Measures, Slaloms, and Forcing Axioms

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A thesis submitted to the School of Mathematics of the University of East Anglia in partial fulfilment of the requirements for the degree of Doctor of Philosophy

August 2018

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Contents

Chapter 1

Introduction

This thesis is, unashamedly, a thesis of two parts. The first and larger part consists of a study of (an aspect of) the *Souslin Hypothesis*, and the second and smaller part concerns a study of (an aspect of) forcing axioms. Given that the very concept of a 'forcing axiom' (an idea due to Martin, [MS70]) arose out of Solovay and Tenenbaum's proof of the consistency of the Souslin Hypothesis [ST71], one could attempt to build some sort of tenuous connection between the two parts, but this connection would exist only in my imagination if it were to be very overactive, so I shall not attempt to make it. A simpler connection which I shall satisfy myself with pointing out is that both of these topics are central topics in set theory, the former on the 'applied' side, the latter on the 'pure' side.

All the topological spaces in this thesis are Hausdorff spaces. The Souslin Hypothesis, broadly interpreted, concerns the difference between two properties enjoyed by sets of reals: having a countable dense set (that is, being *separable*), and having the property that any collection of pairwise disjoint non-empty open sets is countable (that is, being *ccc*, or having the *countable chain condition*). It is easy to see that the former implies the latter, but the former is a property of the topological space as a whole, whereas the latter is purely a property of the topology of the space. It follows that while the former is the stronger property, the latter is a nicer property, in that it is more amenable to testing by purely topological methods.

Before starting with the history of the Souslin Hypothesis and trying to justify this broad interpretation of it, I feel a justification of the study of this problem is in order. Why are separable spaces important? Most objects that mathematicians study are infinite. However, despite there being quite a few infinite cardinalities, most of the infinite objects that mathematicians study are either countable or 'essentially' countable, in that there are some countably many 'characteristics' such that any particular 'piece' of this object can be described to any predetermined precision by describing their behaviour on a large enough finite subset of these characteristics. A guiding example is a real number and its infinite decimal expansion. As any infinite, uncountable, mathematical object typically has a topological structure, and separability is the most stringent definition for 'essentially countable' in a topological context, it follows that the nicest (infinite) topological spaces are the separable ones. On the other hand, the countable chain condition is arguably the least stringent definition for 'essentially countable' in a topological context. One looks forward then, to finding classes of topological spaces where the two coincide, since these are examples of spaces where having a little gives one a lot.

Returning to the history of the Souslin Hypothesis, it starts thus: Cantor showed in that all countable dense linear orders without endpoints are isomorphic. It follows that any complete, dense-in-itself, linearly-ordered compact topological space which has both end points and is separable is in fact homeomorphic to the unit interval $[0,1]$. Souslin wondered if a similar characterisation of $[0,1]$ could be obtained by replacing 'separable' in this statement by 'has the countable chain condition', and the assertion that this does indeed happen has come to be known as 'Souslin's Hypothesis'. Of course, if all such spaces were separable, then one would obtain the desired conclusion. Consequently, one can rephrase Souslin's Hypothesis (henceforth, SH) in the following way:

Question 1. In the category of complete, dense-in-itself, linearly-ordered, compact topological spaces with both endpoints, are the properties of separability and having the ccc equivalent?

It is now known that SH is independent of ZFC. Jech [Jec67] and Tennenbaum [Ten68] showed that the negation of SH is consistent with ZFC, and Solovay and Tennenbaum [ST71] showed that SH is consistent with ZFC. Jensen also showed that the negation of SH holds in Gödel's Constructible Universe [Jen68], and also that SH is consistent with ZFC together with the Generalised Continuum Hypothesis GCH [DJ06]. Each of these results have resulted in important advances in set theory, and SH and Cantor's Continuum Hypothesis, CH, have been the two motivating problems which have driven the growth of the entire subject of modern set theory.

However the setbacks of independence have not stopped topologists from investigating this gap between separability and the countable chain condition. For brevity, by a *Souslinean space*, we shall mean a compact, non-separable, ccc space. One can in fact show that Suslin's hypothesis is equivalent to the following:

Question 2. Is there a linearly ordered Souslinean space?

The most inclusive question one can ask is: which categories of compact topological spaces do not contain any Souslinean spaces? But two more natural questions are: What properties of the unit interval $[0,1]$ ensure that there are no Suslinean spaces with that property? In a given class of spaces, what strengthenings of the countable chain condition ensure separability? While the first question is a very simple investigation of the contexts in which there is no gap between separability and the countable chain condition, the second question treats the Souslin Hypothesis as being an investigation of the nature of the unit interval $[0,1]$; the third question can be seen as an investigation of the 'size' of the gap between separability and the ccc. An excellent reference for this topic is [Tod00], and an older reference is [Tal74].

In this thesis, instances of all three of these questions are considered, though all of the results are negative in that they suggest that this gap is not so easily bridgeable. The results of the relevant chapters are, barring one chapter, based on the paper [BNIar], which is joint work with Piotr Borodulin-Nadzieja. The motivating question for that paper was the following, a positive answer to which Todorčević called "the ultimate version of Suslin hypothesis" in [Tod00]:

Question 3. Is there a Souslinean space which does not map continuously onto $[0,1]^{\omega_1}$?

One can show that no compact linearly-ordered space can be mapped onto $[0,1]^{\omega_1}$ (see Corollary 121), and the property of mapping continuously onto $[0,1]^{\omega_1}$ can be seen as suggesting that a space is large in a certain way that we might discount any Souslinean spaces having this property as not violating the spirit of the Souslin Hypothesis. The property of not being linearly ordered which is present in the statement of Souslin's original hypothesis can be seen as another such discount to largeness. Another example of a discount to largeness is having a π-basis (see Section 2.4) of size greater than or equal to the continuum, $\mathfrak{c}$, or not being first countable (that is, not having a countable local basis at each point). Indeed, Hajnal and Juhász have shown that *Martin's Axiom at* ω_1, MA_{ω_1}, implies that there are no Souslinean spaces having a π-base of size less than $\mathfrak{c}$, [Juh71], and Todorčević and Veličković showed that MA_{ω_1} is equivalent to the statement that there are no Souslinean spaces which are first countable, [TV87].

Returning to our so-called 'ultimate Suslin hypothesis', it was shown by Bell that MA_{ω_1} is consistent with a Souslinean space which maps onto $[0,1]^{\omega_1}$, [Bel96], and later Moore showed that MA implies the existence of such a space, [Moo99]. It was finally shown by Todorčević that such a space exists just in ZFC itself [Tod00].

This led Plebanek and Borodulin-Nadzieja to weaken this 'ultimate Suslin Hypothesis' by strengthening the countable chain condition in the question of Todorčević. They considered the following question [BNP15]:

Question 4. Is there a Souslinean space which supports a measure but does not map continuously onto $[0,1]^{\omega_1}$?

Here, the measures are finitely additive and non-negative, and since the measure of the entire space is finite, so any space carrying a measure necessarily has the countable chain condition. In [Kun81], using CH, Kunen constructed a Souslinean space supporting a measure which is first countable, and hence does not map onto $[0,1]^{\omega_1}$. In [BNP15], Plebanek and Borodulin-Nadzieja were able to construct such a space under MA. The motivating question of [BNIar] was whether such a space could be constructed just in ZFC.

We were not able to find such a ZFC construction. We were however able to weaken the hypothesis needed for the existence of such a space to one involving two cardinal characteristics of the continuum.

Theorem 5. ($\mathrm{add}(\mathcal{N}) = \mathrm{non}(\mathcal{M})$) There is a Souslinean space which supports a measure, and does not map continuously onto $[0,1]^{\omega_1}$. In fact, this space is of the form $K \setminus \omega$ for K a compactification of ω.

In particular, such spaces exist in any model of CH or MA. Our methodology was to analyse the space constructed by Todorčević, and to find a modification of it from this extra hypothesis so that it supports a measure. The existence of the measure is shown in an indirect way, using a result of Kamburelis which characterises when a Boolean algebra supports a measure by its behaviour in forcing extensions in a measure algebra [Kam89]. By varying Todorčević' construction slightly, we were able to obtain several different compactifications of ω which did or did not support a measure. For example

Theorem 6. There is a Souslinean space which supports a measure which is of the form $K \setminus \omega$ for K a compactification of ω.

That there is a Souslinean space of the form $K \setminus \omega$ for K a compactification of ω was first shown by Bell [Bel80]. Our construction, which is the first such space also supporting a measure, answers a question from [DP15]. In more recent work, Borodulin-Nadzieja and Żuchowski have constructed other such examples [BNŻ16]. The Banach spaces of continuous functions of these compactifications are of interest as well. For example

Theorem 7. There is a compatification K of ω such that $K \setminus \omega$ is non-separable and such that the natural copy of c_0 in $C(K)$ is complemented.

Such spaces were studied by Drygier and Plebanek in [DP17], but they could only construct such a space under CH. In particular, no such ZFC example was known before.

The second part of this thesis does not have so many new results, and most of it is related to some work in progress. The main theme underlying it is to gain an understanding of the current limitations of forcing axioms. These limitations are of two types, and both of them have to do with the currently available forcing techniques being unable to handle the cardinal $\aleph_2$.

We start by recalling what a forcing axiom is.

Definition 8. Let $\mathcal{P}$ be a class of partial orders and κ a cardinal. Then $\mathsf{FA}_\kappa(\mathcal{P})$ denotes the statement that for every $\mathbb{P} \in \mathcal{P}$ and any collection of dense subsets of $\mathbb{P}$ of size at most κ, there is a downwards-closed directed subset of $\mathbb{P}$ which meets every element of this collection.

The first such was *Martin's Axiom*, denoted MA, due to Martin and Solovay, [MS70], which is the forcing axiom for the class of partial orders having the *countable chain condition* (that is, partial orders having no uncountable antichains, shortened to ccc) and meeting κ-many dense sets for any $\kappa < \mathfrak{c}$, the size of the continuum. Here it should be noted that if the continuum hypothesis, CH, holds, then MA is trivially true. What Martin and Solovay showed is that MA can hold with the continuum being arbitrarily large. An important fragment of MA is MA_{ω_1}, which is the forcing axiom for the class of partial orders having the ccc and meeting $\aleph_1$-many dense sets, which often suffices for many applications of $\mathsf{MA} + \neg\mathsf{CH}$.

Since then, two other forcing axioms have joined it, the *Proper Forcing Axiom*, denoted PFA, due independently to Shelah[She98] and Baumgartner [Dev83], which is the forcing axiom for the class of *proper* partial orders (see Definition 51) and meeting $\aleph_1$-many dense sets, and *Martin's Maximum*, denoted MM, due to Foreman, Magidor, and Shelah [FMS88], which is the forcing axiom for the class of partial orders which preserve stationary subsets of ω_1 and meeting $\aleph_1$-many dense sets. This order also reflects their ordering by strength: MA_{ω_1} is the weakest of these, whereas MM is the strongest. Also, while MA is consistent with the continuum being arbitrarily large, PFA and MM both imply that $\mathfrak{c} = \aleph_2$.

Since then, these axioms, namely, $\mathsf{MA}_{\omega_1}, \mathsf{PFA}$ and MM, have had very many applications, both inside set theory, as well as outside set theory, to Abelian group theory, measure theory, functional analysis, topology etc. See for example [Fre84, Bau84, Tod13]. The large number of these applications raises two natural questions which we now discuss.

The first is the following. Many of these applications can be thought of as exploiting the 'gap' between countable and uncountable, equivalently, between ω and ω_1. Understanding this gap has proved to be enormously fruitful to set theory, both from attempting to widen it (where one uses statements which hold in Gödel's Constructible Universe to perform diagonalisations for example), and attempts to narrow it (using forcing axioms to glue together approximations to an object coherently). It has also proved fruitful both from the 'pure' standpoint as well as the 'applied' standpoint. What about the other gaps between cardinals? Do each of them have their own interesting combinatorics? Is there some sort of stabilising phenomenon whereby the gap between $\aleph_{\alpha+437}$ and $\aleph_{\alpha+438}$ tend to become similar or less interesting as α increases? Questions about the stability of these gaps is far too advanced a question for us, and we make do with simply trying to understand the most basic unexplored such gap, that between ω_1 and ω_2. While we have so far persisted in this vague choice of terminology, and asked grand questions, we now ask some simpler (but still vague) questions. Does ω_2 have any interesting combinatorics of its own? Can one simply 'pull up one cardinal higher' various results about ω_1 and have them apply to ω_2? Since forcing axioms play an important role in the combinatorics of ω_1, they provide us a way of concretising some of these vague notions. Are there any interesting forcing axioms for meeting ω_2-many dense sets?

Note that MA_{ω_2} already provides a positive answer to this question. However, since PFA and MM imply that $\mathfrak{c} = \aleph_2$, we cannot pull them up one level while avoiding inconsistencies. Furthermore, since ccc partial orders cannot construct very interesting objects of size $\aleph_2$, while MA_{ω_2} does provide a positive answer to our question, it is a rather weak positive answer, so one hopes that one can do better.

The second of these natural questions goes as follows. When one considers the interesting consequences of PFA and MM both within and without set theory, these consequences are shown to be consistent with $\mathfrak{c} = \aleph_2$. But do these consequences themselves impose any limitations on the size of the continuum? In the case of consequences outside of set theory (for example [Far11, Tod06, BJP05, Vel05]), one can think of this as trying to measure how set theoretic these consequences are: we are asking if they have any bearing on the most famous problem in set theory.

Since MA is consistent with the continuum being arbitrarily large, we do have some understanding of the relation between its consequences and the size of the continuum. For the rest however we are ignorant, and so we ask: are there interesting fragments of PFA which are consistent with the continuum being larger than $\aleph_2$?

Both of these questions are in fact instances of the same problem, namely that of preserving ω_2 (and hence, ω_1 as well) in iterated forcing constructions. Both of the two standard approaches do not easily work as shown by the following folklore facts:

Theorem 9. (i) The finite-support iteration of posets which do not have the countable chain condition collapses ω_1 at stages of cofinality ω.

(ii) The countable-support iteration of posets which do not have the $\aleph_2$-cc collapses ω_2 at stages of cofinality ω_1.

(iii) In a countable-support iteration, CH is true at stages of cofinality ω_1, so the countable-support iteration of posets which add a real collapses ω_2 at stages of cofinality ω_2.

The sole possibility left if one persists in doing a countable support iteration is to start with a model of CH and iterate $\aleph_2$-cc posets which do not add a real. A result of Shelah [She98, Appendix, 3.4A] shows that the most natural candidate for a forcing axiom of this sort fails.

Theorem 10. (CH) There is a σ-closed $\aleph_2$-cc partial order $\mathbb{P}$ and $\aleph_2$-many dense subsets of it such that no directed subset of $\mathbb{P}$ can meet all of the dense sets.

Some positive results were obtained (see [She78, KT79]) by strengthening the $\aleph_2$-cc and adding the condition that the posets $\mathbb{P}$ be well-met: given any $p, q \in \mathbb{P}$ which are compatible (there is some $r \in \mathbb{P}$ such that $r \leq p, q$), they in fact have a greatest lower bound. In Chapter 10 we give a simple proof of the following:

Theorem 11. (CH) There is a σ-closed well-met $\aleph_2$-cc partial order $\mathbb{P}$ and $\aleph_2$-many dense subsets of it such that no directed subset of $\mathbb{P}$ can meet all of the dense sets.

In fact our posets satisfy a much stronger chain condition. In particular, not only is the well-met condition necessary to obtain the positive results, one must also strengthen the $\aleph_2$-cc considerably.

On the other hand, if one abandons countable-support iterations, another way of iterating partial orders must be found. Such an approach was taken by Asperó and Mota in [AM15a, AM15b, AM16] where they consider a type of iteration which they call *iterated forcing with finite support using symmetric systems of structures as side conditions*. Using this, they were able to show that certain fragments of PFA are consistent with the continuum being larger than ω_2.

Following this work of Asperó and Mota, in Chapter 8 a fragment of PFA, the *Bounded Graph Ideal Dichotomy*, abbreviated $\mathsf{GID}_{\aleph_1}$, is isolated which it is hoped would be amenable to some modification of their method. If such a modification could be found, one would be able to show that $\mathsf{GID}_{\aleph_1}$ is consistent with the continuum being larger than $\aleph_2$. In Section 8.4 an attempt is made to justify this hope.

This fragment in fact corresponds to restricted forms of a certain class of partial orders which have been used very successfully by Todorčević to discover many interesting consequences of PFA (see for example [Tod13]), the partial orders which have *finite chains of elementary substructure as side conditions*, or the partial orders built using the *side condition method*. The axiom $\mathsf{GID}_{\aleph_1}$ is then an attempt at isolating a 'bounded form of an axiom for the side condition method'. In Chapter 9 several applications are given of $\mathsf{GID}_{\aleph_1}$ (though most of them in fact follow from a previously known statement proved consistent by Todorčević in [Tod85]), and in Section 9.4 an attempt is made at justifying its status as *the* bounded form of an axiom for the side condition method by giving the example of the prototypical application of the side condition method, the *Open Graph Axiom* of Todorčević, abbreviated OGA. Namely, it is shown that $\mathsf{GID}_{\aleph_1}$ implies a very weak form of the OGA which is obtained by restricting two unbounded set quantifiers to sets of size at most $\aleph_1$.

Unfortunately, we have been unable so far to show that $\mathsf{GID}_{\aleph_1}$ is indeed consistent with the continuum being larger than $\aleph_2$, which leaves the importance of the axiom $\mathsf{GID}_{\aleph_1}$ in a state of limbo. We hope that this is merely its temporary abode.

1.1 A personal tour of the thesis

In Chapter 2, various background results are collected together. As the thesis is a thesis of a few parts, I have tried to partition this chapter so that the reader may easily be able to find various results and notation quickly when they need to refer back to this chapter. Nonetheless, I have tried to include in the beginning of every chapter the (non-obvious) notation I have adopted earlier to help with the ease of reading.

In Chapter 3, a proof is given of a measure-theoretic result of Kamburelis which is used later in Chapter 6. The proof I give is somewhat different than the one in the paper of Kamburelis [Kam89], and these differences raise some questions which are also discussed. This chapter is self-contained.

Chapters 4, 5, 6, 7 are joint work with Piotr Borodulin-Nadzieja and from the paper [BNIar]. In Chapter 4, *slaloms* are introduced, and some families of slaloms are constructed. The main result of this chapter is obtained via a modification of a construction of Fremlin and Kunen, [FK91]. While our conclusion is stronger than their conclusion, we require more than ZFC unlike their result.

Chapter 5 describes a functor due to Todorčević [Tod00] which constructs a compact space when given the input of a family of slaloms. The rest of this chapter is devoted to a long list of facts about this functor. Barring Section 5.1, there are no definitions in this chapter, and in particular, the statements (but not the proofs) from Chapter 6, 7 can be understood without reading the rest of this chapter. The reader may then go back to this chapter as and when required. I hope that this might lessen any demotivating effect of the long dry list of facts contained therein. Almost all of these facts have straightforward proofs, however I have tried to not skip any details in order to make them as clear as possible (mostly to myself).

Chapter 6 collects together all of these facts to prove that the spaces obtained by applying Todorčević' functor have the properties that we want. Most of the proofs are obtained by simply composing the various facts from Chapter 5, except for the case of mapping onto $[0,1]^{\omega_1}$, where I have given more details since this was a notion that was previously unknown to me.

Chapter 7 is not self-contained, however it exhibits a degree of independence. It is an application of the compact spaces we have constructed to Banach spaces, and the problem that we have considered is motivated within the chapter itself. Since I was not familiar with this area before, I have also included some (possibly basic) results that the chapter requires within the chapter instead of relegating them to Chapter 2.

Chapters 8 and 9 are unrelated to the previous chapters. They describe some work in progress of mine. The bulk of Chapter 8 concerns itself with formulating and proving consistent an axiom $\mathsf{GID}_{\aleph_1}$ which is a consequence of PFA (but does not require large cardinals for its consistency). This axiom aims to capture some common arguments that can be found in various arguments about proper forcing, and which one hopes might be consistent with the continuum being larger than $\aleph_2$. The reason for this hope is explained in Section 8.4. The arguments from this chapter are modifications of arguments of Todorčević for example from [Tod13]. It might be argued then that the main result of this chapter is simply the formulation of this axiom and the analogy which is explained in Section 8.4. Apart from Chapter 5, this is the other chapter which could be categorised as a 'technical chapter'.

In Chapter 9 some applications are given of $\mathsf{GID}_{\aleph_1}$ which can all be found for example in [Tod13]. In fact most of these are consequences of a fragment of $\mathsf{GID}_{\aleph_1}$ which has been considered before for example in [Tod85]. However, in Section 9.4 another attempt is made at justifying the abstract approach employed in the formulation of $\mathsf{GID}_{\aleph_1}$.

Chapter 10 is self-contained. An elementary (modulo a function constructed by Todorčević) argument is given why a very natural higher forcing axiom fails.

Throughout, I have tried to break chapters down into easily digestible pieces, and I hope this makes the task of the reader easier.

1.2 Acknowledgements

I would like to thank here my supervisors David Asperó and Mirna Džamonja for their great patience with me throughout the duration of my studies. Somebody had recommended Norwich as a place where I would find people willing to give me their time and sit down and explain mathematics to me, and the last four years have proved this to be true. I would also like to thank my examiners Jonathan Kirby and Philip Welch (and my supervisors, and the people at UEA, and ...) for being very understanding of certain delays in the submission of my thesis.

I would also like to thank the many mathematics teachers who were responsible for me discovering an interest in mathematics, in logic, in set theory, and for me deciding to pursue mathematics, logic, and set theory. The list is long, and for many of these people, their influence seems to me as being decisive, causing a certain swerve in my life that leads me here. However, to expound on this seems, in the moment, a combination of inappropriate and indulgent. For this reason, I simply make a list: the teachers and people of Bhaskaracharya Pratishthan, in particular Mrs. Deshpande and Dr. Aditi Phadke; Mrs. Deshpande from my school; Prof. M. Prakash; my friends from Bhaskaracharya Pratishthan and Prakash sir's class, three of them in particular; the students, library, and professors of CMI, in particular Prof. S. P. Suresh; Benedikt Löwe; the people in the mathematics track at the

ILLC during my Master and in particular the participants of the Set Theory Lunch; David Asperó; Mirna Džamonja. While there are still many things that I need to, and would like to, learn from them, they serve to me as examples of the effect that a teacher and a colleague can have on someone. All of these people that I have had the fortune to meet are a part of what I would like to be. And, in turn, I shall continue to have their voices in my head whenever I find myself sitting at a table with a pen and paper.

I would also like to thank my coauthor on the paper whose results are included here, Piotr Borodulin-Nadzieja, and various people who lived at some point or the other in Amsterdam, in Norwich, in Cambridge, and in Norwich once more. On the whole, it will be a great sufficiency to me.

Lastly, I would like to thank my mother, my father, and my brother for, among other things, perhaps being my first teachers in mathematics.

Chapter 2

Preliminaries

2.1 General notation

An equality sign with a triangle on top, $\stackrel{\triangle}{=}$, will be used to establish some notational convention. The positive natural numbers are denoted by $\mathbb{N}^+$. If X is a set, then $\mathcal{P}(X)$ shall denote the powerset of X. If $\mathbb{P}$ is a partial order, we shall usually not be careful with specifying what the carrier set is and what the symbol for the order relation is. In general, if a partial order $\mathbb{P}$ is clear from context, then $\leq$ shall refer to its order relation, and $\mathbb{P}$ shall also be used to denote its carrier set as well.

By a *real number* we will mean an element of Baire space, ω^ω, or an element of some $\prod_{n\in\omega} S_n$, where $S_n \subseteq \omega$, the exact choice of which shall be clear from the context. If $S \subseteq \omega \times \omega$, then $S(n)$ will denote the horizontal section $\{m\colon (n,m) \in S\}$. If A and B are subsets of some set S (which will typically be the set of natural numbers), then $A =^* B$ shall denote that modulo a finite set, A and B are equal; equivalently, that the symmetric difference of A and B is finite. We will similarly talk of $A \subseteq^* B$ etc., which unless another context is specified, shall mean that $A \subseteq B$ modulo a finite set. Similarly, unless another context is specified, $[A]$ shall denote the set $\{B \subseteq S\colon B =^* A\}$.

By Fin we will denote the ideal of finite subsets (of a set which should be clear from the context).

If $f : X \to Y$ is a function and $Z \subseteq Y$, then $f|Z$ shall denote the function from Z into Y obtained by restricting the domain of f. If f is a function and X is a subset of its domain, then $f[X]$ will denote the set of the images of the elements of X under f.

Sequences shall often be confused with their range, and a set of ordinals will also be confused with the sequence of its increasing enumeration.

If $\bar{u}$ and $\bar{v}$ are sequences, then the concatenation of $\bar{u}$ with $\bar{v}$ will be denoted $\bar{u} \bullet \bar{v}$. If X, Y are sets of ordinals and α is an ordinal, then $X < Y, X < \alpha, \alpha < Y$ etc. shall refer to the particular relation holding between every pair of elements from each set, or every element from one set and the ordinal, or the ordinal and every element from the set etc. All graphs are assumed to be undirected. If A and B are sets, then $A \otimes B$ shall denote the collection of two element sets $\{a, b\}$ such that $a \in A$ and $b \in B$. In case A and B are sets of ordinals, then $A \otimes B$ shall denote the collection of ordered pairs $\{\alpha, \beta\}$ such that $\alpha \in A$, $\beta \in B$, and $\alpha < \beta$.

Notation 1. Let $n \in \omega$. Let $\mathcal{F} \subseteq [\omega_1]^n$. If $\bar{u} \in [\omega_1]^m$ where $m < n$, then

$$(\mathcal{F})_{\bar{u}} \stackrel{\triangle}{=} \{\bar{v} \in [\omega_1]^{n-m}\colon \bar{u} < \bar{v}, \bar{u} \bullet \bar{v} \in \mathcal{F}\}.$$

If $\bar{u} = \langle \xi \rangle$ is a sequence of length 1, then we write $(\mathcal{F})_\xi$ instead of $(\mathcal{F})_{\langle \xi \rangle}$.

We now give a clarification of hopefully the most enigmatic term we shall use in this thesis.

Definition 2. Given a set S, a cardinal κ is *large enough with respect to S* or *much larger than S* if $\kappa > (2^{|T|})^+$ where T is the transitive closure of S.

Before starting with specific notation and preliminaries, we list some standard references where the results we quote below can be found. For Boolean algebras and Stone duality, we refer to [KMB89]. For measure theory, we refer to [Fre15], or even our set theory references, which are [Jec03, Kun14]. For set theory of the reals, we refer to [BJ95]. We do not require any knowledge of topology beyond a basic course and the definitions we supply.

2.2 Boolean algebras

If $\mathfrak{B}$ is a Boolean algebra, we shall denote its constants by $0_{\mathfrak{B}}$ and $1_{\mathfrak{B}}$, and its operations by $+_{\mathfrak{B}}$, $-_{\mathfrak{B}}$, its relations by $\leq_{\mathfrak{B}}$ etc. whenever we want to be precise or feel that there is some scope for confusion. Otherwise, we will typically suppress the subscripts, and in fact we shall treat Boolean algebras as being subsets of powerset algebras in the sense that we will usually use $\cup$ instead of $+$, $\setminus$ instead of $-$, $\subseteq$ instead of $\leq$ etc.

Definition 3. Let $\mathfrak{B}$ be a Boolean algebra, and let $\mathcal{A} \subseteq \mathfrak{B}^+$. We say that $\mathcal{A}$ is a *generating set of* $\mathfrak{B}$ if every element of $\mathfrak{B}$ can be expressed as a Boolean expression of some finitely many elements of $\mathcal{A}$.

If $\mathfrak{B}$ is a Boolean algebra and $\mathcal{I}$ is an ideal on it ($\mathcal{U}$ is a filter on it), then we denote the quotient Boolean algebra as $\mathfrak{B}/\mathcal{I}$ ($\mathfrak{B}/\mathcal{U}$). If $b \in \mathfrak{B}$, then $[b]_{\mathcal{I}}$ ($[b]_{\mathcal{U}}$) shall denote the corresponding member of $\mathfrak{B}_{\mathcal{I}}$ ($\mathfrak{B}_{\mathcal{U}}$).

Proposition 4. *Let $\mathfrak{B}$ be a Boolean algebra, and $\mathcal{U}$ a filter on it. If $b \in \mathfrak{B}$ and $u \in \mathcal{U}$, then $[b]_{\mathcal{U}} = [b \cap u]_{\mathcal{U}}$.*

Proposition 5. *Let $\mathfrak{B}$ be a Boolean algebra, and $\mathfrak{A}$ a subalgebra of it. Let $\mathcal{U}$ be an ultrafilter on $\mathfrak{A}$, and $\mathcal{V}$ the ultrafilter on $\mathfrak{B}$ which it generates. Let $b \in \mathfrak{B} \setminus \mathfrak{A}$ be such that for each $a \in \mathcal{U}$, $b \cap a \neq 0_{\mathfrak{B}}$. Then $[b]_{\mathcal{V}} \neq 0_{\mathfrak{B}/\mathcal{V}}$. Indeed, all the elements of $\mathfrak{B}/\mathcal{V}$ are of this form.*

If $\mathfrak{B}$ is a Boolean algebra and $a \in \mathfrak{B}$, then $\mathfrak{B}_a$ shall denote the principal ideal generated by a, which we note is a Boolean algebra as well: its constants are $0_{\mathfrak{B}_a}$ and $1_{\mathfrak{B}_a} = a$, and the operations are given by $b +_{\mathfrak{B}_a} c = b +_{\mathfrak{B}} c$ and $-_{\mathfrak{B}_a} b = a -_{\mathfrak{B}} b$ (the others). Expressions for the other operations and relations can be derived from these.

Proposition 6. *Let $\mathfrak{B}$ be a Boolean algebra, and let $\mathcal{A} \subseteq \mathfrak{B}^+$ be a generating set of it. Let $\mathcal{I} \subseteq \mathfrak{B}$ be an ideal. Then those non-zero elements of the form $[a]_{\mathcal{I}}$ for $a \in \mathcal{A}$ form a generating set of $\mathfrak{B}/\mathcal{I}$.*

Definition 7. Let $\mathfrak{B}$ be a Boolean algebra, and let $\mathcal{A} \subseteq \mathfrak{B}^+$. We say that $\mathcal{A}$ is a *π-base* of $\mathfrak{B}$ if there is an element of $\mathcal{A}$ below every non-zero element of $\mathfrak{B}$.

Theorem 8. *Let $\mathfrak{B}$ be a Boolean algebra, and let $\mathcal{A} \subseteq \mathfrak{B}^+$ be a generating set for it. Then the non-zero elements of the form*

$$(\bigcap_{a \in F} a) \cap (\bigcap_{b \in G} \sim b)$$

for finite subsets F, G of $\mathcal{A}$ form a π-base of $\mathfrak{B}$.

Next, we shall need two fundamental results by Sikorski about Boolean algebras. The first is known as Sikorski's Extension Criterion.

Theorem 9. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras, and let $\mathcal{C} \subseteq \mathfrak{A}^+$ be a generating set for it. Let $f : \mathcal{C} \to \mathfrak{B}$ be any map. Then the following is a necessary and sufficient condition for f to extend to a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$: for any $n \in \omega$, for any $b_1, b_2, \ldots, b_n \in \mathcal{C}$, and for any $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{-1, +1\}$,*

$$\varepsilon_1 b_1 \cap \varepsilon_2 b_2 \cap \ldots \cap \varepsilon_n b_n = 0_{\mathfrak{A}} \implies \varepsilon_1 f(b_1) \cap \varepsilon_2 f(b_2) \cap \ldots \cap \varepsilon_n f(b_n) = 0_{\mathfrak{B}}.$$

The second is known as the Sikorski Extension Theorem.

Theorem 10. *Let $\mathfrak{B}$ be a Boolean algebra, $\mathfrak{A}$ a Boolean subalgebra of $\mathfrak{B}$, and $\mathfrak{C}$ a complete Boolean algebra. Let $f : \mathfrak{A} \to \mathfrak{C}$ be a Boolean homomorphism. Then this can be extended to a Boolean homomorphism $\bar{f} : \mathfrak{B} \to \mathfrak{C}$.*

We end with the following definitions which we will need in the thesis.

