Statement Voting

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December 29, 2018

Abstract

In this work, we introduce a new concept, *statement voting*. Statement voting can be viewed as a natural extension of traditional candidate voting. Instead of defining a fixed election candidate, each voter can define a statement in his or her ballot but leave the vote "undefined" during the voting phase. During the tally phase, the (conditional) actions expressed in the statement will be carried out to determine the final vote.

We provide a comprehensive study of this new concept: under the Universal Composability (UC) framework, we define a class of ideal functionalities for statement voting, and then construct several protocols for realizing these functionalities. Since statement voting covers liquid democracy as a special case, our constructions immediately provide us the first solutions to liquid democracy. We remark that our statement voting can be extended to enable more complex voting and generic ledger-based non-interactive multi-party computation. We believe that the statement voting concept opens a door for constructing a new class of e-voting schemes.

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1 Introduction

Elections/Referendums provide people in each society with the opportunity to express their opinions in the collective decision making process. Unfortunately, the existing election/voting systems have many limitations and it often fails to serve the best interest of the people. For example, to make correct decisions, the voters have to invest tremendous effort to analyze the issues. The cost of identifying the best voting strategy is high, even if we assume that the voter has collected accurate information. In addition, misinformation campaigns often influence the voters to select certain candidates which could be against the voters' own interests. We here ask the following challenging question:

> *Is it possible to introduce new technologies to circumvent the implementation barriers so that more effective democracy can be enabled?*

We very much expect an affirmative answer because from a societal perspective, we need to ensure that these unmotivated/misinformed voters to participate in the process of decision making.

A new concept. We could approach the above problem via multiple angles. In this paper, we propose a new and clean concept: *statement voting*. Statement voting can be viewed as a natural extension of traditional candidate voting. Instead of defining a fixed election candidate, each voter can define a statement in his or her ballot but leave the vote "undefined" during the voting phase. During the tally phase, the (conditional) actions expressed in the statement will be carried out to determine the final vote. Single Transferable Vote (STV) is a special case of statement voting, where the voters rank the election candidates instead of naming only one candidate in their ballots. The ranked candidate list together with the STV tally rule can be viewed as an outcome-dependent statement. Roughly speaking, the statement declares that if my favorite candidate has already won or has no chance to win, then I would like to vote for my second favorite candidate, and so on.

Liquid democracy [For02], a hybrid of direct democracy and representative democracy, is another special case of statement voting; there, the voters can either vote directly on issues, or they can delegate their votes to representatives who vote on their behalf. The vote delegation can be expressed as a target-dependent statement, where a voter can define that his/her ballot is the same as the target voter's ballot.

Careful readers may wonder why this type of natural voting idea has never appeared in the *physical* world. First of all, conventional paper-voting ballot cannot handle complex statements. Moreover, in the reality, the voters care about privacy and anonymity. To ensure anonymity, the voters are not willing to leave their identities in the ballots. If no identities (or equivalences) are included in the ballots, then it is difficult for a statement refers to and depends on a particular voter's ballot, such as liquid democracy. The election committees might assign each voter a temporal ID to achieve anonymity, but a voter needs to obtain the target voter's temporal ID in order to delegate his vote. This requires secure peer-to-peer channels among all the voters, which is not practical. Before presenting our constructions, we first need to clearly define and model our security goal.

Modeling statement voting. We provide a rigorous modeling for statement voting. More concretely, we model statement voting in the well-known Universal Composability (UC) framework, via an ideal functionality $\mathcal{F}_{\mathrm{SV}}$. The functionality interacts with a set of voters, trustees, and consists of preparation phase, ballot casting phase, and tally phase. During the preparation phase, the trustees, need to indicate their presence to $\mathcal{F}_{\mathrm{SV}}$. The election/voting will not start until all the trustees have participated in the preparation.

In our formulation, we introduce a working table $\mathbb{W}$ to trace the voters' behavior. Each entry of the working table is saved for storing one voter's information including the voter's original ID, his alternative ID, and the voting statement he submitted; During the ballot casting phase, each voter can submit his voting statement. These voting statements will be collected and recorded in working table $\mathbb{W}$. If a voter is corrupt, then he is also allowed to revise his own alternative ID in the working table. When all the trustees are corrupted, the functionality $\mathcal{F}_{\mathrm{SV}}$ leaks the voters' information (i.e., $\mathbb{W}$), to the adversary.

The collected information in the working table $\mathbb{W}$ will be used in the tally phase for defining the privacy leakage as well as the final result. More concretely, we compute a new table $\mathbb{U}$ by first eliminating all V_i's in $\mathbb{W}$, and then sorting all the entries lexicographically. This carefully defined table $\mathbb{U}$ can now be used to define (1) the final result via applying a circuit $\mathsf{TallyProcess}$ on $\mathbb{U}$, and (2) certain level of privacy leakage L. Our formulation here allows us to define a *class* of statement voting functionalities. For example, to define a functionality with strong privacy, we can set $L := \mathsf{TallyProcess}(\mathbb{U})$; we can also set $L := \mathbb{U}$ to define a functionality with relatively weaker privacy, or set $L := \mathbb{W}$ to define a functionality without privacy.

We emphasize that in practice, virtually all threshold cryptographic systems cannot achieve fairness; namely, during the opening process of a threshold cryptographic system, the last several share holders can jointly see the content to be opened themselves before hand. Hence, they can decide if they want to actually open the content. However, surprisingly, this subtle issue was rarely modeled in the literature. For instance, the only previously known e-voting functionality [Gro04] fails to address it. During the tally phase, the tally will be released if all the trustees agree to proceed.

Constructions. Our statement voting concept can be implemented via the following different approaches. We assume a trusted *Registration Authority* (RA) to ensure voter eligibility and a consistent *Bulletin Board* (BB) where the voting transactions and result will be announced to.

A fully/somewhat homomorphic encryption based scheme. In this scheme, the trustees first run a distributed key generation protocol to setup the voting public key PK. Each voter V_i then encrypt, sign and submit their *voting statements*, x_i (in forms of $(\mathsf{PID}_i, \mathsf{Enc}_{\mathsf{PK}}(x_i))$) to the BB. To present re-play attacks, zero-knowledge (ZK) proofs are necessary to ensure the voter knows the plaintext included in his/her submitted ciphertext. After that, the tally processing circuit is evaluated over $\{(\mathsf{PID}_i, \mathsf{Enc}_{\mathsf{PK}}(x_i))\}_{i\in[n]}$ by every trustee. The final tally ciphertext is then decrypted by the trustees and the result will be announced on the BB. To ensure universal verifiability, the circuit evaluation and ZK proofs shall be publicly auditable.

A verifiable MPC based scheme. In this scheme, we can adopt BDO publicly auditable MPC [BDO14], where the trustees form the MPC system. They pre-compute sufficiently many correlated randomness (e.g., Beaver triples), and also set up a voting public key. Each voter V_i then encrypt, sign and submit their *voting statements*, x_i (in forms of $(\mathsf{PID}_i, \mathsf{Enc}_{\mathsf{PK}}(x_i))$) to the BB. Again, to present re-play attacks, ZK proofs are necessary to ensure the voter knows the plaintext included in his/her submitted ciphertext. After that, the trustees perform MPC online computation to first decrypt those encrypted ballots and then evaluate the tally processing circuit over the shared ballots. Finally, the tally result will be posted on the BB. Note that during the online phase, BDO MPC scheme also posts audit information on the BB to enable public verifiability.

A mix-net based scheme. In this scheme, the trustees first run a distributed key generation protocol to set up the public key PK of a re-randomizeable encryption scheme. Each voter V_i then encrypt, sign and submit a random temporal ID w_i, in forms of $(\mathsf{PID}_i, \mathsf{Enc}_{\mathsf{PK}}(w_i))$ to the BB. After that each voter will submit an encrypted voting statement where PID_j are replaced with re-randomized encryption τ_j, for all $j \in [n]$. The encrypted statement together with the voter's encrypted temporal ID will then be shuffled via a mix-net. The resulting ciphertexts will be decrypted by the trustees and evaluated by every voter themselves. Note that the tally processing function must be symmetric, otherwise we cannot use mix-net. The privacy that this construction achieves is relatively weaker. However, we emphasize that this level of privacy has been widely accepted and is consistent with all the existing paper-based voting systems.

An immediate application: liquid democracy. In the past decades, the concept of liquid democracy [For02] has been emerging as an alternative decision making model to make better use of collective intelligence. Due to its advantages, liquid democracy has received high attentions since the spread of its concept; however, there is no satisfactory solution in the form of either paper-voting or e-voting yet[1].

[1] All the existing liquid democracy implementations, e.g., Google Votes and Decentralized Autonomous Organization (DAO) do not consider privacy/anonymity. This drawback prevents them from being used in serious elections. Here, we note that straightforward blockchain-based solutions cannot provide good privacy in practice. Although some blockchains such as Zerocash [BCG+14] can

Extensions and further remarks. In this work, we initiate the study of statement voting. We remark that our statement voting concept can be significantly extended to support much richer ballot statements. It opens a door for constructing a new class of e-voting schemes. We also remark that this area of research is far from being completed, and our design and modeling ideas can be further improved. Nevertheless, designing useful and acceptable statement policy is a topic in the computational social choice theory, which is outside the scope of this paper. We finally remark that, voting policies can be heavily influenced by local legal and societal conditions. How to define "right" voting policy itself is a very interesting question. We believe our techniques here have the potential to help people to identify suitable voting policies which can further eliminate the barriers to democracy.

Related work. The concept of liquid democracy (a.k.a. delegative democracy) is emerging over the last decades [Mil69, Alg06, BZ16]. To our best knowledge, Ford [For02] first officially summarized the main characteristics of liquid democracy and brought it to the vision of computer science community. However, in terms of implementation/prototyping, there was no system that can enable liquid democracy until very recently. Google Votes [HL15] is a decision-making system that can support liquid democracy, and it is built on top of social networks, e.g., the internal corporate Google+ network. Decentralized Autonomous Organization (DAO) [DAO17] realized liquid democracy using the blockchain technology, and it has been widely used to vote on expenses and actions of various contracts. Recently, Merkle [Mer16] provide a comprehensive review on DAO, collective intelligence, and liquid democracy. He believes that liquid democracy can utilize the expertise of all the citizens to make high-quality decisions. Nevertheless, all the existing liquid democracy voting systems only focus on the functionality aspect of liquid democracy, and no privacy or some other advanced security properties were considered.

With regards to conventional security oriented e-voting systems, Chaum [Cha81] proposed to use anonymous channels and pseudonyms to achieve voter privacy and verifiability. Benaloh and Yung later showed how to distribute the centralized election authority using threshold cryptography [BY86]. Sako and Kilian [SK95] minimized the assumption needed for a mix-type voting system to achieve so-call receipt-freeness, where the voters can simulate/hide their votes to the coercers. Groth [Gro04] gave the first UC definition for an e-voting system, and he proposed a protocol using (threshold) homomorphic encryption. Moran and Naor [MN06] later studied the privacy and receipt-freeness of an e-voting system in the stand-alone setting. Unruh and Muller-Quade [UMQ10] gave a formal study of e-voting coerciability in the UC framework. Alwen *et al.* [AOZZ15] considered stronger versions of coerciability in the MPC setting under UC framework.

The most widely used e-voting system in practice is Helios [Adi08]. Almost all the end-to-end verifiable e-voting systems [CRS05, CEC+08, KZZ15b, KZZ15a] requires a consistent bulletin board (BB). However, none of them gives a practical realization of BB. Meanwhile, there is some work [KMNV16] on extending Helios to support proxy voting, where the voters can delegate their votes to some pre-defined proxies. Each voter can only delegate once.

On the other hand, Kiayias, Zhou, and Zikas [KZZ16] gives a UC model of the global ledger and discuss about how to use such a ledger to enable fair and robust MPC.

Organization. In Section 2, we present the required preliminaries including a brief overview of UC framework and some useful ideal functionalities. In Section 3, we rigorously model statement voting. In Section 4, we present the details of our statement voting construction via threshold fully homomorphic encryption; additional information such as security proof and building block instantiation can be found in Supplemental material A. In Section 5, we present a statement voting construction via verifiable MPC; additional information can be found in in Supplemental material B. Then in Section 6, we present a more efficient solution via mix-net; additional information can be found in in Supplemental material C. Finally, our application to liquid democracy can be found in Section 7.

be viewed as a global mixer, they implicitly require anonymous channels. While in practice, all the implementations of anonymous channels suffer from time leakage, i.e., the user's ID is only hidden among the other users who are also using the system at the same time. Subsequently, the adversary can easily identify the user during quiet hours.

2 Preliminaries

2.1 The UC framework

Following Canetti's framework [Can01, Can00], a protocol is represented as interactive Turing machines (ITMs), each of which represents the program to be run by a participant. Protocols that securely carry out a given task are defined in three steps, as follows. First, the process of executing a protocol in an adversarial environment is formalized. Next, an "ideal process" for carrying out the task at hand is formalized. The parties have access to an "ideal functionality," which is essentially an incorruptible "trusted party" that is programmed to capture the desired functionality of the task at hand. A protocol is said to securely realize an ideal functionality if the process of running the protocol amounts to "emulating" the ideal process for that ideal functionality. Below we overview the model of protocol execution (called the *real-world model*), the ideal process, and the notion of protocol emulation.

The model for protocol execution. The model of computation consists of the parties running an instance of a protocol π, a network adversary $\mathcal{A}$ that controls the communication among the parties, and an environment $\mathcal{Z}$ that controls the inputs to the parties and sees their outputs. The execution consists of a sequence of *activations,* where in each activation a single participant (either $\mathcal{Z}$, $\mathcal{A}$, or some other ITM) is activated, and may write on a tape of at most *one* other participant, subject to the rules below. Once the activation of a participant is complete, the participant whose tape was written on is activated next.

Let $\mathsf{EXEC}_{\pi,\mathcal{A},\mathcal{Z}}(\lambda, z, r)$ denote the output of the environment $\mathcal{Z}$ when interacting with parties running protocol π on security parameter λ, input z and random input $r = r_{\mathcal{Z}}, r_{\mathcal{A}}, r_1, r_2, ...$ as described above (z and $r_{\mathcal{Z}}$ for $\mathcal{Z}$; $r_{\mathcal{A}}$ for $\mathcal{A}$, r_i for party P_i). Let $\mathsf{EXEC}_{\pi,\mathcal{A},\mathcal{Z}}(k, z)$ denote the random variable describing $\mathsf{EXEC}_{\pi,\mathcal{A},\mathcal{Z}}(k, z, r)$ when r is uniformly chosen. Let $\mathsf{EXEC}_{\pi,\mathcal{A},\mathcal{Z}}$ denote the ensemble $\{\mathsf{EXEC}_{\pi,\mathcal{A},\mathcal{Z}}(k, z)\}_{k\in\mathbb{N},z\in\{0,1\}^*}$.

