Lie semigroup operator algebras

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BA, MA (Cantab)

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Abstract

The parabolic algebra $\mathcal{A}_{\mathrm{p}}$ and the hyperbolic algebra $\mathcal{A}_{\mathrm{h}}$ are nonselfadjoint w*-closed operator algebras which were first considered by A. Katavolos and S. C. Power. In [KP97] and [KP02] they showed that their invariant subspace lattices are homeomorphic to compact connected Euclidean manifolds, and that the parabolic algebra is reflexive in the sense of Halmos.

We give a new proof of the reflexivity of the parabolic algebra through analysis of Hilbert-Schmidt operators. We also show that there are operators in $\mathcal{A}_{\mathrm{p}}$ with nontrivial kernel.

We then consider some natural "companion algebras" of the parabolic algebra which leads to a compact subspace lattice known as the Fourier-Plancherel sphere. We show that the unitary automorphism group of this lattice is isomorphic to a semidirect product of $\mathbb{R}^2$ and $SL_2(\mathbb{R})$.

A proof that the hyperbolic algebra is reflexive follows by an essentially identical analysis of Hilbert-Schmidt operators to that which was used to establish the reflexivity of $\mathcal{A}_{\mathrm{p}}$. We also present a transparent proof of a known result concerning a strong operator topology limit of projections.

Both of the Katavolos-Power algebras are generated as w*-closed operator algebras by the image of a semigroup of a Lie group under a unitary-valued representation. Following [KP02], we call such operator algebras *Lie semigroup operator algebras.* We seek new examples of such algebras by considering the images of the semigroup $SL_2(\mathbb{R}_+)$ of the Lie group $SL_2(\mathbb{R})$ under unitary-valued representations of $SL_2(\mathbb{R})$. We show that a particular Lie semigroup operator algebra $\mathcal{A}_+$ arising in this way is reflexive and that it is the operator algebra leaving a double triangle subspace lattice invariant. Surprisingly, $\mathcal{A}_+$ is generated as a w*-closed algebra by the image of a proper subsemigroup of $SL_2(\mathbb{R}_+)$.

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Acknowledgements

This thesis has not been submitted for the award of a higher degree elsewhere. Except where indicated otherwise, this thesis is my own work. In particular, §5.1 is based on the paper [LP03] co-authored by myself and my supervisor, Professor S. C. Power.

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Chapter 1

Introduction

This thesis is concerned with the properties of certain nonselfadjoint noncommutative weakly closed operator algebras. The original examples of such algebras were described in the two papers [KP97] and [KP02] of Katavolos and Power. Since much of this thesis is intimately connected with these papers, we briefly review their contents here.

In [KP97], the *parabolic algebra* is analysed. This is the w*-closed algebra of bounded operators on the Hilbert space $L^2(\mathbb{R})$ generated by the Hardy space $H^\infty(\mathbb{R})$ acting as multiplication operators and the right shifts.

One version of Beurling's theorem formulated by Lax in [Lax59] asserts that the closed subspaces K of $L^2(\mathbb{R})$ which are invariant under multiplication by functions in $H^\infty(\mathbb{R})$ either take the form $K = L^2(E)$ for some Lebesgue measurable subset E of $\mathbb{R}$, or that there is a unimodular function $u \in L^\infty(\mathbb{R})$ such that $K = uH^2(\mathbb{R})$. In [KP97], the authors use this result, a cocycle argument and the properties of certain inner functions to determine the invariant subspace lattice $\operatorname{Lat}\mathcal{A}_\mathrm{p}$ of the parabolic algebra $\mathcal{A}_\mathrm{p}$. They show that if we give $\operatorname{Lat}\mathcal{A}_\mathrm{p}$ the topology induced by the strong operator topology on the set of orthogonal projections, then $\operatorname{Lat}\mathcal{A}_\mathrm{p}$ is homeomorphic to a closed disk. They also find the unitary automorphism group of this subspace lattice, and show that it is isomorphic to a semidirect product of the additive groups $\mathbb{R}^2$ and $\mathbb{R}$.

An operator algebra $\mathcal{A}$ is said to be *reflexive* if each operator leaving invariant every invariant subspace of $\mathcal{A}$ lies inside $\mathcal{A}$. This terminology is due to Halmos [Hal70]. One well-known class of reflexive operator algebras is the class of *nest algebras*, the algebras consisting of all bounded linear operators which leave a particular chain of subspaces invariant. By showing that every Hilbert-Schmidt operator in $\mathcal{A}_{\mathrm{p}}$ is a certain type of pseudo-differential operator, the parabolic algebra is identified in [KP97] with the intersection of two nest algebras. It follows immediately that $\mathcal{A}_{\mathrm{p}}$ is reflexive.

In Chapter 3, we revisit the question of the reflexivity of the parabolic algebra. We give a different characterisation of the Hilbert-Schmidt operators in $\mathcal{A}_{\mathrm{p}}$ and so obtain a new proof of this result. Motivated by the unresolved question of whether or not $\mathcal{A}_{\mathrm{p}}$ is an integral domain, we also construct non-zero operators in $\mathcal{A}_{\mathrm{p}}$ with non-trivial kernel.

As remarked in [KP02], the invariant subspace lattice of the parabolic algebra sits inside a larger subspace lattice, the *Fourier-Plancherel sphere*. In Chapter 4, we determine the reflexive binest algebras obtained when we pick two nests from this lattice. In the non-degenerate cases, these are all unitarily equivalent to $\mathcal{A}_{\mathrm{p}}$. Using arguments from Lang's book [Lan85], we identify a copy of the Lie group $SL_2(\mathbb{R})$ inside the unitary automorphism group of the Fourier-Plancherel sphere. This automorphism group is then seen to be isomorphic to a semidirect product of $\mathbb{R}^2$ and $SL_2(\mathbb{R})$.

If we replace the right translation semigroup in the definition of $\mathcal{A}_{\mathrm{p}}$ by a certain semigroup of unitary dilation operators, then we arrive at the *hyperbolic algebra* $\mathcal{A}_{\mathrm{h}}$, which is the subject of [KP02]. In this paper, Katavolos and Power determine the invariant subspace lattice Lat $\mathcal{A}_{\mathrm{h}}$ of this nonselfadjoint weakly closed operator algebra, which turns out to be homeomorphic to a compact connected subset of $\mathbb{R}^4$.

In Chapter 5, we give a proof of the reflexivity of $\mathcal{A}_{\mathrm{h}}$, published in [LP03]. This is strikingly similar to the reflexivity proof of Chapter 3. We close the chapter by establishing a certain strong operator topology limit of projections onto subspaces in Lat $\mathcal{A}_{\mathrm{h}}$. This result is also proven in [KP02], as part of the proof of the connectedness of Lat $\mathcal{A}_{\mathrm{h}}$. Our proof emphasises the geometric picture in the spirit of Halmos ([Hal69], [Hal71]) by expressing the subspaces involved as the graphs of unbounded operators.

It is high time for us to explain the title of this thesis. Following comments made in [KP02], we define a *Lie semigroup operator algebra* as follows. Given a Lie group G we will call a subsemigroup G_+ of G a *Lie semigroup.* If we also select a unitary-valued representation ρ of G on some Hilbert space, then the Lie semigroup operator algebra obtained from the triple (G, G_+, ρ) is defined to be the weakly closed operator algebra generated by $\rho(G_+)$. Both of the Katavolos-Power algebras discussed above are examples. In each case the representation ρ maps a Lie group into the set of bounded linear operators on the Hilbert space $L^2(\mathbb{R})$. For the parabolic algebra, the Lie group in question is the Heisenberg group of 3×3 matrices, and the "$ax + b$" Lie group gives rise to the hyperbolic algebra.

Given the broad scope of this definition, it seems natural to hope that the parabolic and hyperbolic algebras are not the only interesting members of this class. In Chapter 6 we begin the search for new examples, focusing on the Lie group $SL_2(\mathbb{R})$ of 2×2 matrices with real entries and determinant $+1$ and the Lie semigroup $SL_2(\mathbb{R}_+)$ consisting of those matrices in $SL_2(\mathbb{R})$ with non-negative entries. We first examine some representations of $SL_2(\mathbb{R})$ on $\mathbb{C}^N$. Since there are no irreducible unitary-valued representations of $SL_2(\mathbb{R})$ on a finite-dimensional Hilbert space, we instead examine the closed algebras generated by $\rho(SL_2(\mathbb{R}_+))$ where ρ is an irreducible representation. Such representations are well-known, and it is easily seen that these algebras are the full matrix algebras $M_N(\mathbb{C})$. In particular, they are reflexive.

Having dealt with this somewhat trivial case, we turn to the principal series representations of $SL_2(\mathbb{R})$ on $L^2(\mathbb{R})$, restricting our attention to the non-irreducible representation ρ in the principal series. Let $\mathcal{A}_+$ denote the Lie semigroup operator algebra obtained from $(SL_2(\mathbb{R}), SL_2(\mathbb{R}_+), \rho)$. We show that the invariant subspace lattice $\operatorname{Lat}\mathcal{A}_+$ of $\mathcal{A}_+$ has three components, being the union of a topological sphere with the two isolated points $\{(0), L^2(\mathbb{R})\}$. The key step is the observation that $\rho(SL_2(\mathbb{R}_+))$ fixes a double triangle subspace lattice; we show that the reflexive closure of this five-element lattice coincides with $\operatorname{Lat}\mathcal{A}_+$. We also show that $\mathcal{A}_+$ is equal to an operator algebra which is unitarily equivalent to a superalgebra of $\mathcal{A}_{\mathrm{h}}$, although it appears at first sight to be a proper subalgebra of $\mathcal{A}_+$. This is done in a similar manner to the reflexivity proof of the previous chapter, and we obtain the reflexivity of $\mathcal{A}_+$ as a consequence.

Chapter 2

Preliminaries

This chapter contains a short account of the elementary theory which we need. Proofs which are readily available elsewhere have been omitted.

2.1 Hardy spaces

The details of this theory may be found in [Hof62], [Gar81] and [Nik02].

Let

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\} \quad \text{and} \quad \mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$$

denote the unit circle and the open unit disk in $\mathbb{C}$ respectively. Further, let $\operatorname{Hol}\mathbb{S}$ be the set of holomorphic functions on an open subset $\mathbb{S}$ of $\mathbb{C}$. For $p > 0$, the Hardy spaces of the disk are

$$H^p(\mathbb{D}) = \left\{ f \in \operatorname{Hol}(\mathbb{D}) \,\middle|\, \sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty \right\}, \quad 0 < p < \infty,$$

$$H^\infty(\mathbb{D}) = \left\{ f \in \operatorname{Hol}(\mathbb{D}) \,\middle|\, \sup_{z \in \mathbb{D}} |f(z)| < \infty \right\}.$$

We write $\mathbb{H}^+$ and $\mathbb{H}^-$ for the open upper and lower half-planes

$$\mathbb{H}^+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}, \quad \mathbb{H}^- = \{z \in \mathbb{C} \mid \operatorname{Im} z < 0\}.$$

The Hardy spaces of the upper half-plane are

$$H^p(\mathbb{H}^+) = \left\{ f \in \mathrm{Hol}(\mathbb{H}^+) \;\middle|\; \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^p \, dx < \infty \right\}, \quad 0 < p < \infty,$$

$$H^\infty(\mathbb{H}^+) = \left\{ f \in \mathrm{Hol}(\mathbb{H}^+) \;\middle|\; \sup_{z \in \mathbb{H}^+} |f(z)| < \infty \right\}.$$

We define quasinorms on these spaces in the obvious manner:

$$\|f\|^p_{H^p(\mathbb{D})} = \sup_{0 \le r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi}, \quad 0 < p < \infty, \; f \in H^p(\mathbb{D}),$$

$$\|f\|^p_{H^p(\mathbb{H}^+)} = \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^p \, dx, \quad 0 < p < \infty, \; f \in H^p(\mathbb{H}^+),$$

$$\|f\|_{H^\infty(\mathbb{D})} = \|f\|_{L^\infty(\mathbb{D})}, \quad f \in H^\infty(\mathbb{D}),$$

$$\|f\|_{H^\infty(\mathbb{H}^+)} = \|f\|_{L^\infty(\mathbb{H}^+)}, \quad f \in H^\infty(\mathbb{H}^+).$$

For $1 \le p \le \infty$, these quasinorms on $H^p(\mathbb{D})$ and $H^p(\mathbb{H}^+)$ are norms.

We now define the Hardy spaces on the circle and the line by

$$H^p(\mathbb{T}) = \left\{ f \in L^p(\mathbb{T}) \;\middle|\; \int_0^{2\pi} e^{in\theta} f(e^{i\theta}) \frac{d\theta}{2\pi} = 0, \; n = 1, 2, 3, \ldots \right\}, \; 1 \le p \le \infty,$$

$$H^p(\mathbb{R}) = \left\{ f \in L^p(\mathbb{R}) \;\middle|\; \int_{\mathbb{R}} \frac{f(t)}{t-z} \, dt = 0, \; z \in \mathbb{H}^- \right\}, \quad 1 \le p < \infty,$$

$$H^\infty(\mathbb{R}) = \left\{ f \in L^\infty(\mathbb{R}) \;\middle|\; \int_{\mathbb{R}} f(t) \left(\frac{1}{t-z} - \frac{1}{t+i} \right) dt = 0, \; z \in \mathbb{H}^- \right\}.$$

These spaces inherit norms from $L^p(\mathbb{T})$ and $L^p(\mathbb{R})$.

Let f and g be measurable functions $\mathbb{T} \to \mathbb{C}$. The convolution of f and g is the function $f * g : \mathbb{T} \to \mathbb{C}$,

$$(f * g)(e^{i\theta}) = \int_0^{2\pi} f(e^{i\varphi}) g(e^{i(\theta - \varphi)}) \frac{d\varphi}{2\pi}, \quad 0 \le \theta < 2\pi$$

whenever this integral exists. Similarly if f and g are measurable functions $\mathbb{R} \to \mathbb{C}$, then their convolution $f * g : \mathbb{R} \to \mathbb{C}$ is given by the following expression whenever the integrand is integrable.

$$(f * g)(x) = \int_{\mathbb{R}} f(y) g(x-y) \, dy, \quad x \in \mathbb{R}$$

Theorem 2.1 (Fatou). *(i). Let $1 \le p \le \infty$ and let E be the operator defined on $H^p(\mathbb{T})$ by*

$$Ef(re^{i\theta}) = (P_r * f)(e^{i\theta}), \quad 0 \le r < 1,\ 0 \le \theta < 2\pi$$

where P_r is the Poisson kernel

$$P_r(e^{i\theta}) = \operatorname{Re} \frac{e^{i\theta} + r}{e^{i\theta} - r}, \quad 0 \le r < 1,\ 0 \le \theta < 2\pi.$$

Then E is an isometric isomorphism of $H^p(\mathbb{T})$ onto $H^p(\mathbb{D})$. Given a function $f \in H^p(\mathbb{D})$, let Bf be the function

$$Bf(e^{i\theta}) = \lim_{z \to e^{i\theta}} f(z). \tag{2.1}$$

The limit in this expression is the nontangential limit as $z \to e^{i\theta}$ with $z \in \mathbb{D}$, and Bf is defined for those $e^{i\theta} \in \mathbb{T}$ for which this limit exists. Then Bf is defined almost everywhere with respect to Lebesgue measure in $\mathbb{T}$ and $B = E^{-1}$. If $0 < p < 1$ and $f \in H^p(\mathbb{D})$ then the function Bf given by (2.1) is defined almost everywhere.

(ii). Let $1 \le p \le \infty$ and let E be the operator defined on $H^p(\mathbb{R})$ by

$$Ef(x + iy) = (P_y * f)(x), \quad x \in \mathbb{R},\ y > 0$$

where P_y is the Poisson kernel

$$P_y(x) = \frac{1}{\pi} \operatorname{Im} \frac{1}{x - iy}, \quad x \in \mathbb{R},\ y > 0.$$

Then E is an isometric isomorphism of $H^p(\mathbb{R})$ onto $H^p(\mathbb{H}^+)$. Given a function $f \in H^p(\mathbb{H}^+)$, let Bf be the function

$$Bf(x) = \lim_{z \to x} f(z). \tag{2.2}$$

The limit in this expression is the nontangential limit as $z \to x$ with $z \in \mathbb{H}^+$, and Bf is defined for those $x \in \mathbb{R}$ for which this limit exists. Then Bf is *defined almost everywhere with respect to Lebesgue measure and $B = E^{-1}$. If $0 < p < 1$ and $f \in H^p(\mathbb{H}^+)$ then the function Bf given by (2.2) is defined almost everywhere.*

In light of Theorem 2.1, when $1 \leq p \leq \infty$ we often make the following identifications.

$$\begin{aligned} H^p(\mathbb{D}) &\longleftrightarrow H^p(\mathbb{T}), \\ f \in H^p(\mathbb{D}) &\longleftrightarrow Bf \in H^p(\mathbb{T}), \\ H^p(\mathbb{H}^+) &\longleftrightarrow H^p(\mathbb{R}), \\ f \in H^p(\mathbb{H}^+) &\longleftrightarrow Bf \in H^p(\mathbb{R}). \end{aligned} \tag{2.3}$$

The function Bf is called the *boundary value function* of f, and the function Ef is called the *Poisson extension* of f.

We pause to extract a useful generalisation of Cauchy's formula and a density result.

Proposition 2.2. *For $1 \leq p < \infty$ and $z \in \mathbb{H}^+$, let*

$$Cf(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t - z}\, dt, \quad f \in H^p(\mathbb{R}).$$

Then Cf is the Poisson extension of f to $\mathbb{H}^+$.

Proof. Let $x \in \mathbb{R}$ and $y > 0$. Then

$$P_y(x) = \frac{1}{\pi} \operatorname{Im} \frac{1}{x - iy} = \frac{1}{2\pi i}\Big(\frac{1}{x - iy} - \frac{1}{x + iy}\Big).$$

So writing $z = x + iy$, the Poisson extension Ef of $f \in H^p(\mathbb{R})$ is given by

$$\begin{aligned} Ef(z) &= \int_{\mathbb{R}} P_y(x - t) f(t)\, dt \\ &= \frac{1}{2\pi i} \int_{\mathbb{R}} \Big(\frac{1}{\bar{z} - t} + \frac{1}{t - z}\Big) f(t)\, dt \\ &= \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t - z}\, dt, \end{aligned}$$

since $f \in H^p(\mathbb{R})$ and by definition,

$$\int_{\mathbb{R}} \frac{f(t)}{t-\bar{z}}\,dt = 0, \quad z \in \mathbb{H}^+. \qquad \square$$

For $v \in \mathbb{C}$, let $b_v : \mathbb{C} \setminus \{v\} \to \mathbb{C}$ be the function

$$b_v(z) = \frac{1}{z-v}, \quad z \in \mathbb{C} \setminus \{v\}.$$

When $v \in \mathbb{H}^-$, the function b_v lies in the Hardy space $H^2(\mathbb{R})$ and in accordance with (2.3) we identify b_v with its boundary value function $Bb_v = b_v|\mathbb{R}$. The last result can then be rewritten in the form

$$Cf(z) = \langle f, b_{\bar{z}} \rangle = Ef(z), \quad z \in \mathbb{H}^+,\ f \in H^p(\mathbb{R}).$$

Lemma 2.3. *The linear spans of the sets*

$$\{b_v \mid v \in \mathbb{H}^-\} \quad \textit{and} \quad \{b_v b_w \mid v, w \in \mathbb{H}^-\}$$

are both dense in $H^2(\mathbb{R})$.

Proof. If $f \in H^2(\mathbb{R})$ and $\langle f, b_v \rangle = 0$ for each $v \in \mathbb{H}^-$, then by Proposition 2.2, the Poisson extension of f to $\mathbb{H}^+$ is the constant function 0, so $f = 0$. So the orthogonal complement in $H^2(\mathbb{R})$ of $\{b_v \mid v \in \mathbb{H}^-\}$ is the zero subspace and so this set does indeed have dense linear span in $H^2(\mathbb{R})$.

Observe that for any distinct complex numbers v and w,

$$b_v(z)b_w(z) = \frac{b_v(z) - b_w(z)}{v-w}, \quad z \in \mathbb{C} \setminus \{v, w\}.$$

So if $f \in H^2(\mathbb{R})$ and $\langle f, b_v b_w \rangle = 0$ for every $v, w \in \mathbb{H}^-$, then when v and w are distinct in $\mathbb{H}^-$,

$$0 = \langle f, b_v b_w \rangle = \frac{\langle f, b_v \rangle - \langle f, b_w \rangle}{\bar{v} - \bar{w}} = \frac{f(\bar{v}) - f(\bar{w})}{\bar{v} - \bar{w}}.$$

So the Poisson extension of f to $\mathbb{H}^+$ has zero derivative, and so is a constant. However, the only constant function in $H^2(\mathbb{H}^+)$ is the zero function, so

$f = 0$. Hence the orthogonal complement in $H^2(\mathbb{R})$ of $\{b_v b_w \mid v, w \in \mathbb{H}^-\}$ is the zero subspace and so this set has dense linear span in $H^2(\mathbb{R})$. $\square$

Let ω be the bijection of $\overline{\mathbb{D}} \setminus \{1\}$ onto the closed upper half-plane $\overline{\mathbb{H}^+}$,

$$\omega(z) = i\frac{1+z}{1-z}, \quad z \in \overline{\mathbb{D}} \setminus \{1\} = \{z \in \mathbb{C} \mid |z| \le 1,\ z \ne 1\}$$

and for $1 \le p \le \infty$ let $U_p : L^p(\mathbb{T}) \to L^p(\mathbb{R})$ be the map

$$U_p f(x) = \Big(\frac{1}{\pi(x+i)^2}\Big)^{1/p} f\big(\omega^{-1}(x)\big), \quad x \in \mathbb{R},\ f \in L^p(\mathbb{T}),\ 1 \le p < \infty,$$

$$U_\infty f(x) = f\big(\omega^{-1}(x)\big), \quad x \in \mathbb{R},\ f \in L^\infty(\mathbb{T}).$$

Theorem 2.4. *For $1 \le p \le \infty$, the map U_p is an isometric isomorphism of $H^p(\mathbb{T})$ onto $H^p(\mathbb{R})$.*

Theorem 2.5 (Szegö). *Let $0 < p \le \infty$ and let $f \in L^p(\mathbb{T})$ be a function with $\log|f| \in L^1(\mathbb{T})$. Let the function $[f] : \mathbb{D} \to \mathbb{C}$ be given by*

$$[f](z) = \exp \int_{\mathbb{R}} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log|f(e^{i\theta})| \frac{d\theta}{2\pi}, \quad z \in \mathbb{D}.$$

Then $[f] \in H^p(\mathbb{D})$ and the boundary value function $B[f]$ satisfies

$$|Bf(e^{i\theta})| = |f(e^{i\theta})| \quad \textit{for almost every } \theta \in [0, 2\pi).$$

Moreover, for every non-zero function $f \in H^p(\mathbb{T})$, the function $\log|f|$ lies in $L^1(\mathbb{T})$.

Corollary 2.6. *Let $1 \le p \le \infty$ and let $f \in H^p(\mathbb{T})$. Then f takes the value 0 on a set of strictly positive Lebesgue measure if and only if $f = 0$. The same is also true if $f \in H^p(\mathbb{R})$.*

Definition 2.7. A function $f \in H^\infty(\mathbb{T})$ is *inner* if f is unimodular, i.e.

$$|f(e^{i\theta})| = 1 \quad \text{for almost every } \theta \in [0, 2\pi).$$

For $0 < p \leq \infty$, a function $f \in H^p(\mathbb{D})$ is *outer* if there is a constant λ with $|\lambda| = 1$ such that

$$f = \lambda[f]$$

where $[f]$ is the function defined in Theorem 2.5. We call $[f]$ the *outer part* of f.

A function $f \in H^\infty(\mathbb{H}^+)$ is *inner* if f is unimodular, i.e.

$$|Bf(x)| = 1 \quad \text{for almost every } x \in \mathbb{R}.$$

For $1 \leq p \leq \infty$, a function $f \in H^p(\mathbb{H}^+)$ is *outer* if the function $U_p^{-1}f$ is outer in $H^p(\mathbb{D})$.

Using the identifications (2.3) also allows us to define inner and outer functions in the spaces $H^p(\mathbb{T})$ and $H^p(\mathbb{R})$ for $1 \leq p \leq \infty$.

The importance of the classes of inner and outer functions lies in the fact that H^p functions have a canonical factorisation into inner and outer parts.

Theorem 2.8 (F. Riesz, V. Smirnov). *Let $0 < p \leq \infty$. If $f \in H^p(\mathbb{D})$ is non-zero, then f has a unique factorisation of the form $f = V[f]$ where $V \in H^\infty(\mathbb{D})$ is inner and $[f] \in H^p(\mathbb{D})$ is the outer part of f.*

If $1 \leq p \leq \infty$ and $f \in H^p(\mathbb{H}^+)$ is non-zero, then f has a factorisation of the form $f = \lambda V W$ where λ is a unimodular constant, $V \in H^\infty(\mathbb{H}^+)$ is inner and $W \in H^p(\mathbb{H}^+)$ is outer. Moreover, this factorisation is unique up to constant unimodular factors.

The following sufficient condition for a function to be outer is Corollary II.4.8 of [Gar81].

Lemma 2.9. *Let $f \in H^p(\mathbb{H}^+)$ for some $p > 0$. If either of the following two conditions is satisfied, f is an outer function.*

(i). $\operatorname{Re} f(z) \geq 0$ *for* $z \in \mathbb{H}^+$.

(ii). There is a continuously differentiable arc Γ terminating at 0 which contains more than one point, and $f(\mathbb{H}^+) \subseteq \mathbb{C} \setminus \Gamma$.

The next two results are in §A.4.2 of [Nik02].

Theorem 2.10 (V. Smirnov). *If a function $h : \mathbb{D} \to \mathbb{C}$ is holomorphic and $\operatorname{Re} h(z) \geq 0$ for each $z \in \mathbb{D}$, then $h \in H^r(\mathbb{D})$ for each $r \in (0,1)$, and h is outer.*

Proposition 2.11. *Let f_1 and f_2 be functions in $H^p(\mathbb{D})$ for some $p > 0$. Then the product $f_1 f_2$ is outer if and only if f_1 and f_2 are both outer. In particular, if $f \in H^p(\mathbb{D})$ and $1/f \in H^q(\mathbb{D})$ for some $p, q > 0$ then f is outer.*

Beurling's theorem characterises the closed subspaces of $L^2(\mathbb{T})$ which are invariant for the shift operator

$$M_z : f(z) \mapsto zf(z), \quad f \in L^2(\mathbb{T}),\ z \in \mathbb{T}.$$

Theorem 2.12 (Beurling's theorem). *Suppose that K is a closed subspace of the Hardy space $H^2(\mathbb{T})$ and that the set $M_z K$ is contained in K. Then $K = uH^2(\mathbb{T})$ for some inner function $u \in H^\infty(\mathbb{T})$.*

Transferring this result to the line, we arrive at the following result, formulated by Lax in [Lax59].

Theorem 2.13 (Beurling-Lax theorem). *Suppose that K is a closed subspace of the Hardy space $H^2(\mathbb{R})$ and that the set $e^{i\lambda x}K$ is contained in K for each $\lambda \geq 0$. Then $K = uH^2(\mathbb{R})$ for some inner function $u \in H^\infty(\mathbb{R})$.*

Given inner functions f and g, we say that g *divides* f if there is an inner function h such that $f = gh$. The *greatest common inner divisor* of a family $\{f_i \mid i \in I\}$ of inner functions, if it exists, is an inner function g such that g divides f_i for each $i \in I$ and if another inner function g' divides f_i

for each $i \in I$ then g' divides g. If $1 \leq p \leq \infty$ and we are given a set $S = \{h_i \mid i \in I\} \subseteq H^p$ where $H^p = H^p(\mathbb{T})$ or $H^p = H^p(\mathbb{R})$, then by Theorem 2.8,

$$S = \{\lambda_i V_i W_i \mid i \in I\}$$

where $\lambda_i \in \mathbb{C}$, V_i is inner and W_i is outer for each $i \in I$. The *greatest common inner divisor* of S is the greatest common inner divisor of the set $\{V_i \mid i \in I\}$, if it exists. This is defined up to a unimodular constant.

Proposition 2.14. *Suppose that $1 \leq p \leq \infty$ and that K is a subset of $H^p(\mathbb{T})$ or $H^p(\mathbb{R})$ containing a non-zero function. Then the greatest common inner divisor of K exists.*

Proof. As shown in [Ber88], when K is a subset of $H^\infty(\mathbb{T})$ this result follows from Beurling's theorem. The case $K \subseteq H^\infty(\mathbb{R})$ follows immediately using Theorem 2.4, and by Theorem 2.8 the proof is complete. □

We use the following result, which is Corollary A.6.5.5 of [Nik02].

Proposition 2.15. *Let Σ be a subset of $H^2(\mathbb{R})$ containing a non-zero function and let κ be the greatest common inner divisor of Σ. Then the closed linear span in $L^2(\mathbb{R})$ of $\bigcup_{s\geq 0} e^{isx}\Sigma$ is $\kappa H^2(\mathbb{R})$.*

2.2 Fourier transforms

We work on the Hilbert space $L^2(\mathbb{R})$. Let F denote the Fourier-Plancherel transform on this space, which is given by the continuous extension of

$$Ff(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) e^{-ixy}\, dy \tag{2.4}$$

from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. As is well-known, F is a unitary operator on $L^2(\mathbb{R})$ and $F^2 f(x) = f(-x)$, $F^3 = F^*$ and $F^4 = I$. Moreover, on the intersection $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, F coincides with the Fourier transform on $L^1(\mathbb{R})$. The Paley-Wiener theorem explicitly identifies the Hardy space $H^2(\mathbb{R})$ using the Fourier transform; for a proof, see [Hof62] or [Nik02].

Theorem 2.16 (Paley-Wiener theorem). $FH^2(\mathbb{R}) = L^2(\mathbb{R}_+)$.

For K a subset of $L^2(\mathbb{R})$, let us write $\overline{K}$ for the set of complex conjugates $\overline{f}$ of functions $f \in K$. From (2.4), it is easy to see that for $f \in L^2(\mathbb{R})$,

$$\overline{F^* f} = F\overline{f}, \quad \text{so} \quad \overline{f} = F^*\overline{F^* f}.$$

Since $F^2 = (F^*)^2$ satisfies $F^2 L^2(\mathbb{R}_\pm) = L^2(\mathbb{R}_\mp)$, by Theorem 2.16,

$$\begin{aligned} \overline{H^2(\mathbb{R})} &= F^*\overline{F^* H^2(\mathbb{R})} = F^*\overline{F^2 L^2(\mathbb{R}_+)} = F^* L^2(\mathbb{R}_-) \\ &= F^*\big(L^2(\mathbb{R}_+)^\perp\big) = \big(F^* L^2(\mathbb{R}_+)\big)^\perp = H^2(\mathbb{R})^\perp. \end{aligned}$$

The next two results are both proven in [Kat76].

Proposition 2.17. *A function $f \in L^1(\mathbb{R})$ has zero Fourier transform if and only if $f = 0$.*

Given a function $f \in L^1(\mathbb{T})$ and $n \in \mathbb{Z}$, the *nth Fourier coefficient of f* is the complex number

$$c_n(f) = \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} \, \frac{d\theta}{2\pi}.$$

Proposition 2.18. *A function $f \in L^1(\mathbb{T})$ satisfies $c_n(f) = 0$ for each $n \in \mathbb{Z}$ if and only if $f = 0$. Consequently, if $f \in L^1(\mathbb{T})$ satisfies $c_n(f) = 0$ for $n = \pm 1, \pm 2, \pm 3, \ldots$, then $f(e^{i\theta}) = c_0(f)$ for almost every $\theta \in \mathbb{T}$.*

Corollary 2.19. *Suppose that $u \in L^\infty(\mathbb{R})$ is a unimodular function such that $uH^2(\mathbb{R}) = H^2(\mathbb{R})$. Then there is a constant $\lambda \in \mathbb{C}$ such that $u(x) = \lambda$ for almost every $x \in \mathbb{R}$.*

Proof. We use the notation of Theorem 2.4. The inverse of the isomorphism $U_2 : H^2(\mathbb{T}) \to H^2(\mathbb{R})$ is

$$\begin{aligned} U_2^{-1}g(z) &= \pi^{1/2}(\omega(z)+i)g\big(\omega(z)\big) \\ &= 2i\pi^{1/2}(1-z)^{-1}g\big(\omega(z)\big), \quad z \in \mathbb{T}\setminus\{1\},\ g \in H^2(\mathbb{R}). \end{aligned}$$

Thus

$$H^2(\mathbb{T}) = U_2^{-1}\big(H^2(\mathbb{R})\big) = U_2^{-1}\big(uH^2(\mathbb{R})\big) = (u \circ \omega)H^2(\mathbb{T}).$$

In particular, since the constant function 1 lies in $H^2(\mathbb{T})$, the unimodular function $u \circ \omega$ lies in $H^2(\mathbb{T}) \cap L^\infty(\mathbb{T}) = H^\infty(\mathbb{T})$. Moreover, since $\overline{u} = 1/u$,

$$H^2(\mathbb{R}) = \overline{u}uH^2(\mathbb{R}) = \overline{u}H^2(\mathbb{R}),$$

so the unimodular functions u and $\overline{u}$ enter the argument symmetrically. Thus $\overline{u} \circ \omega$ must also lie in $H^\infty(\mathbb{T})$ and so

$$\int_0^{2\pi} e^{in\theta}(u \circ \omega)(e^{i\theta})\,\frac{d\theta}{2\pi} = 0, \quad n = \pm 1, \pm 2, \pm 3, \ldots.$$

By Proposition 2.18, $u \circ \omega$ must be equal to $c_0(u \circ \omega)$ almost everywhere on $\mathbb{T}$. Since ω has finite derivative on $\mathbb{T}\setminus\{1\}$, it maps null subsets of $\mathbb{T}\setminus\{1\}$ to null subsets of $\mathbb{R}$. Moreover, ω maps $\mathbb{T}\setminus\{1\}$ onto $\mathbb{R}$, so $u(x) = c_0(u \circ \omega)$ for almost every $x \in \mathbb{R}$. $\square$

2.3 Hilbert-Schmidt operators

Given a separable infinite-dimensional Hilbert space $\mathcal{H}$, we write $\mathcal{L}(\mathcal{H})$ for the set of bounded linear operators on $\mathcal{H}$. Let $\{e_n\}_{n\in\mathbf{N}}$ be an orthonormal basis for $\mathcal{H}$. The set $\mathcal{C}_2(\mathcal{H})$ of Hilbert-Schmidt operators on $\mathcal{H}$ is defined by

$$\mathcal{C}_2(\mathcal{H}) = \Big\{K \in \mathcal{L}(\mathcal{H}) \,\Big|\, \sum_{n\in\mathbf{N}} \|Ke_n\|^2 < \infty\Big\}.$$

This definition and the quantity

$$\|K\|_{\mathcal{C}_2(\mathcal{H})} = \Big(\sum_{n\in\mathbf{N}} \|Ke_n\|^2\Big)^{1/2}, \quad K \in \mathcal{C}_2(\mathcal{H})$$

are both independent of the choice of orthonormal basis $\{e_n\}_{n\in\mathbf{N}}$, and the map $K \mapsto \|K\|_{\mathcal{C}_2(\mathcal{H})}$ is a Banach space norm on $\mathcal{C}_2(\mathcal{H})$, called the *Hilbert-Schmidt norm.*

Let us write $\mathcal{C}_2$ for the set of Hilbert-Schmidt operators on $L^2(\mathbb{R})$. As shown in [Rin71], these operators can be explicitly identified as

$$\mathcal{C}_2 = \{\operatorname{Int} k \mid k \in L^2(\mathbb{R}^2)\}$$

where

$$(\operatorname{Int} k)f(x) = \int_{\mathbb{R}} k(x,y)f(y)\,dy, \quad f \in L^2(\mathbb{R}),\ x \in \mathbb{R}.$$

The Hilbert-Schmidt operators form a two-sided ideal of compact operators in $\mathcal{L}\big(L^2(\mathbb{R})\big)$ and the Hilbert-Schmidt norm $\|\ \|_{\mathcal{C}_2}$ satisfies

$$\|\operatorname{Int} k\|_{\mathcal{C}_2} = \|k\|_{L^2(\mathbb{R}^2)}, \quad k \in L^2(\mathbb{R}^2), \tag{2.5}$$

so the bijection $\mathcal{C}_2 \to L^2(\mathbb{R})$, $\operatorname{Int} k \mapsto k$ is isometric with respect to this norm. Moreover, the Hilbert-Schmidt norm dominates the operator norm on $\mathcal{C}_2$.

