)$$

where $\varphi = \varphi^a + \varphi^b$.

4. Deflection corresponding to the steady waves propagating at the load speed

By the method of separating variables in the equation (13) we arrive at the following differential equations for the principle coordinates $a_n(\xi, t)$,

$$\frac{d^2 a_n}{dt^2} + \omega_n^2(\xi) a_n = H_n(\xi) e^{-iU\xi t}, \quad H_n(\xi) = \frac{\int_{-L}^L P^F(\xi, y) \psi_n(\xi, y) dy}{\rho_l L \int_{-L}^L (\alpha \psi_n(\xi, y) + \varphi_n(\xi, y, 0)) \psi_n dy}$$

where $\alpha = (\rho_i h_i / \rho_l L)$ and $\varphi_n(\xi, y, z)$ is the flow potential corresponding to the mode $\psi_n(\xi, y)$. This equation is solved analytically subject to the initial conditions.

The ice deflection $w^a(x, y, t)$ is given by the inverse Fourier transform applied to (11), where

$$\begin{aligned} \int_{-\infty}^{\infty} a_n(\xi, t) \psi_n(\xi, y) e^{i\xi x} d\xi &= \int_{-\infty}^0 a_n(\xi, t) \psi_n(\xi, y) e^{i\xi x} d\xi + \int_0^{\infty} a_n(\xi, t) \psi_n(\xi, y) e^{i\xi x} d\xi = \\ &= \int_0^{\infty} H_n \psi_n \left(\frac{\xi U}{2\omega_n^2} \left[\frac{e^{i(\xi x + \omega_n t)}}{\omega_n + \xi U} - \frac{e^{i(\xi x - \omega_n t)}}{\omega_n - \xi U} + \frac{e^{-i(\xi x + \omega_n t)}}{\omega_n + \xi U} - \frac{e^{-i(\xi x - \omega_n t)}}{\omega_n - \xi U} \right] + \frac{\cos(\xi(x - Ut))}{2(\omega_n^2 - \xi^2 U^2)} \right) d\xi. \end{aligned} \quad (15)$$

In the numerical calculations, the series (11) for $w^a(\xi, y, t)$ is truncated to N_{mod} terms. Using moving coordinate system (X, y, z) we find asymptotic behaviour of each integral in (15) for each mode number n for large times, $t \rightarrow \infty$ and $X = O(1)$. Depending on the load speed U some integrals in (15) have poles at the points where $\omega_n(\xi) = \xi U$. Note that the integral (15) is regular and can be written as the sum of five integrals which are understood as Cauchy principal value integrals, in general. If an integral does not have a pole then it is regular and its contribution is of order $O(1/t)$ as $t \rightarrow \infty$. The large-time contributions of non-regular integrals consist of two parts: (a) a part which is even in X and decaying as $|X| \rightarrow \infty$, (b) waves (12) with amplitudes A^n propagating behind and in front of the load. In these waves, $k = \xi_m^n$, $m = 1, 2$, where ξ_m^n are solutions of the equation $\omega_n(\xi) = \xi U$, $\xi_1^n < \xi_2^n$, and the amplitudes A_m^n are given by

$$A_m^n = \frac{2\pi H_n(\xi_m^n)}{\xi_m^n U (c_g^n(\xi_m^n) - U)},$$

where c_g^n is the group velocity of the n -th mode of hydroelastic wave in the channel. The phase speeds of these waves is equal to the speed of the load, U . These waves do not propagate with respect to the load in the moving coordinate system. The long waves with ξ_1^n are placed behind the load, and the short waves with ξ_2^n are in front of the load.

5. Deflection corresponding to the unsteady waves propagating from the load

By the method of separating variables in the equation (14) we arrive at the following differential equations for the principle coordinates $b_n(\xi, t)$,

$$\frac{d^2 b_n}{dt^2} + \omega_n^2(\xi) b_n = H_n(\xi) \left(\frac{e^{i\omega' t - i\xi U t} + e^{-i\omega' t - i\xi U t}}{2} \right),$$

This equation is solved analytically subject to the initial conditions.