Ideal functionalities and ideal protocols. Security of protocols is defined via comparing the protocol execution to an *ideal protocol* for carrying out the task at hand. A key ingredient in the ideal protocol is the *ideal functionality* that captures the desired functionality, or the specification, of that task. The ideal functionality is modeled as another ITM (representing a "trusted party") that interacts with the parties and the adversary. More specifically, in the ideal protocol for functionality $\mathcal{F}$ all parties simply hand their inputs to an ITM instance running $\mathcal{F}$.

Securely realizing an ideal functionality. We say that a protocol π *emulates* protocol ϕ if for any network adversary $\mathcal{A}$ there exists an adversary (also known as simulator) $\mathcal{S}$ such that no environment $\mathcal{Z}$, on any input, can tell with non-negligible probability whether it is interacting with A and parties running π, or it is interacting with $\mathcal{S}$ and parties running ϕ. This means that, from the point of view of the environment, running protocol π is "just as good" as interacting with ϕ. We say that π *securely realizes* an ideal functionality $\mathcal{F}$ if it emulates the ideal protocol for $\mathcal{F}$. More precise definitions follow. A distribution ensemble is called *binary* if it consists of distributions over $\{0, 1\}$.

Definition 2.1. *Let π and ϕ be protocols, and $\mathcal{F}$ be an ideal functionality. We say that π* UC-emulates *ϕ if for any adversary $\mathcal{A}$ there exists an adversary $\mathcal{S}$ such that for any environment $\mathcal{Z}$ that obeys the rules of interaction for UC security we have $\mathsf{EXEC}_{\phi,\mathcal{S},\mathcal{Z}} \approx \mathsf{EXEC}_{\pi,\mathcal{A},\mathcal{Z}}$. We say that π* UC-realizes *$\mathcal{F}$ if π* UC-emulates *the ideal protocol for functionality $\mathcal{F}$.*

Hybrid protocols. Hybrid protocols are protocols where, in addition to communicating as usual as in the standard model of execution, the parties also have access to (multiple copies of) an ideal functionality. Hybrid protocols represent protocols that use idealizations of underlying primitives, or alternatively make *trust*

assumptions on the underlying network. They are also instrumental in stating the universal composition theorem. Specifically, in an $\mathcal{F}$-hybrid protocol (i.e., in a hybrid protocol with access to an ideal functionality $\mathcal{F}$), the parties may give inputs to and receive outputs from an unbounded number of copies of $\mathcal{F}$. The definition of a protocol securely realizing an ideal functionality is extended to hybrid protocols in the natural way.

2.2 Ideal functionalities

Bulletin board functionality The public bulletin board (BB) is modeled as a global functionality $\bar{\mathcal{G}}_{\text{BB}}$. Formal description can be found in Fig. 1. The functionality is parameterized with a predicate Validate that ensures all the newly posted messages are consistent with the existing BB content w.r.t. Validate. Any party can use $(\text{SUBMIT}, \mathsf{sid}, \mathsf{msg})$ and $(\text{READ}, \mathsf{sid})$ to write/read the BB. We remark that our $\bar{\mathcal{G}}_{\text{BB}}$ can be much simplified version of the global public ledger functionality $\bar{\mathcal{G}}_{\text{LEDGER}}$ recently defined by Kiayias et al [KZZ16].

Functionality $\bar{\mathcal{G}}_{\text{BB}}$

The shared functionality $\bar{\mathcal{G}}_{\text{BB}}$ is globally available to all the parties and the adversary $\mathcal{S}$. It is parameterized with a predicate Validate, and variable state. Initially, state $:= \varepsilon$.

- Upon receiving $(\text{SUBMIT}, \mathsf{sid}, \mathsf{msg})$ from a party P or the adversary $\mathcal{S}$, if $\mathsf{Validate}(\mathsf{state}, \mathsf{msg}) = 1$, then set $\mathsf{state} := \mathsf{state}||\mathsf{msg}$.
- Upon receiving $(\text{READ}, \mathsf{sid})$ from a party P or the adversary $\mathcal{S}$, return $(\text{READ}, \mathsf{sid}, \mathsf{state})$ to the requestor.

Figure 1: The public bulletin board functionality.

Certificate functionality We present the multi-session version of certificate functionality following the modeling of [Can03]. The multi-session certificate functionality $\widehat{\mathcal{F}}_{\text{CERT}}$ can provide direct binding between a signature for a message and the identity of the corresponding signer. This corresponds to providing signatures accompanied by "certificates" that bind the verification process to the signers' identities. For completeness, we recap $\widehat{\mathcal{F}}_{\text{CERT}}$ in Fig. 2.

2.3 Non-interactive zero-knowledge proofs/arguments

Here we briefly introduce non-interactive zero-knowledge (NIZK) schemes in the Random Oracle (RO) model. Let $\mathcal{R}$ be an efficiently computable binary relation. For pairs $(x, w) \in \mathcal{R}$ we call x the statement and w the witness. Let $\mathcal{L}_{\mathcal{R}}$ be the language consisting of statements in $\mathcal{R}$, i.e. $\mathcal{L}_{\mathcal{R}} = \{x | \exists w \text{ s.t. } (x, w) \in \mathcal{R}\}$. An NIZK scheme includes following algorithms: a PPT algorithm Prov that takes as input $(x, w) \in \mathcal{R}$ and outputs a proof π; a polynomial time algorithm Verify takes as input (x, π) and outputs 1 if the proof is valid and 0 otherwise.

Definition 2.2 (NIZK Proof of Membership in the RO Model). $\mathsf{NIZK}^{RO}_{\mathcal{R}}.\{\mathsf{Prov}, \mathsf{Verify}, \mathsf{Sim}, \mathsf{Ext}\}$ *is an NIZK Proof of Membership scheme for the relation* $\mathcal{R}$ *if the following properties hold:*

- *Completeness: For any* $(x, w) \in \mathcal{R}$,

$$\Pr\left[\ \zeta \leftarrow \{0,1\}^{\lambda}; \pi \leftarrow \mathsf{Prov}^{RO}(x, w; \zeta) : \mathsf{Verify}^{RO}(x, \pi) = 0\ \right] \leq \mathsf{negl}(\lambda).$$

- *Zero-knowledge: If for any* PPT *distinguisher* $\mathcal{A}$ *we have*

$$\left|\ \Pr[\mathcal{A}^{RO, \mathcal{O}_1}(1^{\lambda}) = 1] - \Pr[\mathcal{A}^{RO, \mathcal{O}_2}(1^{\lambda}) = 1]\ \right| \leq \mathsf{negl}(\lambda).$$

  The oracles are defined as follows: $\mathcal{O}_1$ *on query* $(x, w) \in \mathcal{R}$ *returns* π*, where* $(\pi, aux) \leftarrow \mathsf{Sim}^{RO}(x)$*;* $\mathcal{O}_2$ *on query* $(x, w) \in \mathcal{R}$ *returns* π*, where* $\pi \leftarrow \mathsf{Prov}^{RO}(x, w; \zeta)$ *and* $\zeta \leftarrow \{0,1\}^{\lambda}$.

Functionality $\widehat{\mathcal{F}}_{\text{CERT}}$

The functionality $\widehat{\mathcal{F}}_{\text{CERT}}$ interacts with a set of signers $\{\mathsf{S}_1, \ldots, \mathsf{S}_k\}$, and a set of verifiers $\{\mathsf{R}_1, \ldots, \mathsf{R}_n\}$, and the adversary $\mathcal{S}$.

- Upon receiving $(\textsc{Sign}, \mathsf{sid}, \mathsf{ssid}, m)$ from a signer $P \in \{\mathsf{S}_1, \ldots, \mathsf{S}_k\}$, verify that $\mathsf{ssid} = (P, \mathsf{ssid}')$ for some ssid'. If not, ignore the request. Otherwise, send $(\textsc{SignNotify}, \mathsf{sid}, \mathsf{ssid}, m)$ to the adversary $\mathcal{S}$. Upon receiving $(\textsc{Signature}, \mathsf{sid}, \mathsf{ssid}, m, \sigma)$ from $\mathcal{S}$, verify that no entry $(\mathsf{ssid}, m, \sigma, 0)$ is recorded. If it is, then return $(\textsc{Error})$ to P and halt. Else, return $(\textsc{Signature}, \mathsf{sid}, \mathsf{ssid}, m, \sigma)$ to P, and record the entry $(\mathsf{ssid}, m, \sigma, 1)$.
- Upon receiving $(\textsc{Verify}, \mathsf{sid}, \mathsf{ssid}, m, \sigma)$ from any party $P \in \{\mathsf{R}_1, \ldots, \mathsf{R}_n\}$, send $(\textsc{VerifyNotify}, \mathsf{sid}, \mathsf{ssid}, m)$ to the adversary $\mathcal{S}$. Upon receiving $(\textsc{Verified}, \mathsf{sid}, \mathsf{ssid}, m, b^*)$ from $\mathcal{S}$, do:
  - If $(\mathsf{ssid}, m, \sigma, 1)$ is recorded then set $b = 1$.
  - Else, if the signer of subsession ssid is not corrupted, and no entry $(\mathsf{ssid}, m, \cdot, 1)$ is recorded, then set $b = 0$ and record the entry $(\mathsf{ssid}, m, \sigma, 0)$.
  - Else, if there is an entry $(\mathsf{ssid}, m, \sigma, b')$ recorded, then set $b := b'$.
  - Else, set $b := b^*$, and record the entry $(\mathsf{ssid}, m, \sigma, b^*)$.

  Output $(\textsc{Verified}, \mathsf{sid}, \mathsf{ssid}, m, b)$ to P.

Figure 2: The multi-session functionality for certificate.

- *Soundness: For all PPT adversary $\mathcal{A}$,*

$$\Pr\left[\ (x, \pi) \leftarrow \mathcal{A}^{RO}(1^\lambda) : x \notin \mathcal{L}_R \wedge \mathsf{Verify}^{RO}(x, \pi) = 1\ \right] \leq \mathsf{negl}(\lambda).$$

Definition 2.3 (NIZK Proof of Knowledge in the RO Model)**.** $\mathsf{NIZK}^{RO}_{\mathcal{R}}.\{\mathsf{Prov}, \mathsf{Verify}, \mathsf{Sim}, \mathsf{Ext}\}$ *is an NIZK Proof of Knowledge scheme for the relation $\mathcal{R}$ if the completeness, zero-knowledge, and extraction properties hold, where the extraction is defined as follows.*

- *Extractability: For all PPT adversary $\mathcal{A}$,*

$$\Pr\left[\begin{array}{l}(x, \pi) \leftarrow \mathcal{A}^{RO}(1^\lambda); w \leftarrow \mathsf{Ext}^{RO}(x, \pi) : \\ (x, w) \in \mathcal{R} \textit{ if } \mathsf{Verify}^{RO}(x, \pi) = 1\end{array}\right] \geq 1 - \mathsf{negl}(\lambda).$$

We need non-interactive zero-knowledge proofs/arguments of knowledge and non-interactive zero-knowledge proofs/arguments of membership. For simplicity, we will drop RO from the superscript if the context is clear.

We use $(\mathsf{NIZK}_{\mathcal{R}_i}.\mathsf{Verify}, \mathsf{NIZK}_{\mathcal{R}_i}.\mathsf{Sim})$ to denote the corresponding verification algorithm and simulator.

3 Modeling

Our statement voting system consists of a set of trustees $\mathbb{T} := \{\mathsf{T}_1, \ldots, \mathsf{T}_k\}$, and a set of voters $\mathbb{V} := \{\mathsf{V}_1, \ldots, \mathsf{V}_n\}$. In this section, we will define an ideal functionality for statement voting. In next sections (Sections 4, 5 and 6), we will construct several protocols for realizing the ideal statement voting functionality.

The statement voting functionality. The ideal functionality for statement voting, denoted as $\mathcal{F}_{\text{SV}}$, is formally described in Fig. 3. The functionality interacts with n number of voters, k number of trustees. It consists of three phases—Preparation, Ballot Casting, and Tally. The functionality uses a working table $\mathbb{W}$ to trace the

Functionality $\mathcal{F}_{\text{SV}}$

The functionality $\mathcal{F}_{\text{SV}}$ interacts with a set of voters $\mathbb{V} := \{\mathsf{V}_1, \ldots, \mathsf{V}_n\}$, a set of trustees $\mathbb{T} := \{\mathsf{T}_1, \ldots, \mathsf{T}_k\}$, and the adversary $\mathcal{S}$. Let $\mathbb{V}_{\mathsf{honest}}$, $\mathbb{V}_{\mathsf{corrupt}}$ and $\mathbb{T}_{\mathsf{honest}}$, $\mathbb{T}_{\mathsf{corrupt}}$ denote the set of honest/corrupt voters and trustees, respectively.
Functionality $\mathcal{F}_{\text{SV}}$ is parameterized by an algorithm $\mathsf{TallyProcess}$ (see Figure 4), a working table $\mathbb{W}$, and variables $result$, T_1, T_2, and B_i for all $i \in [n]$.

Initially, set $result := \emptyset$, $T_1 := \emptyset$, $T_2 := \emptyset$; for $i \in [n]$, set $B_i := \emptyset$.
Table $\mathbb{W}$ consists of n entries, and each entry consists of voter's real ID, voter's alternative ID, and the statement that the voter submitted; for all $i \in [n]$, the ith entry $\mathbb{W}[i] := (\mathsf{V}_i, w_i, statement_i)$, where $w_i \leftarrow \{0,1\}^\lambda$, $statement_i := \emptyset$.

Preparation:

1. Upon receiving input $(\textsc{InitialTrustee}, \mathsf{sid})$ from the trustee $\mathsf{T}_j \in \mathbb{T}$, set $T_1 := T_1 \cup \{\mathsf{T}_j\}$, and send a notification message $(\textsc{InitialTrusteeNotify}, \mathsf{sid}, \mathsf{T}_j)$ to the adversary $\mathcal{S}$.

Ballot Casting:

1. Upon receiving input $(\textsc{Cast}, \mathsf{sid}, (s_i, w_i^*))$ from the voter $\mathsf{V}_i \in \mathbb{V}$, if $|T_1| < k$, ignore the input.
   Otherwise,
   - if V_i is honest (now $w_i^* := \perp$), then update $\mathbb{W}[i] := (\mathsf{V}_i, w_i, s_i)$; send a message $(\textsc{CastNotify}, \mathsf{sid}, \mathsf{V}_i)$ to the adversary $\mathcal{S}$.
   - if V_i is corrupt, then update $\mathbb{W}[i] := (\mathsf{V}_i, w_i^*, s_i)$.