Definition 11. Let $\mathfrak{B}$ be a Boolean algebra. If $\mathcal{C} \subseteq \mathfrak{B}$ is a subset of $\mathfrak{B}$ which consists of pairwise disjoint sets and such that the only element of $\mathfrak{B}$ greater than all of them is $1_{\mathfrak{B}}$, then we call $\mathcal{C}$ a *partition of unity* in $\mathfrak{B}$.

2.3 Measure theory

We are interested in measures on Boolean algebras. However, these come in many forms, so we clarify.

Definition 12. Let $\mathfrak{B}$ be a Boolean algebra. A function $\mu : \mathfrak{B} \to [0, \infty)$ is called a *measure* on $\mathfrak{B}$ if

(i) $\mu(0_{\mathfrak{B}}) = 0$;

(ii) it is *finitely additive*: if $a, b \in \mathfrak{B}$ are disjoint, then $\mu(a \cup b) = \mu(a) + \mu(b)$.

It is said to be *strictly positive* if $\mu(a) = 0$ implies that $a = 0_{\mathfrak{B}}$. It is said to be *countably additive* or *σ-additive* if for any sequence $\langle a_n \colon n \in \omega \rangle \subseteq \mathfrak{B}$ consisting of pairwise disjoint elements, if $\cup_{n\in\omega} a_n \in \mathfrak{B}$, then $\mu(\cup_{n\in\omega} a_n) = \Sigma_{n\in\omega} \mu(a_n)$.

We shall in fact need another type of measures as well, see Definition 22.

Note that the homomorphisms between Boolean algebras which support a measure will typically be assumed to preserve the measure, though if we need to underline this aspect of theirs, we might be explicit in referrring to them as *measure-preserving homomorphisms*. In this thesis, by default, all measures are assumed to be strictly positive. Note that if μ is a measure on $\mathcal{B}$, then by rescaling, we can assume that there is a *probability measure* ν on $\mathfrak{B}$. That is, ν is a measure such that $\nu(1_{\mathfrak{B}}) = 1$. The class of atomless Boolean algebras on which there is a strictly positive finitely additive measure is denoted $\mathbf{M}$. Kelley's Theorem, Theorem 65, gives a criterion for when a Boolean algebra is in $\mathbf{M}$. Note that if μ is a measure on an atomless Boolean algebra $\mathfrak{B}$, then if $a \in \mathfrak{B}^+$, then there is some $b \leq a$ such that $\mu(a) > \mu(b) > 0$.

If μ is a measure on the Boolean algebra $\mathfrak{B}$, and $a \in \mathfrak{B}$, then we can define the restriction of μ to the Boolean algebra $\mathfrak{B}_a$. Typically, we shall perform this procedure when μ is a probability measure, and so we normalise this restriction to obtain a probability measure μ_a on $\mathfrak{B}_a$. It is easy to show that μ_a inherits all the properties of μ such as strict positivity, countable additivity etc.

If $\mathfrak{B} \in \mathbf{M}$ as witnessed by a strictly positive finitely additive measure μ, this allows us to define a notion of distance d on $\mathfrak{B}$ by defining $d(A, B) = \mu(A \triangle B)$. This makes $(\mathfrak{B}, d)$ into a (not necessarily complete) metric space. If κ is the density character of this metric space, we say that the *density character*, or simply *density* or *μ-density* if we wish to be careful, of $\mathfrak{B}$ is κ as well.

Theorem 13. *Let μ be a measure on $\mathfrak{B}$. Let $\mathcal{I}$ be an ideal of $\mathfrak{B}$ such that μ vanishes on $\mathcal{I}$:*

$$b \in \mathcal{I} \implies \mu(b) = 0.$$

Then the function $[\mu]_{\mathcal{I}}$ on $\mathfrak{B}/\mathcal{I}$ defined by: for $b \in \mathfrak{B}$,

$$[\mu]_{\mathcal{I}}([b]_{\mathcal{I}}) = \mu(b)$$

is well-defined and a measure on $\mathfrak{B}/\mathcal{I}$. If μ is countably additive, so is $[\mu]_{\mathcal{I}}$. Furthermore, if $\mathcal{I} = \{b \in \mathfrak{B} \colon \mu(b) = 0\}$, then $[\mu]_{\mathcal{I}}$ is strictly positive.

Also important to us will be the class of measure algebras.

Definition 14. Let $\mathfrak{B}$ be a Boolean σ-algebra. If there is a strictly positive σ-additive measure μ on it, then we refer to $(\mathfrak{B}, \mu)$ as a *measure algebra*. If further $\mu(1_{\mathfrak{B}}) = 1$, then the term *probability algebra* is also used.

Note that homomorphisms between measure algebras are the Boolean algebra homomorphisms which preserve the σ-algebra structure (that is, countable unions, countable intersections etc.) as well as the measure. One can also show that if $(\mathfrak{B}, \mu)$ is a measure algebra and $a \in \mathfrak{B}$, then $(\mathfrak{B}_a, \mu_a)$ is a measure algebra as well. Again, when we do this, we shall normally start with a probability algebra, and note that $(\mathfrak{B}_a, \mu_a)$ is then a probability algebra as well.

If $(\mathfrak{B}, \mu)$ is a measure algebra, then $\mathfrak{B} \in \mathbf{M}$ as well, so we can again define a notion of distance d on $\mathfrak{B}$ by defining $d(A, B) = \mu(A \triangle B)$. This makes $(\mathfrak{B}, d)$ not only into a metric space, but into a complete metric space, the completeness requiring the countable additivity of μ. Naturally, we can then talk about the *density character*, or simply *density* or *μ-density* of $\mathfrak{B}$. If the density character of a measure algebra is $\aleph_0$, we say that it is a *separable* measure algebra. If $(\mathfrak{B}, \mu)$ has density character κ, and for each $a \in \mathfrak{B}$, $(\mathfrak{B}_a, \mu_a)$ has density character κ as well, then we say that $(\mathfrak{B}, \mu)$ is *homogenous*.

The most important way of constructing measure algebras is the following: one starts with a set X, $\mathcal{A}$ a σ-algebra on it, and μ a not necessarily strictly positive σ-additive measure on $\mathcal{A}$. Such a triple $(X, \mathcal{A}, \mu)$ is called a *measure space*. Then we consider the σ-ideal of null elements of $\mathcal{A}$ (which we shall refer to as the *null ideal of the measure space* or the *null ideal of* $\mathcal{A}$):

$$\mathcal{N} = \{Y \in \mathcal{A} \colon \mu(Y) = 0\},$$

and then on the quotient Boolean algebra $[\mathcal{A}]_{\mathcal{N}}$ we can define a measure $[\mu]_{\mathcal{N}}$ in the following way: if $Y \in \mathcal{A}$, then $\mu_{\mathcal{N}}([Y]_{\mathcal{N}}) = \mu(Y)$. Using Theorem 13 one check that $\mu_{\mathcal{N}}$ is well-defined, and a countably additive strictly positive measure on $[\mathcal{A}]_{\mathcal{N}}$.

Often however, it is not so easy to obtain a measure space in the first place. The following result lets one construct them. We shall refer to it as the Hahn-Kolmogorov Theorem, though it or a closely related result is often attributed to Caratheodory as well. Caratheodory's Theorem is typically much more explicit about the construction, and proceeds through the construction of outer measures etc. Here we do not have any use of this explicitness.

Theorem 15. *Let $(X, \mathcal{A}, \mu)$ be a triple such that*

(i) X is a set;

(ii) $\mathcal{A}$ is an algebra on it (not necessarily a σ-algebra);

(iii) μ is a countably additive measure on $\mathcal{A}$.

Let $\mathcal{B}$ be the σ-algebra on X which $\mathcal{A}$ generates. Then there is a unique extension $\tilde{\mu}$ of μ to $\mathcal{B}$. Furthermore, the elements of $\mathcal{A}$ form a dense subset of $\mathcal{B}$: for every $\varepsilon > 0$ and $B \in \mathcal{B}$, there is some $A \in \mathcal{A}$ such that $\mu(A \triangle B) < \varepsilon$.

The Hahn-Kolmogorov Theorem or Caratheodory's Theorem allow us to construct many more measure algebras, including the most important examples of measure algebras. Let I be an infinite set. We can then define on the algebra of basic clopen subsets of the product space 2^I (that is, $[0,1]^I$ with the product topology) a strictly positive finitely additive measure μ (sometimes referred to as the *coin toss measure*) in the following way: if $\sigma : F \to 2$ is a function whose domain F is a finite subset of I, then we specify that

$$\mu(\{f \in 2^I : f \restriction F = \sigma\}) = \frac{1}{2^{|F|}}.$$

It is easy to verify that this is a finitely additive measure, and because the space 2^I is compact, it is in fact σ-additive for trivial reasons. Since the σ-algebra that the clopen subsets of 2^I generate is the σ-algebra $\mathrm{Bor}(2^I)$ of Borel subsets of 2^I, we can extend μ to a measure $\tilde{\mu}$ on $\mathrm{Bor}(2^I)$. By doing so, we obtain a measure space $(2^I, \mathrm{Bor}(2^I), \tilde{\mu})$. Let $\mathcal{N}_I$ be the null ideal of this measure space. The final step is important enough to this thesis that we enshrine it in an official definition.

Definition 16. Let I be an infinite index set. The measure algebra $(\mathfrak{R}_I, \mu_I)$ is the following:

(i) $\mathfrak{R}_I = \mathrm{Bor}(2^I)/\mathcal{N}_I$, and

(ii) $\mu_I = [\tilde{\mu}]_{\mathcal{N}_I}$ where $\tilde{\mu}$ is the extension of the coin toss measure on the basic clopen subsets of 2^I to $\mathrm{Bor}(2^I)$.

Note that $(\mathfrak{R}_I, \mu_I)$ is an atomless probability algebra, and one can also show that it is homogenous with density character $|I|$. By a deep result of Maharam, the above construction is generic for this class of Boolean algebras.

Theorem 17. *Let κ be an infinite cardinal. Let $(\mathfrak{B}, \mu)$ be a homogenous atomless probability algebra of density character κ. Then $(\mathfrak{B}, \mu)$ is isomorphic as a measure algebra to $(\mathfrak{R}_\kappa, \mu_\kappa)$.*

This allowed Maharam to classify all atomless measure algebras. We state this result here in the language of probability algebras.

Theorem 18. *Let $(\mathfrak{B}, \mu)$ be a an atomless probability algebra. Then there is a countable sequence of cardinals $\langle \kappa_n : n \in \omega \rangle$ and a countable partition of $1_{\mathfrak{B}}$, $\langle a_n : n \in \omega \rangle$, such that for each $n \in \omega$, $(\mathfrak{B}_{a_n}, \mu_{|a_n})$ is isomorphic to $(\mathfrak{R}_{\kappa_n}, \mu_{\kappa_n})$.*

We shall need the following basic facts about the measure algebras $(\mathfrak{R}_I, \mu_I)$.

Proposition 19. *Let $\kappa < \lambda$ be cardinals. Then $(\mathfrak{R}_\kappa, \mu_\kappa)$ is a measure subalgebra of $(\mathfrak{R}_\lambda, \mu_\lambda)$.*

Note that because of the nature of the construction of $(\mathfrak{R}_I, \mu_I)$, any particular element of the algebra is the quotient of an element of $\mathrm{Bor}(2^I)$, and measure of the latter in the appropriate sense is equal to the μ_I-measure of the former. This allows us to be sloppy in sense that we may talk about the μ_I-measures of the elements of $\mathrm{Bor}(2^I)$ without doing any undue harm. We shall be very sloppy (in this sense) throughout this thesis. A first example follows.

Proposition 20. *Let I be an index set. Let $X, Y \subseteq I$ be such that X and Y are disjoint. If $A \in \mathrm{Bor}(2^X)$, then let $E_A = \{f \in 2^I : f|_X \in A\}$.*

(i) Let $A \in \mathrm{Bor}(2^X)$. Then $\mu_I(E_A) = \mu_X(A)$.

(ii) Let $A \in \mathrm{Bor}(2^X)$ and $B \in \mathrm{Bor}(2^Y)$. Then E_A and E_B are independent sets in $\mathrm{Bor}(2^I)$.

(iii) In fact, if $\langle X_n : n \in \omega \rangle$ are pairwise disjoint subsets of I and $\langle A_n : n \in \omega \rangle$ is such that $A_n \in \mathrm{Bor}(2^{X_n})$, then $\langle E_{X_n} : n \in \omega \rangle$ is a mutually independent sequence of events.

The following is the Borel-Cantelli Lemma and its converse which we shall need.

Theorem 21. *Let $\langle E_n : n \in \omega \rangle$ be a sequence of events in a probability space.*

(i) If $\Sigma_{n\in\omega}\mathrm{Pr}(E_n) < \infty$, then the probability that infinitely many of them occur is 0:

$$\mathrm{Pr}(\mathrm{limsup}_{\mathrm{n}\to\infty} E_n) = 0.$$

(ii) Further, if $\langle E_n : n \in \omega \rangle$ form a mutually independent sequence of events, and $\Sigma_{n\in\omega}\mathrm{Pr}(E_n) = \infty$, then the probability that infinitely many of them occur is 1:

$$\mathrm{Pr}(\mathrm{limsup}_{\mathrm{n}\to\infty} E_n) = 1.$$

In fact, it is sufficient for $\langle E_n : n \in \omega \rangle$ to only be a sequence of pairwise independent sequence of events for this conclusion to hold.

While the first part is easy to prove, it should be pointed out that proving the converse assuming only pairwise independence is much harder than assuming mutual independence.

We end this section with another type of measures which we shall need. The chief difference with the kinds of measures we have considered so far is that they are allowed to take negative values as well.

Definition 22. Let $\mathfrak{B}$ be a Boolean algebra. A function $\mu : \mathfrak{B} \to (-\infty, +\infty)$ is called a *signed measure* on $\mathfrak{B}$ if

(i) $\mu(0_{\mathfrak{B}}) = 0$;

(ii) it is *finitely additive*: if $a, b \in \mathfrak{B}$ are disjoint, then $\mu(a \cup b) = \mu(a) + \mu(b)$.

It is said to be *bounded* if $\sup_{b\in\mathfrak{B}}|\mu(b)|$ is finite.

An important fact about bounded signed measures is that they can be uniquely represented as the difference of (positive) measures. This is known as the *Jordan decomposition* of a signed measure into (positive) measures.

Theorem 23. *Let $\mathfrak{B}$ be a Boolean algebra, and μ a bounded signed measure on it. Then there are measures ν_1 and ν_2 on $\mathfrak{B}$ such that $\mu(b) = \nu_1(b) - \nu_2(b)$ for each $b \in \mathfrak{B}$. This representation is moreover unique.*

Definition 24. Let μ be a signed measure on a Boolean algebra $\mathfrak{B}$ and let $\mu = \nu_1 - \nu_2$ be its Jordan decomposition, then the *variance* of μ, denoted $|\mu|$ is defined as follows: for each $b \in \mathfrak{B}$,

$$|\mu|(b) = \nu_1(b) + \nu_2(b).$$

Proposition 25. *If μ is a signed measure on a Boolean algebra $\mathfrak{B}$, then $|\mu|$ is a (positive) measure on $\mathfrak{B}$.*

The collection of bounded signed measures on a Boolean algebra form a Banach space, where the norm is given by the variation.

2.4 Topology

All topological spaces are assumed to be Hausdorff. We shall need the following standard fact from topology.

Theorem 26. *Let X be a compact space and Y any (Hausdorff) space. Let $f : X \to Y$ be a continuous map. Then it is a closed map: the image of a closed set is a closed set.*

Definition 27. Let X be a topological space. If L is a compact topological space such that $X \subseteq L$ and X is dense in L, then we say that L is a *compactification* of X. We also call $L \setminus X$ the *remainder* of X in L, or a *growth* of X.

In this thesis, we shall only consider the above situation where K is the set of natural numbers ω with the discrete topology, so we shall be talking about growths of ω.

Definition 28. Let X be a topological space, and let $x \in X$. We say that x has countable π-*character at x* if there is a countable collection of non-empty open sets $\mathcal{U}$ (not necessarily containing x) such that for any open subset V of X containing x, some $U \in \mathcal{U}$ is contained in V.

2.5 Stone duality

Definition 29. Let $\mathfrak{B}$ be a Boolean algebra. The *Stone space* of $\mathfrak{B}$ is the topological space which is described by the following:

(i) the carrier set of this space is $\mathrm{Ult}(\mathfrak{B})$, the collection of all the ultrafilters on $\mathfrak{B}$, and

(ii) the basis for the topology on it consists of the following sets: for each $b \in \mathfrak{B}$,

$$\{\mathcal{U} \in \mathrm{Ult}(\mathfrak{B}) : b \in \mathcal{U}\}.$$

We point out that we do not always use this $\mathrm{Ult}(\mathfrak{B})$ terminology for the Stone space of $\mathfrak{B}$. This space is compact, Hausdorff and zero-dimensional, and one can show that all spaces of this sort are obtained as the Stone space of some Boolean algebra. Indeed, the Stone functor is a contravariant functor between the category of Boolean algebras with Boolean homomorphisms, and compact, Hausdorff, zero-dimensional topological spaces with continuous maps, and one can in fact show that this is an equivalence of categories. The functor in the other direction being the one which takes a compact Hausdorff zero-dimensional topological space and returns the Boolean algebra of its clopen subsets.

The properties of the Boolean algebra and the properties of its Stone space are strongly intertwined. Important to this thesis are the following:

Theorem 30. *Let $\mathfrak{B}$ be a Boolean algebra and K its Stone space.*

(i) K is separable iff $\mathfrak{B}$ is σ-centered.

(ii) K is metrisable iff $\mathfrak{B}$ is countable.

The former can be shown using elementary Stone duality, whereas the latter requires, for example, the Urysohn Metrisation Theorem.

Theorem 31. *Let $\mathfrak{B}$ be a Boolean algebra, and let $\mathfrak{A}$ be a Boolean subalgebra of it. Let K be the Stone space of $\mathfrak{B}$ and let L be the Stone space of $\mathfrak{A}$. Then the map $f : K \to L$ given by*

$$f(\mathcal{U}) = \mathcal{U} \cap \mathfrak{A}$$

is a continuous surjection. Furthermore, for each $\mathcal{V} \in L$, $f^{-1}[\{\mathcal{V}\}]$ is the Stone space of $\mathfrak{B}/\mathcal{U}$ where $\mathcal{U}$ is the (not necessarily maximal) filter on $\mathfrak{B}$ that $\mathcal{V}$ generates.

Checking the continuity is simply a matter of applying the definition, whereas checking the surjectivity requires us to extend the filters on $\mathfrak{B}$ that ultrafilters on $\mathfrak{A}$ generate to ultrafilters on $\mathfrak{B}$ (in particular, some form of the Axiom of Choice is needed to guarantee that such extensions can be found). Checking the last sentence is also just a matter of applying the definitions.

We shall need the following fact about Boolean algebras with generating sets of a particular type.

Theorem 32. *Let K be the Stone space of a Boolean algebra $\mathfrak{B}$ which is generated by a chain $\mathcal{C} \subseteq \mathfrak{B}$. Then there is a linear order on K which induces its topology. That is, K is a compact linearly ordered topological space.*

We shall need some basic facts about the Stone-Čech compactification of a space. What follows can be done much more generally, but we do not have any need for this generality.

Definition 33. Let X be a discrete topological space. The *Stone-Čech compactification* of X, denoted βX is the Stone space of $\mathcal{P}(X)$. The topology is the usual topology on Stone spaces.

One can naturally identify X with the principal ultrafilters of $\mathcal{P}(X)$, and this gives an embedding of X into βX. In fact, X is dense in βX. Even more, any compact space in which X is dense is the continuous image of βX.

Theorem 34. *Let $\mathcal{U} \subseteq \mathcal{P}(\omega)/\mathrm{Fin}$ and $\hat{\mathcal{U}} \subseteq \mathcal{P}(\omega)$ be its saturation with respect to* Fin*:*

$$A \in \hat{\mathcal{U}} \iff [A] \in \mathcal{U}.$$

Then if $\mathcal{U}$ is an ultrafilter of $\mathcal{P}(\omega)/\mathrm{Fin}$, then $\hat{\mathcal{U}}$ is a non-principal ultrafilter of $\mathcal{P}(\omega)$. Furthermore, every non-principal ultrafilter of $\mathcal{P}(\omega)$ is obtained in such a way.

Corollary 35. *$\beta\omega \setminus \omega$ is the Stone space of $\mathcal{P}(\omega)/\mathrm{Fin}$.*

A similar statement is true for subalgebras of $\mathcal{P}(\omega)/\mathrm{Fin}$ and subalgebras of $\mathcal{P}(\omega)$ as well.

Theorem 36. *Let $\mathfrak{B}$ be a subalgebra of $\mathcal{P}(\omega)/\mathrm{Fin}$ and $\hat{\mathfrak{B}} \subseteq \mathcal{P}(\omega)$ its saturation with respect to* Fin*:*

$$A \in \hat{\mathfrak{B}} \iff [A] \in \mathfrak{B}.$$

Then $\hat{\mathfrak{B}}$ is a Boolean algebra, and every Boolean subalgebra of $\mathcal{P}(\omega)$ containing all the finite sets is obtained in this way. Also, if K is the Stone space of $\hat{\mathfrak{B}}$, then $K \setminus \omega$ is the Stone space of $\mathfrak{B}$.

2.6 Banach space theory

All Banach spaces are assumed to be real Banach spaces. Recall that if K is a compact space, then $C(K)$, the space of continuous real-valued functions with domain K, is a Banach space, the operations being defined pointwise, and the norm being the supremum norm: for $f \in C(K)$,

$$\|f\| = \sup\{|f(x)| \colon x \in K\}.$$

There is a strong correlation between the properties of a compact space K and the properties of its space of continuous functions $C(K)$. Relevant to us is the next theorem. Note that the equivalence of the first two statements is just the second part of Theorem 30.

Theorem 37. *Let K be the Stone space of an infinite Boolean algebra $\mathfrak{B}$. Then the following are equivalent:*

(i) $\mathfrak{B}$ is countable;

(ii) K is metrisable;

(iii) $C(K)$ is separable.

If X and Y are Banach spaces and $T : X \to Y$ is an isomorphic embedding of X into Y, that is, an injective bounded (equivalently, continuous) linear operator between X and Y, then we can talk about $\mathrm{ran}(T)$ as being a *copy* of X in Y. One can show that this is a closed subspace of Y. There is however a more important class of closed subspaces of a Banach space.

Definition 38. Let X, Y be Banach spaces, with X a closed subspace of Y. We say that X is *complemented in* Y, if there is a bounded linear operator $P : Y \to Y$ such that $P^2 = P$ (that is, a *projection*) such that $\mathrm{ran}(P) = X$.

This gives rise to an obvious notion of a complemented copy of a Banach space in another. Note that since $P^2 = P$, it follows that P is the identity on X in the above. If X is a complemented subspace of Y via a projection P, let Z be the kernel of P. One can show that Z is as well a closed subspace of Y, and in fact the operator $I - P$ witnesses that it is as well complemented. We also have a decomposition of X, $X = Y \oplus Z$. That is, every element of Y can be uniquely expressed as the sum of an element of X and an element of Z. In this sense, one can say that X is not merely a closed subspace of Y, it is in fact nicely embedded as a closed subspace of Y.

Theorem 39. *Let K and L be compact spaces such that there is a continuous surjection $T : K \to L$. Then $\tilde{T} : C(L) \to C(K)$ is an isometric embedding where for any $f \in C(L)$,*

$$\tilde{T}(f)(x) = f(Tx).$$

Using this and the universal property of the Stone-Čech compactification, we can prove the following universality result. We state it only for the cases of it that we shall need.

Theorem 40. *Let K be a compactification of ω. Then it can be isometrically embedded into $C(\beta\omega)$.*

Since l_∞ is isometric to $C(\beta\omega)$, this shows that $C(K)$ for K a compactification of ω embeds isometrically into l_∞.

2.7 Set theory

We assume that the reader is familiar with for example the first 15 chapters of [Jec03], as well as some basic knowledge about set theory of the reals and proper forcing. We do however give some basic definitions here so as to clarify notation.

We shall not need any extensive knowledge of stationary sets or closed unbounded sets (barring their definition) in this thesis. The only exception will be Fodor's Lemma which we shall need at one point, and we recap for the reader.

Theorem 41. *Let κ be an uncountable regular cardinal and $S \subseteq \kappa$ a stationary subset of it. Let $f : S \to \kappa$ be a regressive function: for every $\alpha \in S$, $f(\alpha) < \alpha$. Then f is constant on a stationary subset of κ.*

In fact we shall only need thaat a function as above is constant on a set of size κ.

We shall also need the following standard fact.

Proposition 42. *There is an independent family of size $\mathfrak{c}$ in $\mathcal{P}(\omega)/\mathrm{Fin}$.*

2.7.1 Notation relating to ideals on sets

In Chapter 8 we shall be extensively dealing with ideals on sets, so here we recall some relevant terminology and results.

Let S be a set. Recall that a collection $\mathcal{I}$ of subsets of S is called an *ideal* if it contains all the singletons, and is closed under taking subsets and finite unions. We say that it is *non-trivial* if $S \notin \mathcal{I}$. If it is also closed under countable unions, then we say that it is a *σ-ideal.* A collection $\mathcal{J} \subseteq \mathcal{I}$ is called a *generating set* for $\mathcal{I}$ if their closure under subsets and finite unions is $\mathcal{I}$. If $\mathcal{I}$ has a generating set of size $\aleph_1$, then we say that it is *ω_1-generated.* Similarly, if the closure of $\mathcal{J}$ under subsets and countable unions is $\mathcal{I}$, then we shall say that $\mathcal{J}$ is a *σ-generating* set for $\mathcal{I}$, and if $\mathcal{I}$ has a σ-generating set of size $\aleph_1$, then we shall say that it is *ω_1-generated as a σ-ideal.* Usually when we will talk about a σ-ideal being ω_1-generated, we shall mean that it is ω_1-generated as a σ-ideal. All ideals will be assumed to be non-trivial unless otherwise mentioned.

Recall also that if $\mathcal{I}$ is a non-trivial ideal on S, then $\mathcal{U}$, the collection of all $S \setminus I$ such that $I \in \mathcal{I}$ forms a *filter* on S, which contains all the co-finite sets and is closed under supersets and intersections.

2.7.2 Countable elementary structures

We shall have occasion to talk about countable elementary substructures of $H(\kappa)$ for κ an uncountable regular cardinal very often in Chapters 8 and 9. We remind the reader that this refers to the collection of all elements of the universe which are hereditarily of size less than κ; equivalently, $x \in H(\kappa)$ if x and every element in its transitive closure has size less than κ. In particular, $H(\kappa)$ is a transitive set, and the set of ordinals in $H(\kappa)$ is exactly κ.

First a useful piece of notation. If N is a set of ordinals, then δ_N shall denote $\sup(N \cap \omega_1)$. In practice, N will be a countable elementary substructure of some $H(\kappa)$ for some large enough regular κ, and so $\delta_N = N \cap \omega_1$ in this case.

We mention a basic fact about such $H(\kappa)$ which we shall need to ensure that various calculations and manipulations which we perform with elements of these structures can in fact be performed inside the structure itself (as opposed to needing the entire universe of sets).

Theorem 43. *Let κ be an infinite regular cardinal. Then $(H(\kappa), \in)$ is a model of all of* ZFC *with the possible exception of the Axiom of Powerset.*

When we talk about such structures we shall in fact always actually be referring to the structure $(H(\kappa), \in, \lhd)$ where $\lhd$ will be some suitable well-ordering of the structure. One important use of the well-ordering $\lhd$ is that it provides a canonical witness to existential statements which are true in the structure.

We shall also talk about the *correctness* or *elementarity* of structures, referring simply to their being elementary substructures of $H(\kappa)$, or elementary substructures of the universe with respect to a particular fragment of ZFC etc.

We shall need the following simple fact at one point.

Proposition 44. *Let $\lambda < \kappa$ be uncountable regular cardinals. Let $M \prec H(\kappa)$ be such that $\lambda \in M$. Then $M \cap H(\lambda) \prec H(\lambda)$.*

We shall also need certain types of sequences of elementary structures.

Definition 45. A sequence of countable structures $\langle N_\xi \colon \xi < \omega_1 \rangle$ is *increasing* or an *increasing chain* if for every $\xi < \omega_1$, $\langle N_\nu \colon \nu < \xi \rangle \in N_\xi$.

2.7.3 Forcing

Our forcing notation is standard as can be found in [Jec03] or [Kun14]. In particular, if q is a stronger condition than p, then we write $q \leq p$. Also, forcing partial orders will typically be assumed to be separative.

We assume that the reader is familiar with forcing up to the statement (but not the consistency proof of) the Proper Forcing Axiom. We point out however that this thesis does not contain any iterated forcing constructions, though some are described in Chapter 8 while referring to the literature. We also provide a recap of some standard definitions and in particular the basic results from proper forcing.

First we mention the most important abbreviation we shall make use of. At various points we talk about a certain statements being consistent. What we shall mean is that if ZFC is consistent, then the conjuction of ZFC and these statements is consistent. Similarly, we shall talk about a statement being consistent assuming the consistency of another statement. Here too, we shall actually mean that if the conjunction of ZFC and the latter statement is consistent, then the conjunction of ZFC and the former is consistent.

Next, some standard definitions.

Definition 46. Let $\mathbb{P}$ be a partial order and $p, q \in \mathbb{P}$.

(i) We say that p and q are *compatible*, denoted $p \parallel q$ if there is an $r \in \mathbb{P}$ such that $r \leq p, q$.

(ii) We say that p and q are *incompatible*, denoted $p \perp q$ if they are not incompatible.

(iii) If $A \subseteq \mathbb{P}$ is such that for each $r, s \in A$, $r \perp s$, then we say that A is an *antichain* of $\mathbb{P}$.

(iv) If κ is a cardinal such that all antichains of $\mathbb{P}$ have size less than κ, then we say that $\mathbb{P}$ has the *κ-cc*.

(v) If κ is a cardinal such that for any sequence of conditions $\langle p_\alpha \colon \alpha < \kappa \rangle$ such that $\alpha < \beta < \kappa$ implies that $p_\beta \leq p_\alpha$, there is some $q \in \mathbb{P}$ such that $q \leq p_\alpha$ for each $\alpha < \kappa$, then we say that $\mathbb{P}$ is *κ-closed*.

We remind the reader that the countable chain condition is the same as the $\aleph_1$-cc, and that σ-closed is the same as $\aleph_0$-closed.

Proposition 47. *Let $\mathbb{P}$ be a poset and κ be a cardinal.*

(i) If $\mathbb{P}$ is κ-cc, then it does not collapse cardinals greater than or equal to κ

(ii) If $\mathbb{P}$ is κ-closed, then it does not collapse cardinals less than or equal to κ^+.

Definition 48. *Martin's Axiom*, abbreviated MA, is the statement that for any ccc partial order $\mathbb{P}$ as well as $< \mathfrak{c}$-many dense subsets of $\mathbb{P}$, there is a directed subset of $\mathbb{P}$ which meets all of these dense sets. Also, if κ is a cardinal, then MA_κ is the statement that for any ccc partial order $\mathbb{P}$ as well as κ-many dense subsets of $\mathbb{P}$, there is a directed subset of $\mathbb{P}$ which meets all of these dense sets.

We remind the reader of the standard consequence of the Baire Category Theorem that $\mathsf{MA}_{\aleph_0}$ always holds, and hence CH implies MA. The following is due to Solovay and Tennenbaum [ST71].

Theorem 49. *(*GCH*) For any regular cardinal κ, there is a partial order which preserves cardinals and forces $\mathsf{MA} + \mathfrak{c} > \kappa$.*

We now provide a recap of proper forcing.