2.4 Weak operator topologies

Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space, let $T \in \mathcal{L}(\mathcal{H})$ and let $x, y \in \mathcal{H}$. Set

$$\mathcal{W}(T,x,y) = \{A \in \mathcal{L}(\mathcal{H}) \mid |\langle (T-A)x, y\rangle| < 1\},$$

$$\mathcal{S}(T,x) = \{A \in \mathcal{L}(\mathcal{H}) \mid \|(T-A)x\| < 1\}.$$

The *weak operator topology* and *strong operator topology* are the topologies on $\mathcal{L}(\mathcal{H})$ with bases

$$\{\bigcap_{i=1}^{n} \mathcal{W}(T_i, x_i, y_i) \mid n \in \mathbb{N},\ T_i \in \mathcal{L}(\mathcal{H}),\ x_i, y_i \in \mathcal{H}\} \quad \text{and}$$
$$\{\bigcap_{i=1}^{n} \mathcal{S}(T_i, x_i) \mid n \in \mathbb{N},\ T_i \in \mathcal{L}(\mathcal{H}),\ x_i \in \mathcal{H}\} \quad \text{respectively.}$$

Let $K \in \mathcal{L}(\mathcal{H})$. Then K is *trace class* if the operator $|K|^{1/2}$ is a Hilbert-Schmidt operator. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. If K is trace class then the sum

$$\operatorname{Tr} K = \sum_{n\in\mathbb{N}} \langle K e_n, e_n \rangle$$

converges absolutely and is independent of the choice of orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. We call $\operatorname{Tr} K$ the *trace* of K. The set $\mathcal{C}_1(\mathcal{H})$ of trace class operators is a two-sided ideal of compact operators in $\mathcal{L}(\mathcal{H})$ and the dual space of $\mathcal{C}_1(\mathcal{H})$ is seen to be $\mathcal{L}(\mathcal{H})$ when we identify $T \in \mathcal{L}(\mathcal{H})$ with the linear functional

$$\psi_T : \mathcal{C}_1(\mathcal{H}) \to \mathbb{C}, \quad K \mapsto \operatorname{Tr}(TK).$$

The *w^*-topology* on $\mathcal{L}(\mathcal{H})$ is the weak-star topology arising from this duality.

Parts (i)–(iii) of the next result may be found in [Dav96] and parts (iii) and (iv) are proven in [Con91].

Proposition 2.20. *(i). The norm topology on $\mathcal{L}(\mathcal{H})$ is stronger than the strong operator topology, which is stronger than the weak operator topology. The w^*-topology is stronger than the weak operator topology and it is weaker than the norm topology.*

(ii). Multiplication on the left by a fixed bounded linear operator is continuous in the strong and weak operator topologies, as is multiplication on the right. Moreover, multiplication is jointly strong operator topology continuous on bounded sets; that is, if $S_\alpha \to S$ and $T_\alpha \to T$ are strong operator

topology convergent bounded nets, then $S_\alpha T_\alpha \to ST$ in the strong operator topology.

(iii). The closures of a convex subset of $\mathcal{L}(\mathcal{H})$ in the strong and weak operator topologies coincide.

(iv). The weak operator topology and the w^*-topology agree on bounded subsets of $\mathcal{L}(\mathcal{H})$.

A *strong operator topology continuous one-parameter semigroup* of operators on a Hilbert space $\mathcal{H}$ is a set $\mathcal{S} = \{S_t \mid t \in \mathbb{R}_+\} \subseteq \mathcal{L}(\mathcal{H})$ such that the mapping S of the additive semigroup $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ into $\mathcal{L}(\mathcal{H})$ given by $S(t) = S_t$ is a strong operator topology continuous homomorphism. A *strong operator topology continuous one-parameter group* is a set $\mathcal{S} = \{S_t \mid t \in \mathbb{R}\} \subseteq \mathcal{L}(\mathcal{H})$ such that $\mathcal{S}_+ = \{S_t \mid t \in \mathbb{R}_+\}$ is a strong operator topology continuous one-parameter semigroup, and $S_{-t} = S_t^{-1}$ for $t \in \mathbb{R}$.

We will use the notation

$$T = \textsc{sot}\text{-}\lim T_\alpha$$

to indicate that T is the strong operator topology limit of the net T_α.

2.5 Subspace lattices

We adopt the convention that a *subspace* of a normed vector space is a closed linear manifold. Given a set $\mathcal{L}$ of subspaces of a Hilbert space $\mathcal{H}$, we write

$$\operatorname{Alg} \mathcal{L} = \{A \in \mathcal{L}(\mathcal{H}) \mid AK \subseteq K \text{ for every } K \in \mathcal{L}\}.$$

Dually, given a set of operators $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$, we define the set $\operatorname{Lat} \mathcal{A}$ by

$$\operatorname{Lat} \mathcal{A} = \{K \text{ a subspace of } \mathcal{H} \mid AK \subseteq K \text{ for every } A \in \mathcal{A}\}.$$

We give the set of subspaces of $\mathcal{H}$ the following natural lattice operations. Given two subspaces K and L, $K \vee L$ is their closed linear span and $K \wedge L$ is their intersection. It is easy to see that $\operatorname{Lat} \mathcal{A}$ is always closed under these operations, so it is a lattice. We call $\operatorname{Lat} \mathcal{A}$ the *invariant subspace lattice* of $\mathcal{A}$.

A *subspace lattice* is a sublattice of the lattice of subspaces of $\mathcal{H}$. When present, the top element of a subspace lattice is $\mathcal{H}$ and the bottom element is the zero subspace (0). A *proper subspace* of $\mathcal{H}$ is a subspace which does not lie in the trivial lattice $\{(0), L^2(\mathbb{R})\}$. Plainly, $\{(0), L^2(\mathbb{R})\}$ is always contained in $\operatorname{Lat} \mathcal{A}$.

An *operator algebra* $\mathcal{A}$ is a norm-closed subspace of $\mathcal{L}(\mathcal{H})$ such that whenever A_1 and A_2 are elements of $\mathcal{A}$, their product $A_1 A_2$ also lies in $\mathcal{A}$. For any subspace lattice $\mathcal{L}$, $\operatorname{Alg} \mathcal{L}$ is a weak operator topology closed operator algebra. For $\operatorname{Alg} \mathcal{L}$ is closed under taking products, since if A_1 and A_2 lie in $\operatorname{Alg} \mathcal{L}$, then for $K \in \mathcal{L}$,

$$A_1 A_2 K \subseteq A_1 K \subseteq K.$$

To see that $\operatorname{Alg} \mathcal{L}$ is weak operator topology closed, let $K \in \mathcal{L}$, $k \in K$, $j \in K^{\perp}$ and let A_α be a net in $\operatorname{Alg} \mathcal{L}$ converging in the weak operator topology to a bounded linear operator A. Then

$$0 = \lim_{\alpha} \langle A_\alpha k, j \rangle \to \langle Ak, j \rangle,$$

so $\langle Ak, j \rangle = 0$ and so $A \in \operatorname{Alg} \mathcal{L}$.

We will frequently adopt the convention of identifying a subspace K with $[K]$, the orthogonal projection onto K, and a subspace lattice $\mathcal{L}$ with

$$[\mathcal{L}] = \{[K] \mid K \in \mathcal{L}\},$$

the set of orthogonal projections onto subspaces in $\mathcal{L}$. Using this identification, $\mathcal{L}$ becomes a set of projections which we endow with the strong operator topology. An invariant subspace lattice $\operatorname{Lat}\mathcal{A}$ is always closed in this topology; for let $[K_\alpha]$ be a net of projections in $\operatorname{Lat}\mathcal{A}$ converging in the strong operator topology. Since the set of projections in $\mathcal{L}(\mathcal{H})$ is closed in the strong operator topology, $[K_\alpha] \to [K]$ strongly for some subspace K of $\mathcal{H}$. A subspace J lies in $\operatorname{Lat}\mathcal{A}$ if and only if $[J]A[J] = A[J]$ for every $A \in \mathcal{A}$. Fix $A \in \mathcal{A}$; then the nets $[K_\alpha]$ and $A[K_\alpha]$ are bounded and so by Proposition 2.20(ii), $[K_\alpha]A[K_\alpha] \to [K]A[K]$ and $A[K_\alpha] \to A[K]$ strongly. So $[K]A[K] = A[K]$ and $K \in \operatorname{Lat}\mathcal{A}$.

Following Halmos [Hal70], we say that an operator algebra $\mathcal{A}$ is *reflexive* if it takes the form $\mathcal{A} = \operatorname{Alg}\mathcal{L}$ for some subspace lattice $\mathcal{L}$. Equivalently, $\mathcal{A}$ is reflexive if $\mathcal{A} = \operatorname{Alg}\operatorname{Lat}\mathcal{A}$. Similarly, a subspace lattice $\mathcal{L}$ is reflexive when $\mathcal{L} = \operatorname{Lat}\operatorname{Alg}\mathcal{L}$ which happens precisely when $\mathcal{L} = \operatorname{Lat}\mathcal{A}$ for some operator algebra $\mathcal{A}$. The equivalence of these characterisations of reflexivity follows from the following assertions, which hold for arbitrary collections of bounded linear operators $\mathcal{A}$ and $\mathcal{A}'$ and arbitrary sets of subspaces $\mathcal{L}$ and $\mathcal{L}'$. Their proof is elementary.

$$\begin{array}{llll} \text{If} & \mathcal{A}' \subseteq \mathcal{A} & \text{then} & \operatorname{Lat}\mathcal{A}' \supseteq \operatorname{Lat}\mathcal{A}; \\ \text{if} & \mathcal{L}' \subseteq \mathcal{L} & \text{then} & \operatorname{Alg}\mathcal{L}' \supseteq \operatorname{Alg}\mathcal{L}; \\ \multicolumn{2}{l}{\mathcal{A} \subseteq \operatorname{Alg}\operatorname{Lat}\mathcal{A};} & \multicolumn{2}{l}{\mathcal{L} \subseteq \operatorname{Lat}\operatorname{Alg}\mathcal{L}.} \end{array}$$

The *reflexive closure* of a set $\mathcal{A}$ of bounded linear operators is the reflexive operator algebra $\operatorname{Alg}\operatorname{Lat}\mathcal{A}$. The *reflexive closure* of a set $\mathcal{L}$ of subspaces of $\mathcal{H}$ is the reflexive subspace lattice $\operatorname{Lat}\operatorname{Alg}\mathcal{L}$.

2.6 Tensor products

Let X and Y be σ-finite measure spaces. Given a subspace S of $L^2(X)$ and a subspace T of $L^2(Y)$, we write

$$S \otimes T = \bigvee \{s \otimes t : (x, y) \mapsto s(x)t(y) \mid s \in S,\ t \in T\}$$

where $\bigvee$ denotes the closed linear span in the function space $L^2(X \times Y)$. The function space $S \otimes T$ is the *tensor product* of S and T, and it is a Hilbert space in its own right under the inner product given by the continuous sesquilinear extension of

$$\langle s_1 \otimes t_1, s_2 \otimes t_2 \rangle_{S \otimes T} = \langle s_1, s_2 \rangle_S \cdot \langle t_1, t_2 \rangle_T$$

to all of $S \otimes T$.

Lemma 2.21. *Let S and T be subspaces of $L^2(\mathbb{R})$. Suppose that $f \in L^2(\mathbb{R}^2)$ satisfies the following conditions:*

$$\textit{for almost every } y \in \mathbb{R},\ x \mapsto f(x, y) \in S \quad \textit{and} \tag{2.6a}$$

$$\textit{for almost every } x \in \mathbb{R},\ y \mapsto f(x, y) \in T. \tag{2.6b}$$

Then $f \in S \otimes T$. Conversely, every $f \in S \otimes T$ satisfies conditions (2.6).

Remark. Elements of $L^2(X)$ are, strictly speaking, equivalence classes of functions which agree almost everywhere on X. To make explicit the correct interpretation of this lemma, let us temporarily write $\langle\!\langle f \rangle\!\rangle_{L^2(X)}$ for the equivalence class in $L^2(X)$ containing the function f. The desired meaning of (2.6a) is

$$\text{for each } g \in \langle\!\langle f \rangle\!\rangle_{L^2(\mathbb{R}^2)}, \text{ for almost every } y \in \mathbb{R},\ \langle\!\langle x \mapsto g(x, y) \rangle\!\rangle_{L^2(\mathbb{R})} \in S.$$

The rest of the statement and proof of this lemma must be similarly interpreted.

Proof. Let $(s_i)_{i\in I}$ and $(t_j)_{j\in J}$ be orthonormal bases for S and T respectively. Extend them to orthonormal bases $(s_i)_{i\in I'}$ and $(t_j)_{j\in J'}$ for $L^2(\mathbb{R})$ where I' and J' are index sets containing I and J respectively. A basis for $S\otimes T$ is

$$\{s_i \otimes t_j \mid i \in I, t \in J\},$$

and a basis for $L^2(\mathbb{R}^2) = L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ is

$$\{s_i \otimes t_j \mid i \in I', t \in J'\}.$$

Let K be the index set

$$K = (I' \times J') \setminus (I \times J).$$

Suppose $f \in L^2(\mathbb{R}^2)$ and that f satisfies (2.6). If $f \notin S \otimes T$ then there is a pair $(i_0, j_0) \in K$ such that $\langle f, s_{i_0} \otimes t_{j_0}\rangle \neq 0$. Without loss of generality, suppose $i_0 \in I' \setminus I$. By Fubini's theorem,

$$\begin{aligned}\langle f, s_{i_0} \otimes t_{j_0}\rangle &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\overline{s_{i_0}(x)t_{j_0}(y)}\,dx\,dy\\ &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(x,y)\overline{s_{i_0}}(x)\,dx\right)\overline{t_{j_0}}(y)\,dy,\end{aligned}$$

and since $\{S_i \mid i \in I' \setminus I\} \subseteq S^{\perp}$, this is zero by hypothesis. So $f \in S\otimes T$ as desired.

Conversely, suppose $f \in S \otimes T$. For $s \in S$, $t \in T$ and $(i,j) \in K$,

$$\langle s \otimes t, s_i \otimes t_j\rangle = \langle s, s_i\rangle\langle t, t_j\rangle = 0.$$

Since $S \otimes T$ is the closed linear span of such simple tensors $s \otimes t$,

$$\langle f, s_i \otimes t_j\rangle = 0 \quad \text{for all } (i,j) \in K. \tag{2.7}$$

For $y \in \mathbb{R}$, let $f_y : \mathbb{R} \to \mathbb{C}$ be the function $f_y(x) = f(x,y)$. Suppose that the set

$$N = \{y \in \mathbb{R} \mid f_y \notin S\}$$

has strictly positive measure. For each $y \in N$, we can find $i(y) \in I' \setminus I$ such that $\langle f_y, s_{i(y)} \rangle \neq 0$. Since N has positive measure and I' is countable, without loss of generality we may assume that $i(y) = i$ for every $y \in N$, for some $i \in I' \setminus I$. By (2.7), for any $g \in L^2(\mathbb{R})$,

$$0 = \langle f, s_i \otimes g \rangle = \int \langle f_y, s_i \rangle \overline{g(y)} \, dy.$$

Let $g \in L^2(\mathbb{R})$ be a function vanishing on a null set. By multiplying g by a suitable unimodular function we may ensure that $\langle f_y, s_i \rangle g(y) = |\langle f_y, s_i \rangle g(y)|$ for almost every y. Then

$$0 = \int \langle f_y, s_i \rangle \overline{g(y)} \, dy = \int_N |\langle f_y, s_i \rangle| g(y) \, dy > 0.$$

This contradiction shows that f must satisfy (2.6a), and by symmetry it must also satisfy (2.6b). $\square$

Chapter 3

The parabolic algebra

The parabolic algebra $\mathcal{A}_{\mathrm{p}}$ was introduced by Katavolos and Power in [KP97] as a binest algebra, the intersection of two nest algebras. In order to emphasise the parallels between $\mathcal{A}_{\mathrm{p}}$ and the hyperbolic algebra $\mathcal{A}_{\mathrm{h}}$, we give a different proof of its reflexivity to that found in [KP97]. We will do this by characterising the Hilbert-Schmidt operators in $\mathcal{A}_{\mathrm{p}}$. Although their existence is not immediately apparent, we also construct examples of non-zero operators in $\mathcal{A}_{\mathrm{p}}$ with non-trivial kernel.

We consider two strong operator topology continuous one-parameter groups $\{M_\lambda \mid \lambda \in \mathbb{R}\}$ and $\{D_\mu \mid \mu \in \mathbb{R}\}$ of unitary operators on $L^2(\mathbb{R})$, where for $f \in L^2(\mathbb{R})$,

$$\begin{aligned} M_\lambda f(x) &= e^{i\lambda x} f(x), \quad \lambda \in \mathbb{R} \quad \text{and} \\ D_\mu f(x) &= f(x-\mu), \quad \mu \in \mathbb{R}. \end{aligned}$$

We write χ_S for the indicator function of the Borel subset S of $\mathbb{R}^n$. Let $\operatorname{supp} f$ denote the essential support of the function $f : \mathbb{R}^n \to \mathbb{C}$; that is, $\operatorname{supp} f$ is the smallest closed subset K of $\mathbb{R}^n$ such that $f(x) = f(x)\chi_K(x)$ for almost every $x \in \mathbb{R}^n$. The existence of such a set K follows from Exercise 2.11 of [Rud66]. For $1 \leq p \leq \infty$, we write $L^p(S)$ for the subspace of functions $f \in L^p(\mathbb{R}^n)$ with $\operatorname{supp} f$ contained in S. We define two subspace lattices $\mathcal{N}_{\mathrm{v}}$ and $\mathcal{N}_{\mathrm{a}}$ by

$$\mathcal{N}_{\mathrm{v}} = \{D_\mu L^2(\mathbb{R}_+) \mid \mu \in \mathbb{R}\} \cup \{(0), L^2(\mathbb{R})\},$$
$$\mathcal{N}_{\mathrm{a}} = \{M_\lambda H^2(\mathbb{R}) \mid \lambda \in \mathbb{R}\} \cup \{(0), L^2(\mathbb{R})\}.$$

By the Paley-Wiener theorem,

$$FH^2(\mathbb{R}) = L^2(\mathbb{R}_+). \tag{3.1}$$

Moreover, for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, by (2.4),

$$\begin{aligned} FM_\lambda f(x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\lambda y} f(y) e^{-ixy}\, dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) e^{-iy(x-\lambda)}\, dy \\ &= D_\lambda F f(x). \end{aligned}$$

Since $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, we have $FM_\lambda = D_\lambda F$ and so

$$FM_\lambda F^* = D_\lambda, \quad \lambda \in \mathbb{R}. \tag{3.2}$$

Thus by (3.1),

$$FM_\lambda H^2(\mathbb{R}) = FM_\lambda F^* F H^2(\mathbb{R}) = D_\lambda L^2(\mathbb{R}_+). \tag{3.3}$$

In particular, $F\mathcal{N}_{\mathrm{a}} = \mathcal{N}_{\mathrm{v}}$.

The lattices $\mathcal{N}_{\mathrm{v}}$ and $\mathcal{N}_{\mathrm{a}}$ are totally ordered by inclusion and are closed under the lattice operations of taking intersections and closed linear spans. Such lattices are called *nests*, and the operator algebras that leave them invariant are called *nest algebras.* These algebras have a well-developed theory; see [Dav88] for a comprehensive treatment. Terminology from this theory gives rise to the name the *Volterra nest* for $\mathcal{N}_{\mathrm{v}}$. We call $\mathcal{N}_{\mathrm{a}}$ the *analytic nest.* We also write $\mathcal{A}_{\mathrm{v}} = \operatorname{Alg}\mathcal{N}_{\mathrm{v}}$ and $\mathcal{A}_{\mathrm{a}} = \operatorname{Alg}\mathcal{N}_{\mathrm{a}}$ for the *Volterra nest algebra* and the *analytic nest algebra* respectively.

Given a set $\mathcal{S}$ of bounded operators on a Hilbert space, let

$$\text{w}^*\text{-alg}\,\mathcal{S}$$

denote the w*-closed operator algebra generated by $\mathcal{S}$; this is the closure in the w*-topology of the algebra generated by $\mathcal{S}$.

Definition 3.1. The *parabolic algebra* $\mathcal{A}_{\mathrm{p}}$ is the w*-closed operator algebra

$$\mathcal{A}_{\mathrm{p}} = \text{w}^*\text{-alg}\{M_\lambda, D_\mu \mid \lambda, \mu \geq 0\}.$$

The *Fourier binest algebra* $\mathcal{A}_{\mathrm{FB}}$ is the operator algebra

$$\mathcal{A}_{\mathrm{FB}} = \operatorname{Alg}(\mathcal{N}_{\mathrm{a}} \cup \mathcal{N}_{\mathrm{v}}).$$

Plainly $\mathcal{A}_{\mathrm{FB}} = \mathcal{A}_{\mathrm{a}} \cap \mathcal{A}_{\mathrm{v}}$.

We call $\mathcal{A}_{\mathrm{p}}$ *parabolic* since the generators D_μ of $\mathcal{A}_{\mathrm{p}}$ are implemented by a parabolic action, translation, on the upper half-plane. The remaining generators M_λ are also parabolic in the sense that the Fourier transform implements a unitary equivalence between M_λ and D_λ by (3.2).

Given $g \in L^\infty(\mathbb{R})$, let M_g denote the multiplication operator

$$M_g f(x) = g(x) f(x), \quad f \in L^2(\mathbb{R}).$$

Then M_g is a bounded linear operator on $L^2(\mathbb{R})$. Moreover, $\mathcal{A}_{\mathrm{p}}$ contains every operator M_h for $h \in H^\infty(\mathbb{R})$ since the w*-closed linear span of the set $\{M_\lambda \mid \lambda \geq 0\}$ is equal to $\{M_h \mid h \in H^\infty(\mathbb{R})\}$.

A *binest* is the union of two nests. Since $F\mathcal{N}_{\mathrm{a}} = \mathcal{N}_{\mathrm{v}}$ we call the binest $\mathcal{N}_{\mathrm{a}} \cup \mathcal{N}_{\mathrm{v}}$ the *Fourier binest* and $\mathcal{A}_{\mathrm{FB}}$ the Fourier binest algebra.

The Fourier binest $\mathcal{N}_a \cup \mathcal{N}_v$ is not commutative. Indeed, any two of the subspaces

$$L^2(\mathbb{R}_+), \quad L^2(\mathbb{R}_+)^\perp = L^2(\mathbb{R}_-), \quad H^2(\mathbb{R}), \quad H^2(\mathbb{R})^\perp = \overline{H^2(\mathbb{R})}$$

intersect in the zero subspace by Corollary 2.6. So if $f \in L^2(\mathbb{R})$ satisfies

$$[H^2(\mathbb{R})][L^2(\mathbb{R}_+)]f = [L^2(\mathbb{R}_+)][H^2(\mathbb{R})]f \tag{3.4}$$

then $[H^2(\mathbb{R})][L^2(\mathbb{R}_+)]f \in L^2(\mathbb{R}_+) \cap H^2(\mathbb{R}) = (0)$, so $[L^2(\mathbb{R}_+)]f$ lies in the intersection $\overline{H^2(\mathbb{R})} \cap L^2(\mathbb{R}_+) = (0)$, so $f \in L^2(\mathbb{R}_-)$. By symmetry, f must also lie in $\overline{H^2(\mathbb{R})}$, so $f \in L^2(\mathbb{R}_-) \cap \overline{H^2(\mathbb{R})} = (0)$. So (3.4) is not satisfied for *any* non-zero $f \in L^2(\mathbb{R})$ and so the Fourier binest is not commutative. Thus the Fourier binest and any lattice containing it cannot be treated using the powerful methods developed for commutative subspace lattices such as those found in [Arv74].

3.1 Reflexivity

As in [KP97], our first task is to show that $\mathcal{A}_p = \mathcal{A}_{FB}$. Our treatment is different to that in [KP97]. We first observe that the inclusion

$$\mathcal{A}_p \subseteq \mathcal{A}_{FB} \tag{3.5}$$

is trivial since each of the generators of $\mathcal{A}_p$ leaves each subspace in the Fourier binest $\mathcal{N}_a \cup \mathcal{N}_v$ invariant. Therefore every operator in the weak operator topology closed algebra $\mathcal{A}$ generated by $\{M_\lambda, D_\lambda \mid \lambda, \mu \geq 0\}$ leaves every subspace in the Fourier binest invariant. The w*-topology is stronger than the weak operator topology, so

$$\mathcal{A}_p \subseteq \mathcal{A} \subseteq \mathcal{A}_{FB}.$$

To show that $\mathcal{A}_{\mathrm{p}} = \mathcal{A}_{\mathrm{FB}}$, we will explicitly describe the sets of Hilbert-Schmidt operators $\mathcal{A}_{\mathrm{p}} \cap \mathcal{C}_2$ and $\mathcal{A}_{\mathrm{FB}} \cap \mathcal{C}_2$ in each algebra and show that they are w*-dense in $\mathcal{A}_{\mathrm{p}}$ and $\mathcal{A}_{\mathrm{FB}}$ respectively.

Given $k \in L^2(\mathbb{R}^2)$, let $\Theta_{\mathrm{p}}(k) : \mathbb{R}^2 \to \mathbb{C}$ be the function

$$\Theta_{\mathrm{p}}(k)(x,t) = k(x, x-t), \quad x, t \in \mathbb{R}.$$

Lemma 3.2. *The mapping $\Theta_{\mathrm{p}} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, $k \mapsto \Theta_{\mathrm{p}}(k)$ is unitary.*

Proof. The map Θ_{p} is plainly linear $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$. It is easy to see that the inverse mapping Θ_{p}^{-1} is given by

$$\Theta_{\mathrm{p}}^{-1}(j)(x,y) = j(x, x-y), \quad x, y \in \mathbb{R}, \tag{3.6}$$

so Θ_{p} is bijective. Moreover,

$$\begin{aligned} \|\Theta_{\mathrm{p}}(k)\|^2_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}} \int_{\mathbb{R}} |k(x, x-t)|^2 \, dx \, dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} |k(x,y)|^2 \, dx \, dy \\ &= \|k\|^2_{L^2(\mathbb{R}^2)}. \end{aligned}$$

Hence the mapping Θ_{p} is a bijective isometry, so Θ_{p} is indeed unitary $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$. □

Let $\mathbb{H}^-_{\mathbb{Q}}$ denote the countable set of points in $\mathbb{H}^-$ with rational real and imaginary parts.

Lemma 3.3. *For each $t \in \mathbb{R}$, the countable set*

$$\Lambda_t = \{ b_v D_t b_w \mid v, w \in \mathbb{H}^-_{\mathbb{Q}} \}$$

has dense linear span in $H^2(\mathbb{R})$.

Proof. By Lemma 2.3, the set $B = \{b_v b_w \mid v, w \in \mathbb{H}^-\}$ has dense linear span in $H^2(\mathbb{R})$. Observe that for $t \in \mathbb{R}$,

$$D_t b_w(x) = \frac{1}{x - (w+t)} = b_{w+t}(x), \quad x \in \mathbb{R}.$$

As $v \to v_0$ in $\mathbb{H}^-$, the functions b_v tend to b_{v_0} pointwise and so in $L^2(\mathbb{R})$, by dominated convergence with dominating function $(2b_{v_0 - i\operatorname{Im} v_0/2})^2$; similarly, $b_v^2 \to b_{v_0}^2$ in $L^2(\mathbb{R})$. Since $(v-w)b_v b_w = b_v - b_w$, it follows that Λ_t is dense in B and so Λ_t also has dense linear span in $H^2(\mathbb{R})$. $\square$

Proposition 3.4. *Let $k \in L^2(\mathbb{R}^2)$.*

(i). If $\operatorname{Int} k \in \mathcal{A}_{\mathrm{a}}$*, then* $\Theta_{\mathrm{p}}(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R})$.

(ii). If $\operatorname{Int} k \in \mathcal{A}_{\mathrm{v}}$*, then* $\Theta_{\mathrm{p}}(k) \in L^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+)$.

In particular,

$$\mathcal{A}_{\mathrm{FB}} \cap \mathcal{C}_2 \subseteq \{\operatorname{Int} k \mid \Theta_{\mathrm{p}}(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+)\}. \tag{3.7}$$

Proof. (i). By Lemma 3.2, $\Theta_{\mathrm{p}}(k) \in L^2(\mathbb{R}^2)$, so $t \mapsto \Theta_{\mathrm{p}}(k)(x,t) \in L^2(\mathbb{R})$ for almost every $x \in \mathbb{R}$. Let $k \in L^2(\mathbb{R}^2)$ with $\operatorname{Int} k \in \mathcal{A}_{\mathrm{a}} \cap \mathcal{C}_2$. For every $\lambda \in \mathbb{R}$,

$$(\operatorname{Int} k) M_\lambda H^2(\mathbb{R}) \subseteq M_\lambda H^2(\mathbb{R}) = (M_\lambda \overline{H^2(\mathbb{R})})^\perp.$$

So if f and g are functions in $H^2(\mathbb{R})$, then

$$\begin{aligned}
0 &= \langle (\operatorname{Int} k) M_\lambda f, M_\lambda \overline{g} \rangle \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} k(x,y) e^{i\lambda y} f(y) e^{-i\lambda x} g(x)\, dy\, dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \Theta_{\mathrm{p}}(k)(x,t) e^{-i\lambda t} f(x-t) g(x)\, dx\, dt \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \Theta_{\mathrm{p}}(k)(x,t)\, g(x) D_t f(x)\, dx\, e^{-i\lambda t}\, dt.
\end{aligned} \tag{3.8}$$

Let $\Phi : \mathbb{R} \to \mathbb{C}$ be the function

$$\Phi(t) = \int_{\mathbb{R}} \Theta_{\mathrm{p}}(k)(x,t)\, g(x) D_t f(x)\, dx, \quad t \in \mathbb{R}.$$

Since $\operatorname{Int}|k| \in \mathcal{C}_2$ and $|f|, |g| \in L^2(\mathbb{R})$,

$$\begin{aligned}\int_{\mathbb{R}} |\Phi(t)|\, dt &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |\Theta_{\mathrm{p}}(k)(x,t)\, g(x) D_t f(x)|\, dx\, dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} |k(x,y) g(x) f(y)|\, dx\, dy \\ &= \langle (\operatorname{Int}|k|)|f|, |g|\rangle < \infty,\end{aligned}$$

so the function Φ is integrable. By (3.8),

$$\int_{\mathbb{R}} \Phi(t) e^{-i\lambda t} dt = 0 \qquad \text{for every } \lambda \text{ in } \mathbb{R},$$

so Φ has zero Fourier transform. Since Φ is in $L^1(\mathbb{R})$, it follows by Proposition 2.17 that $\Phi = 0$; that is, for almost every t,

$$\int_{\mathbb{R}} \Theta_{\mathrm{p}}(k)(x,t)\, g(x) D_t f(x)\, dx = 0. \tag{3.9}$$

Thus for every pair (f, g) with $f, g \in H^2(\mathbb{R})$, there is a conull set $T(f,g) \subseteq \mathbb{R}$ such that (3.9) holds for $t \in T(f,g)$. Let Γ be the set of pairs of $H^2(\mathbb{R})$ functions

$$\Gamma = \{(b_w, b_v) \mid v, w \in \mathbb{H}_{\mathbb{Q}}^-\}.$$

The set Γ is countable, so

$$T = \bigcap_{(f,g)\in\Gamma} T(f,g)$$

is conull and (3.9) holds for every $t \in T$ and every $(f,g) \in \Gamma$. Fix $t \in T$; then

$$\Lambda_t = \{g D_t f \mid (f,g) \in \Gamma\}$$

has dense linear span in $H^2(\mathbb{R})$ by Lemma 3.3. Moreover,

$$\int_{\mathbb{R}} \Theta_{\mathrm{p}}(k)(x,t) h(x)\, dx = 0 \qquad \text{for every } h \in \Lambda_t.$$

Thus for every $t \in T$,

$$x \mapsto \Theta_{\mathrm{p}}(k)(x,t) \in \left(\overline{H^2(\mathbb{R})}\right)^{\perp} = H^2(\mathbb{R}).$$

By Lemma 2.21 we must have $\Theta_{\mathrm{p}}(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R})$.

(ii). It is routine to show that the condition $\operatorname{Int} k \in \mathcal{A}_{\mathrm{v}}$ is equivalent to the essential support of k being contained in the closed lower half-plane bounded by the line $y = x$. Hence the essential support of $\Theta_{\mathrm{p}}(k)$ must be contained in the upper half-plane $\{(x, t) \mid t \geq 0\}$, and Lemma 3.2 shows that $\Theta_{\mathrm{p}}(k) \in L^2(\mathbb{R}^2)$. Since

$$\{j \in L^2(\mathbb{R}^2) \mid \operatorname{supp} j \subseteq \{(x,t) \mid t \geq 0\}\} = L^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+),$$

this completes the proof of (ii).

Finally, since $\mathcal{A}_{\mathrm{FB}} \cap \mathcal{C}_2 = (\mathcal{A}_{\mathrm{a}} \cap \mathcal{C}_2) \cap (\mathcal{A}_{\mathrm{v}} \cap \mathcal{C}_2)$, (i) and (ii) together establish (3.7). $\square$

We will show that the inclusion (3.7) is in fact equality. With the density of the Hilbert-Schmidt operators in each algebra, this will suffice to show that $\mathcal{A}_{\mathrm{p}} = \mathcal{A}_{\mathrm{FB}}$. We start with an approximation lemma with we will call upon repeatedly.

Lemma 3.5. *Let $\{S_t \mid t \in \mathbb{R}\}$ be a set of contractions on $L^2(\mathbb{R})$ such that the map $t \mapsto S_t$ from $\mathbb{R}$ to $\mathcal{L}(\mathcal{H})$ is strong operator topology continuous and let $\varphi \in L^1(\mathbb{R})$. Let*

$$\sigma_\varphi(f,g) = \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(t) S_t f(x) \overline{g(x)}\, dx\, dt, \qquad f, g \in L^2(\mathbb{R}).$$

Then σ_φ is a bounded sesquilinear form and there is a unique bounded linear operator T on $L^2(\mathbb{R})$ such that $\langle Tf, g\rangle = \sigma_\varphi(f,g)$ for every $f, g \in L^2(\mathbb{R})$. Moreover, $\|T\| \leq \|\varphi\|_{L^1(\mathbb{R})}$.

Suppose further that φ has compact essential support contained in $[a,b]$ for real numbers $a < b$. Then

$$T \in \text{w*-alg}\{S_t \mid t \in [a,b]\}.$$

Proof. The sesquilinear form σ_φ is bounded, since for f and g in $L^2(\mathbb{R})$,

$$\sigma_\varphi(f,g) = \int_{\mathbb{R}} \varphi(t) \int_{\mathbb{R}} S_t f(x)\overline{g(x)}\,dx\,dt = \int_{\mathbb{R}} \varphi(t)\langle S_t f, g\rangle\,dt,$$

so

$$|\sigma_\varphi(f,g)| \le \int_{\mathbb{R}} |\varphi(t)\langle S_t f, g\rangle|\,dt \le \|\varphi\|_{L^1(\mathbb{R})}\,\|f\|\,\|g\|.$$

So σ_φ defines a unique bounded linear operator T with $\|T\| \le \|\varphi\|_{L^1(\mathbb{R})}$ such that $\sigma_\varphi(f,g) = \langle Tf, g\rangle$ for every f and g in $L^2(\mathbb{R})$.

Suppose now that φ has compact essential support contained in $[a,b]$ for real numbers $a < b$. For integers n and m with $n \ge 1$ and $0 \le m < n$, let $\tau(m,n) = a + m(b-a)/n$,

$$\alpha_{m,n} = \int_{\tau(m,n)}^{\tau(m+1,n)} \varphi(s)\,ds \quad \text{and} \quad T_n = \sum_{m=0}^{n-1} \alpha_{m,n} S_{\tau(m,n)}.$$

We claim that the sequence $(T_n)_{n\ge1}$ is norm-bounded and converges in the weak operator topology to T. Let

$$\rho_n(t) = a + \frac{b-a}{n}\left\lfloor \frac{(t-a)n}{b-a} \right\rfloor, \qquad t \in [a,b)$$

so that $\rho_n(t)$ is the lower end of the interval $[\tau(m,n), \tau(m+1,n))$ in which t lies. For f and g in $L^2(\mathbb{R})$,

$$\begin{aligned}
\langle (T-T_n)f, g\rangle &= \int_{\mathbb{R}} \overline{g(x)}\Big(\int_{\mathbb{R}} \varphi(t) S_t f(x)\,dt - \sum_{m=0}^{n-1} \alpha_{m,n} S_{\tau(m,n)} f(x)\Big)\,dx \\
&= \int_{\mathbb{R}} \overline{g(x)} \sum_{m=0}^{n-1} \int_{\tau(m,n)}^{\tau(m+1,n)} \varphi(t)\big(S_t f(x) - S_{\tau(m,n)} f(x)\big)\,dt\,dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(t)\big(S_t f(x) - S_{\rho_n(t)} f(x)\big)\overline{g(x)}\,dt\,dx \\
&= \int_{\mathbb{R}} \varphi(t) \int_{\mathbb{R}} \big(S_t f(x) - S_{\rho_n(t)} f(x)\big)\overline{g(x)}\,dx\,dt \\
&= \int_{\mathbb{R}} \varphi(t)\big(\langle S_t f, g\rangle - \langle S_{\rho_n(t)} f, g\rangle\big)\,dt.
\end{aligned}$$

For any $t \in \mathbb{R}$, $|\langle S_t f, g\rangle| \le \|S_t f\|\,\|g\| \le \|f\|\,\|g\|$, and $S_{\rho_n(t)} \to S_t$ in the strong operator topology (and so also in the weak operator topology) as

$n \to \infty$. Since φ is integrable and each S_t is a contraction, it follows by dominated convergence with dominating function $2\|f\|\,\|g\|\varphi(t)$ that $T_n \to T$ in the weak operator topology as $n \to \infty$. For every n,

$$\|T_n\| \leq \|\varphi\|_{L^1(\mathbb{R})} < \infty,$$

so $(T_n)_{n\geq 1}$ is a norm-bounded sequence.