   If $|\mathbb{T}_{\mathsf{corrupt}}| = k$, then additionally send a message $(\textsc{Leak}, \mathsf{sid}, \mathbb{W}[i])$ to the adversary $\mathcal{S}$.

Tally:

1. Upon receiving input $(\textsc{Tally}, \mathsf{sid})$ from the trustee $\mathsf{T}_j \in \mathbb{T}$, set $T_2 := T_2 \cup \{\mathsf{T}_j\}$ and do the following:
   - set $\mathbb{U} := \mathbb{W}$; then eliminate all V_i's in $\mathbb{U}$; finally sort the entries in $\mathbb{U}$ lexicographically.
   - define L. For example, set $L := \mathsf{TallyProcess}(\mathbb{U})$ or $L := \mathbb{U}$ or $L := \mathbb{W}$.

   Send a notification message $(\textsc{TallyNotify}, \mathsf{sid}, \mathsf{T}_j)$ to $\mathcal{S}$.
   If $|T_2 \cap \mathbb{T}_{\mathsf{honest}}| + |\mathbb{T}_{\mathsf{corrupt}}| = k$, send a leakage message $(\textsc{Leak}, \mathsf{sid}, L)$ to $\mathcal{S}$.
   If $|T_2| = k$, compute $result \leftarrow \mathsf{TallyProcess}(\mathbb{U})$.
2. Upon receiving input $(\textsc{ReadResult}, \mathsf{sid})$ from a voter $\mathsf{V}_i \in \mathbb{V}$, if $result = \emptyset$, ignore the input.
   Else, return $(\textsc{ResultReturn}, \mathsf{sid}, result)$ to V_i.

Figure 3: The voting functionality.

voters' behavior during the entire ideal execution. Each entry of the working table is saved for storing one voter's information including the voter's original ID, his alternative ID, and the voting statement that he submitted;

Preparation phase. During the preparation phase, the trustees, playing the role of voting organizers, need to indicate their presence to $\mathcal{F}_{\text{SV}}$ by sending $(\textsc{InitialTrustee}, \mathsf{sid})$ to it. The election/voting will not start until all the trustees have participated in the preparation.

Ballot Casting phase. During the ballot casting phase, each voter can submit his voting statement, and this voting statement will be recorded in the corresponding entry. If a voter is corrupt, then he is also allowed to revise his own alternative ID in the working table. More concretely, based on the input $(\textsc{Cast}, \mathsf{sid}, (s_i, w_i^*))$ from voter V_i, the corresponding entry will be updated, i.e., $\mathbb{W}[i] := (\mathsf{V}_i, w_i, s_i)$ if the voter is honest, and $\mathbb{W}[i] := (\mathsf{V}_i, w_i^*, s_i)$ if V_i is corrupt. When all the trustees are corrupted, the functionality $\mathcal{F}_{\text{SV}}$ leaks the voters' information (i.e., $\mathbb{W}$), to the adversary.

Tally phase. Voters' information in the working table $\mathbb{W}$ will be used in the tally phase for defining the privacy leakage as well as the final result. More concretely, we compute a new table $\mathbb{U}$ by first eliminating all V_i's in $\mathbb{W}$, and then sorting all the entries lexicographically. This carefully defined table $\mathbb{U}$ can now be used to define (1) the final result via applying a circuit $\mathsf{TallyProcess}$ on $\mathbb{U}$, and (2) certain level of privacy leakage L. Our formulation here allows us to define a *class* of statement voting functionalities. For example, to define a functionality with strong privacy, we can set $L := \mathsf{TallyProcess}(\mathbb{U})$; we can also set $L := \mathbb{U}$ to define a functionality with relatively weaker privacy, or set $L := \mathbb{W}$ to define a functionality without privacy.

Let $\mathsf{TallyAlg}$ be a deterministic symmetric election tally function that takes $\mathcal{V}$ and outputs the tally result. The concrete functionality of $\mathsf{TallyAlg}$ depends on the actual election, and we do not have any restriction on $\mathsf{TallyAlg}$. Take a simple 1-out-of-m election where there are m candidates $\mathcal{C} := (C_1, \ldots, C_m)$ as an example. Each vote $v_i \in \mathcal{V}$ is an element in $\mathcal{C}$. The tally result is an m-vector $\mathbb{Z}_+^m$ whose i-th coordinate is equal to the number of times C_i was chosen in $\mathcal{V}$. In the rest of this paper, we use $v_i = \perp$ to indicate *blank* vote, and $\mathsf{TallyAlg}$ should ignore those inputs.

$\mathsf{TallyProcess}$

Input: a set of ballots $\mathcal{B} := (B_1, \ldots, B_n)$

Output: the tally result *result*

Statement interpretation:

- Compute $(v_1, \ldots, v_n) \leftarrow \mathsf{StatementProcess}(B_1, \ldots, B_n)$, where $\mathsf{StatementProcess}$ takes input as the set of statements and outputs the voters' final votes.

Tally computation:

- Compute $result \leftarrow \mathsf{TallyAlg}(v_1, \ldots, v_n)$, where $\mathsf{TallyAlg}(\cdot)$ is the tally algorithm that takes input as the votes and outputs the tally result.
- Return *result*.

Figure 4: The extended tally processing algorithm.

4 Homomorphic Encryption based construction

To illustrate the main concept, we first present a (key-homomorphic) threshold fully homomorphic encryption (FHE) based scheme. A key-homomorphic public key encryption scheme allows the user to deterministically combine several public keys $\mathtt{pk}_1, \ldots, \mathtt{pk}_n$ into a combined public key $\mathtt{pk}$; meanwhile, the corresponding secret keys $\mathtt{sk}_1, \ldots, \mathtt{sk}_n$ can be combined into the secret key $\mathtt{sk}$ for $\mathtt{pk}$. This property can enable efficient distributed

key generation. In a nutshell, the scheme works as follows. During the preparation phase, each trustee $\mathsf{T}_j \in \mathbb{T}$ generates a public key $\mathtt{pk}_j$ and posts it on the $\bar{\mathcal{G}}_{\mathrm{BB}}$. The voters $\mathsf{V}_i \in \mathbb{V}$ can then combine the posted $\mathtt{pk}_j$'s to the election public key $\mathtt{pk}$. During the ballot casting phase, the voters $\mathsf{V}_i \in \mathbb{V}$ submit their encrypted statement to the $\bar{\mathcal{G}}_{\mathrm{BB}}$. After that, all the parties can evaluate the (public deterministic) TallyProcess circuit over the encrypted data. During the tally phase, the trustees $\mathsf{T}_j \in \mathbb{T}$ then jointly decrypt the final tally ciphertext(s) to reveal the election outcome. We will now introduce the main primitive, publicly-evaluable key-homomorphic *threshold fully homomorphic encryption* (TFHE).

4.1 Key-homomorphic threshold fully homomorphic encryption

A publicly evaluable key-homomorphic threshold fully homomorphic encryption scheme TFHE consists of a tuple of algorithms: (Setup, Keygen, Enc, Eval, Dec, CombinePK, CombineSK, ShareDec, ShareCombine) as follows.

- $\mathsf{param} \leftarrow \mathsf{Setup}(1^\lambda)$. The algorithm Setup takes input as the security parameter λ, and outputs public parameters param. All the other algorithms implicitly take param as input.
- $(\mathtt{pk}, \mathtt{sk}) \leftarrow \mathsf{Keygen}(\mathsf{param})$. The algorithm Keygen takes input as the public parameter param, and outputs a public key $\mathtt{pk}$, a secret key $\mathtt{sk}$.
- $c \leftarrow \mathsf{Enc}(\mathtt{pk}, m)$. The algorithm Enc takes input as the public key $\mathtt{pk}$ and the message m, and outputs the ciphertext c.
- $c' := \mathsf{Eval}(\mathtt{pk}, \mathcal{F}, c_1, \ldots, c_n)$. The algorithm Eval takes input as the public/evaluation key $\mathtt{pk}$, the description of the evaluation function (circuit) $\mathcal{F}$, and a set of ciphertexts $c_1, \ldots, c_n$, and outputs the result ciphertext c'.
- $m \leftarrow \mathsf{Dec}(\mathtt{sk}, c)$. The algorithm Dec takes input as the secret key $\mathtt{sk}$ and a ciphertext c, and outputs the decrypted plaintext m.
- $\mathtt{pk} := \mathsf{CombinePK}(\mathtt{pk}_1, \ldots, \mathtt{pk}_k)$. The algorithm CombinePK takes input as a set of public keys $(\mathtt{pk}_1, \ldots, \mathtt{pk}_k)$, and outputs a combined public key $\mathtt{pk}$.
- $\mathtt{sk} \leftarrow \mathsf{CombineSK}(\mathtt{sk}_1, \ldots, \mathtt{sk}_k)$. The algorithm CombineSK takes input as a set of secret key $(\mathtt{sk}_1, \ldots, \mathtt{sk}_k)$, and outputs combined secret key $\mathtt{sk}$.
- $\mu_i \leftarrow \mathsf{ShareDec}(\mathtt{sk}_i, c)$. The algorithm ShareDec takes input as the secret key $\mathtt{sk}_i$ and a ciphertext c, and outputs a decryption share μ_i.
- $m \leftarrow \mathsf{ShareCombine}(c, \mu_1, \ldots, \mu_k)$. The algorithm ShareCombine takes input as a ciphertext c and k decryption shares $(\mu_1, \ldots, \mu_k)$, and outputs a plaintext m.
- $c' \leftarrow \mathsf{Trans}(c, \{\mathtt{sk}_i\}_{i \in [k] \setminus \{j\}})$. The algorithm Trans takes input as a ciphertext $c \leftarrow \mathsf{TFHE.Enc}(\mathtt{pk}_j, m)$ and a set of secret keys $\{\mathtt{sk}_i\}_{i \in [k] \setminus \{j\}}$, and outputs a ciphertext c'.
- $\{\mu_j\}_{j \in [k] \setminus \mathcal{I}} \leftarrow \mathsf{SimShareDec}(c, m, \{\mu_i\}_{i \in \mathcal{I}})$. The algorithm SimShareDec takes as input a ciphertext c, a plaintext m, and a set of decryption shares $\{\mu_i\}_{i \in \mathcal{I}}$ and outputs a set of decryption shares $\{\mu_j\}_{j \in [k] \setminus \mathcal{I}}$. Here $\mathcal{I} \subsetneq [k]$.

Definition 4.1. *We say* $\mathsf{TFHE} = \{\mathsf{Setup}, \mathsf{Keygen}, \mathsf{Enc}, \mathsf{Eval}, \mathsf{Dec}, \mathsf{CombinePK}, \mathsf{CombineSK}, \mathsf{ShareDec}, \mathsf{ShareCombine}\}$ *is a* secure *key-homomorphic threshold fully homomorphic encryption if the following properties hold:*

- Key combination correctness: *If* $\{(\mathtt{pk}_i, \mathtt{sk}_i)\}_{i\in[k]}$ *are all valid key pairs,* $\mathtt{pk} := \mathsf{TFHE.CombinePK}(\{\mathtt{pk}_i\}_{i\in[k]})$ *and* $\mathtt{sk} := \mathsf{TFHE.CombineSK}(\{\mathtt{sk}_i\}_{i\in[k]})$*, then* $(\mathtt{pk}, \mathtt{sk})$ *is also a valid key pair.*

  For all ciphertext $c \in \mathcal{C}_{\mathtt{pk}}$*, where* $\mathcal{C}_{\mathtt{pk}}$ *is the ciphertext-space defined by* pk*, we have*

$$\mathsf{TFHE.Dec}(\mathtt{sk}, c) = \mathsf{TFHE.ShareCombine}(c, \mathsf{TFHE.ShareDec}(\mathtt{sk}_1, c), \ldots, \mathsf{TFHE.ShareDec}(\mathtt{sk}_k, c)) \ .$$

- Ciphertext transformative indistinguishability: *We say that a* TFHE *scheme achieves* ciphertext transformative indistinguishability*, if for all message* m*, for any* $j \in [k]$*, there exists a* PPT *algorithm* Trans *such that*

$$\big(\mathsf{param}, \mathsf{TFHE.Trans}(c, \{\mathtt{sk}_i\}_{i\in[k]\setminus\{j\}})\big) \approx \big(\mathsf{param}, \mathsf{TFHE.Enc}(\mathtt{pk}, m)\big)$$

  where $\{(\mathtt{pk}_i, \mathtt{sk}_i)\}_{i\in[k]}$ *are all valid key pairs,* $\mathtt{pk} := \mathsf{TFHE.CombinePK}(\{\mathtt{pk}_i\}_{i\in[k]})$ *and* $\mathtt{sk} := \mathsf{TFHE.CombineSK}(\{\mathtt{sk}_i\}_{i\in[k]})$*.*

- Share-simulation indistinguishability: *We say* TFHE *scheme achieves* share-simulation indistinguishability *if there exists a* PPT *simulator* SimShareDec *such that for all valid key pairs* $\{(\mathtt{pk}_i, \mathtt{sk}_i)\}_{i\in[k]}$*, all subsets* $\mathcal{I} \subsetneq [k]$*, all message* m*, the following two distributions are computationally indistinguishable:*

$$\big(\mathsf{param}, c, \mathsf{SimShareDec}(c, m, \{\mu_i\}_{i\in\mathcal{I}})\big) \approx \big(\mathsf{param}, c, \{\mu_j\}_{j\in[k]\setminus\mathcal{I}}\big)$$

  where $\mathsf{param} \leftarrow \mathsf{TFHE.Setup}(1^\lambda)$*,* $c \leftarrow \mathsf{TFHE.Enc}(\mathtt{pk}, m)$ *and* $\mu_j \leftarrow \mathsf{TFHE.ShareDec}(\mathtt{sk}_j, c)$ *for* $j \in [k] \setminus \mathcal{I}$*.*

The above TFHE is adopted from Lopez-Alt *et al.* [LTV11, AJL+12], with the following modification: we use pk as the evaluation key. This modification allow us to evaluate the ciphertexts publicly. In [LTV11, AJL+12], only distinguished players (i.e., the ones with secret keys) after a joint protocol, can obtain the evaluation key. This feature of public evaluation of ciphertexts is critical for e-voting. We also provide instantiation in Supplementary material A.2. Our instantiation is based on the well-known FHE construction by Gentry, Sahai and Waters [GSW13].

4.2 Protocol description

In this section, we formally describe our TFHE-based construction for statement voting. The protocol is designed in the $\{\bar{\mathcal{G}}_{\mathrm{BB}}, \widehat{\mathcal{F}}_{\mathrm{CERT}}\}$-hybrid world and it consists of three phases: preparation, ballot casting, and tally. For the sake of notation simplicity, we omit the processes of filtering invalid messages on $\bar{\mathcal{G}}_{\mathrm{BB}}$. In practice, $\bar{\mathcal{G}}_{\mathrm{BB}}$ contains many messages with invalid signatures, and all those messages should be ignored.