Definition 50. Let $\mathbb{P}$ be a partial order and κ a large enough regular cardinal. Let $M \prec H(\kappa)$ be countable such that $\mathbb{P} \in M$. Let $p^* \in \mathbb{P}$. Then p^* is a $(M, \mathbb{P})$-*master condition* or $(M, \mathbb{P})$-*generic condition* or a *master condition for M* if for every dense subset D of $\mathbb{P}$ in M,

$$p \Vdash \text{“}D \cap M \cap G \neq \emptyset.\text{”}$$

Definition 51. A partial order $\mathbb{P}$ is said to be proper if for all large enough regular cardinals κ, for all countable $M \prec H(\kappa)$ containing $\mathbb{P}$, for all $p \in \mathbb{P} \cap M$, there is some $p^* \leq p$ which is a $(M, \mathbb{P})$-master condition.

We remind the reader that ccc posets are proper, and that proper posets do not collapse ω_1.

Definition 52. The *Proper Forcing Axiom*, abbreviated PFA, is the statement that for any proper partial order $\mathbb{P}$ as well as ω_1-many dense subsets of $\mathbb{P}$, there is a directed subset of $\mathbb{P}$ which meets all of these dense sets.

Assuming the consistency of *supercompact cardinals*, PFA is consistent, a result independently proved by Baumgartner [Dev83] and Shelah [She98]. It is also known that some large cardinals are required for the consistency of PFA, the best currently known lower bounds being due to Todorčević [Tod84].

We list two more facts about proper forcing which are important to this thesis. The first is due independently to Todorčević [Tod97] and Veličković [Vel92].

Theorem 53. *(*PFA*)* $\mathfrak{c} = \aleph_2$*.*

In particular, one cannot hope for a forcing axiom for proper posets which meets more than $\aleph_1$-many dense sets unlike MA_{ω_2}. The second is folklore.

Theorem 54. *If* $\mathbb{P}$ *is a proper poset, then it preserves stationary subsets of* ω_1*.*

Definition 55. *Martin's Maximum*, abbreviated MM, is the statement that for any poset $\mathbb{P}$ which preserves stationary subsets of ω_1 as well as $\aleph_1$-many dense subsets of $\mathbb{P}$, there is a directed subset of $\mathbb{P}$ which meets all of these dense sets.

The previous theorem then tells us the following.

Theorem 56. *(*MM*)* $\mathsf{PFA} + \mathfrak{c} = \aleph_2$*.*

Therefore, MM as well does not have any analogue for meeting more than ω_1-many dense sets. Assuming the consistency of *supercompact cardinals*, MM is consistent, a result due to Foreman, Magidor, and Shelah [FMS88].

We end this subsection with the following two simple results which we shall need in Chapter 10.

Theorem 57. *There is a separative* σ*-closed* $\aleph_2$*-cc partial order if and only if* CH*.*

Theorem 58. *Let* κ *be an infinite cardinal. Let* $\mathbb{P}$ *be a partial order of size* κ^+ *then after forcing with the* $\leq \kappa$*-support product of* κ^+*-many copies of* $\mathbb{P}$*, we have that* $\mathbb{P}$ *is* κ*-centered.*

Note that above we have not mentioned anything about the preservation of cardinals.

2.7.4 Set theory of the reals

Let S be a set. If $\mathcal{I}$ is an ideal of subsets of S, then the following are some cardinals associated with it.

$$\operatorname{add}(\mathcal{I}) = \min\{|\mathcal{A}| \colon \mathcal{A} \subseteq \mathcal{I},\ \bigcup \mathcal{A} \notin \mathcal{I}\},$$
$$\operatorname{non}(\mathcal{I}) = \min\{|X| \colon X \subseteq S,\ X \notin \mathcal{I}\},$$
$$\operatorname{cov}(\mathcal{I}) = \min\{|\mathcal{A}| \colon \mathcal{A} \subseteq \mathcal{I},\ \bigcup \mathcal{A} = S\}.$$

By $\mathcal{N}$ we will mean the σ-ideal of Lebesgue null sets, by $\mathcal{M}$, the σ-ideal of meager sets, and by $\mathcal{N}_{\omega_1}$ the σ-ideal of λ_{ω_1}-null sets.

The *bounding number* is defined by

$$\mathfrak{b} = \min\{|\mathcal{F}| \colon \mathcal{F} \subseteq \omega^\omega,\ \forall g \in \omega^\omega\ \exists f \in \mathcal{F}\ f \not\leq^* g\}.$$

Here $f \leq^* g$ means $f(n) \leq g(n)$ for all but finitely many n's.

We shall need the following standard facts about these cardinals.

Proposition 59. *The cardinal* $\mathrm{add}(\mathcal{N})$ *is uncountable and regular. Also,* $\mathrm{add}(\mathcal{N}) \leq \mathfrak{b}$*.*

Proposition 60.

$$\omega_1 \leq \mathrm{add}(\mathcal{N}), \mathrm{non}(\mathcal{N}), \mathrm{cov}(\mathcal{N}), \mathrm{add}(\mathcal{M}), \mathrm{non}(\mathcal{M}), \mathrm{cov}(\mathcal{M}), \mathfrak{b}.$$

Proposition 61. *(*CH *or* MA*)*

$$\mathrm{add}(\mathcal{N}) = \mathrm{non}(\mathcal{N}) = \mathrm{cov}(\mathcal{N}) = \mathrm{add}(\mathcal{M}) = \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mathfrak{b} = \mathfrak{c}.$$

We warn the reader that the above definitely does not even scratch the surface of what is known about models with various patterns of equalities and inequalities holding between the above cardinals.

Chapter 3

Kamburelis' Theorem

The aim of this chapter is to give a self-contained proof of a theorem of Kamburelis from [Kam89], Theorem 67, which is crucial to our understanding of the existence, non-existence and, permitting temporarily some poetic license, substance of the measures on the algebras that we shall construct in Chapter 5. Kamburelis' Theorem characterises Boolean algebras which support a strictly positive measure as those which are σ-centred in some forcing extension by a measure algebra. One direction of our proof is new in that it uses the Kelley Intersection Theorem, Theorem 65, instead of the more involved argument from Kamburelis' paper in order to conclude that Boolean algebras which are σ-centred in forcing extensions by a measure algebra support a strictly positive measure. This approach allows us to show that depending on the size of the measure algebra, one can extract some extra information on the types of measures that the Boolean algebra supports.

In Section 3.1 we recall some standard facts from measure theory which we shall use as well as recalling the results of Kelley [Kel59] that we shall require. In Section 3.2 we give our proof of Kamburelis' Theorem. In Section 3.3 we describe the extra information that one can extract from our proof of this result, and conclude in Section 3.4 with some questions which this raises.

Before we start, we remind the reader of some notation we have established.

(i) $\mathbb{N}^+$ denotes the positive natural numbers.

(ii) $\mathbf{M}$ is the class of Boolean algebras supporting a strictly positive finitely additive measure.

(iii) If I is an infinite set, then $(\mathfrak{R}_I, \mu_I)$ is the homogenous measure algebra of character $|I|$. Here we are using Maharam's Theorem, Theorem 18, which tells us that all homogenous measure algebras of density character κ (for κ an infinite cardinal) are isomorphic as measure algebras to justify our use of the article 'the'.

3.1 Prerequisites

We shall need the following measure-theoretic lemma which can be proved using elementary Stone duality; see for example [Kel59, Proposition 1].

Lemma 62. *Let $\mathfrak{B} \in \mathbf{M}$. Let $k \in \mathbb{N}^+$ and $\mathcal{C} \subseteq \mathfrak{B}$ be such that if $a \in \mathcal{C}$ then $\mu(a) > \frac{1}{k}$. Then, if $|\mathcal{C}| > nk$ for some natural number n, then there is $\mathcal{D} \subseteq \mathcal{C}$ of size greater than n such that $\bigcap \mathcal{D} \in \mathfrak{B}^+$.*

Crucial to Kelley's characterisation theorem for Boolean algebras supporting strictly positive finitely additive measures, that is, Boolean algebras in $\mathbf{M}$ is the following notion.

Definition 63. Let $\mathfrak{B}$ be a Boolean algebra. Let $\mathcal{C} \subseteq \mathfrak{B}$. If $\varepsilon > 0$ is a real number, then we say that the *intersection number* of $\mathcal{C}$ is at least ε if for any $k \in \mathbb{N}^+$ and sequence $\langle a_i : i < k \rangle$ of not necessarily distinct elements of $\mathcal{C}$, there is some $F \subseteq \{0, 1, \dots k-1\}$ of size at least εk such that $\bigcap_{i \in F} a_i \in \mathfrak{B}^+$. The *intersection number* of $\mathcal{C}$ is defined to be the supremum of all the ε such that the intersection number of $\mathcal{C}$ is at least ε. We adopt the convention that the supremum of an infinite set is 0.

Note that Lemma 62 can be expressed in this newly acquired terminology as saying that if $\mathfrak{B}$ is a Boolean algebra in $\mathbf{M}$, then for any $k \in \mathbb{N}^+$, if $\mathcal{C}$ is a subset of $\mathfrak{B}$ consisting solely of elements of measure greater than $\frac{1}{k}$, then $\mathcal{C}$ has intersection number at least k.

Definition 64. Let $\mathfrak{B}^+$ be a Boolean algebra. We say that it satisfies *Kelley's criterion* if there is a fragmentation $\mathfrak{B}^+ = \bigcup_{n \in \omega} \mathcal{C}_n$ such that each $\mathcal{C}_n$ has positive intersection number.

We are now in a position to state Kelley's characterisation theorem, [Kel59, Theorem 4].

Theorem 65. *Let $\mathfrak{B}$ be a Boolean algebra. Then $\mathfrak{B} \in \mathbf{M}$ iff $\mathfrak{B}$ satisfies Kelley's criterion.*

If $\mathfrak{B} \in \mathbf{M}$ (with μ being a witnessing measure), it is easy to show that $\mathfrak{B}$ satisfies Kelley's criterion using the observation we have just made. Namely, for each $k \in \mathbb{N}^+$, let

$$\mathcal{C}_k = \{a \in \mathfrak{B} : \mu(a) > \frac{1}{k}\}.$$

Then by Lemma 62, each $\mathcal{C}_k$ has intersection number at least k, from which it follows that $\mathfrak{B}^+ = \bigcup_{n \in \mathbb{N}^+} \mathcal{C}_n$ is a fragmentation of $\mathfrak{B}^+$ into countably many sets of positive intersection number which witnesses that $\mathfrak{B}$ satisfied Kelley's criterion. The other direction is considerably harder. Kelley, for example, uses the Hahn-Banach Theorem, and for the role of the Axiom of Choice in the matter, see [HR98, Form 52].

We shall also need a consequence of Maharam's Theorem, Theorem 18 which almost appears explicitly in [Kam89, Lemma 3.5]. Maharam's Theorem tells us that an arbitrary measure algebra can be decomposed into countably many canonical measure algebras, and from this we can conclude that every measure algebra can be embedded into a canonical measure algebra. We are however interested also in bounding the size of this target canonical measure algebra. Equipped with some extra information about the original measure algebra, this follows by a simple application of Maharam's Theorem. The following lemma does exactly this, with the added ingredient that it starts not with a measure algebra, but in fact with a Boolean algebra supporting a strictly positive finitely additive measure.

Lemma 66. *Let $\mathfrak{B}$ be a Boolean algebra and κ a cardinal. Suppose that there is a strictly positive finitely additive measure μ on $\mathfrak{B}$ such that $\mathfrak{B}$ has a μ-dense subset of size at most κ. Then $\mathfrak{B}$ is a subalgebra of $\mathfrak{R}_\kappa$.*

Proof. Let K denote the Stone space of $\mathfrak{B}$. Then K is a compact space, and $\mathrm{Clop}(K)$ is isomorphic to $\mathfrak{B}$. Consequently, we can consider ν to be a measure on $\mathrm{Clop}(K)$ such that $(\mathfrak{B}, \mu)$ and $(\mathrm{Clop}(K), \nu)$ are measure isomorphic. Let $\mathcal{A}$ be the σ-algebra of subsets of K that $\mathrm{Clop}(K)$ generates, and we can extend ν to $\tilde{\nu}$ such that $(K, \mathcal{A}, \tilde{\nu})$ is a measure space using the Hahn-Kolmogorov Theorem, Theorem 15. Since $(\mathfrak{B}, \mu)$ has a dense subset of size κ, so does $(\mathrm{Clop}(K), \nu)$, and since this algebra is dense in $(\mathcal{A}, \tilde{\nu})$, it follows that the latter has a dense subset of size κ as well. It follows that the measure algebra $(\mathcal{A}/\mathcal{N}, [\tilde{\nu}]_{\mathcal{N}})$ has a dense subset of size κ as well where $\mathcal{N}$ denotes the null ideal of $\mathcal{A}$.

For notational convenience, we denote $(\mathcal{A}/\mathcal{N}, [\tilde{\nu}]_{\mathcal{N}})$ by $(\mathfrak{C}, \theta)$. Now, Maharam's Theorem, Theorem 18, implies that there is a partition of $1_{\mathfrak{C}}$, $\{c_n : n \in \omega\}$, and a countable sequence of cardinals $\{\kappa_n : n \in \omega\}$ such that $(\mathfrak{C}_{c_n}, \theta_{c_n})$ is isomorphic as a measure algebra to $(\mathfrak{R}_{\kappa_n}, \mu_{\kappa_n})$. Since $\mathfrak{C}$ contains a ν-dense subset of size κ, it follows that each $\mathfrak{C}_{c_n}$ also contains a θ_{c_n}-dense subset of size at most κ, and hence so do all of the $\mathfrak{R}_{\kappa_n}$ have μ_{κ_n}-dense subsets of size at most κ. Consequently, $\kappa_n \leq \kappa$ for each $n \in \omega$. By considering a partition of κ into countably many sets each of size κ, we can embed each of the $(\mathfrak{R}_{\kappa_n}, \mu_{\kappa_n})$ into a different piece. Since each of the $(\mathfrak{R}_{\kappa_n}, \mu_{\kappa_n})$ are isomorphic to $(\mathfrak{C}_{c_n}, \theta_{c_n})$, we can, by appropriately gluing together these embeddings obtain an embedding of $(\mathfrak{C}, \theta)$ in $(\mathfrak{R}_\kappa, \mu_\kappa)$. Since $(\mathfrak{B}, \mu)$ can itself be embedded in $(\mathfrak{C}, \theta)$, it follows that $(\mathfrak{B}, \mu)$ is a measure subalgebra of $(\mathfrak{R}_\kappa, \mu_\kappa)$. □

3.2 Kamburelis' Theorem

Theorem 67. *Let $\mathfrak{B}$ be a Boolean algebra. The following are equivalent:*

(i) $\mathfrak{B}$ is in $\mathbf{M}$;

(ii) There is a measure algebra $\mathfrak{R}$ such that $V^{\mathfrak{R}} \models$ "$\check{\mathfrak{B}}$ is σ-centered".

Proof. We start with (i) implies (ii). Let $\kappa = |\mathfrak{B}|$. We shall show that $\check{\mathfrak{B}}$ is σ-centered after forcing with $\mathfrak{R}_{\omega\times\kappa}$. For $r \in 2^{\omega\times\kappa}$, let $(r)_n \in 2^\kappa$ be defined by $(r)_n(\alpha) = r(n, \alpha)$.

We know by Lemma 66 that $\mathfrak{B}$ embeds into $\mathfrak{R}_\kappa$, and we identify it with its image in $\mathfrak{R}_\kappa$. Let

$$\{A_\alpha : \alpha < \kappa\} \subseteq \mathrm{Bor}(2^\kappa)$$

be such that

$$\mathfrak{B}^+ = \{[A_\alpha] : \alpha < \kappa\},$$

and note that $\mu_\kappa(A_\alpha)$ is necessarily non-zero for each $\alpha < \kappa$. For $n \in \omega$, let

$$E_{\alpha,n} = \{r \in 2^{\omega\times\kappa} : (r)_n \in A_\alpha\},$$

and

$$D_\alpha = \{r \in 2^{\omega\times\kappa} : \forall n \in \omega[(r)_n \notin A_\alpha]\}.$$

By Lemma 20, we have that $\mu_{\omega\times\kappa}(E_{\alpha,n}) = \mu_\kappa(A_\alpha) > 0$, and hence

$$\Sigma_{n\in\omega}\mu_{\omega\times\kappa}(E_{\alpha,n}) = \Sigma_{n\in\omega}\mu_\kappa(A_\alpha) = \infty.$$

Since $\langle E_{\alpha,n} : n \in \omega\rangle$ forms a mutually independent sequence of events and

$$D_\alpha = 2^{\omega\times\kappa} \setminus \bigcup_{n\in\omega} E_{\alpha,n},$$

we have by the converse of the Borel-Cantelli Lemma, Theorem 21, that $\mu_{\omega\times\kappa}(D_\alpha) = 0$ for each $\alpha < \kappa$. Finally, let G be a V-generic subset of $\mathfrak{R}_{\omega\times\kappa}$. Then in $V[G]$, there is some $g \in 2^{\omega\times\kappa}$ such that $g \notin N$ for any $N \in (\mathcal{N}_{\omega\times\kappa})^V$. In particular, for each $\alpha < \kappa$, $g \notin D_\alpha$, so for each $\alpha < \kappa$, there is some $n_\alpha \in \omega$ such that $g \in E_{\alpha,n_\alpha}$. Then for $n \in \omega$, let

$$\mathcal{C}_n = \{[A_\alpha] \colon g \in E_{\alpha,n}\},$$

and notice that each $\mathcal{C}_n$ is an ultrafilter on $\mathfrak{B}$, and since $\mathfrak{B}^+ = \bigcup_{n\in\omega} \mathcal{C}_n$, we have that $\mathfrak{B}$ is σ-centered in $V[G]$.

Now we show that (ii) implies (i). The proof of this direction is different from that in Kamburelis' paper. Let $\mathfrak{R}$ be a measure algebra which forces that $\check{\mathfrak{B}}$ is σ-centered. By using the Axiom of Choice in the extension, we can assume that $\check{\mathfrak{B}}^+$ is in fact fragmented into countably many ultrafilters. For $n \in \omega$, let $\dot{\mathcal{C}}_n$ be an $\mathfrak{R}$-name such that

$$V^{\mathfrak{R}} \vDash \text{“}\check{\mathfrak{B}}^+ = \bigcup_{n\in\omega} \dot{\mathcal{C}}_n \text{ and each } \dot{\mathcal{C}}_n \text{ is an ultrafilter”}.$$

For $n \in \omega$ and $k \in \mathbb{N}^+$, let

$$\mathcal{D}^n_k = \{a \in \mathfrak{B}^+ \colon \mu_{\mathfrak{R}}(\|\check{a} \in \dot{\mathcal{C}}_n\|) > \frac{1}{k}\},$$

and

$$\mathcal{E}^n_k = \{\|\check{a} \in \dot{\mathcal{C}}_n\| \colon a \in \mathcal{D}^n_k\}.$$

Note that if $F \subseteq \mathcal{D}^n_k$ is such that $\bigcap_{a\in F} \|\check{a} \in \dot{\mathcal{C}}_n\| \in \mathfrak{R}^+$, then since

$$V^{\mathfrak{R}} \vDash \text{“}\dot{\mathcal{C}}_n \text{ is an ultrafilter”},$$

we must have that $\bigcap F \in \mathfrak{B}^+$. Therefore, the intersection number of $\mathcal{D}^n_k$ is at least the intersection number of $\mathcal{E}^n_k$. Now, to finish, note that by Lemma 62, each $\mathcal{E}^n_k$ has intersection number at least $\frac{1}{k}$, and so each $\mathcal{D}^n_k$ must also have intersection number at least $\frac{1}{k}$. Since $\mathfrak{B}^+ = \bigcup_{n\in\omega,k\in\mathbb{N}^+} \mathcal{D}^n_k$, we have by Theorem 65 that $\mathfrak{B} \in \mathbf{M}$. $\square$

3.3 Strengthenings

Since Kamburelis' Theorem characterises Boolean algebras $\mathfrak{B}$ in $\mathbf{M}$ in terms of an extra parameter, the measure algebra $\mathfrak{R}$, it is natural to wonder if particular properties of $\mathfrak{R}$ give us any extra information about $\mathfrak{B}$. In this section we show that some information can indeed be extracted in this way, but while we are able to extract some extra information from both directions of the equivalence, we are not able to actually obtain a strengthened equivalence. A clarification follows.

First, note that combining the proof of Theorem 3.2 with Lemma 66, we are able to strengthen one half of Kamburelis' Theorem.

Theorem 68. *Let κ be a cardinal and $\mathfrak{B}$ be a Boolean algebra which supports a strictly positive measure μ such that $\mathfrak{B}$ contains a μ-dense subset of size κ. Then $V^{\mathfrak{R}_{\kappa\times\omega}} \vDash$“$\check{\mathfrak{B}}$ is σ-centered”. In particular, $\mathfrak{B}$ is σ-centered in the forcing extension by a measure algebra which has a dense subset of size κ.*

To strengthen the other half, we need a property which was first considered by Talagrand [Tal80] for the case of $\kappa = \aleph_0$ and by Džamonja-Plebanek [DP08] more generally.

Definition 69. Let $\mathfrak{B}$ be a Boolean algebra and κ a cardinal. We say that $\mathfrak{B}$ is *κ-approximable* if there is a sequence of (not necessarily strictly positive) measures $\langle \mu_\alpha \colon \alpha < \kappa \rangle$ such that for every $a \in \mathfrak{B}^+$, there is some $\alpha < \kappa$ such that $\mu_\alpha(a) > \frac{1}{2}$. If $\kappa = \aleph_0$, then we use the term *approximable.*

We point out that by a result of Talagrand, see [MN80], when considering $\aleph_0$-approximability, replacing $\frac{1}{2}$ above by any δ, $0 < \delta < 1$, gives rise to an equivalent definition. We shall unfortunately not need this nice fact in the sequel.

We shall need the following lemmas in what follows. The first is a simple modification of Lemma 62.

Lemma 70. *Let $\mathfrak{B} \in \mathbf{M}$. Let $a \in \mathfrak{B}^+$. Let k be a positive natural number and $\mathcal{C} \subseteq \mathfrak{B}$ be such that if $b \in \mathcal{C}$ then $\frac{\mu(a \cap b)}{\mu(a)} > \frac{1}{k}$. Then, if $|\mathcal{C}| > nk$ for some natural number n, then there is $\mathcal{D} \subseteq \mathcal{C}$ of size greater than n such that $\bigcap_{b \in \mathcal{D}} (a \cap b) \in \mathfrak{B}^+$, and hence, $\bigcap \mathcal{D} \in \mathfrak{B}^+$.*

Proof. Apply Lemma 62 to the Boolean algebra $\mathfrak{B}_a$. □

The next lemma is [Kel59, Theorem 2].

Lemma 71. *Let $\mathfrak{B}$ be a Boolean algebra, and let $\mathcal{C} \subseteq \mathfrak{B}$ have intersection number δ. Then there is a (not necessarily strictly positive) measure μ on $\mathfrak{B}$ such that*

$$\inf\{\mu(a) \colon a \in \mathcal{C}\} = \delta.$$

Theorem 72. *Let $\mathfrak{B}$ be a Boolean algebra and κ a cardinal. Suppose that there is a measure algebra $\mathfrak{R}$ with a dense subset of size κ such that $V^{\mathfrak{R}} \vDash$"$\check{\mathfrak{B}}$ is σ-centered". Then $\mathfrak{B}$ is in $\mathbf{M}$ and is κ-approximable.*

Proof. Let $\mathfrak{R}$ be the promised measure algebra, and let μ denote its measure, and let $\langle a_\alpha \colon \alpha < \kappa \rangle$ be a μ-dense subset of $\mathfrak{R}$. For $n \in \omega$, let $\dot{\mathcal{C}}_n$ be a $\mathfrak{R}$-name such that

$$V^{\mathfrak{R}} \vDash \text{"}\check{\mathfrak{B}}^+ = \bigcup_{n \in \omega} \dot{\mathcal{C}}_n \text{ and each } \dot{\mathcal{C}}_n \text{ is an ultrafilter"}.$$

For each $m \in \omega$, $k \in \mathbb{N}^+$ and $\alpha < \kappa$, let

$$\mathcal{D}_{m,k,\alpha} = \{b \in \mathfrak{B}^+ \colon \mu(\|\check{b} \in \dot{\mathcal{C}}_m\|) > \frac{1}{k}, \mu(a_\alpha \triangle \|\check{b} \in \dot{\mathcal{C}}_m\|) < \frac{1}{3k}\}.$$

The following claim, when combined with Lemma 71 finishes the proof of the theorem since

$$\mathfrak{B}^+ = \bigcup_{m \in \omega, k \in \mathbb{N}^+, \alpha < \kappa} \mathcal{D}_{m,k,\alpha}.$$

Claim. *For each $m \in \omega$, $k \in \mathbb{N}^+$ and $\alpha < \kappa$, $\mathcal{D}_{m,k,\alpha}$ has intersection number at least $\frac{1}{2}$.*

Proof. Let $m \in \omega$, $k \in \mathbb{N}^+$ and $\alpha < \kappa$ be fixed, and for notational convenience we denote $\mathcal{D}_{m,k,\alpha}$ by $\mathcal{D}$. For even more notational convenience, let $a = a_\alpha$. For $b \in \mathcal{D}$, let $e_b = \|\check{b} \in \dot{\mathcal{C}}_m\|$, and let $\mathcal{E} = \{e_b \colon b \in \mathcal{D}\}$. Note that if $F \subseteq \mathcal{D}$ and $\bigcap_{b \in F} e_b \in \mathfrak{R}^+$, then $\bigcap F \in \mathfrak{B}^+$ since

$$V^{\mathfrak{R}} \vDash \text{"}\dot{\mathcal{C}}_n \subseteq \check{\mathfrak{B}}^+ \text{ is an ultrafilter"}.$$

It follows that if we show that $\mathcal{E}$ has intersection number at least $\frac{1}{2}$, then $\mathcal{D}$ has intersection number at least $\frac{1}{2}$ as well. So we aim for the former.

Note that if $e \in \mathcal{E}$, then $\mu(e) > \frac{1}{k}$, and $\mu(a \triangle e) < \frac{1}{3k}$. Therefore,

$$\frac{\mu(a \cap e)}{\mu(a)} = \frac{\mu(e) - \mu(e \setminus a)}{\mu(e) - \mu(e \setminus a) + \mu(a \setminus e)} \geq \frac{\mu(e) - \mu(e \setminus a) - \mu(a \setminus e)}{\mu(e) + \mu(e \setminus a) + \mu(a \setminus e)} = \frac{\mu(e) - \mu(a \triangle e)}{\mu(e) + \mu(a \triangle e)}.$$

Since $\mu(e) > 3\mu(a \triangle e)$, we have that $2(\mu(e) - \mu(a \triangle e)) > \mu(e) + \mu(a \triangle e)$, and so

$$\frac{\mu(a \cap e)}{\mu(a)} \geq \frac{\mu(e) - \mu(a \triangle e)}{\mu(e) + \mu(a \triangle e)} > \frac{1}{2}.$$

Finally, applying Lemma 70, we see that $\mathcal{E}$ has intersection number at least $\frac{1}{2}$, which proves the claim. $\dashv$

$\square$

Corollary 73. *Let $\mathfrak{B}$ be a Boolean algebra and κ a cardinal. Then the first implies the second which implies the third:*

(i) *$\mathfrak{B}$ supports a strictly positive measure μ such that $\mathfrak{B}$ contains a μ-dense subset of size κ;*

(ii) *$\mathfrak{B}$ is σ-centered in the forcing extension by a measure algebra which has a dense subset of size κ;*

(iii) *$\mathfrak{B}$ is in $\mathbf{M}$ and is κ-approximable.*

We point out that the fact that the first of the above implies the third is well-known, see [DP08], so the interpolation of the two by the second is the real content of the corollary.

3.4 Some questions

Corollary 73 raises the natural question about whether any of the three statements are in fact equivalent to the others. The question of whether the third statement implies the first was considered by Talagrand in [Tal80], and it was shown that under CH, there is an approximable Boolean algebra which does note support any separable measures. In [DP08], such a Boolean algebra was shown to exist solely from ZFC. It follows that at least for the case of $\kappa = \aleph_0$, the first and the third statement from Corollary 73 cannot be equivalent. This suggests the following two questions:

Question 74. *Let $\mathfrak{B}$ be a Boolean algebra and κ a cardinal. Are the following equivalent?*

(i) *There is a measure algebra with a dense set of size κ after forcing with which $\mathfrak{B}$ is σ-centered.*

(ii) *$\mathfrak{B}$ supports a measure of density κ.*

Question 75. *Let $\mathfrak{B}$ be a Boolean algebra and κ a cardinal. Are the following equivalent?*

(i) *There is a measure algebra with a dense set of size κ after forcing with which $\mathfrak{B}$ is σ-centered.*

(ii) *$\mathfrak{B}$ is κ-approximable.*

A general instance of the first question would be to understand the relation between the possible densities of measures on a Boolean algebra when compared with this same set as interpreted in a forcing extension by a measure algebra.

One might also ask, as Džamonja-Plebanek do [DP08, Question 3.6]:

Question 76. *Let κ be an uncountable cardinal and $\mathfrak{B} \in \mathbf{M}$ be κ-approximable. Does $\mathfrak{B}$ support a strictly positive measure of density κ?*

Chapter 4

Slaloms and their destructibility

The aim of this chapter is to introduce *slaloms*. We shall be interested in certain families of slaloms and particularly whether they are *bounded* in an appropriate sense, or whether they can be bounded in an extension by a measure algebra. We shall also be interested in whether these families of slaloms form a chain under an appropriate order. We shall use these families of slaloms to construct some compact spaces in Chapter 5, and the properties of these compact spaces will be tightly connected to the properties of the families of slaloms that we started with.

In Section 4.1 we introduce slaloms as well as the families of slaloms that we will be interested in in this thesis. In Section 4.2, we modify a construction of Kunen and Fremlin from [FK91] to construct a chain of slaloms from a particular family of slaloms. In Section 4.3, we show that this family of slaloms is *random destructible*, a property that will be crucially used in Chapter 6 to ensure that the compact spaces that we construct in Chapter 5 will support a measure.

Before we begin, we remind the reader of some notation.

(i) A natural number n will sometimes also denote the set of its predecessors $\{0, 1, \ldots, n-1\}$.

(ii) If $A \subseteq \omega \times \omega$, then by $A(n)$ we denote the set $\{m \in \omega\colon (n, m) \in A\}$.

4.1 Slaloms

Definition 77. For $g \in \omega^\omega$ the set of *g-slaloms*, denoted $\mathcal{S}_g$, is defined as follows.

$$\mathcal{S}_g = \{S \subseteq \omega \times \omega\colon \forall n(|S(n)| < g(n))\}.$$

If $h \in \omega^\omega$ is the exponential function, that is, $h(n) = 2^n$, then we write $\mathcal{S}$ for $\mathcal{S}_h$, and we simply call h-slaloms *slaloms*.

For the purposes of this thesis, any increasing function $g \in \omega^\omega$ such that $\sum_n \frac{1}{g(n)}$ is finite could have been used instead of h.

First, we define two orders on the collection of slaloms.

Definition 78. Let $A, B \in \mathcal{S}$. We say that A *contains* B, denoted $A \subseteq B$, if for each $n \in \omega$, $A(n) \subseteq B(n)$. We say that A *almost contains* B, denoted $A \subseteq^* B$, if this happens for all but finitely many $n \in \omega$.

Definition 79. Let $g \in \omega^\omega$.

(i) A family $\mathcal{F} \subseteq \omega^\omega$ is *localised* by $\mathcal{S}_g$ if there is an $S \in \mathcal{S}_g$ such that for every $f \in \mathcal{F}$, for all but finitely many $n \in \omega$, we have that $f(n) \in S(n)$ (we shall denote this by $f \subseteq^* S$).

(ii) A family $\mathcal{A} \subseteq \mathcal{S}_g$ is *$\subseteq^*$-bounded by $\mathcal{S}_g$*, or simply, *bounded by $\mathcal{S}_g$*, if there is $S \in \mathcal{S}_g$ such that for every $A \in \mathcal{A}$, $A \subseteq^* S$, that is, for all but finitely many $n \in \omega$ we have that $A(n) \subseteq S(n)$. A family is *unbounded by $\mathcal{S}_g$* when it is not bounded by $\mathcal{S}_g$.

If g is the exponential function $n \mapsto 2^n$, then we simply say localised, bounded etc.