Let $\mathcal{S}_{[a,b]} = \text{w*-alg}\{S_t \mid t \in [a,b]\}$. The w*-topology and the weak operator topology agree on bounded sets, so $T_n \to T$ in the w*-topology as well. Since T_n is a finite sum of elements of $\mathcal{S}_{[a,b]}$, we have $T_n \in \mathcal{S}_{[a,b]}$ for every n. Hence $T \in \mathcal{S}_{[a,b]}$. □

Remark. Lemma 3.5 is a variant of a classical result (see Proposition 7.1.4 of [Ped79]) which gives a correspondence between strong operator topology continuous unitary-valued representations of $\mathbb{R}$ and certain representations of $L^1(\mathbb{R})$.

The map $t \mapsto D_t$ is plainly strongly continuous, so applying Lemma 3.5 to the unitary group $\{D_t \mid t \in \mathbb{R}\}$ and the semigroup $\{D_t \mid t \in \mathbb{R}_+\} \subseteq \mathcal{A}_\mathrm{p}$, we may define operators Δ_φ as follows.

Proposition 3.6. *Let $\varphi \in L^1(\mathbb{R}_+)$. Then the sesquilinear form*

$$\tau_\varphi(f,g) = \int_\mathbb{R}\int_\mathbb{R} \varphi(t) D_t f(x)\overline{g(x)}\,dx\,dt, \qquad f,g \in L^2(\mathbb{R})$$

is bounded, and there is a unique bounded linear operator Δ_φ such that $\langle \Delta_\varphi f, g\rangle = \tau_\varphi(f,g)$ for every f and g in $L^2(\mathbb{R})$. Moreover, $\|\Delta_\varphi\| \leq \|\varphi\|_{L^1(\mathbb{R})}$. If φ has compact essential support then $\Delta_\varphi \in \mathcal{A}_\mathrm{p}$. □

Proposition 3.7. *Let $h \in H^2(\mathbb{R})$ and $\varphi \in L^2(\mathbb{R}_+)$. Let $h \otimes \varphi$ denote the function $(x,t) \mapsto h(x)\varphi(t)$ and let $k = \Theta_\mathrm{p}^{-1}(h \otimes \varphi)$. Then* $\operatorname{Int} k \in \mathcal{A}_\mathrm{p} \cap \mathcal{C}_2$. *Moreover, if h is also in $H^\infty(\mathbb{R})$ and φ is also in $L^1(\mathbb{R})$, then* $\operatorname{Int} k = M_h \Delta_\varphi$.

Proof. By (2.5) and Lemma 3.2,

$$\|\operatorname{Int} k\|_{\mathcal{C}_2} = \|k\|_{L^2(\mathbb{R}^2)} = \|h \otimes \varphi\|_{L^2(\mathbb{R}^2)} = \|h\|_{L^2(\mathbb{R})}\|\varphi\|_{L^2(\mathbb{R})},$$

so if $h_n \to h$ in $L^2(\mathbb{R})$, $\varphi_n \to \varphi$ in $L^2(\mathbb{R})$ and $k_n = \Theta_{\mathrm{p}}^{-1}(h_n \otimes \varphi_n)$, then by Lemma 3.2 the operators $\operatorname{Int} k_n$ converge to $\operatorname{Int} k$ in Hilbert-Schmidt norm and so in operator norm. Since $H^2(\mathbb{R}) \cap H^\infty(\mathbb{R})$ is dense in $H^2(\mathbb{R})$ and $\mathcal{A}_{\mathrm{p}}$ is norm-closed, we may therefore assume that $h \in H^2(\mathbb{R}) \cap H^\infty(\mathbb{R})$. Moreover, we may assume that φ has compact support, since the sequence $\varphi_n = \varphi\chi_{[0,n]}$ converges to φ in $L^2(\mathbb{R})$ and each φ_n is compactly supported. Since the Cauchy-Schwarz inequality implies that every compactly supported function in $L^2(\mathbb{R})$ lies in $L^1(\mathbb{R}_+)$, the function φ lies in $L^1(\mathbb{R}_+)$.

If f and g are in $L^2(\mathbb{R})$, then by (3.6) and Proposition 3.6,

$$\begin{aligned}
\langle (\operatorname{Int} k)f, g\rangle &= \int_{\mathbb{R}}\int_{\mathbb{R}} \big(\Theta_{\mathrm{p}}^{-1}(h \otimes \varphi)\big)(x,y) f(y)\, dy\, \overline{g(x)}\, dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} h(x)\varphi(x-y) f(y)\overline{g(x)}\, dy\, dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} h(x)\varphi(t) D_t f(x)\overline{g(x)}\, dt\, dx \\
&= \langle M_h \Delta_\varphi f, g\rangle.
\end{aligned}$$

So $\operatorname{Int} k = M_h\Delta_\varphi$. Moreover, as remarked earlier,

$$\text{w}^*\text{-alg}\{M_\lambda \mid \lambda \geq 0\}$$

contains each operator M_h for $h \in H^\infty(\mathbb{R})$, so certainly $M_h \in \mathcal{A}_{\mathrm{p}}$. By Proposition 3.6, $\Delta_\varphi \in \mathcal{A}_{\mathrm{p}}$ so we also have $\operatorname{Int} k = M_h\Delta_\varphi \in \mathcal{A}_{\mathrm{p}}$. □

Proposition 3.8. *If* $\nu \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+)$ *then* $\operatorname{Int}\Theta_{\mathrm{p}}^{-1}(\nu) \in \mathcal{A}_{\mathrm{p}} \cap \mathcal{C}_2$. *So*

$$\{\operatorname{Int} k \mid \Theta_{\mathrm{p}}(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+)\} \subseteq \mathcal{A}_{\mathrm{p}} \cap \mathcal{C}_2.$$

Proof. If ν is a simple tensor of the form $\nu = h \otimes \varphi$ for some $h \in H^2(\mathbb{R})$ and $\varphi \in L^2(\mathbb{R}_+)$, then Proposition 3.7 shows that $\operatorname{Int}\Theta_{\mathrm{p}}^{-1}(\nu) \in \mathcal{A}_{\mathrm{p}} \cap \mathcal{C}_2$.

Any $\nu \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+)$ may be approximated in $L^2(\mathbb{R}^2)$ by finite sums of the form $\sum \lambda_n h_n \otimes \varphi_n$ where $\lambda_n \in \mathbb{C}$, $h_n \in H^2(\mathbb{R})$ and $\varphi_n \in L^2(\mathbb{R}_+)$. Thus by (2.5) and Lemma 3.2, $\operatorname{Int}\Theta_{\mathrm{p}}^{-1}(\nu)$ may be approximated in Hilbert-Schmidt norm by the corresponding sums

$$\operatorname{Int}\Theta_{\mathrm{p}}^{-1}\big(\sum \lambda_n h_n \otimes \varphi_n\big) = \sum \lambda_n \operatorname{Int}\Theta_{\mathrm{p}}^{-1}(h_n \otimes \varphi_n).$$

By Proposition 3.7, each summand is in $\mathcal{A}_{\mathrm{p}}$, the Hilbert-Schmidt norm dominates the operator norm and $\mathcal{A}_{\mathrm{p}}$ is norm-closed, so $\operatorname{Int}\Theta_{\mathrm{p}}^{-1}(\nu) \in \mathcal{A}_{\mathrm{p}}$.

Since Θ_{p} is a bijection by Lemma 3.2, the final assertion follows. □

Corollary 3.9. *The parabolic and Fourier binest algebras contain the same Hilbert-Schmidt operators. Explicitly,*

$$\mathcal{A}_{\mathrm{p}} \cap \mathcal{C}_2 = \mathcal{A}_{\mathrm{FB}} \cap \mathcal{C}_2 = \{\operatorname{Int} k \mid \Theta_{\mathrm{p}}(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+)\}.$$

Proof. As seen in (3.5), the inclusion $\mathcal{A}_{\mathrm{p}} \subseteq \mathcal{A}_{\mathrm{FB}}$ is clear. So by Proposition 3.4 and Proposition 3.8,

$$\mathcal{A}_{\mathrm{p}} \cap \mathcal{C}_2 \subseteq \mathcal{A}_{\mathrm{FB}} \cap \mathcal{C}_2 \subseteq \{\operatorname{Int} k \mid \Theta_{\mathrm{p}}(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+)\} \subseteq \mathcal{A}_{\mathrm{p}} \cap \mathcal{C}_2. \quad \square$$

Proposition 3.10. *The parabolic algebra contains a bounded approximate identity of Hilbert-Schmidt operators. In other words, there is a norm-bounded sequence $(X_n)_{n\geq 1}$ of operators in $\mathcal{A}_{\mathrm{p}} \cap \mathcal{C}_2$ such that $X_n \to I$ in the strong operator topology.*

Proof. For $n \in \mathbb{N}$, let

$$h_n(x) = \frac{ni}{x + ni}, \qquad \varphi_n(t) = n\chi_{[0,n^{-1}]}(t).$$

Then $h_n \in H^\infty(\mathbb{R}) \cap H^2(\mathbb{R})$ and φ_n lies in $L^2(\mathbb{R}_+) \cap L^1(\mathbb{R})$ and has compact support. Let $k_n = \Theta_{\mathrm{p}}^{-1}(h_n \otimes \varphi_n)$ and let $X_n = \operatorname{Int} k_n$. By Proposition 3.7, $X_n = M_{h_n}\Delta_{\varphi_n} \in \mathcal{A}_{\mathrm{p}} \cap \mathcal{C}_2$ for every n. Moreover,

$$\|X_n\| \leq \|M_{h_n}\|\,\|\Delta_{\varphi_n}\| \leq \|h_n\|_{L^\infty(\mathbb{R})}\,\|\varphi_n\|_{L^1(\mathbb{R})} \leq 1,$$

so $(X_n)_{n\geq 1}$ is a bounded sequence in $\mathcal{L}(L^2(\mathbb{R}))$. Since $h_n(x) \to 1$ uniformly on compact subsets of the real line, $M_{h_n} \to I$ in the strong operator topology. If we pick a continuous compactly supported function f, then the integral

$$I_n(x) = \int_{\mathbb{R}} \varphi_n(t) D_t f(x)\, dt$$

converges for every x to define a continuous compactly supported function I_n, which by Proposition 3.6 coincides with $\Delta_{\varphi_n} f$. Now

$$\|\Delta_{\varphi_n} f - f\|^2 = \int_{\mathbb{R}} \left| \int_0^{1/n} n D_t f(x)\, dt - f(x) \right|^2 dx$$

and

$$\left| \int_0^{1/n} n D_t f(x)\, dt - f(x) \right|^2 \leq 4\|f\|_\infty^2 \chi_S(x)$$

where S is the compact set

$$S = \{x + \tau \mid x \in \operatorname{supp} f,\ \tau \in [0,1]\}.$$

So by dominated convergence, $\Delta_{\varphi_n} f \to f$. Since such functions f are dense in $L^2(\mathbb{R})$, it follows that $\Delta_{\varphi_n} \to I$ boundedly in the strong operator topology. Multiplication is jointly strong operator topology continuous on bounded sets of operators, so $X_n = M_{h_n} \Delta_{\varphi_n} \to I$ in the strong operator topology as well. □

Corollary 3.11. *The Fourier binest algebra and the parabolic algebra are equal.*

Proof. Since by Corollary 3.9 the Hilbert-Schmidt operators in the Fourier binest algebra and the parabolic algebra coincide, the bounded approximate identity of Hilbert-Schmidt operators $(X_n)_{n\geq 1}$ of Proposition 3.10 is common to $\mathcal{A}_{\mathrm{p}}$ and $\mathcal{A}_{\mathrm{FB}}$. So if $T \in \mathcal{A}_{\mathrm{p}}$, then $T = \textsc{sot}\text{-}\lim(TX_n) \in \mathcal{A}_{\mathrm{FB}}$ since $TX_n \in \mathcal{A}_{\mathrm{p}} \cap \mathcal{C}_2 = \mathcal{A}_{\mathrm{FB}} \cap \mathcal{C}_2$ and $\mathcal{A}_{\mathrm{FB}}$ is closed in the strong operator topology. Thus $\mathcal{A}_{\mathrm{FB}} \subseteq \mathcal{A}_{\mathrm{p}}$ and since we already have the reverse inclusion (3.5) the proof is complete. □

Recall that an operator algebra $\mathcal{A}$ is reflexive precisely if it has the form $\mathcal{A} = \operatorname{Lat}\mathcal{L}$ for some subspace lattice $\mathcal{L}$. In particular, $\mathcal{A}_{\text{FB}}$ is reflexive so we have also proven:

Theorem 3.12. *The parabolic algebra is reflexive.* □

We will henceforth write $\mathcal{A}_{\text{p}}$ in preference to $\mathcal{A}_{\text{FB}}$.

3.2 Non-injective operators in $\mathcal{A}_{\text{p}}$

We end this chapter with some examples of operators in $\mathcal{A}_{\text{p}}$ with non-trivial kernel. Our motivation lies in the algebraic question of whether or not $\mathcal{A}_{\text{p}}$ is an integral domain; that is, whether $ST = 0$ with S and T in $\mathcal{A}_{\text{p}}$ necessarily implies that $S = 0$ or $T = 0$.

Conjecture 3.13. *The parabolic algebra $\mathcal{A}_{\text{p}}$ is an integral domain.*

An easy sufficient condition for an operator algebra $\mathcal{A}$ to be an integral domain is for every non-zero operator in $\mathcal{A}$ to have trivial kernel. Finite rank operators are plentiful in nest algebras and they are clearly not injective, but as observed in [KP97], the parabolic algebra contains no finite-rank operators. Nevertheless, the following constructions show that $\mathcal{A}_{\text{p}}$ does indeed contain non-injective operators, and we must look elsewhere to resolve the conjecture.

Proposition 3.14. *There is a non-zero operator $T \in \mathcal{A}_{\text{p}}$ and a non-zero function $f \in H^2(\mathbb{R})$ such that $Tf = 0$.*

Proof. Let $\mathbb{T}$ be the circle

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$$

and let g be the piecewise constant function $\mathbb{T} \to \mathbb{C}$ given by

$$g(e^{i\theta}) = \begin{cases} 1/2 & 0 \leq \theta < \pi, \\ 3/2 & \pi \leq \theta < 2\pi, \end{cases} \qquad \theta \in [0, 2\pi).$$

For z in the open unit disk $\mathbb{D}$, let $[g]$ be the function

$$[g](z) = \exp \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |g(e^{i\theta})| \frac{d\theta}{2\pi}.$$

Since $\log |g| \in L^1(\mathbb{T})$, by Theorem 2.5 there is no problem making this definition, and $[g]$ is an outer function in $H^\infty(\mathbb{D})$. Moreover, $|[g]| = |g|$ almost everywhere on $\mathbb{T}$.

We now transfer this function to the upper half-plane. Let

$$h(z) = [g]\left(\frac{z - i}{z + i}\right), \qquad \operatorname{Im} z > 0.$$

By Theorem 2.4, the boundary value function of h, which we also call h, lies in $H^\infty(\mathbb{R})$. Observe that

$$|h(x)| = \begin{cases} 1/2 & x < 0, \\ 3/2 & x > 0. \end{cases}$$

Define $f \in L^2(\mathbb{R})$ as follows. For $x \in [0, 1)$, set $f(x) = 1$ and for $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, let

$$f(x + n) = \prod_{j=1}^{n} h(x + j)^{-1}, \quad f(x - n) = \prod_{j=0}^{n-1} h(x - j), \quad x \in [0, 1).$$

Then $f \in L^2(\mathbb{R})$, since

$$\begin{aligned} \int_{\mathbb{R}} |f(x)|^2 \, dx &= 1 + \sum_{n \in \mathbf{N}} \int_0^1 |f(x + n)|^2 \, dx + \sum_{n \in \mathbf{N}} \int_0^1 |f(x - n)|^2 \, dx \\ &= 1 + \sum_{n \in \mathbf{N}} (2/3)^{2n} + \sum_{n \in \mathbf{N}} (1/2)^{2(n-1)} < \infty. \end{aligned}$$

Since $h \in H^\infty(\mathbb{R})$, the operator $T = M_h - D_1 \in \mathcal{A}_{\mathrm{p}}$. Moreover, the definition of the non-zero function f ensures that it is in $\ker T$. For if

$x \in [0, 1)$ and $n \in \mathbb{N}$, then

$$Tf(x) = h(x)f(x) - f(x-1) = h(x).1 - h(x) = 0;$$

$$\begin{aligned} Tf(x+n) &= h(x+n)f(x+n) - f(x+n-1) \\ &= h(x+n)\prod_{j=1}^{n} h(x+j)^{-1} - \prod_{j=1}^{n-1} h(x+j)^{-1} = 0; \text{ and} \end{aligned}$$

$$\begin{aligned} Tf(x-n) &= h(x-n)f(x-n) - f\big(x-(n+1)\big) \\ &= h(x-n)\prod_{j=0}^{n-1} h(x-j) - \prod_{j=0}^{n} h(x-j) = 0. \end{aligned}$$

So $M_h - D_1$ does indeed have non-trivial kernel. □

For every invariant subspace $K \in \mathcal{L}_{\mathrm{FB}}$, consider the set

$$\mathcal{A}_{\mathrm{p}}|K = \{S|K \mid S \in \mathcal{A}_{\mathrm{p}}\}$$

of operators in $\mathcal{A}_{\mathrm{p}}$ restricted to K. This is an operator algebra and it is of interest to exhibit non-injective operators in these algebras.

Proposition 3.15. *For each non-trivial subspace K in $\mathcal{N}_{\mathrm{a}} \cup \mathcal{N}_{\mathrm{v}}$ there is a non-zero operator $T \in \mathcal{A}_{\mathrm{p}} \cap \mathcal{C}_2$ and a non-zero function $f \in K$ such that $Tf = 0$.*

Proof. We first treat the case $K = L^2(\mathbb{R}_+)$. Let $\chi : \mathbb{R} \to \mathbb{C}$ be the indicator function of the unit interval $[0, 1]$ and let

$$h_1(x) = (x+i)^{-2}, \qquad h_2(x) = -xh_1(x)/2,$$

$$\varphi_1(x) = x\chi(x), \quad \varphi_2 = \chi \quad \text{and} \quad f = \chi.$$

Then $h_1, h_2 \in H^2(\mathbb{R})$ and $f, \varphi_1, \varphi_2 \in L^2(\mathbb{R}_+)$. Let $j : \mathbb{R}^2 \to \mathbb{C}$ be the function

$$j = h_1 \otimes \varphi_1 + h_2 \otimes \varphi_2$$

and let $k = \Theta_{\mathrm{p}}^{-1}(j)$. By Corollary 3.9, $\operatorname{Int} k \in \mathcal{A}_{\mathrm{p}} \cap \mathcal{C}_2$.

Let $T = \operatorname{Int} k$; we claim that $Tf = 0$. Let

$$
\begin{aligned}
I(x) &= [0,1] \cap [x-1, x] = \{y \in [0,1] \mid x - y \in [0,1]\} \\
&= \begin{cases} [0,x] & x \in [0,1], \\ [x-1,1] & x \in [1,2], \\ \emptyset & x < 0 \text{ or } x > 2. \end{cases}
\end{aligned}
$$

Then

$$
\begin{aligned}
Tf(x) &= \int_{\mathbb{R}} k(x,y)\chi(y)\,dy \\
&= \int_{[0,1]} k(x,y)\,dy \\
&= \int_{[0,1]} h_1(x)\varphi_1(x-y) + h_2(x)\varphi_2(x-y)\,dy \\
&= (x+i)^{-2} \int_{[0,1]} (x-y)\chi(x-y) - x\chi(x-y)/2\,dy \\
&= \tfrac{1}{2}(x+i)^{-2} \int_{[0,1]} (x-2y)\,\chi(x-y)\,dy \\
&= \tfrac{1}{2}(x+i)^{-2} \int_{I(x)} x - 2y\,dy \\
&= \tfrac{1}{2}(x+i)^{-2} \big[y(x-y)\big]_{\partial I(x)}.
\end{aligned}
$$

Hence

$$
Tf(x) = \begin{cases} \frac{1}{2}(x+i)^{-2}[y(x-y)]_0^x = 0 & x \in [0,1], \\ \frac{1}{2}(x+i)^{-2}[y(x-y)]_{x-1}^1 = 0 & x \in [1,2], \\ 0 & x < 0 \text{ or } x > 2. \end{cases} \tag{3.10}
$$

If K is any nontrivial subspace in $\mathcal{N}_{\mathrm{v}}$ then $K = D_\mu L^2(\mathbb{R}_+)$ for some $\mu \in \mathbb{R}$, and $(D_\mu T D_{-\mu}) D_\mu f = D_\mu T f = 0$. Since D_μ leaves $\mathcal{N}_{\mathrm{a}} \cup \mathcal{N}_{\mathrm{v}}$ invariant and $\mathcal{A}_{\mathrm{p}} = \operatorname{Alg}(\mathcal{N}_{\mathrm{a}} \cup \mathcal{N}_{\mathrm{v}})$, the operator $D_\mu T D_{-\mu}$ lies in $\mathcal{A}_{\mathrm{p}} \cap \mathcal{C}_2$ and plainly $D_\mu f$ is a non-zero function in K.

For the case $K = H^2(\mathbb{R})$, consider the algebra

$$
\mathcal{A} = \operatorname{Alg}(\mathcal{N}_{\mathrm{v}} \cup \mathcal{N}_{\mathrm{a}}^{\perp}) = \operatorname{Alg} F(\mathcal{N}_{\mathrm{a}} \cup \mathcal{N}_{\mathrm{v}}) = F \mathcal{A}_{\mathrm{p}} F^*.
$$

A parallel argument to that of §3.1 shows that

$$\mathcal{A} \cap \mathcal{C}_2 = \{\operatorname{Int} k \mid \Theta_{\mathrm{p}}(k) \in \overline{H^2(\mathbb{R})} \otimes L^2(\mathbb{R}_+)\}.$$

Let

$$h_1(x) = (x-i)^{-2}, \qquad h_2(x) = -x h_1(x)/2,$$

$$\varphi_1(x) = x\chi(x), \quad \varphi_2 = \chi \quad \text{and} \quad f = \chi.$$

Then $h_1, h_2 \in \overline{H^2(\mathbb{R})}$ and $f, \varphi_1, \varphi_2 \in L^2(\mathbb{R}_+)$. Let $j : \mathbb{R}^2 \to \mathbb{C}$ be the function

$$j = h_1 \otimes \varphi_1 + h_2 \otimes \varphi_2,$$

let $k = \Theta_{\mathrm{p}}^{-1}(j)$ and let $T = \operatorname{Int} k \neq 0$. Then $T \in \mathcal{A} \cap \mathcal{C}_2$ and

$$Tf(x) = \tfrac{1}{2}(x-i)^{-2}\big[y(x-y)\big]_{\partial I(x)} = 0$$

just as in (3.10). Let $\tilde{T}$ be the non-zero operator $\tilde{T} = F^*TF \in \mathcal{A}_{\mathrm{p}} \cap \mathcal{C}_2$ and $\tilde{f} = F^*f$. Then $\tilde{T}\tilde{f} = F^*Tf = 0$ and $\tilde{f}$ is a non-zero function in $H^2(\mathbb{R})$.

Finally, if K is any nontrivial subspace in $\mathcal{N}_{\mathrm{a}}$ then $K = M_\lambda H^2(\mathbb{R})$ for some $\lambda \in \mathbb{R}$ and $M_\lambda \mathcal{A}_{\mathrm{p}} M_{-\lambda} \subseteq \mathcal{A}_{\mathrm{p}}$. Since $(M_\lambda \tilde{T} M_{-\lambda})(M_\lambda \tilde{f}) = M_\lambda \tilde{T}\tilde{f} = 0$, we see that $M_\lambda \tilde{f} \in K$ is non-zero and lies in the kernel of the operator $M_\lambda \tilde{T} M_{-\lambda}$, which lies in $\mathcal{A}_{\mathrm{p}} \cap \mathcal{C}_2$. □

Chapter 4

The Fourier-Plancherel sphere

In [KP97], the invariant subspace lattice of the parabolic algebra was shown to be a union of nests, any pair of which have trivial intersection. We can extend $\operatorname{Lat} \mathcal{A}_{\mathrm{p}}$ in a natural way to give a larger union of nests, the *Fourier-Plancherel sphere*. Since the parabolic algebra is a binest algebra, it is natural to ask which operator algebras leave other binests invariant. In this chapter we answer this question for all binests taken from the Fourier-Plancherel sphere. We show that in the non-degenerate cases, the algebra of operators leaving such a binest invariant is unitarily equivalent to $\mathcal{A}_{\mathrm{p}}$. We also tackle the related problem of finding the unitary automorphism group of the Fourier-Plancherel sphere, and show that this group is isomorphic to a semidirect product of $\mathbb{R}^2$ and $SL_2(\mathbb{R})$.

4.1 Other binest algebras

We begin with some facts proven in Theorems 3.2 and 4.6 of [KP97].

Theorem 4.1. *Let* $\mathcal{L}_{\mathrm{FB}} = \operatorname{Lat}\mathcal{A}_{\mathrm{p}}$ *be the reflexive closure of the Fourier binest* $\mathcal{N}_{\mathrm{a}} \cup \mathcal{N}_{\mathrm{v}}$. *Then*

(i). $\mathcal{L}_{\mathrm{FB}}$ *decomposes as the union of nests*

$$\mathcal{L}_{\mathrm{FB}} = \mathcal{N}_{\mathrm{a}} \cup \mathcal{N}_{\mathrm{v}} \cup \bigcup_{s>0} \mathcal{N}_s$$

where $\mathcal{N}_s$ *is the nest* $M_{\phi_s}\mathcal{N}_{\mathrm{a}}$ *and* $\phi_s(x) = e^{-isx^2/2}$. *Any pair of distinct nests in this union intersect in* $\{(0), L^2(\mathbb{R})\}$ *only.*

(ii). The supremum and infimum of any two proper subspaces in distinct nests contained in $\mathcal{L}_{\mathrm{FB}}$ *are* $L^2(\mathbb{R})$ *and the zero subspace respectively.*

(iii). $\mathcal{L}_{\mathrm{FB}}$ *is a reflexive subspace lattice.*

(iv). If we view $\mathcal{L}_{\mathrm{FB}}$ *as a set of projections with the strong operator topology, then* $\mathcal{L}_{\mathrm{FB}}$ *is homeomorphic to the closed unit disk. In particular, it is compact. The topological boundary of* $\mathcal{L}_{\mathrm{FB}}$ *is the Fourier binest* $\mathcal{N}_{\mathrm{a}} \cup \mathcal{N}_{\mathrm{v}}$. *Moreover, the map* $\eta : [0,1] \to \mathcal{L}_{\mathrm{FB}}$,

$$\eta(t) = \begin{cases} [\phi_{-\log t} H^2(\mathbb{R})] & t \in (0,1], \\ [L^2(\mathbb{R}_+)] & t = 0 \end{cases}$$

is a homeomorphism onto its range.

We remark only that the identification of $\mathcal{L}_{\mathrm{FB}}$ uses the Beurling-Lax theorem and a cocycle argument; the topological assertions follow through careful examination of various strong operator topology limits. Figure 4.1 illustrates the identification of $\mathcal{L}_{\mathrm{FB}}$ with a closed disk.

For S an invertible operator in $\mathcal{L}(L^2(\mathbb{R}))$, let us define the mapping $\operatorname{Ad} S : \mathcal{L}(L^2(\mathbb{R})) \to \mathcal{L}(L^2(\mathbb{R}))$ by

$$(\operatorname{Ad} S)T = STS^{-1}, \quad T \in \mathcal{L}(L^2(\mathbb{R})).$$

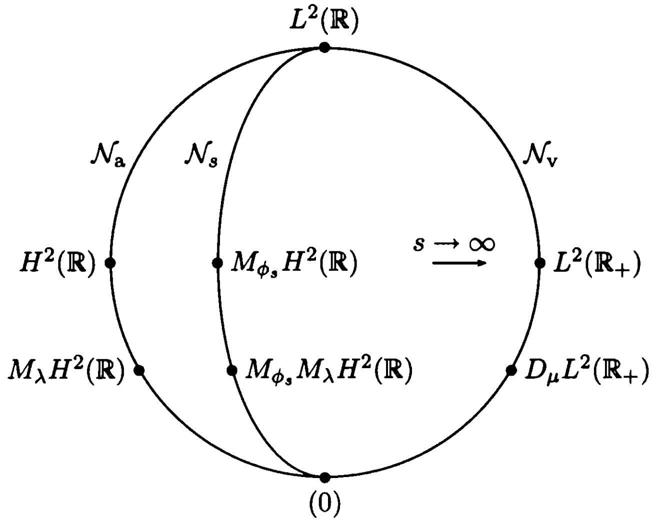


Figure 4.1: The lattice $\mathcal{L}_{\text{FB}}$. The parameters λ and μ corresponding to the marked subspaces are both positive.

If U is unitary, then $\operatorname{Ad} U$ is an automorphism of $\mathcal{L}(L^2(\mathbb{R}))$. Moreover, it is easy to see that for any subspace lattice $\mathcal{L}$ and any operator algebra $\mathcal{A}$,

$$(\operatorname{Ad} U) \operatorname{Alg} \mathcal{L} = \operatorname{Alg}(U\mathcal{L}) \quad \text{and} \tag{4.1}$$

$$U \operatorname{Lat} \mathcal{A} = \operatorname{Lat}\big((\operatorname{Ad} U)\mathcal{A}\big). \tag{4.2}$$

We now show how $\mathcal{L}_{\text{FB}}$ is embedded in a larger subspace lattice which was introduced in [KP02] as the *Fourier-Plancherel sphere* $\widehat{\mathcal{L}}_{\text{FB}}$.

One way in which $\widehat{\mathcal{L}}_{\text{FB}}$ arises is by considering some "companion algebras" obtained from $\mathcal{A}_{\text{p}}$ by a change of coordinates. Given a set $\mathcal{L}$ of subspaces, let $\mathcal{L}^\perp = \{K^\perp \mid K \in \mathcal{L}\}$. Since $\mathcal{A}_{\text{p}} = \operatorname{Alg}(\mathcal{N}_{\text{a}} \cup \mathcal{N}_{\text{v}})$, it is natural to examine the algebras

$$\mathcal{A}_1 = \operatorname{Alg}(\mathcal{N}_{\text{v}} \cup \mathcal{N}_{\text{a}}^\perp), \quad \mathcal{A}_2 = \operatorname{Alg}(\mathcal{N}_{\text{a}}^\perp \cup \mathcal{N}_{\text{v}}^\perp), \quad \mathcal{A}_3 = \operatorname{Alg}(\mathcal{N}_{\text{v}}^\perp \cup \mathcal{N}_{\text{a}}).$$

We also write $\mathcal{A}_0 = \mathcal{A}_{\text{p}}$. By (3.3), the Fourier transform F maps

$$\mathcal{N}_{\text{a}} \mapsto \mathcal{N}_{\text{v}} \mapsto \mathcal{N}_{\text{a}}^\perp \mapsto \mathcal{N}_{\text{v}}^\perp \mapsto \mathcal{N}_{\text{a}}, \tag{4.3}$$

so

$$\mathcal{A}_j = \operatorname{Alg}\big(F^j(\mathcal{N}_{\mathrm{a}} \cup \mathcal{N}_{\mathrm{v}})\big), \quad j = 0, 1, 2, 3.$$

By (4.1),

$$\mathcal{A}_j = F^j \mathcal{A}_{\mathrm{p}} F^{-j}, \quad j = 0, 1, 2, 3$$

and by (4.2),

$$\mathcal{L}_j = \operatorname{Lat} \mathcal{A}_j = F^j \operatorname{Lat} \mathcal{A}_{\mathrm{p}}, \quad j = 0, 1, 2, 3.$$

We now define the *Fourier-Plancherel sphere* $\widehat{\mathcal{L}}_{\mathrm{FB}}$ by

$$\widehat{\mathcal{L}}_{\mathrm{FB}} = \bigcup_{j=0}^{3} \mathcal{L}_j = \bigcup_{j=0}^{3} F^j \operatorname{Lat} \mathcal{A}_{\mathrm{p}}.$$

The next result is stated in [KP02].

Proposition 4.2. *With the strong operator topology, the Fourier-Plancherel sphere is homeomorphic to a sphere in* $\mathbb{R}^3$. *Moreover, the Fourier transform acts on* $\widehat{\mathcal{L}}_{\mathrm{FB}}$ *in the following manner. For* $s > 0$ *and* $\lambda \in \mathbb{R}$,

$$\begin{aligned} M_\lambda M_{\varphi_s} H^2(\mathbb{R}) \mapsto M_{-\lambda/s} M_{\varphi_{-1/s}} \overline{H^2(\mathbb{R})} \mapsto M_{-\lambda} M_{\varphi_s} \overline{H^2(\mathbb{R})} \\ \mapsto M_{\lambda/s} M_{\varphi_{-1/s}} H^2(\mathbb{R}) \mapsto M_\lambda M_{\varphi_s} H^2(\mathbb{R}), \end{aligned}$$

and for $\lambda \in \mathbb{R}$,

$$M_\lambda H^2(\mathbb{R}) \mapsto D_\lambda L^2(\mathbb{R}_+) \mapsto M_{-\lambda} \overline{H^2(\mathbb{R})} \mapsto D_{-\lambda} L^2(\mathbb{R}_-) \mapsto M_\lambda H^2(\mathbb{R}).$$

Thus $\mathcal{N}_{\mathrm{v}}^\perp = F^2 \mathcal{N}_{\mathrm{v}}$ *and* $\mathcal{N}_s^\perp = F^2 \mathcal{N}_s$ *for* $s \in \mathbb{R}$, *and*

$$\widehat{\mathcal{L}}_{\mathrm{FB}} = \mathcal{N}_{\mathrm{v}} \cup \mathcal{N}_{\mathrm{v}}^\perp \cup \bigcup_{s \in \mathbb{R}} (\mathcal{N}_s \cup \mathcal{N}_s^\perp). \tag{4.4}$$

Finally, for $s > 0$, *the Fourier transform* F *acts on the nests* $\mathcal{N}_s$ *as follows:*

$$\mathcal{N}_s \mapsto \mathcal{N}_{-1/s}^\perp \mapsto \mathcal{N}_s^\perp \mapsto \mathcal{N}_{-1/s} \mapsto \mathcal{N}_s.$$

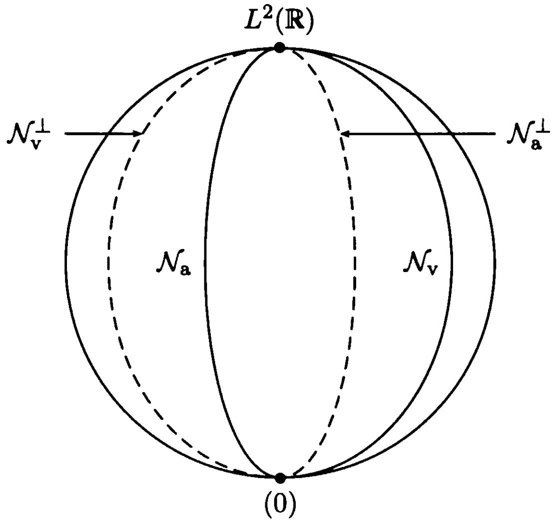


Figure 4.2: The lattice $\widehat{\mathcal{L}}_{\text{FB}}$

Proof. Since F and F^* are certainly continuous in the strong operator topology, F restricts to give homeomorphisms

$$\mathcal{L}_0 \mapsto \mathcal{L}_1 \mapsto \mathcal{L}_2 \mapsto \mathcal{L}_3 \mapsto \mathcal{L}_0$$

so these lattices are all homeomorphic to a closed disk by Theorem 4.1(iv). From (4.3) it follows that $\mathcal{L}_j$ and $F\mathcal{L}_j$ have one boundary nest in common and that the union $\widehat{\mathcal{L}}_{\text{FB}}$ has the topology of a sphere in $\mathbb{R}^3$.

The remaining assertions follow rapidly from a routine calculation using (3.2) and Lemma 4.2 of [KP97], which states that for $s > 0$,

$$FM_{\phi_s}H^2(\mathbb{R}) = M_{\phi_{-1/s}}\overline{H^2(\mathbb{R})} \quad \text{and} \quad FM_{\phi_{-s}}H^2(\mathbb{R}) = M_{\phi_{1/s}}H^2(\mathbb{R}). \quad \square$$

Remark. From (4.3) and Proposition 4.2, we see that F acts as a rotation on the set of nests in $\widehat{\mathcal{L}}_{\text{FB}}$.

We now identify all binest algebras obtained from binests in the lattice $\widehat{\mathcal{L}}_{\text{FB}}$. We say that a binest which is the union of two distinct nests $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$ is *degenerate* if $\mathcal{N}^{(2)} = (\mathcal{N}^{(1)})^\perp$, and that it is *non-degenerate* otherwise.