The ice deflection $w^b(x, y, t)$ is given by the inverse Fourier transform applied to (11), where

$$\begin{aligned} \int_{-\infty}^{\infty} b_n(\xi, t) \psi_n(\xi, y) e^{i\xi x} d\xi &= \int_{-\infty}^0 b_n(\xi, t) \psi_n(\xi, y) e^{i\xi x} d\xi + \int_0^{\infty} b_n(\xi, t) \psi_n(\xi, y) e^{i\xi x} d\xi = \\ &= \int_0^{\infty} H_n \psi_n \left\{ \frac{1}{4\omega_n^2} \left(\left[\frac{\xi U + \omega'}{\omega_n + (\xi U + \omega')} + \frac{\xi U - \omega'}{\omega_n + (\xi U - \omega')} \right] e^{i(\xi x + \omega_n t)} - \right. \right. \\ &- \left. \left[\frac{\xi U - \omega'}{\omega_n - (\xi U - \omega')} + \frac{\xi U + \omega'}{\omega_n - (\xi U + \omega')} \right] e^{i(\xi x - \omega_n t)} + \left[\frac{\xi U + \omega'}{\omega_n + (\xi U + \omega')} + \frac{\xi U - \omega'}{\omega_n + (\xi U - \omega')} \right] e^{-i(\xi x + \omega_n t)} - \right. \\ &- \left. \left. \left[\frac{\xi U - \omega'}{\omega_n - (\xi U - \omega')} + \frac{\xi U + \omega'}{\omega_n - (\xi U + \omega')} \right] e^{-i(\xi x - \omega_n t)} \right) + \right. \\ &\left. \frac{e^{i(\xi x - \xi U t) + i\omega' t}}{2(\omega_n^2 - (\xi U - \omega')^2)} + \frac{e^{i(\xi x - \xi U t) - i\omega' t}}{2(\omega_n^2 - (\xi U + \omega')^2)} + \frac{e^{-i(\xi x - \xi U t) + i\omega' t}}{2(\omega_n^2 - (\xi U + \omega')^2)} + \frac{e^{-i(\xi x - \xi U t) - i\omega' t}}{2(\omega_n^2 - (\xi U - \omega')^2)} \right\} d\xi. \end{aligned} \quad (16)$$

The last four terms in the equation (16) describe unsteady hydroelastic waves propagating from the load with the wave number ξ and frequency ω' . The numerators of these terms can be written in the form $\cos(\xi X + \omega' t)$ and $\cos(\xi X - \omega' t)$.

Asymptotic analysis, done for the steady waves, can be repeated in this case. First, note that the integrals in the equation (16) can be singular in four cases: $\omega_n = -(\xi U + \omega')$, $\omega_n = -(\xi U - \omega')$, $\omega_n = (\xi U - \omega')$ and $\omega_n = (\xi U + \omega')$. We omit the first case since it is not physical. Other cases give

$$c_n + c' = U, \quad c_n - c' = U, \quad c' - c_n = U, \quad (17)$$

where c' is the phase speed of the wave propagating along the channel from the load, $c' = \omega'/\xi$. If there are singularities at the point ξ_* in some terms in the equation for $b_n(\xi, t)$, then these singularities will always be in two terms, one in term with $1/(4\omega_n^2)$ and one in the four last terms in the equation (16). These singularities counter each other and the total integral for $b_n(\xi, t)$ is regular. Following the same asymptotic analysis, equation (17) gives that, in contrast to the steady waves, waves caused by the load oscillations occur where the speed of the load is equal to the phase speed of the hydrodynamic wave shifted by the phase speed c' , which is a hyperbola for $\omega' \neq 0$. In total there can be no more than four such waves for each mode ψ_n . Figure 1 shows the curves $c_n + c'$, $c_n - c'$ and $c' - c_n$ as functions of the wave number for the first mode in the considered channel. Calculations are performed for the freshwater ice with $h_i = 10\text{cm}$, $L = 10\text{m}$, $H = 5\text{m}$ and $\omega' = 1\text{ s}^{-1}$. The dotted line shows the example of the speed of the load, for which there will be maximum number of the unsteady waves propagating from the load for the given mode. For an example, let us calculate the wave parameters in the case of a singular point ξ_* , $\omega_n(\xi_*) = -\xi_* U + \omega'$. In which direction this wave runs, forward or backward from the load, depends on the sign of the derivative $\omega_n + \xi U - \omega'$. The sign is greater than zero, hence, in this case the wave is formed before the load. This wave has the form