One reason why slaloms are of interest in set theory is that they allow us to give combinatorial characterisations of some of the cardinal invariants of the continuum. Of interest to us is the following result of Bartoszyński.

Theorem 80. *[BJ95, Theorem 2.3.9] Let $g \in \omega^\omega$ be such that $\lim_n g(n) = \infty$. Then*

$$\mathrm{add}(\mathcal{N}) = \min\{|\mathcal{F}| \colon \mathcal{F} \subseteq \omega^\omega, \mathcal{F} \textit{ is not localised by } \mathcal{S}_g\}.$$

We now come to the main property of families of slaloms that we shall be interested in.

Definition 81. Let $g \in \omega^\omega$, and $\mathbb{P}$ a forcing poset. Let $\mathcal{F} \subseteq \omega^\omega$ be a family which is not localised by $\mathcal{S}_g$. We say that $\mathcal{F}$ is *g-destructible* by $\mathbb{P}$ if

$$\Vdash_{\mathbb{P}} \text{“}\check{\mathcal{F}} \text{ is localised by } \dot{\mathcal{S}}_g\text{”}.$$

Similarly, if $\mathcal{A} \subseteq \mathcal{S}_g$ is a family not bounded by any element of $\mathcal{S}_g$, then we say that it is *g-destructible* by $\mathbb{P}$ if

$$\Vdash_{\mathbb{P}} \text{“}\check{\mathcal{A}} \text{ is bounded by } \dot{\mathcal{S}}_g\text{”}.$$

As before *destructible* means h-destructible (recall that $h \in \omega^\omega$ is given by $h(n) = 2^n$). If $\mathbb{P}$ is the partial order to add a random real, equivalently, the Lebesgue measure algebra or any separable measure algebra, then we shall simply say *random g-destructible* or *random destructible*.

Lemma 82. *Let $g \in \omega^\omega$ and $\mathcal{F} \subseteq \omega^\omega$ be a family which is not localised by $\mathcal{S}_g$. Then $\mathcal{F}$ is g-destructible by some measure algebra iff it is g-destructible by a separable measure algebra, equivalently, is random g-destructible.*

Proof. If $\mathcal{F}$ is g-destructible by a separable measure algebra, it is trivially destructible by a measure algebra, so one of the directions is trivial. For other directions, suppose that $\mathcal{F}$ is g-destructible by a measure algebra $\mathbb{B}$, and suppose that $\dot{S}$ is an $\mathbb{B}$-name for a g-slalom witnessing this. In particular, in the forcing extension by $\mathbb{B}$, for each $n \in \omega$, $\dot{S}(n) \in [\omega]^{<g(n)}$. For each $n \in \omega$, let $\dot{C}_n$ be a nice name for $\dot{S}(n)$, so in particular,

$$\Vdash_{\mathbb{B}} \text{“}\dot{S}(n) = \dot{C}_n\text{”}.$$

Note that since $\mathbb{B}$ has the ccc, for each $n \in \omega$, $\dot{C}_n$ only refers to countably many elements of $\mathbb{B}$. Let $\dot{T}$ be the following name for a function with domain ω:

$$\dot{T} = \{(1_{\mathbb{B}}, \dot{C}_n) \colon n \in \omega\}.$$

Clearly, $\dot{T}$ also only refers to countably many elements of $\mathbb{B}$, and by looking at the measure subalgebra generated by this countable subset of $\mathbb{B}$ we get a separable measure algebra localising $\mathcal{F}$, hence establishing the other direction. $\square$

A similar proof gives the following.

Lemma 83. *Let $g \in \omega^\omega$ and $\mathcal{A} \subseteq \mathcal{S}_g$ a family which is not bounded by $\mathcal{S}_g$. Then $\mathcal{A}$ is g-destructible by some measure algebra iff it is g-destructible by a separable measure algebra, equivalently, is random g-destructible.*

We shall also be interested in particular collections of slaloms.

Definition 84. Let
$$\mathcal{Z} = \{S \subseteq \omega \times \omega \colon S \in \mathcal{S} \text{ and } \lim_n \frac{1}{2^n}|S(n)| = 0\}.$$

Our choice of the letter 'Z' above comes from the fact that the slaloms in $\mathcal{Z}$ all have density 0 with respect to the exponential function $h \in \omega^\omega$. The elements of $\mathcal{Z}$ however do not lie in a region of the Baire space which is below any particular function $f \in \omega^\omega$, a property that we would like to ensure in the families of slaloms which we study in this thesis. Such regions of the Baire space shall be interesting to us since we can perform the Haar measure construction on this region, a fact that shall be crucial when studying the effect of forcing with a measure algebra on sets of slaloms. We make things precise.

Definition 85. Let $g \in \omega^\omega$.

(i) Let $\mathcal{X}_g = \prod g(n)$.

(ii) We will consider $\mathcal{X}_g$ equipped with the product topology, and let μ_g be the standard Haar measure on $\mathrm{Bor}(\mathcal{X}_g)$, the Borel subsets of $\mathcal{X}_g$, so in particular, if $i < g(n)$,
$$\mu(\{f \in \mathcal{X}_g \colon f(n) = i\}) = \frac{1}{g(n)}.$$

(iii) If $h \in \omega^\omega$ is the exponential function $n \mapsto 2^n$, then $\mathcal{X} = \mathcal{X}_h$, and $\mu = \mu_h$.

It is clear that these spaces $\mathcal{X}_g$ are homoemorphic to the Cantor set. We shall be interested in slaloms on such spaces, and in fact we restrict our attention to $\mathcal{X}$.

Definition 86. (i) Let
$$\mathcal{I} = \{S \subseteq \omega \times \omega \colon S(n) \subseteq 2^n \text{ for each } n \text{ and } \sum_n \frac{1}{2^n}|S(n)| < \infty\}.$$

(ii) Let
$$\mathcal{W} = \mathcal{I} \cap \mathcal{S}.$$

That is, $\mathcal{I}$ consists of those elements of $\mathcal{S}$ which are bounded by the exponential function h, and which are further *summable*. Also, $\mathcal{W}$ consists of elements of $\mathcal{I}$ which miss at least one point of $\{n\} \times 2^n$ for each n. That is, $S \in \mathcal{W}$ iff $S \in \mathcal{I}$ and for every $n \in \omega$, $S(n)$ is a proper subset of 2^n.

Notice also that if $f \colon \{(n,i) \colon n \in \omega, i < 2^n\} \to \omega$ is the natural enumeration function (sending $\{n\} \times 2^n$ to $[2^n, 2^{n+1})$ for each n), then $I \in \mathcal{I}$ if and only if $f[I] \in \mathtt{I}_{1/n}$ where $\mathtt{I}_{1/n}$, the classical summable ideal on ω (see for example [Far00]), is defined as follows
$$\mathtt{I}_{1/n} = \{A \subseteq \omega \colon \sum_n \frac{|A \cap [2^n, 2^{n+1})|}{2^n} < \infty\}.$$

4.2 A chain of slaloms

In [FK91] a subfamily of ω^ω which cannot be localised by $\mathcal{S}$ was used to construct a family of elements of $\mathcal{Z}$ which is not $\subseteq^*$-bounded in $\mathcal{S}$. (Note that in [FK91] the authors considered $\mathcal{S}_g$ for $g(n) = n$ instead of $\mathcal{S}$ but it does not make any difference for their results.)

Theorem 87. *[FK91, Theorem 4] There is a $\subseteq^*$-chain $\{A_\alpha \colon \alpha < \mathrm{add}(\mathcal{N})\} \subseteq \mathcal{Z}$ which is not bounded by $\mathcal{S}$.*

The main result of this section is to perform a similar construction for the space $\mathcal{X}$, but which is not only unbounded by $\mathcal{S}$, but also random destructible. Unfortunately, this requires us to go beyond ZFC.

Before we proceed, we make the simple observation that if $f \in \mathcal{X}$, then

$$\{(n, f(n)) \colon n > 0\} \in \mathcal{W},$$

and also, if $S \in \mathcal{S}$ is such that $S(n) \subseteq 2^n$ for every n, then there is a $f \in \mathcal{X}$ such that $S \subseteq T$, where $T(n) = 2^n \setminus \{f(n)\}$ for every n. We shall use these observations in what follows.

Proposition 88 (folklore)**.** *There is a family $\mathcal{F} \subseteq \mathcal{X}$ of size* $\mathrm{non}(\mathcal{M})$ *which is not localised by $\mathcal{S}$.*

Proof. For $g \in \mathcal{X}$, let

$$A_g = \{f \in \mathcal{X} \colon \exists^\infty n\ (f(n) = g(n))\}.$$

Notice that for each $g \in \mathcal{X}$ the set A_g is comeager. Indeed, for $g \in \mathcal{X}$ and $n \in \omega$, let

$$A_g^n = \{f \in \mathcal{X} \colon f(n) = g(n)\}.$$

Each A_g^n is open and $\bigcup_{n>m} A_g^n$ is dense for each $m \in \omega$, and

$$A_g = \bigcap_m \bigcup_{n>m} A_g^n.$$

Let $\{f_\alpha \colon \alpha < \mathrm{non}(\mathcal{M})\} \subseteq \mathcal{X}$ be a family witnessing $\mathrm{non}(\mathcal{M})$, that is, be non-meagre of the smallest possible size. Then for each $g \in \mathcal{X}$ there is an α such that $f_\alpha \in A_g$ and so $f_\alpha(n) = g(n)$ for infinitely many n.

The family $\{f_\alpha \colon \alpha < \mathrm{non}(\mathcal{M})\}$ is not localised by $\mathcal{S}$, because for every $S \in \mathcal{S}$ there is $g_S \in \mathcal{X}$ such that $g_S(n) \notin S(n)$ for each n. Hence, there is an α such that $f_\alpha(n) = g_S(n)$ for infinitely many n. So, for each $S \in \mathcal{S}$ there is an $\alpha < \mathrm{non}(\mathcal{M})$ such that $\{n \colon f_\alpha(n) \notin S(n)\}$ is infinite. □

Now, as in [FK91, Theorem 4], we will use a set of reals as above to find a $\subseteq^*$-chain in $\mathcal{W}$ which is not $\subseteq^*$-bounded in $\mathcal{S}$. The proof is essentially the same as there, with some minor modifications.

Theorem 89. *($\mathrm{add}(\mathcal{N}) = \mathrm{non}(\mathcal{M})$) There is a $\subseteq^*$-chain $\{A_\alpha \colon \alpha < \mathrm{add}(\mathcal{N})\} \subseteq \mathcal{W}$ such that for every $S \in \mathcal{S}$ there is $\alpha < \mathrm{add}(\mathcal{N})$ such that $A_\alpha \not\subseteq^* S$.*

Proof. Let $\mathcal{F} = \{f_\alpha \colon \alpha < \mathrm{add}(\mathcal{N})\}$ be as in Proposition 88.

Let $A_0 = f_0 \cap ([1, \infty) \times \omega)$ and assume that we have constructed the A_α for $\alpha < \beta$. For each $\alpha < \beta$ fix a function $g_\alpha \colon \omega \to \omega$ such that

$$\sum_{i \geq g_\alpha(n)} \frac{1}{2^i} |A_\alpha(i)| < \frac{1}{2^n}.$$

As $\beta < \text{add}(\mathcal{N}) \leq \mathfrak{b}$, there is a function $g\colon \omega \to \omega$ which is strictly increasing and which $\leq^*$-dominates $\{g_\alpha\colon \alpha < \beta\}$. For each $\alpha < \beta$, fix m_α such that $g(n) \geq g_\alpha(n)$ for each $n \geq m_\alpha$.

For $\alpha < \beta$, define $F_\alpha\colon \omega \to [\omega \times \omega]^{<\omega}$ such that

$$F_\alpha(n) = \begin{cases} A_\alpha \cap [g(n), g(n+1)) \times \omega \text{ if } n \geq m_\alpha, \\ \emptyset \text{ otherwise.} \end{cases}$$

Now, since $[\omega \times \omega]^{<\omega}$ is countable and $\beta < \text{add}(\mathcal{N})$, by Theorem 80 applied to the space $\omega^{[\omega\times\omega]^{<\omega}}$ we see that there is an f-slalom $\Phi \subseteq \omega \times [\omega \times \omega]^{<\omega}$ for $f \in \omega^\omega$ given by $f(n) = n+1$ which localises all of the F_α. That is,

(i) $\{n\colon F_\alpha(n) \notin \Phi(n)\}$ is finite,

(ii) $|\{I\colon (n, I) \in \Phi\}| \leq n$.

Additionally, throwing out some elements if needed, we can assume that

(iii) for each $(n, I) \in \Phi$ there is $\alpha < \beta$ such that $F_\alpha(n) = I$.

The last condition implies that whenever $(n, I) \in \Phi$, then $I \subseteq [g(n), g(n+1)) \times \omega$ and

$$\sum_{i \geq g(n)} \frac{1}{2^i} |I(i)| < \frac{1}{2^n}.$$

Also, if I is such that $(n, I) \in \Phi$ and $(k, l) \in I$, then $(k, l) \in A_\alpha$ for some $\alpha < \beta$ and as we will see at the end of the proof, therefore there is a $\gamma \leq \alpha$ such that $l = f_\gamma(k)$. Let

$$A = \bigcup \{I\colon \exists n \ (n, I) \in \Phi\}.$$

Notice that

$$\sum_{g(n) \leq i < g(n+1)} \frac{1}{2^i} |A(i)| < \frac{n}{2^n}$$

and since $\sum_n \frac{n}{2^n} = 2$, we have that $A \in \mathcal{W}$. Moreover, for each $\alpha < \beta$ there is $m \geq m_\alpha$ such that $(n, F_\alpha(n)) \in \Phi$ for every $n \geq m$. So, $A_\alpha \subseteq A \cup ([0, g(m)] \times \omega$ and it follows that $A_\alpha \subseteq^* A$.

Now, it is easy to see that there is a $k < \omega$ such that $(A \cup f_\beta) \cap ([k, \infty) \times \omega) \in \mathcal{W}$. Put

$$A_\beta = (A \cup f_\beta) \cap ([k, \infty) \times \omega).$$

We have now finished the construction. To see that $\{A_\alpha\colon \alpha < \text{add}(\mathcal{N})\}$ is not $\subseteq^*$-bounded by any slalom in $\mathcal{S}$, notice that the family $\mathcal{F}$ was chosen so as to not be localised by any slalom in $\mathcal{S}$, and since every real from this family is $\subseteq^*$-contained in some A_α (to be more specific, we simply have that if $\alpha < \text{add}(\mathcal{N})$ then $f_\alpha \subseteq^* A_\alpha$), the former family also inherits this property.

Clearly

$$\bigcup_{\alpha < \text{add}(\mathcal{N})} A_\alpha \subseteq \bigcup_{\alpha < \text{add}(\mathcal{N})} f_\alpha,$$

so $A_\alpha(n) \subseteq 2^n$ for each n and $\alpha < \text{add}(\mathcal{N})$.

$\square$

Remark 90. Let
$$\kappa = \min\{|\mathcal{D}| \colon \mathcal{D} \subseteq \mathcal{W}, \neg\exists S \in \mathcal{S}\ \forall D \in \mathcal{D}\ D \subseteq^* S\}.$$
The reader may notice that Proposition 88 amounts to a proof that $\kappa \leq \mathrm{non}(\mathcal{M})$, and that Theorem 89 can actually be proved from the assumption that $\mathrm{add}(\mathcal{N}) = \kappa$.

In fact, there is a better upper bound for κ (than $\mathrm{non}(\mathcal{M})$). Recall that if $\mathcal{I}$ is an ideal on ω, then $\mathrm{cov}^*(\mathcal{I})$ is the minimal size of a subfamily of $\mathcal{I}$ such that for every infinite $X \subseteq \omega$ there is an element of the family intersecting X on an infinite set (see e.g. [HHH07]). In this setting κ is the minimal size of a subfamily of $\mathcal{I}_{1/n}$ such that for each subset of ω intersecting each interval $[2^n, 2^{n+1})$ at least once, there is an element of this family intersecting it infinitely many times. Clearly then, $\kappa \leq \mathrm{cov}^*(\mathcal{I}_{1/n})$.

It is also not hard to see that $\mathrm{cov}(\mathcal{N}) \leq \kappa$. Indeed, for $W \in \mathcal{W}$ let $A_W = \{f \in \mathcal{X} \colon \exists^\infty n\ f(n) \in W(n)\}$. By the Borel-Cantelli Lemma, Theorem 21, $\lambda(A_W) = 0$ for each $W \in \mathcal{W}$. If $\mathcal{F} \subseteq \mathcal{W}$ is not bounded by any slalom, then each $f \in \mathcal{X}$ is in some A_F, $F \in \mathcal{F}$ (since we can in particular consider the slalom which on every n is exactly $2^n \setminus \{f(n)\}$). Hence, if $\mathcal{F}$ witnesses κ, then the family $\{A_F \colon F \in \mathcal{F}\}$ covers $\mathcal{X}$, and hence has size at least $\mathrm{cov}(\mathcal{N})$.

In fact, if $\mathrm{cov}(\mathcal{N}) < \mathfrak{b}$, then $\kappa = \mathrm{cov}(\mathcal{N})$ (see [Bar88, Theorem 2.2]). It seems likely that consistently $\mathrm{cov}(\mathcal{N}) < \kappa$, but we were not able to prove it.

4.3 A random destructible family of slaloms

For this section, let $\mathbb{B}$ denote the following separable measure algebra: $\mathbb{B} = \mathrm{Bor}(\mathcal{X})/\mathcal{N}_\mu$.

Definition 91. For $n > 0$, $k < 2^n$ let $I^n_k = \{f \in \mathcal{X} \colon f(n) = k\}$. Define a $\mathbb{B}$-name $\dot{S}$ for a subset of $\omega \times \omega$ in the following way:
$$[\![k \in \dot{S}(n)]\!] = \mathcal{X} \setminus I^n_k.$$

Clearly, $\dot{S}$ is an $\mathbb{B}$-name for a slalom. We will call $\dot{S}$ *the canonical name for a slalom.* Notice that
$$\Vdash_{\mathbb{B}} \text{“}\exists \dot{f} \in \dot{\mathcal{X}}\ \dot{S}(n) = 2^n \setminus \{\dot{f}(n)\}\text{”},$$
and $\dot{f}$ is a name for a random real.

We will prove that the family $\mathcal{W}$ is destructible by $\mathbb{B}$.

Proposition 92. *For every $W \in \mathcal{W}$*
$$\Vdash_{\mathbb{B}} \text{“}\check{W} \subseteq^* \dot{S}\text{”},$$
where $\dot{S}$ is as in Definition 91.

Proof. Fix a $W \in \mathcal{W}$ and a $b \in \mathbb{B}$ of positive measure, and let $\epsilon > 0$ be such that $\mu(b) > \epsilon$. Take n such that $\sum_{i>n} \frac{1}{2^i}|W(i)| < \epsilon$. Clearly,
$$\sum_{i>n} \mu\Big(\bigcup_{k \in W(i)} I^i_k\Big) < \epsilon$$
and so if
$$c = \bigcup_{i>n} \bigcup_{k \in W(i)} I^i_k,$$

then $\mu(c) < \epsilon$. So we finish by noticing that $b \setminus c \in \mathbb{B}^+$ and

$$b \setminus c \Vdash \text{“}\forall i > n\ \check{W}(i) \in \dot{S}(i)\text{”}.$$

$\square$

Corollary 93. *(*$\mathrm{add}(\mathcal{N}) = \mathrm{non}(\mathcal{M})$*) There is a* $\subseteq^*$*-chain* $\{A_\alpha \colon \alpha < \mathrm{add}(\mathcal{N})\} \subseteq \mathcal{W}$ *which is unbounded but random destructible.*

Chapter 5

Todorčević' Construction

The aim of this chapter is to study a method due to Todorčević [Tod00] of constructing compactifications of the natural numbers using families of slaloms. The most important feature of this method for us is that there is a strong correspondence between the properties of these families (especially from the point of view of their boundedness) and the compactification obtained from them by applying the Todorčević machinery to it. Of greatest interest to us will be the property of whether the growths obtained from these compactifications support a measure or not and the nature of any measures so obtained. We shall also be interested in whether these growths map onto $[0,1]^{\omega_1}$. In this chapter, we shall describe the Todorčević machinery, and study the correspondence between the nature of the families of slaloms which are supplied to the machinery and the properties of the spaces which we obtain. The concrete applications will be given in Chapter 6

In Section 5.1 we describe the basic idea of Todorčević. In Section 5.2 we collect several facts about the components of the spaces which are used to construct this space. In Section 5.3 we look at how the choice of the families of slaloms which is supplied to the machinery affects the spaces which are obtained.

Before we start, let us first recall some notation and terminology from the previous chapter.

Notation 94. (i) $h \in \omega^\omega$ denotes the exponential function $n \mapsto 2^n$;

(ii) $\mathcal{X}$ denotes the product space $\prod h(n)$, and we define Haar measure on $\mathrm{Bor}(\mathcal{X})$ in the standard way, and in particular if $i < h(n)$, then

$$\mu(\{f \in \mathcal{X}_h \colon f(n) = i\}) = \frac{1}{h(n)};$$

(iii) $\mathcal{S}$ denotes the set of slaloms $S \subseteq \omega \times \omega$ such that $|S(n)| < 2^n$ for each $n \in \omega$;

(iv) If $A, B \in \mathcal{S}$, then $A \subseteq B$ if $A(n) \subseteq B(n)$ for every $n \in \omega$, and $A \subseteq^* B$ if this happens for all but finitely many $n \in \omega$;

(v) $\mathcal{V}$ denotes the collection of those $S \in \mathcal{S}$ such that $S(n) \subseteq 2^n$ for each $n \in \omega$ and which have density 0 in the sense that $\lim_n \frac{1}{2^n}|S(n)| = 0$;

(vi) $\mathcal{W}$ denotes the collection of those $S \in \mathcal{S}$ such that $S(n) \subseteq 2^n$ for each $n \in \omega$ and which are summable in the sense that $\sum_n \frac{1}{2^n}|S(n)| < \infty$;

(vii) If $\mathcal{A} \subseteq \mathcal{S}$ is such that there is some $S \in \mathcal{S}$ such that for every $A \in \mathcal{A}$, $A \subseteq^* S$, then we say that $\mathcal{A}$ is bounded, or to be more precise, bounded by S;

(viii) If $\mathcal{A} \subseteq \mathcal{S}$ is such that $\check{\mathcal{A}}$ is bounded by some element of $\dot{\mathcal{S}}$ in an extension by some measure algebra (equivalently, by the separable measure algebra), then we say that $\mathcal{A}$ is random destructible.

5.1 The construction

In this section we will explain some details of the construction from [Tod00, Theorem 8.4]. The space that Todorčević constructs is the Stone space of a Boolean algebra $\mathcal{P}(\Omega)/\mathrm{Fin}$. This Boolean algebra has two types of generators. We get down to the details.

Definition 95. (i) Let $\Omega = \{(S, n) \colon n \in \omega,\ S \in \mathcal{S},\ S \subseteq (n \times 2^n)\}$.

(ii) For each $A \subseteq \omega \times \omega$ define

$$T_A = \{(T, n) \in \Omega \colon A \cap (n \times 2^n) \subseteq T\}.$$

(iii) For $(S, n) \in \Omega$ let

$$T_{(S,n)} = \{(T, m) \in \Omega \colon m \geq n, T \cap (n \times 2^n) = S\}.$$

(iv) For $\mathcal{A} \subseteq \mathcal{P}(\omega \times \omega)$, let $\mathfrak{T}_{\mathcal{A}}$ be the subalgebra of $\mathcal{P}(\Omega)$ generated by

$$\{T_A \colon A \in \mathcal{A}\} \cup \{T_{(S,n)} \colon (S, n) \in \Omega\}.$$

(v) For $\mathcal{A} \subseteq \mathcal{P}(\omega \times \omega)$, let $\mathfrak{T}^*_{\mathcal{A}}$ be the Boolean subalgebra of $\mathfrak{T}_{\mathcal{A}}$ generated only by $\{T_A \colon A \in \mathcal{A}\}$;

(vi) For $\mathcal{A} \subseteq \mathcal{P}(\omega \times \omega)$, let $K_{\mathcal{A}}$ be the Stone space of $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$.

Now, by varying the family $\mathcal{A} \subseteq \mathcal{P}(\omega \times \omega)$ that we supply to the Todorčević construction, we can get compact spaces $K_{\mathcal{A}}$ having a variety of different properties. In all the applications considered in this thesis, $\mathcal{A}$ is in fact a subset of $\mathcal{S}$.

5.2 Facts about the generators

In order to show the effect that supplying different families $\mathcal{A} \subseteq \mathcal{S}$ to the Todorčević machinery has on the output spaces $K_{\mathcal{A}}$, we shall need a number of observations about the generators of these algebras, and we next collect them all in one place. The proofs of the lemmas in this section are mostly routine verifications, however, it is hoped that separating these observations from the main results improves the readability of this chapter.

Lemma 96. *Let $A \subseteq \omega \times \omega$. Then T_A is infinite if and only if $A \in \mathcal{S}$.*

Proof. If $A \in \mathcal{S}$, then $(A \cap (n \times 2^n), n) \in T_A$ for each $n \in \omega$. If $A \notin \mathcal{S}$, there is some $n \in \omega$ such that $|A(n)| \geq 2^n$. Therefore, $(S, m) \in T_A$ implies that $m \leq n$. This tells us that T_A is finite since for any natural number n, the set $\{(S, m) \in \Omega \colon m \leq n\}$ is finite. □

Lemma 97. *Let $A \subseteq \omega \times \omega$. Then $\sim T_A$ is infinite if and only if there is some $n \in \omega$ such that $A(n)$ is non-empty.*

Proof. Clear. $\square$

Lemma 98. *Let $A, B \in \mathcal{S}$. Then*

(i) $T_{(A \cup B)} = T_A \cap T_B$;

(ii) if $A \subseteq B$, then $T_B \subseteq T_A$.

Proof. Clear. $\square$

Lemma 99. *Let $(S, n), (T, m) \in \Omega$. Then either $T_{(S,n)} \cap T_{(T,m)}$ is empty or one of $T_{(S,n)}, T_{(T,m)}$ contains the other.*

Proof. Clear. $\square$

Lemma 100. *Let $m < n$ and $(S, m) \in \Omega$. Then there is a finite $F \subseteq \Omega$ such that $(T, i) \in F$ implies that $i = n$, and such that $T_{(S,m)} \subseteq \bigcup_{(T,i) \in F} T_{(T,i)}$ modulo a finite set.*

Proof. Let F be the set of all $(T, n) \in \Omega$ such that $T \cap (m \times 2^m) = S$. Only the (finitely many) $(R, i) \in T_{(S,m)}$ such that $i < n$ are not in $\bigcup_{(T,i) \in F} T_{(T,i)}$. $\square$

Lemma 101. *Let $\mathcal{A} \subseteq \mathcal{S}$ be such that for any $\mathcal{F} \subseteq \mathcal{A}$ finite, $\bigcap_{A \in \mathcal{F}} T_A$ is infinite. Then there is some $S \in \mathcal{S}$ such that $A \subseteq S$ for each $A \in \mathcal{A}$.*

Proof. In fact, we claim that $S \subseteq \omega \times \omega$ defined by $S(n) = \bigcup_{A \in \mathcal{A}} A(n)$ for each $n \in \omega$ does the trick. To verify this, we only need to check that for each $n \in \omega$, $|S(n)| < 2^n$. If this does not happen, then there is already some finite $\mathcal{F} \subseteq \mathcal{A}$ such that $|\bigcup_{A \in \mathcal{F}} A(n)| \geq 2^n$. But then in that case $T_{\bigcup \mathcal{F}}$ is finite, which is not possible by Lemma 96. $\square$

Lemma 102. *Let $S \in \mathcal{S}$ and $\mathcal{A} \subseteq \mathcal{S}$ be such that $A \in \mathcal{A}$ implies that $A \subseteq S$. Then $\bigcap_{A \in \mathcal{A}} T_A$ is infinite.*

Proof. This is clear since for each $n \in \omega$ and $A \in \mathcal{A}$, $(S \cap (n \times 2^n), n) \in T_A$. $\square$

Lemma 103. *Let $\mathcal{A} \subseteq \mathcal{S}$ be such that for any $\mathcal{F} \subseteq \mathcal{A}$ finite, $\bigcap_{A \in \mathcal{F}} T_A$ is infinite. Also, let (T, m) be such that for each $A \in \mathcal{A}$, $T_A \cap T_{(T,m)}$ is infinite. Then for any $\mathcal{F} \subseteq \mathcal{A}$ finite, $\bigcap_{A \in \mathcal{F}} (T_A \cap T_{(T,m)})$ is infinite.*

Proof. Let $S \in \mathcal{S}$ be such that $A \subseteq S$ for each $A \in \mathcal{A}$, which we know exists by Lemma 101. Let $Q \subseteq \omega \times \omega$ be defined as follows:

(i) $Q \cap ([0, m) \times \omega) = T$, and

(ii) $Q \cap ([m, \infty) \times \omega) = S \cap ([m, \infty) \times \omega)$.

Then for each $n \in \omega$, $(Q \cap (n \times 2^n), n) \in \bigcap_{A \in \mathcal{F}} (T_A \cap T_{(T,m)})$. $\square$

Lemma 104. *Let $F \subseteq \mathcal{S}$ be a finite set such that $\bigcap_{A \in F} (\sim T_A)$ is non-empty. Then the latter is infinite, and in fact if $(S, n) \in \bigcap_{A \in F} (\sim T_A)$, then $T_{(S,n)} \subseteq \bigcap_{A \in F} (\sim T_A)$.*

Proof. Suppose that $(S, n) \notin T_A$. This means that $A \cap (n \times 2^n) \nsubseteq S$, so there is some $i < n$ such that $A(i) \nsubseteq S(i)$. But if $(T, m) \in T_{(S,n)}$, then $T \cap (n \times 2^n) = S$, so $A(i) \nsubseteq T(i)$ as well. It follows that $(T, m) \notin T_A$. The result follows. $\square$

Lemma 105. *Let $A, B, C \in \mathcal{S}$ be such that $A, B \subseteq C$, and such that $A \subseteq^* B$. Then there is some $n \in \omega$ such that $T_A \cap T_{(C \cap (n \times 2^n), n)} \supseteq T_B \cap T_{(C \cap (n \times 2^n), n)}$.*

Proof. Let $n \in \omega$ be such that $m \geq n$ implies that $A(n) \subseteq B(n)$. Then if $(S, m) \in T_B \cap T_{(C \cap (n \times 2^n), n)}$, then

(i) $m \geq n$,

(ii) $S \cap (n \times 2^n) = C \cap (n \times 2^n)$, and

(iii) $B \cap (m \times 2^m) \subseteq T$.

Since $C \supseteq A$, and for each $i \in [n, m)$, $A(i) \subseteq B(i)$, it is clear that in such a situation $(S, m) \in T_A \cap T_{(C \cap (m \times 2^m), m)}$ as well.