4.2.1 Preparation phase

As depicted in Figure 5, in the preparation phase, each trustee T_j, first picks a randomness generates α_j and generates a partial public key using $(\overline{\mathtt{pk}}_j, \overline{\mathtt{sk}}_j) \leftarrow \mathsf{TFHE.Keygen}(\mathsf{param}; \alpha_j)$. It then generates an NIZK proof

$$\pi_j^{(1)} \leftarrow \mathsf{NIZK}_{\mathcal{R}_1}\ \{\ (\overline{\mathtt{pk}}_j), (\alpha_j, \overline{\mathtt{sk}}_j) : (\overline{\mathtt{pk}}_j, \overline{\mathtt{sk}}_j) = \mathsf{TFHE.Keygen}(\mathsf{param}; \alpha_j)\ \}$$

to show that this process is executed correctly; namely, it shows knowledge of $(\alpha_j, \overline{\mathtt{sk}}_j)$ w.r.t. to the generated partial public key $\overline{\mathtt{pk}}_j$. It then signs and posts $(\overline{\mathtt{pk}}_j, \pi_j^{(1)})$ to $\bar{\mathcal{G}}_{\mathrm{BB}}$.

Preparation:

Upon receiving $(\text{INITIALTRUSTEE}, \mathsf{sid})$ from the environment $\mathcal{Z}$, the trustee T_j, $j \in [k]$, operates as the follows:

- Generate $(\overline{\mathtt{pk}}_j, \overline{\mathtt{sk}}_j) \leftarrow \mathsf{TFHE.Keygen}(\mathsf{param}; \alpha_j)$ where α_j is the fresh randomness, and then compute

$$\pi_j^{(1)} \leftarrow \mathsf{NIZK}_{\mathcal{R}_1} \left\{ \ (\overline{\mathtt{pk}}_j), (\alpha_j, \overline{\mathtt{sk}}_j) : (\overline{\mathtt{pk}}_j, \overline{\mathtt{sk}}_j) = \mathsf{TFHE.Keygen}(\mathsf{param}; \alpha_j) \ \right\}$$

- Send $(\text{SIGN}, \mathsf{sid}, \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}))$ to $\widehat{\mathcal{F}}_{\text{CERT}}$ and receives $(\text{SIGNATURE}, \mathsf{sid}, \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), \sigma_j^{(1)})$ from $\widehat{\mathcal{F}}_{\text{CERT}}$, where $\mathsf{ssid} = (\mathsf{T}_j, \mathsf{ssid}')$ for some ssid'.
- Send $(\text{SUBMIT}, \mathsf{sid}, \langle \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), \sigma_j^{(1)} \rangle)$ to $\bar{\mathcal{G}}_{\text{BB}}$.

Figure 5: TFHE based statement voting scheme $\Pi_{\text{FHE-SV}}$ in the $\{\bar{\mathcal{G}}_{\text{BB}}, \widehat{\mathcal{F}}_{\text{CERT}}\}$-hybrid world (Part I).

4.2.2 Ballot casting phase

As depicted in Figure 6, in the ballot casting phase, each voter V_i, $i \in [n]$ first fetches the partial public keys $\{\overline{\mathtt{pk}}_j\}_{j\in[k]}$ from $\bar{\mathcal{G}}_{\text{BB}}$. After checking their corresponding NIZK proofs, the voter V_i combines them to the election public key $\mathtt{pk} := \mathsf{TFHE.CombinePK}(\{\overline{\mathtt{pk}}_j\}_{j=1}^k)$. V_i then encrypts his ballot (V_i, s_i) as $c_i = \mathsf{TFHE.Enc}(\mathtt{pk}, (\mathsf{V}_i, s_i))$. Here V_i is abused as the voter's PID and s_i is her statement. The voter then posts the ciphertext c_i on the $\bar{\mathcal{G}}_{\text{BB}}$ together with the corresponding NIZK proof showing that c_i is indeed generated by the voter V_i.

Ballot Casting:

Upon receiving $(\text{CAST}, \mathsf{sid}, s_i)$ from the environment $\mathcal{Z}$, the voter V_i, $i \in [n]$ operates as the follows:

- Send $(\text{READ}, \mathsf{sid})$ to $\bar{\mathcal{G}}_{\text{BB}}$, and obtain $(\text{READ}, \mathsf{sid}, \mathit{state})$ from $\bar{\mathcal{G}}_{\text{BB}}$. If $\left\{\langle \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), \sigma_j^{(1)} \rangle\right\}_{j\in[k]}$ is contained in state, then for $j \in [k]$, send $(\text{VERIFY}, \mathsf{sid}, \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), \sigma_j^{(1)})$ to $\widehat{\mathcal{F}}_{\text{CERT}}$, and receive $(\text{VERIFIED}, \mathsf{sid}, \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), b_j^{(1)})$ from $\widehat{\mathcal{F}}_{\text{CERT}}$; If $\prod_{j=1}^k b_j^{(1)} = 1$, check $\mathsf{NIZK}_{\mathcal{R}_1}.\mathsf{Verify}(\overline{\mathtt{pk}}_j, \pi_j^{(1)}) = 1$ for $j \in [k]$. If any of the checks is invalid, halt.
- Compute and store $\mathtt{pk} := \mathsf{TFHE.CombinePK}(\{\overline{\mathtt{pk}}_j\}_{j=1}^k)$.
- Encrypt $c_i \leftarrow \mathsf{TFHE.Enc}(\mathtt{pk}, (\mathsf{V}_i, s_i); \beta_i)$ where β_i is the fresh randomness, and then compute

$$\pi_i^{(2)} \leftarrow \mathsf{NIZK}_{\mathcal{R}_2} \left\{ \ (\overline{\mathtt{pk}}, c_i), (\mathsf{V}_i, s_i, \beta_i) : c_i = \mathsf{TFHE.Enc}(\mathtt{pk}, (\mathsf{V}_i, s_i); \beta_i) \ \right\}$$

- Send $(\text{SIGN}, \mathsf{sid}, \mathsf{ssid}, (c_i, \pi_i^{(2)})$ to $\widehat{\mathcal{F}}_{\text{CERT}}$, where $\mathsf{ssid} = (\mathsf{V}_i, \mathsf{ssid}')$ for some ssid', and receive $(\text{SIGNATURE}, \mathsf{sid}, \mathsf{ssid}, (c_i, \pi_i^{(2)}), \sigma_i^{(2)})$ from $\widehat{\mathcal{F}}_{\text{CERT}}$.
- Send $(\text{SUBMIT}, \mathsf{sid}, \langle \mathsf{ssid}, (c_i, \pi_i^{(2)}), \sigma_i^{(2)} \rangle)$ to $\bar{\mathcal{G}}_{\text{BB}}$.

Figure 6: TFHE based statement voting scheme $\Pi_{\text{FHE-SV}}$ in the $\{\bar{\mathcal{G}}_{\text{BB}}, \widehat{\mathcal{F}}_{\text{CERT}}\}$-hybrid world (Part II).

4.2.3 Tally phase

The tally phase is depicted in Figure 7. The trustee $\mathsf{T}_j \in \mathbb{T}$, $j \in [k]$ fetches the posted encrypted ballots $\{c_i\}_{i\in[n]}$ from $\bar{\mathcal{G}}_{\text{BB}}$. It checks the corresponding NIZK proofs and removes the invalid ones. Each of the trustees $\mathsf{T}_j \in \mathbb{T}$ then evaluates the TallyProcess circuit as $c := \mathsf{TFHE.Eval}(\mathtt{pk}, \mathsf{TallyProcess}, c_1, \ldots, c_n)$. After that, all the trustees jointly decrypt c to the final tally τ, attached with necessary NIZK proofs. Finally, all the voters $\mathsf{V}_i \in \mathbb{V}$ can read the tally result τ from $\bar{\mathcal{G}}_{\text{BB}}$.

Tally:
Upon receiving $(\textsc{Tally}, \mathsf{sid})$ from the environment $\mathcal{Z}$, the trustee T_j, where $j \in [k]$, operates as the follows:

- Send $(\textsc{Read}, \mathsf{sid})$ to $\bar{\mathcal{G}}_{\text{BB}}$, and obtain $(\textsc{Read}, \mathsf{sid}, \mathit{state})$ from $\bar{\mathcal{G}}_{\text{BB}}$. If $\left\{\langle \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), \sigma_j^{(1)}\rangle\right\}_{j\in[k]}$ is contained in state, then for $j \in [k]$, send $(\textsc{Verify}, \mathsf{sid}, \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), \sigma_j^{(1)})$ to $\widehat{\mathcal{F}}_{\textsc{Cert}}$, and receive $(\textsc{Verified}, \mathsf{sid}, \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), b_j^{(1)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$; If $\prod_{j=1}^{k} b_j^{(1)} = 1$, check $\mathsf{NIZK}_{\mathcal{R}_1}.\mathsf{Verify}(\overline{\mathtt{pk}}_j, \pi_j^{(1)}) = 1$ for $j \in [k]$. If any of the checks is invalid, halt.
- Compute $\mathtt{pk} \leftarrow \mathsf{TFHE.CombinePK}(\{\overline{\mathtt{pk}}_j\}_{j=1}^{k})$.
- For $i \in [n]$, if $\langle \mathsf{ssid}, (c_i, \pi_i^{(2)}), \sigma_i^{(2)}\rangle$ is contained in state, then send $(\textsc{Verify}, \mathsf{sid}, \mathsf{ssid}, (c_i, \pi_i^{(2)}), \sigma_i^{(2)})$ to $\widehat{\mathcal{F}}_{\textsc{Cert}}$, and receive $(\textsc{Verified}, \mathsf{sid}, \mathsf{ssid}, (c_i, \pi_i^{(2)}), b_i^{(2)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$; if $b_i^{(2)} = 1$, check $\mathsf{NIZK}_{\mathcal{R}_2}.\mathsf{Verify}((\overline{\mathtt{pk}}, c_i), \pi_i^{(2)}) = 1$. If any of the above checks is invalid, reset $c_i := \perp$.
- Compute $c := \mathsf{TFHE.Eval}(\mathtt{pk}, \mathsf{TallyProcess}, c_1, \ldots, c_n)$.
- Compute $\overline{\tau}_j \leftarrow \mathsf{TFHE.ShareDec}(\overline{\mathtt{sk}}_j, c)$ together with

$$\pi_j^{(3)} \leftarrow \mathsf{NIZK}_{\mathcal{R}_3}\left\{\begin{array}{l}(c, \overline{\tau}_j, \overline{\mathtt{pk}}_j), (\overline{\mathtt{sk}}_j, \alpha_j) : \\ (\overline{\mathtt{pk}}_j, \overline{\mathtt{sk}}_j) = \mathsf{TFHE.Keygen}(\mathsf{param}; \alpha_j) \ \wedge \ \overline{\tau}_j = \mathsf{TFHE.ShareDec}(\overline{\mathtt{sk}}_j, c)\end{array}\right\}$$

- Send $(\textsc{Sign}, \mathsf{sid}, \mathsf{ssid}, (\overline{\tau}_j, \pi_j^{(3)}))$ to $\widehat{\mathcal{F}}_{\textsc{Cert}}$ and receives $(\textsc{Signature}, \mathsf{sid}, \mathsf{ssid}, (\overline{\tau}_j, \pi_j^{(3)}), \sigma_j^{(3)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$, where $\mathsf{ssid} = (\mathsf{T}_j, \mathsf{ssid}')$ for some ssid'.
- Send $(\textsc{Submit}, \mathsf{sid}, \langle \mathsf{ssid}, (\overline{\tau}_j, \pi_j^{(3)}), \sigma_j^{(3)}\rangle)$ to $\bar{\mathcal{G}}_{\text{BB}}$.

Upon receiving $(\textsc{ReadResult}, \mathsf{sid})$ from the environment $\mathcal{Z}$, the voter V_i, $i \in [n]$ operates as the follows:

- Send $(\textsc{Read}, \mathsf{sid})$ to $\bar{\mathcal{G}}_{\text{BB}}$, and obtain $(\textsc{Read}, \mathsf{sid}, \mathit{state})$ from $\bar{\mathcal{G}}_{\text{BB}}$. If $\left\{\langle \mathsf{ssid}, (\overline{\tau}_j, \pi_j^{(3)}), \sigma_j^{(3)}\rangle\right\}_{j\in[k]}$ is contained in state, then for $j \in [k]$, send $(\textsc{Verify}, \mathsf{sid}, \mathsf{ssid}, (\overline{\tau}_j, \pi_j^{(3)}), \sigma_j^{(3)})$ to $\widehat{\mathcal{F}}_{\textsc{Cert}}$, and receive $(\textsc{Verified}, \mathsf{sid}, \mathsf{ssid}, (\overline{\tau}_j, \pi_j^{(3)}), b_j^{(3)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$; If $\prod_{j=1}^{k} b_j^{(3)} = 1$, check $\mathsf{NIZK}_{\mathcal{R}_3}.\mathsf{Verify}((c, \overline{\tau}_j, \overline{\mathtt{pk}}_j), \pi_j^{(3)}) = 1$ for $j \in [k]$. If any of the checks is invalid, halt.
- Compute $\tau \leftarrow \mathsf{TFHE.ShareCombine}(\{\overline{\tau}_j\}_{j=1}^{k})$.
- Return $(\textsc{ReadResultReturn}, \mathsf{sid}, \tau)$ to the environment $\mathcal{Z}$.

Figure 7: TFHE based statement voting scheme $\Pi_{\text{FHE-SV}}$ in the $\{\bar{\mathcal{G}}_{\text{BB}}, \widehat{\mathcal{F}}_{\textsc{Cert}}\}$-hybrid world (Part III).

4.3 Security

Now we show our construction can realize statement voting functionality with strong privacy. That is, the leakage L is minimal (i.e., the tally result). We note that a simplified version of our statement voting functionality with strong privacy can be found in Figure 16 in Supplementary material A.1. More formally, we have the following the theorem.

Theorem 4.2. *Protocol $\Pi_{\text{FHE-SV}}$ described in Figure 5, Figure 6 and Figure 7 UC-realizes $\mathcal{F}_{\text{SV}}$ in the $\{\bar{\mathcal{G}}_{\text{BB}}, \widehat{\mathcal{F}}_{\text{CERT}}\}$-hybrid world against static corruption.*

The security proof can be found in Supplementary material A.1.

5 MPC based construction

The presented TFHE-based construction is used to illustrate the core idea. In practice, FHE schemes may still be hundreds times slower than the state-of-the-art MPC protocols, especially when NIZK proofs are involved. In fact, the construction described in Section 4 can be viewed as a special case of an MPC protocol in the server-client setting, where the trustees $\mathbb{T}$ form the MPC players. The voters submit their statements, and the trustees then jointly evaluate the TallyProcess circuit.