Lemma 4.3. *Suppose that $A \in \mathcal{L}(L^2(\mathbb{R}))$ commutes with every projection $[L^2([\mu,\infty))]$ for $\mu \in \mathbb{R}$. Then A commutes with M_g for $g \in L^\infty(\mathbb{R})$.*

Proof. Recall that the Banach space dual of $L^1(\mathbb{R})$ is $L^\infty(\mathbb{R})$, where we identify $g \in L^\infty(\mathbb{R})$ with the functional $\psi_g \in (L^1(\mathbb{R}))^*$,

$$\psi_g(f) = \int_{\mathbb{R}} f(x)\overline{g(x)}\,dx, \quad f \in L^1(\mathbb{R}).$$

We claim that the weak operator topology on $\{M_g \mid g \in L^\infty(\mathbb{R})\}$ coincides with the weak-star topology on $L^\infty(\mathbb{R})$ with respect to this duality, where we identify $g \in L^\infty(\mathbb{R})$ with M_g. For a net g_α in $L^\infty(\mathbb{R})$ converges weak-star to $g \in L^\infty(\mathbb{R})$ if and only if

$$\psi_{g_\alpha}(f) \to \psi_g(f) \text{ for every } f \in L^1(\mathbb{R}). \tag{4.5}$$

Since

$$L^1(\mathbb{R}) = \{\overline{f_1} f_2 \mid f_1, f_2 \in L^2(\mathbb{R})\},$$

condition (4.5) is equivalent to

$$\langle g_\alpha f_1, f_2\rangle \to \langle g f_1, f_2\rangle \text{ for } f_1, f_2 \in L^2(\mathbb{R}),$$

that is, the convergence of the net M_{g_α} to M_g in the weak operator topology.

Let S be the linear span of the functions $\chi_{[\mu,\infty)}$ for $\mu \in \mathbb{R}$. Since S is a convex linear manifold, by the bipolar theorem (Corollary V.1.9 of [Con85]), the weak-star closure of S is the set

$$\{h \in L^\infty(\mathbb{R}) \mid (f \in L^1(\mathbb{R}) \text{ and } \psi_g(f) = 0 \text{ for } g \in S) \Longrightarrow \psi_h(f) = 0\}.$$

If $f \in L^1(\mathbb{R})$ and $\psi_g(f) = 0$ for $g \in S$, then in particular,

$$\int_I f(x)\,dx = 0 \quad \text{for any bounded interval } I \subseteq \mathbb{R}.$$

By the Lebesgue differentiation theorem (Corollary 3.14 of [SW71]), the function f is almost everywhere zero, so the weak-star closure of S is all of $L^\infty(\mathbb{R})$. Since $[L^2([\mu,\infty))] = M_{\chi_{[\mu,\infty)}}$ for $\mu \in \mathbb{R}$, this shows that the linear span of the set $\{[L^2([\mu,\infty))] \mid \mu \in \mathbb{R}\}$ is weak operator topology dense in $\{M_g \mid g \in L^\infty(\mathbb{R})\}$.

Suppose that A commutes with $[L^2([\mu,\infty))] = M_{\chi_{[\mu,\infty)}}$ for $\mu \in \mathbb{R}$. Then A commutes with M_f for $f \in S$. Let g_α be a net in S converging in the weak-star topology to $g \in L^\infty(\mathbb{R})$. By Proposition 2.20(ii), the net $AM_{g_\alpha} = M_{g_\alpha}A$ converges in the weak operator topology to $AM_g = M_gA$. Thus A commutes with M_g for $g \in L^\infty(\mathbb{R})$. □

Theorem 4.4. *Let $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$ be distinct nests in $\widehat{\mathcal{L}}_{\mathrm{FB}}$. Then the intersection of $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$ is trivial. If the binest $\mathcal{N}^{(1)} \cup \mathcal{N}^{(2)}$ is degenerate, then $\mathrm{Alg}(\mathcal{N}^{(1)} \cup \mathcal{N}^{(2)})$ is unitarily equivalent to*

$$\mathcal{A}_\infty = \{M_g \mid g \in L^\infty(\mathbb{R})\};$$

in particular, $\mathrm{Alg}(\mathcal{N}_{\mathrm{v}} \cup \mathcal{N}_{\mathrm{v}}^\perp) = \mathcal{A}_\infty$. If the binest $\mathcal{N}^{(1)} \cup \mathcal{N}^{(2)}$ is non-degenerate then $\mathrm{Alg}(\mathcal{N}^{(1)} \cup \mathcal{N}^{(2)})$ is unitarily equivalent to $\mathcal{A}_{\mathrm{p}}$.

Proof. We first show that $\mathrm{Alg}(\mathcal{N}_{\mathrm{v}} \cup \mathcal{N}_{\mathrm{v}}^\perp) = \mathcal{A}_\infty$. If $A \in \mathrm{Alg}(\mathcal{N}_{\mathrm{v}} \cup \mathcal{N}_{\mathrm{v}}^\perp)$, then A commutes with each of the orthogonal projections $[L^2([\mu,\infty))]$ for $\mu \in \mathbb{R}$, so by Lemma 4.3, A commutes with M_g for $g \in L^\infty(\mathbb{R})$. So $A \in \mathcal{A}_\infty'$; but by Lemma XI.3 of [Lan85], $\mathcal{A}_\infty' = \mathcal{A}_\infty$. Hence $\mathrm{Alg}(\mathcal{N}_{\mathrm{v}} \cup \mathcal{N}_{\mathrm{v}}^\perp) \subseteq \mathcal{A}_\infty$. The inclusion $\mathcal{A}_\infty \subseteq \mathrm{Alg}(\mathcal{N}_{\mathrm{v}} \cup \mathcal{N}_{\mathrm{v}}^\perp)$ is trivial, so we have equality. Plainly, $\mathcal{N}_{\mathrm{v}} \cap \mathcal{N}_{\mathrm{v}}^\perp = \{(0), L^2(\mathbb{R})\}$.

For any $s \in \mathbb{R}$, since $\mathcal{N}_0 = \mathcal{N}_{\mathrm{a}}$ and $M_{\phi_s}\mathcal{N}_t = \mathcal{N}_{s+t}$,

$$\mathrm{Alg}\left(\mathcal{N}_s \cup \mathcal{N}_s^\perp\right) = \mathrm{Alg}(M_{\phi_s}(\mathcal{N}_{\mathrm{a}} \cup \mathcal{N}_{\mathrm{a}}^\perp)) = \mathrm{Alg}\left(M_{\phi_s}F(\mathcal{N}_{\mathrm{v}} \cup \mathcal{N}_{\mathrm{v}}^\perp)\right),$$

so $\mathrm{Alg}(\mathcal{N}_s \cup \mathcal{N}_s^\perp) = \mathrm{Ad}(M_{\phi_s}F)\mathcal{A}_\infty$ and

$$\mathcal{N}_s \cap \mathcal{N}_s^\perp = M_{\phi_s}F(\mathcal{N}_{\mathrm{v}} \cap \mathcal{N}_{\mathrm{v}}^\perp) = \{(0), L^2(\mathbb{R})\}.$$

The non-degenerate cases run as follows. Let s and t be real numbers with $s > t$. Then

$$\mathrm{Alg}\left(\mathcal{N}_{\mathrm{v}} \cup \mathcal{N}_s\right) = \mathrm{Alg}\left(M_{\phi_s}(\mathcal{N}_{\mathrm{v}} \cup \mathcal{N}_{\mathrm{a}})\right) = \mathrm{Ad}(M_{\phi_s})\mathcal{A}_{\mathrm{p}};$$

$$\begin{aligned}\mathrm{Alg}\left(\mathcal{N}_{\mathrm{v}} \cup \mathcal{N}_s^{\perp}\right) &= \mathrm{Alg}\left(M_{\phi_s}(\mathcal{N}_{\mathrm{v}} \cup \mathcal{N}_{\mathrm{a}}^{\perp})\right)\\ &= \mathrm{Alg}\left(M_{\phi_s}F(\mathcal{N}_{\mathrm{a}} \cup \mathcal{N}_{\mathrm{v}})\right)\\ &= \mathrm{Ad}(M_{\phi_s}F)\mathcal{A}_{\mathrm{p}};\end{aligned}$$

$$\begin{aligned}\mathrm{Alg}\left(\mathcal{N}_s \cup \mathcal{N}_t\right) &= \mathrm{Alg}\left(M_{\phi_t}(\mathcal{N}_{s-t} \cup \mathcal{N}_{\mathrm{a}})\right)\\ &= \mathrm{Alg}\left(M_{\phi_t}F^*(F\mathcal{N}_{s-t} \cup \mathcal{N}_{\mathrm{v}})\right)\\ &= \mathrm{Alg}\left(M_{\phi_t}F^*(\mathcal{N}_{-1/(s-t)}^{\perp} \cup \mathcal{N}_{\mathrm{v}})\right)\\ &= \mathrm{Ad}(M_{\phi_t}F^*M_{\phi_{-1/(s-t)}}F)\mathcal{A}_{\mathrm{p}};\end{aligned}$$

$$\begin{aligned}\mathrm{Alg}\left(\mathcal{N}_s^{\perp} \cup \mathcal{N}_t^{\perp}\right) &= \mathrm{Alg}\left(F^2(\mathcal{N}_s \cup \mathcal{N}_t)\right)\\ &= \mathrm{Ad}(F^2M_{\phi_t}F^*M_{\phi_{-1/(s-t)}}F)\mathcal{A}_{\mathrm{p}}\\ &= \mathrm{Ad}(M_{\phi_t}FM_{\phi_{-1/(s-t)}}F)\mathcal{A}_{\mathrm{p}};\end{aligned}$$

$$\begin{aligned}\mathrm{Alg}\left(\mathcal{N}_s \cup \mathcal{N}_t^{\perp}\right) &= \mathrm{Alg}\left(M_{\phi_s}(\mathcal{N}_{\mathrm{a}} \cup \mathcal{N}_{t-s}^{\perp})\right)\\ &= \mathrm{Alg}\left(M_{\phi_s}F^*(\mathcal{N}_{\mathrm{v}} \cup F\mathcal{N}_{t-s}^{\perp})\right)\\ &= \mathrm{Alg}\left(M_{\phi_s}F^*(\mathcal{N}_{\mathrm{v}} \cup \mathcal{N}_{-1/(t-s)}^{\perp})\right)\\ &= \mathrm{Ad}(M_{\phi_s}F^*M_{\phi_{-1/(t-s)}}F)\mathcal{A}_{\mathrm{p}};\end{aligned}$$

$$\begin{aligned}\mathrm{Alg}\left(\mathcal{N}_s^{\perp} \cup \mathcal{N}_t\right) &= \mathrm{Alg}\left(F^2(\mathcal{N}_s \cup \mathcal{N}_t^{\perp})\right)\\ &= \mathrm{Ad}(F^2M_{\phi_s}F^*M_{\phi_{-1/(t-s)}}F)\mathcal{A}_{\mathrm{p}}\\ &= \mathrm{Ad}(M_{\phi_s}FM_{\phi_{-1/(t-s)}}F)\mathcal{A}_{\mathrm{p}};\end{aligned}$$

$$\begin{aligned}\mathrm{Alg}\left(\mathcal{N}_{\mathrm{v}}^{\perp} \cup \mathcal{N}_s\right) &= \mathrm{Alg}\left(F^2(\mathcal{N}_{\mathrm{v}} \cup \mathcal{N}_s^{\perp})\right) = \mathrm{Ad}(F^2M_{\phi_s}F)\mathcal{A}_{\mathrm{p}}\\ &= \mathrm{Ad}(M_{\phi_s}F^*)\mathcal{A}_{\mathrm{p}};\end{aligned}$$

$$\operatorname{Alg}\left(\mathcal{N}_{\mathrm{v}}^{\perp} \cup \mathcal{N}_{s}^{\perp}\right) = \operatorname{Alg}\left(F^2(\mathcal{N}_{\mathrm{v}} \cup \mathcal{N}_s)\right) = \operatorname{Ad}(F^2 M_{\phi_s})\mathcal{A}_{\mathrm{p}}.$$

Finally, to show that $\mathcal{N}^{(1)} \cap \mathcal{N}^{(2)}$ is trivial when $\mathcal{N}^{(1)} \neq (\mathcal{N}^{(2)})^{\perp}$, it suffices to show that $\mathcal{N}_{\mathrm{v}} \cap \mathcal{N}_s$ is trivial for $s \in \mathbb{R}$, since the calculations above are easily adapted to show that there is a unitary mapping $\mathcal{N}^{(1)} \cap \mathcal{N}^{(2)}$ onto $\mathcal{N}_{\mathrm{v}} \cap \mathcal{N}_s$. However, every function in a proper subspace in $\mathcal{N}_{\mathrm{v}}$ has proper support and every non-zero function in a proper subspace in $\mathcal{N}_s$ has full support, so their intersection is indeed trivial. □

Remark. The reflexive closures

$$\operatorname{Lat}\operatorname{Alg}(\mathcal{N}^{(1)} \cup \mathcal{N}^{(2)})$$

of binests which are contained in $\widehat{\mathcal{L}}_{\mathrm{FB}}$ can be computed from Theorem 4.1(i), equation (4.2) and the proof of Theorem 4.4. Since the degenerate binests are commutative subspace lattices, they are reflexive as shown in [Arv74], §1.6. The reflexive closures of the non-degenerate lattices may be described as follows. Let B be a bijection from the set of nests in $\widehat{\mathcal{L}}_{\mathrm{FB}}$ onto the circle $\mathbb{T}$ such that

(i). the maps $s \mapsto B(\mathcal{N}_s)$ and $s \mapsto B(\mathcal{N}_s^{\perp})$ are continuous $\mathbb{R} \to \mathbb{T}$;

(ii). $B(\mathcal{N}^{\perp}) = -B(\mathcal{N})$ for every nest $\mathcal{N} \subseteq \widehat{\mathcal{L}}_{\mathrm{FB}}$;

(iii). $B(\mathcal{N}_s) \to B(\mathcal{N}_{\mathrm{v}})$ and $B(\mathcal{N}_{-s}) \to B(\mathcal{N}_{\mathrm{v}}^{\perp})$ as $s \to +\infty$.

Then any non-degenerate binest $\mathcal{N}^{(1)} \cup \mathcal{N}^{(2)}$ is of the form

$$B^{-1}(e^{i\theta}) \cup B^{-1}(e^{i\psi})$$

for some real numbers θ and ψ with $0 < \theta - \psi < \pi$. Performing the computations described above reveals the reflexive closure of this binest to be

$$\operatorname{Lat}\operatorname{Alg}\left(B^{-1}(e^{i\theta}) \cup B^{-1}(e^{i\psi})\right) = \{B^{-1}(e^{i\gamma}) \mid \gamma \in [\psi, \theta]\}.$$

4.2 Unitary automorphism groups

We define the *unitary automorphism group* $\mathcal{U}(\mathcal{L}_{\text{FB}})$ of $\mathcal{L}_{\text{FB}}$ as

$$\mathcal{U}(\mathcal{L}_{\text{FB}}) = \{U \in \mathcal{L}(L^2(\mathbb{R})) \mid U \text{ is unitary and } U\mathcal{L}_{\text{FB}} = \mathcal{L}_{\text{FB}}\}.$$

Since $\mathcal{A}_{\text{p}}$ is reflexive, if $U \in \mathcal{U}(\mathcal{L}_{\text{FB}})$ then by (4.1),

$$(\operatorname{Ad} U)\mathcal{A}_{\text{p}} = (\operatorname{Ad} U) \operatorname{Alg} \mathcal{L}_{\text{FB}} = \operatorname{Alg}(U\mathcal{L}_{\text{FB}}) = \operatorname{Alg} \mathcal{L}_{\text{FB}} = \mathcal{A}_{\text{p}}.$$

Conversely, if U is unitary and $(\operatorname{Ad} U)\mathcal{A}_{\text{p}} = \mathcal{A}_{\text{p}}$ then by (4.2),

$$U\mathcal{L}_{\text{FB}} = \operatorname{Lat}\big((\operatorname{Ad} U)\mathcal{A}_{\text{p}}\big) = \mathcal{L}_{\text{FB}}.$$

Thus $\mathcal{U}(\mathcal{L}_{\text{FB}})$ has an alternative description as

$$\mathcal{U}(\mathcal{L}_{\text{FB}}) = \{U \in \mathcal{L}(L^2(\mathbb{R})) \mid U \text{ is unitary and } (\operatorname{Ad} U)\mathcal{A}_{\text{p}} = \mathcal{A}_{\text{p}}\}. \tag{4.6}$$

In [KP97], $\mathcal{U}(\mathcal{L}_{\text{FB}})$ is identified by finding the group isomorphism ρ between the matrix group

$$\left\{ m(\lambda,\mu,t) = \begin{pmatrix} e^t & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & \mu & e^{-t} \end{pmatrix} \,\middle|\, \lambda,\mu,t \in \mathbb{R} \right\}$$

and the group

$$\operatorname{Ad}\big(\mathcal{U}(\mathcal{A}_{\text{p}})\big) = \{\operatorname{Ad} U \mid U \in \mathcal{U}(\mathcal{L}_{\text{FB}})\}$$

given by

$$\rho : m(\lambda,\mu,t) \mapsto \mathbf{M}_\lambda \mathbf{D}_\mu \mathbf{V}_t$$

where $\mathbf{M}_\lambda = \operatorname{Ad}(M_\lambda)$, $\mathbf{D}_\mu = \operatorname{Ad}(D_\mu)$, $\mathbf{V}_t = \operatorname{Ad}(V_t)$ and V_t is the unitary operator on $L^2(\mathbb{R})$ given by

$$V_t f(x) = e^{t/2} f(e^t x), \quad f \in L^2(\mathbb{R}),\ x,t \in \mathbb{R}.$$

We call V_t a *dilation operator*. For any pair U_1, U_2 of unitary operators, $\operatorname{Ad} U_1 = \operatorname{Ad} U_2$ if and only if $U_1^* U_2$ lies in the centre of $\mathcal{L}(L^2(\mathbb{R}))$, or equivalently $U_1 = e^{i\psi} U_2$ for some $\psi \in \mathbb{R}$. Hence we have the following generator description of $\mathcal{U}(\mathcal{L}_{\text{FB}})$:

$$\mathcal{U}(\mathcal{L}_{\text{FB}}) = \langle M_\lambda, D_\mu, V_t, e^{i\psi} I \mid \lambda, \mu, t, \psi \in \mathbb{R} \rangle. \tag{4.7}$$

We define the unitary automorphism group of $\widehat{\mathcal{L}}_{\text{FB}}$ by

$$\mathcal{U}(\widehat{\mathcal{L}}_{\text{FB}}) = \{U \in \mathcal{L}(L^2(\mathbb{R})) \mid U \text{ is unitary and } U\widehat{\mathcal{L}}_{\text{FB}} = \widehat{\mathcal{L}}_{\text{FB}}\}.$$

We seek generators for this group and show how $\mathcal{U}(\mathcal{L}_{\text{FB}})$ sits inside it.

Lemma 4.5. *For $t \in \mathbb{R}$, the dilation operator V_t lies in the operator algebra generated by $\{M_{\phi_s}, F, e^{i\psi} I \mid s, \psi \in \mathbb{R}\}$. In fact,*

$$\begin{aligned} V_t &= e^{-i\pi/4} M_{\phi_{-\exp(t)}} F^* M_{\phi_{-\exp(-t)}} F^* M_{\phi_{-\exp(t)}} F^* \\ &= e^{i\pi/4} M_{\phi_{\exp(t)}} F M_{\phi_{\exp(-t)}} F M_{\phi_{\exp(t)}} F. \end{aligned}$$

Proof. Let us write S_g for the operation of convolution with a function $g \in L^\infty(\mathbb{R})$, defined on the Schwartz space $\mathcal{S}(\mathbb{R})$; that is,

$$S_g f(x) = \int_{\mathbb{R}} g(x-t) f(t)\, dt, \quad f \in \mathcal{S}(\mathbb{R}),\ x \in \mathbb{R}.$$

For ζ a non-negative complex number, let $\zeta^{1/2}$ denote the square root of ζ with non-negative real part. For $b \in \mathbb{R} \setminus \{0\}$, let h_b and $\hat{h}_b$ be the bounded functions

$$h_b = \phi_{2\pi b}, \qquad \hat{h}_b = \frac{1}{(ib)^{1/2}} h_{-1/b}.$$

Let $\tilde{F}$ be the alternate Fourier transform defined on $\mathcal{S}(\mathbb{R})$ by

$$\tilde{F} f(x) = \int_{\mathbb{R}} f(y) e^{-2\pi i x y}\, dy, \quad f \in \mathcal{S}(\mathbb{R}),\ x \in \mathbb{R}.$$

Observe that $\tilde{F} = V_{\log 2\pi} F | \mathcal{S}(\mathbb{R})$. In §XI.1 of [Lan85] it is shown that

$$\tilde{F} S_{h_b} = M_{\hat{h}_b} \tilde{F}, \quad b \in \mathbb{R} \setminus \{0\}. \tag{4.8}$$

Now

$$h_{-1/b}(x/2\pi) = \phi_{-1/2\pi b}(x),$$

so

$$V_{-\log 2\pi} M_{\hat{h}_b} V_{\log 2\pi} = \frac{1}{(ib)^{1/2}} M_{\phi_{-1/2\pi b}}.$$

So (4.8) gives

$$S_{\phi_s} f = \frac{1}{(is/2\pi)^{1/2}} F^* M_{\phi_{-1/s}} F f, \quad f \in \mathcal{S}(\mathbb{R}),\ s \in \mathbb{R} \setminus \{0\}. \tag{4.9}$$

Observe that for $x, s, t \in \mathbb{R}$,

$$\phi_s(x-t) = e^{isxt} \phi_s(x) \phi_s(t).$$

Hence for $s, x \in \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned} S_{\phi_s} f(x) &= \int_{\mathbb{R}} \phi_s(x-t) f(t)\, dt \\ &= \phi_s(x) \int_{\mathbb{R}} e^{isxt} \phi_s(t) f(t)\, dt \\ &= (2\pi)^{1/2} \phi_s(x) (F^* M_{\phi_s} f)(sx). \end{aligned} \tag{4.10}$$

When $s > 0$, this yields

$$S_{\phi_s} f = (2\pi/s)^{1/2} M_{\phi_s} V_{\log s} F^* M_{\phi_s} f, \quad f \in \mathcal{S}(\mathbb{R}). \tag{4.11}$$

Since $\mathcal{S}(\mathbb{R})$ has dense linear span in $L^2(\mathbb{R})$, equations (4.9) and (4.11) imply that

$$e^{-i\pi/4} F^* M_{\phi_{-1/s}} F = M_{\phi_s} V_{\log s} F^* M_{\phi_s}, \quad s > 0;$$

so

$$V_{\log s} = e^{-i\pi/4} M_{\phi_{-s}} F^* M_{\phi_{-1/s}} F M_{\phi_{-s}} F, \quad s > 0.$$

Let $R = F^2$ be the operator $Rf(x) = f(-x)$. Then R commutes with V_t for $t \in \mathbb{R}$, and since ϕ_s is even, R also commutes with M_{ϕ_s} for each $t \in \mathbb{R}$. In particular, for any $\sigma \in \mathbb{R}$,

$$F^* M_{\phi_\sigma} F^* = F^* M_{\phi_\sigma} R F = F^* R M_{\phi_\sigma} F = F M_{\phi_\sigma} F. \tag{4.12}$$

Thus

$$V_{\log s} = e^{-i\pi/4} M_{\phi_{-s}} F^* M_{\phi_{-1/s}} F^* M_{\phi_{-s}} F^*, \quad s > 0. \tag{4.13}$$

For $s < 0$ and $f \in \mathcal{S}(\mathbb{R})$, (4.10) yields

$$S_{\phi_s} f = \big(2\pi/(-s)\big)^{1/2} M_{\phi_s} V_{\log(-s)} F M_{\phi_s} f.$$

Thus by (4.9),

$$\begin{aligned} V_{\log(-s)} &= e^{i\pi/4} M_{\phi_{-s}} F^* M_{\phi_{-1/s}} F M_{\phi_{-s}} F^* \\ &= e^{i\pi/4} M_{\phi_{-s}} F M_{\phi_{-1/s}} F M_{\phi_{-s}} F, \quad s < 0. \end{aligned} \tag{4.14}$$

Setting $t = \log s$ in (4.13) and $t = \log(-s)$ in (4.14) completes the proof. □

Lemma 4.6. *(i). Let m denote Lebesgue measure on the interval $[0,1]$ and let $\mathcal{N}$ be the nest of subspaces of $L^2([0,1])$ given by*

$$\mathcal{N} = \{L^2([0,t]) \mid t \in [0,1]\}.$$

If U is a unitary operator on $L^2([0,1])$ and $U\mathcal{N} = \mathcal{N}$, then there is a unimodular function $\varphi \in L^\infty([0,1])$ and an almost everywhere differentiable order-preserving bijection $g : [0,1] \to [0,1]$ such that the operator C_g defined by

$$C_g f(x) = \big(g'(x)\big)^{1/2} (f \circ g)(x), \quad f \in L^2([0,1]),\ x \in [0,1] \tag{4.15}$$

is unitary on $L^2([0,1])$ and $U = M_\varphi C_g$.

(ii). Let U be a unitary operator on $L^2(\mathbb{R})$ satisfying $U\mathcal{N}_{\mathrm{v}} = \mathcal{N}_{\mathrm{v}}$. Then there is a unimodular function $\varphi \in L^\infty(\mathbb{R})$ and an almost everywhere differentiable order-preserving bijection $g : \mathbb{R} \to \mathbb{R}$ such that the operator C_g defined by

$$C_g f(x) = \big(g'(x)\big)^{1/2} (f \circ g)(x), \quad f \in L^2(\mathbb{R}),\ x \in \mathbb{R} \tag{4.16}$$

is unitary on $L^2(\mathbb{R})$ and $U = M_\varphi C_g$.

Proof. (i). For $t \in [0,1]$, let P_t be the projection $P_t = [L^2([0,t])]$ and let $h(t)$ be the unique element of $[0,1]$ satisfying $(\operatorname{Ad} U)P_t = P_{h(t)}$. Then the function $h : [0,1] \to [0,1]$ is 1–1 and strictly increasing. Let $\mathbf{1}$ denote the constant function on $[0,1]$ taking the value 1 and let $g = h^{-1}$. Then $P_{g(t)} = (\operatorname{Ad} U^*)P_t$ for $t \in [0,1]$, so

$$g(t) = \langle P_{g(t)}\mathbf{1}, \mathbf{1}\rangle = \langle P_t U\mathbf{1}, U\mathbf{1}\rangle = \int_0^t |U\mathbf{1}(x)|^2\, dm(x), \quad t \in [0,1]. \tag{4.17}$$

Consider the measure $m \circ g$ defined by $(m \circ g)(E) = m\big(g(E)\big)$ for E a measurable subset of $[0,1]$. By (4.17), when E is an interval,

$$(m \circ g)(E) = \int_E |U\mathbf{1}(x)|^2\, dm(x) \tag{4.18}$$

It follows that (4.18) holds for each measurable subset $E \subseteq [0,1]$. Hence $m \circ g \ll m$ and by the Lebesgue differentiation theorem (in the form of Theorem 8.17 of [Rud66]), g is differentiable almost everywhere with

$$g'(x) = \frac{d(m \circ g)}{dm}(x) = |U\mathbf{1}(x)|^2 \quad \text{for almost every } x \in [0,1].$$

For $f \in L^2([0,1])$,

$$\|C_g f\|^2 = \int_0^1 |(f \circ g)(x)|^2 \frac{d(m \circ g)}{dm}(x)\, dm(x) = \int_0^1 |f(x)|^2\, dm(x) = \|f\|^2,$$

so C_g is an isometry in $\mathcal{L}\big(L^2([0,1])\big)$. By symmetry, h is also differentiable almost everywhere and we may define an isometry $C_h \in \mathcal{L}\big(L^2([0,1])\big)$ by (4.15) with h in place of g. By the chain rule,

$$g'\big(h(x)\big)\, h'(x) = 1 \quad \text{for almost every } x \in [0,1].$$

It follows that $C_h = C_g^{-1}$ and $(\operatorname{Ad} C_g)P_t = P_{h(t)}$. Hence C_g is unitary.

Let V be the unitary operator $V = UC_g^*$. Since

$$(\operatorname{Ad} V)P_t = (\operatorname{Ad} U)(\operatorname{Ad} C_h)P_t = (\operatorname{Ad} U)P_{g(t)} = P_t,$$

the operator V reduces each subspace in $\mathcal{N}$, and so V lies in $\mathrm{Alg}(\mathcal{N} \cup \mathcal{N}^{\perp})$. Theorem 4.4 shows that $\mathrm{Alg}(\mathcal{N}_{\mathrm{v}} \cup \mathcal{N}_{\mathrm{v}}^{\perp}) = \{M_f \mid f \in L^{\infty}(\mathbb{R})\}$, and an identical argument shows that $\mathrm{Alg}(\mathcal{N} \cup \mathcal{N}^{\perp}) = \{M_{\varphi} \mid \varphi \in L^{\infty}([0,1])\}$. Since V is unitary, $V = M_{\varphi}$ for φ a unimodular function in $L^{\infty}([0,1])$.

(ii). Let $Z : L^2(\mathbb{R}) \to L^2([0,1])$ be an operator of the form

$$Zf(x) = |\alpha'(x)|^{1/2} f(\alpha(x)), \quad f \in L^2(\mathbb{R}),\ x \in [0,1]$$

where $\alpha : (0,1) \to \mathbb{R}$ is a smooth order-reversing bijection. It is easily verified that Z is a unitary map of $L^2(\mathbb{R})$ onto $L^2([0,1])$ and that $[Z\mathcal{N}_{\mathrm{v}}] = [\mathcal{N}]$. Hence ZUZ^{-1} satisfies the hypotheses of part (i), so

$$U = (Z^{-1} M_{\tilde{\varphi}} Z)(Z^{-1} C_{\tilde{g}} Z)$$

where $\tilde{\varphi}$ is a unimodular function in $L^{\infty}([0,1])$ and $\tilde{g} : [0,1] \to [0,1]$ is an almost everywhere differentiable order-preserving bijection which defines a unitary operator $C_{\tilde{g}} \in \mathcal{L}(L^2([0,1]))$ of the form (4.15). It is simple to see that

$$Z^{-1} M_{\tilde{\varphi}} Z = M_{\tilde{\varphi} \circ \alpha^{-1}};$$

plainly $\varphi = \tilde{\varphi} \circ \alpha^{-1}$ is a unimodular function in $L^{\infty}(\mathbb{R})$. Moreover, the function $g = \alpha \circ \tilde{g} \circ \alpha^{-1}$ is an almost everywhere differentiable order-preserving bijection $\mathbb{R} \to \mathbb{R}$ and the unitary operator $Z^{-1} C_{\tilde{g}} Z$ is equal to the operator C_g defined by (4.16), so $U = M_{\varphi} C_g$. □

Remark. Another proof of Lemma 4.6 may be obtained from Proposition 3 of [AK95].

Theorem 4.7. *The unitary automorphism group of $\widehat{\mathcal{L}}_{\mathrm{FB}}$ is*

$$\mathcal{U}(\widehat{\mathcal{L}}_{\mathrm{FB}}) = \langle M_{\phi_s}, M_{\lambda}, F, e^{i\psi} I \mid s, \lambda, \psi \in \mathbb{R} \rangle.$$

Proof. Let $\mathcal{G}_{\mathrm{a}}$, $\mathcal{G}_{\mathrm{v}}$ and $\mathcal{G}_s$ denote the "great circles"

$$\mathcal{G}_{\mathrm{a}} = \mathcal{N}_{\mathrm{a}} \cup \mathcal{N}_{\mathrm{a}}^{\perp}, \quad \mathcal{G}_{\mathrm{v}} = \mathcal{N}_{\mathrm{v}} \cup \mathcal{N}_{\mathrm{v}}^{\perp}, \quad \mathcal{G}_s = \mathcal{N}_s \cup \mathcal{N}_s^{\perp} \text{ for } s \in \mathbb{R}.$$

Let U be a unitary automorphism of $\widehat{\mathcal{L}}_{\mathrm{FB}}$. If $U\mathcal{G}_{\mathrm{a}} = \mathcal{G}_{\mathrm{v}}$, then FU fixes $\mathcal{G}_{\mathrm{a}}$; if $U\mathcal{G}_{\mathrm{a}} = \mathcal{G}_s$ for some $s \in \mathbb{R}$, then $M_{\phi_{-s}}U$ fixes $\mathcal{G}_{\mathrm{a}}$. So we may assume that $U\mathcal{G}_{\mathrm{a}} = \mathcal{G}_{\mathrm{a}}$.

If $U\mathcal{G}_{\mathrm{v}} \neq \mathcal{G}_{\mathrm{v}}$, then $U\mathcal{G}_{\mathrm{v}} = \mathcal{G}_s$ for some $s \neq 0$. If $s > 0$ then by Proposition 4.2, $F\mathcal{G}_s = \mathcal{G}_{-1/s}$ and $F\mathcal{G}_{\mathrm{a}} = \mathcal{G}_{\mathrm{v}}$, so the operator $U' = F^* M_{\phi_{1/s}} FU$ satisfies $U'\mathcal{G}_{\mathrm{a}} = \mathcal{G}_{\mathrm{a}}$ and $U'\mathcal{G}_{\mathrm{v}} = \mathcal{G}_{\mathrm{v}}$. So we may assume that $U\mathcal{G}_{\mathrm{v}} = \mathcal{G}_{\mathrm{v}}$ and $U\mathcal{G}_{\mathrm{a}} = \mathcal{G}_{\mathrm{a}}$.

We now have four cases to consider:

1. $U\mathcal{N}_{\mathrm{a}} = \mathcal{N}_{\mathrm{a}}$, $U\mathcal{N}_{\mathrm{v}} = \mathcal{N}_{\mathrm{v}}$; 2. $U\mathcal{N}_{\mathrm{a}} = \mathcal{N}_{\mathrm{a}}^{\perp}$, $U\mathcal{N}_{\mathrm{v}} = \mathcal{N}_{\mathrm{v}}^{\perp}$;
3. $U\mathcal{N}_{\mathrm{a}} = \mathcal{N}_{\mathrm{a}}^{\perp}$, $U\mathcal{N}_{\mathrm{v}} = \mathcal{N}_{\mathrm{v}}$; 4. $U\mathcal{N}_{\mathrm{a}} = \mathcal{N}_{\mathrm{a}}$, $U\mathcal{N}_{\mathrm{v}} = \mathcal{N}_{\mathrm{v}}^{\perp}$.

Substituting the operator F^2U in place of U interchanges cases 1 and 2 and also interchanges cases 3 and 4, so it is sufficient to consider cases 1 and 3 only.

Suppose that case 3 holds, so that $U\mathcal{N}_{\mathrm{a}} = \mathcal{N}_{\mathrm{a}}^{\perp}$ and $U\mathcal{N}_{\mathrm{v}} = \mathcal{N}_{\mathrm{v}}$. We claim that $U\mathcal{N}_1 = \mathcal{N}_{-s}^{\perp}$ for some $s > 0$. To see this, let $\mathfrak{N}$ be the set

$$\mathfrak{N} = \{\mathcal{N}_{\mathrm{v}}, \mathcal{N}_{\mathrm{v}}^{\perp}\} \cup \{\mathcal{N}_s, \mathcal{N}_s^{\perp} \mid s \in \mathbb{R}\}.$$

Observe that every element of $\mathfrak{N}$ is a nest contained in $\widehat{\mathcal{L}}_{\mathrm{FB}}$; by (4.4), $\widehat{\mathcal{L}}_{\mathrm{FB}}$ is the union of all nests in $\mathfrak{N}$. Since U is unitary, it is invertible and order-preserving. So U maps nests onto nests and induces a bijection of $\mathfrak{N}$ by $U : \mathcal{N} \mapsto U\mathcal{N}$.

Let $\mathcal{T}$ be the "equator" of $\widehat{\mathcal{L}}_{\mathrm{FB}}$,

$$\mathcal{T} = \{L^2(\mathbb{R}_+), L^2(\mathbb{R}_-)\} \cup \{M_{\phi_s} H^2(\mathbb{R}), M_{\phi_s} \overline{H^2(\mathbb{R})} \mid s \in \mathbb{R}\}.$$

The set $\mathcal{T}$ contains exactly one subspace from each nest in $\mathfrak{N}$, so the action $U : \mathcal{T} \to U\mathcal{T}$, $K \mapsto UK$ of U on $\mathcal{T}$ determines the action of U on $\mathfrak{N}$. Moreover, U is a homeomorphism between $\mathcal{T}$ and $U\mathcal{T}$ considered as sets of projections with the strong operator topology, and $\mathcal{T}$ is itself homeomorphic to the circle $\mathbb{T}$ by Theorem 4.1(iv) and Proposition 4.2. Let us give $\mathfrak{N}$ the topology induced by the topology on $\mathcal{T}$. Then the bijective action of U on $\mathfrak{N}$ is a homeomorphism.