$\square$

Lemma 106. *Let $\mathfrak{B}$ be the subalgebra of $\mathcal{P}(\Omega)$ generated only by $\{T_{(S,n)} \colon (S, n) \in \Omega\}$. Let $\mathcal{U}$ be an ultrafilter on $\mathfrak{B}/\mathrm{Fin}$. Then there is some some $A \in \mathcal{S}$ which completely determines $\mathcal{U}$ in the sense that for any $(S, n) \in \Omega$, $[T_{(S,n)}] \in \mathcal{U}$ if and only if $S = A \cap (n \times 2^n)$. Such an A can in fact be found such that for each $n \in \omega$, $A(n) \subseteq 2^n$.*

Proof. First, observe that for each $m \in \omega$, there is exactly one $(S, m) \in \mathcal{U}$. Then, let $A \subseteq \omega \times \omega$ be defined by $A(m) = S(n)$ for some $m < n$ and $(S, n) \in \mathcal{U}$. It is easy to see that A is well-defined (by Lemma 99) and as required. $\square$

Lemma 107. *Let $\mathfrak{B}$ be the subalgebra of $\mathcal{P}(\Omega)$ generated only by $\{T_{(S,n)} \colon (S, n) \in \Omega\}$. Let $\mathfrak{A} = \mathfrak{B}/\mathrm{Fin}$. Then every element of $\mathfrak{A}^+$ contains $[T_{(S,n)}]$ for some $(s, n) \in \Omega$. That is, $\{[T_{(S,n)}] \colon (S, n) \in \Omega\}$ is a π-base of $\mathfrak{A}$.*

Proof. Since $\{[T_{(S,n)}] \colon (S, n) \in \Omega\}$ generates $\mathfrak{A}$, for any element of $\mathfrak{A}^+$, there is some finite set $F \subseteq \Omega$ such that this element can be represented as a Boolean expression of $\{[T_{(S,n)}] \colon (S, n) \in F\}$. Pick an arbitrary element of $\mathfrak{A}^+$, and fix such a set F of generators for it, along with a particular Boolean expression for it. Let $X \in \mathcal{P}(\Omega)$ be obtained by applying the same Boolean expression, but replacing every occurrence of $[T_{(S,n)}]$ by $T_{(S,n)}$ etc. It follows that $[X]$ is then the element that we have in mind. Note that X is an infinite set, so for arbitrarily large $m \in \omega$, there is some $(T, m) \in X$. Let $(T, m) \in X$ be such that $m > n$ for any $(S, n) \in F$. We will have proved our lemma once we verify the following claim.

Claim 108. $T_{(T,m)} \subseteq X$.

Proof. Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{B}$ generated by $\{T_{(S,n)} \colon (S, n) \in F\}$, and consider the following equivalence relation $\approx$ on the set $\{(R, i) \in \Omega \colon i \geq m\}$:

$$(Q, i) \approx (R, j) \iff i = j \text{ and } Q \cap (m \times 2^m) = R \cap (m \times 2^m).$$

It is easy to see that for each $(S,n) \in \Omega$, $T_{(S,n)}$ is saturated with respect to $\approx$ in the following sense: if $(Q,i) \approx (R,j)$, then

$$(Q,i) \in T_{(S,n)} \iff (R,j) \in T_{(S,n)}.$$

That is, all the generators of $\mathfrak{C}$ have this saturation property. It is also easy to see that this saturation property is preserved by the Boolean operations between them. It follows that every element of $\mathfrak{C}$ has this property. Since $X \in \mathfrak{C}$ and $(T,m) \in X$, the claim follows. $\dashv$

$\square$

5.3 The influence of the inputs

We are now in a position to see how the properties of the families $\mathcal{A}$ affect the properties of the space $K_{\mathcal{A}}$.

5.3.1 π-base

Lemma 109. *Let $\mathcal{A} \subseteq \mathcal{S}$ be closed under finite unions (so long as they belong to $\mathcal{S}$). Then the non-zero elements of the form $[T_A \cap T_{(S,n)}]$ form a π-base for $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$.*

Proof. The set $\{[T_A] \colon A \in \mathcal{A}\} \cup \{[T_{(S,n)}] \colon (S,n) \in \Omega\}$ generates the Boolean algebra $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$. By Theorem 8, the non-zero elements of $\mathfrak{T}_A/\mathrm{Fin}$ which are intersections of the generators and their complements form a π-base of $\mathfrak{T}_A/\mathrm{Fin}$. Consequently, if we can prove that each of them contains a non-zero element of the form $[T_A \cap T_{(S,n)}]$, we would be done.

First, notice that by Lemma 107, for any Boolean term formed from only using elements of the form $[T_{(S,n)}]$, there is some $[T_{(T,m)}]$ contained in it. Second, notice that for any non-empty element which is the intersection of finitely many terms of the form $[\sim T_{(S,n)}]$, there is some $(T,m) \in \Omega$ such that $[T_{(T,m)}]$ is contained in this element. It follows that any non-zero term which is formed by taking an intersection of finitely many terms of the form $[T_{(S,n)}], [\sim T_{(T,m)}], [\sim T_A]$ contains some term of the form $[T_{(R,i)}]$.

Third, any non-zero term which is the intersection of finitely many terms of the form $[T_A]$ is actually of the form $[T_B]$ for some $B \in \mathcal{A}$. Here we first use Lemma 98 to obtain B, and then Lemma 96 to conclude that $B \in \mathcal{S}$, and then the closure of $\mathcal{A}$ under finite unions to conclude that $B \in \mathcal{A}$. It follows that any non-zero term which is formed by taking an intersection of finitely many terms of the form $[T_{(S,n)}], [\sim T_{(T,m)}], [T_A], [\sim T_B]$ contains some term of the form $[T_C \cap T_{(R,i)}]$ for some $C \in \mathcal{A}$. $\square$

5.3.2 Chain conditions

Lemma 110. *If $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$ is σ-centred, there are $\langle S_n \colon n \in \omega\rangle \subseteq \mathcal{S}$ such that for each $A \in \mathcal{A}$ there is some $n \in \omega$ such that $A \subseteq S_n$.*

Proof. It follows from the hypothesis that there is a sequence $\langle \mathcal{C}_n \colon n \in \omega\rangle \subseteq \mathcal{A}$ such that $\bigcup_{n\in\omega} \mathcal{C}_n = \mathcal{A}$ and such that for each $n \in \omega$, $\{[T_A] \colon A \in \mathcal{C}_n\}$ is centred. Then by Lemma 101 we can obtain $\langle S_n \colon n \in \omega\rangle \subseteq \mathcal{S}$ as required. $\square$

Lemma 111. *Let $\mathcal{A} \subseteq \mathcal{S}$ be a $\subseteq^*$-increasing chain whose cofinality is uncountable such that there is no $S \in \mathcal{S}$ such that for each $A \in \mathcal{A}$, $A \subseteq^* S$. Then $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$ is not σ-centred.*

Proof. Suppose to the contrary that $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$ is σ-centred. Then we can obtain a sequence $\langle \mathcal{C}_n : n \in \omega \rangle \subseteq \mathcal{A}$ and $\langle S_n : n \in \omega \rangle \subseteq \mathcal{S}$ as in the proof of the previous lemma. Since $\mathcal{A}$ is a chain of uncountable cofinality, it follows that one of the $\mathcal{C}_n$ is cofinal in $\mathcal{A}$. Now, given any $A \in \mathcal{A}$, there is some $B \in \mathcal{C}_n$ such that $A \subseteq^* B$, and $B \subseteq S_n$. It follows that for each $A \in \mathcal{A}$, $A \subseteq^* S_n$. Contradiction. $\square$

Lemma 112. *Let $\mathcal{A} \subseteq \mathcal{S}$ be such that for every $f \in \mathcal{X}$, $S_f \in \mathcal{A}$ where $S_f \subseteq \omega \times \omega$ is defined by $S_f(n) = \{f(n)\}$. Then $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$ is not σ-centred.*

Proof. Again, suppose to the contrary that $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$ is σ-centred and obtain a sequence $\langle \mathcal{C}_n : n \in \omega \rangle \subseteq \mathcal{A}$ and $\langle S_n : n \in \omega \rangle \subseteq \mathcal{S}$. For each $n \in \omega$, there is some $f_n \in \mathcal{X}$ such that for each $m \in \omega$, $f_n(m) \notin S_n(m)$ (equivalently, such that for each $m \in \omega$, $S_n(m) \subseteq 2^m \setminus \{f_n(m)\}$). Now, let $g \in \mathcal{X}$ be such that for each $n \in \omega$, there are infinitely many $m \in \omega$ such that $g(m) = f_n(m)$. Consider S_g, which we know from the hypothesis is in $\mathcal{A}$. Clearly, $S_g \not\subseteq^* S_n$ for each $n \in \omega$, which is a contradiction. $\square$

Lemma 113. *Let $\mathcal{A} \subseteq \mathcal{S}$ be be closed under finite unions (so long as they belong to $\mathcal{S}$) and such that there is some $f \in \mathcal{X}$ such that for each $A \in \mathcal{A}$, for all but finitely many $n \in \omega$, $f(n) \notin A(n)$, or equivalently, such that there is some $S \in \mathcal{S}$ such that for each $A \in \mathcal{A}$, for all but finitely many $n \in \omega$, $A(n) \subseteq S(n)$ (that is, $A \subseteq^* S$). Then $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$ is σ-centred.*

Proof. We first make a simple observation. Let $\mathfrak{B}$ be a Boolean algebra, and $\mathcal{B} \subseteq \mathfrak{B}^+$ be such that for each element $a \in \mathfrak{B}^+$, there is some $b \in \mathcal{B}$ such that $b \leq a$ (that is, $\mathcal{B}$ is a π-base of $\mathfrak{B}$). Suppose that we can find a countable fragmentation $\langle \mathcal{C}_n : n \in \omega \rangle$ of $\mathcal{B}$, $\bigcup_{n\in\omega} \mathcal{C}_n$, such that each $\mathcal{C}_n$ is centred. Then we can find a countable fragmentation $\langle \mathcal{D}_n : n \in \omega \rangle$ of $\mathfrak{B}^+$ into centred sets as well: for each $n \in \omega$, let

$$\mathcal{D}_n = \{a \in \mathfrak{B}^+ : \exists b \in \mathcal{C}_n [b \leq a]\}.$$

To summarise, if the π-base of a Boolean algebra is σ-centred, the Boolean algebra itself is σ-centred. This simplifies our task in this proof.

Let $\langle S_n : n \in \omega \rangle$ enumerate all the possible countable modifications of S. It follows by the hypothesis that for each $A \in \mathcal{A}$, there is some $n \in \omega$ such that $A \subseteq S_n$. For each $n \in \omega$, let $\mathcal{C}_n = \{A \in \mathcal{A} : A \subseteq S_n\}$. It is clear that $\{[T_A] : A \in \mathcal{C}_n\}$ is a centred set for each $n \in \omega$.

Now, let $m \in \omega$ and $(R, n) \in \Omega$ be arbitrary. Consider the set

$$\mathcal{B} = \{A \in \mathcal{C}_m : T_A \cap T_{(R,i)} \text{ is infinite}\}.$$

By Lemma 103, it is clear that $\{[T_A \cap T_{(R,n)}] : A \in \mathcal{C}_m\}$ is a centred set. It follows that for each $m \in \omega$ and $(R, n) \in \Omega$ we have constructed a centred subset of the π-base of $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$ which we obtained in Lemma 109. It is easy to see that every element of the π-base is in one such centred set: if $A \in \mathcal{C}_m$, just pick some $(R, n) \in \Omega$ such that $T \cap (n \times 2^n) = R$, for example.

Summarising, a π-base of our Boolean algebra is σ-centred. By the observation we made in the early days of this proof, this suffices for our purposes. $\square$

5.3.3 Fibres

Lemma 114. *Let $\mathcal{A} \subseteq \mathcal{S}$ be a $\subseteq^*$-increasing chain. Let $\mathfrak{B}$ be the Boolean subalgebra of $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$ generated by $\{[T_{S,n}] : (S, n) \in \Omega\}$, let $\mathcal{U}$ be an ultrafilter on it, and let $\mathcal{V}$ be the (not necessarily maximal) filter on $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$ that it generates. Then $(\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin})/\mathcal{V}$ contains a chain which generates* *it. Furthermore, this chain is order-isomorphic to a suborder of the reverse of* $(\mathcal{A}, \subseteq^*)$ *(that is, the order obtained by reversing all the relations).*

Proof. Before we start the proof proper, recall from Lemma 98 that if $A, B \in \mathcal{S}$, then $T_B \subseteq T_A$. This is why we have to reverse the order on $\mathcal{A}$.

Now, onwards to the proof. Since $\{[T_A] \colon A \in \mathcal{A}\} \cup \{[T_{(S,n)}] \colon (S,n) \in \Omega\}$ is a generating set for $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$, and $(\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin})/\mathcal{V}$ is a quotient of the latter by the filter $\mathcal{V}$, it follows from Proposition 6 that $\{[T_A]_{\mathcal{V}} \colon A \in \mathcal{A}\} \cup \{[T_{(S,n)}]_{\mathcal{V}} \colon (S,n) \in \Omega\}$ is a generating set for this quotient algebra. All that is left to do now is to examine which elements of this generating set are non-zero, and to show that they form a chain in the algebra.

By Lemma 106, there is some $S \in \mathcal{S}$ which completely determines $\mathcal{U}$ in the following sense:

$$[T_{(T,m)}] \in \mathcal{U} \iff S \cap (m \cap 2^m) = T.$$

Since $\mathcal{V}$ is generated by $\mathcal{U}$, this tells us that

$$[T_{(T,m)}]_{\mathcal{V}} = 0_{(\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin})/\mathcal{V}} \iff S \cap (m \cap 2^m) \neq T,$$

and

$$[T_{(T,m)}]_{\mathcal{V}} = 1_{(\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin})/\mathcal{V}} \iff S \cap (m \cap 2^m) = T.$$

Next, suppose that $A \in \mathcal{A}$ is such that $A \nsubseteq S$. It follows that there is some $n \in \omega$ such that $T_A \cap T_{(S \cap (n \times 2^n), n)}$ is finite, and hence $[T_A \cap T_{(S \cap (n \times 2^n), n)}]$ is the zero element of $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$, and hence of $(\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin})/\mathcal{V}$ as well. Since $[T_{(S \cap (n \times 2^n), n)}] \in \mathcal{U}$ and $\mathcal{V}$ is the filter on $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$ which $\mathcal{U}$ generates, it follows that $[T_A]_{\mathcal{V}}$ is the zero element of $(\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin})/\mathcal{V}$. This, combined with Proposition 5 tells us that for $A \in \mathcal{A}$,

$$[T_A]_{\mathcal{V}} \neq 0_{(\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin})/\mathcal{V}} \iff A \subseteq S.$$

Summarising, we have that the only elements of the generating set which are not $0_{(\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin})/\mathcal{V}}$ or $1_{(\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin})/\mathcal{V}}$ are those $[T_A]_{\mathcal{V}}$ for the $A \in \mathcal{A}$ such that $A \subseteq S$. Let $\mathcal{B}$ denote the set of these A. We know that $\mathcal{A}$ is $\subseteq^*$-increasing.

Claim 115. *If* $A \subseteq^* B$ *are in* $\mathcal{B}$*, then* $[T_B]_{\mathcal{V}}$ *is contained in* $[T_A]_{\mathcal{V}}$ *(in the order on* $(\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin})/\mathcal{V}$*).*

Proof. By Lemma 105, there is some $n \in \omega$ such that

$$T_A \cap T_{(S \cap (n \times 2^n), n)} \supseteq T_B \cap T_{(S \cap (n \times 2^n), n)}.$$

Since $[T_{(S \cap (n \times 2^n), n)}]$ is in $\mathcal{U}$, it is in $\mathcal{V}$ as well. It follows by Proposition 4 that $[T_B]_{\mathcal{V}} \subseteq [T_A]_{\mathcal{V}}$. $\dashv$

It follows that $\{[T_A]_{\mathcal{V}} \colon A \in \mathcal{B}\}$ is a linearly ordered generating set for $\mathcal{B}$. The fact that the order is isomorphic to a suborder of the reverse of $(\mathcal{A}, \subseteq^*)$ is clear.

□

Lemma 116. *Let* $\mathcal{A} \subseteq \mathcal{S}$ *be a* $\subseteq^*$*-increasing chain. Let* $\mathfrak{B}$ *be the Boolean subalgebra of* $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$ *generated by* $\{[T_{S,n}] \colon (S,n) \in \Omega\}$*. Let* L *be its Stone space. Let* $f : K_{\mathcal{A}} \to L$ *be the obvious continuous surjection: for* $\mathcal{U}$ *and ultrafilter on* $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$*,*

$$f(\mathcal{U}) = \mathcal{U} \cap \mathfrak{B}.$$

Then for each $\mathcal{V}$ *an ultrafilter on* $\mathfrak{B}$*,* $f^{-1}[\{\mathcal{V}\}]$ *is a linearly-ordered topological space.*

Proof. Let $\mathcal{V}$ be an ultrafilter on $\mathfrak{B}$. From Theorem 31, we know that $f^{-1}[\{\mathcal{V}\}]$ is the Stone space of $(\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin})/\mathcal{V}$. From Lemma 114, we know that $(\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin})/\mathcal{V}$ is generated by a chain, and that this chain is order-isomorphic to a suborder of the reverse of $(\mathcal{A}, \subseteq^*)$. By Theorem 32, it follows that the Stone space of $(\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin})/\mathcal{V}$ is a linearly ordered topological space. $\square$

Chapter 6

Some compactifications of the natural numbers

In this chapter we combine the results from Chapters 3, 4, and 5 to construct some compactifications of the natural numbers. Namely, we shall apply Todorčević' method from Chapter 5 to the families of slaloms we constructed in Chapter 4 to obtain some compact spaces. The properties of the families of slaloms that we were most interested in was whether or not they were bounded, and whether they can be bounded in forcing extensions by a measure algebra. By results from Chapter 5, this will correspond to the compact spaces being σ-centered, and whether they can be made σ-centered in forcing extensions by a measure algebra. This in turn by Kamburelis' Theorem, Theorem 3.2 from Chapter 3, will tell us whether these compact spaces support a measure or not.

In Section 6.1, we show that consistently, there are Souslinean growths of ω, which support measures but do not map continuously onto $[0,1]^{\omega_1}$. In Section 6.2, we show in ZFC that there are Souslinean growths of ω which support measures which map continuously onto $[0,1]^{\mathfrak{c}}$.

Before we begin, we remind readers of some notation and terminology.

(i) If K is a compact space which contains the natural numbers ω as a dense subspace, we call it a compactification of ω. Also, $K \setminus \omega$ is called the remainder of K, and spaces of this form are called growths of ω.

(ii) By Stone duality, if K is the Stone space of a Boolean algebra $\mathfrak{B} \subseteq \mathcal{P}(\omega)$ containing all the finite subsets of ω, then $K \setminus \omega$ is the Stone space of the quotient of $\mathfrak{B}$ by the ideal Fin of finite subsets of ω, the latter being a Boolean subalgebra of $\mathcal{P}(\omega)/\mathrm{Fin}$.

We also make a little disclaimer: in all that follows, we shall not actually construct Boolean algebras of subsets of ω or equivalence classes of them etc. Instead, we shall construct Boolean algebras of subsets of Ω or equivalence classes of them etc. Since the latter is countable as well, we can easily transfer these constructions to ω if required, preserving all the properties we want.

6.1 Small Souslinean growths of ω which support measures, consistently

We start this section by giving a sufficient condition on a topological space to ensure that it does not map continuously onto $[0,1]^{\omega_1}$, see for example [Moo99, Section 4]. We shall then give a proof that this condition is sufficient.

Definition 117. A space K is said to be *linearly-fibred* if there is a compact metric space M and a continuous surjection $f : K \to M$ such that for each $x \in M$, there is a linear order which induces the subspace topology on $f^{-1}[\{x\}]$.

We shall need the following lemma before we can see establish the sufficiency.

Lemma 118. *Let K be an compact linearly ordered topological space. Then it contains a point of countable π-character.*

Proof. This is clear for finite space. So, suppose that K is infinite. Let $<_K$ denote the linear order on K. Since K is infinite, then either $(K, <_K)$ contains a copy of $(\omega, <)$, or a copy of its reverse. In either case, let $S \subseteq K$ be such a copy, and notice that it has the following property: if I is an interval of K which contains infinitely many elements of S, then it contains cofinitely many elements of S.

Claim 119. *There is some $y \in K$ such that each open interval of K containing y contains an infinite subset of S.*

Proof. If not, for each $y \in K$, there is an open interval U_y of it such that $U_y \cap S$ is finite. Then $\langle U_y \colon y \in K\rangle$ is an open cover of K which does not contain a finite subcover of K since S is infinite. $\dashv$

Now, let y be such a point, and notice that the countable collection of intervals (x, y) such that $x <_K y$ are both in S witnesses that y has countable π-character. $\square$

The following theorem, which will be crucial to showing the smallness of our spaces, is due to Shapirovskii [Sha80].

Theorem 120. *Let K be a compact space. Then the following are equivalent:*

(i) There is no continuous mapping of K onto $[0,1]^{\omega_1}$;

(ii) Every closed subpace of K contains a point of countable π-character.

Corollary 121. *Let K be a compact linearly fibred space. Then K does not map continuously onto $[0,1]^{\omega_1}$.*

Proof. Let M be a compact metric space and $f : K \to M$ a continuous surjection witnessing that K is linearly fibred. Let L be a closed subspace of K. Now, any continuous mapping between a compact space and a Hausdorff space is a closed mapping (that is, the images of closed sets are closed sets, see Theorem 26). It follows that the image of L under f is a closed subset of the compact metric space M, and hence a compact metric space itself. By looking in the preimage of any point in the image of L under f, we obtain a closed, and hence compact, linearly ordered topological subspace of L. By Lemma 118, such spaces always contain a point of countable π-character. $\square$

Armed with that, we can prove the main technical result of this section.

Theorem 122. *Let $\mathcal{A} \subseteq \mathcal{S}$ be such that*

(i) $\mathcal{A}$ is closed under finite unions as long as they belong to $\mathcal{S}$;

(ii) *$\mathcal{A}$ is well-ordered by $\subseteq^*$ and its cofinality is uncountable;*

(iii) *$\mathcal{A}$ is unbounded in $\mathcal{S}$: there is no $S \in \mathcal{S}$ such that for each $A \in \mathcal{A}$, $A \subseteq^* S$;*

(iv) *$\mathcal{A}$ is random destructible: after forcing with a measure algebra, there is some $S \in \mathcal{S}$ such that for all $A \in \mathcal{A}$, $A \subseteq^* S$.*

Then $K_{\mathcal{A}}$, the Stone space of $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$ is non-separable, supports a strictly positive finitely additive measure, and is linearly fibred.

Proof. From Lemma 111, $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$ is not σ-centred. By Lemma 113, $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$ is σ-centred in an extension by a measure algebra. It follows by Kamburelis' Theorem, Theorem 3.2, that $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$ supports a strictly positive finitely additive measure.

We shall now show that $K_{\mathcal{A}}$ is linearly-fibred, which by Theorem 120 will let us conclude that it does not map continuously onto $[0,1]^{\omega_1}$. Let $\mathfrak{B}$ be the Boolean subalgebra of $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$ generated by $\{[T_{S,n}] \colon (S,n) \in \Omega\}$, and let L be its Stone space. Let $f : K_{\mathcal{A}} \to L$ be the obvious continuous surjection: for $\mathcal{U}$ and ultrafilter on $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$,

$$f(\mathcal{U}) = \mathcal{U} \cap \mathfrak{B}.$$

First, note that $\mathfrak{B}$ has a countable generating set, and hence is a countable Boolean algebra. It follows by Theorem 30 that its Stone space L is metrisable, and so $f : K \to L$ is a map from K into a metric space. Next, we know by Theorem 116 that for each $\mathcal{V} \in L$, $f^{-1}[\{\mathcal{V}\}]$ is a linearly ordered topological space. It follows that K is linearly-fibred, and hence by Corollary 121, does not map continuously onto $[0,1]^{\omega_1}$. $\square$

Corollary 123. *($\mathrm{add}(\mathcal{N}) = \mathrm{non}(\mathcal{M})$) There is a Souslinean growth of ω which supports a strictly positive finitely additive measure and is linearly fibred.*

Proof. By Theorem 89, there is a $\subseteq^*$-chain $\{A_\alpha \colon \alpha < \mathrm{add}(\mathcal{N})\} \subseteq \mathcal{W}$ such that for every $S \in \mathcal{S}$, there is an $\alpha < \mathrm{add}(\mathcal{N})$ such that $A_\alpha \nsubseteq^* S$. Let $\mathcal{B}$ be its closure under finite modifications as long as they belong to $\mathcal{S}$. Since for each $A \in \mathcal{A}$, there are countably many elements of $B \in \mathcal{S}$ such that $A =^* B$, it follows that $\mathcal{B}$ is still a $\subseteq^*$-chain. Since $\mathrm{add}(\mathcal{N})$ is an uncountable regular cardinal, it follows that this chain also has uncountable cofinality. Note that $\mathcal{B} \subseteq \mathcal{W}$ as well.

Claim 124. *$\mathcal{B}$ is closed under finite unions as long as they belong to $\mathcal{S}$.*

Proof. Let $A_1 \subseteq^* A_2 \subseteq^* \ldots \subseteq^* A_n$ be in $\mathcal{B}$. Suppose that their union is still in $\mathcal{S}$. Then their union is actually a finite modification of A_n. Since $\mathcal{B}$ is closed under finite modifications as long as they belong to $\mathcal{S}$, it follows that their union is in $\mathcal{B}$. $\dashv$

Also, by Proposition 92, all of $\mathcal{W}$ is random destructible, and so $\mathcal{B}$ is as well. To summarise, $\mathcal{B}$ is a $\subseteq^*$-chain of uncountable cofinality which is closed under finite unions as long as they belong to $\mathcal{S}$. It is unbounded, but random destructible. It follows by the theorem that $K_{\mathcal{B}}$, which is a growth of ω, is Souslinean and supports a strictly positive finitely additive measure and does not map continuously onto $[0,1]^{\omega_1}$. $\square$

6.2 Large Souslinenean growths of ω supporting a measure

Theorem 125. *Let $\mathcal{A} \subseteq \mathcal{S}$ be such that*

(i) $\mathcal{A}$ is closed under finite unions as long as they belong to $\mathcal{S}$;

(ii) $\mathcal{A}$ is unbounded in $\mathcal{S}$: there is no $S \in \mathcal{S}$ such that for each $A \in \mathcal{A}$, $A \subseteq^* S$;

(iii) $\mathcal{A}$ is random destructible: after forcing with a measure algebra, there is some $S \in \mathcal{S}$ such that for all $A \in \mathcal{A}$, $A \subseteq^* S$.

Then $K_{\mathcal{A}}$, the Stone space of $\mathfrak{T}_{\mathcal{A}}/\mathrm{Fin}$ is non-separable and supports a strictly positive finitely additive measure.

Proof. This proof is a proper subset of the proof of Theorem 122. □

Corollary 126. *There is a Souslinean growth of ω which supports a strictly positive finitely additive measure.*

Proof. We shall show that $K_{\mathcal{W}}$, the Stone space of $\mathfrak{T}_{\mathcal{W}}/\mathrm{Fin}$, is as required. First, note that $\mathcal{W} \subseteq \mathcal{S}$ is closed under finite unions as long as they belong to $\mathcal{S}$, and by Proposition 92 is random destructible. It is also unbounded, since for any $f \in \mathcal{X}$, $S_f \in \mathcal{W}$ where $S_f \subseteq \omega \times \omega$ is given by $S_f(n) = \{f(n)\}$. Since for any $S \in \mathcal{S}$ there is some $f \in \mathcal{X}$ such that for each $n \in \omega$, $S_f(n) \cap S(n) = \emptyset$, unboundedness follows. □

Given that the results from the previous section needed an axiom beyond ZFC, it is natural to wonder if the space we have constructed maps continuously onto $[0,1]^{\omega_1}$ or not. Unfortunately, this space even maps continuously onto $[0,1]^{\mathfrak{c}}$. To prove this, we shall need some preliminary work. Before we start that, note that there is a continuous surjection from $[0,1]^{\mathfrak{c}}$ onto $[0,1]^{\omega_1}$: let $S \subseteq \mathfrak{c}$ be a subset of size ω_1, and then consider $\Phi : [0,1]^{\mathfrak{c}} \to [0,1]^{\omega_1}$ given by

$$f \in [0,1]^{\mathfrak{c}} \implies \Phi(f) = f|S.$$

We shall need the following result of Shapirovskii [Sha80].

Theorem 127. *Let X be a compact Hausdorff space. Then the following are equivalent:*

(i) X maps onto $[0,1]^{\mathfrak{c}}$;

(ii) There is a dyadic system $\langle (C_0^\alpha, C_1^\alpha) \colon \alpha < \mathfrak{c}\rangle$ of closed subsets of X such that for each F a finite subset of $\mathfrak{c}$ and $p : F \to 2$,

$$\bigcap_{\alpha \in F} C^\alpha_{p(\alpha)} \neq \emptyset.$$

Using this, one can give a simple condition on a Boolean algebra which implies that its Stone space maps continuously onto $[0,1]^{\mathfrak{c}}$.

Lemma 128. *Let $\mathfrak{B}$ be a Boolean algebra which contains an independent set of size $\mathfrak{c}$. Let K be the Stone space of $\mathfrak{B}$. Then there is a continuous surjection $f : K \to [0,1]^{\mathfrak{c}}$.*

Proof. Let $\langle b_\alpha \colon \alpha < \mathfrak{c}\rangle$ be an independent set in $\mathfrak{B}$. For $\alpha < \mathfrak{c}$, let N_0^α be the basic clopen neighbourhood of K that b_α generates

$$N_0^\alpha = \{\mathcal{U} \in K \colon b_\alpha \in \mathcal{U}\},$$

and N_1^α be the basic clopen neighbourhood of K that $\sim b_\alpha$ generates

$$N_1^\alpha = \{\mathcal{U} \in K \colon \sim b_\alpha \in \mathcal{U}\}.$$

Then $\langle (N_0^\alpha, N_1^\alpha) \colon \alpha < \mathfrak{c}\rangle$ is a dyadic system as Theorem 127 requires. □

Proposition 129. *The space $K_{\mathcal{W}}$ can be mapped continuously onto $[0,1]^{\mathfrak{c}}$.*

Proof. We will show that $\mathfrak{T}_{\mathcal{W}}/\mathrm{Fin}$ contains an independent family of size $\mathfrak{c}$, which we have seen suffices. By Theorem 42, there is an independent family of size $\mathfrak{c}$ in $\mathcal{P}(\omega)/\mathrm{Fin}$. Let $\{X_\alpha \colon \alpha < \mathfrak{c}\}$ be such that $\{[X_\alpha] \colon \alpha < \mathfrak{c}\}$ is such an independent family. Note that this in particular implies that all of the X_α are infinite. Let $S \in \mathcal{W}$ be such that for every $n > 1$, $|S(n)| \geq 2$ (and $S(0) = S(1) = \emptyset$). Also, for each $n > 1$, let Z_0^n, Z_1^n be non-empty pairwise disjoint subsets of $S(n)$. For each $\alpha < \mathfrak{c}$, we shall define $S_\alpha \in \mathcal{W}$ as follows:

$$S_\alpha(n) = \begin{cases} Z_1^n \text{ if } n \in X_\alpha, \\ Z_0^n \text{ otherwise.} \end{cases}$$

It is clear that each S_α is contained in S (and hence is in $\mathcal{W}$) and is infinite. The following claim will then complete the proof of the theorem.

Claim 130. *The family $\{T_{S_\alpha} \colon \alpha < \mathfrak{c}\}/\mathrm{Fin}$ is an independent family of $\mathfrak{T}_{\mathcal{W}}/\mathrm{Fin}$.*

Proof. Let $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ be pairwise distinct ordinals less than $\mathfrak{c}$. We need to show that

$$|\bigcap_{1\leq i\leq m} T_{S_{\alpha_i}} \cap \bigcap_{1\leq j\leq m} (\sim T_{S_{\beta_j}})| = \aleph_0.$$

Now, since $\{X_\alpha \colon \alpha < \mathfrak{c}\}/\mathrm{Fin}$ is an independent family, we can find an infinite $Y \subseteq \omega$ such that

$$Y \subseteq \bigcap_{1\leq i\leq m} X_{\alpha_i} \cap \bigcap_{1\leq j\leq m} (\sim X_{\beta_j}).$$

Then, let $T \subseteq \omega \times \omega$ be defined as follows:

$$T(n) = \begin{cases} Z_1^n \text{ if } n \in Y, \\ S(n) \text{ otherwise.} \end{cases}$$

Notice that T is infinite, and also, since $T \subseteq S$, that $T \in \mathcal{W}$. Also, for $1 \leq i \leq m$, $S_{\alpha_i} \subseteq T$, since the only $n < \omega$ when $T(n) \neq S(n)$ are the $n \in Y$, in which case $T(n) = S_1^n = S_{\alpha_i}(n)$ since $Y \subseteq X_{\alpha_i}$. But also, since $Y \subseteq (\sim X_{\beta_j})$ for each $1 \leq j \leq n$, we have that for $n \in Y$, $T(n) \cap S_{\beta_j}(n) = \emptyset$, with the latter set being non-empty.