In the following, we show how to eliminate the needs of a FHE using so-called publicly auditable MPC. Note that the main difference between a conventional MPC protocol and an e-voting system is that the e-voting system should still ensure the integrity of an election process even when all the trustees are corrupted. Whereas, a conventional MPC protocol does not ensure computation correctness when all the players are corrupted.

Baum et al. [BDO14], proposed a publicly auditable MPC in the $\bar{\mathcal{G}}_{\text{BB}}$-hybrid model. Their scheme is based on SPDZ [DPSZ12, DKL+13], but it can be extended to support most other later SPDZ variants along this line of research. The general idea to make an MPC system publicly auditable is to attach each shared value with a (Pedersen) commitment so that the same linear/opening operations of the shared value can be carried out on the corresponding commitments. Those commitments are posted on the $\bar{\mathcal{G}}_{\text{BB}}$, so that everyone can perform the same MPC online phase circuit on the commitments and check if the opening of the resulting commitment is consistent with the MPC output. Hereby, due to space limitation, we will omit the MPC construction and refer interested readers to [BDO14] for details. In the following, we first give another building block.

5.1 Threshold PKE

We would like to adopt a key-homomorphic threshold PKE scheme TE. It consists of a tuple of algorithms: (Setup, Keygen, Enc, Dec, CombinePK, CombineSK, ShareDec, ShareCombine) as follows.

- $\mathsf{param} \leftarrow \mathsf{Setup}(1^\lambda)$. The algorithm Setup takes input as the security parameter λ, and outputs public parameters param. All the other algorithms implicitly take param as input.
- $(\mathtt{pk}, \mathtt{sk}) \leftarrow \mathsf{Keygen}(\mathsf{param})$. The algorithm Keygen takes input as the public parameter param, and outputs a public key pk, a secret key sk.
- $c \leftarrow \mathsf{Enc}(\mathtt{pk}, m)$. The algorithm Enc takes input as the public key pk and the message m, and outputs the ciphertext c.
- $c' \leftarrow \mathsf{ReRand}(\mathtt{pk}, c)$. The algorithm ReRand takes input as the public key pk and a ciphertext c, and outputs a re-randomized ciphertext c'.
- $m \leftarrow \mathsf{Dec}(\mathtt{sk}, c)$. The algorithm Dec takes input as the secret key sk and a ciphertext c, and outputs the decrypted plaintext m.

- $\mathtt{pk} := \mathsf{CombinePK}(\mathtt{pk}_1, \ldots, \mathtt{pk}_k)$. The algorithm CombinePK takes input as a set of public keys $(\mathtt{pk}_1, \ldots, \mathtt{pk}_k)$, and outputs a combined public key $\mathtt{pk}$.
- $\mathtt{sk} \leftarrow \mathsf{CombineSK}(\mathtt{sk}_1, \ldots, \mathtt{sk}_k)$. The algorithm CombineSK takes input as a set of secret key $(\mathtt{sk}_1, \ldots, \mathtt{sk}_k)$, and outputs combined secret key $\mathtt{sk}$.
- $\mu_i \leftarrow \mathsf{ShareDec}(\mathtt{sk}_i, c)$. The algorithm ShareDec takes input as the secret key $\mathtt{sk}_i$ and a ciphertext c, and outputs a decryption share μ_i.
- $m \leftarrow \mathsf{ShareCombine}(c, \mu_1, \ldots, \mu_k)$. The algorithm ShareCombine takes input as a ciphertext c and k decryption shares $(\mu_1, \ldots, \mu_k)$, and outputs a plaintext m.
- $c' \leftarrow \mathsf{Trans}(c, \{\mathtt{sk}_i\}_{i\in[k]\setminus\{j\}})$. The algorithm Trans takes input as a ciphertext $c \leftarrow \mathsf{TE.Enc}(\mathtt{pk}_j, m)$ and a set of secret keys $\{\mathtt{sk}_i\}_{i\in[k]\setminus\{j\}}$, and outputs a ciphertext c'.
- $\{\mu_j\}_{j\in[k]\setminus\mathcal{I}} \leftarrow \mathsf{SimShareDec}(c, m, \{\mu_i\}_{i\in\mathcal{I}})$. The algorithm SimShareDec takes as input a ciphertext c, a plaintext m, and a set of decryption shares $\{\mu_i\}_{i\in\mathcal{I}}$ and outputs a set of decryption shares $\{\mu_j\}_{j\in[k]\setminus\mathcal{I}}$. Here $\mathcal{I} \subsetneq [k]$.

Definition 5.1. *We say* $\mathsf{TE} = \{\mathsf{Setup}, \mathsf{Keygen}, \mathsf{Enc}, \mathsf{Dec}, \mathsf{CombinePK}, \mathsf{CombineSK}, \mathsf{ShareDec}, \mathsf{ShareCombine}\}$ *is a* secure *key-homomorphic threshold public key encryption if the following properties hold:*

Key combination correctness: *If* $\{(\mathtt{pk}_i, \mathtt{sk}_i)\}_{i\in[k]}$ *are all valid key pairs,* $\mathtt{pk} := \mathsf{TE.CombinePK}(\{\mathtt{pk}_i\}_{i\in[k]})$ *and* $\mathtt{sk} := \mathsf{TE.CombineSK}(\{\mathtt{sk}_i\}_{i\in[k]})$*, then* $(\mathtt{pk}, \mathtt{sk})$ *is also a valid key pair.*

For all ciphertext $c \in \mathcal{C}_{\mathtt{pk}}$*, where* $\mathcal{C}_{\mathtt{pk}}$ *is the ciphertext-space defined by* $\mathtt{pk}$*, we have*

$$\mathsf{TE.Dec}(\mathtt{sk}, c) = \mathsf{TE.ShareCombine}(c, \mathsf{TE.ShareDec}(\mathtt{sk}_1, c), \ldots, \mathsf{TE.ShareDec}(\mathtt{sk}_k, c)) \ .$$

Ciphertext transformative indistinguishability: *There exists a* PPT *algorithm* Trans *such that if* $\{(\mathtt{pk}_i, \mathtt{sk}_i)\}_{i\in[k]}$ *are all valid key pairs,*

$\mathtt{pk} := \mathsf{TE.CombinePK}(\{\mathtt{pk}_i\}_{i\in[k]})$ *and* $\mathtt{sk} := \mathsf{TE.CombineSK}(\{\mathtt{sk}_i\}_{i\in[k]})$*, then for all message* m*, for any* $j \in [k]$*, the following holds.*

$$\big(\mathsf{param}, \mathsf{TE.Trans}(c, \{\mathtt{sk}_i\}_{i\in[k]\setminus\{j\}})\big) \ \approx \ \big(\mathsf{param}, \mathsf{TE.Enc}(\mathtt{pk}, m)\big)$$

IND-CPA security: *We say that a* TE *scheme achieves* indistinguishability under plaintext attacks (IND-CPA) *if for any* PPT *adversary* $\mathcal{A}$ *the following advantage* AdvCPA *is negligible.*

$\underline{\textit{EXPERIMENT}^{\mathsf{CPA}}(1^\lambda)}$

1. *Run* $\mathsf{param} \leftarrow \mathsf{TE.Setup}(1^\lambda)$.
2. *Run* $(\mathtt{pk}, \mathtt{sk}) \leftarrow \mathsf{TE.Keygen}(\mathsf{param})$*;*
4. $\mathcal{A}(\mathtt{pk})$ *outputs* m_0, m_1 *of equal length;*
5. *Pick* $b \leftarrow \{0, 1\}$*; Run* $c \leftarrow \mathsf{TE.Enc}(\mathtt{pk}, m_b)$*;*
6. $\mathcal{A}(c)$ *outputs* b^**; It returns* 1 *if* $b = b^*$*; else, returns* 0.

We define the advantage of $\mathcal{A}$ *as*

$$\mathsf{AdvCPA}_{\mathcal{A}}(1^\lambda) = \left|\Pr[\textit{EXPERIMENT}^{\mathsf{CPA}}(1^\lambda) = 1] - \frac{1}{2}\right| \ .$$

Share-simulation indistinguishability: *We say* TE *scheme achieves* share-simulation indistinguishability *if there exists a* PPT *simulator* SimShareDec *such that for all valid key pairs* $\{(\mathtt{pk}_i, \mathtt{sk}_i)\}_{i\in[k]}$*, all subsets* $\mathcal{I} \subsetneq [k]$*, all message* m*, the following two distributions are computationally indistinguishable:*

$$\left(\mathsf{param}, c, \mathsf{SimShareDec}(c, m, \{\mu_i\}_{i\in\mathcal{I}})\right) \approx \left(\mathsf{param}, c, \{\mu_j\}_{j\in[k]\setminus\mathcal{I}}\right)$$

where $\mathsf{param} \leftarrow \mathsf{TE.Setup}(1^\lambda)$*,* $c \leftarrow \mathsf{TE.Enc}(\mathtt{pk}, m)$ *and* $\mu_j \leftarrow \mathsf{TE.ShareDec}(\mathtt{sk}_j, c)$ *for* $j \in [k] \setminus \mathcal{I}$*.*

5.2 Protocol description

In this section, we formally describe our MPC-based construction for statement voting. we assume there exists an MPC protocol Π_{MPC} that UC-realize $\mathcal{F}_{\text{MPC}}^{\mathcal{C}}$, where $\mathcal{F}_{\text{MPC}}^{\mathcal{C}}$ is the MPC functionality (as described in Fig. 1 of [BDO14]) and $\mathcal{C}$ is the statement voting circuit depicted in Fig. 8, below. $\mathcal{C}$ takes public input as each trustee T_i's partial public key $\overline{\mathtt{pk}}_i$, and a set of encrypted ballots $\{c_j \leftarrow \mathsf{TE.Enc}(\mathtt{pk}, (\mathsf{V}_i, s_i))\}_{j\in[n]}$, where s_i is the voter V_i's statement. Meanwhile, $\mathcal{C}$ also takes private inputs as a random coin α_j from each trustee T_j, $j \in [k]$. $\mathcal{C}$ first uses α_j to generate $(\hat{\mathtt{pk}}_j, \hat{\mathtt{sk}}_j) \leftarrow \mathsf{TE.Keygen}(\mathsf{param}; \alpha_j)$; it then checks if $\hat{\mathtt{pk}}_j = \overline{\mathtt{pk}}_j$, $j \in [k]$. If verified, $\mathcal{C}$ uses $\{\hat{\mathtt{sk}}_j\}_{j\in[k]}$ to decrypt the ciphertexts $\{c_i\}_{i\in[n]}$ to obtains the ballots $\{B_i\}_{i\in[n]}$. It then computes and outputs the tally $\tau \leftarrow \mathsf{TallyProcess}(B_1, \ldots, B_n)$.

Public input: $\{\overline{\mathtt{pk}}_j\}_{j\in[k]}$ and $\{c_i\}_{i\in[n]}$
Private input: $\{\alpha_j\}_{j\in[k]}$
Public output: τ

- For $j \in [k]$, generate $(\hat{\mathtt{pk}}_j, \hat{\mathtt{sk}}_j) \leftarrow \mathsf{TE.Keygen}(\mathsf{param}; \alpha_j)$.
- Check if $\hat{\mathtt{pk}}_j = \overline{\mathtt{pk}}_j$, $j \in [k]$; otherwise, return $\perp$.
- Set $\mathtt{sk} := \mathsf{TE.CombineSK}(\hat{\mathtt{sk}}_1, \ldots, \hat{\mathtt{sk}}_k)$.
- For $i \in [n]$, compute $B_i \leftarrow \mathsf{TE.Dec}(\mathtt{sk}, c_i)$.
- Return $\tau \leftarrow \mathsf{TallyProcess}(B_1, \ldots, B_n)$.

Figure 8: Statement voting circuit $\mathcal{C}$

The protocol is designed in the $\{\bar{\mathcal{G}}_{\text{BB}}, \widehat{\mathcal{F}}_{\text{CERT}}, \mathcal{F}_{\text{MPC}}^{\mathcal{C}}\}$-hybrid world and it consists of three phases: preparation, ballot casting, and tally. Again, for the sake of notation simplicity, we omit the processes of filtering invalid messages on $\bar{\mathcal{G}}_{\text{BB}}$. In practice, $\bar{\mathcal{G}}_{\text{BB}}$ contains many messages with invalid signatures, and all those messages should be ignored. We assume all the parties implicitly have a common input $\mathsf{param} \leftarrow \mathsf{Setup}(1^\lambda)$.

5.2.1 Preparation phase

As depicted in Fig. 9, the preparation phase is the same as the TFHE based construction, except it uses TE instead.

5.2.2 Ballot casting phase

As depicted in Figure 10, the ballot casting phase is also the same as the TFHE scheme, except TE is used instead.

5.2.3 Tally phase

The tally phase is depicted in Figure 11. The trustees $\mathsf{T}_j \in \mathbb{T}$ fetches all the posted encrypted ballots on the $\bar{\mathcal{G}}_{\text{BB}}$, denoted as $\{c_i\}_{i\in[n]}$. It also checks their corresponding NIZK proofs. After that, each of the trustees

Preparation:

Upon receiving $(\textsc{InitialTrustee}, \mathsf{sid})$ from the environment $\mathcal{Z}$, the trustee T_j, $j \in [k]$, operates as the follows:

- Generate $(\overline{\mathtt{pk}}_j, \overline{\mathtt{sk}}_j) \leftarrow \mathsf{TE.Keygen}(\mathsf{param}; \alpha_j)$ where α_j is the fresh randomness, and then compute

$$\pi_j^{(1)} \leftarrow \mathsf{NIZK}_{\mathcal{R}_{11}} \left\{ \ (\overline{\mathtt{pk}}_j), (\alpha_j, \overline{\mathtt{sk}}_j) : (\overline{\mathtt{pk}}_j, \overline{\mathtt{sk}}_j) = \mathsf{TE.Keygen}(\mathsf{param}; \alpha_j) \ \right\}$$

- Send $(\textsc{Sign}, \mathsf{sid}, \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}))$ to $\widehat{\mathcal{F}}_{\textsc{Cert}}$ and receives $(\textsc{Signature}, \mathsf{sid}, \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), \sigma_j^{(1)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$, where $\mathsf{ssid} = (\mathsf{T}_j, \mathsf{ssid}')$ for some ssid'.
- Send $(\textsc{Submit}, \mathsf{sid}, \langle \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), \sigma_j^{(1)} \rangle)$ to $\bar{\mathcal{G}}_{\text{BB}}$.

Figure 9: MPC based statement voting scheme $\Pi_{\text{MPC-SV}}$ in the $\{\bar{\mathcal{G}}_{\text{BB}}, \widehat{\mathcal{F}}_{\textsc{Cert}}, \mathcal{F}_{\text{MPC}}^{\mathcal{C}}\}$-hybrid world (Part I).