It follows that the set

$$[\mathcal{N}_{\mathrm{a}}, \mathcal{N}_{\mathrm{v}}] = \bigcup_{s \geq 0} \mathcal{N}_s \cup \mathcal{N}_{\mathrm{v}}$$

must be mapped by U onto one of the sets

$$[\mathcal{N}_{\mathrm{a}}^{\perp}, \mathcal{N}_{\mathrm{v}}] = \bigcup_{s \geq 0} \mathcal{N}_s^{\perp} \cup \mathcal{N}_{\mathrm{v}}^{\perp} \cup \bigcup_{s \in \mathbb{R}} \mathcal{N}_s \cup \mathcal{N}_{\mathrm{v}},$$

or

$$[\mathcal{N}_{\mathrm{v}}, \mathcal{N}_{\mathrm{a}}^{\perp}] = \mathcal{N}_{\mathrm{v}} \cup \bigcup_{s \leq 0} \mathcal{N}_s^{\perp}.$$

If $U[\mathcal{N}_{\mathrm{a}}, \mathcal{N}_{\mathrm{v}}] = [\mathcal{N}_{\mathrm{a}}^{\perp}, \mathcal{N}_{\mathrm{v}}]$ then there is some $s > 0$ such that $U\mathcal{N}_s = \mathcal{N}_{\mathrm{v}}^{\perp}$. Since $U\mathcal{N}_{\mathrm{v}} = \mathcal{N}_{\mathrm{v}}$ and U is unitary,

$$\mathcal{N}_{\mathrm{v}}^{\perp} = (U^* \mathcal{N}_{\mathrm{v}})^{\perp} = U^* \mathcal{N}_{\mathrm{v}}^{\perp} = \mathcal{N}_s.$$

However, $\mathcal{N}_{\mathrm{v}}^{\perp} \cap \mathcal{N}_s = \{(0), L^2(\mathbb{R})\}$ since every function in a proper subspace in $\mathcal{N}_{\mathrm{v}}^{\perp}$ has proper support whereas every function f in a proper subspace in $\mathcal{N}_s$ has $\operatorname{supp} f = \mathbb{R}$ by Corollary 2.6. This contradiction shows that $U[\mathcal{N}_{\mathrm{a}}, \mathcal{N}_{\mathrm{v}}]$ cannot be equal to $[\mathcal{N}_{\mathrm{a}}^{\perp}, \mathcal{N}_{\mathrm{v}}]$.

Hence $U[\mathcal{N}_{\mathrm{a}}, \mathcal{N}_{\mathrm{v}}] = [\mathcal{N}_{\mathrm{v}}, \mathcal{N}_{\mathrm{a}}^{\perp}]$ and so $U\mathcal{N}_1 = \mathcal{N}_{-s}^{\perp}$ for some $s > 0$. Since $U\mathcal{N}_{\mathrm{v}} = \mathcal{N}_{\mathrm{v}}$, by Lemma 4.6 there exist a unimodular function $\varphi \in L^{\infty}(\mathbb{R})$ and an order-preserving almost everywhere differentiable bijection $g : \mathbb{R} \to \mathbb{R}$ such that $U = M_{\varphi} C_g$ where C_g is defined by (4.16). Thus

$$UM_{\phi_1} H^2(\mathbb{R}) = M_{\varphi} C_g M_{\phi_1} H^2(\mathbb{R}) = M_{\phi_{-s}} M_{\lambda} \overline{H^2(\mathbb{R})}$$

for some $\lambda \in \mathbb{R}$. Moreover, $U\mathcal{N}_{\mathrm{a}} = \mathcal{N}_{\mathrm{a}}^{\perp}$, so $UH^2(\mathbb{R}) = M_\mu \overline{H^2(\mathbb{R})}$ for some $\mu \in \mathbb{R}$. Since $C_g M_f = M_{f \circ g} C_g$ for $f \in L^\infty(\mathbb{R})$,

$$M_{\phi_1 \circ g} M_\varphi C_g H^2(\mathbb{R}) = M_{\phi_1 \circ g} M_\mu \overline{H^2(\mathbb{R})} = M_{\phi_{-s}} M_\lambda \overline{H^2(\mathbb{R})}.$$

Taking orthogonal complements, we see that

$$M_{\phi_s} M_{-\lambda} M_{\phi_1 \circ g} M_\mu H^2(\mathbb{R}) = H^2(\mathbb{R}). \tag{4.19}$$

Let $u : \mathbb{R} \to \mathbb{C}$ be the unimodular function

$$u(x) = \phi_s(x) e^{-i\lambda x} (\phi_1 \circ g)(x) e^{i\mu x}, \quad x \in \mathbb{R}.$$

Then u is unimodular, and rewriting (4.19) we obtain

$$uH^2(\mathbb{R}) = H^2(\mathbb{R}).$$

By Corollary 2.19,

$$u(x) = \exp i\big[-(g(x)^2 + sx^2)/2 + (\mu - \lambda)x\big]$$

must be an almost everywhere constant function of x. But $g(x) \to \infty$ as $x \to \infty$, so this is impossible.

So we are reduced to case 1: U fixes both the analytic nest and the Volterra nest. So

$$\begin{aligned}(\operatorname{Ad} U)\mathcal{A}_{\mathrm{p}} &= (\operatorname{Ad} U) \operatorname{Alg}(\mathcal{N}_{\mathrm{a}} \cup \mathcal{N}_{\mathrm{v}}) = \operatorname{Alg}\big(U(\mathcal{N}_{\mathrm{a}} \cup \mathcal{N}_{\mathrm{v}})\big) \\ &= \operatorname{Alg}(\mathcal{N}_{\mathrm{a}} \cup \mathcal{N}_{\mathrm{v}}) = \mathcal{A}_{\mathrm{p}};\end{aligned}$$

so $U \in \mathcal{U}(\mathcal{L}_{\mathrm{FB}})$ by (4.6). Thus by (4.7),

$$U \in \langle M_\lambda, D_\mu, V_t, e^{i\psi} I \mid \lambda, \mu, t, \psi \in \mathbb{R} \rangle.$$

Applying (3.2) and Lemma 4.5 now shows that

$$\mathcal{U}(\widehat{\mathcal{L}}_{\mathrm{FB}}) \subseteq \langle M_{\phi_s}, M_\lambda, F, e^{i\psi} I \mid s, \lambda, \psi \in \mathbb{R} \rangle.$$

The reverse inclusion is clear, so the proof is complete. □

We now consider a matrix representation of the group

$$\mathrm{Ad}\left(\mathcal{U}(\widehat{\mathcal{L}}_{\mathrm{FB}})\right) = \{\mathrm{Ad}(U) \mid U \in \mathcal{U}(\widehat{\mathcal{L}}_{\mathrm{FB}})\}.$$

Since $\mathrm{Ad}(e^{i\psi}I) = \mathrm{Ad}\,I$, considering this rather than dealing directly with $\mathcal{U}(\widehat{\mathcal{L}}_{\mathrm{FB}})$ affords a slight simplification.

We write $\mathbf{M}_{\phi_s}$ and $\mathbf{F}$ for $\mathrm{Ad}(M_{\phi_s})$ and $\mathrm{Ad}(F)$ respectively. By Theorem 4.7,

$$\mathrm{Ad}\left(\mathcal{U}(\widehat{\mathcal{L}}_{\mathrm{FB}})\right) = \langle \mathbf{M}_{\phi_s}, \mathbf{M}_\lambda, \mathbf{F} \mid s, \lambda \in \mathbb{R}\rangle.$$

Given a subset $\mathcal{S}$ of a group, let $W(\mathcal{S})$ be a finite word in the elements of $\mathcal{S}$; that is,

$$W(\mathcal{S}) \in \{S_1 S_2 \ldots S_n \mid n = 0, 1, 2, 3, \ldots,\ \ S_i \in \mathcal{S} \text{ for } i = 1, 2, \ldots, n\}.$$

The relations

$$\begin{gathered}
\mathbf{M}_\lambda \mathbf{F} = \mathbf{F}\mathbf{D}_{-\lambda}, \quad \mathbf{D}_\mu \mathbf{F} = \mathbf{F}\mathbf{M}_\mu, \quad \mathbf{M}_\lambda \mathbf{D}_\mu = \mathbf{D}_\mu \mathbf{M}_\lambda, \\
\mathbf{M}_\lambda \mathbf{M}_{\phi_s} = \mathbf{M}_{\phi_s}\mathbf{M}_\lambda, \qquad \mathbf{D}_\mu \mathbf{M}_{\phi_s} = \mathbf{M}_{\phi_s}\mathbf{M}_{\mu s}\mathbf{D}_\mu, \\
\mathbf{M}_\lambda \mathbf{M}_\mu = \mathbf{M}_{\lambda+\mu}, \qquad \mathbf{D}_\lambda \mathbf{D}_\mu = \mathbf{D}_{\lambda+\mu},
\end{gathered} \tag{4.20}$$

are easily verified, and using them we can put any element of $\mathrm{Ad}\left(\mathcal{U}(\widehat{\mathcal{L}}_{\mathrm{FB}})\right)$ into the form

$$W(\{\mathbf{F}, \mathbf{M}_{\phi_s} \mid s \in \mathbb{R}\})\mathbf{M}_\lambda \mathbf{D}_\mu. \tag{4.21}$$

Recall that $SL_2(\mathbb{R})$ is the Lie group of 2×2 matrices with real entries and determinant $+1$. The following theorem is proven in §XI.2 of [Lan85].

Theorem 4.8. *The group $SL_2(\mathbb{R})$ is isomorphic to the quotient Q of the free group Γ with generators*

$$\{\tilde{u}(b), \tilde{w} \mid b \in \mathbb{R}\}$$

by the relations

(i). $\tilde{u}$ *is an additive homomorphism;*

(ii). *if* $\tilde{v}(a) = \tilde{w}\tilde{u}(a^{-1})\tilde{w}\tilde{u}(a)\tilde{w}\tilde{u}(a^{-1})$ *for each non-zero real number* a*, then* $\tilde{v}$ *is a multiplicative homomorphism;*

(iii). $\tilde{w}^2 = \tilde{v}(-1)$*;*

(iv). $\tilde{v}(a)\tilde{u}(b)\tilde{v}(a^{-1}) = \tilde{u}(ba^2)$*.*

Let $\psi : \Gamma \to Q$ *be the quotient map of* Γ *onto* Q *and let* $u(b) = \psi(\tilde{u}(b))$*,* $v(a) = \psi(\tilde{v}(a))$ *and* $w = \psi(\tilde{w})$*. If* ι *is the mapping*

$$\iota(u(b)) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad \iota(w) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then ι *extends to an isomorphism* $\iota : Q \to SL_2(\mathbb{R})$ *and* $\iota(v(a)) = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$*.*

Let us identify Q *and* $SL_2(\mathbb{R})$ *using* ι*. Then every element of* $SL_2(\mathbb{R})$ *is uniquely expressible in one of the two forms*

$$u(a)v(b) \quad or \quad u(a)v(b)wu(c)$$

for real numbers $a, b, c \in \mathbb{R}$ *with* $b \neq 0$*.*

Remark. The decomposition described in Theorem 4.8 is called the *Bruhat decomposition.* In practice, it easy to determine which of the two factorisations a given matrix in $SL_2(\mathbb{R})$ has, since the matrices of the form

$$\iota(u(a)v(b)) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} b & ab^{-1} \\ 0 & b^{-1} \end{pmatrix}$$

are precisely those matrices in $SL_2(\mathbb{R})$ with a zero in the bottom left-hand corner.

Proposition 4.9. *Let* G *be the group*

$$G = \langle \mathbf{F}, \mathbf{M}_{\phi_s} \mid s \in \mathbb{R} \rangle.$$

Then the mapping π *on* $\{\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right) \mid s \in \mathbb{R}\}$ *given by*

$$\pi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathbf{F}, \qquad \pi\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \mathbf{M}_{\phi_s}$$

extends to an isomorphism between $SL_2(\mathbb{R})$ *and* G.

Proof. Let $w = \mathbf{F}$ and $u(b) = \mathbf{M}_{\phi_b}$ for $b \in \mathbb{R}$. We claim that $u(b)$ and w satisfy relations (i)–(iv) of Theorem 4.8, so that G is isomorphic to a quotient of $SL_2(\mathbb{R})$.

Since $\phi_s\phi_t = \phi_{s+t}$ for $s, t \in \mathbb{R}$, the map u is an additive homomorphism. Let

$$v(a) = wu(a^{-1})wu(a)wu(a^{-1}) \quad \text{for non-zero } a \in \mathbb{R}.$$

Recall that $R = F^2$ is the self-adjoint unitary flip operator $Rf(x) = f(-x)$ and that R commutes with V_t and M_{ϕ_s} for each $s, t \in \mathbb{R}$. Either by Lemma 4.5 or by direct computation, is easy to see that

$$FV_tF^* = V_t^* = V_{-t}, \quad t \in \mathbb{R}. \tag{4.22}$$

Hence for $a > 0$, by (4.13),

$$\begin{aligned} v(a) &= \mathrm{Ad}(FM_{\phi_{1/a}}FM_{\phi_a}FM_{\phi_{1/a}}) \\ &= \mathrm{Ad}\left((M_{\phi_{-1/a}}F^*M_{\phi_{-a}}F^*M_{\phi_{-1/a}}F^*)^*\right) \\ &= \mathrm{Ad}\left((V_{\log 1/a})^*\right) \\ &= \mathrm{Ad}(V_{\log a}). \end{aligned}$$

For $a < 0$, by (4.12) and (4.14),

$$\begin{aligned} v(a) &= \mathrm{Ad}(FM_{\phi_{1/a}}FM_{\phi_a}FM_{\phi_{1/a}}) \\ &= \mathrm{Ad}\left((M_{\phi_{-1/a}}F^*M_{\phi_{-a}}F^*M_{\phi_{-1/a}}F^*)^*\right) \\ &= \mathrm{Ad}(R(M_{\phi_{-1/a}}FM_{\phi_{-a}}FM_{\phi_{-1/a}}F)^*) \end{aligned}$$

$$= \mathrm{Ad}(R(V_{\log(-1/a)})^*)$$
$$= \mathrm{Ad}(RV_{\log|a|}).$$

Thus

$$v(a) = \begin{cases} \mathrm{Ad}(V_{\log a}) & a > 0, \\ \mathrm{Ad}(RV_{\log|a|}) & a < 0. \end{cases}$$

Now

$$v(xy) = \begin{cases} \mathrm{Ad}(V_{\log xy}) = \mathrm{Ad}(V_{\log x})\,\mathrm{Ad}(V_{\log y}) & x, y > 0, \\ \mathrm{Ad}(V_{\log xy}) = \mathrm{Ad}(RV_{\log|x|})\,\mathrm{Ad}(RV_{\log|y|}) & x, y < 0, \\ \mathrm{Ad}(RV_{\log|xy|}) = \mathrm{Ad}(RV_{\log|x|})\,\mathrm{Ad}(V_{\log y}) & x < 0 < y, \end{cases}$$

which is equal in each case to $v(x)v(y)$. So $v(xy) = v(x)v(y)$ for non-zero real numbers x and y, so v is a multiplicative homomorphism. Moreover, $v(-1) = \mathrm{Ad}(R) = w^2$, and since $V_t M_{\phi_s} V_{-t} = M_{\phi_{e^{2t}s}}$ for $s \in \mathbb{R}$, we have

$$v(a)u(b)v(a^{-1}) = u(ba^2) \quad \text{for } a > 0,\ b \in \mathbb{R}.$$

Since R commutes with M_{ϕ_s} and V_t for $s, t \in \mathbb{R}$, for $a, b \in \mathbb{R}$ with $a \neq 0$ it follows that $w^2 = v(-1)$ commutes with $u(b)$ and $v(a)$. So for $a < 0$,

$$v(a)u(b)v(a^{-1}) = w^2 v(|a|)u(b)w^2 v(|a|^{-1}) = v(|a|)u(b)v(|a|^{-1}) = u(ba^2).$$

Hence conditions (i)–(iv) of Theorem 4.8 are satisfied, so G is isomorphic to a quotient of $SL_2(\mathbb{R})$.

We claim that G is actually isomorphic to $SL_2(\mathbb{R})$ itself. This will follow from Theorem 4.8 if we show that G has a Bruhat decomposition: we must show that every element of G can be written uniquely in one of the forms

$$u(a)v(b) \quad \text{or} \quad u(a)v(b)wu(c)$$

for real numbers $a, b, c \in \mathbb{R}$ with $b \neq 0$. Since G is a homomorphic image of $SL_2(\mathbb{R})$ every element of G can indeed be expressed in such a way by Theorem 4.8. It remains to demonstrate uniqueness. For the remainder of this proof, let a, b, c, a', b', c' denote real numbers with $b, b' \neq 0$.

If $u(a)v(b) = u(a')v(b')$ then $\mathrm{Ad}(R^n M_{\phi_{a-a'}} V_{\log|b|-\log|b'|}) = \mathrm{Ad}(I)$ where $n = 1$ if $bb' < 0$ and $n = 0$ otherwise, so $R^n M_{\phi_{a-a'}} V_{\log|b|-\log|b'|} = \zeta I$ for some unimodular constant ζ. Considering the images under these operators of indicator functions of intervals in $\mathbb{R}$ rapidly leads to the conclusion that $|b| = |b'|$, $a = a'$ and $n = 0$. The signs of b and b' differ only if $n = 1$, so $b = b'$.

If $u(a)v(b)wu(c) = u(a')v(b')wu(c')$, then $v(b)wu(c) = u(a' - a)v(b')w$. So

$$V_t F M_{\phi_{c-c'}} = \zeta R^n M_{\phi_{a'-a}} V_{t'} F \tag{4.23}$$

where $t = \log|b|$, $t' = \log|b'|$, $n = 1$ if $bb' < 0$ and $n = 0$ otherwise, and ζ is a unimodular constant. Let $f = F^* \chi_{[0,1]}$; then $f \in H^2(\mathbb{R})$. The image of f under the right-hand side of (4.23) has support $[0, e^{-t'}]$ if $n = 0$ and $[-e^{-t'}, 0]$ if $n = 1$. By Proposition 4.2, the image of f under the left-hand side of (4.23) is

$$V_t F M_{\phi_{c-c'}} f \in V_t F M_{\phi_{c-c'}} H^2(\mathbb{R}) = \begin{cases} V_t M_{\phi_{-1/(c-c')}} \overline{H^2(\mathbb{R})} & \text{if } c > c', \\ V_t M_{\phi_{-1/(c-c')}} H^2(\mathbb{R}) & \text{if } c < c', \\ V_t L^2(\mathbb{R}_+) & \text{if } c = c'. \end{cases}$$

Since $\zeta R^n V_{t'} F f$ has proper support, we must have $c = c'$ and since $V_t F f$ has support equal to $[0, e^{-t}]$, also $t = t'$ and $n = 0$, so $b = b'$. So $\zeta M_{\phi_{a'-a}} = I$, so $a = a'$ and $\zeta = 1$.

Suppose that $u(a)v(b)wu(c) = u(a')v(b')$. It follows from relation (iv) of Theorem 4.8 that

$$u(a' - a)v(b') = v(b')u\big((a' - a)/b'^2\big).$$

Hence

$$v(b)wu(c) = u(a' - a)v(b') = v(b')u\big((a' - a)/b'^2\big),$$

so

$$v(b/b')w = u\big((a' - a)/b'^2 - c\big).$$

Let $\gamma = (a' - a)/b'^2 - c$. Then

$$V_t F = \zeta R^n M_{\phi_\gamma}$$

for $t = \log|b/b'|$, some integer n and some unimodular complex number ζ. The operator $V_t F$ maps the compactly supported function $\chi_{[0,1]}$ to a function with full support whereas the operator $\zeta R^n M_{\phi_\gamma}$ cannot map a compactly supported function to a function with full support. So this case cannot occur.

So G does indeed have the unique decomposition property and so G is isomorphic to $SL_2(\mathbb{R})$. That the map π implements this isomorphism follows from Theorem 4.8. □

Our description (4.21) of a general element of $\mathrm{Ad}\left(\mathcal{U}(\widehat{\mathcal{L}}_{\mathrm{FB}})\right)$ now becomes

$$\pi(K)\mathbf{M}_\lambda \mathbf{D}_\mu$$

where K is an arbitrary element of $SL_2(\mathbb{R})$ and $\lambda, \mu \in \mathbb{R}$. It is not hard to see that this "normal form" for elements of $\mathrm{Ad}\left(\mathcal{U}(\widehat{\mathcal{L}}_{\mathrm{FB}})\right)$ is unique: for if $\pi(K')\mathbf{M}_{\lambda'}\mathbf{D}_{\mu'} = \pi(K'')\mathbf{M}_{\lambda''}\mathbf{D}_{\mu''}$, then $\pi(K) = \mathbf{M}_\lambda \mathbf{D}_\mu$ for $\lambda = \lambda'' - \lambda'$, $\mu = \mu'' - \mu'$ and $K = K''^{-1}K'$. Now $\pi(K) = \mathrm{Ad}\, U$ for U a unitary in the group $\langle F, M_{\phi_s}, e^{i\psi} I \mid s, \psi \in \mathbb{R}\rangle$ and such operators U map the equator

$$\mathcal{T} = \{L^2(\mathbb{R}_+), L^2(\mathbb{R}_-)\} \cup \{M_{\phi_s} H^2(\mathbb{R}), M_{\phi_s}\overline{H^2(\mathbb{R})} \mid s \in \mathbb{R}\}$$

onto itself. On the other hand, $\mathbf{M}_\lambda \mathbf{D}_\mu = \mathrm{Ad}(M_\lambda D_\mu)$, and scalar multiples of $M_\lambda D_\mu$ do not preserve $\mathcal{T}$ unless $\lambda = \mu = 0$. So $U = e^{i\psi} I$ and also $K = \pi^{-1}(\mathrm{Ad}\, U) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Let N and H be groups and let α be a homomorphism from H into the automorphism group of N. We follow the conventions of [AB95] and define the *semidirect product* $N \rtimes_\alpha H$ of N and H with respect to α as the group with elements $N \times H$ and group operation

$$(n_1, h_1)(n_2, h_2) = \big(n_1 \alpha(h_1) n_2, h_1 h_2\big). \tag{4.24}$$

Given a matrix or column vector m, we write m^{t} for the transpose of m.

Theorem 4.10. *Let* $\theta : \mathrm{Ad}\,\big(\mathcal{U}(\widehat{\mathcal{L}}_{\mathrm{FB}})\big) \to SL_3(\mathbb{R})$ *be the map*

$$\theta : \pi(K)\mathbf{M}_\lambda \mathbf{D}_\mu \mapsto \left(\begin{array}{c|cc} 1 & \mu & \lambda \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{K}} \\ 0 & & \end{array}\right).$$

Then θ is a well defined injective homomorphism which implements an isomorphism between $\mathrm{Ad}\,\big(\mathcal{U}(\widehat{\mathcal{L}}_{\mathrm{FB}})\big)$ *and*

$$\theta\big(\mathrm{Ad}(\mathcal{U}(\widehat{\mathcal{L}}_{\mathrm{FB}}))\big) = \left\{ \left(\begin{array}{c|cc} 1 & \mu & \lambda \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{K}} \\ 0 & & \end{array}\right) \,\middle|\, \lambda, \mu \in \mathbb{R},\ K \in SL_2(\mathbb{R}) \right\}.$$

This matrix group is isomorphic to the semidirect product $\mathbb{R}^2 \rtimes_\alpha SL_2(\mathbb{R})$ *of the additive group* $\mathbb{R}^2$ *and* $SL_2(\mathbb{R})$, *where* $\alpha : SL_2(\mathbb{R}) \to \mathrm{Aut}\,\mathbb{R}^2$ *is the homomorphism*

$$\alpha(K)v = (K^{-1})^{\mathrm{t}} v, \quad v \in \mathbb{R}^2,\ K \in SL_2(\mathbb{R}).$$

Moreover, the restriction of θ to $\mathrm{Ad}\,\big(\mathcal{U}(\mathcal{L}_{\mathrm{FB}})\big) = \{\mathrm{Ad}\, U \mid U \in \mathcal{U}(\mathcal{L}_{\mathrm{FB}})\}$ *is an isomorphism between* $\mathrm{Ad}\,\big(\mathcal{U}(\mathcal{L}_{\mathrm{FB}})\big)$ *and*

$$\theta\big(\mathrm{Ad}(\mathcal{U}(\mathcal{L}_{\mathrm{FB}}))\big) = \left\{ \left(\begin{array}{c|cc} 1 & \mu & \lambda \\ \hline 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{array}\right) \,\middle|\, \lambda, \mu, t \in \mathbb{R} \right\}.$$

Proof. The discussion preceding the theorem statement shows that θ is well defined on $\mathrm{Ad}\,\big(\mathcal{U}(\widehat{\mathcal{L}}_{\mathrm{FB}})\big)$. We claim that θ is a homomorphism. In fact, for λ, μ, s in $\mathbb{R}$ and J, K in $SL_2(\mathbb{R})$, we have

$$\theta(\mathbf{M}_\lambda)\theta(\mathbf{F}) = \left(\begin{array}{c|cc} 1 & 0 & \lambda \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -1 & 0 \end{array}\right) = \left(\begin{array}{c|cc} 1 & -\lambda & 0 \\ \hline 0 & 0 & 1 \\ 0 & -1 & 0 \end{array}\right)$$

$$= \theta(\mathbf{F})\theta(\mathbf{D}_{-\lambda}) = \theta(\mathbf{F}\mathbf{D}_{-\lambda});$$

$$\theta(\mathbf{D}_\mu)\theta(\mathbf{F}) = \left(\begin{array}{c|cc} 1 & \mu & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & -1 & 0 \end{array}\right) = \left(\begin{array}{c|cc} 1 & 0 & \mu \\ \hline 0 & 0 & 1 \\ 0 & -1 & 0 \end{array}\right)$$

$$= \theta(\mathbf{F})\theta(\mathbf{M}_\mu) = \theta(\mathbf{F}\mathbf{M}_\mu);$$

$$\theta(\mathbf{D}_\mu)\theta(\mathbf{M}_\lambda) = \left(\begin{array}{c|cc} 1 & \mu & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) \left(\begin{array}{c|cc} 1 & 0 & \lambda \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|cc} 1 & \mu & \lambda \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

$$= \theta(\mathbf{M}_\mu)\theta(\mathbf{D}_\lambda) = \theta(\mathbf{M}_\mu\mathbf{D}_\lambda);$$

$$\theta(\mathbf{M}_\lambda)\theta(\mathbf{M}_{\phi_s}) = \left(\begin{array}{c|cc} 1 & 0 & \lambda \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & s \\ 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|cc} 1 & 0 & \lambda \\ \hline 0 & 1 & s \\ 0 & 0 & 1 \end{array}\right)$$

$$= \theta(\mathbf{M}_{\phi_s})\theta(\mathbf{M}_\lambda) = \theta(\mathbf{M}_{\phi_s}\mathbf{M}_\lambda);$$

$$\theta(\mathbf{D}_\mu)\theta(\mathbf{M}_{\phi_s}) = \left(\begin{array}{c|cc} 1 & \mu & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & s \\ 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|cc} 1 & \mu & \mu s \\ \hline 0 & 1 & s \\ 0 & 0 & 1 \end{array}\right)$$

$$= \theta(\mathbf{M}_{\phi_s})\theta(\mathbf{M}_{\mu s})\theta(\mathbf{D}_\mu) = \theta(\mathbf{M}_{\phi_s}\mathbf{M}_{\mu s}\mathbf{D}_\mu);$$

$$\theta(\mathbf{M}_\lambda)\theta(\mathbf{M}_\mu) = \left(\begin{array}{c|cc} 1 & 0 & \lambda \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) \left(\begin{array}{c|cc} 1 & 0 & \mu \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|cc} 1 & 0 & \lambda+\mu \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

$$= \theta(\mathbf{M}_\mu)\theta(\mathbf{M}_\lambda) = \theta(\mathbf{M}_{\lambda+\mu});$$

$$\theta(\mathbf{D}_\lambda)\theta(\mathbf{D}_\mu) = \left(\begin{array}{c|cc} 1 & \lambda & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) \left(\begin{array}{c|cc} 1 & \mu & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|cc} 1 & \lambda+\mu & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

$$= \theta(\mathbf{D}_\mu)\theta(\mathbf{D}_\lambda) = \theta(\mathbf{D}_{\lambda+\mu});$$

$$\theta(\pi(K))\theta(\pi(J)) = \theta(\pi(KJ));$$

$$\theta(\pi(K)\mathbf{M}_\lambda\mathbf{D}_\mu) = \theta(\pi(K))\theta(\mathbf{M}_\lambda)\theta(\mathbf{D}_\mu).$$

In short, θ respects equations (4.20) and so the process of "putting into normal form". It follows that θ is indeed a homomorphism. Plainly θ has trivial kernel, so θ is injective and so is a group isomorphism onto its range.

Let us define the map $\psi : \mathbb{R}^2 \rtimes_\alpha SL_2(\mathbb{R}) \to \theta\big(\mathrm{Ad}(\mathcal{U}(\widehat{\mathcal{L}}_{\mathrm{FB}}))\big)$ by

$$\psi : (v, K) \mapsto \left(\begin{array}{c|c} 1 & v^{\mathrm{t}}K \\ \hline 0 & K \end{array}\right), \quad v \in \mathbb{R}^2,\ K \in SL_2(\mathbb{R}).$$

Then ψ is plainly injective and since

$$\psi\big(((K^{-1})^{\mathrm{t}}v, K)\big) = \left(\begin{array}{c|c} 1 & v^{\mathrm{t}} \\ \hline 0 & K \end{array}\right), \quad v \in \mathbb{R}^2,\ K \in SL_2(\mathbb{R})$$

it follows that ψ is a bijection. Let $v_1, v_2 \in \mathbb{R}^2$ and let $K_1, K_2 \in SL_2(\mathbb{R})$. By (4.24),

$$\begin{aligned} \psi\big((v_1, K_1)(v_2, K_2)\big) &= \psi\big((v_1 + (K_1^{-1})^{\mathrm{t}}v_2, K_1K_2)\big) \\ &= \left(\begin{array}{c|c} 1 & v_1^{\mathrm{t}}K_1K_2 + v_2^{\mathrm{t}}K_2 \\ \hline 0 & K_1K_2 \end{array}\right) \\ &= \left(\begin{array}{c|c} 1 & v_1^{\mathrm{t}}K_1 \\ \hline 0 & K_1 \end{array}\right)\left(\begin{array}{c|c} 1 & v_2^{\mathrm{t}}K_2 \\ \hline 0 & K_2 \end{array}\right) \\ &= \psi\big((v_1, K_1)\big)\psi\big((v_2, K_2)\big). \end{aligned}$$

Thus ψ is an isomorphism of $\mathbb{R}^2 \rtimes_\alpha SL_2(\mathbb{R})$ onto $\theta\big(\mathrm{Ad}(\mathcal{U}(\widehat{\mathcal{L}}_{\mathrm{FB}}))\big)$.

By (4.7),

$$\mathrm{Ad}\,\big(\mathcal{U}(\mathcal{L}_{\mathrm{FB}})\big) = \langle \mathbf{M}_\lambda, \mathbf{D}_\mu, \mathbf{V}_t \mid \lambda, \mu, t \in \mathbb{R}\rangle.$$

Moreover, by Lemma 4.5,

$$\begin{aligned} \pi^{-1}(\mathbf{V}_t) &= \pi^{-1}(\mathbf{M}_{\phi_{\exp(t)}}\mathbf{F}\mathbf{M}_{\phi_{\exp(-t)}}\mathbf{F}\mathbf{M}_{\phi_{\exp(t)}}\mathbf{F}) \\ &= \begin{pmatrix} 1 & e^t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & e^{-t} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & e^t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}. \end{aligned}$$

Thus θ restricts to give a group isomorphism between

$$\theta\big(\mathrm{Ad}(\mathcal{U}(\mathcal{L}_{\mathrm{FB}}))\big) = \left\{ \left(\begin{array}{c|cc} 1 & \mu & \lambda \\ \hline 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{array}\right) \,\middle|\, \lambda, \mu, t \in \mathbb{R} \right\}$$

and $\mathrm{Ad}\,\big(\mathcal{U}(\mathcal{L}_{\mathrm{FB}})\big)$, as claimed. $\square$

Chapter 5

The hyperbolic algebra

Recall that for $t \in \mathbb{R}$, the dilation operator V_t is the unitary operator on $L^2(\mathbb{R})$ given by

$$V_t f(x) = e^{t/2} f(e^t x), \quad f \in L^2(\mathbb{R}),\ x \in \mathbb{R}.$$

In this chapter, we consider the algebra obtained when we replace the translation semigroup $\{D_\mu \mid \mu \geq 0\}$ of generators for the parabolic algebra with the dilation semigroup $\{V_t \mid t \geq 0\}$. This algebra was first studied in [KP02]. We show that it is reflexive and give an alternative proof of a strong operator topology limit of projections established in [KP02].

Definition 5.1. The *hyperbolic algebra* $\mathcal{A}_\mathrm{h}$ is the w*-closed operator algebra

$$\mathcal{A}_\mathrm{h} = \text{w*-alg}\{M_\lambda, V_t \mid \lambda, t \geq 0\}. \tag{5.1}$$

For $s \in \mathbb{R}$, let $d_s : \mathbb{R} \to \mathbb{C}$ be the unimodular function $d_s(x) = |x|^{is}$, and let $\mathcal{L}_M$ and $\mathcal{L}_S$ be the subspace lattices

$$\mathcal{L}_M = \{L^2([-a, b]) \mid a, b \in [0, \infty]\},$$
$$\mathcal{L}_S = \{d_s H^2(\mathbb{R}) \mid s \in \mathbb{R}\} \cup \{(0), L^2(\mathbb{R}_+), L^2(\mathbb{R}_-), L^2(\mathbb{R})\}.$$

The *dilation lattice* is the subspace lattice $\mathcal{L}_M \cup \mathcal{L}_S$. We will also write $\mathcal{A}_M = \operatorname{Alg}\mathcal{L}_M$ and $\mathcal{A}_S = \operatorname{Alg}\mathcal{L}_S$. The *dilation lattice algebra* $\mathcal{A}_{\mathrm{DL}}$ is the operator algebra

$$\mathcal{A}_{\mathrm{DL}} = \operatorname{Alg}(\mathcal{L}_M \cup \mathcal{L}_S).$$

Plainly $\mathcal{A}_{\mathrm{DL}} = \mathcal{A}_M \cap \mathcal{A}_S$.

We use the name "hyperbolic" for $\mathcal{A}_{\mathrm{h}}$ since the generators V_t are implemented by dilation, a hyperbolic action on the upper half-plane. Observe that for $s, t \in \mathbb{R}$ and $a, b \in [0, \infty)$, the subspaces $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$ are reduced by V_t,

$$V_t L^2([-a, b]) = L^2([-ae^{-t}, be^{-t}]) \quad \text{and} \quad V_t\big(d_s H^2(\mathbb{R})\big) = d_s H^2(\mathbb{R}).$$

So if $t \geq 0$, every subspace in the dilation lattice is invariant for the dilation operator V_t, which explains our nomenclature.

5.1 Reflexivity

We follow the argument of [LP03]. Observe that

$$\mathcal{A}_{\mathrm{h}} \subseteq \mathcal{A}_{\mathrm{DL}}, \tag{5.2}$$

since each of the generators of $\mathcal{A}_{\mathrm{h}}$ leaves each subspace in the dilation lattice invariant.

As in Chapter 3, the key to identifying the two algebras will be to show that they contain the same Hilbert-Schmidt operators and a bounded approximate identity of Hilbert-Schmidt operators; this will show that the Hilbert-Schmidt operators are dense in each algebra and so the algebras are equal.

Let Q be the first and third quadrants of the plane,

$$Q = \{(x, y) \in \mathbb{R}^2 \mid xy \geq 0\}.$$

We first show that considering functions supported on Q is a natural thing for us to do.

Lemma 5.2. *Let* $\operatorname{Int} k$ *be a Hilbert-Schmidt operator leaving both of the subspaces* $L^2(\mathbb{R}_+)$ *and* $L^2(\mathbb{R}_-)$ *invariant. Then* $\operatorname{supp} k \subseteq Q$.

Proof. Let $(e_n)_{n\geq 1}$ be a basis for $L^2(\mathbb{R}_+)$ and extend it to a basis $(e_n)_{n\in\mathbb{Z}^*}$ for $L^2(\mathbb{R})$ where $\mathbb{Z}^*$ is the set of non-zero integers. A basis for $L^2(\mathbb{R}^2)$ is

$$\{e_m \otimes e_n \mid m, n \in \mathbb{Z}^*\}$$

and a basis for $L^2(Q)$ is

$$\{e_m \otimes e_n \mid m, n \in \mathbb{Z}^* \text{ and } m, n \text{ are either both positive or both negative}\}.$$

Thus if the essential support of the non-zero function $k \in L^2(\mathbb{R}^2)$ is not contained in Q, there is a pair (m, n) of distinct non-zero integers such that $\langle k, e_m \otimes e_n \rangle \neq 0$ and exactly one of m and n is positive and the other is negative. So

$$\begin{aligned}\langle k, e_m \otimes e_n \rangle &= \int_{\mathbb{R}} \int_{\mathbb{R}} k(x,y) \overline{e_m(x) e_n(y)}\, dy\, dx \\ &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} k(x,y) \overline{e_n}(y)\, dy \Big) \overline{e_m}(x)\, dx \neq 0.\end{aligned}$$

So

$$\langle (\operatorname{Int} k)\overline{e_n}, e_m \rangle \neq 0.$$

If $n > 0$, then $\overline{e_n} \in L^2(\mathbb{R}_+)$ and $e_m \in L^2(\mathbb{R}_-)$, so $\operatorname{Int} k$ does not leave $L^2(\mathbb{R}_+)$ invariant, contrary to hypothesis. Similarly, if $n < 0$, then $\overline{e_n} \in L^2(\mathbb{R}_-)$ and $e_m \in L^2(\mathbb{R}_+)$, so $\operatorname{Int} k$ cannot leave $L^2(\mathbb{R}_-)$ invariant. So such a pair (m, n) does not exist and k must be supported on Q. □

Let p and q be the functions defined on the non-zero real numbers by

$$p(x) = \begin{cases} x^{1/2} & x > 0, \\ i|x|^{1/2} & x < 0, \end{cases} \qquad q(x) = \begin{cases} x^{-1/2} & x > 0, \\ -i|x|^{-1/2} & x < 0. \end{cases}$$

Observe that p and q are the restrictions to $\mathbb{R}\setminus\{0\}$ of branches of the analytic functions $z \mapsto z^{\pm 1/2}$ defined on the cut plane $\mathbb{C} \setminus i\mathbb{R}_-$ and that $q = p^{-1}$.