It follows that if $l < \omega$ is the least element of Y, then for every $k > l$

$$(T \cap (k \times 2^k), k) \in \bigcap_{1\leq i\leq m} T_{S_{\alpha_i}} \cap \bigcap_{1\leq j\leq m} (\sim T_{S_{\beta_j}}),$$

thus yielding that the latter set is infinite and finishing the proof of the claim. ⊣

□

Chapter 7

An application to Banach space theory

The aim of this chapter is to give an application of the compact spaces constructed in Chapter 6 to Banach space theory. Since this is thematically somewhat separated from the content of Chapter 6, we chose to devote a separate chapter to it. This is also why we spend some time on introducing the problem in Section 7.1, the problem of constructing a non-trivial *smooth compactification* of ω. In Section 7.2 we describe the main technical tool that we shall use to solve this problem as well as showing how a positive answer to this problem allows us to construct a space as in Section 6.2 in Chapter 6. In Section 7.2, we construct such smooth compactifications using the techniques of Chapter 4 and Chapter 5.

7.1 Introduction to the problem

Recall that c_0 is the Banach space of sequences of real numbers converging to 0, and l_∞ is the Banach space of bounded real numbers. In both cases, the norm is the supremum norm. Clearly, c_0 is a closed subspace of l_∞. It is natural to wonder in this case: is c_0 complemented in l_∞? The following, Phillips' Lemma, answers this exact question.

Theorem 131. *There is no projection on l_∞ whose range is c_0. That is, no copy of c_0 is complemented in l_∞.*

One can contrast this with Sobczyk's Theorem.

Theorem 132. *Let X be a separable Banach space. Let $T : c_0 \to X$ be an isomorphic embedding. Then $T[c_0]$ is complemented in X. That is, in a separable Banach space, every copy of c_0 is complemented.*

Let us consider this question more abstractly. First, recall that l_∞ is isometric to $C(\beta\omega)$. If K is a compactification of ω, then c_0, the space of sequences of real numbers converging to 0, can be identified with a subspace Y of $C(K)$, which is called the *natural copy* of c_0 in $C(K)$: if $f \in C(K)$, then

$$f \in Y \iff f|(K \setminus \omega) = 0.$$

Note that the natural copy of c_0 in K is an isometric embedding.

Now, by the universal property of $\beta\omega$, we know that if K is any compactification of ω, then it is the continuous image of $\beta\omega$. By Theorem 39, we have that there is an isometric embedding of $C(K)$ into $C(\beta\omega)$, and hence into l_∞ as well, since the latter is isometric to l_∞. Altogether, we have the following commutative diagram where all the arrows are isometries:

$$\begin{tikzcd} c_0 \arrow[r] \arrow[dr] & C(K) \arrow[d] \\ & l_\infty \end{tikzcd}$$

Here the arrows out of c_0 are the natural embeddings, whereas the arrow from $C(K)$ to l_∞ is the one given by Theorem 39, that is, the one obtained from the universal property of the Stone-Čech compactification.

Now, we know that the natural copy of c_0 is not complemented in l_∞. We can ask then:

Question 133. *Is there a compactification K of ω such that the natural copy of c_0 in $C(K)$ is complemented?*

Such compactifications are called *smooth*. Smoothness of a compactification of ω can be seen as a 'smallness' property of a compactification. The following result justifies this.

Proposition 134. *Let K, L be compactifications of ω such that there is a continuous surjection $T : K \to L$ which is the identity on ω. Then if K is smooth, then L is smooth.*

Proof. Let $\Phi : c_0 \to C(K)$ be the isometric embedding of c_0 into the natural copy of c_0 in $C(K)$, and $\Psi : c_0 \to C(L)$ the same for $C(L)$. Let $\tilde{T} : C(L) \to C(K)$ be the isometric embedding where for any $f \in C(L)$,

$$\tilde{T}(f)(x) = f(Tx).$$

The hypothesis tells us that $\Phi = \tilde{T} \circ \Psi$. We can then think of c_0 as a closed subspace of $C(L)$ which itself is a closed subspace of $C(K)$. It is easy to verify that if $P : C(K) \to C(K)$ is a projection witnessing that $\Phi[c_0]$ is complemented in $C(K)$, then $P|C(L) : C(L) \to C(L)$ is a projection witnessing that $\Psi[c_0]$ is complemented in $C(L)$: the crucial step being that $\mathrm{ran}(P) = c_0$. $\square$

It follows that if we think of an order on the collection of compactifications of ω where $L \leq K$ if there is a continuous surjection from K onto L which is the identity on ω, then the smooth compactifications of ω is downwards-closed: if K is smooth and $L \leq K$, then L is smooth as well. On the other hand, Phillips' Lemma asserts that $C(K)$ for K the largest possible compactification of ω is not smooth. Here the universal property of the Stone-Čech compactification is used to see that $\beta\omega$ is the largest possible compactification of ω.

Returning to Question 133, it is easy to see that in this form it has a positive answer.

Proposition 135. *Let $\mathfrak{B} \subseteq \mathcal{P}(\omega)$ be a countable Boolean algebra containing all the finite subsets of ω. Then, if K is its Stone space, every copy of c_0 in $C(K)$ it is complemented.*

Proof. By Theorem 37, the hypothesis implies that $C(K)$ is separable. Then use Sobczyk's Theorem to finish. $\square$

However, this is a trivial answer: it shows that in the order on compactifications of ω that we defined above, the class of metrisable compactifications (here we use Theorem 37) which itself is downwards-closed in the order we defined is completely contained in the class of smooth compactifications, but Sobczyk's Theorem already tells us this. A less trivial question asks whether these classes are actually equivalent:

Question 136. *Is there a non-metrisable compactification K of ω such that the natural copy of c_0 in $C(K)$ is complemented?*

An even stronger question is

Question 137. *Is there a compactification K of ω such that its remainder is non-separable and the natural copy of c_0 in $C(K)$ is complemented?*

As we shall show, the answer to the stronger question, and hence the weaker question as well, is 'yes'. Previously Drygier and Plebanek in [DP17] had constructed such a compactification assuming CH. We make do with just ZFC. Namely, we shall construct a Boolean subalgebra $\mathfrak{B}$ of $\mathcal{P}(\omega)$ which contains all the finite sets and which is not only uncountable (which is equivalent to its Stone space being non-metrisable), but such that $\mathfrak{B}/\mathrm{Fin}$ is in fact not even σ-centered (which is equivalent to its Stone space being non-separable).

7.2 Preliminary results

While there is a thematic affinity between the problem we are considering in this chapter and the problems we considered in Chapter 6, that of constructing 'small big' compactifications of ω, there is a much more explicit connection between this problem and the problems from Chapter 6. However, before we explain this connnection, we give the main technical result which we shall use in this chapter. The following is [DP17, Lemma 3.1]. For the rest of this section, for $n \in \omega$, δ_n will be the Dirac measure on $\mathcal{P}(\omega)$ supported at n: if $A \in \mathcal{P}(\omega)$, then

$$\delta_n(A) = 1 \iff n \in A.$$

Theorem 138. *Let $\mathfrak{A}$ be a subalgebra of $\mathcal{P}(\omega)$ containing all finite sets and let K be its Stone space. The following conditions are equivalent:*

(i) the natural copy of c_0 is complemented in $C(K)$,

(ii) there is a bounded sequence of signed measures $(\nu_n)_n$ on $\mathfrak{A}$ such that each $|\nu_n|$ vanishes on finite subsets and

$$\lim_{n\to\infty} \big(\nu_n(A) - \delta_n(A)\big) = 0$$

for every $A \in \mathfrak{A}$.

We are now in a position to make the connection between the constructions of Chapter 6 and the contents of the current chapter. The following fact is attributed to Kubiś in [DP17].

Corollary 139. *Let K be a compactification of ω such that the natural copy of c_0 is complemented in $C(K)$, then $K \setminus \omega$ supports a measure.*

Proof. Let $\mathfrak{B} \subseteq P(\omega)$ be such that K is its Stone space and let $(\nu_n)_n$ be a sequence as in Theorem 138. Consider the function ν with domain $\mathfrak{B}$ given by: for $A \in \mathfrak{B}$

$$\nu(A) = \sum_n \frac{1}{2^{n+1}} |\nu_n|(A).$$

It is easy to see that ν is a (positive) measure on $\mathfrak{B}$: it only takes non-zero values since each of the $|\nu_n|$ are positive measures, and it only takes finite values since there is some K such that $|\nu_n| < K$ for each $n \in \omega$. Also, since each of the $|\nu_n|$ vanish on the finite sets, it vanishes on finite sets as well.

Claim 140. *ν vanishes on exactly the finite sets.*

Proof. Let $A \in \mathfrak{B}$ be infinite. Then $\delta_n(A) = 1$ for infinitely many $n \in \omega$. Since

$$\lim_{n\to\infty} \big(\nu_n(A) - \delta_n(A)\big) = 0,$$

it follows that there must be some $n \in \omega$ such that $\nu_n(A) \neq 0$. From the Jordan decomposition of signed measures, Theorem 23, it is clear that for each signed measure μ on $\mathfrak{B}$, $\mu(A) \leq |\mu|(A)$ for each $A \in \mathfrak{B}$. Therefore, $|\nu_n|(A) \neq 0$ for some $n \in \omega$, and since the $|\nu_n|(A)$ is always positive, it follows that $|\nu_n|(A) > 0$ for some $n \in \omega$, and hence $\nu(A) > 0$. $\dashv$

It follows that $\mathrm{Fin} = \{A \in \mathfrak{B} \colon \nu(A) = 0\}$, and hence by Theorem 13, $[\nu]_{\mathrm{Fin}}$ is a strictly positive measure on $\mathfrak{B}/\mathrm{Fin}$. It follows that if K is the Stone space of $\mathfrak{B}$, then $K \setminus \omega$, which is the Stone space of $\mathfrak{B}/\mathrm{Fin}$, supports a strictly positive measure. $\square$

In [DP17, Theorem 5.1] the authors show that under CH there is a compactification K of ω with a non-separable growth such that the natural copy of c_0 in $C(K)$ is complemented. It follows that the growth $K \setminus \omega$ supports a strictly positive measure. In the next section we construct such a space in ZFC.

7.3 A smooth compactification

The space that we construct will be a slight modification of the space from Corollary 126. We remind the reader of some terminology.

(i) $\mathcal{X}$ denotes the product space $\prod 2^n$, and we define Haar measure on $\mathrm{Bor}(\mathcal{X})$ in the standard way, and in particular if $i < 2^n$, then

$$\mu(\{f \in \mathcal{X} \colon f(n) = i\}) = \frac{1}{2^n};$$

(ii) $\mathbb{B} = \mathrm{Bor}(\mathcal{X})/\mathcal{N}_\mu$;

(iii) $\mathcal{S}$ denotes the set of slaloms $S \subseteq \omega \times \omega$ such that $|S(n)| < 2^n$ for each $n \in \omega$;

(iv) If $A, B \in \mathcal{S}$, then $A \subseteq B$ if $A(n) \subseteq B(n)$ for every $n \in \omega$, and $A \subseteq^* B$ if this happens for all but finitely many $n \in \omega$;

(v) $\mathcal{W}$ denotes the collection of those $S \in \mathcal{S}$ such that $S(n) \subseteq 2^n$ for each $n \in \omega$ and which are summable in the sense that $\sum_n \frac{1}{2^n} |S(n)| < \infty$.

Also, a reminder of the relevant details of the Todorčević construction.

(i) $\Omega = \{(S,n)\colon n \in \omega,\ S \in \mathcal{S},\ S \subseteq (n \times 2^n)\}$.

(ii) For each $A \subseteq \omega \times \omega$,
$$T_A = \{(T,n) \in \Omega\colon A \cap (n \times 2^n) \subseteq T\}.$$

(iii) For $(S,n) \in \Omega$,
$$T_{(S,n)} = \{(T,m) \in \Omega\colon m \geq n, T \cap (n \times 2^n) = S\}.$$

(iv) For $\mathcal{A} \subseteq \mathcal{P}(\omega \times \omega)$, let $\mathfrak{T}_\mathcal{A}$ be the subalgebra of $\mathcal{P}(\Omega)$ generated by
$$\{T_A\colon A \in \mathcal{A}\} \cup \{T_{(S,n)}\colon (S,n) \in \Omega\}.$$

(v) For $\mathcal{A} \subseteq \mathcal{P}(\omega \times \omega)$, let $\mathfrak{T}^*_\mathcal{A}$ be the Boolean subalgebra of $\mathfrak{T}_\mathcal{A}$ generated only by $\{T_A\colon A \in \mathcal{A}\}$.

However, unlike Chapter 6, we shall be interested in the Stone spaces of Boolean algebras $\mathfrak{T}^*_\mathcal{A}/\mathrm{Fin}$ for $\mathcal{A} \subseteq \mathcal{P}(\omega \times \omega)$. More specifically, we shall be interested in $\mathfrak{T}^*_\mathcal{W}/\mathrm{Fin}$ and its Stone space.

For this section, let $\mathbb{B}$ denote the following separable measure algebra: $\mathbb{B} = \mathrm{Bor}(\mathcal{X})/\mathcal{N}_\mu$. Let $\dot{U}$ be an $\mathbb{B}$-name for a slalom, that is, an element of $\mathcal{S}$. Define a function
$$f_{\dot{U}}\colon \{T_W\colon W \in \mathcal{W}\}/\mathrm{Fin} \to \mathbb{B}$$
in the following way:
$$f_{\dot{U}}([T_W]) = [\![\check{W} \subseteq \dot{U}]\!].$$
We will show that it can be extended to a homomorphism.

Proposition 141. *For each $\mathbb{B}$-name $\dot{U}$ for a slalom the function $f_{\dot{U}}$ can be extended to a homomorphism $\phi_{\dot{U}}\colon \mathfrak{T}^*_\mathcal{W}/\mathrm{Fin} \to \mathbb{B}$.*

Proof. We shall use Sikorski's Extension Criterion, Theorem 9. Since $\{T_W\colon W \in \mathcal{W}\}/\mathrm{Fin}$ generates $\mathfrak{T}^*_\mathcal{W}/\mathrm{Fin}$, it suffices to check that if $A_0, \dots, A_k, B_0, \dots, B_l \in \mathcal{W}$ and
$$C = T_{A_0} \cap \dots \cap T_{A_k} \cap (\sim T_{B_0}) \cap \dots \cap (\sim T_{B_l})$$
is finite, then
$$C' = f_{\dot{U}}([T_{A_0}]) \cap \dots \cap f_{\dot{U}}([T_{A_k}]) \cap (\sim f_{\dot{U}}([T_{B_0}])) \cap \dots \cap (\sim f_{\dot{U}}([T_{B_l}])) = \emptyset.$$
First, we will look at two particular cases:

(i) There is n such that $|\bigcup_{i \leq k} A_i(n)| = 2^n$. Then
$$C' \subseteq \bigcap_{i \leq k} [\![\check{A}_i(n) \subseteq \dot{U}(n)]\!] = [\![\bigcup_{i \leq k} \check{A}_i(n) \subseteq \dot{U}(n)]\!] = \emptyset,$$
since $\dot{U}$ is a name for a slalom.

(ii) There is $j \leq l$ such that $B_j \subseteq \bigcup_{i \leq k} A_i$. Then
$$C' \subseteq [\![\bigcup_{i \leq k} \check{A}_i \subseteq \dot{U}]\!] \cap [\![\check{B}_j \not\subseteq \dot{U}]\!] = \emptyset.$$

Assume now that neither of the above is satisfied. In this case $(\bigcup_{i\leq k} A_i \cap (n\times 2^n), n) \in \Omega$ for each n and

$$C \supseteq \{(\bigcup_{i\leq k} A_i \cap (n\times 2^n), n) \colon n\in\omega\}.$$

The latter set is clearly infinite and we are done. $\square$

Thanks to Proposition 141 we can induce measures by names for slaloms. Note that usually these measures need not be positive on all infinite elements of $\mathfrak{T}^*_{\mathcal{W}}$.

Corollary 142. *Let $\dot{U}$ be an $\mathbb{B}$-name for a slalom. The following formula uniquely defines a measure on $\mathfrak{T}^*_{\mathcal{W}}$:*

$$\nu(T_W) = \mu(f_{\dot{U}}([T_W])) = \mu([\![\check{W} \subseteq \dot{U}]\!]).$$

This measure only takes non-negative values and vanishes on finite sets.

Proof. Since sets of the form T_W generate $\mathfrak{T}^*_{\mathcal{W}}$, we get uniqueness. Since the measure of every element of $\mathfrak{T}^*_{\mathcal{W}}$ is the measure of some element in $\mathbb{B}$, it is clear that it only takes non-negative values. Since the equivalence class of finite sets is $0_{\mathbb{B}}$, it is clear that ν vanishes on finite sets. $\square$

We are going to show that there is a sequence of measures defined on $\mathfrak{T}^*_{\mathcal{W}}$ as in Theorem 138. We will use the canonical name for a slalom $\dot{S}$ that we gave in Definition 91. Recall that for $n > 0$, $k < 2^n$, $\dot{S}$ is a $\mathbb{B}$-name for a subset of $\omega\times\omega$ defined in the following way:

$$[\![k \in \dot{S}(n)]\!] = \mathcal{X} \setminus I^n_k$$

where $I^n_k = \{f \in \mathcal{X} \colon f(n) = k\}$.

Before we proceed, we remind the reader of the disclaimer from the beginning of Chapter 6. Instead of constructing Boolean subalgebras of $\mathcal{P}(\omega)$, we are constructing Boolean subalgebras of $\mathcal{P}(\Omega)$. We are identifying the two sets ω and Ω since they have the same size. This identification will require us to suitably modify Theorem 138, by replacing occurences of, for example, the Dirac delta function for elements of ω by the Dirac delta function for elements of Ω.

For $(T,m)\in\Omega$ define an $\mathbb{B}$-name $\dot{S}_{(T,m)}$ in the following way:

$$[\![k \in \dot{S}_{(T,m)}(n)]\!] = \begin{cases} [\![k\in\dot{S}(n)]\!] \text{ if } n\geq m, \\ 1_{\mathbb{B}} \text{ if } n < m \text{ and } k\in T(n), \\ 0_{\mathbb{B}} \text{ if } n<m \text{ and } k\notin T(n). \end{cases}$$

Then $\dot{S}_{(T,m)}$ is a name for a slalom for each $(T,m)\in\Omega$ since T is a slalom and $\dot{S}$ is a name for a slalom: this ensures that for each $n\in\omega$, $[\![|\dot{S}_{(T,m)}(n)| < 2^n]\!] = 1_{\mathbb{B}}$.

For $W\in\mathcal{W}$ and $(T,m)\in\Omega$, we then use Corollary 142 to obtain $\nu_{(T,m)}$ on $\mathfrak{T}^*_{\mathcal{W}}$ by setting

$$\nu_{(T,m)}(T_W) = \mu([\![\check{W} \subseteq \dot{S}_{(T,m)}]\!]),$$

and this $\nu_{(T,m)}$ only takes non-negative values and vanishes on finite sets. Since the Jordan decomposition, Theorem 23, is unique and $\nu_{(T,m)}$ is itself a measure, it follows that $|\nu_{(T,m)}| = \nu_{(T,m)}$, so in particular $|\nu_{(T,m)}|$ vanishes on finite sets too.

Proposition 143. *For every $A \in \mathfrak{T}^*_{\mathcal{W}}$*

$$\lim_{(T,m)\in\Omega}(\nu_{(T,m)}(A) - \delta_{(T,m)}(A)) = 0.$$

Proof. Let $W \in \mathcal{W}$.

Claim 144. $\lim_{(T,m)\in T_W} \nu_{(T,m)}(T_W) = 1$.

Proof. Let $\epsilon > 0$. There is m such that $\mu([\![\forall n > m\ \check{W}(n) \subseteq \dot{S}(n)]\!]) > 1-\epsilon$. So, for each $n > m$ and $(T,n) \in T_W$

$$\nu_{(T,n)}(T_W) = \mu([\![\check{W} \subseteq \dot{S}_{(T,n)}]\!]) \geq \mu\left([\![\check{W} \cap (n \times 2^n) \subseteq \check{T}]\!] \cap [\![\forall i \geq n\ \check{W}(i) \subseteq \dot{S}(i)]\!]\right) =$$

$$= \mu([\![\forall i \geq n\ \check{W}(i) \subseteq \dot{S}(i)]\!]) > 1-\epsilon.$$

$\dashv$

Claim 145. $\lim_{(T,m)\notin T_W} \nu_{(T,m)}(T_W) = 0$.

Proof. In fact, if $(T,m) \notin T_W$, then

$$\nu_{(T,m)}(T_W) = \mu([\![\check{W} \subseteq \dot{S}_{(T,m)}]\!]) \leq \mu([\![\check{W} \cap (m \times 2^m) \subseteq \check{T}]\!]) = 0.$$

$\dashv$

In this way we have proved that $\lim_{(T,n)}(\nu_{(T,n)}(T_W) - \delta_{(T,n)}(T_W)) = 0$ for each $W \in \mathcal{W}$. Each element of $\mathfrak{T}^*_{\mathcal{W}}$ is a finite Boolean combination of elements of this form, so the convergence for arbitrary elements of the algebra easily follows. $\square$

Corollary 146. *There is a compactification L of ω such that $L \setminus \omega$ is non-separable and the natural copy of c_0 is complemented in $C(L)$.*

Proof. Let L be the Stone space of $\mathfrak{T}^*_{\mathcal{W}}$. By the proof of Lemma 112, it is easy to see that $\mathfrak{T}^*_{\mathcal{W}}/\mathrm{Fin}$ is not σ-centered. So, the space $L \setminus \omega$ is non-separable. Proposition 143 and Theorem 138 ensure that the natural copy of c_0 is complemented in $C(L)$.

$\square$

Remark 147. In the Section 6.2 we proved that the algebra $\mathfrak{T}_{\mathcal{W}}/\mathrm{Fin}$ supports a measure without giving an example of such a measure. We can combine Proposition 143 with Corollary 139 to obtain an explicit example of such a measure.

Remark 148. If $\mathrm{add}(\mathcal{N}) = \mathrm{non}(\mathcal{M})$, then instead of $\mathfrak{T}^*_{\mathcal{W}}$ we can use $\mathfrak{T}^*_{\mathcal{B}}$, where $\mathcal{B}$ is defined as in Theorem 6.1. It enjoys all the properties of the former, but in addition by [BNP15, Theorem 3.1], it only carries measures of countable Maharam type. By [DP17, Theorem 8.4], this implies we have that $C(L)$ is *hereditarily separably Sobczyk*. That is, every isomorphic copy of c_0 in $C(L)$ contains a further copy of c_0 which is complemented.

Chapter 8

The Graph Ideal Dichotomy

The aim of this chapter is to introduce an axiom, the *Graph Ideal Dichotomy* (Definition 160), denoted $\mathsf{GID}_{\aleph_1}$, to give some applications of it, and to prove that its consistency with ZFC (assuming the consistency of ZFC). This axiom also follows from the Proper Forcing Axiom, PFA. This axiom is a more general form of the partition relation $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$, see Definition 166, which was considered by Todorčević ([Tod83]) in order to prove, among other things, that it is consistent that there are no S-spaces. The only source of the generality comes from replacing the σ-ideal on ω_1 generated by the countable ordinals by an arbitrary non-trivial ω_1-generated σ-ideal on ω_1, and the corresponding changes that this imposes.

The structure of this chapter is as follows. In Section 8.1 we define two basic objects that we shall use: coideals and their Fubini powers. Since we shall be talking a lot about particular types of ideals, we shall also recall some terminology here. In Section 8.2 we shall consider a particular way of relating graphs on a particular vertex set and ideals on that same vertex set. In Section 8.3 we define our axiom $\mathsf{GID}_{\aleph_1}$ and prove in Theorem 161 that it follows from PFA as well as the fact that no large cardinals are required for its consistency. In Section 8.4 we attempt to justify our study of this axiom. Primarily, this is done by relating the proof of Theorem 161 to *symmetric systems*, which have recently found use in work of Asperó and Mota [AM15a, AM16] in proving that some consequences of PFA are consistent with the continuum being arbitrarily large. Before we start, we recall some terminology and facts which we shall make heavy use of.

(i) If S is a set of ordinals and $n \in \omega$, then $[S]^n$ denotes the set of n-element increasing sequences of elements of S.

(ii) If $\bar{u}$ and $\bar{v}$ are sets of ordinals, then $\bar{u} \otimes \bar{v}$ denotes the set of pairs (α, β) such that $\alpha \in \bar{u}$, $\beta \in \bar{v}$, and $\alpha < \beta$.

(iii) If $\bar{u}$ and $\bar{v}$ are sequences, then the concatenation of $\bar{u}$ with $\bar{v}$ will be denoted $\bar{u} \bullet \bar{v}$.

(iv) If S is a set of ordinals, $n \in \omega$ and $\mathcal{F} \subseteq [S]^n$, then for any $\xi \in S$,

$$(\mathcal{F})_\xi = \{\bar{u} \in [S]^{n-1} \colon \xi < \bar{u}, \langle \xi \rangle \bullet \bar{u} \in S\}.$$

(v) For κ an uncountable regular cardinal, $H(\kappa)$ denotes the collection of sets which are hereditarily of size less than κ. This satisfies all the axioms of ZFC except for the Axiom of Powersets. We shall typically be talking about elementary substructures of $H(\kappa)$, and in this case we shall always actually be referring to the structure $(H(\kappa), \in, \triangleleft)$ where $\triangleleft$ will be some suitable well-ordering of the structure.

(vi) We shall say that a sequence of countable structures $\langle N_\xi \colon \xi < \omega_1 \rangle$ is increasing or an increasing chain if for every $\xi < \omega_1$, $\langle N_\nu \colon \nu < \xi \rangle \in N_\xi$.

8.1 Coideals and Fubini powers

Let S be a set. Recall that an ideal on S is *non-trivial* if $S \notin \mathcal{I}$. If it is closed under countable unions, then we say that it is a *σ-ideal*. A collection $\mathcal{J} \subseteq \mathcal{I}$ is called a *generating set* for $\mathcal{I}$ if their closure under subsets and finite unions is $\mathcal{I}$. If $\mathcal{I}$ has a generating set of size $\aleph_1$, then we say that it is *ω_1-generated*. Similarly, if the closure of $\mathcal{J}$ under subsets and countable unions is $\mathcal{I}$, then we shall say that $\mathcal{J}$ is a *σ-generating* set for $\mathcal{I}$, and if $\mathcal{I}$ has a σ-generating set of size $\aleph_1$, then we shall say that it is *ω_1-generated as a σ-ideal*. Usually when we will talk about a σ-ideal being ω_1-generated, we shall mean that it is ω_1-generated as a σ-ideal. All ideals will be assumed to be non-trivial unless otherwise mentioned. Also, we shall always assume that an ideal $\mathcal{I}$ on an infinite set S contains all the finite subsets of S.

Recall also that if $\mathcal{I}$ is a non-trivial ideal on S, then $\mathcal{U}$, the collection of all $S \setminus I$ such that $I \in \mathcal{I}$ forms a *filter* on S, which contains all the co-finite sets and is closed under supersets and intersections. There is another type of object which we can associate with an ideal which is not as common as the filter associated with an ideal.

Definition 149. Let $\mathcal{I}$ be an ideal of subsets of S. The set $\mathcal{H}$ of all subsets of S which are not in S is called the *coideal associated with* $\mathcal{I}$.

A standard analogy for an ideal and its dual filter on a set is, respectively, that of 'measure zero' and 'measure one' subsets of that set. The same analogy can be extended to coideals, where they correspond to 'positive measure' sets. We now give some elementary properties of coideals.

Proposition 150. *Let $\mathcal{I}$ be an ideal of subsets of S, and let $\mathcal{F}$ and $\mathcal{H}$ be, respectively, the filter and the coideal associated with it.*

(i) If $X \in \mathcal{H}$ and $X \subseteq Y \subseteq S$, then $Y \in \mathcal{H}$.

(ii) If $X \in \mathcal{H}$ and $X = Y \cup Z$, then either Y or Z is in $\mathcal{H}$.

(iii) If $X \in \mathcal{H}$ and $Y \in \mathcal{U}$, then $X \cap Y \in \mathcal{H}$.

(iv) If $\mathcal{I}$ is a σ-ideal, then for any $X \in \mathcal{H}$, if $X = \bigcup_{n\in\omega} X_n$, then $X_n \in \mathcal{H}$ for some $n \in \omega$.

We point out that the first two properties characterise all coideals on a set, and when the fourth property is added to them, they characterise all coideal of σ-ideals on a set. Next, we consider a product operation on coideals. The nomenclature is, we hope, suggestive given the analogy discussed above.

Definition 151. Let $\mathcal{I}$ be an ideal of subsets of ω_1. Let $\mathcal{H}$ be the coideal associated with $\mathcal{I}$. For $n \in \omega$, we define by induction the *Fubini power*, $\mathcal{H}^n$, of the coideal:

(i) If $n = 0$, then $\mathcal{H}^0 = \{\emptyset\}$.

(ii) If $n = k+1$, then $\mathcal{H}^{k+1}$ consists of those $\mathcal{F} \subseteq [\omega_1]^{k+1}$ such that

$$\{\xi < \omega_1 \colon (\mathcal{F})_\xi \in \mathcal{H}^k\} \in \mathcal{H}.$$

A quirk of our definition is that $\mathcal{H}^1 \neq \mathcal{H}$, but note that $S \in \mathcal{H}^1$ iff $\bigcup S \in \mathcal{H}$.

Example 152. We feel that an example would be illuminating here. Let $\mathcal{I}$ be the ideal of countable subsets of ω_1. It is clear that it is a σ-ideal, and that the countable ordinals σ-generate it. Consequently, it is an ω_1-generated σ-ideal. Notice also that $\mathcal{H}$, the associated coideal, is simply the collection of all uncountable subsets of ω_1. Let us now consider the Fubini powers of this coideal. If $n \in \omega$, then $\mathcal{F} \subseteq [\omega_1]^n$ is in $\mathcal{H}^n$ if and only if $\mathcal{F}$ has ordertype ω_1^n under the lexicographic ordering (here for notational convenience we assume that $\omega_1^0 = 1$).

This example guides all of our definitions and proofs. This extra level of abstraction might seem artificial, but we shall attempt to justify it in Section 9.4.

Definition 153. Let $\mathcal{I}$ be a non-trivial σ-ideal on ω_1.

(i) Let $\xi < \omega_1$ and $M \prec H(\omega_2)$ countable. We say that ξ is *far from M* if $\xi \notin \bigcup(M \cap \mathcal{I})$.

(ii) Let $n \in \omega$ and $\bar{u} = \langle \xi_0, \xi_1, \ldots, \xi_{n-1} \rangle$ be an increasing sequence of countable ordinals. We say that a finite $\in$-chain $M_0 \in M_1 \in \ldots \in M_k$ of countable elementary substructures of $H(\omega_2)$ *separates* $\bar{u}$ if $\bar{u} \cap M_0 = \emptyset$ and for any distinct $\xi, \nu \in \bar{u}$, there is some $0 < j \leq k$ such that $\xi_i \in M_j$ and ξ_{i+1} is far from M_j.

The choice of ω_2 does not make any difference in the above definition in the sense that if κ was another regular cardinal greater than ω_1 such that $M \prec H(\kappa)$, and $\xi < \omega_1$, then (the left side defined in the obvious way)

$$\xi \text{ is far from } M \iff \xi \text{ is far from } M \cap H(\omega_2).$$

Implicit in the above statement is that $\omega_2 \in M$ and a use of Proposition 44 to guarantee that $M \cap H(\lambda) \prec H(\lambda)$. We also point out that our choice in the definition above that the smallest node of the $\in$-chain, M_0, not contain any element of $\bar{u}$ is not standard. We make two elementary but crucial observations which the main result of this section, Lemma 157, shall be a combination of.

Proposition 154. *Let $\mathcal{I}$ be a non-trivial σ-ideal on ω_1. Let $\mathcal{H}$ and $\mathcal{U}$ be, respectively, its associated coideal and filter.*

(i) If $M \prec H(\omega_2)$ is countable, then the set of $\xi < \omega_1$ which are far from M is in $\mathcal{H}$.