Ballot Casting:

Upon receiving $(\textsc{Cast}, \mathsf{sid}, s_i)$ from the environment $\mathcal{Z}$, the voter V_i, $i \in [n]$ operates as the follows:

- Send $(\textsc{Read}, \mathsf{sid})$ to $\bar{\mathcal{G}}_{\text{BB}}$, and obtain $(\textsc{Read}, \mathsf{sid}, \mathit{state})$ from $\bar{\mathcal{G}}_{\text{BB}}$. If $\left\{ \langle \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), \sigma_j^{(1)} \rangle \right\}_{j \in [k]}$ is contained in state, then for $j \in [k]$, send $(\textsc{Verify}, \mathsf{sid}, \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), \sigma_j^{(1)})$ to $\widehat{\mathcal{F}}_{\textsc{Cert}}$, and receive $(\textsc{Verified}, \mathsf{sid}, \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), b_j^{(1)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$; If $\prod_{j=1}^{k} b_j^{(1)} = 1$, check $\mathsf{NIZK}_{\mathcal{R}_1}.\mathsf{Verify}(\overline{\mathtt{pk}}_j, \pi_j^{(1)}) = 1$ for $j \in [k]$. If any of the checks is invalid, halt.
- Compute and store $\mathtt{pk} := \mathsf{TE.CombinePK}(\{\overline{\mathtt{pk}}_j\}_{j=1}^{k})$.
- Encrypt $c_i \leftarrow \mathsf{TE.Enc}(\mathtt{pk}, (\mathsf{V}_i, s_i); \beta_i)$ where β_i is the fresh randomness, and then compute

$$\pi_i^{(2)} \leftarrow \mathsf{NIZK}_{\mathcal{R}_{12}} \left\{ \ (\overline{\mathtt{pk}}, c_i), (\mathsf{V}_i, s_i, \beta_i) : c_i = \mathsf{TE.Enc}(\mathtt{pk}, (\mathsf{V}_i, s_i); \beta_i) \ \right\}$$

- Send $(\textsc{Sign}, \mathsf{sid}, \mathsf{ssid}, (c_i, \pi_i^{(2)})$ to $\widehat{\mathcal{F}}_{\textsc{Cert}}$, where $\mathsf{ssid} = (\mathsf{V}_i, \mathsf{ssid}')$ for some ssid', and receive $(\textsc{Signature}, \mathsf{sid}, \mathsf{ssid}, (c_i, \pi_i^{(2)}), \sigma_i^{(2)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$.
- Send $(\textsc{Submit}, \mathsf{sid}, \langle \mathsf{ssid}, (c_i, \pi_i^{(2)}), \sigma_i^{(2)} \rangle)$ to $\bar{\mathcal{G}}_{\text{BB}}$.

Figure 10: MPC based statement voting scheme $\Pi_{\text{MPC-SV}}$ in the $\{\bar{\mathcal{G}}_{\text{BB}}, \widehat{\mathcal{F}}_{\textsc{Cert}}, \mathcal{F}_{\text{MPC}}^{\mathcal{C}}\}$-hybrid world (Part II).

Tally:
Upon receiving $(\textsc{Tally}, \mathsf{sid})$ from the environment $\mathcal{Z}$, the trustee T_j, where $j \in [k]$, operates as the follows:

- Send $(\textsc{Read}, \mathsf{sid})$ to $\bar{\mathcal{G}}_{\text{BB}}$, and obtain $(\textsc{Read}, \mathsf{sid}, \mathit{state})$ from $\bar{\mathcal{G}}_{\text{BB}}$. If $\left\{\langle \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), \sigma_j^{(1)}\rangle\right\}_{j\in[k]}$ is contained in state, then for $j \in [k]$, send $(\textsc{Verify}, \mathsf{sid}, \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), \sigma_j^{(1)})$ to $\widehat{\mathcal{F}}_{\textsc{Cert}}$, and receive $(\textsc{Verified}, \mathsf{sid}, \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), b_j^{(1)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$; If $\prod_{j=1}^{k} b_j^{(1)} = 1$, check $\mathsf{NIZK}_{\mathcal{R}_1}.\mathsf{Verify}(\overline{\mathtt{pk}}_j, \pi_j^{(1)}) = 1$ for $j \in [k]$. If any of the checks is invalid, halt.
- For $i \in [n]$, if $\langle \mathsf{ssid}, (c_i, \pi_i^{(2)}), \sigma_i^{(2)}\rangle$ is contained in state, then send $(\textsc{Verify}, \mathsf{sid}, \mathsf{ssid}, (c_i, \pi_i^{(2)}), \sigma_i^{(2)})$ to $\widehat{\mathcal{F}}_{\textsc{Cert}}$, and receive $(\textsc{Verified}, \mathsf{sid}, \mathsf{ssid}, (c_i, \pi_i^{(2)}), b_i^{(2)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$; if $b_i^{(2)} = 1$, check $\mathsf{NIZK}_{\mathcal{R}_2}.\mathsf{Verify}((\overline{\mathtt{pk}}, c_i), \pi_i^{(2)}) = 1$. If any of the above checks is invalid, reset $c_i := \perp$.
- Send $(\textsc{Input}, \mathsf{sid}, \alpha_j, \{\mathtt{pk}_\ell\}_{\ell\in[k]}, \{c_i\}_{i\in[n]})$ to $\mathcal{F}_{\text{MPC}}^{\mathcal{C}}$, and obtain τ.
- Send $(\textsc{Sign}, \mathsf{sid}, \mathsf{ssid}, \tau)$ to $\widehat{\mathcal{F}}_{\textsc{Cert}}$ and receives $(\textsc{Signature}, \mathsf{sid}, \mathsf{ssid}, \tau), \sigma_j^{(3)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$, where $\mathsf{ssid} = (\mathsf{T}_j, \mathsf{ssid}')$ for some ssid'.
- Send $(\textsc{Submit}, \mathsf{sid}, \langle \mathsf{ssid}, \tau, \sigma_j^{(3)}\rangle)$ to $\bar{\mathcal{G}}_{\text{BB}}$.

Upon receiving $(\textsc{ReadResult}, \mathsf{sid})$ from the environment $\mathcal{Z}$, the voter V_i, $i \in [n]$ operates as the follows:

- Send $(\textsc{Read}, \mathsf{sid})$ to $\bar{\mathcal{G}}_{\text{BB}}$, and obtain $(\textsc{Read}, \mathsf{sid}, \mathit{state})$ from $\bar{\mathcal{G}}_{\text{BB}}$.
- Fetch τ from state and return $(\textsc{ReadResultReturn}, \mathsf{sid}, \tau)$ to the environment $\mathcal{Z}$.

Figure 11: MPC based statement voting scheme $\Pi_{\text{MPC-SV}}$ in the $\{\bar{\mathcal{G}}_{\text{BB}}, \widehat{\mathcal{F}}_{\textsc{Cert}}, \mathcal{F}_{\text{MPC}}^{\mathcal{C}}\}$-hybrid world (Part III).

sends $(\textsc{Input}, \mathsf{sid}, \alpha_j, \{\mathtt{pk}_\ell\}_{\ell\in[k]}, \{c_i\}_{i\in[n]})$ to $\mathcal{F}_{\text{MPC}}^{\mathcal{C}}$. After the computation, the trustee $\mathsf{T}_j \in \mathbb{T}$ receives the tally result τ and posts it to the $\bar{\mathcal{G}}_{\text{BB}}$.

5.3 Security

Similar to the TFHE based solution, our construction here can also achieve the statement voting functionality with strong privacy. We have the following the theorem.

Theorem 5.2. *Protocol $\Pi_{\text{MPC-SV}}$ described in Figure 9, Figure 10 and Figure 11 UC-realizes $\mathcal{F}_{\text{SV}}$ in the $\{\bar{\mathcal{G}}_{\text{BB}}, \widehat{\mathcal{F}}_{\textsc{Cert}}, \mathcal{F}_{\text{MPC}}^{\mathcal{C}}\}$-hybrid world against static corruption.*

The security proof can be found in Supplementary material B.1.

6 Mix-net based construction

In this section, we will construct a much more efficient statement voting scheme based on mix-net. The privacy that this construction achieves is relatively weaker. However, we emphasize that this level of privacy has been widely accepted and is consistent with all the existing paper-based voting systems.

The intuition is as follows. At the beginning of scheme, each voter V_i, $i \in [n]$, is assigned with a temporal random ID, denoted as ID_i. Let $\mathcal{I} := \{\mathsf{ID}_1, \ldots, \mathsf{ID}_n\}$ be the set of all the voter's random IDs. The voter's statement takes input as a subset of $\mathcal{I}$, denoted as $\mathcal{D}$, and uses $\mathsf{ID} \in \mathcal{D}$ as references to point to those voters' ballots. For instance, the statement could be "*If both voter ID_x and voter ID_y vote for 'Yes', then my vote is 'Yes'; otherwise, my vote is 'No'.*" The ballot of a voter V_i is in forms of $B_i := (\mathsf{ID}_i, \mathsf{statement}_i)$, where ID_i is the voter's temporal ID, and $\mathsf{statement}_i$ is the voter's statement.

To ensure privacy, the voters cannot post their temporal IDs publicly on the bulletin board $\bar{\mathcal{G}}_{\text{BB}}$; however, the voters should be allowed to freely refer to any voter's ID. To address this challenge, we introduce the following technique. At the beginning of the protocol execution, each voter picks a random ID and posts the encryption of the ID on the $\bar{\mathcal{G}}_{\text{BB}}$. If a voter wants to refer to another voter in the statement, he/she simply copies the ciphertext of the corresponding voter's ID.

We emphasize that in practice the mix-net servers can be different from talliers (a.k.a. decrypters). As such, they could have different threshold requirements. For notation simplicity, we combine both roles to the same set of parties, trustees, in the protocol description.

6.1 Threshold re-randomizable encryption

A threshold re-randomizable encryption scheme TRE consists of a tuple of algorithms: (Setup, Keygen, Enc, Dec, CombinePK, CombineSK, ShareDec, ShareCombine, ReRand) as follows.

- $\mathsf{param} \leftarrow \mathsf{Setup}(1^\lambda)$. The algorithm Setup takes input as the security parameter λ, and outputs public parameters param. All the other algorithms implicitly take param as input.
- $(\mathtt{pk}, \mathtt{sk}) \leftarrow \mathsf{Keygen}(\mathsf{param})$. The algorithm Keygen takes input as the public parameter param, and outputs a public key pk, a secret key sk.
- $c \leftarrow \mathsf{Enc}(\mathtt{pk}, m)$. The algorithm Enc takes input as the public key pk and the message m, and outputs the ciphertext c.
- $c' \leftarrow \mathsf{ReRand}(\mathtt{pk}, c)$. The algorithm ReRand takes input as the public key pk and a ciphertext c, and outputs a re-randomized ciphertext c'.
- $m \leftarrow \mathsf{Dec}(\mathtt{sk}, c)$. The algorithm Dec takes input as the secret key sk and a ciphertext c, and outputs the decrypted plaintext m.
- $\mathtt{pk} := \mathsf{CombinePK}(\mathtt{pk}_1, \ldots, \mathtt{pk}_k)$. The algorithm CombinePK takes input as a set of public keys $(\mathtt{pk}_1, \ldots, \mathtt{pk}_k)$, and outputs a combined public key pk.
- $\mathtt{sk} \leftarrow \mathsf{CombineSK}(\mathtt{sk}_1, \ldots, \mathtt{sk}_k)$. The algorithm CombineSK takes input as a set of secret key $(\mathtt{sk}_1, \ldots, \mathtt{sk}_k)$, and outputs combined secret key sk.
- $\mu_i \leftarrow \mathsf{ShareDec}(\mathtt{sk}_i, c)$. The algorithm ShareDec takes input as the secret key $\mathtt{sk}_i$ and a ciphertext c, and outputs a decryption share μ_i.
- $m \leftarrow \mathsf{ShareCombine}(c, \mu_1, \ldots, \mu_k)$. The algorithm ShareCombine takes input as a ciphertext c and k decryption shares $(\mu_1, \ldots, \mu_k)$, and outputs a plaintext m.
- $c' \leftarrow \mathsf{Trans}(c, \{\mathtt{sk}_i\}_{i\in[k]\setminus\{j\}})$. The algorithm Trans takes input as a ciphertext $c \leftarrow \mathsf{TRE.Enc}(\mathtt{pk}_j, m)$ and a set of secret keys $\{\mathtt{sk}_i\}_{i\in[k]\setminus\{j\}}$, and outputs a ciphertext c'.
- $\{\mu_j\}_{j\in[k]\setminus\mathcal{I}} \leftarrow \mathsf{SimShareDec}(c, m, \{\mu_i\}_{i\in\mathcal{I}})$. The algorithm SimShareDec takes as input a ciphertext c, a plaintext m, and a set of decryption shares $\{\mu_i\}_{i\in\mathcal{I}}$ and outputs a set of decryption shares $\{\mu_j\}_{j\in[k]\setminus\mathcal{I}}$. Here $\mathcal{I} \subsetneq [k]$.