$$w(x, y, t) = \text{Re} \left[A' \psi_n(\xi_*, y) e^{i(\xi_* X + \omega' t)} \right], \quad (18)$$

where the amplitude A is given by

$$A' = \frac{\pi H_n(\xi_*)}{(\omega' - \xi_* U)(c_g^n(\xi_*) + U)},$$

6. Conclusion

The obtained exact decomposition of the ice deflection to the symmetric, long and short wave parts for large times makes it possible to estimate the maximum strains under the load, far behind and in front of the load. The analytical formulas are obtained for these waves. It is shown that effect of the oscillation of the magnitude of the load leads to the unsteady waves propagating from the load with speeds different from the speed of the load. Number of these waves is limited by four for each mode.

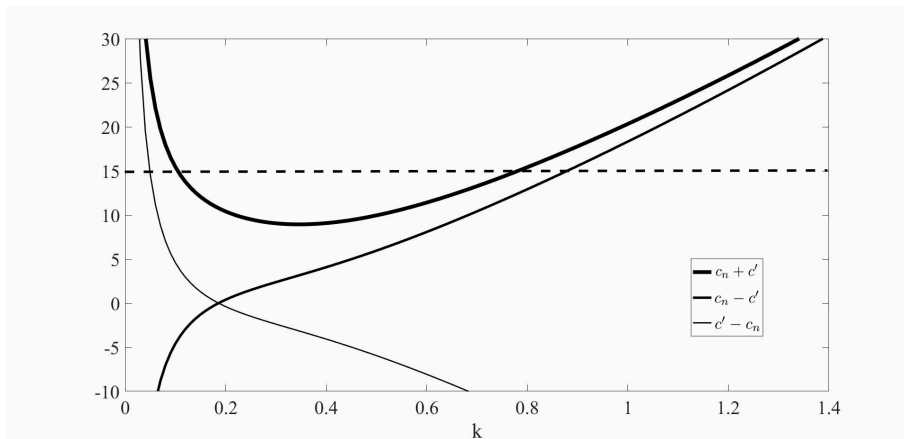


Figure 1. The curves $c_n + c' = U$, $c_n - c' = U$, and $c' - c_n = U$, where c_n is the phase speed of the first mode, $c' = \omega'/k$, $\omega' = 1 \text{ s}^{-1}$. The dotted line shows the speed of the load.

Acknowledgments

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References

- [1] Squire V, Hosking R, Kerr A and Langhorne P 1996 *Moving loads on ice* (Amsterdam: Kluwer Academic Publishers)
- [2] Guyenne P and Parau E 2012 *Journal of Fluid Mechanics* **713** 307–29
- [3] Hosking R J, Sneyd A D and Waugh D W 1988 *Journal of Fluid Mechanics* **196** 409–30
- [4] Korobkin A, Khabakhpasheva T and Papin A 2014 *European Journal of Mechanics B/Fluids* **47** 166–75
- [5] Shishmarev K, Khabakhpasheva T and Korobkin A 2016 *Applied Ocean Research* **59** 313–26
- [6] Schulkes R and Sneyd A 1988) *Journal of Fluid Mechanics* **186** 25–46
- [7] Khabakhpasheva T I 2006 *Fluid Dynamics* **41(3)** 424–33.
- [8] Sturova I V 2017 *Journal of Applied Mechanics and Technical Physics* **58** 80–8