(ii) If $F \in \mathcal{H}$ and $M \prec H(\omega_2)$ is such that $F \in M$, then there is some $\xi \in F$ such that ξ is far from M.

(iii) If $F \in \mathcal{H}$ and $U \in \mathcal{U}$ then $F \cap U \in \mathcal{H}$.

Proof. (i) Since M is countable, $M \cap \mathcal{I}$ is a countable set. Also,

$$(\bigcup(M \cap \mathcal{I})) \cup \{\xi < \omega_1 \colon \xi \text{ is far from } M\} = \omega_1.$$

So, by non-triviality of $\mathcal{I}$ we can finish.

(ii) Since $\mathcal{I}$ is a σ-ideal and $F \in \mathcal{H}$, $\bigcup(M \cap \mathcal{I}) \in \mathcal{I}$, so there is some $\xi \in F \setminus \bigcup(M \cap \mathcal{I})$.

(iii) This is a repetition of the third part of Proposition 150.

□

The next observation is the first point at which we shall use the fact that our ideals are ω_1-generated.

Proposition 155. *Let $\mathcal{I}$ be a non-trivial σ-ideal on ω_1 generated as a σ-ideal by $\langle I_\alpha \colon \alpha < \omega_1 \rangle$. Let $F \in \mathcal{H}$ and let $\langle M_\xi \colon \xi < \omega_1 \rangle$ be an increasing chain of elementary substructures of $H(\omega_2)$ such that*

$$F, \langle I_\alpha \colon \alpha < \omega_1 \rangle \in M_0.$$

For each $\xi < \omega_1$, let

$$x_\xi \in F \cap (M_{\xi+1} \setminus \bigcup(\mathcal{I} \cap M_\xi)).$$

Then $\{x_\xi \colon \xi < \omega_1\} \in \mathcal{H}$ and is a subset of F.

Proof. We first consider the situation in Example 152. In this case, we have some $F \subseteq \omega_1$ uncountable, and we pick for each $\xi < \omega_1$ an $x_\xi \in F \cap (M_{\xi+1} \setminus M_\xi)$. It is clear that we get an uncountable subset of F as a result, since for each $\alpha < \omega_1$, $M_{\alpha+1} \cap \omega_1 > \alpha$, and so we have that $x_{\alpha+1} > \alpha$.

Now, for the general case. Suppose if possible that $\{x_\xi \colon \xi < \omega_1\} \notin \mathcal{H}$. This implies that there is some $\alpha < \omega_1$ such that $\{x_\xi \colon \xi < \omega_1\} \subseteq I_\alpha$. Here we use that $\langle I_\alpha \colon \alpha < \omega_1 \rangle$ generates $\mathcal{I}$ as a σ-ideal. However, since for each $\xi < \omega_1$ we have that $\langle I_\alpha \colon \alpha < \omega_1 \rangle \in M_\xi$, so $I_\alpha \in M_{\alpha+1}$, and so $x_{\alpha+1} \notin I_\alpha$. Contradiction. □

Equipped with those, we make a first step towards Lemma 157, which is a simple extension of Proposition 154 to Fubini products.

Proposition 156. *Let $\mathcal{I}$ be a non-trivial σ-ideal on ω_1. Let $n \in \omega$ and $\mathcal{F} \subseteq [\omega_1]^n$.*

(i) $\mathcal{F} \in \mathcal{H}^n$ if and only if for every $M_0 \prec H(\omega_2)$ such that $\mathcal{F} \in M_0$, there is $\bar{u} \in \mathcal{F}$ and $M_1 \in \cdots \in M_k$ elementary substructures of $H(\omega_2)$ such that $M_0 \in M_1 \in \cdots \in M_k$ separate $\bar{u}$.

(ii) If $M_0 \in M_1 \in \cdots \in M_k$ are all countable elementary substructures of $H(\omega_2)$ which separate $\bar{u}$, then for every $\mathcal{F} \subseteq [\omega_1]^n$ in M_0 such that $\bar{u} \in \mathcal{F}$, $\mathcal{F} \in \mathcal{H}^n$.

(iii) Suppose that $\mathcal{F} \in \mathcal{H}^n$. Let U be in the filter dual to $\mathcal{I}$. Then $\mathcal{F} \cap [U]^n \in \mathcal{H}^n$.

We now prove the main result of this section. As we have said before, it combines the idea from Proposition 154, or even more accurately, Proposition 156, with the one from Proposition 155. It allows us to quite strongly refine elements of the Fubini power of a coideal. As in Proposition 155, it is crucial that we are working with an ω_1-generated σ-ideal.

Lemma 157. *Suppose $\mathcal{I}$ is a non-trivial ideal on ω_1 which is σ-generated by $\langle I_\alpha \colon \alpha < \omega_1 \rangle$. Let $n \in \omega, \mathcal{F} \in \mathcal{H}^n$ and let $\langle M_\xi \colon \xi < \omega_1 \rangle$ and $\langle U_\xi \colon \xi < \omega_1 \rangle$ be such that*

(i) $\langle M_\xi \colon \xi < \omega_1 \rangle$ is an increasing chain of elementary substructures of $H(\omega_2)$ such that $\mathcal{F}, \langle I_\alpha \colon \alpha < \omega_1 \rangle \in M_0$;

(ii) $U_\xi \in M_{\xi+1}$ is in the filter dual to $\mathcal{I}$ and $U_\xi \cap \bigcup(M_\xi \cap \mathcal{I}) = \emptyset$.

For $\xi < \omega_1$ let $V_\xi = U_\xi \cap M_{\xi+1}$, and let $V = \bigcup_{\xi<\omega_1} V_\xi$ and $\mathcal{E} = \mathcal{F} \cap [V]^n$. Then $\mathcal{E} \in \mathcal{H}^n$.

Proof. Note that in the proof of Proposition 155, we first considered what was essentially a very special case of this statement already. Namely, we had $n = 1$, $I_\alpha = \alpha$, and $U_\xi = \omega_1 \setminus M_\xi$. In [Tod98, Lemma 8], this was generalised to the case of arbitrary $n \in \omega$, and now we shall do essentially the same proof for the case of an arbitrary non-trivial ω_1-generated σ-ideal.

The proof is by induction on n. For $n = 1$, suppose towards a contradiction that $\bigcup \mathcal{E} \notin \mathcal{H}$. That is, $\bigcup \mathcal{E} \in \mathcal{I}$. Let $\xi < \omega_1$ be such that $\bigcup \mathcal{E} \subseteq \bigcup_{\nu<\xi} I_\nu$, and let $\zeta < \omega_1$ be such that $\xi \leq \delta_\zeta$. Now, $\mathcal{F} \in M_{\zeta+1} \cap \mathcal{H}^1$, so

$$M_{\zeta+1} \models \text{“}\bigcup \mathcal{F} \cap U_\zeta \in \mathcal{H},\text{”}$$

and in particular, $\bigcup \mathcal{F} \cap V_\zeta \neq \emptyset$, whereas $\bigcup \mathcal{E} \cap U_\zeta = \emptyset$, a contradiction.

We do the inductive step from k to $k+1$. We have that $\mathcal{F} \in \mathcal{H}^{k+1}$, so

$$S = \{\xi < \omega_1 \colon (\mathcal{F})_\xi \in \mathcal{H}^k\} \in \mathcal{H}.$$

Also, $S \in M_0$, so by the base case of the induction, $S \cap V \in \mathcal{H}$. But if $\xi \in S \cap V$ and $\nu < \omega_1$ is such that $\xi \in M_\nu$, then $(\mathcal{F})_\xi \in \mathcal{H}^k$, and we can apply the inductive hypothesis to

$$\langle M_\zeta \colon \nu \leq \zeta < \omega_1 \rangle \text{ and } \langle U_\zeta \colon \nu \leq \zeta < \omega_1 \rangle$$

to see that

$$(\mathcal{F})_\xi \cap [V']^k \in \mathcal{H}^k, \text{ where } V' = \bigcup_{\nu \leq \zeta < \omega_1} V_\zeta.$$

But $V' \subseteq V$, so $(\mathcal{F})_\xi \cap [V]^k \in \mathcal{H}^k$ as well. So we have shown that $\{\xi < \omega_1 \colon (\mathcal{E})_\xi \in \mathcal{H}^k\} \in \mathcal{H}$, and we can finish. □

8.2 Graphs and ideals

The aim of this section is to consider a way of relating graphs on ω_1 and ω_1-generated σ-ideals on ω_1.

Definition 158. Let $\mathcal{G} = (\omega_1, E)$ be a graph. Let $\langle I_\alpha \colon \alpha < \omega_1 \rangle$ be subsets of ω_1 including all the singletons and let $\mathcal{I}$ be the ideal that they σ-generate.

(i) Let $n \in \omega$. If $\mathcal{F} \subseteq [\omega_1]^n$ is an uncountable set such that for every disjoint $\bar{u} < \bar{v} \in \mathcal{F}$, $\bar{u} \otimes \bar{v} \not\subseteq E$, then we say that $\mathcal{F}$ is a *bad set*.

(ii) We say that the graph is *proper with respect to* $\mathcal{I}$ if for every $n \in \omega$ and $\mathcal{F} \in \mathcal{H}^n$, there are $\bar{u} < \bar{v} \in \mathcal{F}$ such that $\bar{u} \otimes \bar{v} \subseteq E$ (that is, $\mathcal{F}$ is not a bad set).

We pause here to acknowledge the unfortunate choice of terminology given that proper forcing will play such a central role in what follows. However, as we shall see in the proof of Theorem 161, if a graph is proper with respect to an ideal, then there is a very natural forcing notion to add an uncountable clique to the graph which is proper in the forcing sense. We hope that this will soothe some of the annoyance on the part of the reader.

Next, we prove the main technical result of this section, and indeed, of this chapter. It is an abstract form of the main technical argument in several applications of the ‘side condition method’ as described in, for example, [Tod13].

Lemma 159. *Let $\mathcal{G} = (\omega_1, E)$ be a graph and $\mathcal{I}$ a nontrivial ideal on ω_1 which is σ-generated by $\langle I_\alpha \colon \alpha < \omega_1 \rangle$ such that $\mathcal{G}$ is proper with respect to $\mathcal{I}$. Let $\mathcal{F} \subseteq [\omega_1]^n$ for some non-zero $n \in \omega$ and $\bar{v} \in \mathcal{F}$ be such that there are $M_0 \in M_2 \in \cdots \in M_{n-1}$ which separate $\bar{v}$, and such that $\mathcal{G}, \mathcal{F} \in M_0$ and for each i, $M_i \prec H(\omega_2)$. Then there is $\bar{u} \in \mathcal{F} \cap M_0$ such that $\bar{u} \otimes \bar{v} \subseteq E$.*

Proof. First, note that as $\bar{v}$ is separated by the M_i, we have by Proposition 156 that for every $\mathcal{E} \subseteq [\omega_1]^n$ in M_0 such that $\bar{v} \in \mathcal{E}$, $\mathcal{E} \in \mathcal{H}^n$. Now, suppose that we cannot find such a $\bar{u}$. For every $\xi < \delta = M_0 \cap \omega_1$, let

$$\mathcal{E}_\xi = \{\bar{w} \in \mathcal{F} \colon \bar{w} > \xi, \forall \bar{u} \in \mathcal{F} \cap [\xi]^n, \bar{u} \otimes \bar{w} \not\subseteq E\}.$$

Note that for every $\xi < \delta$, $\mathcal{E}_\xi \in M_0$, and since $\bar{v} \in \mathcal{E}_\xi$, $\mathcal{E}_\xi \in \mathcal{H}^n$. It follows that

$$M_0 \models \text{“}\forall \xi < \omega_1 \exists \mathcal{E}_\xi \in \mathcal{H}^n \forall \bar{u} \in \mathcal{F} \cap [\xi]^n \forall \bar{w} \in \mathcal{E}_\xi [\xi < \bar{w}, \bar{u} \otimes \bar{w} \not\subseteq E]\text{”}.$$

So by the correctness of M_0, this statement is actually true. Let $\langle \mathcal{E}_\xi \colon \xi < \omega_1 \rangle$ denote a sequence witnessing this statement.

Now, we shall recursively build a sequence $\langle (N_\xi, \eta_\xi) \colon \xi < \omega_1 \rangle$ from which we can extract a contradiction. We do the case $n = 1$ first to give the basic idea of the proof.

Case 1: $\mathbf{n = 1}$. Our sequence $\langle (N_\xi, \eta_\xi) \colon \xi < \omega_1 \rangle$ will be such that

(i) $\mathcal{G}, \langle \mathcal{E}_\nu \colon \nu < \omega_1 \rangle \in N_\xi \prec H(\omega_2)$;

(ii) $\langle N_\nu \colon \nu < \xi \rangle \in N_\xi$;

(iii) η_ξ is far from N_ξ;

(iv) $\eta_\xi \in \mathcal{E}_{\delta_\xi}$, where $\delta_\xi = N_\xi \cap \omega_1$;

(v) $\eta_\xi \in N_{\xi+1}$.

Now, note that if $\xi < \nu < \omega_1$, then $\eta_\xi \in N_\nu$, so $\eta_\xi < \delta_\nu$, so $\{\eta_\xi, \eta_\nu\} \notin E$ as $\eta_\nu \in \mathcal{E}_{\delta_\nu}$. It follows that the set $S = \{\eta_\xi \colon \xi < \omega_1\}$ is such that $[S]^2 \cap E = \emptyset$. But we also have that $S \in \mathcal{H}$ by Lemma 157, which contradicts that $\mathcal{G}$ is proper with respect to $\mathcal{I}$.

Case 2: $\mathbf{n > 1}$. This will be an elaboration on the previous case. Our aim is to obtain $S \in \mathcal{H}$, $\mathcal{C} \in \mathcal{H}^n$, and $\{\delta_\xi \colon \xi \in S\}$ such that $\delta_\xi < \xi$, and if $\bar{u} \in \mathcal{C}$ then $\bar{u} \in \mathcal{F}$ and $\bar{u} = \langle \xi \rangle \bullet \bar{u}'$ for some $\xi \in S$. We also want that if $\xi \in S$ and $\bar{u} < \xi$ and is in $\mathcal{C}$, then in fact $\bar{u} < \delta_\xi$. This will ensure that if $\langle \xi \rangle \bullet \bar{v}' \in \mathcal{E}_{\delta_\xi}$, then $\bar{u} \otimes \langle \xi \rangle \bullet \bar{v}' \not\subseteq E$.

We build $\langle (N_\xi, \eta_\xi) \colon \xi < \omega_1 \rangle$ such that

(i) $\mathcal{G}, \langle \mathcal{E}_\nu \colon \nu < \omega_1 \rangle \in N_\xi \prec H(\omega_2)$;

(ii) $\langle N_\nu \colon \nu < \xi \rangle \in N_\xi$;

(iii) η_ξ is far from N_ξ;

(iv) $\eta_\xi \in N_{\xi+1}$;

(v) η_ξ is such that $(\mathcal{E}_{\delta_\xi})_{\eta_\xi} \in \mathcal{H}^{n-1}$ where $\delta_\xi = N_\xi \cap \omega_1$.

As in the previous case we have that $S = \{\eta_\xi \colon \xi < \omega_1\} \in \mathcal{H}$. Now, consider the set $\mathcal{D} \subseteq [\omega_1]^n$ given by

$$\mathcal{D} = \{\langle \eta_\xi \rangle \bullet \bar{u}' \colon \xi < \omega_1, \bar{u}' \in (\mathcal{E}_{\delta_\xi})_{\eta_\xi}\}.$$

Since $S \in \mathcal{H}$ and for every $\xi < \omega_1$, $(\mathcal{D})_{\eta_\xi} = (\mathcal{E}_{\delta_\xi})_{\eta_\xi}$ and $(\mathcal{E}_{\delta_\xi})_{\eta_\xi} \in \mathcal{H}^{n-1}$, we have that $\mathcal{D} \in \mathcal{H}^n$. We shall now refine $\mathcal{D}$ using Lemma 157 to get a $\mathcal{C} \in \mathcal{H}^n$ which will lead us to our desired contradiction.

For $\xi < \omega_1$, let

$$J_\xi = \bigcup_{\nu < \delta_\xi} I_\nu = \bigcup (N_\xi \cap \mathcal{I}).$$

Note that $J_\xi \in \mathcal{I}$ and $J_\xi \in N_{\xi+1}$ for each $\xi < \omega_1$. Let $T_\xi = N_{\xi+1} \cap (\omega_1 \setminus J_\xi)$ and $T = \bigcup_{\xi < \omega_1} T_\xi$, and finally, let $\mathcal{C} = \mathcal{D} \cap [T]^n$.

Claim. *$\mathcal{C} \in \mathcal{H}^n$.*

Proof. Follows by Lemma 157. ⊣

It is clear that $\mathcal{C}$ achieves the aim we had set for the construction. Namely, if $\bar{u} < \bar{v}$ are both in $\mathcal{C}$, then $\bar{u} \otimes \bar{v} \not\subseteq E$. This contradicts that $\mathcal{G}$ is proper with respect to $\mathcal{I}$. □

8.3 The axiom

We can finally state our axiom and prove that it is consistent.

Definition 160. The *Bounded Graph-Ideal Dichotomy*, denoted $\mathsf{GID}_{\aleph_1}$, is the statement that for every graph $\mathcal{G} = (\omega_1, E)$, if there is an ω_1-sequence of sets $\langle I_\alpha \colon \alpha < \omega_1 \rangle$ generating a non-trivial σ-ideal $\mathcal{I}$ such that $\mathcal{G}$ is proper with respect to $\mathcal{I}$, then $\mathcal{G}$ has an uncountable clique.

Note that we have not phrased this axiom as a dichotomy but as saying that if one alternative fails (a certain σ-ideal is non-trivial) then the other alternative holds (there is an uncountable clique). Also, the boundedness is in the graph having ω_1 as its carrier set and the ideal being ω_1-generated.

Theorem 161. *(*PFA*)* $\mathsf{GID}_{\aleph_1}$.

Proof. Let $\mathcal{G} = (\omega_1, E)$ be a graph and $\mathcal{I}$ a nontrivial ideal on ω_1 which is σ-generated by $\langle I_\alpha \colon \alpha < \omega_1 \rangle$ such that $\mathcal{G}$ is proper with respect to $\mathcal{I}$. We show that there is a proper partial order which adds an uncountable clique to $\mathcal{G}$. The partial order itself is the simplest one in the given context: an element $p \in \mathbb{P}$ is given by $p = (\bar{w}_p, \mathcal{N}_p)$ where

(i) $\bar{w}_p \subseteq \omega_1$ is a finite clique;

(ii) $\mathcal{N}_p$ is a finite $\in$-chain of elementary substructures of $H(\omega_2)$ all of whom contain $\mathcal{G}$ and $\langle I_\alpha \colon \alpha < \omega_2 \rangle$;

(iii) $\mathcal{N}_p$ separates $\bar{w}_p$.

The ordering is given by co-ordinatewise containment. If $p \in \mathbb{P}$ and $M^* \prec H(\kappa)$ where κ is some large regular cardinal such that $p, \mathbb{P} \in M^*$, then $p^* = (\bar{w}_p, \mathcal{N}_p \cup \{M\})$ is a master condition for M^* below p, where $M = M^* \cap H(\omega_2)$. We see this using Proposition 156 and Lemma 159. Namely, if $D \in M^*$ is a dense set and $q \leq p^*$ is in D, then consider the set

$$\mathcal{F} = \{\bar{v} \in [\omega_1]^n \colon \exists r \in D, r \leq p, \bar{v} = \bar{w}_r \setminus (\bar{w}_q \cap M)\},$$

where $n = |\bar{w}|$, where $\bar{w} = \bar{w}_q \setminus M$.

Clearly, $\mathcal{F} \in M^*$, and hence $\mathcal{F} \in M$. So, it follows from Proposition 159 and the fact that $\bar{w} \in \mathcal{F}$, that $\mathcal{F} \in M$, and that $\bar{w}$ is separated by an $\in$-chain of structures each of which contain M, that $\mathcal{F} \in \mathcal{H}^n$. So then by Lemma 159, we can find $\bar{v} \in \mathcal{F} \cap M$ such that $\bar{v} \cup \bar{w}$ is a clique, and hence also $\bar{v} \cup (\bar{w}_q \cap M) \cup \bar{w}$ is a clique. So, in M^*, if r witnesses that $\bar{v} \in \mathcal{F}$, then $r \parallel q$, and so we finish. $\square$

Remark 162. Note that this proof in fact tells us that we can obtain a stronger axiom. Given a non-trivial σ-ideal $\mathcal{I}$ with a generating set $\langle I_\alpha \colon \alpha < \omega_1 \rangle$, we say that a set $X \subseteq \omega_1$ is *progressive* with respect to $\langle I_\alpha \colon \alpha < \omega_1 \rangle$ if for all $\beta \in X$, $\beta \notin \bigcup_{\alpha \in X \cap \beta} I_\alpha$. Then we can consider the strengthening of $\mathsf{GID}_{\aleph_1}$ where the conclusion states that we get a clique of $\mathcal{G}$ which is progressive.

No large cardinals are required for the consistency of $\mathsf{GID}_{\aleph_1}$. There are two possible forcing constructions achieving this, both of which are respectively obtained by suitably modifying forcing constructions from [Tod85] and [Tod83] where statements closely related to $\mathsf{GID}_{\aleph_1}$ are shown to be consistent using them.

The approach followed in [Tod85] is to perform a countable-support iteration of length ω_2 taking care of all the suitable pairs of graphs and ideals. Since each such pair can be coded by a subset of ω_1, so long as we ensure that the ω_2-length iteration has have the $\aleph_2$-cc, we can ensure that after ω_2-many steps, any requirement in the resulting model would have been taken care of at some intermediate stage. To ensure this, we need to start with a model of CH and modify our forcings so that where certain finite *matrices of models* are used instead of finite $\in$-chains of models. Then, the $\aleph_2$-*pic* of Shelah [She98] is used to ensure the $\aleph_2$-cc of the entire iteration.

In [Tod83], the special case of $\mathsf{GID}_{\aleph_1}$ corresponding to Example 152 is shown to be consistent with ZFC using a different approach. Here, one starts with a model of GCH and performs a two step forcing construction with both steps in fact consisting of an iteration. In the first step, one performs a *mixed-support iteration* of length ω_2 obtained by alternately forcing with *Jensen's club poset* and the poset to add $\aleph_1$-many Cohen reals. Then, one performs a finite-support iteration of certain ccc posets which correspond to the appropriate pairs of graphs and ideals. The first iteration preserves ω_1 and has the $\aleph_2$-cc, and so the entire construction preserves ω_1 and has the $\aleph_2$-cc.

Since both these proofs are just simple variations of the arguments from [Tod83] and [Tod85], we omit them.

8.4 Symmetric systems

So far we have formulated a somewhat abstract axiom $\mathsf{GID}_{\aleph_1}$ and shown that it follows from PFA and discussed how it does not require any large cardinals for its consistency. In Chapter 9 we shall see some applications of it, though they can all be divided into two parts: the first consists of applications of the special case of $\mathsf{GID}_{\aleph_1}$ when we consider only the situation in Example 152, see Theorem 168, and (very) weakenings of important consequences of PFA, see Section 9.4. In this section we attempt to justify this activity.

For notational convenience, if κ is some fixed uncountable regular cardinal, then given $T \subseteq H(\kappa)$ and $N \in [H(\kappa)]^{\aleph_0}$, we shall denote the structure $(N, T \cap N)$ by (N, T).

Definition 163. Let $T \subseteq H(\kappa)$ and $\mathcal{N}$ be a finite set of countable subsets of $H(\kappa)$. We say that $\mathcal{N}$ is a T-*symmetric system* if the following hold:

(i) if $N \in \mathcal{N}$, then $(N, \in, T) \prec (H(\kappa), \in T)$;

(ii) if $N, N' \in \mathcal{N}$ are such that $\delta_N = \delta_{N'}$, then there is a unique isomorphism

$$\Psi_{N,N'} : (N, \in, T) \to (N', \in, T),$$

and this isomorphism is the identity on $N \cap N'$;

(iii) if $N, N', M \in \mathcal{N}$ are such that $M \in N$ and $\delta_N = \delta_{N'}$, then $\Psi_{N,N'}(M) \in \mathcal{N}$;

(iv) if $M, N \in \mathcal{N}$ are such that $\delta_M < \delta_N$, then there is some $N' \in \mathcal{N}$ such that $\delta_N = \delta_{N'}$ and $M \in N'$.

These objects are similar to (but not the same as) the matrices that we mentioned at the end of the previous section when discussing the proof of $\mathsf{GID}_{\aleph_1}$ which one can extract from [Tod85]. More importantly for us, they were used by Asperó and Mota in some recent papers [AM15a, AM16] in order to show that certain consequences of PFA are consistent with the continuum being arbitrarily large. To be precise (and vague at the same time), their framework can be considered as 'iterated forcing with finite symmetric systems as side conditions'. The following observation indicates that there is some compatibility between symmetric systems and the axiom $\mathsf{GID}_{\aleph_1}$.

Observation 164. Let $\mathcal{G} = (\omega_1, E)$ be a graph and $\mathcal{I}$ a nontrivial ideal on ω_1 which is σ-generated by $\langle I_\alpha \colon \alpha < \omega_1 \rangle$ such that $\mathcal{G}$ is proper with respect to $\mathcal{I}$. Let $T \subseteq H(\omega_2)$ be a predicate such that for any $(N, \in, T) \prec (H(\omega_2), \in, T)$, $\mathcal{G}, \langle I_\alpha \colon \alpha < \omega_1 \rangle \in N$. Let $\mathbb{P}$ be the partial order consisting of $p = (\bar{w}_p, \mathcal{N}_p)$ where

(i) $\bar{w}_p \subseteq \omega_1$ is a finite clique;

(ii) $\mathcal{N}_p$ is a T-symmetric system;

(iii) $\mathcal{N}_p$ separates $\bar{w}_p$:

 (a) for every $\xi < \nu$ in $\bar{w}_p$, there is some $N \in \mathcal{N}_p$ such that $\xi \in N$ and $\xi \notin N$;

 (b) if $\nu \in \bar{w}_p$ is such that $\nu \notin N$ for some $N \in \mathcal{N}_p$, then ν is far from N.

Then $\mathbb{P}$ is proper (and adds an uncountable clique of $\mathcal{G}$), and in fact for any $N^* \prec H(\kappa)$ where κ is some large regular cardinal such that $\mathbb{P} \in N^*$, if $N = N^* \cap H(\omega_2)$ and $p \in \mathbb{P}$ is such that $N \in \mathcal{N}_p$, then p is $(N^*, \mathbb{P})$-generic. Verifying this proceeds in much the same way as the proof of Theorem 161 using these two observations:

(i) If N, N' are countable elementary substructures of $H(\omega_2)$ such that $\delta_N = \delta_{N'}$, both of which contain $\mathcal{G}$ and $\langle I_\alpha \colon \alpha < \omega_1 \rangle$ and $\xi < \omega_1$, then ξ is far from N iff ξ is far from N'.

(ii) If $\mathcal{N}$ is a T-symmetric system, $N \in \mathcal{N}$ and $\bar{u} \in [\omega_1]^{<\omega}$ is separated by $\mathcal{N}$ and such that $\bar{u} \cap N = \emptyset$, then there is an $\in$-chain $\mathcal{M} \subseteq \mathcal{N}$ whose lowest model is N which also separates $\bar{u}$.

Before we proceed, we point out that the former crucially requires that $\mathcal{G}$ is a graph on ω_1 and that $\mathcal{I}$ is ω_1-generated. The latter also requires the specific argument we needed in Lemma 159.

It follows that for any $\mathcal{N}$ a T-symmetric system and $p \in \mathbb{P}$ such that $p \in N$ for $N \in \mathcal{N}$ a model of minimal height, if $\mathcal{M}$ is the $\subseteq$-least symmetric system containing $\mathcal{N}$ and $\mathcal{N}_p$ (which can easily be shown to exist), then $p^* = (\bar{w}_p, \mathcal{M})$ is not only a condition in $\mathbb{P}$, but is $(M^*, \mathbb{P})$-generic for every $M^* \prec H(\kappa)$ containing $\mathbb{P}$ such that $M^* \cap H(\omega_2) \in \mathcal{M}$, where κ is some large enough regular cardinal.

This observation is the reason we feel that $\mathsf{GID}_{\aleph_1}$ would be amenable to being shown consistent with the continuum being arbitrarily large using some refinements of the Asperó-Mota method. Unfortunately we have not been able to achieve this so far.

Question 165. *Is $\mathsf{GID}_{\aleph_1}$ consistent with the continuum being arbitrarily large?*

There is another reason why we feel that $\mathsf{GID}_{\aleph_1}$ is an interesting axiom in its own right. The side condition method, of which Theorem 161 was a prime example is one of the most fruitful methods of establishing consequences of PFA. But if one wants to isolate what its limitations are, one needs to first understand what its abstract form is. Such an attempt was made by Zapletal in [Zap97], which was carried on in [Yor15]. The framework given by Zapletal was called *ideal-based forcings*. While it does allow one to handle $\mathsf{GID}_{\aleph_1}$ restricted to the situation in Example 152, so in particular for all the applications in Chapter 9 except for those in Section 9.4, it does not encapsulate any clear fragment of OGA.

The axiom $\mathsf{GID}_{\aleph_1}$ does handle this situation, though the burden of generality here is that we are unable to obtain results similar to the ones in [Zap97, Yor15] for it. We hope this situation will change in the future.

Chapter 9

Applications of the Graph Ideal Dichotomy

The aim of this chapter is to give some applications of the axiom $\mathsf{GID}_{\aleph_1}$ which we have isolated. In Section 9.1 we show that $\mathsf{GID}_{\aleph_1}$ implies a certain partition relation $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$, and give some applications of $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$. One important application is that $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$ implies that $\mathfrak{b} > \omega_1$, and hence the continuum hypothesis fails, Corollary 169. In Section 9.2 we show that $\mathsf{GID}_{\aleph_1}$, and in fact $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$, implies that there are no Souslin trees on ω_1. In Section 9.3 we show that $\mathsf{GID}_{\aleph_1}$, and in fact $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$, implies that there are no destructible (ω_1, ω_1)-gaps. In Section 9.4 we consider a very weak form of the Open Graph Axiom and show that it follows from $\mathsf{GID}_{\aleph_1}$. We are not aware if it follows already from $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$.

9.1 S-spaces and partition relations

Definition 166. The partition relation $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$ means that for every graph $\mathcal{G} = (\omega_1, E)$ on ω_1, one of the following happens:

(i) There is $A \in [\omega_1]^{\aleph_1}$ such that $[A]^2 \subseteq E$, that is, A is an uncountable clique;

(ii) There is $A \in [\omega_1]^{\aleph_1}$ and an uncountable $\mathcal{B}$ consisting of pairwise disjoint finite subsets of ω_1 such that for all $\alpha \in A$ and for all $F \in \mathcal{B}$ such that $\alpha < F$, there is $\beta \in F$ such that $\{\alpha, \beta\} \notin E$. In this case $(A, \mathcal{B})$ are called a *bad pair* for this graph.

Note that this partition relation is referred to as (P) in [Tod89]. In [Tod83], among other consequences of $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$, the following was proved, which settled an important problem in general topology (see [Tod89]).

Theorem 167. *($\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$)*

(i) Every hereditarily separable regular space is hereditarily Lindelöf. That is, there are no S-spaces*.*

(ii) ($\mathfrak{p} > \omega_1$) $\omega_1 \to (\omega_1, \alpha)^2$ holds for every $\alpha < \omega_1$.

Note that the conclusion of the second part of the above theorem can also be obtained from $\mathfrak{b} > \omega_1$ instead of $\mathfrak{p} > \omega_1$ as is done in [Tod98].

Theorem 168. *($\mathsf{GID}_{\aleph_1}$)* $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$.