Definition 6.1. *We say* TRE $=$ {Setup, Keygen, Enc, Dec, CombinePK, CombineSK, ShareDec, ShareCombine, ReRand} *is a* secure *threshold re-randomizable public key encryption if the following properties hold:*

Key combination correctness: *If* $\{(\mathtt{pk}_i, \mathtt{sk}_i)\}_{i\in[k]}$ *are all valid key pairs,* $\mathtt{pk} := \mathsf{TRE.CombinePK}(\{\mathtt{pk}_i\}_{i\in[k]})$ *and* $\mathtt{sk} := \mathsf{TRE.CombineSK}(\{\mathtt{sk}_i\}_{i\in[k]})$*, then* $(\mathtt{pk}, \mathtt{sk})$ *is also a valid key pair.*

For all ciphertext $c \in \mathcal{C}_{\mathtt{pk}}$*, where* $\mathcal{C}_{\mathtt{pk}}$ *is the ciphertext-space defined by* pk*, we have*

$$\mathsf{TRE.Dec}(\mathtt{sk}, c) = \mathsf{TRE.ShareCombine}(c, \mathsf{TRE.ShareDec}(\mathtt{sk}_1, c), \ldots, \mathsf{TRE.ShareDec}(\mathtt{sk}_k, c)) \ .$$

Ciphertext transformative indistinguishability: *There exists a* PPT *algorithm* Trans *such that if* $\{(\mathtt{pk}_i, \mathtt{sk}_i)\}_{i\in[k]}$ *are all valid key pairs,*

$\mathtt{pk} := \mathsf{TRE.CombinePK}(\{\mathtt{pk}_i\}_{i\in[k]})$ *and* $\mathtt{sk} := \mathsf{TRE.CombineSK}(\{\mathtt{sk}_i\}_{i\in[k]})$*, then for all message* m*, for any* $j \in [k]$*, the following holds.*

$$\big(\mathsf{param}, \mathsf{TRE.Trans}(c, \{\mathtt{sk}_i\}_{i\in[k]\setminus\{j\}})\big) \approx \big(\mathsf{param}, \mathsf{TRE.Enc}(\mathtt{pk}, m)\big)$$

Unlinkability: *We say a* TRE *scheme is* unlinkable *if for any* PPT *adversary* $\mathcal{A}$ *the following advantage* AdvUnlink *is negligible.*

$\underline{\textit{EXPERIMENT}^{\mathsf{Unlink}}(1^\lambda)}$

1. $\mathcal{A}$ *outputs a set* $\mathcal{I} \subset \{1, \ldots, k\}$ *of up to* $k-1$ *corrupted indices.*
2. *For* $i = [n]$*, run* $(\overline{\mathtt{pk}}_i, \overline{\mathtt{sk}}_i) \leftarrow \mathsf{TRE.Keygen}(1^\lambda; \omega_i)$*;*
3. $\mathcal{A}(\{\mathtt{pk}_j\}_{j\in[k]\setminus\mathcal{I}})$ *outputs* c_0, c_1*;*
4. $b \leftarrow \{0,1\}$*;* $c' \leftarrow \mathsf{TRE.ReRand}(\mathtt{pk}, c_b; \omega)$*;*
5. $\mathcal{A}(c')$ *outputs* b^**; It returns* 1 *if* $b = b^*$*; else, returns* 0*.*

We define the advantage of $\mathcal{A}$ *as*

$$\mathsf{AdvUnlink}_{\mathcal{A}}(1^\lambda) = \left|\Pr[\textit{EXPERIMENT}^{\mathsf{Unlink}}(1^\lambda) = 1] - \frac{1}{2}\right| .$$

Share-simulation indistinguishability: *We say* TRE *scheme achieves* share-simulation indistinguishability *if there exists a* PPT *simulator* SimShareDec *such that for all valid key pairs* $\{(\mathtt{pk}_i, \mathtt{sk}_i)\}_{i\in[k]}$*, all subsets* $\mathcal{I} \subsetneq [k]$*, all message* m*, the following two distributions are computationally indistinguishable:*

$$\big(\mathsf{param}, c, \mathsf{SimShareDec}(c, m, \{\mu_i\}_{i\in\mathcal{I}})\big) \approx \big(\mathsf{param}, c, \{\mu_j\}_{j\in[k]\setminus\mathcal{I}}\big)$$

where $\mathsf{param} \leftarrow \mathsf{TRE.Setup}(1^\lambda)$*,* $c \leftarrow \mathsf{TRE.Enc}(\mathtt{pk}, m)$ *and* $\mu_j \leftarrow \mathsf{TRE.ShareDec}(\mathtt{sk}_j, c)$ *for* $j \in [k]\setminus\mathcal{I}$*.*

6.2 Protocol description

In this section, we formally describe our mix-net based construction for statement voting. The protocol is designed in the $\{\bar{\mathcal{G}}_{\mathrm{BB}}, \widehat{\mathcal{F}}_{\mathrm{CERT}}\}$-hybrid world and it consists of three phases: preparation, ballot casting, and tally.

We will use threshold re-randomizable encryption (TRE) as building block. A threshold re-randomizable encryption scheme TRE consists of a tuple of algorithms: (Setup, Keygen, Enc, Dec, CombinePK, CombineSK, ShareDec, ShareCombine, ReRand). TRE is related to TFHE except that, Eval algorithm is disabled in TRE; instead, we will allow the ciphertexts can be re-randomized via ReRand algorithm. More details can be found in Supporting material 6.1.

6.2.1 Preparation phase

As depicted in Fig. 12, in the preparation phase, each trustee T_j, $j \in [k]$ first picks a randomness generates α_j and generates a partial public key using $(\overline{\mathtt{pk}}_j, \overline{\mathtt{sk}}_j) \leftarrow \mathsf{TRE.Keygen}(\mathsf{param}; \alpha_j)$. It then generates an NIZK proof

$$\pi_j^{(1)} \leftarrow \mathsf{NIZK}_{\mathcal{R}_4}\left\{ \ (\overline{\mathtt{pk}}_j), (\alpha_j, \overline{\mathtt{sk}}_j) : (\overline{\mathtt{pk}}_j, \overline{\mathtt{sk}}_j) = \mathsf{TRE.Keygen}(\mathsf{param}; \alpha_j) \ \right\}$$

to show that this process is executed correctly; namely, it shows knowledge of $(\alpha_j, \overline{\mathtt{sk}}_j)$ w.r.t. to the generated partial public key $\overline{\mathtt{pk}}_j$. It then signs and posts $(\overline{\mathtt{pk}}_j, \pi_j^{(1)})$ to $\bar{\mathcal{G}}_{\mathrm{BB}}$.

Preparation:
Upon receiving $(\textsc{InitialTrustee}, \mathsf{sid})$ from the environment $\mathcal{Z}$, the trustee T_j, $j \in [k]$, operates as the follows:

- Generate $(\overline{\mathtt{pk}}_j, \overline{\mathtt{sk}}_j) \leftarrow \mathsf{TRE.Keygen}(\mathsf{param}; \alpha_j)$ where α_j is the fresh randomness, and then compute

$$\pi_j^{(1)} \leftarrow \mathsf{NIZK}_{\mathcal{R}_4} \left\{ \ (\overline{\mathtt{pk}}_j), (\alpha_j, \overline{\mathtt{sk}}_j) : (\overline{\mathtt{pk}}_j, \overline{\mathtt{sk}}_j) = \mathsf{TRE.Keygen}(\mathsf{param}; \alpha_j) \ \right\}$$

- Send $(\textsc{Sign}, \mathsf{sid}, \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}))$ to $\widehat{\mathcal{F}}_{\textsc{Cert}}$ and receives $(\textsc{Signature}, \mathsf{sid}, \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), \sigma_j^{(1)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$, where $\mathsf{ssid} = (\mathsf{T}_j, \mathsf{ssid}')$ for some ssid'.
- Send $(\textsc{Submit}, \mathsf{sid}, \langle \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), \sigma_j^{(1)} \rangle)$ to $\bar{\mathcal{G}}_{\text{BB}}$.

Figure 12: Mix-net based statement voting scheme $\Pi_{\text{MIX-SV}}$ in the $\{\bar{\mathcal{G}}_{\text{BB}}, \widehat{\mathcal{F}}_{\textsc{Cert}}\}$-hybrid world (Part I).

6.2.2 Ballot casting phase

As depicted in Fig. 13, the ballot casting phase consists of two rounds. In the first round, each voter V_i, $i \in [n]$ first fetches the trustees' partial public keys $\{\overline{\mathtt{pk}}_j\}_{j=1}^k$ from $\bar{\mathcal{G}}_{\text{BB}}$. She then checks the validity of their attached NIZK proofs. If all the NIZK proofs are verified, she computes and stores the election public key as $\mathtt{pk} \leftarrow \mathsf{TRE.CombinePK}(\{\overline{\mathtt{pk}}_j\}_{j=1}^k)$. In addition, the voter V_i picks a random temporal ID $w_i \leftarrow \{0,1\}^\lambda$. She then uses the election public key $\mathtt{pk}$ to encrypt w_i as $W_i \leftarrow \mathsf{TRE.Enc}(\mathtt{pk}, w_i; \beta_i)$ with fresh randomness β_i. She also computes

$$\pi_i^{(2)} \leftarrow \mathsf{NIZK}_{\mathcal{R}_5}\{(\mathtt{pk}, W_i), (\beta_i, w_i) : W_i = \mathsf{TRE.Enc}(\mathtt{pk}, w_i; \beta_i)\}$$

to show that she is the creator of this ciphertext. Voter V_i then signs and posts $(W_i, \pi_i^{(2)})$ to $\bar{\mathcal{G}}_{\text{BB}}$.

In the second round, each voter V_i, $i \in [n]$ first fetches all the posted encrypted temporal IDs from $\bar{\mathcal{G}}_{\text{BB}}$, and checks their attached NIZK proofs. For any missing or invalid (encrypted) temporal IDs, the voters replace them with $\mathsf{TRE.Enc}(\mathtt{pk}, \perp; 0)$, which is encryption of $\perp$ with trivial randomness. Moreover, the voters also defines $W_0 \leftarrow \mathsf{TRE.Enc}(\mathtt{pk}, \perp; 0)$.

For the sake of uniformity, our scheme restricts that each voter's statement can refer up to $\lambda_1 \in \mathbb{N}$ the other voters' IDs, and the size of the statement should fit in the plaintext space. For a voter V_i, $i \in [n]$, denote $\mathcal{D}_i \subseteq [n]$ as the set of indices of the referenced voters' IDs. Let $\mathcal{W}_i := \{W_j \,|\, j \in \mathcal{D}_i\}$ be the set of ciphertexts of the corresponding referenced voters' IDs. The voter V_i re-randomizes all the ciphertexts in $\mathcal{W}_i$ and pads re-randomized W_0's to form a ciphertext vector of size λ_1, denoted as $(U_1, \ldots, U_{\lambda_1})$. The voter V_i then replaces the voter IDs in the statement as the pointers to U_j, $j \in [\lambda_1]$. Denote the modified statement as s_i'. It then encrypts s_i' to ciphertext S_i. Of course, to ensure correctness, NIZK proofs are generated to show (i) U_j, $j \in [\lambda_1]$ is indeed re-randomized from one of the ciphertexts in $(W_0, \ldots, W_n)$, and (ii) S_i is indeed created by the voter himself/herself. The voter V_i then signs and posts $(U_1, \ldots, U_{\lambda_1})$ and S_i together with the corresponding NIZK proofs to $\bar{\mathcal{G}}_{\text{BB}}$.

6.2.3 Tally phase

The tally phase is depicted in Fig. 14 and Fig. 15. The trustees first fetches $(W_i, (U_1, \ldots, U_{\lambda_1}), S_i)$ (which is viewed as the submitted ballot for voter V_i) from $\bar{\mathcal{G}}_{\text{BB}}$ and check their attached NIZK proofs. All the invalid ballots will be discard. Let n' be the number of valid ballots. All the trustees then jointly shuffle the ballots via a re-encryption mix-net. More specifically, each trustee sequentially permutes $(W_i, (U_1, \ldots, U_{\lambda_1}), S_i)$ as a bundle using shuffle re-encryption. To ensure correctness, the trustee also produces a NIZK proof showing the correctness of the shuffle re-encryption process. After that, upon receiving $(\textsc{Tally}, \mathsf{sid})$ from the environment,

Ballot Casting:

Upon receiving $(\textsc{Cast}, \mathsf{sid}, s_i)$ from $\mathcal{Z}$, the voter V_i operates as the follows:

○ Round 1:

- Send $(\textsc{Read}, \mathsf{sid})$ to $\bar{\mathcal{G}}_{\text{BB}}$, and obtain $(\textsc{Read}, \mathsf{sid}, \mathit{state})$ from $\bar{\mathcal{G}}_{\text{BB}}$. If $\left\{\langle \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), \sigma_j^{(1)}\rangle\right\}_{j\in[k]}$ is contained in state, then for $j \in [k]$, send $(\textsc{Verify}, \mathsf{sid}, \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), \sigma_j^{(1)})$ to $\widehat{\mathcal{F}}_{\textsc{Cert}}$, and receive $(\textsc{Verified}, \mathsf{sid}, \mathsf{ssid}, (\overline{\mathtt{pk}}_j, \pi_j^{(1)}), b_j^{(1)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$; If $\prod_{j=1}^{k} b_j^{(1)} = 1$, check $\mathsf{NIZK}_{\mathcal{R}_4}.\mathsf{Verify}(\overline{\mathtt{pk}}_j, \pi_j^{(1)}) = 1$ for $j \in [k]$. If any of the checks is invalid, halt.
- Compute and store $\mathtt{pk} \leftarrow \mathsf{TRE.CombinePK}(\{\overline{\mathtt{pk}}_j\}_{j=1}^{k})$.
- Randomly selects $w_i \leftarrow \{0,1\}^{\lambda}$ and compute $W_i \leftarrow \mathsf{TRE.Enc}(\mathtt{pk}, w_i; \beta_i)$ with fresh randomness β_i together with $\pi_i^{(2)} \leftarrow \mathsf{NIZK}_{\mathcal{R}_5}\left\{\ (\mathtt{pk}, W_i), (\beta_i, w_i) : W_i = \mathsf{TRE.Enc}(\mathtt{pk}, w_i; \beta_i)\ \right\}$.
- Send $(\textsc{Sign}, \mathsf{sid}, \mathsf{ssid}, (W_i, \pi_i^{(2)}))$ to $\widehat{\mathcal{F}}_{\textsc{Cert}}$, and receive $(\textsc{Signature}, \mathsf{sid}, \mathsf{ssid}, (W_i, \pi_i^{(2)}), \sigma_i^{(2)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$, where $\mathsf{ssid} = (\mathsf{V}_i, \mathsf{ssid}')$ for some ssid'.
- Send $(\textsc{Submit}, \mathsf{sid}, \langle \mathsf{ssid}, (W_i, \pi_i^{(2)}), \sigma_i^{(2)}\rangle)$ to $\bar{\mathcal{G}}_{\text{BB}}$.