Given $k \in L^2(Q)$, let $\Theta_{\mathrm{h}}(k) : \mathbb{R}^2 \to \mathbb{C}$ be the function

$$\Theta_{\mathrm{h}}(k)(x,t) = \overline{p}(x)e^{t/2}k(x, e^t x), \quad x, t \in \mathbb{R}.$$

Lemma 5.3. *The mapping $\Theta_{\mathrm{h}} : L^2(Q) \to L^2(\mathbb{R}^2)$, $k \mapsto \Theta_{\mathrm{h}}(k)$ is unitary.*

Proof. The map Θ_{h} is plainly linear $L^2(Q) \to L^2(\mathbb{R}^2)$. The inverse mapping Θ_{h}^{-1} is given by

$$\Theta_{\mathrm{h}}^{-1}(j)(x,y) = \begin{cases} \overline{q}(x)\sqrt{\frac{x}{y}}\, j\big(x, \log(y/x)\big) & \text{if } xy > 0, \\ 0 & \text{otherwise,} \end{cases} \qquad x, y \in \mathbb{R}, \tag{5.3}$$

so Θ_{h} is bijective. Since $|\overline{p}(x)|^2 = |x|$ for x a non-zero real number,

$$\begin{aligned} \|\Theta_{\mathrm{h}}(k)\|^2_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}}\int_{\mathbb{R}} |\overline{p}(x)e^{t/2}k(x,e^t x)|^2\, dt\, dx \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} |k(x,e^t x)|^2\, e^t|x| dt\, dx \\ &= \iint_Q |k(x,y)|^2\, dy\, dx \\ &= \|k\|^2_{L^2(Q)}. \end{aligned}$$

Hence the mapping Θ_{h} is a bijective isometry, so Θ_{h} is indeed unitary $L^2(Q) \to L^2(\mathbb{R}^2)$. □

Recall that for $v \in \mathbb{H}^-$, $b_v \in H^2(\mathbb{R})$ is the function $b_v(x) = (x-v)^{-1}$. For $t \in \mathbb{R}$, let Σ_t be the countable set of functions

$$\Sigma_t = \{p(x)e^{isx}b_v(x)(V_t b_w)(x) \mid s \geq 0,\ s \in \mathbb{Q},\ v, w \in \mathbb{H}^-_{\mathbb{Q}}\}.$$

It is trivial to verify that functions in Σ_t are the boundary value functions of functions which lie in $H^2(\mathbb{H}^+)$, so $\Sigma_t \subseteq H^2(\mathbb{R})$.

Lemma 5.4. *For each $t \in \mathbb{R}$, the countable set Σ_t has dense linear span in $H^2(\mathbb{R})$. Moreover, the set*

$$\mathcal{Z} = \{h \in H^2(\mathbb{R}) \mid ph \in H^\infty(\mathbb{R})\}$$

is dense in $H^2(\mathbb{R})$.

Proof. Let Σ be the set

$$\Sigma = \{p(x)e^{isx}b_v(x)b_w(x) \mid s \geq 0,\ v, w \in \mathbb{H}^-\}.$$

Plainly the closure of Σ_0 contains Σ, so the lemma will follow for $t = 0$ if we can show that Σ has dense linear span in $H^2(\mathbb{R})$. Let κ be the greatest common inner divisor of the set Σ. By Proposition 2.15, the linear span of Σ is dense in $\kappa H^2(\mathbb{R})$.

We claim that κ is a constant unimodular function. This will follow if we show that for $v, w \in \mathbb{H}^-$, the function pb_vb_w is outer. Using notation from §2.1, let U_2 be the unitary isomorphism $U_2 : H^2(\mathbb{T}) \to H^2(\mathbb{R})$,

$$U_2 f(x) = \frac{1}{\pi^{1/2}(x+i)} f\big(\omega^{-1}(x)\big), \quad f \in H^2(\mathbb{T}),\ x \in \mathbb{R}$$

where $\omega : \overline{\mathbb{D}} \setminus \{1\} \to \overline{\mathbb{H}^+}$ is the bijection

$$\omega(z) = i\frac{1+z}{1-z}, \quad z \in \overline{\mathbb{D}} \setminus \{1\} = \{z \in \mathbb{C} \mid |z| \leq 1,\ z \neq 1\}$$

with biholomorphic restriction $\mathbb{D} \to \mathbb{H}^+$. Then ω^{-1} is given by

$$\omega^{-1}(z) = \frac{z-i}{z+i}, \quad z \in \overline{\mathbb{H}^+}$$

and $\omega|\mathbb{T} \setminus \{1\}$ is a bijection $\mathbb{T} \setminus \{1\} \to \mathbb{R}$. The inverse of U_2 is

$$\begin{aligned} U_2^{-1} g(z) &= \pi^{1/2}(\omega(z)+i)g\big(\omega(z)\big) \\ &= 2i\pi^{1/2}(1-z)^{-1} g\big(\omega(z)\big), \quad z \in \mathbb{T} \setminus \{1\},\ g \in H^2(\mathbb{R}). \end{aligned}$$

By definition, a function $g \in H^2(\mathbb{R})$ is outer if and only if $U_2^{-1}g \in H^2(\mathbb{T})$ is outer. Fix $v, w \in \mathbb{H}^-$ and let $g = pb_vb_w$ and $f = U_2^{-1}g$. Then

$$f(z) = 2i\pi^{1/2}(1-z)^{-1}p\big(\omega(z)\big)b_v\big(\omega(z)\big)b_w\big(\omega(z)\big), \quad z \in \mathbb{T}.$$

Since $\operatorname{Re} p\big(\omega(z)\big) \geq 0$ for each $z \in \mathbb{D}$, by Theorem 2.10, $p \circ \omega \in H^r(\mathbb{T})$ for each $r \in (0,1)$ and $p \circ \omega$ is outer. Moreover, writing $\lambda = \omega^{-1}(v)$,

$$b_v\big(\omega(z)\big) = \frac{1}{\omega(z) - \omega(\lambda)} = \frac{i(1-z)(1-\lambda)}{2(\lambda - z)}.$$

Thus

$$U_2^{-1}b_v(z) = \pi^{1/2}\frac{\lambda - 1}{\lambda - z}.$$

The function $z \mapsto \lambda - z$ is certainly in $H^2(\mathbb{T})$. Since $b_v \in H^2(\mathbb{R})$ and U_2^{-1} maps $H^2(\mathbb{R}) \to H^2(\mathbb{T})$, the function $U_2^{-1}b_v$ is in $H^2(\mathbb{T})$. So $U_2^{-1}b_v$ and $1/(U_2^{-1}b_v)$ both lie in $H^2(\mathbb{T})$. By Proposition 2.11, $U_2^{-1}b_v$ is outer.

We now claim that $b_w \circ \omega$ is also outer; for if $\mu = \omega^{-1}(w)$, then $|\mu| > 1$, so

$$(b_w \circ \omega)(z) = \frac{i(1-z)(1-\mu)}{2(\mu - z)} \in H^\infty(\mathbb{T})$$

and by Theorem 2.10, the function $z \mapsto 1 - z$ is outer. Since $U_2^{-1}b_w$ is outer and $(b_w \circ \omega)(z)$ is a non-zero constant multiple of $(1-z)U_2^{-1}b_w(z)$, it follows that $b_w \circ \omega$ is also outer.

By Lemma 2.11, the product f of these three outer functions is also outer, so g is outer in $H^2(\mathbb{R})$. It follows that κ is trivial and Σ does indeed have dense linear span in $H^2(\mathbb{R})$, as does Σ_0.

For $t \in \mathbb{R}$, since

$$V_t b_w(x) = \frac{e^{t/2}}{e^t x - w} = e^{-t/2}b_{e^{-t}w},$$

the closed linear span of Σ_t contains the closed linear span of Σ, so Σ_t also has dense linear span in $H^2(\mathbb{R})$.

Finally, let $\Lambda_0 \subseteq H^2(\mathbb{R})$ be the set

$$\Lambda_0 = \{b_v b_w \mid v, w \in \mathbb{H}_{\mathbb{Q}}^-\}.$$

Each $h \in \Lambda_0$ satisfies $ph \in H^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) = H^2(\mathbb{R}) \cap H^\infty(\mathbb{R})$. By Lemma 3.3, Λ_0 has dense linear span in $H^2(\mathbb{R})$. Since $\mathcal{Z}$ contains the linear span of Λ_0, the proof is complete. □

Proposition 5.5. *Let $k \in L^2(Q)$.*

(i). If $\operatorname{Int} k \in \mathcal{A}_S$, then $\Theta_{\mathrm{h}}(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R})$.

(ii). If $\operatorname{Int} k \in \mathcal{A}_M$, then $\Theta_{\mathrm{h}}(k) \in L^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+)$.

In particular,

$$\mathcal{A}_{\mathrm{DL}} \cap \mathcal{C}_2 \subseteq \{\operatorname{Int} k \mid \Theta_{\mathrm{h}}(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+)\}. \tag{5.4}$$

Proof. (i). By Lemma 5.3, $\Theta_{\mathrm{h}}(k) \in L^2(\mathbb{R}^2)$, so $t \mapsto \Theta_{\mathrm{h}}(k)(x,t) \in L^2(\mathbb{R})$ for almost every $x \in \mathbb{R}$. Let $k \in L^2(Q)$ with $\operatorname{Int} k \in \mathcal{A}_S \cap \mathcal{C}_2$. Then for every $s \in \mathbb{R}$,

$$(\operatorname{Int} k) d_s H^2(\mathbb{R}) \subseteq d_s H^2(\mathbb{R}) = \big(d_s \overline{H^2(\mathbb{R})}\big)^\perp.$$

So if f and g are functions in $H^2(\mathbb{R})$,

$$\begin{aligned}
0 &= \langle (\operatorname{Int} k) d_s f, d_s \overline{g} \rangle \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} k(x,y) |y|^{is} f(y) |x|^{-is} g(x)\, dy\, dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{q}(x) e^{-t/2} \Theta_{\mathrm{h}}(k)(x,t) e^{ist} f(e^t x) g(x)\, e^t |x|\, dx\, dt \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \Theta_{\mathrm{h}}(k)(x,t)\, p(x) g(x) V_t f(x)\, dx\, e^{ist}\, dt.
\end{aligned} \tag{5.5}$$

Note that we have used the condition $\operatorname{supp} k \subseteq Q$ to make the change of variables $y = e^t x$ in the third line of this calculation. Let $\Phi : \mathbb{R} \to \mathbb{C}$ be the function

$$\Phi(t) = \int_{\mathbb{R}} \Theta_{\mathrm{h}}(k)(x,t)\, p(x) g(x) V_t f(x)\, dx, \quad t \in \mathbb{R}.$$

Since $\operatorname{Int}|k| \in \mathcal{C}_2$ and $|f|, |g| \in L^2(\mathbb{R})$,

$$\begin{aligned}\int_{\mathbb{R}} |\Phi(t)|\, dt &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} |\Theta_{\mathrm{h}}(k)(x,t)\, p(x) g(x) V_t f(x)|\, dx\, dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} |k(x,y) g(x) f(y)|\, dx\, dy \\ &= \langle (\operatorname{Int}|k|)|f|, |g|\rangle < \infty,\end{aligned}$$

so the function Φ is integrable. By (5.5),

$$\int_{\mathbb{R}} \Phi(t) e^{ist} dt = 0 \qquad \text{for every } s \text{ in } \mathbb{R},$$

so Φ has zero Fourier transform. Since Φ is in $L^1(\mathbb{R})$, it follows by Proposition 2.17 that $\Phi = 0$; that is, for almost every t,

$$\int_{\mathbb{R}} \Theta_{\mathrm{h}}(k)(x,t)\, p(x) g(x) V_t f(x)\, dx = 0. \tag{5.6}$$

Thus for every pair (f, g) with $f, g \in H^2(\mathbb{R})$, there is a conull set $T(f,g) \subseteq \mathbb{R}$ such that (5.6) holds for $t \in T(f,g)$. Let Γ be the set of pairs of $H^2(\mathbb{R})$ functions

$$\Gamma = \{(b_w, e^{isx} b_v) \mid s \geq 0,\ s \in \mathbb{Q},\ v, w \in \mathbb{H}^-_{\mathbb{Q}}\}.$$

The set Γ is countable, so

$$T = \bigcap_{(f,g)\in\Gamma} T(f,g)$$

is conull and (5.6) holds for every $t \in T$ and every $(f,g) \in \Gamma$. Fix $t \in T$; then

$$\Sigma_t = \{p g V_t f \mid (f,g) \in \Gamma\}$$

has dense linear span in $H^2(\mathbb{R})$ by Lemma 5.4. Moreover,

$$\int_{\mathbb{R}} \Theta_{\mathrm{h}}(k)(x,t) h(x)\, dx = 0 \qquad \text{for every } h \in \Sigma_t.$$

Thus for every $t \in T$,

$$x \mapsto \Theta_{\mathrm{h}}(k)(x,t) \in \left(\overline{H^2(\mathbb{R})}\right)^{\perp} = H^2(\mathbb{R}).$$

By Lemma 2.21 we must have $\Theta_{\mathrm{h}}(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R})$.

(ii). It is routine to show that the condition $\operatorname{Int} k \in \mathcal{A}_M$ is equivalent to the essential support of k being contained in the cone swept out by the y-axis and the lines $y = \nu x$ for ν in $[1, \infty)$. Hence the essential support of $\Theta_{\mathrm{h}}(k)$ must be contained in the upper half-plane $\{(x,t) \mid t \geq 0\}$, and Lemma 5.3 shows that $\Theta_{\mathrm{h}}(k) \in L^2(\mathbb{R}^2)$. Since

$$\{j \in L^2(\mathbb{R}^2) \mid \operatorname{supp} j \subseteq \{(x,t) \mid t \geq 0\}\} = L^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+),$$

this completes the proof of (ii).

Finally, since $\mathcal{A}_{\mathrm{DL}} \cap \mathcal{C}_2 = (\mathcal{A}_M \cap \mathcal{C}_2) \cap (\mathcal{A}_S \cap \mathcal{C}_2)$, (i) and (ii) together establish (5.4). $\square$

As in Chapter 3, we will show that the inclusion (5.4) is in fact an equality. The next two results are analogues of Proposition 3.6 and Proposition 3.7.

First, we apply Lemma 3.5 to the strong operator topology continuous one-parameter group of unitaries $\{V_t \mid t \in \mathbb{R}\}$ and the semigroup

$$\{V_t \mid t \geq 0\} \subseteq \mathcal{A}_{\mathrm{h}}$$

to define operators V_φ as follows.

Proposition 5.6. *Let $\varphi \in L^1(\mathbb{R}_+)$. Then the sesquilinear form*

$$\tau_\varphi(f,g) = \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(t) V_t f(x) \overline{g(x)}\, dx\, dt, \quad f, g \in L^2(\mathbb{R})$$

is bounded, and there is a unique bounded linear operator V_φ such that $\langle V_\varphi f, g\rangle = \tau_\varphi(f,g)$ for f and g in $L^2(\mathbb{R})$. Moreover, $\|V_\varphi\| \leq \|\varphi\|_{L^1(\mathbb{R})}$. If φ has compact essential support then $V_\varphi \in \mathcal{A}_{\mathrm{h}}$. $\square$

Proposition 5.7. *Let $h \in H^2(\mathbb{R})$ and $\varphi \in L^2(\mathbb{R}_+)$. Let $h \otimes \varphi$ denote the function $(x,t) \mapsto h(x)\varphi(t)$ and let $k = \Theta_{\mathrm{h}}^{-1}(h \otimes \varphi)$. Then $\operatorname{Int} k \in \mathcal{A}_{\mathrm{h}} \cap \mathcal{C}_2$. Moreover, if also ph is in $H^\infty(\mathbb{R})$ and $\varphi \in L^1(\mathbb{R})$, then $\operatorname{Int} k = M_{ph} V_\varphi$.*

Proof. By (2.5) and Lemma 5.3,

$$\|\operatorname{Int} k\|_{\mathcal{C}_2} = \|k\|_{L^2(\mathbb{R}^2)} = \|h \otimes \varphi\|_{L^2(\mathbb{R}^2)} = \|h\|_{L^2(\mathbb{R})}\|\varphi\|_{L^2(\mathbb{R})},$$

so if $h_n \to h$ in $L^2(\mathbb{R})$ and $\varphi_n \to \varphi$ in $L^2(\mathbb{R})$ then by Lemma 5.3 the operators $\operatorname{Int} \Theta_{\mathrm{h}}^{-1}(h_n \otimes \varphi_n)$ converge to $\operatorname{Int} k$ in Hilbert-Schmidt norm and so in operator norm. By Lemma 5.4, we may therefore assume that $h \in H^2(\mathbb{R})$ with $ph \in H^\infty(\mathbb{R})$. Moreover, we may assume that φ has compact support, since the sequence $\varphi_n = \varphi\chi_{[0,n]}$ converges to φ in $L^2(\mathbb{R})$. By the Cauchy-Schwarz inequality, the function φ lies in $L^1(\mathbb{R}_+)$.

If f and g are in $L^2(\mathbb{R})$, then by (5.3) and Proposition 5.6,

$$\begin{aligned}
\langle (\operatorname{Int} k)f, g\rangle &= \int_{\mathbb{R}}\int_{\mathbb{R}} \big(\Theta_{\mathrm{h}}^{-1}(h\otimes\varphi)\big)(x,y) f(y)\overline{g(x)}\,dy\,dx \\
&= \int_{\mathbb{R}}\int_{\{y \mid xy>0\}} \overline{q}(x)\sqrt{\frac{x}{y}}\,h(x)\varphi\big(\log(y/x)\big) f(y)\overline{g(x)}\,dy\,dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} p(x)h(x)\varphi(t)V_t f(x)\overline{g(x)}\,dt\,dx \\
&= \langle M_{ph}V_\varphi f, g\rangle.
\end{aligned}$$

So $\operatorname{Int} k = M_{ph}V_\varphi$. Since $ph \in H^\infty(\mathbb{R})$, $M_{ph} \in \mathcal{A}_{\mathrm{h}}$. By Proposition 5.6, $V_\varphi \in \mathcal{A}_{\mathrm{h}}$. So the product $\operatorname{Int} k = M_{ph}V_\varphi$ is also in $\mathcal{A}_{\mathrm{h}}$. □

Proposition 5.8. *If* $\nu \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+)$ *then* $\operatorname{Int}\Theta_{\mathrm{h}}^{-1}(\nu) \in \mathcal{A}_{\mathrm{h}} \cap \mathcal{C}_2$. *So*

$$\{\operatorname{Int} k \mid \Theta_{\mathrm{h}}(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+)\} \subseteq \mathcal{A}_{\mathrm{h}} \cap \mathcal{C}_2.$$

Proof. This follows from Lemma 5.3 and Proposition 5.7 exactly as in the proof of Proposition 3.8. □

Corollary 5.9. *The hyperbolic and dilation lattice algebras contain the same Hilbert-Schmidt operators. Explicitly,*

$$\mathcal{A}_{\mathrm{h}} \cap \mathcal{C}_2 = \mathcal{A}_{\mathrm{DL}} \cap \mathcal{C}_2 = \{\operatorname{Int} k \mid \Theta_{\mathrm{h}}(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+)\}.$$

Proof. The inclusion $\mathcal{A}_{\mathrm{h}} \subseteq \mathcal{A}_{\mathrm{DL}}$ was established in (5.2). So by Proposition 5.5 and Proposition 5.8,

$$\mathcal{A}_{\mathrm{h}} \cap \mathcal{C}_2 \subseteq \mathcal{A}_{\mathrm{DL}} \cap \mathcal{C}_2 \subseteq \{\operatorname{Int} k \mid \Theta_{\mathrm{h}}(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+)\} \subseteq \mathcal{A}_{\mathrm{h}} \cap \mathcal{C}_2. \quad \square$$

Proposition 5.10. *The hyperbolic algebra contains a bounded approximate identity of Hilbert-Schmidt operators. In other words, there is a norm-bounded sequence $(X_n)_{n\geq 1}$ of operators in $\mathcal{A}_{\mathrm{h}} \cap \mathcal{C}_2$ such that $X_n \to I$ in the strong operator topology.*

Proof. For $n \in \mathbb{N}$, let

$$h_n(x) = \frac{-n^2 p(x)}{(x+n^{-1}i)(x+ni)^2}, \qquad \varphi_n(t) = n\chi_{[0,n^{-1}]}(t)$$

and let $g_n(x) = p(x)h_n(x)$. Then $|g_n(x)| \leq 1$ for $x \in \mathbb{R}$ and $g_n(x) \to 1$ uniformly on compact subsets of $\mathbb{R} \setminus \{0\}$, so $(M_{g_n})_{n\geq 1}$ is a sequence of contractions whose strong operator topology limit is the identity. Observe that $h_n \in H^2(\mathbb{R})$ and $\varphi_n \in L^2(\mathbb{R}_+)$ for each $n \geq 1$. By Proposition 5.6, $\|V_{\varphi_n}\| \leq \|\varphi_n\|_{L^1(\mathbb{R})} = 1$ for each $n \geq 1$, so $(V_{\varphi_n})_{n\geq 1}$ is a sequence of contractions. If we pick a continuous compactly supported function f, then the integral

$$I_n(x) = \int_{\mathbb{R}} \varphi_n(t) V_t f(x)\, dt$$

converges for every x to define a continuous compactly supported function I_n, which by Proposition 5.6 coincides with $V_{\varphi_n} f$. Thus by dominated convergence,

$$\|V_{\varphi_n} f - f\|^2 = \int \left| \int_0^{1/n} n V_t f(x)\, dt - f(x) \right|^2 dx \to 0 \text{ as } n \to \infty.$$

Since such functions f are dense in $L^2(\mathbb{R})$, the sequence $V_{\varphi_n} \to I$ boundedly in the strong operator topology.

Let $X_n = \Theta_{\mathrm{h}}^{-1}(h_n \otimes \varphi_n)$. Then $X_n = M_{g_n} V_{\varphi_n} \in \mathcal{A}_{\mathrm{h}} \cap \mathcal{C}_2$ by Proposition 5.8, and $\|X_n\| \leq \|M_{g_n}\| \, \|V_{\varphi_n}\| \leq 1$. Multiplication is jointly strong operator topology continuous on bounded sets of operators, so

$$X_n = M_{g_n} V_{\varphi_n} \to I$$

in the strong operator topology as well. □

Corollary 5.11. *The dilation lattice algebra and the hyperbolic algebra are equal.*

Proof. This follows from (5.2), Corollary 5.9 and Proposition 5.10 by the same argument used to prove Corollary 3.11. □

Since $\mathcal{A}_{\mathrm{DL}}$ is plainly reflexive, we have also proven:

Theorem 5.12. *The hyperbolic algebra is reflexive.* □

We will write $\mathcal{A}_{\mathrm{h}}$ in preference to $\mathcal{A}_{\mathrm{DL}}$.

5.2 A strong operator topology limit

In [KP02], a cocycle argument is used to show that the invariant subspace lattice of the hyperbolic algebra is

$$\operatorname{Lat} \mathcal{A}_{\mathrm{h}} = \{K_{\zeta,\lambda,\mu} \mid \zeta \in \mathbb{C}^*, \ \lambda, \mu \geq 0\} \cup \mathcal{L}_M$$

where

$$K_{\zeta,\lambda,\mu} = v_\zeta e_{\lambda,\mu} H^2(\mathbb{R}), \qquad e_{\lambda,\mu}(x) = \exp i(\lambda x + \mu x^{-1})$$

and for $\zeta \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $v_\zeta : \mathbb{R} \to \mathbb{C}$ is the two-valued function

$$v_\zeta(x) = \begin{cases} 1 & x > 0, \\ \zeta & x \leq 0. \end{cases}$$

It is also shown that this lattice is compact and connected as a lattice of projections with the strong operator topology. It is not immediately apparent from this presentation that $\operatorname{Lat}\mathcal{A}_{\mathrm{h}}$ contains $\mathcal{L}_S$, although since $\mathcal{A}_{\mathrm{h}} = \operatorname{Alg}(\mathcal{L}_M \cup \mathcal{L}_S)$, we know that this must indeed be the case. This lemma, whose proof we take from [KP02], gives the explicit correspondence.

Lemma 5.13. *For $s \in \mathbb{R}$,*

$$d_s H^2(\mathbb{R}) = v_{\exp(\pi s)} H^2(\mathbb{R}).$$

Thus

$$\begin{aligned}\mathcal{L}_S &= \{v_\sigma H^2(\mathbb{R}) \mid \sigma > 0\} \cup \{(0), L^2(\mathbb{R}_+), L^2(\mathbb{R}_-), L^2(\mathbb{R})\} \\ &= \{K_{\sigma,0,0} \mid \sigma > 0\} \cup \{(0), L^2(\mathbb{R}_+), L^2(\mathbb{R}_-), L^2(\mathbb{R})\}. \qquad (5.7)\end{aligned}$$

Proof. For $s \in \mathbb{R}$, let $g_s : \mathbb{R} \to \mathbb{C}$ be the bounded function

$$g_s(x) = v_{\exp(-\pi s)}(x) d_s(x) = \begin{cases} x^{is} & x > 0, \\ e^{-\pi s}|x|^{is} & x \leq 0. \end{cases}$$

Then g_s is the boundary value function of $z \mapsto z^{is}$, a bounded holomorphic function on the upper half-plane, so $g_s \in H^\infty(\mathbb{R})$. Moreover, g_s is invertible in $H^\infty(\mathbb{R})$, so $g_s H^2(\mathbb{R}) = H^2(\mathbb{R})$. Hence

$$d_s H^2(\mathbb{R}) = v_{\exp(\pi s)} g_s H^2(\mathbb{R}) = v_{\exp(\pi s)} H^2(\mathbb{R}).$$

The result follows. $\square$

In this section we give an alternative proof of one result needed to establish the connectivity of $\operatorname{Lat}\mathcal{A}_{\mathrm{h}}$. The techniques we use owe much to two papers of Halmos ([Hal69], [Hal71]) and will resurface in the next chapter.

We recall some standard terminology and notation from the theory of unbounded operators. We use the terms *manifold* and *submanifold* to refer to linear subspaces of Hilbert spaces which are not necessarily closed; *subspace* will always mean a closed subspace. An *unbounded operator* between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ is a linear map T with domain $\mathcal{D}(T)$ and range $\mathcal{R}(T)$ where $\mathcal{D}(T)$ and $\mathcal{R}(T)$ are submanifolds of $\mathcal{H}$ and $\mathcal{K}$ respectively. In the remainder of this section, the term *operator* will typically mean an unbounded operator unless it is explicitly said to be bounded. The *graph* of T is the set

$$\operatorname{graph}(T) = \{(x, Tx) \mid x \in \mathcal{D}(T)\};$$

this is a submanifold of the Hilbert space $\mathcal{H} \oplus \mathcal{K}$. The operator T is said to be *closed* if $\operatorname{graph}(T)$ is closed in $\mathcal{H} \oplus \mathcal{K}$.

Let $T : \mathcal{D}(T) \to \mathcal{H}$ with $\mathcal{D}(T)$ a submanifold of $\mathcal{H}$. Then T is said to be *positive* if $\langle Tx, x\rangle \geq 0$ for every $x \in \mathcal{D}(T)$. If $\mathcal{D}(T)$ is dense in $\mathcal{H}$ then the *adjoint* of T is the unique operator $T^* : \mathcal{D}(T^*) \to \mathcal{H}$ satisfying

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \qquad \text{for all } x \in \mathcal{D}(T),\ y \in \mathcal{D}(T^*)$$

where $\mathcal{D}(T^*)$ is the set of vectors $y \in \mathcal{H}$ such that there exists some $y^* \in \mathcal{H}$ such that $\langle Tx, y\rangle = \langle x, y^*\rangle$ for every $x \in \mathcal{D}(T)$. Note that $\mathcal{D}(T^*)$ could be the zero subspace. If $T = T^*$ then T is said to be *self-adjoint.*

The aim of this section is to prove Proposition 4.4 of [KP02] in a transparent systematic manner. Although there is a proof in [KP02] which is considerably shorter than this one, the length of the proof presented here is largely due to technicalities arising from the use of unbounded operators.

Proposition 5.14 (Proposition 4.4 of [KP02]). *The following strong operator topology limit holds:*

$$\underset{0\leq\sigma\to\infty}{\text{SOT-lim}}[v_\sigma H^2(\mathbb{R})] = [L^2(\mathbb{R}_-)].$$

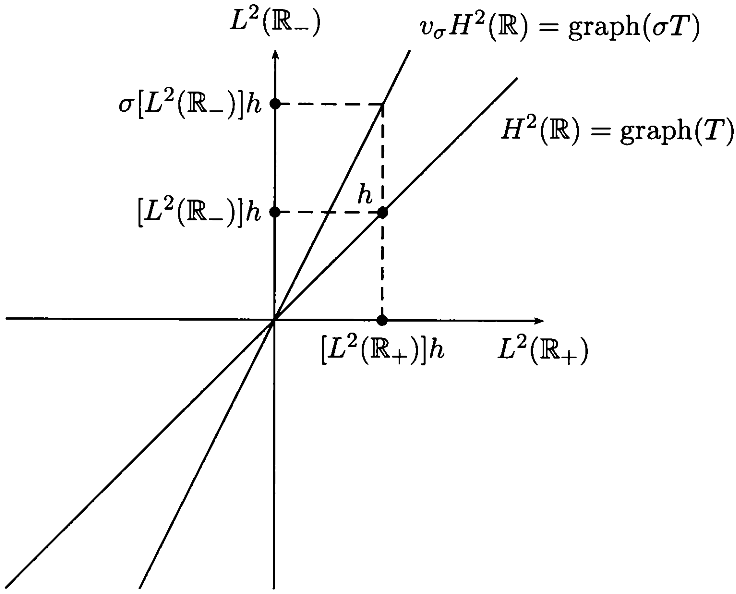


Figure 5.1: The graph picture

We take a moment to explain the intuitive picture behind our proof. Let T be the unbounded operator

$$T : [L^2(\mathbb{R}_+)]H^2(\mathbb{R}) \to L^2(\mathbb{R}_-), \quad [L^2(\mathbb{R}_+)]h \mapsto [L^2(\mathbb{R}_-)]h, \quad h \in H^2(\mathbb{R}).$$

By Corollary 2.6, each $h \in H^2(\mathbb{R})$ is determined by $[L^2(\mathbb{R}_\pm)]h = h \cdot \chi_{\mathbb{R}_\pm}$, so the operator T is well defined. It is easy to see that for $\sigma \in \mathbb{C}^*$,

$$v_\sigma H^2(\mathbb{R}) = \text{graph}(\sigma T).$$

We guide our intuition using graphs of functions $\mathbb{R} \to \mathbb{R}$, which are subsets of the plane $\mathbb{R}^2$. The operator T maps a dense subset of $L^2(\mathbb{R}_+)$ onto a dense subset of $L^2(\mathbb{R}_-)$, so we identify the x-axis with $L^2(\mathbb{R}_+)$ and the y-axis with $L^2(\mathbb{R}_-)$. Let ℓ be the graph of the function $y(x) = x$; we identify ℓ with the subspace $H^2(\mathbb{R})$. Then T corresponds to the function $y(x) = x$; for if we take a point $h \in \ell$ then $[x\text{-axis}]h$ is simply the x-coordinate of ℓ and $[y\text{-axis}]h$ is the y-coordinate of ℓ. Similarly, $\text{graph}(\sigma T)$ is identified with the graph ℓ_σ of the function $y(x) = \sigma x$. This setup is shown in Figure 5.1. In the limit $\sigma \to \infty$, the line ℓ_σ tends to the y-axis in some sense, so we also expect graph(σT) to approach $L^2(\mathbb{R}_-)$ as $\sigma \to \infty$.

Our first lemma concerns some basic properties of certain unbounded linear operators.

Lemma 5.15. *(i). Let $\mathcal{H}$ be a Hilbert space and let T be a closed linear operator $T : \mathcal{D}(T) \to \mathcal{H}$ where $\mathcal{D}(T) \subseteq \mathcal{H}$. Then* $\operatorname{graph}(T)^\perp = A \operatorname{graph}(T^*)$, *where $A \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ is the unitary operator defined by*

$$A\big((x, y)\big) = (-y, x), \quad x, y \in \mathcal{H}.$$

(ii). If $\mathcal{D}(T)$ and $\mathcal{R}(T)$ are both dense in $\mathcal{H}$ and T has trivial kernel, then $J = (I + T^*T)^{-1} : \mathcal{H} \to \mathcal{H}$ is a self-adjoint contraction in $\mathcal{L}(\mathcal{H})$, and the range of J is $\mathcal{D}(T^*T)$.

(iii). If T is also self-adjoint and positive, then T^{-1} is self-adjoint and positive.

Proof. Statements (i) and (ii) are proven in §XII.1.5 and §XII.7.1 respectively of [DS63]. For (iii), since T has trivial kernel, the operator T^{-1} exists, $\mathcal{D}(T^{-1}) = \mathcal{R}(T)$ and $\mathcal{R}(T^{-1}) = \mathcal{D}(T)$. Moreover, T is self-adjoint and $\mathcal{R}(T)$ is dense in $\mathcal{H}$, so by Lemma XII.I.6 of [DS63], $T^{-1} = (T^*)^{-1} = (T^{-1})^*$, so T^{-1} is self-adjoint. To see that T^{-1} is positive, observe that for each $x \in \mathcal{D}(T^{-1}) = \mathcal{R}(T)$,

$$\langle T^{-1}x, x\rangle = \langle T^{-1}x, TT^{-1}x\rangle = \langle T(T^{-1}x), T^{-1}x\rangle \geq 0$$

by the positivity of T. □

Lemma 5.16. *Let $\mathcal{H}$ be a Hilbert space, let $\{R_s \mid s \geq 0\} \subseteq \mathcal{L}(\mathcal{H})$ and let $\mathcal{H}_0$ be a dense subset of $\mathcal{H}$. Suppose that there is a number $C \geq 0$ such that*

$\|R_s\| \le C$ for every $s \ge 0$ and that $R_s x \to R_0 x$ as $s \to 0$ for every $x \in \mathcal{H}_0$. Then $R_s \to R_0$ in the strong operator topology as $s \to 0$.

Proof. Fix $y \in \mathcal{H}$. For $s > 0$ pick $y_s \in \mathcal{H}_0$ such that $y_s \to y$ as $s \to 0$. Then

$$\begin{aligned}\|(R_s - R_0)y\| &\le \|R_s y - R_s y_s\| + \|R_s y_s - R_0 y_s\| + \|R_0 y_s - R_0 y\| \\ &\le 2C\|y - y_s\| + \|(R_s - R_0)y_s\| \to 0 \text{ as } s \to 0. \qquad \square\end{aligned}$$

Lemma 5.17. *Let $\mathcal{H}$ be a Hilbert space and let T be a closed self-adjoint positive linear operator on $\mathcal{H}$ with trivial kernel and dense domain and image. For s a real number, set $T_s = sT$. Let J_s be the operator*

$$J_s = (I + T_s^2)^{-1}.$$

Then J_s, $T_s J_s$ and $T_s^2 J_s$ are bounded operators on $\mathcal{H}$, and the following strong operator topology limits hold as $s \to 0$.

$$\textit{(i).}\ J_s \to I, \qquad \textit{(ii).}\ T_s J_s \to 0 \qquad \textit{and} \qquad \textit{(iii).}\ T_s^2 J_s \to 0.$$

Proof. (i). By Lemma 5.15(ii), J_s is a contraction for each $s \in \mathbb{R}$. Fix $x \in \mathcal{D}(T) = \mathcal{D}(T^2)$ and $y \in \mathcal{H}$. Then

$$\langle x, y\rangle = \langle J_s(I + T_s^2)x, y\rangle = \langle J_s x, y\rangle + s^2\langle J_s T^2 x, y\rangle,$$

so

$$\langle (I - J_s)x, y\rangle = s^2\langle J_s T^2 x, y\rangle.$$

If we set $y = (I - J_s)x$, then we see that

$$\|(I - J_s)x\|^2 = s^2|\langle J_s T^2 x, (I - J_s)x\rangle| \le 2s^2\|T^2 x\|\,\|x\| \to 0 \text{ as } s \to 0.$$

Since J_s is a contraction for every $s \in \mathbb{R}$ and $\mathcal{D}(T)$ is dense in $\mathcal{H}$, Lemma 5.16 shows that $J_s \to I$ in the strong operator topology as $s \to 0$.