Proof. The proof is the same as the proof of [Tod98, Lemma 7]. Let $\mathcal{G} = (\omega_1, E)$ be a graph on ω_1 and let $\mathcal{I}$ be the (ω_1-generated) ideal of countable subsets of ω_1. Suppose that $\mathcal{G}$ has no uncountable cliques. It follows by $\mathsf{GID}_{\aleph_1}$ that $\mathcal{G}$ is not proper with respect to $\mathcal{I}$. Let $\mathcal{F} \in \mathcal{H}^n$ witness this for some $n \in \omega$. Let $M_0 \prec H(\omega_2)$ be countable and contain $\mathcal{G}$ and $\mathcal{F}$. Using Proposition 156, let $M_1 \in M_2 \in \dots M_k$ be countable elementary substructures of $H(\omega_2)$ all containing M_0, and let $\bar{v} \in \mathcal{F}$ be separated by $\{M_0, M_1, \dots M_k\}$. We know that there is no $\bar{u} \in \mathcal{F} \cap M_0$ such that $\bar{u} \otimes \bar{v} \subseteq E$. It follows that any attempts to build such a $\bar{u}$ must fail, so there is some $\bar{u}' \in M_0$ such that $(\mathcal{F})_{\bar{u}'} \in \mathcal{H}^{n-m}$ where $m = |\bar{u'}|$ such that $\bar{u}' \otimes \bar{v} \subseteq E$ and there is no $\nu > \bar{u}'$ in M_0 such that $(\mathcal{F})_{\bar{u}' \otimes \langle \nu \rangle} \in \mathcal{H}^{n-m-1}$ and $\{\nu\} \otimes \bar{v} \subseteq E$.

Let
$$S = \{\nu < \omega_1 \colon (\mathcal{F})_{\bar{u}' \otimes \langle \nu \rangle} \in \mathcal{H}^{n-m-1}\}.$$
Then M_0 sees that for every $\xi < \omega_1$, there is some $\mathcal{E}_\xi \subseteq \mathcal{F}$ such that $\mathcal{E}_\xi \in \mathcal{H}^n$, and for every $\nu \in S \cap \xi$ and $F \in \mathcal{E}_\xi$, $\nu < F$ and $\{\nu\} \otimes F \nsubseteq E$. So, by correctness, there is some sequence $\langle \mathcal{E}_\xi \colon \xi < \omega_1 \rangle$ with this property. Now, construct recursively $\langle (N_\xi, \eta_\xi, F_\xi) \colon \xi < \omega_1 \rangle$ such that

(i) $\langle N_\xi \colon \xi < \omega_1 \rangle$ is an increasing chain of elementary substructures of $H(\omega_2)$ each of whom contains $\langle \mathcal{E}_\xi \colon \xi < \omega_1 \rangle$ and $\mathcal{G}$;

(ii) $F_\xi, \eta_\xi \in N_{\xi+1}$;

(iii) $F_\xi \in \mathcal{E}_{\delta_\xi}$ where $\delta_\xi = \delta_{N_\xi}$;

(iv) $\eta_\xi \in S$;

(v) $F_\xi < \eta_\xi$.

Then $\{\eta_\xi \colon \xi < \omega_1\}$ and $\{F_\xi \colon \xi < \omega_1\}$ witness the second alternative of $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$. □

In fact, it is possible to show that $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$ is equivalent to the version of $\mathsf{GID}_{\aleph_1}$ where we only consider the ideal of countable subsets of ω_1. This statement has implicitly been considered in [Tod98].

Corollary 169. *($\mathsf{GID}_{\aleph_1}$)* $\mathfrak{b} > \omega_1$.

Proof. In [Tod89] it is shown that if $\mathfrak{b} = \omega_1$, then there is an S-space. □

In particular, $\mathsf{GID}_{\aleph_1}$ implies that the continuum hypothesis fails.

9.2 Souslin trees

Recall that an ω_1-*tree* is a tree of height ω_1 with countable levels such that each node has infinitely many successors in all levels larger than its own. It is *Aronszajn* if it has no uncountable chain, and *Souslin* if it has no uncountable antichains.

Theorem 170. *($\mathsf{GID}_{\aleph_1}$) There are no Souslin trees.*

Proof. Let $T = (\omega_1, \leq_T)$ be an Aronszajn tree, and suppose that it does not have any uncountable antichains. Let $\mathcal{G} = (\omega_1, E)$ be given by $\{\alpha, \beta\} \in E$ if $\alpha \perp_T \beta$. Let $\mathcal{I}$ be the ideal of all countable subsets of ω_1. By $\mathsf{GID}_{\aleph_1}$, there is some $n < \omega$ and $\mathcal{F} \in \mathcal{H}^n$ such that for any $\bar{u}, \bar{v} \in \mathcal{F}$ such that $\bar{u} < \bar{v}$, $\bar{u} \otimes \bar{v} \nsubseteq E$. Let $\langle F_\alpha \colon \alpha < \omega_1 \rangle \subseteq \mathcal{F}$ be such that $\alpha < \beta$ implies for every $\gamma \in F_\alpha$ and $\delta \in F_\beta$ we have $\gamma < \delta$ and also $\mathrm{ht}(\gamma) < \mathrm{ht}(\delta)$, and let $\mathcal{U}$ be a uniform ultrafilter on ω_1. Also, for each $\alpha < \omega_1$, fix $\langle F_\alpha(i) \colon i < n \rangle$ an increasing enumeration of F_α. Then by standard arguments related to Baumgartner's poset for specialising an ω_1-tree with no uncountable chains [BMR70], there is $X \subseteq \omega_1$ uncountable, $i, j < n$, and for each $\alpha \in X$, $Y_\alpha \in \mathcal{U}$ such that for every $\alpha \in X$ and $\beta \in Y_\alpha$, $\alpha < \beta$ and $\{F_\alpha(i), F_\beta(j)\} \notin E$, so $F_\alpha(i) <_T F_\beta(j)$. Hence, the set $\{F_\alpha(i) \colon \alpha \in X\}$ is an uncountable chain of T. So T is not Aronszajn, a contradiction. $\square$

The above theorem could in fact have been proved indirectly from Theorem 168 and the result of M.E. Rudin that one can construct an S-space from a Suslin tree ([Rud75, Chapter V], but see also [Tod89, Chapter 5] for another construction).

9.3 Destructible gaps

Recall that a double sequence $(a_\alpha, b_\alpha \colon \alpha < \omega_1)$ is called a (ω_1, ω_1)-*pregap* if for every $\alpha < \beta < \omega_1$

(i) $a_\alpha \subseteq \omega$ and $b_\alpha \subseteq \omega$;

(ii) $a_\alpha \cap b_\alpha = \emptyset$;

(iii) $a_\alpha \cap b_\beta =^* \emptyset$ and $a_\beta \cap b_\alpha =^* \emptyset$;

(iv) $a_\alpha \subseteq^* a_\beta$ and $b_\alpha \subseteq^* b_\beta$.

If $c \subseteq \omega$ is such that for every $\alpha < \omega_1$ we have $a_\alpha \subseteq^* c$ and $b_\alpha \cap c =^* \emptyset$, then we say that c *separates* it. If there is no $c \subseteq \omega$ which separates it, we say that it is an (ω_1, ω_1)-*gap*. If a gap can be separated in some ω_1-preserving forcing extension, we say that it is *destructible*.

Lemma 171. *Let $(a_\alpha, b_\alpha \colon \alpha < \omega_1)$ be an (ω_1, ω_1)-pregap.*

(i) Then $(a_\alpha, b_\alpha \colon \alpha < \omega_1)$ is a gap iff for every (not necessarily distinct) $X \in [\omega_1]^{\aleph_1}$ and $Y \in [\omega_1]^{\aleph_1}$, there are $\alpha \in X$ and $\beta \in Y$ such that $(a_\alpha \cap b_\beta) \cup (b_\alpha \cap a_\beta) \neq \emptyset$.

(ii) Then $(a_\alpha, b_\alpha \colon \alpha < \omega_1)$ is destructible iff for every $X \in [\omega_1]^{\aleph_1}$, there are $\alpha < \beta$ in X such that $(a_\alpha \cap b_\beta) \cup (b_\alpha \cap a_\beta) = \emptyset$.

Proof. See for example [Hir03, Lemma 1.5, Lemma 1.8]. $\square$

Theorem 172. *($\mathsf{GID}_{\aleph_1}$) There are no destructible (ω_1, ω_1)-gaps.*

Proof. Let $(a_\alpha, b_\alpha \colon \alpha < \omega_1)$ be an (ω_1, ω_1)-gap. Let $\mathcal{G} = (\omega_1, E)$ be the graph given by $\{\alpha, \beta\} \in E$ if $(a_\alpha \cap b_\beta) \cup (b_\alpha \cap a_\beta) \neq \emptyset$. Let $\mathcal{I}$ be the ideal of all countable subsets of ω_1. It is easy to show inductively that for every $n < \omega_1$ and $\mathcal{F} \in \mathcal{H}^n$ there are $\bar{u}, \bar{v} \in \mathcal{F}$ such that $\bar{u} < \bar{v}$ and $\bar{u} \otimes \bar{v} \subseteq E$: for $n = 1$ it follows by using the characterisation of being a gap from Lemma 171 for the special case of $X = Y = \bigcup \mathcal{F}$, and for larger n we use the general case using the $n = 1$ case as a base step. Then, by an application of $\mathsf{GID}_{\aleph_1}$, we get an uncountable clique in $\mathcal{G}$. But this implies that $(a_\alpha, b_\alpha \colon \alpha < \omega_1)$ is an indestructible gap by the characterisation from Lemma 171. $\square$

9.4 More applications

Note that all of the applications of $\mathsf{GID}_{\aleph_1}$ that we have mentioned so far have actually been applications of $\mathsf{GID}_{\aleph_1}$ restricted to the ideal of countable subsets of ω_1, which as we mentioned after Theorem 168 is actually equivalent to $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$. This raises the question as to why we consider the general form at all. The reason for this is to highlight the commonality between all of the applications of the side condition method in the literature. For example, browsing through [Tod13] it is easy to find suitable restrictions of many of the statements considered there and show that they follow from $\mathsf{GID}_{\aleph_1}$. We give an example.

Recall that the *Open Graph Axiom* is the following statement: If X is a separable metric space and $\mathcal{G} = (X, E)$ is an open graph on X, then this graph is either countably chromatic or contains an uncountable clique. There are two standard approaches to forcing an instance of OGA. To explain these, let X be a separable metric space and $\mathcal{G} = (X, E)$ is an open graph on X which is not countably chromatic.

The first approach, as seen for example in [TF95], requires showing that in the extension of the universe by the poset which collapses $\mathfrak{c}$ to $\aleph_1$ with countable conditions, the poset of finite cliques of $\mathcal{G}$ has the countable chain condition.

The second approach, which interests us more and can be found for example in [Tod89] or [Tod13] is more interesting to us because it uses the side condition method in the same way as Theorem 161. Namely, the poset used is the following: let κ be a large enough regular cardinal compared to $|X|$. The poset $\mathbb{P}$ consists of elements p such that $p = (\bar{w}_p, \mathcal{N}_p)$ where

(i) $\bar{w}_p \subseteq X$ is a finite clique of $\mathcal{G}$;

(ii) $\mathcal{N}_p$ is a finite $\in$-chain of elementary substructures of $H(\kappa)$ all of whom contain X and $\mathcal{G}$;

(iii) $\mathcal{N}_p$ separates $\bar{w}_p$: for any $x, y \in \bar{w}_p$, there is some $N \in \mathcal{N}_p$ such that $|\{x, y\} \cap N| = 1$, and for any $x \in \bar{w}_p$ and $N \in \mathcal{N}_p$ such that $x \notin N$, x is not in any discrete subset of $\mathcal{G}$ in N.

The ordering is given by co-ordinatewise containment. One then shows that this poset is proper. It is clear that the above poset is very similar to the poset we used in Theorem 161 (and indeed, the latter was heavily inspired by the former), and in fact from the usual proofs of properness of $\mathbb{P}$ one can extract the following result.

Lemma 173. *Let X be a separable metric space of size $\aleph_1$ and $\mathcal{G} = (X, E)$ an open graph on X which is not countably chromatic. Let $\mathcal{I}$ be the σ-ideal generated by the discrete subsets of $\mathcal{G}$. Then $\mathcal{G}$ is proper with respect to $\mathcal{I}$.*

It follows that if we restrict the size of X to be $\aleph_1$ and also restrict the colouring to be one so that $\mathcal{I}$ were ω_1-generated as a σ-ideal, we have a statement that follows from $\mathsf{GID}_{\aleph_1}$.

Definition 174. Let the *very weak Open Graph Axiom*, denoted vwOGA, be the statement that for every X a separable metric space of size $\aleph_1$, if $\mathcal{G} = (X, E)$ is an open graph on X such that the σ-ideal of countably chromatic subsets of X is non-trivial and ω_1-generated, then X has an uncountable clique.

Here note that the σ-ideal generated by the discrete subsets of X is exactly the σ-ideal of countably chromatic subsets of X.

Theorem 175. *($\mathsf{GID}_{\aleph_1}$) vwOGA.*

In a similar fashion, one can extract weakenings of various statements considered in [Tod13, Chapter 8] which follow from $\mathsf{GID}_{\aleph_1}$ by essentially the same proof.

We point out here that while we have not been able to show that $\mathsf{GID}_{\aleph_1}$ is consistent with the continuum being larger than ω_2, Farah has shown in [Far95] that the following statement which is intermediate between OGA and vwOGA is consistent with the continuum being larger than ω_2: if X is a separable metric space of size $\aleph_1$ and $\mathcal{G} = (X, E)$ is an open graph on X, then this graph is either countably chromatic or contains an uncountable clique.

However, Farah's method, which is related to the first approach to proving OGA consistent, does not allow us to prove that $\mathsf{GID}_{\aleph_1}$ is also consistent with the continuum being larger than ω_2 because it makes crucial use of the fact that we are dealing with a separable metric space, and some properties of separable metric spaces in countably-closed forcing extensions.

Chapter 10

An inconsistent higher forcing axiom

The contents of this chapter are somewhat unrelated to the contents of the rest of the chapters of this thesis, though there is a shared philosophy between this chapter and the motivation behind formulating the axiom $\mathsf{GID}_{\aleph_1}$ in Chapter 8. Namely, one of exploring the current limits of forcing axioms. The main result of this chapter, Theorem 181, shows that for any uncountable regular cardinal κ, a very natural forcing axiom for meeting κ^+-many dense sets is inconsistent.

Due to it being somewhat separate from the rest of this thesis, we have written it so as to be self-contained. In Section 10.1 we introduce the problem and give some historical background. In Section 10.2 we present the technical results that we shall need in the proof of Theorem 181. The only non-trivial such is a function with some very strong unboundedness properties constructed by Todorčević. In Section 10.3 we give a proof of the main result, Theorem 181.

Before we begin, we recall some standard terminology about partial orders.

Definition 176. Let $\mathbb{P}$ be a partial order and κ a cardinal.

(i) We say that $\mathbb{P}$ is *well-met* if given any $p, q \in \mathbb{P}$ such that $p \parallel q$, they have a greatest lower bound r: $r \leq p, q$, and for any $s \leq p, q$, $s \leq r$.

(ii) We say that a subset of $\mathbb{P}$ is 2-*linked* or simply *linked* if its elements are pairwise compatible. We can define n-*linked* for other $n \in \omega$ in a similar way.

(iii) We say that a subset of $\mathbb{P}$ is *centred* if any finite subset of it consists of mutually compatible conditions.

(iv) We say that $\mathbb{P}$ has *precaliber* κ if any family of κ-many conditions of $\mathbb{P}$ contains a subfamily of size κ which is centred.

(v) We say that $\mathbb{P}$ is κ-2-*linked* or simply κ-*linked* if we can decompose it into $\mathbb{P} = \bigcup_{\alpha\in\kappa} \mathbb{Q}_\alpha$ such that all of the $\mathbb{Q}_\alpha$ are linked. We can also define κ-n-*linked* for other $n \in \omega$ in the obvious way.

(vi) We say that $\mathbb{P}$ is κ-*centred* if we can decompose it into $\mathbb{P} = \bigcup_{\alpha\in\kappa} \mathbb{Q}_\alpha$ such that all of the $\mathbb{Q}_\alpha$ are centred.

(vii) $\mathsf{FA}_\kappa(\mathbb{P})$ denotes the forcing axiom for the poset $\mathbb{P}$ for meeting κ-many dense sets. If $\mathcal{P}$ is a class of partial orders, then $\mathsf{FA}_\kappa(\mathcal{P})$ denotes the forcing axiom $\mathsf{FA}_\kappa(\mathbb{Q})$ for every $\mathbb{Q} \in \mathcal{P}$.

10.1 Introduction to the problem

As we have discussed before, historically, the first forcing axiom was Martin's Axiom [MS70], which is $\mathsf{FA}_{<\mathfrak{c}}(\mathcal{P})$ where $\mathcal{P}$ is the class of all partial orders with the countable chain condition. Since then, two other forcing axioms have come to occupy a central place in set theory with it, the Proper Forcing Axiom [She98, Dev83], which is $\mathsf{FA}_{\aleph_1}(\mathcal{P})$ where $\mathcal{P}$ is the class of proper partial orders, and Martin's Maximum [FMS88], which is $\mathsf{FA}_{\aleph_1}(\mathcal{P})$ where $\mathcal{P}$ is the class of partial orders which preserve stationary subsets of ω_1. Most importantly, all of these forcing axioms have had very many applications both within and without set theory (see for example [Fre84, Bau84, Tod13]).

However, most of these applications have focused on the gap between the countable and uncountable, or equivalently, the gap between ω and ω_1. Given the abundance of cardinals beyond ω_1 (tongue firmly in cheek), it is natural to wonder about whether the independence phenomenon is so rampant even at higher levels of the cardinal hierarchy. Since forcing axioms have been so crucial to our understanding of independence in the gap between the countable and the uncountable, it is natural to wonder: are there forcing axioms at higher cardinals as well? More formally,

Question 177. *Are there interesting classes of partial orders $\mathcal{P}$ and uncountable cardinals $\kappa > \aleph_1$ such that $\mathsf{FA}_\kappa(\mathcal{P})$ holds?*

For the moment, we shall focus on the smallest case of this problem, where $\kappa = \aleph_2$. It should be pointed out that the Martin-Solovay proof of Martin's Axiom already gives us the consistency of $\mathsf{FA}_{\aleph_2}(\mathcal{P})$ where $\mathcal{P}$ is the class of all partial orders with the countable chain condition. However, ccc partial orders can force much fewer interesting objects of size $\aleph_2$ than they can of size $\aleph_1$, so the class of ccc partial orders fails the 'interesting test' in Question 177. It is also known that $\mathsf{FA}_{\aleph_2}(\mathcal{P})$ where $\mathcal{P}$ is either the class of proper partial orders or that of partial orders which preserve stationary subsets of ω_1 is inconsistent.

Indeed, if $\mathbb{P}$ is a partial order such that $\mathsf{FA}_{\aleph_2}(\mathbb{P})$ holds, then $\mathbb{P}$ must necessarily not collapse ω_1 and ω_2. These restrictions on $\mathbb{P}$ naturally suggest an 'interesting' class of partial orders.

Question 178. *(CH) Let $\mathcal{P}$ be the class of σ-closed $\aleph_2$-cc partial orders. Is $\mathsf{FA}_{\aleph_2}(\mathcal{P})$ consistent?*

The reason for our assumption of CH is that if $\mathfrak{c} > \aleph_1$, then the above class $\mathcal{P}$ is empty (see Theorem 57). Shelah has answered this question negatively, see [She98, Appendix, 3.4A].

Theorem 179. *(CH) There is a σ-closed $\aleph_1$-centred non-wellmet partial order $\mathbb{P}$ such that $\mathsf{FA}_{\aleph_2}(\mathbb{P})$ is inconsistent.*

On the other hand, in [She78], Shelah proved the following:

Theorem 180. *Let $\mathcal{P}$ be the class of partial orders $\mathbb{P}$ such that*

(i) $|\mathbb{P}| < 2^{\aleph_1}$*;*

(ii) $\mathbb{P}$ *is σ-closed;*

(iii) $\mathbb{P}$ *is well-met;*

(iv) $\mathbb{P}$ *has the $\aleph_2$-stationary-cc: for any $\langle p_\alpha \colon \alpha < \omega_2\rangle \subseteq \mathbb{P}$, there is a club $E \subseteq \omega_2$ and a regressive function $f : E \to \omega_2$ such that*

$$\forall \alpha, \beta \in E,\ f(\alpha) = f(\beta) \implies p_\alpha \parallel_{\mathbb{P}} p_\beta.$$

Then $\mathsf{CH} + 2^{\aleph_1} > \aleph_2 + \mathsf{FA}_{<2^{\aleph_1}}(\mathcal{P})$ *is consistent.*

Independently, Baumgartner (see [KT79]) as well as Laver proved weaker versions of the above where the $\aleph_2$-stationary-cc is replaced by being $\aleph_1$-2-linked. See the remarks at the end of [She78]. It should be pointed out that one can replace $\aleph_1$ by an arbitrary uncountable regular cardinal κ (with appropriate modifications) in these results.

Comparing Theorem 179 and Theorem 180 suggests that the condition of being well-met is an important one: even though the chain condition in Theorem 179 is stronger, it is a negative result, whereas Theorem 180 has a weaker chain condition but has the extra property of being well-met, and is a positive result. The main result of this chapter show that the $\aleph_2$-stationary-cc cannot be relaxed by much if one wants a positive result, even with the property of being well-met. This, together with Theorem 180 partially answers a question of Tall (see the end of [Tal94]).

Theorem 181. *If κ is an uncountable regular cardinal such that $\kappa^{<\kappa} = \kappa$, then there is a well-met $< \kappa$-closed precaliber κ^+ poset $\mathbb{Q}$ of size κ^+ such that $\mathsf{FA}_{\kappa^+}(\mathbb{Q})$ fails.*

Not that by Fodor's Lemma, Theorem 41, κ^+-stationary-cc posets also have precaliber κ^+. Independently, Lücke has obtained the weaker conclusion where the partial order only has the κ^+-cc, under the extra hypothesis that κ^+ is not weakly compact in $\mathbf{L}$ [Lüc17]. A related paper is [She13], which concerns itself with showing that the well-met condition is important, in a similar sense as in Theorem 179.

We finish by saying that we have by no means given a complete account of the development of higher forcing axioms. We have left out many advances since the original Shelah results. A small sample of the papers in a similar spirit as those, using σ-closed partial orders, is [RS11, Eis03, She17]. More recently, there have also been attempts at obtaining higher forcing axioms at ω_2 without requiring that the partial orders be σ-closed [Mit09, GN16, Vel15, GK17]. There are also ongoing attempts to find forcing axioms at singular cardinals [CDM$^+$17]. It is hoped that they succeed.

10.2 The technical tools

We now describe the three combinatorial tools we shall use in the proof of our main theorem. The first two being standard, the third not so.

First, we shall need a standard generalisation of the Δ-system Lemma, also due to Erdős and Rado, see [HH99, Theorem 13.1] or [Kun14, Theorem II.1.6].

Theorem 182. *Let κ be an infinite cardinal such that $\kappa^{<\kappa} = \kappa$. Then for any collection $\mathcal{A}$ of size κ^+ consisting of sets of size $< \kappa$, there is some $\mathcal{B} \subseteq \mathcal{A}$ of size κ^+ and some set r of size $\leq \kappa$ such that for each $a, b \in \mathcal{B}$ distinct, $a \cap b = r$. That is, $\mathcal{B}$ is a Δ-system of size κ^+ with root r.*

Proof. Let $\lambda \gg \kappa^+$ be a regular cardinal, and let $M \prec H(\lambda)$ be an elementary submodel of size κ which contains $\kappa, \kappa^+, \mathcal{A}$ and such that $\kappa \subseteq M \cap \kappa^+ \in \kappa^+$, and which is closed under $< \kappa$ sequences. To see that such an M exists, we use that $\kappa^{<\kappa} = \kappa$. Let $a \in \mathcal{A}$ be such that $a \not\subseteq M$. We can find such an a since if b is any set of size $< \kappa$ such that $b \subseteq M$, then $b \in M$, and since $\mathcal{A}$ has size κ^+ whereas M has size κ.

Let $r = a \cap M$, and note that $r \in M$. In M, let $\mathcal{B} \subseteq \mathcal{A}$ be maximal such that for every $b, c \in \mathcal{B}$, $b \cap c = r$. By the elementarity of M, $\mathcal{B}$ is indeed maximal even when considered in the entire universe. Now, if $\mathcal{B}$ has size $< \kappa^+$, then M can see this by elementarity. In this case, $\mathcal{B} \subseteq M$, since

$\kappa \in M$ and $\kappa \subseteq M$. But then note that $\mathcal{B} \cup \{a\}$ is a strictly larger subfamily of $\mathcal{A}$ such that for any $b, c \in \mathcal{A}$, $b \cap c = r$. Here we are using that $a \cap M = r$. This contradicts the maximality of $\mathcal{B}$, and hence $\mathcal{B}$ must have size κ^+. □

The second is a simple counting argument.

Theorem 183. *Let κ be an infinite regular cardinal such that $\kappa^{<\kappa} = \kappa$. Then the $< \kappa$-support product of κ-many κ-centred posets is κ-centred.*

Proof. Let $\langle \mathbb{P}_\alpha \colon \alpha < \kappa \rangle$ be given, and suppose that for each $\alpha < \kappa$ we have that $f_\alpha : \mathbb{P}_\alpha \to \kappa$ is a function such that for each $\beta < \kappa$, $f^{-1}[\{\beta\}]$ is centred. Let $\mathbb{P}$ be the $< \kappa$-support product of $\langle \mathbb{P}_\alpha \colon \alpha < \kappa \rangle$. We shall define a function $F : \mathbb{P} \to \kappa^{<\kappa}$ such that for each $g \in \kappa^{<\kappa}$, $F^{-1}[\{g\}]$ is centred. Since $\kappa^{<\kappa} = \kappa$, this will allow us to finish.

Let $p \in \mathbb{P}$ have domain $D \in [\kappa]^{<\kappa}$. Then $F(p)$ will also have domain D. Also, for each $\beta \in D$, $F(p)(\beta) = f_\beta(p(\beta))$. It is easy to check that F is as required. □

We now come to the main technical tool we shall use in the proof of our main result, which is a function constructed by Todorčević. The following is [Tod10, Theorem 8.6] or [Tod07, Theorem 6.3.6].

Theorem 184. *Let κ be an infinite cardinal. Then there is a function $\rho_2 : [\kappa^+]^2 \to \omega$ with the following property: for every family $\mathcal{A}$ of pairwise disjoint subsets of κ^+ of size $< \kappa$ and every $n \in \omega$, there is some $\mathcal{B} \subseteq \mathcal{A}$ of size κ^+ such that for every $a, b \in \mathcal{B}$ distinct, and $\alpha \in a$ and $\beta \in b$, $\rho_2(\alpha, \beta) > n$.*

This function ρ_2 is the *number of steps function over a C-sequence on the cardinal κ* (see [Tod07]). Using a function with such a strong unboundedness property allows us to prove our main result without needing anything more than very elementary arguments, the deep combinatorics underlying our proof all being hidden by our use of this function of Todorčević.

10.3 Forcing axiom failure

This entire section is devoted to the proof of Theorem 181. We shall prove it in a sequence of steps. For the rest of this section, let κ be an uncountable regular cardinal such that $\kappa^{<\kappa} = \kappa$, and let $\rho_2 : [\kappa^+]^2 \to \omega$ be a function as supplied to us by Theorem 184. We first define for each $n \in \omega$ two partial orders.

Definition 185. For $n \in \omega$, the partial order $\mathbb{P}_n$ consists of subsets of κ^+ of size $< \kappa$ such that $\rho_2[a \otimes a] > n$ for each $a \in \mathbb{P}_n$. The order is reverse inclusion.

Definition 186. For $n \in \omega$, the partial order $\mathbb{Q}_n$ is the $< \kappa$-support product of κ-many copies of $\mathbb{P}_n$.

In Corollary 190 we shall show that one of the $\mathbb{Q}_n$ shall witness the statement of Theorem 181. Naturally, for this we need to show that they have the properties that this statement requires.

Theorem 187. *For each $n \in \omega$, $\mathbb{Q}_n$ is well-met, $< \kappa$-closed, and has precaliber κ^+.*

Proof. The only thing that is not obvious is that $\mathbb{Q}_n$ has precaliber κ^+. So, let $\langle q_\alpha\colon \alpha < \kappa^+\rangle \subseteq \mathbb{Q}_n$ be given. For $\alpha < \kappa^+$, let D_α be the domain of q_α. By refining using the Δ-system Lemma, Theorem 182, we can assume that D_α form a Δ-system with root D. In fact, since we are interested in showing that many of the q_α are compatible with each other, we can assume that each $D_\alpha = D$, the general case following from this.

So, we can assume that we are given $\langle q_\alpha\colon \alpha < \kappa^+\rangle \subseteq \mathbb{Q}_n$, all of whom have the same domain D. For each $\alpha < \kappa^+$, let $S_\alpha = \bigcup_{\xi\in D} q_\alpha(\xi)$. Note that each of the S_α have size $< \kappa$. By refining using the Δ-system Lemma, Theorem 182, we can assume that the S_α form a Δ-system with root R. Note that R has size $< \kappa$. To each $\alpha < \kappa^+$, we can associate the following function $f_\alpha : D \to \mathcal{P}(R)$: for $\xi \in D$,

$$f_\alpha(\xi) = R \cap q_\alpha(\xi).$$

Since $\kappa^{<\kappa} = \kappa$, there are at most κ-many admissible functions of this sort, and hence, by refining, we can assume that there is some function $f : D \to \mathcal{P}(R)$ such that for each $\alpha < \kappa^+$, $f_\alpha = f$. In this case, the compatibility between the q_α is equivalent to the compatibility between the q_α with the contribution of R to their components ignored. In simpler English, we can assume without losing any generality that R is empty.

We summarise our progress so far. By successive refinements, we have seen that proving the precaliber κ^+ in the following special case suffices: we are given $\langle q_\alpha\colon \alpha < \kappa^+\rangle \subseteq \mathbb{Q}_n$, all of whom have domain D, and such that $\langle S_\alpha\colon \alpha < \kappa^+\rangle$ consists of pairwise disjoint sets. This is easily accomplished by the unboundedness property of ρ_2 from Theorem 184 applied to $\langle S_\alpha\colon \alpha < \kappa^+\rangle$. $\square$

Corollary 188. *For each $n < \omega$, if $\mathsf{FA}_{\kappa^+}(\mathbb{Q}_n)$ holds, then $\mathbb{P}_n$ is κ-centred.*

Proof. Clear since $\mathbb{P}_n$ has size κ^+. $\square$

Now, we wrap things up. We first make a simple observation.

Proposition 189. *The full support product $\mathbb{P} = \Pi_{n\in\omega}\mathbb{P}_n$ does not have the κ^+-cc.*

Proof. For $\alpha < \kappa^+$ consider $q_\alpha \in \mathbb{P}$ defined by $q_\alpha(n) = \{\alpha\}$. For any $\alpha < \beta < \kappa^+$, there is some n such that $\rho_2(\alpha, \beta) = n$. It follows that $q_\alpha(n) \perp_{\mathbb{P}_n} q_\beta(n)$, and hence, that $q_\alpha \perp_{\mathbb{P}} q_\beta$. Consequently, $\langle q_\alpha\colon \alpha < \kappa^+\rangle$ is a κ^+-sized antichain of $\mathbb{P}$. $\square$

Corollary 190. *For some $n \in \omega$, $\mathsf{FA}_{\kappa^+}(\mathbb{Q}_n)$ does not hold.*

Proof. Since κ is uncountable, by Theorem 183, the $< \kappa$-product of κ-centred posets is κ-centred, and in particular the full support product of countably many κ-centred posets is κ-centred. If $\mathsf{FA}_{\kappa^+}(\mathbb{Q}_n)$ holds for each $n \in \omega$, then each of the $\mathbb{P}_n$ are κ-centred. Therefore, their full support product $\mathbb{P} = \Pi_{n\in\omega}\mathbb{P}_n$ is κ-centred. But this contradicts Proposition 189. $\square$

Note that the $\mathbb{Q}_n$ actually satisfy a much stronger chain condition than precaliber κ^+: they have $(\kappa^+, \kappa^+, < \kappa)$ as a precaliber. That is, any κ^+-sized family of conditions contains a subfamily of size κ^+ such that any $< \kappa$-many of its members are mutually compatible.

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