(ii). First observe that by Lemma 5.15(ii),

$$\mathcal{R}(J_s) = \mathcal{D}(T_s^* T_s) = \mathcal{D}(T_s^2) \subseteq \mathcal{D}(T_s),$$

so $T_s J_s$ has domain $\mathcal{H}$. By §XII.1.6 of [DS63], since J_s is bounded, $(T_s J_s)^* = J_s^* T_s^* = J_s T_s$. Thus $(T_s J_s)^* T_s J_s = J_s T_s^2 J_s$ also has domain $\mathcal{H}$ and

$$(T_s J_s)^* T_s J_s = J_s T_s^2 J_s = J_s(I + T_s^2) J_s - J_s^2 = J_s(I - J_s). \tag{5.8}$$

Moreover, since J_s is a contraction for each $s > 0$,

$$\|J_s(I - J_s)\| \leq \|J_s\| \, \|I - J_s\| \leq 2.$$

Thus for each $x \in \mathcal{H}$,

$$\begin{aligned} \|T_s J_s x\|^2 &= \langle T_s J_s x, T_s J_s x\rangle = \langle J_s(I - J_s)x, x\rangle \\ &\leq \|J_s^* x\| \, \|(I - J_s)x\| \leq \|x\| \, \|(I - J_s)x\| \\ &\leq 2\|x\|^2. \end{aligned} \tag{5.9}$$

So $T_s J_s \in \mathcal{L}(\mathcal{H})$ and (5.9) and (i) also show that $T_s J_s \to 0$ in the strong operator topology as $s \to \infty$.

(iii). Simply observe that

$$T_s^2 J_s = (I + T_s^2) J_s - J_s = I - J_s.$$

So part (iii) is equivalent to part (i). □

Lemma 5.18. *Using the hypotheses and notation of the previous lemma, let*

$$Q_s = \begin{pmatrix} J_s & T_s J_s \\ T_s J_s & T_s^2 J_s \end{pmatrix}, \quad s \in \mathbb{R}.$$

Then $Q_s \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ and Q_s is the orthogonal projection onto the graph of T_s. Moreover,

$$\operatorname*{SOT-lim}_{s \to 0} Q_s = I \oplus 0 \quad \textit{and} \quad \operatorname*{SOT-lim}_{s \to \infty} Q_s = 0 \oplus I.$$

Proof. The matrix operator Q_s is defined on all of $\mathcal{H} \oplus \mathcal{H}$ since J_s is defined on all of $\mathcal{H}$, and it is bounded since each entry is bounded by Lemma 5.17. To simplify notation, we first treat the case $s = 1$. Let $Q = Q_1$ and let $f \in \mathcal{D}(T)$. Then

$$Q\big((f, Tf)\big) = \big((J + TJT)f, T(J + TJT)f\big) = (f, Tf)$$

since $(J + TJT)f = \big(I + TJT(I + T^2)\big)Jf = (I + T^2)Jf = f$. Similarly,

$$Q\big(((-Tf), f)\big) = \big((-JT + TJ)f, T(-JT + TJ)f\big) = (0, 0)$$

since $(-JT+TJ)f = J(-T+(I+T^2)TJ)f = 0$. So Q agrees with $[\mathrm{graph}(T)]$ on $\mathrm{graph}(T)$ and on $\mathrm{graph}(T)^\perp$ by Lemma 5.15(i), so $Q = [\mathrm{graph}(T)]$. Adding a liberal sprinkling of subscripts to these calculations shows that $Q_s = [\mathrm{graph}(T_s)]$.

By Lemma 5.17, for $f, g \in \mathcal{H}$,

$$Q_s\big((f, g)\big) = (J_s f + T_s J_s g, T_s J_s f + T_s^2 J_s g) \to (f, 0) \text{ as } s \to 0.$$

So $I \oplus 0$ is indeed the strong operator topology limit of Q_s as $s \to 0$.

Finally, let $\Phi \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ be the unitary operator

$$\Phi : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}, \quad \Phi\big((f, g)\big) = (g, f).$$

Since T has trivial kernel the operator $T^{-1} : \mathcal{R}(T) \to \mathcal{H}$ exists. The operator T is self-adjoint and positive, so by Lemma 5.15(iii), T^{-1} is self-adjoint and positive too. Moreover,

$$\begin{aligned}
\mathrm{graph}(T_{s^{-1}}) &= \{(f, s^{-1}Tf) \mid f \in \mathcal{D}(T)\} \\
&= \{(sf, Tf) \mid f \in \mathcal{D}(T)\} \\
&= \{(sT^{-1}g, g) \mid g \in \mathcal{D}(T^{-1})\} \\
&= \Phi\, \mathrm{graph}(sT^{-1}).
\end{aligned}$$

In particular, T^{-1} is closed. Since T^{-1} is self-adjoint and positive with trivial kernel and dense domain and image, what we have already proven shows that

$$\underset{s\to 0}{\text{SOT-lim}}[\operatorname{graph}(sT^{-1})] = I \oplus 0.$$

Hence by Proposition 2.20(ii),

$$\begin{aligned}
\underset{s\to\infty}{\text{SOT-lim}}[\operatorname{graph}(T_s)] &= \underset{s\to 0}{\text{SOT-lim}}[\operatorname{graph}(T_{s^{-1}})] \\
&= \Phi\ \underset{s\to 0}{\text{SOT-lim}}[\operatorname{graph}(sT^{-1})]\, \Phi^{-1} \\
&= \Phi(I \oplus 0)\Phi^{-1} \\
&= 0 \oplus I.
\end{aligned}$$

□

Recall that a closed operator T on a Hilbert space $\mathcal{H}$ with dense domain can be written uniquely as $T = WA$ where $W \in \mathcal{L}(\mathcal{H})$ is a partial isometry with initial domain $\overline{T^*\mathcal{H}}$ and A is a positive self-adjoint operator such that $\overline{A\mathcal{H}} = \overline{T^*\mathcal{H}}$ (see §XII.7.7 of [DS63]). This factorisation is called the *polar decomposition* of T.

Corollary 5.19. *Let T be a closed linear operator on $\mathcal{H}$ with trivial kernel and dense domain and range. For $s \geq 0$ let $T_s = sT$, $K_s = \operatorname{graph}(T_s)$ and $Q_s = [K_s]$. Then*

$$\underset{0\leq s\to 0}{\text{SOT-lim}}\, Q_s = I \oplus 0 \quad \textit{and} \quad \underset{0\leq s\to\infty}{\text{SOT-lim}}\, Q_s = 0 \oplus I.$$

Proof. Let $T = WA$ be the polar decomposition of T where W is a partial isometry and A is self-adjoint and positive. Since T has dense domain and range, W must be unitary. By uniqueness, the polar decomposition of sT

for $s \geq 0$ is $sT = W(sA)$. Since $\mathcal{D}(A) = \mathcal{D}(T)$,

$$\begin{aligned}(I \oplus W^*)K_s &= \{(f, W^*(sT)f) \mid f \in \mathcal{D}(T)\} \\ &= \{(f, sAf) \mid f \in \mathcal{D}(A)\} \\ &= \operatorname{graph}(sA),\end{aligned}$$

and A must be closed with dense domain and range. Moreover, $A = W^*T$ has trivial kernel, so the previous lemma shows that

$$\underset{s \to 0}{\text{SOT-lim}}[(I \oplus W^*)K_s] = I \oplus 0 \quad \text{and} \quad \underset{s \to \infty}{\text{SOT-lim}}[(I \oplus W^*)K_s] = 0 \oplus I.$$

Since $[(I \oplus W^*)K_s] = (I \oplus W^*)Q_s(I \oplus W)$, the result follows. □

Proof of Proposition 5.14. Let $\sigma \in \mathbb{R}$ and let $\mathcal{H}$ be the Hilbert space $L^2(\mathbb{R}_+)$. Define the linear map $\tilde{T} : [L^2(\mathbb{R}_+)]H^2(\mathbb{R}) \to L^2(\mathbb{R}_-)$ by

$$\tilde{T}([L^2(\mathbb{R}_+)]h) = [L^2(\mathbb{R}_-)]h, \quad h \in H^2(\mathbb{R}).$$

Then $\mathcal{D}(\tilde{T}) = [L^2(\mathbb{R}_+)]H^2(\mathbb{R})$ is a dense subset of $L^2(\mathbb{R}_+)$. Observe that $\tilde{T}$ has trivial kernel and is well defined by Corollary 2.6. Moreover, if f is a function in $L^2(\mathbb{R}_-) \cap \mathcal{R}(\tilde{T})^\perp$, then for each $h \in H^2(\mathbb{R})$,

$$0 = \langle f, [L^2(\mathbb{R}_-)]h \rangle = \langle f, h \rangle$$

so $f \in \overline{H^2(\mathbb{R})} \cap L^2(\mathbb{R}_-) = (0)$. Hence $\mathcal{R}(\tilde{T}) = [L^2(\mathbb{R}_-)]H^2(\mathbb{R})$ is dense in $L^2(\mathbb{R}_-)$. Let us identity $(f, g) \in L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_-)$ with $f + g \in L^2(\mathbb{R})$ and so identify the spaces $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_-)$ and $L^2(\mathbb{R})$. Then it becomes apparent that the operator $\tilde{T}$ is closed, since

$$v_\sigma H^2(\mathbb{R}) = \operatorname{graph}(\sigma \tilde{T}) \subseteq L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_-) = L^2(\mathbb{R})$$

is closed.

Let U and V be the unitary operators

$$U : L^2(\mathbb{R}_-) \to \mathcal{H}, \quad Uf(x) = f(-x),$$
$$V : L^2(\mathbb{R}_+) \to \mathcal{H}, \quad Vf = f.$$

The operator $W = V \oplus U$ is a unitary mapping $L^2(\mathbb{R})$ onto $\mathcal{H} \oplus \mathcal{H}$ and $W\big(0 \oplus L^2(\mathbb{R}_-)\big) = 0 \oplus \mathcal{H}$. Let $T : \mathcal{H} \to \mathcal{H}$ be the operator $T = U\tilde{T}$. Then T is closed with trivial kernel and dense domain and range, and

$$\begin{aligned} W\big(v_\sigma H^2(\mathbb{R})\big) &= W\big(\mathrm{graph}(\sigma\tilde{T})\big) \\ &= W\{(f, \sigma\tilde{T}f) \mid f \in L^2(\mathbb{R}_+)\} \\ &= \{(Vf, \sigma U\tilde{T}f) \mid f \in L^2(\mathbb{R}_+)\} \\ &= \{(f, \sigma U\tilde{T}f) \mid f \in \mathcal{H}\} \\ &= \mathrm{graph}(\sigma T). \end{aligned}$$

We are now poised to apply Corollary 5.19. If

$$K_\sigma = \mathrm{graph}(\sigma T) = W\big(v_\sigma H^2(\mathbb{R})\big),$$

then as $\sigma \to \infty$,

$$[W\big(v_\sigma H^2(\mathbb{R})\big)] = [K_\sigma] \to [0 \oplus \mathcal{H}] = [W\big(0 \oplus L^2(\mathbb{R}_-)\big)]$$

in the strong operator topology. Since W is unitary, it follows that

$$\underset{s\to\infty}{\text{SOT-lim}}[v_\sigma H^2(\mathbb{R})] = [0 \oplus L^2(\mathbb{R}_-)] = [L^2(\mathbb{R}_-)]. \qquad \square$$

Chapter 6

Lie semigroup operator algebras from $SL_2(\mathbb{R}_+)$

We begin by recalling some terminology introduced in [KP02]. Let G be a Lie group and let ρ be a unitary-valued representation of G on a Hilbert space $\mathcal{H}$; that is,

$$\rho : G \to \mathcal{U}(\mathcal{H}) \subseteq \mathcal{L}(\mathcal{H})$$

and ρ is a $*$-homomorphism of G into $\mathcal{U}(\mathcal{H})$, the group of unitary operators on $\mathcal{H}$. We will call a semigroup of a Lie group a *Lie semigroup.* Let G_+ be a Lie semigroup of G and let $\mathcal{S} = \rho(G_+)$. We call w*-alg($\mathcal{S}$) a *Lie semigroup operator algebra.*

The parabolic algebra and the hyperbolic algebra are Lie semigroup operator algebras. Indeed, if we let H be the Heisenberg group of 3×3 matrices

$$H = \left\{ \left(\begin{smallmatrix} 1 & \lambda & t \\ 0 & 1 & \mu \\ 0 & 0 & 1 \end{smallmatrix} \right) \mid \lambda, \mu, t \in \mathbb{R} \right\}$$

and H_+ the Lie semigroup

$$H_+ = \left\{ \left(\begin{smallmatrix} 1 & \lambda & t \\ 0 & 1 & \mu \\ 0 & 0 & 1 \end{smallmatrix} \right) \mid \lambda, \mu \geq 0,\ t \in \mathbb{R} \right\}$$

then the unitary-valued representation

$$\rho : H \to \mathcal{U}(L^2(\mathbb{R})), \quad \left(\begin{smallmatrix} 1 & \lambda & t \\ 0 & 1 & \mu \\ 0 & 0 & 1 \end{smallmatrix} \right) \mapsto e^{it} D_\mu M_\lambda, \quad \lambda, \mu, t \in \mathbb{R}$$

satisfies w*-alg $\big(\rho(H_+)\big) = \mathcal{A}_{\mathrm{p}}$. Similarly, if

$$G = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix} \right) \mid a > 0,\ b \in \mathbb{R} \right\}, \qquad G_+ = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix} \right) \mid a \geq 1,\ b > 0 \right\},$$

$$\rho : \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix} \right) \mapsto M_b V_{\log a}, \quad a > 0,\ b \in \mathbb{R}$$

then w*-alg $\big(\rho(G_+)\big) = \mathcal{A}_{\mathrm{h}}$.

As shown in [KP97], [KP02] and [LP03], the parabolic algebra and the hyperbolic algebra share several interesting properties. We use the term *Euclidean manifold* to refer to a subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

1. They are *doubly nonselfadjoint*; that is, there are strong operator topology continuous one-parameter semigroups S_1 and S_2 of unitary operators satisfying

   $$\mathcal{A} = \text{w*-alg}(S_1 \cup S_2), \quad S_i \cap S_i^* = \{I\} \quad \text{and} \quad \mathcal{A} \neq \text{w*-alg}\, S_i, \quad i = 1, 2.$$

2. They have trivial intersection with their adjoints: $\mathcal{A} \cap \mathcal{A}^* = \mathbb{C}I$.

3. They contain no finite-rank operators.

4. They contain a bounded approximate identity of Hilbert-Schmidt operators, so the Hilbert-Schmidt operators in $\mathcal{A}$ are dense.

5. Endowed with the strong operator topology, $\operatorname{Lat} \mathcal{A}$ is homeomorphic to a compact connected Euclidean manifold,

6. They are reflexive.

To the author's knowledge, these two algebras are the only examples of Lie semigroup operator algebras studied in the literature. In this chapter we begin the search for interesting new examples.

We focus on the Lie group $SL_2(\mathbb{R})$ of 2×2 matrices with real entries and determinant $+1$, and its Lie semigroup

$$SL_2(\mathbb{R}_+) = \{\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in SL_2(\mathbb{R}) \mid \alpha, \beta, \gamma, \delta \geq 0\}.$$

This choice of Lie semigroup is a natural one; indeed, $SL_2(\mathbb{R}_+)$ is a distinguished semigroup of $SL_2(\mathbb{R})$ as pointed out in [HHL89], §V.4.

Observe that $SL_2(\mathbb{R}_+)$ is generated as a semigroup by elements of the form

$$r_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \quad u_\beta = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \quad l_\gamma = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \tag{6.1}$$

for $\alpha > 0$ and $\beta, \gamma \geq 0$. In fact, for arbitrary $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in SL_2(\mathbb{R}_+)$,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \alpha^{-1}(\beta\gamma + 1) \end{pmatrix} = l_{\alpha^{-1}\gamma} u_{\alpha\beta} r_\alpha.$$

Let j be the matrix $j = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then the full group $SL_2(\mathbb{R})$ is generated as a semigroup by $SL_2(\mathbb{R}_+) \cup \{j\}$.

To get our hands on a Lie semigroup operator algebra we must now select a unitary-valued representation of $SL_2(\mathbb{R})$. We consider representations on finite-dimensional spaces before looking at some infinite-dimensional representations.

6.1 Finite-dimensional representations

Following the standard terminology of [Sal76], we say that a representation ρ of a group G on a Hilbert space $\mathcal{H}$ is *irreducible* if $\operatorname{Lat}\big(\rho(G)\big) = \{(0), \mathcal{H}\}$. As explained in [Don97], there are no unitary-valued irreducible representations of $SL_2(\mathbb{R})$ on a finite-dimensional space, and the irreducible representations of $SL_2(\mathbb{R})$ on finite-dimensional spaces may be described in the following manner. For $N \in \mathbb{N}$, let $\mathbb{P}_N$ be the $(N+1)$-dimensional space of complex homogeneous polynomials in two variables of degree N; that is,

$$\mathbb{P}_N = \left\{ P : (z_1, z_2)^{\mathrm{t}} \mapsto \sum_{n=0}^{N} a_n z_1^n z_2^{N-n} \,\middle|\, a_n \in \mathbb{C} \right\}.$$

Then a finite-dimensional irreducible representation ρ of $SL_2(\mathbb{R})$ on $\mathbb{P}_N$ is given by

$$\left(\rho\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} P\right)(z_1, z_2)^{\mathrm{t}} = P\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} (z_1, z_2)^{\mathrm{t}}\right).$$

We compute the images under ρ of the generators r_α, u_β and l_γ defined in (6.1). Let $\{e_n \mid n = 0, 1, \ldots, N\}$ be the natural basis of $\mathbb{P}_N$ where

$$e_n : (z_1, z_2)^{\mathrm{t}} \mapsto z_1^n z_2^{N-n},$$

and let P be a polynomial in $\mathbb{P}_N$ so that $P = \sum_{n=0}^{N} a_n e_n$ for some complex numbers a_n, $0 \leq n \leq N$. Then

$$\begin{aligned}
\rho(r_\alpha)P\ (z_1, z_2)^{\mathrm{t}} &= P(\alpha^{-1} z_1, \alpha z_2)^{\mathrm{t}} \\
&= \sum_{n=0}^{N} \alpha^{N-2n} a_n e_n (z_1, z_2)^{\mathrm{t}},
\end{aligned}$$

$$\begin{aligned}
\rho(u_\beta)P\ (z_1, z_2)^{\mathrm{t}} &= P(z_1 - \beta z_2, z_2)^{\mathrm{t}} \\
&= \sum_{n=0}^{N} a_n (z_1 - \beta z_2)^n z_2^{N-n} \\
&= \sum_{0 \leq m \leq n \leq N} \binom{n}{m} (-\beta)^{n-m} a_n e_m (z_1, z_2)^{\mathrm{t}},
\end{aligned}$$

$$\begin{aligned}
\rho(l_\gamma)P\ (z_1, z_2)^{\mathrm{t}} &= P(z_1, z_2 - \gamma z_1)^{\mathrm{t}} \\
&= \sum_{n=0}^{N} a_n z_1^n (z_2 - \gamma z_1)^{N-n} \\
&= \sum_{0 \leq n \leq m \leq N} \binom{N-n}{N-m} (-\gamma)^{m-n} a_n e_m (z_1, z_2)^{\mathrm{t}}.
\end{aligned}$$

Let θ be the vector space isomorphism

$$\theta : \mathbb{P}_N \to \mathbb{C}^{N+1}, \quad \sum_{n=0}^{N} a_n e_n \mapsto (a_0, a_1, \ldots, a_N)^{\mathrm{t}}.$$

Then the representation

$$\pi = (\operatorname{Ad}\theta) \circ \rho : SL_2(\mathbb{R}) \to M_{N+1}(\mathbb{C}), \quad K \mapsto \theta\rho(K)\theta^{-1}$$

maps r_α, u_β and l_γ to the following $(N+1) \times (N+1)$ matrices.

$$\pi(r_\alpha) = (\alpha_{m,n})_{m,n=0}^{N}, \quad \alpha_{m,n} = \begin{cases} \alpha^{N-2n} & m = n, \\ 0 & \text{otherwise}, \end{cases} \tag{6.2a}$$

$$\pi(u_\beta) = (\beta_{m,n})_{m,n=0}^{N}, \quad \beta_{m,n} = \begin{cases} \binom{n}{m}(-\beta)^{n-m} & m \le n, \\ 0 & \text{otherwise}, \end{cases} \tag{6.2b}$$

$$\pi(l_\gamma) = (\gamma_{m,n})_{m,n=0}^{N}, \quad \gamma_{m,n} = \begin{cases} \binom{N-n}{N-m}(-\gamma)^{m-n} & n \le m, \\ 0 & \text{otherwise}. \end{cases} \tag{6.2c}$$

Although π is not unitary-valued, it nevertheless makes sense to ask what $\text{w}^*\text{-alg}\left(\pi(SL_2(\mathbb{R}_+))\right)$ is.

Proposition 6.1. *The algebra* $\text{w}^*\text{-alg}\left(\pi(SL_2(\mathbb{R}_+))\right)$ *is* $M_{N+1}(\mathbb{C})$*, the algebra of all* $(N+1) \times (N+1)$ *matrices with entries in* $\mathbb{C}$*.*

Proof. Let $\mathcal{A} = \text{w}^*\text{-alg}\left(\pi(SL_2(\mathbb{R}_+))\right)$. Since $\mathcal{A}$ is contained in a finite dimensional set of matrices, it is equal to the norm-closed algebra generated by $\pi\left(SL_2(\mathbb{R}_+)\right)$. For $m, n = 0, 1, \ldots, N$, let us write $\varepsilon_{m,n}$ for the matrix unit $\varepsilon_{m,n} = (\delta_{im}\delta_{jn})_{i,j=0}^{N}$. It suffices to show that $\varepsilon_{m,n} \in \mathcal{A}$ for $m, n = 0, 1, \ldots, N$. We use the following notation for diagonal matrices:

$$\operatorname{diag}(\alpha_0, \alpha_1, \ldots, \alpha_N) = \sum_{n=0}^{N} \alpha_n \varepsilon_{n,n}.$$

By (6.2a), for $\alpha > 0$,

$$\alpha^{-N}\pi(r_\alpha) = \operatorname{diag}(1, \alpha^{-2}, \alpha^{-4}, \ldots, \alpha^{-2N+2}, \alpha^{-2N}).$$

As $\alpha \to \infty$, the diagonal matrix $\alpha^{-N}\pi(r_\alpha)$ tends in norm to the matrix unit $\varepsilon_{0,0}$, so $\varepsilon_{0,0} \in \mathcal{A}$. Now

$$\alpha^{-(N-2)}\big(\pi(r_\alpha) - \alpha^N \varepsilon_{0,0}\big) = \operatorname{diag}(0, 1, \alpha^{-2}, \dots, \alpha^{-2N}, \alpha^{-2N-2})$$

tends in norm to $\varepsilon_{1,1}$ as $\alpha \to \infty$, so $\varepsilon_{1,1} \in \mathcal{A}$. Continuing in this manner, we see that the diagonal matrix units $\varepsilon_{n,n}$ lie in $\mathcal{A}$ for $n = 0, 1, \dots, N$.

Fix $\beta, \gamma > 0$ and let $T = (t_{m,n})_{m,n=0}^N = \pi(u_\beta) + \pi(l_\gamma)$. By (6.2b) and (6.2c), $t_{m,n} \neq 0$ for $m, n = 0, 1, \dots, N$. So

$$t_{m,n}^{-1}\varepsilon_{m,m}T\varepsilon_{n,n} = \varepsilon_{m,n} \in \mathcal{A}. \qquad \square$$

6.2 The principal series representations

In this section we consider the representations $\rho_{h,s}$ of $SL_2(\mathbb{R})$ on $L^2(\mathbb{R})$ given by

$$\rho_{h,s}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} f(x) = \frac{\operatorname{sgn}(\beta x + \delta)^h |\beta x + \delta|^{is}}{|\beta x + \delta|} f\big(\frac{\alpha x + \gamma}{\beta x + \delta}\big) \tag{6.3}$$

where $h \in \{0, 1\}$, $s \in \mathbb{R}$ and $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in SL_2(\mathbb{R})$. This family of representations is called the *principal series*. As is well known (see for example [Sal76]), $\rho_{h,s}$ is a unitary-valued representation on $L^2(\mathbb{R})$ for each $h \in \{0, 1\}$ and $s \in \mathbb{R}$. It is irreducible—that is, $\operatorname{Lat}\rho_{h,s}\big(SL_2(\mathbb{R})\big)$ is trivial—unless $h = 1$ and $s = 0$.

Let us fix values of $h \in \{0, 1\}$ and $s \in \mathbb{R}$, and write $\mathcal{A}_+$ for the w*-closed algebra generated by $\rho_{h,s}\big(SL_2(\mathbb{R}_+)\big)$. Then

$$\mathcal{A}_+ = \text{w*-alg}\{\rho_{h,s}(r_\alpha),\ \rho_{h,s}(l_\gamma),\ \rho_{h,s}(u_\beta) \mid \alpha > 0,\ \beta, \gamma \geq 0\}.$$

For $\alpha > 0$ and $\gamma \in \mathbb{R}$,

$$\rho_{h,s}(r_\alpha)f(x) = \alpha^{-is} \cdot \alpha f(\alpha^2 x) \quad \text{and} \quad \rho_{h,s}(l_\gamma)f(x) = f(x + \gamma),$$

so

$$\rho_{h,s}(r_\alpha) = \alpha^{-is}V_{2\log\alpha} \quad \text{and} \quad \rho_{h,s}(l_\gamma) = D_{-\gamma}. \tag{6.4}$$

In contrast, for $\beta \in \mathbb{R}$ the expression

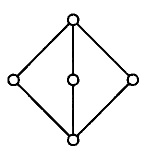
$$\rho_{h,s}(u_\beta)f(x) = \frac{\operatorname{sgn}(\beta x+1)^h|\beta x+1|^{is}}{|\beta x+1|} f\left(\frac{x}{\beta x+1}\right) \tag{6.5}$$

looks unpleasantly complicated. However, if we recall that $j = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and observe that $u_\beta = jl_{-\beta}j^{-1}$, then by (6.4),

$$\rho_{h,s}(u_\beta) = \rho_{h,s}(j)\rho_{h,s}(l_{-\beta})\rho_{h,s}(j)^{-1} = YD_\beta Y^* \tag{6.6}$$

where Y is the unitary operator $Y = Y_{h,s} = \rho_{h,s}(j)$.

In sections 6.2.1 and 6.2.2, we fix $h = 1$, $s = 0$ and write $\rho = \rho_{1,0}$ and $Y = Y_{1,0}$. We will show that, in this exceptional case, $\mathcal{A}_+$ actually belongs to a known class of reflexive operator algebras ([Lon83], [LL92]). These are algebras of the form $\operatorname{Alg}\mathcal{D}$ where $\mathcal{D}$ is a *double triangle lattice*, i.e. $\mathcal{D}$ is isomorphic as a lattice to the 5-element subspace lattice with the following Hasse diagram.



This analysis also gives the unexpected result that $\mathcal{A}_+$ is generated as a w*-closed algebra by $\rho(\mathcal{S})$ where $\mathcal{S}$ is the strict subsemigroup of $SL_2(\mathbb{R}_+)$ generated by $\{r_\alpha, l_\gamma \mid \alpha > 0,\ \gamma \geq 0\}$. In contrast, the corresponding norm-closed algebras generated by $\rho(\mathcal{S})$ and $\rho\big(SL_2(\mathbb{R}_+)\big)$ are distinct.

6.2.1 Invariant subspace lattices

We briefly recall the description of $\operatorname{Lat}\mathcal{A}_{\mathrm{h}}$ from §5.2. For $\lambda, \mu \in \mathbb{R}$ and $\zeta \in \mathbb{C}^*$, the functions $e_{\lambda,\mu}, v_\zeta : \mathbb{R} \to \mathbb{C}$ are given by

$$e_{\lambda,\mu}e_{\lambda,\mu}(x) = e^{i(\lambda x+\mu x^{-1})},\quad v_\zeta(x) = \begin{cases} 1 & x > 0, \\ \zeta & x \leq 0 \end{cases}$$

and $K_{\zeta,\lambda,\mu}$ is the subspace $K_{\zeta,\lambda,\mu} = v_\zeta e_{\lambda,\mu} H^2(\mathbb{R})$. The invariant subspace lattice $\operatorname{Lat}\mathcal{A}_{\mathrm{h}}$ is given by

$$\operatorname{Lat}\mathcal{A}_{\mathrm{h}} = \{K_{\zeta,\lambda,\mu} \mid \zeta \in \mathbb{C}^*,\ \lambda,\mu \geq 0\} \cup \mathcal{L}_M. \tag{6.7}$$

Let $\mathcal{A}_\ell$ be the "lower triangular" subalgebra of $\mathcal{A}_+$

$$\mathcal{A}_\ell = \text{w}^*\text{-alg}\{\rho(r_\alpha), \rho(l_\gamma) \mid \alpha > 0,\ \gamma \geq 0\}.$$

Armed with equation (6.7), an expression for $\operatorname{Lat}\mathcal{A}_\ell$ is fairly easy to come by. Recall that a double triangle lattice of subspaces of $\mathcal{H}$ is a five-element lattice $\mathcal{L} = \{(0), K, L, M, \mathcal{H}\}$ such that $K \cap L = L \cap M = M \cap K = (0)$ and $K \vee L = L \vee M = M \vee K = \mathcal{H}$ where $\vee$ denotes the closed linear span. Let $\mathcal{E}$ be the set of subspaces

$$\mathcal{E} = \{(0), H^2(\mathbb{R}), L^2(\mathbb{R}_-), \overline{H^2(\mathbb{R})}, L^2(\mathbb{R})\}.$$

It is easy to see from Corollary 2.6 that $\mathcal{E}$ is a double triangle lattice.

Lemma 6.2. *The invariant subspace lattice of* $\mathcal{A}_\ell$ *is*

$$\operatorname{Lat}\mathcal{A}_\ell = \{F^*\big(v_\zeta H^2(\mathbb{R})\big) \mid \zeta \in \mathbb{C}^*\} \cup \{(0), H^2(\mathbb{R}), \overline{H^2(\mathbb{R})}, L^2(\mathbb{R})\}. \tag{6.8}$$

In particular, $F \operatorname{Lat}\mathcal{A}_\ell \supseteq \mathcal{L}_S$ *and the double triangle lattice* $\mathcal{E}$ *is contained in* $\operatorname{Lat}\mathcal{A}_\ell$.

Proof. Recall from (6.4) that for $\alpha > 0$ and $\gamma \in \mathbb{R}$, $\rho(r_\alpha) = V_{2\log\alpha}$ and $\rho(l_\gamma) = D_{-\gamma}$. Thus

$$\mathcal{A}_\ell = \text{w}^*\text{-alg}\{D_{-\lambda}, V_t \mid \lambda \geq 0,\ t \in \mathbb{R}\}.$$

By (4.22), $FV_tF^* = V_t^* = V_{-t}$, and by (3.2), $FD_{-\lambda}F^* = M_\lambda$. So

$$F\mathcal{A}_\ell F^* = \text{w}^*\text{-alg}\{V_t, M_\lambda \mid \lambda \geq 0,\ t \in \mathbb{R}\}. \tag{6.9}$$

Comparing this to the generator description (5.1) of the hyperbolic algebra $\mathcal{A}_{\mathrm{h}}$, we see that $F\mathcal{A}_\ell F^*$ is a superalgebra of $\mathcal{A}_{\mathrm{h}}$ and that

$$\operatorname{Lat} F\mathcal{A}_\ell F^* = \{K \in \operatorname{Lat}\mathcal{A}_{\mathrm{h}} \mid V_t K \subseteq K \text{ for each } t < 0\}.$$

Let $\zeta \in \mathbb{C}^*$ and $\lambda, \mu, t \in \mathbb{R}$. Then $V_t K_{\zeta,\lambda,\mu} = K_{\zeta,e^t\lambda,e^{-t}\mu}$, and for $t < 0$ and $\lambda, \mu \geq 0$ this is contained in $K_{\zeta,\lambda,\mu}$ only if $\lambda = \mu = 0$. Similarly, if $a, b \in [0, \infty]$, then when $t < 0$, the subspace $V_t L^2([-a, b])$ is contained in $L^2([-a, b])$ only if $a, b \in \{0, \infty\}$. Thus

$$\operatorname{Lat} F\mathcal{A}_\ell F^* = \{v_\zeta H^2(\mathbb{R}) \mid \zeta \in \mathbb{C}^*\} \cup \{(0), L^2(\mathbb{R}_+), L^2(\mathbb{R}_-), L^2(\mathbb{R})\}.$$

Since $\operatorname{Lat} F\mathcal{A}_\ell F^* = F \operatorname{Lat}\mathcal{A}_\ell$ by (4.2), we can apply F^* to both sides of this equation to obtain (6.8). To see that $\mathcal{E} \subseteq \operatorname{Lat}\mathcal{A}_\ell$, observe that $F^*\big(v_1 H^2(\mathbb{R})\big) = F^* H^2(\mathbb{R}) = L^2(\mathbb{R}_-)$. □

In fact, $\mathcal{E}$ is a sublattice not only of $\operatorname{Lat}\mathcal{A}_\ell$ but also of the apparently smaller lattice $\operatorname{Lat}\mathcal{A}_+$.

Lemma 6.3. $\mathcal{E} \subseteq \operatorname{Lat}\mathcal{A}_+$.

Proof. Let $\mathcal{A}_u$ be the w*-closed operator algebra generated by the one parameter semigroup $\{\rho(u_\beta) \mid \beta \geq 0\}$. Since $\mathcal{A}_+ = \text{w*-alg}(\mathcal{A}_\ell \cup \mathcal{A}_u)$, we have $\operatorname{Lat}\mathcal{A}_+ = \operatorname{Lat}\mathcal{A}_\ell \cap \operatorname{Lat}\mathcal{A}_u$. Let $\beta \geq 0$. By (6.6), $\rho(u_\beta) = Y D_\beta Y^*$. Since

$$Yf(x) = x^{-1} f(-x^{-1}), \tag{6.10}$$

it follows that $Y^* = -Y$. Moreover, (6.10) shows that $H^2(\mathbb{R})$ reduces Y and so $H^2(\mathbb{R})$ and $\overline{H^2(\mathbb{R})}$ are invariant under $\rho(u_\beta)$. Equation (6.10) also implies that $YL^2(\mathbb{R}_\pm) = Y^* L^2(\mathbb{R}_\pm) = L^2(\mathbb{R}_\mp)$, so

$$\rho(u_\beta)L^2(\mathbb{R}_-) = Y D_\beta Y^* L^2(\mathbb{R}_-) = Y D_\beta L^2(\mathbb{R}_+) \subseteq Y L^2(\mathbb{R}_+) = L^2(\mathbb{R}_-).$$

This shows that $\mathcal{E} \subseteq \operatorname{Lat}\mathcal{A}_u$ and we have already seen in Lemma 6.2 that $\mathcal{E}$ is a sublattice of $\operatorname{Lat}\mathcal{A}_\ell$. Hence $\mathcal{E} \subseteq \operatorname{Lat}\mathcal{A}_\ell \cap \operatorname{Lat}\mathcal{A}_u = \operatorname{Lat}\mathcal{A}_+$. □

The next theorem shows that $\mathcal{E}$ is not a reflexive lattice, so the inclusion of Lemma 6.3 is proper. As in §5.2, many of the ideas in the proof come from [Hal69] and [Hal71].

Theorem 6.4. *Let M and N be subspaces of a Hilbert space $\mathcal{H}$. Suppose that M and N are in generic position; that is, M and N are proper subspaces of $\mathcal{H}$ and the intersections*

$$M \cap N, \quad M \cap N^{\perp}, \quad M^{\perp} \cap N \quad \text{and} \quad M^{\perp} \cap N^{\perp}$$

are all equal to the zero subspace. Then the lattice $\mathcal{D} = \{(0), M, N, M^{\perp}, \mathcal{H}\}$ is not reflexive; in fact, Lat Alg $\mathcal{D}$ *contains the "ball lattice"*

$$\mathcal{B} = \{N_\zeta \mid \zeta \in \mathbb{C}^*\} \cup \{(0), M, M^{\perp}, \mathcal{H}\},$$

where the closed subspace N_ζ is given by

$$N_\zeta = ([M] + \zeta[M^{\perp}])N \quad \text{for } \zeta \in \mathbb{C}^*.$$

Moreover, the infimum and supremum of any two distinct elements of $\mathcal{B}$ are the zero subspace and $\mathcal{H}$ respectively.

Proof. Let $P = [M]|N$ and $M' = PN$. If $Pn = 0$ for some $n \in N$ then $n \in N \cap M^{\perp} = (0)$, so P is 1–1. Moreover, if $m \in M \cap (M')^{\perp}$ then for any $n \in N$, $0 = \langle m, Pn\rangle = \langle m, n\rangle$ so $m \in M \cap N^{\perp} = (0)$; so M' is dense in M.

We define the unbounded linear operator $T : M' \to M^{\perp}$ by

$$T(Pg) = (I - P)g, \quad g \in N.$$

Since P is 1–1, Pg determines g so T is well defined. By symmetry, the same argument we used to show that M' is dense in M shows that the image of T is dense in $M^{\perp}$. Now decompose $\mathcal{H}$ as

$$\mathcal{H} = M \oplus M^{\perp} = \{(f, g) \mid f \in M, \ g \in M^{\perp}\}.$$

Then $N = \{(f, Tf) \mid f \in M'\} = \operatorname{graph}(T)$. Clearly

$$\operatorname{Alg} \mathcal{D} \subseteq \operatorname{Alg}(\{M, M^{\perp}\}) = \mathcal{L}(M) \oplus \mathcal{L}(M^{\perp});$$

so let $A = B \oplus C \in \operatorname{Alg} \mathcal{D}$ where $B \in \mathcal{L}(M)$ and $C \in \mathcal{L}(M^{\perp})$. Since A leaves $N = \operatorname{graph}(T)$ invariant,

$$(B \oplus C)\{(f, Tf) \mid f \in M'\} = \{(Bf, CTf) \mid f \in M'\} \subseteq \{(g, Tg) \mid g \in M'\}. \tag{6.11}$$

Thus we must have

$$BM' \subseteq M' \text{ and } CT = T(B|M'). \tag{6.12}$$

Conversely, if (6.12) is satisfied for some $B \in \mathcal{L}(M)$ and $C \in \mathcal{L}(M^{\perp})$ then (6.11) is too, so $B \oplus C \in \operatorname{Alg} \mathcal{D}$. Thus

$$\operatorname{Alg} \mathcal{D} = \{B \oplus C \in \mathcal{L}(M) \oplus \mathcal{L}(M^{\perp}) \mid (6.12) \text{ holds}\}. \tag{6.13}$$

Since $\operatorname{graph}(T) = N$, the operator T is closed, so for any $\zeta \in \mathbb{C}^*$, the operator $\zeta T : M' \to M^{\perp}$ is also closed. Consider the reflexive closure $\operatorname{Lat} \operatorname{Alg} \mathcal{D}$ of $\mathcal{D}$. For each $\zeta \in \mathbb{C}^*$, the linear manifold

$$N_{\zeta} = \{(f, \zeta Tf) \mid f \in M'\}$$

is the graph of the closed operator ζT and so is a closed subspace. Moreover, N_{ζ} is left invariant by every operator $B \oplus C \in \operatorname{Alg} \mathcal{D}$ by (6.13), so $N_{\zeta} \in \operatorname{Lat} \operatorname{Alg} \mathcal{D}$. If $\zeta \in \mathbb{C}^* \setminus \{1\}$ then $N_{\zeta} \notin \mathcal{D}$, so $\operatorname{Lat} \operatorname{Alg} \mathcal{D} \supsetneq \mathcal{D}$ and $\mathcal{D}$ is not reflexive.

Let $\zeta \in \mathbb{C}^*$. To see that $M \cap N_{\zeta} = (0)$, suppose that $m \in M \cap N_{\zeta}$. Then $m = Pn + \zeta(I - P)n$ for some $n \in N$. So $0 = [M^{\perp}]m = \zeta(I - P)n$. The operator $P : N \to \mathcal{H}$ is 1–1, so by symmetry $I - P : N \to \mathcal{H}$ is also 1–1. So $n = m = 0$ and $M \cap N_{\zeta} = (0)$. By symmetry, $M^{\perp} \cap N_{\zeta} = (0)$.

If ζ_1 and ζ_2 are distinct points in $\mathbb{C}^*$ and $x \in N_{\zeta_1} \cap N_{\zeta_2}$, then

$$x = Pn_1 + \zeta_1(I-P)n_1 = Pn_2 + \zeta_2(I-P)n_2$$

for some $n_1, n_2 \in N$. So $[M]x = Pn_1 = Pn_2$; the operator P is 1–1, so $n_1 = n_2 = n$, say. So $(\zeta_1 - \zeta_2)(I-P)n = 0$ and since $I - P$ is also 1–1, $n = 0$. So $N_{\zeta_1} \cap N_{\zeta_2} = (0)$.

By considering $\mathcal{D}' = \{(0), M, N^\perp, M^\perp, \mathcal{H}\}$ in place of $\mathcal{D}$ we see that for $\zeta, \zeta_1, \zeta_2 \in \mathbb{C}^*$ with $\zeta_1 \neq \zeta_2$, the intersections

$$M^\perp \cap N_\zeta^\perp, \quad M \cap N_\zeta^\perp, \quad N_{\zeta_1}^\perp \cap N_{\zeta_2}^\perp$$

are all equal to the zero subspace, so their orthogonal complements

$$M \vee N_\zeta, \quad M^\perp \vee N_\zeta, \quad N_{\zeta_1} \vee N_{\zeta_2}$$

are all equal to $\mathcal{H}$. □

Remark. Longstaff gives a nice short proof of the essentials of Theorem 6.4 in [Lon83], where it is presented as a "Folk theorem". The proof he gives does not explicitly describe the extra subspaces in the reflexive closure of the double triangle lattice as graphs of unbounded operators.

Remark. In fact, we always have $\operatorname{Lat}\operatorname{Alg}\mathcal{D} = \mathcal{B}$ in Theorem 6.4. This is an immediate consequence of a result of Lambrou and Longstaff (Corollary 2.1 in [LL92]), which they prove in the greater generality of a Banach space setting. The corresponding Hilbert space version which applies in our setting is attributed by them to an earlier result of H. K. Middleton. However, knowing that $\operatorname{Lat}\operatorname{Alg}\mathcal{D} \supseteq \mathcal{B}$ will suffice for what follows.

It is natural to write $N_0 = M$ and $N_\infty = M^\perp$. Indeed, if we do so then when viewed as a set of projections and endowed with the strong operator topology, it is easy to see that $\operatorname{Lat}\operatorname{Alg}\mathcal{D}$ becomes the disjoint union of a topological sphere $\{N_\zeta \mid \zeta \in \mathbb{C} \cup \{\infty\}\}$ with the two points $\{(0), \mathcal{H}\}$. Let us henceforth write N_ζ for the subspaces so obtained in the case $\mathcal{D} = \mathcal{E}$, $M = H^2(\mathbb{R})$, $M^\perp = \overline{H^2(\mathbb{R})}$, $N = L^2(\mathbb{R}_-)$ and $\mathcal{H} = L^2(\mathbb{R})$; that is,

$$N_\zeta = \begin{cases} \big([H^2(\mathbb{R})] + \zeta[\overline{H^2(\mathbb{R})}]\big)L^2(\mathbb{R}_-) & \zeta \in \mathbb{C}^*, \\ H^2(\mathbb{R}) & \zeta = 0, \\ \overline{H^2(\mathbb{R})} & \zeta = \infty. \end{cases}$$

We will also write $\mathcal{B}$ for the "ball lattice"

$$\mathcal{B} = \{N_\zeta \mid \zeta \in \mathbb{C} \cup \{\infty\}\} \cup \{(0), L^2(\mathbb{R})\}.$$

By Theorem 6.4,

$$\mathcal{B} \subseteq \operatorname{Lat} \operatorname{Alg} \mathcal{E}. \tag{6.14}$$

Lemma 6.5. *For each* $\zeta \in \mathbb{C}^*$, $F^*(v_\zeta H^2(\mathbb{R})) = N_\zeta$. *Thus* $\operatorname{Lat} \mathcal{A}_\ell = \mathcal{B}$.

Proof. Let $\zeta \in \mathbb{C}^*$. Since $v_\zeta = \chi_{\mathbb{R}_+} + \zeta\chi_{\mathbb{R}_-}$,

$$v_\zeta H^2(\mathbb{R}) = \big([L^2(\mathbb{R}_+)] + \zeta[L^2(\mathbb{R}_-)]\big)H^2(\mathbb{R}).$$

If U is unitary then $U[K]U^* = [UK]$ for any subspace K, so $U[K] = [UK]U$. So

$$\begin{aligned} FN_\zeta &= F\big([H^2(\mathbb{R})] + \zeta[\overline{H^2(\mathbb{R})}]\big)L^2(\mathbb{R}_-) \\ &= \big([FH^2(\mathbb{R})] + \zeta[F\overline{H^2(\mathbb{R})}]\big)FL^2(\mathbb{R}_-) \\ &= \big([L^2(\mathbb{R}_+)] + \zeta[L^2(\mathbb{R}_-)]\big)H^2(\mathbb{R}) \\ &= v_\zeta H^2(\mathbb{R}). \end{aligned}$$

So $N_\zeta = F^*(v_\zeta H^2(\mathbb{R}))$, and by Lemma 6.2,

$$\operatorname{Lat} \mathcal{A}_\ell = \{N_\zeta \mid \zeta \in \mathbb{C}^*\} \cup \{(0), H^2(\mathbb{R}), \overline{H^2(\mathbb{R})}, L^2(\mathbb{R})\} = \mathcal{B}. \qquad \square$$

Remark. In [KP02], the subspaces $d_s H^2(\mathbb{R})$ for $s \in \mathbb{R}$ are introduced and are then shown to be invariant under $\mathcal{A}_{\mathrm{h}}$. On the other hand, Theorem 6.4

and Lemma 6.5 together show that the subspaces $v_\zeta H^2(\mathbb{R})$ for $\zeta \in \mathbb{C}^*$ lie in the reflexive closure $\operatorname{Lat}\operatorname{Alg} F\mathcal{E}$ of the double triangle lattice

$$F\mathcal{E} = \{(0), L^2(\mathbb{R}_+), L^2(\mathbb{R}_-), H^2(\mathbb{R}), L^2(\mathbb{R})\}$$

and by Lemma 5.13, $d_s H^2(\mathbb{R}) = v_{\exp(\pi s)} H^2(\mathbb{R})$ for $s \in \mathbb{R}$. It is easy to see that $F\mathcal{E} \subseteq \operatorname{Lat}\mathcal{A}_{\mathrm{h}}$, so we also have $\operatorname{Lat}\operatorname{Alg} F\mathcal{E} \subseteq \operatorname{Lat}\mathcal{A}_{\mathrm{h}}$. Thus we obtain a transparent argument showing that each subspace $d_s H^2(\mathbb{R})$ lies in $\operatorname{Lat}\mathcal{A}_{\mathrm{h}}$.

Corollary 6.6. $\operatorname{Lat}\operatorname{Alg}\mathcal{E} = \operatorname{Lat}\mathcal{A}_+ = \operatorname{Lat}\mathcal{A}_\ell = \mathcal{B}$.

Proof. Since $\mathcal{A}_\ell \subseteq \mathcal{A}_+$, $\operatorname{Lat}\mathcal{A}_+ \subseteq \operatorname{Lat}\mathcal{A}_\ell$. By Lemma 6.3, $\mathcal{E} \subseteq \operatorname{Lat}\mathcal{A}_+$, so by (6.14) and Lemma 6.5 we have

$$\mathcal{B} \subseteq \operatorname{Lat}\operatorname{Alg}\mathcal{E} \subseteq \operatorname{Lat}\operatorname{Alg}(\operatorname{Lat}\mathcal{A}_+) = \operatorname{Lat}\mathcal{A}_+ \subseteq \operatorname{Lat}\mathcal{A}_\ell = \mathcal{B}. \qquad \square$$

6.2.2 Reflexivity

We show that $\mathcal{A}_+$ is a reflexive operator algebra. Our method is somewhat surprising: we identify $\mathcal{A}_+$ with what appears at first sight to be the proper subalgebra $\mathcal{A}_\ell$. Let $\mathcal{A}_\mathcal{B}$ be the reflexive operator algebra $\mathcal{A}_\mathcal{B} = \operatorname{Alg}\mathcal{B}$. Since $\operatorname{Lat}\mathcal{A}_+ = \mathcal{B}$, it follows that

$$\mathcal{A}_\ell \subseteq \mathcal{A}_+ \subseteq \operatorname{Alg}\operatorname{Lat}\mathcal{A}_+ = \mathcal{A}_\mathcal{B}. \tag{6.15}$$

We will show that these inclusions are actually equalities.

Remark. Observe that by Corollary 6.6,

$$\mathcal{A}_\mathcal{B} = \operatorname{Alg}\operatorname{Lat}\operatorname{Alg}\mathcal{E} = \operatorname{Alg}\mathcal{E},$$

so $\mathcal{A}_\mathcal{B}$ is a "double triangle operator algebra", an operator algebra of the form $\operatorname{Alg}\mathcal{D}$ where $\mathcal{D}$ is a double triangle subspace lattice. This accounts for the title of [Lev04].

Proposition 6.7. *Let* $\operatorname{Int} k$ *be a Hilbert-Schmidt operator leaving invariant* $L^2(\mathbb{R}_+)$, $L^2(\mathbb{R}_-)$ *and* $v_a H^2(\mathbb{R})$ *for* $a > 0$. *Then* $\Theta_{\mathrm{h}}(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R})$. *In particular,*

$$F(\mathcal{A}_{\mathcal{B}} \cap \mathcal{C}_2)F^* \subseteq \{\operatorname{Int} k \mid \Theta_{\mathrm{h}}(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R})\}.$$

Proof. If $\operatorname{Int} k$ leaves the subspaces $L^2(\mathbb{R}_+)$, $L^2(\mathbb{R}_-)$ and $v_a H^2(\mathbb{R})$ invariant for $a > 0$, then by (5.7), $\operatorname{Int} k$ leaves every subspace in $\mathcal{L}_S$ invariant and so $\operatorname{Int} k \in \mathcal{A}_S$. Applying Lemma 5.2 and Proposition 5.5(i), we see that $\Theta_{\mathrm{h}}(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R})$.

Since $F\{H^2(\mathbb{R}), \overline{H^2(\mathbb{R})}\} = \{L^2(\mathbb{R}_+), L^2(\mathbb{R}_-)\}$, (6.8) and Corollary 6.6 show that

$$F\mathcal{B} = F \operatorname{Lat} \mathcal{A}_\ell \supseteq \mathcal{L}_S.$$

Thus by (4.1),

$$\begin{aligned} F(\mathcal{A}_{\mathcal{B}} \cap \mathcal{C}_2)F^* &= F(\operatorname{Alg} \mathcal{B})F^* \cap \mathcal{C}_2 \\ &= \operatorname{Alg}(F\mathcal{B}) \cap \mathcal{C}_2 \\ &\subseteq \{\operatorname{Int} k \mid \Theta_{\mathrm{h}}(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R})\}. \end{aligned}$$ □

The next result follows by applying Lemma 3.5 to the strongly continuous group of unitary operators $\{V_t \mid t \in \mathbb{R}\}$, since by (6.9),

$$F\mathcal{A}_\ell F^* \supseteq \text{w}^*\text{-alg}\{V_t \mid t \in \mathbb{R}\}.$$

Proposition 6.8. *Let* φ *be in* $L^1(\mathbb{R})$. *Then the sesquilinear form*

$$\tau_\varphi(f, g) = \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(t) V_t f(x) \overline{g(x)}\, dx\, dt, \quad f, g \in L^2(\mathbb{R})$$

is bounded, and there is a unique bounded linear operator V_φ *such that* $\langle V_\varphi f, g\rangle = \tau_\varphi(f, g)$ *for every* f *and* g *in* $L^2(\mathbb{R})$. *Moreover,* $\|V_\varphi\| \leq \|\varphi\|_{L^1(\mathbb{R})}$. *If* φ *has compact essential support, then* $V_\varphi \in F\mathcal{A}_\ell F^*$. □

Proposition 6.9. *Let $h \in H^2(\mathbb{R})$ and $\varphi \in L^2(\mathbb{R})$. Let $h \otimes \varphi$ denote the function $(x,t) \mapsto h(x)\varphi(t)$ and let $k = \Theta_{\mathrm{h}}^{-1}(h \otimes \varphi)$. Then $\operatorname{Int} k \in F\mathcal{A}_\ell F^* \cap \mathcal{C}_2$. Moreover, if also ph is in $H^\infty(\mathbb{R})$ and $\varphi \in L^1(\mathbb{R})$, then $\operatorname{Int} k = M_{ph}V_\varphi$.*

Proof. Using the generator description (6.9) of $F\mathcal{A}_\ell F^*$, this follows from Proposition 6.8 using the argument of Proposition 5.7. □

Proposition 6.10. *If $\nu \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ then $\operatorname{Int} \Theta_{\mathrm{h}}^{-1}(\nu) \in F(\mathcal{A}_\ell \cap \mathcal{C}_2)F^*$. So*

$$\{\operatorname{Int} k \mid \Theta_{\mathrm{h}}(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R})\} \subseteq F(\mathcal{A}_\ell \cap \mathcal{C}_2)F^*.$$

Proof. This follows from Lemma 5.3 and Proposition 6.9 exactly as in the proof of Proposition 3.8. □

Lemma 6.11. *The Hilbert-Schmidt operators in $\mathcal{A}_\ell$ are w^*-dense in $\mathcal{A}_\ell$, and the Hilbert-Schmidt operators in $\mathcal{A}_\mathcal{B}$ are w^*-dense in $\mathcal{A}_\mathcal{B}$.*

Proof. By (6.9), $\mathcal{A}_{\mathrm{h}} \subseteq F\mathcal{A}_\ell F^*$. Since the Hilbert-Schmidt operators $\mathcal{C}_2$ form an ideal in $\mathcal{L}(L^2(\mathbb{R}))$ and F is unitary, by (6.15),

$$\mathcal{A}_{\mathrm{h}} \cap \mathcal{C}_2 \subseteq (F\mathcal{A}_\ell F^*) \cap \mathcal{C}_2 = F(\mathcal{A}_\ell \cap \mathcal{C}_2)F^* \subseteq F(\mathcal{A}_\mathcal{B} \cap \mathcal{C}_2)F^*.$$

It follows by Proposition 5.10 that both $F(\mathcal{A}_\ell \cap \mathcal{C}_2)F^*$ and $F(\mathcal{A}_\mathcal{B} \cap \mathcal{C}_2)F^*$ contain a bounded approximate identity. The argument of Corollary 3.11 now shows that $F(\mathcal{A}_\ell \cap \mathcal{C}_2)F^*$ and $F(\mathcal{A}_\mathcal{B} \cap \mathcal{C}_2)F^*$ are dense in $F\mathcal{A}_\ell F^*$ and $F\mathcal{A}_\mathcal{B} F^*$ respectively. Since F is unitary, we can conjugate these dense sets with F^* to reach the desired conclusion. □

Theorem 6.12. *$\mathcal{A}_\ell = \mathcal{A}_+ = \mathcal{A}_\mathcal{B}$. In particular, $\mathcal{A}_+$ is reflexive.*

Proof. We know from (6.15) that $\mathcal{A}_\ell \subseteq \mathcal{A}_+ \subseteq \mathcal{A}_\mathcal{B}$. Hence by Proposition 6.7 and Proposition 6.10,

$$\mathcal{A}_\ell \cap \mathcal{C}_2 = (\operatorname{Ad} F^*)\{\operatorname{Int} k \mid \Theta_{\mathrm{h}}(k) \in H^2(\mathbb{R}) \otimes L^2(\mathbb{R})\} = \mathcal{A}_\mathcal{B} \cap \mathcal{C}_2. \quad (6.16)$$

By Lemma 6.11, this set of Hilbert-Schmidt operators in w*-dense in each of the w*-closed algebras $\mathcal{A}_\ell$ and $\mathcal{A}_\mathcal{B}$, so $\mathcal{A}_\ell = \mathcal{A}_\mathcal{B} = \mathcal{A}_+$. Since $\mathcal{A}_\mathcal{B} = \operatorname{Alg}\mathcal{B}$ is plainly reflexive, the proof is complete. □

Remark. As in [KP02], let us write $\mathcal{L}_V$ for the subspace lattice

$$\mathcal{L}_V = \{v_\zeta H^2(\mathbb{R}) \mid \zeta \in \mathbb{C}^*\} \cup \{(0), L^2(\mathbb{R}_+), L^2(\mathbb{R}_-), L^2(\mathbb{R})\}.$$

We can use Theorem 6.12 to obtain the following intrinsic expression for $\operatorname{Alg}\mathcal{L}_V$. Since $\mathcal{A}_\ell = \mathcal{A}_+ = \mathcal{A}_\mathcal{B}$, by Lemma 6.2, $F\mathcal{B} = \mathcal{L}_V$. Thus by (6.9),

$$\operatorname{Alg}\mathcal{L}_V = (\operatorname{Ad}F)\mathcal{A}_\mathcal{B} = \text{w*-alg}\{V_t, M_\lambda \mid \lambda \geq 0,\ t \in \mathbb{R}\}.$$

Observe that $\mathcal{L}_S \subseteq \mathcal{L}_V$. We claim that $\mathcal{A}_S = \operatorname{Alg}\mathcal{L}_S$ is equal to $\operatorname{Alg}\mathcal{L}_V$. The inclusion $\mathcal{A}_S \supseteq \operatorname{Alg}\mathcal{L}_V$ is immediate, and by Proposition 5.5(i) and (6.16),

$$\mathcal{A}_S \cap \mathcal{C}_2 \subseteq (\operatorname{Ad}F)(\mathcal{A}_\mathcal{B} \cap \mathcal{C}_2),$$

so

$$\mathcal{A}_S \cap \mathcal{C}_2 = (\operatorname{Alg}\mathcal{L}_V) \cap \mathcal{C}_2.$$

As observed in the proof of Lemma 6.11, $(\operatorname{Alg}\mathcal{L}_V) \cap \mathcal{C}_2$ contains a bounded approximate identity, so applying the well-worn argument of Corollary 3.11 one shows that $\mathcal{A}_S \cap \mathcal{C}_2 = (\operatorname{Alg}\mathcal{L}_V) \cap \mathcal{C}_2$ is w*-dense in $\mathcal{A}_S = \operatorname{Alg}\mathcal{L}_V$.

Question 6.13. It is shown in [LL92] that $\mathcal{A}_\mathcal{B}$ contains operators of every even rank and their ranges are dense in $L^2(\mathbb{R})$. Is there an alternative proof of Theorem 6.12 in which these finite-rank operators fulfil the role played by the Hilbert-Schmidt operators above?

Corollary 6.14. *Let $\mathcal{A}_\mathrm{u}$ be the "upper triangular" algebra*

$$\mathcal{A}_\mathrm{u} = \text{w*-alg}\{\rho(r_\alpha), \rho(u_\beta) \mid \alpha > 0,\ \beta \geq 0\}.$$

Then $\mathcal{A}_+ = \mathcal{A}_\mathrm{u}$.

Proof. Let $U \in \mathcal{L}(L^2(\mathbb{R}))$ be the unitary operator $Uf(x) = x^{-1}f(x^{-1})$. Then $U^* = U$,

$$\begin{aligned} U\rho(r_\alpha)U^*f(x) &= x^{-1}U\rho(r_\alpha)f(x^{-1}) \\ &= x^{-1}\alpha U f(\alpha^2 x^{-1}) \\ &= \alpha^{-1}f(\alpha^{-2}x) \\ &= \rho(r_{\alpha^{-1}})f(x) \end{aligned}$$

and

$$\begin{aligned} U\rho(u_\beta)U^*f(x) &= x^{-1}U\rho(u_\beta)f(x^{-1}) \\ &= x^{-1}Uf(\beta + x^{-1}) \\ &= \frac{1}{\beta x+1}f\Big(\frac{x}{\beta x+1}\Big) \\ &= \rho(l_\beta)f(x). \end{aligned}$$

So

$$U\rho(r_\alpha)U^* = \rho(r_{\alpha^{-1}}), \quad U\rho(u_\beta)U^* = \rho(l_\beta) \quad \text{and} \quad U\rho(l_\gamma)U^* = \rho(u_\gamma);$$

so by Theorem 6.12, $\mathcal{A}_{\mathrm{u}} = U\mathcal{A}_\ell U^* = U\mathcal{A}_+U^* = \mathcal{A}_+$. $\square$

We are now in a position to give the properties of $\mathcal{A}_+$ corresponding to those of $\mathcal{A}_{\mathrm{p}}$ and $\mathcal{A}_{\mathrm{h}}$ listed on page 92.

1. We have the generator description

$$\mathcal{A}_+ = \text{w*-alg}(\{D_{-\lambda} \mid \lambda \geq 0\} \cup \{V_t \mid t \in \mathbb{R}\});$$

the generating set is the union of the strong operator topology continuous one-parameter semigroup S_1 and the strong operator topology continuous one-parameter group S_2, where

$$S_1 = \{D_{-\lambda} \mid \lambda \geq 0\} \quad \text{and} \quad S_2 = \{V_t \mid t \in \mathbb{R}\}.$$

The translation semigroup S_1 satisfies $S_1 \cap S_1^* = \{I\}$, but the dilation group S_2 is selfadjoint in the sense that $S_2 = S_2^*$. Thus $\mathcal{A}_+$ is not *doubly* nonselfadjoint.

2. Consequently, $\mathcal{A}_+ \cap \mathcal{A}_+^*$ is not trivial, containing the dilation group $\{V_t \mid t \in \mathbb{R}\}$.

3. $\mathcal{A}_+$ contains finite-rank operators of every even rank.

4. We again have a bounded approximate identity of Hilbert-Schmidt operators.

5. Lat $\mathcal{A}_+$ is homeomorphic to the disjoint union of a Euclidean manifold (a sphere) with two points coming from the trivial lattice.

6. $\mathcal{A}_+$ is reflexive.

Let $\mathcal{S}_\ell$ and $\mathcal{S}_+$ be the sets of operators

$$\mathcal{S}_\ell = \{\rho(r_\alpha), \rho(l_\gamma) \mid \alpha > 0,\ \gamma \geq 0\} \quad \text{and}$$
$$\mathcal{S}_+ = \{\rho(r_\alpha), \rho(l_\gamma), \rho(u_\beta) \mid \alpha > 0,\ \beta, \gamma \geq 0\}.$$

Theorem 6.12 says that w*-alg$(\mathcal{S}_\ell)$ = w*-alg$(\mathcal{S}_+)$, although at first sight it is not at all clear why the generators $\rho(u_\beta)$ should lie in w*-alg$(\mathcal{S}_\ell)$. The algebra generated by $\mathcal{S}_\ell$ "fills out" all of $\mathcal{S}_+$ when the w*-closure is taken. It is interesting to ask in which topologies this phenomenon occurs. We show that the norm-closed algebras generated by $\mathcal{S}_\ell$ and $\mathcal{S}_+$ are not equal. The idea behind the proof of this result is due to S. C. Power.

Lemma 6.15. *Fix $\beta > 0$ and for $t > 0$ let $J_t = (-\infty, -t] \cup [t, \infty)$. Then there is a $t > 0$ such that whenever $f \in L^2(\mathbb{R})$ with $\operatorname{supp} f \subseteq J_t$, the intersection $J_t \cap \operatorname{supp} \rho(u_\beta) f$ is empty. In fact, $t = 3\beta^{-1}$ will do.*

Proof. For $S \subseteq \mathbb{R}$, let us write $\operatorname{cl} S$ for the closure of S. By (6.5),

$$\rho(u_\beta)f(x) = g_\beta(x) f\big((\beta + x^{-1})^{-1}\big), \quad x \in \mathbb{R} \setminus \{0, -\beta^{-1}\}$$

where $g_\beta : \mathbb{R} \to \mathbb{C}$ is an almost everywhere non-zero function. Hence

$$\operatorname{supp} \rho(u_\beta) f = \operatorname{cl} \{ (y^{-1} - \beta)^{-1} \mid y \in \operatorname{supp} f \setminus \{0, \beta^{-1}\} \}.$$

A calculation reveals that if $t = 3\beta^{-1}$, then $0, \beta^{-1} \notin J_t$ and

$$\operatorname{cl} \{ (y^{-1} - \beta)^{-1} \mid y \in J_t \} = [\tfrac{3}{2}\beta^{-1}, \tfrac{3}{4}\beta^{-1}]$$

and this does indeed have empty intersection with J_t. □

Proposition 6.16. *Let $\mathcal{A}_\ell^{\mathrm{n}}$ and $\mathcal{A}_+^{\mathrm{n}}$ denote the norm-closed operator algebras generated by $\mathcal{S}_\ell$ and $\mathcal{S}_+$ respectively. Then $\mathcal{A}_\ell^{\mathrm{n}} \subsetneq \mathcal{A}_+^{\mathrm{n}}$.*

Proof. Fix $\beta > 0$. Intuitively, elements of $\mathcal{S}_\ell$ "fix ∞" whereas $\rho(u_\beta)$ is a "shift through ∞". We exploit this perspective to show that $\rho(u_\beta) \notin \mathcal{A}_\ell^{\mathrm{n}}$.

Given $t > 0$, again let $J_t = (-\infty, -t] \cup [t, \infty)$. Let $\mathcal{A}_\ell'$ denote the algebra generated by $\mathcal{S}_\ell$, so that $\mathcal{A}_\ell'$ is the set of finite sums of finite products of elements of $\mathcal{S}_\ell$.

We claim that for any $t > 0$ and for any $T \in \mathcal{A}_\ell'$, there is a real number $s = s(T, t)$ such that whenever $g \in L^2(\mathbb{R})$ and $\operatorname{supp} g \subseteq J_s$, we have $\operatorname{supp} Tg \subseteq J_t$. If $\alpha > 0$ and $T = \rho(r_\alpha)$, then $T = V_{2\log\alpha}$, so $s = t\alpha^{-2}$ suffices. If $\gamma \geq 0$ and $T = \rho(l_\gamma)$ then $T = D_{-\gamma}$, so $s = t + \gamma$ suffices. A simple induction argument establishes the claim for $T = \rho(a_1 a_2 \ldots a_n)$ where $a_i \in \mathcal{S}_\ell$ for $i = 1, 2, \ldots, n$. Another induction shows that the claim holds for a finite sum of such operators.

Fix $t = 3\beta^{-1}$ and $T \in \mathcal{A}_\ell'$. Let $s = s(T, t)$ and let $g \in L^2(\mathbb{R})$ with $\|g\| = 1$ and $\operatorname{supp} g \subseteq J_s \cap J_t = J_{\max(s,t)}$. Then

$$\operatorname{supp} Tg \subseteq J_t \quad \text{and} \quad J_t \cap \operatorname{supp} \rho(u_\beta) g = \emptyset$$

by Lemma 6.15, so Tg and $\rho(u_\beta)g$ are orthogonal in $L^2(\mathbb{R})$. Moreover, $\rho(u_\beta)$ is unitary, so

$$\|T - \rho(u_\beta)\|^2 \geq \|Tg - \rho(u_\beta)g\|^2 = \|Tg\|^2 + \|\rho(u_\beta)g\|^2 \geq \|\rho(u_\beta)g\|^2 = 1.$$

Since $\mathcal{A}'_\ell$ is norm-dense in $\mathcal{A}^{\mathrm{n}}_\ell$, this shows that $\operatorname{dist}(\rho(u_\beta), \mathcal{A}^{\mathrm{n}}_\ell) \geq 1$ and so $\rho(u_\beta) \notin \mathcal{A}^{\mathrm{n}}_\ell$. □

6.2.3 Questions

Fix $(h, s) \neq (1, 0)$. Let $\rho_{h,s}$ be the irreducible representation in the principal series given by (6.3) and let $\mathcal{A}_+$ be the w*-closed operator algebra generated by $\rho_{h,s}\big(SL_2(\mathbb{R}_+)\big)$. Now Lemma 6.2 still holds for $\mathcal{A}_+$; indeed by (6.4), the subalgebra

$$\mathcal{A}_\ell = \text{w*-alg}\{\rho_{h,s}(r_\alpha), \rho_{h,s}(l_\gamma) \mid \alpha > 0,\ \gamma \geq 0\}$$

is independent of our choice of h and s. However, the author has been unable to find an analogue of Lemma 6.3 since $Y_{h,s} = \rho_{h,s}(j)$ is no longer reduced by $H^2(\mathbb{R})$ and the only proper subspace obviously invariant for $\mathcal{A}_+$ is $L^2(\mathbb{R}_-)$. This prompts the following two questions.

Question 6.17. We know that $\operatorname{Lat}\mathcal{A}_+ \supseteq \{(0), L^2(\mathbb{R}_-), L^2(\mathbb{R})\}$. Do we in fact have equality?

Question 6.18. Is $\mathcal{A}_+$ reflexive?

If the answer to both of these questions is in the affirmative, then $\mathcal{A}_+$ is a somewhat uninteresting nest algebra in the irreducible case.

On a more general theme, we pose the following. Recall that when $(h, s) = (1, 0)$, the lattice $\operatorname{Lat}\mathcal{A}_+$ with the strong operator topology is homeomorphic to the disjoint union of a Euclidean manifold and the two points coming from the trivial lattice $\{(0), L^2(\mathbb{R})\}$. We call such a lattice a *nearly Euclidean lattice.* Of the three Lie semigroup operator algebras $\mathcal{A}_{\mathrm{p}}$, $\mathcal{A}_{\mathrm{h}}$ and $\mathcal{A}_+$ that we have seen, all are reflexive and all have nearly Euclidean invariant subspace lattices.

Question 6.19. Which operator algebras do other unitary-valued representations of $SL_2(\mathbb{R}_+)$ lead to? Are they reflexive, and are their invariant subspace lattices nearly Euclidean?

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References

[AB95] J. L. Alperin and Rowen B. Bell. *Groups and representations*, volume 162 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[AK95] M. Anoussis and A. Katavolos. Unitary actions on nests and the Weyl relations. *Bull. London Math. Soc.*, 27(3):265–272, 1995.

[Arv74] William Arveson. Operator algebras and invariant subspaces. *Ann. of Math.*, 100(2):433–532, 1974.

[Ber88] Hari Bercovici. *Operator theory and arithmetic in* H^∞, volume 26 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1988.

[Con85] John B. Conway. *A course in functional analysis*. Springer, 1985.

[Con91] John B. Conway. *The theory of subnormal operators*, volume 36 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1991.

[Dav88] Kenneth R. Davidson. *Nest algebras*, volume 191 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1988.

[Dav96] Kenneth R. Davidson. *C^*-algebras by example*, volume 6 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 1996.

[Don97] Robert W. Donley, Jr. Irreducible representations of $SL(2, \mathbf{R})$. In *Representation theory and automorphic forms (Edinburgh, 1996)*, volume 61 of *Proc. Sympos. Pure Math.*, pages 51–59. Amer. Math. Soc., Providence, RI, 1997.

[DS63] Nelson Dunford and Jacob T. Schwartz. *Linear operators. Part II: Spectral theory. Self adjoint operators in Hilbert space.* With the assistance of William G. Bade and Robert G. Bartle. Interscience Publishers John Wiley & Sons New York-London, 1963.

[Gar81] John B. Garnett. *Bounded analytic functions*, volume 96 of *Pure and Applied Mathematics*. Academic Press Inc., New York, 1981.

[Hal69] P. R. Halmos. Two subspaces. *Trans. Amer. Math. Soc.*, 144:381–389, 1969.

[Hal70] P. R. Halmos. Ten problems in Hilbert space. *Bull. Amer. Math. Soc.*, 76:887–933, 1970.

[Hal71] P. R. Halmos. Reflexive lattices of subspaces. *J. London Math. Soc. (2)*, 4:257–263, 1971.

[HHL89] Joachim Hilgert, Karl Heinrich Hofmann, and Jimmie D. Lawson. *Lie groups, convex cones, and semigroups*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1989. Oxford Science Publications.

[Hof62] Kenneth Hoffman. *Banach spaces of analytic functions*. Prentice-Hall series in modern mathematics. Prentice Hall, 1962.

[Kat76] Yitzhak Katznelson. *An introduction to harmonic analysis*. Dover Publications Inc., New York, corrected edition, 1976.

[KP97] A. Katavolos and S. C. Power. The Fourier binest algebra. *Math. Proc. Camb. Phil. Soc.*, 122(3):525–539, 1997.

[KP02] A. Katavolos and S. C. Power. Translation and dilation invariant subspaces of $L^2(\mathbb{R})$. *J. Reine Angew. Math.*, 552:101–129, 2002.

[Lan85] Serge Lang. $SL_2(\mathbb{R})$, volume 105 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1985. Reprint of the 1975 edition.

[Lax59] Peter D. Lax. Translation invariant spaces. *Acta Math.*, 101:163–178, 1959.

[Lev04] R. H. Levene. A double triangle operator algebra from $SL_2(\mathbb{R}_+)$, 2004. Preprint at `http://www.maths.lancs.ac.uk/~levene/preprints/`.

[LL92] M. S. Lambrou and W. E. Longstaff. Finite rank operators leaving double triangles invariant. *J. London Math. Soc. (2)*, 45(1):153–168, 1992.

[Lon83] W. E. Longstaff. Nonreflexive double triangles. *J. Austral. Math. Soc. Ser. A*, 35(3):349–356, 1983.

[LP03] R. H. Levene and S. C. Power. Reflexivity of the translation-dilation algebras on $L^2(\mathbb{R})$. *Int. J. Math.*, 14(10):1081–1090, 2003.

[Nik02] Nikolai K. Nikolski. *Operators, functions, and systems: an easy reading. Vol. 1*, volume 92 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

Hardy, Hankel, and Toeplitz, Translated from the French by Andreas Hartmann.

[Ped79] Gert K. Pedersen. *C^*-algebras and their automorphism groups*, volume 14 of *London Mathematical Society Monographs*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1979.

[Rin71] John R. Ringrose. *Compact non-self-adjoint operators*. Van Nostrand Reinhold Co., London, 1971.

[Rud66] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, 1966.

[Sal76] Paul J. Sally, Jr. Harmonic analysis and group representations. In *Studies in harmonic analysis (Proc. Conf., DePaul Univ., Chicago, Ill., 1974)*, pages 224–256. MAA Stud. Math., Vol. 13. Math. Assoc. Amer., Washington, D.C., 1976.

[SW71] Elias M. Stein and Guido Weiss. *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.