○ Round 2:

- Send $(\textsc{Read}, \mathsf{sid})$ to $\bar{\mathcal{G}}_{\text{BB}}$, and obtain $(\textsc{Read}, \mathsf{sid}, \mathit{state})$ from $\bar{\mathcal{G}}_{\text{BB}}$.
  For $\ell \in [n]$, if $\langle \mathsf{ssid}, (W_\ell, \pi_\ell^{(2)}), \sigma_\ell^{(2)}\rangle$ is contained in state, then send $(\textsc{Verify}, \mathsf{sid}, \mathsf{ssid}, (W_\ell, \pi_\ell^{(2)}), \sigma_\ell^{(2)})$ to $\widehat{\mathcal{F}}_{\textsc{Cert}}$, and receive $(\textsc{Verified}, \mathsf{sid}, \mathsf{ssid}, (W_\ell, \pi_\ell^{(2)}), b_j^{(2)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$;
  For $\ell \in [n]$, set $W_\ell \leftarrow \mathsf{TRE.Enc}(\mathtt{pk}, \perp; 0)$ if W_ℓ is missing or $b_\ell^{(2)} = 0$ or $\mathsf{NIZK}_{\mathcal{R}_5}.\mathsf{Verify}((\mathtt{pk}, W_\ell), \pi_\ell^{(2)}) = 0$.
- Set $\ell = 1$. Scan though the statement s_i, for each referenced voter V_j, compute
  - $U_{i,\ell} \leftarrow \mathsf{TRE.ReRand}(\mathtt{pk}, W_j; \gamma_{i,\ell})$ with a fresh randomness $\gamma_{i,\ell}$ and $\pi_{i,\ell}^{(3)} \leftarrow \mathsf{NIZK}_{\mathcal{R}_6}\left\{\begin{array}{l}(\mathtt{pk}, (W_0, \ldots, W_n), U_{i,\ell}), (\gamma_{i,\ell}, j) : \\ U_{i,\ell} = \mathsf{RTE.ReRand}(\mathtt{pk}, W_j; \gamma_{i,\ell})\end{array}\right\}$.
  - Replace V_j with ℓ in the statement s_i. $\ell := \ell + 1$ and repeat the above process for all the voter IDs in s_i.
  - If $\ell < \lambda_1$, compute
    - $U_{i,\ell} \leftarrow \mathsf{TRE.ReRand}(\mathtt{pk}, W_0; \gamma_{i,\ell})$ with a fresh randomness $\gamma_{i,\ell}$ and $\pi_{i,\ell}^{(3)} \leftarrow \mathsf{NIZK}_{\mathcal{R}_6}\left\{\begin{array}{l}(\mathtt{pk}, (W_0, \ldots, W_n), U_{i,\ell}), (\gamma_{i,\ell}, 0) : \\ U_{i,\ell} = \mathsf{TRE.ReRand}(\mathtt{pk}, W_0; \gamma_{i,\ell})\end{array}\right\}$.
    - Repeat the above process until $\ell = \lambda_1$.
  - Denote the modified statement as s_i'. Compute $S_i \leftarrow \mathsf{TRE.Enc}(\mathtt{pk}, s_i'; \delta_i)$ and $\pi_i^{(4)} \leftarrow \mathsf{NIZK}_{\mathcal{R}_5}\left\{\ ((\mathtt{pk}, S_i), (\delta_i, s_i') : S_i = \mathsf{TRE.Enc}(\mathtt{pk}, s_i'; \delta_i)\ \right\}$.
- Send $(\textsc{Sign}, \mathsf{sid}, \mathsf{ssid}, ((U_{i,\ell}, \pi_{i,\ell}^{(3)})_{\ell=1}^{\lambda_1}, S_i, \pi_i^{(4)}))$ to $\widehat{\mathcal{F}}_{\textsc{Cert}}$, where $\mathsf{ssid} = (\mathsf{V}_i, \mathsf{ssid}')$ for some ssid', and receive $(\textsc{Signature}, \mathsf{sid}, \mathsf{ssid}, ((U_{i,\ell}, \pi_{i,\ell}^{(3)})_{\ell=1}^{\lambda_1}, S_i, \pi_i^{(4)}), \sigma_i^{(3)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$.
- Send $(\textsc{Submit}, \mathsf{sid}, \langle \mathsf{ssid}, ((U_{i,\ell}, \pi_{i,\ell}^{(3)})_{\ell=1}^{\lambda_1}, S_i, \pi_i^{(4)}), \sigma_i^{(3)}\rangle)$ to $\bar{\mathcal{G}}_{\text{BB}}$.

Figure 13: Mix-net based statement voting scheme $\Pi_{\text{MIX-SV}}$ in the $\{\bar{\mathcal{G}}_{\text{BB}}, \widehat{\mathcal{F}}_{\textsc{Cert}}\}$-hybrid world (Part II).

all the trustees T_j check the correctness of the entire mix-net and then jointly decrypt the mixed ballots using TRE.ShareDec. More specifically, each trustee will sign and post its decryption shares to $\bar{\mathcal{G}}_{\text{BB}}$.

Tally:
Upon receiving $(\textsc{Tally}, \mathsf{sid})$ from the environment $\mathcal{Z}$, the trustee T_j, where $j \in [k]$, operates as the follows:

∘ Round 1 to k:

– If $j = 1$, send $(\textsc{Read}, \mathsf{sid})$ to $\bar{\mathcal{G}}_{\text{BB}}$, and obtain $(\textsc{Read}, \mathsf{sid}, \mathit{state})$ from $\bar{\mathcal{G}}_{\text{BB}}$. For $x \in [n]$:

- If $\langle \mathsf{ssid}, (W_x, \pi_x^{(2)}), \sigma_x^{(2)} \rangle$ is contained in state, then send $(\textsc{Verify}, \mathsf{sid}, \mathsf{ssid}, (W_x, \pi_x^{(2)}), \sigma_x^{(2)})$ to $\widehat{\mathcal{F}}_{\textsc{Cert}}$, and receive $(\textsc{Verified}, \mathsf{sid}, \mathsf{ssid}, (W_x, \pi_x^{(2)}), b_x^{(2)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$;
- If $\langle \mathsf{ssid}, ((U_{x,\ell}, \pi_{x,\ell}^{(3)})_{\ell=1}^{\lambda_1}, S_x, \pi_x^{(4)}), \sigma_x^{(3)} \rangle$, is contained in state, then send

$$(\textsc{Verify}, \mathsf{sid}, \mathsf{ssid}, ((U_{x,\ell}, \pi_{x,\ell}^{(3)})_{\ell=1}^{\lambda_1}, S_x, \pi_x^{(4)}), \sigma_x^{(3)})$$

to $\widehat{\mathcal{F}}_{\textsc{Cert}}$, receive $(\textsc{Verified}, \mathsf{sid}, \mathsf{ssid}, ((U_{x,\ell}, \pi_{x,\ell}^{(3)})_{\ell=1}^{\lambda_1}, S_x, \pi_x^{(4)}), b_x^{(3)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$;

Set $i := 0$. For $\ell \in [n]$, define $e_i^{(0)} := (W_x, (U_{x,\ell})_{\ell=1}^{\lambda_1}, S_x)$ and $i := i + 1$ if the following holds:

- $W_x, (U_{x,\ell})_{\ell=1}^{\lambda_1}, S_x$ exist in state and $b_x^{(2)} \cdot b_x^{(3)} = 1$;
- $\mathsf{NIZK}_{\mathcal{R}_5}.\mathsf{Verify}((\mathtt{pk}, W_x), \pi_x^{(2)}) = 1$;
- For all $\ell \in [\lambda_1]$, $\mathsf{NIZK}_{\mathcal{R}_6}.\mathsf{Verify}((\mathtt{pk}, (W_0, \ldots, W_n), U_{x,\ell}), \pi_{x,\ell}^{(3)}) = 1$;
- $\mathsf{NIZK}_{\mathcal{R}_5}.\mathsf{Verify}((\mathtt{pk}, S_x), \pi_x^{(4)}) = 1$;

Assume $i = n'$ after the above process.

– (If $j > 1$, T_j sends $(\textsc{Read}, \mathsf{sid})$ to $\bar{\mathcal{G}}_{\text{BB}}$, and obtain $(\textsc{Read}, \mathsf{sid}, \mathit{state})$ from $\bar{\mathcal{G}}_{\text{BB}}$; T_j then fetches $(e_i^{(j-2)})_{i=1}^{n'}, e_i^{(j-1)})_{i=1}^{n'}, \pi_{j-1}^{(5)}$ from state and check $\mathsf{NIZK}_{\mathcal{R}_7}.\mathsf{Verify}((\mathtt{pk}, (e_1^{(j-2)}, \ldots, e_{n'}^{(j-2)}), (e_1^{(j-1)}, \ldots, e_{n'}^{(j-1)})), \pi_{j-1}^{(5)})$.)
T_j randomly picks a permutation Π_j over [n']; For $i \in [n']$, for $\ell \in [\lambda_2]$: set $e_{i,\ell}^{(j)} \leftarrow \mathsf{TRE.ReRand}(\mathtt{pk}, e_{\Pi_j(i),\ell}^{(j-1)}; r_{i,\ell}^{(j)})$, where $r_{i,\ell}^{(j)}$ are fresh randomness. Compute

$$\pi_j^{(5)} \leftarrow \mathsf{NIZK}_{\mathcal{R}_7}\left\{ \begin{array}{c} \left(\mathtt{pk}, (e_1^{(j-1)}, \ldots, e_{n'}^{(j-1)}), (e_1^{(j)}, \ldots, e_{n'}^{(j)})\right), \left(\Pi_j, (r_{i,\ell}^{(j)})_{i\in[n'],\ell\in[\lambda_1+2]}\right) : \\ \forall i \in [n'] \; \forall \ell \in [\lambda_1 + 2] \; : \; e_{i,\ell}^{(j)} = \mathsf{TRE.ReRand}\left(\mathtt{pk}, e_{\Pi_j(i),1}^{(j-1)}; r_{i,\ell}^{(j)}\right) \end{array} \right\}$$

– Send $(\textsc{Sign}, \mathsf{sid}, \mathsf{ssid}, (e_i^{(j)})_{i=1}^{n'}, \pi_j^{(5)}))$ to $\widehat{\mathcal{F}}_{\textsc{Cert}}$ and receive $(\textsc{Signature}, \mathsf{sid}, \mathsf{ssid}, (e_i^{(j)})_{i=1}^{n'}, \pi_j^{(5)}), \sigma_j^{(4)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$, where $\mathsf{ssid} = (\mathsf{T}_j, \mathsf{ssid}')$ for some ssid'.

– Send $(\textsc{Submit}, \mathsf{sid}, \langle \mathsf{ssid}, (e_i^{(j)})_{i=1}^{n'}, \pi_j^{(5)}, \sigma_j^{(4)} \rangle)$ to $\bar{\mathcal{G}}_{\text{BB}}$.

Figure 14: Mix-net based statement voting scheme $\Pi_{\text{MIX-SV}}$ in the $\{\bar{\mathcal{G}}_{\text{BB}}, \widehat{\mathcal{F}}_{\textsc{Cert}}\}$-hybrid world (Part III).

Each voter can then compute the tally result as follows. The voter first fetches all the decryption shares and checks their validity using $\mathsf{NIZK}_{\mathcal{R}_8}.\mathsf{Verify}$. Upon success, the voter uses TRE.ShareCombine to reconstruct the messages. She then use TallyProcess as described in Fig. 4 to calculate the final tally.

6.3 Security

We have the following the theorem.

Theorem 6.2. *Protocol $\Pi_{\text{MIX-SV}}$ described in Figure 12, Figure 13, Figure 14 and Figure 15 UC-realizes $\mathcal{F}_{\text{SV}}$ in the $\{\bar{\mathcal{G}}_{\text{BB}}, \widehat{\mathcal{F}}_{\textsc{Cert}}\}$-hybrid world against static corruption.*

The security proof can be found in Supplemental material C.1.

◦ Round $k+1$:

- Send $(\textsc{Read}, \mathsf{sid})$ to $\bar{\mathcal{G}}_{\text{BB}}$, and obtain $(\textsc{Read}, \mathsf{sid}, \mathit{state})$ from $\bar{\mathcal{G}}_{\text{BB}}$. For $j \in [k]$, if $\langle \mathsf{ssid}, (e_i^{(j)})_{i=1}^{n'}, \pi_j^{(5)}, \sigma_j^{(4)} \rangle$ is contained in state, then send $(\textsc{Verify}, \mathsf{sid}, \mathsf{ssid}, (e_i^{(j)})_{i=1}^{n'}, \pi_j^{(5)}), \sigma_j^{(4)})$ to $\widehat{\mathcal{F}}_{\textsc{Cert}}$, and receive $(\textsc{Verified}, \mathsf{sid}, \mathsf{ssid}, (e_i^{(j)})_{i=1}^{n'}, \pi_j^{(5)}), b_j^{(4)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$; if $b_j^{(4)} = 1$, check $\mathsf{NIZK}_{\mathcal{R}_4}.\mathsf{Verify}((\mathtt{pk}, (e_1^{(j-1)}, \ldots, e_{n'}^{(j-1)}), (e_1^{(j)}, \ldots, e_{n'}^{(j)})), \pi_j^{(5)}) = 1$. If any of the above checks is invalid, halt.
- For $i \in [n']$, $\ell \in [\lambda_1 + 2]$, compute $\overline{m}_{i,\ell}^{(j)} \leftarrow \mathsf{TRE.ShareDec}(\overline{\mathtt{sk}}_j, e_{i,\ell}^{(k)})$. and

$$\pi_{i,j,\ell}^{(6)} \leftarrow \mathsf{NIZK}_{\mathcal{R}_8} \left\{ \begin{array}{l} (e_{i,\ell}^{(k)}, \overline{m}_{i,\ell}^{(j)}, \overline{\mathtt{pk}}_j), (\overline{\mathtt{sk}}_j, \alpha_j) : \\ (\overline{\mathtt{pk}}_j, \overline{\mathtt{sk}}_j) = \mathsf{TRE.Keygen}(\mathsf{param}; \alpha_j) \\ \quad \wedge \overline{m}_{i,\ell}^{(j)} = \mathsf{TRE.ShareDec}(\overline{\mathtt{sk}}_j, e_{i,\ell}^{(k)}) \end{array} \right\}$$

- Send $(\textsc{Sign}, \mathsf{sid}, \mathsf{ssid}, (\overline{m}_{i,\ell}^{(j)}, \pi_{i,j,\ell}^{(6)})_{i \in [n'], \ell \in [\lambda_1+2]})$ to $\widehat{\mathcal{F}}_{\textsc{Cert}}$ and receives $(\textsc{Signature}, \mathsf{sid}, \mathsf{ssid}, (\overline{m}_{i,\ell}^{(j)}, \pi_{i,j,\ell}^{(6)})_{i \in [n'], \ell \in [\lambda_1+2]}, \sigma_j^{(5)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$, where $\mathsf{ssid} = (\mathsf{T}_j, \mathsf{ssid}')$ for some ssid'.
- Send $(\textsc{Submit}, \mathsf{sid}, \langle \mathsf{ssid}, (\overline{m}_{i,\ell}^{(j)}, \pi_{i,j,\ell}^{(6)})_{i \in [n'], \ell \in [\lambda_1+2]}, \sigma_j^{(5)} \rangle)$ to $\bar{\mathcal{G}}_{\text{BB}}$.

Upon receiving $(\textsc{ReadResult}, \mathsf{sid})$ from the environment $\mathcal{Z}$, the voter V_i, where $i \in [n]$, operates as the follows:

- Send $(\textsc{Read}, \mathsf{sid})$ to $\bar{\mathcal{G}}_{\text{BB}}$, and and obtain $(\textsc{Read}, \mathsf{sid}, \mathit{state})$ from $\bar{\mathcal{G}}_{\text{BB}}$.
  For $j \in [k]$, if $\langle \mathsf{ssid}, (\overline{m}_{i,\ell}^{(j)}, \pi_{i,j,\ell}^{(6)})_{i \in [n'], \ell \in [\lambda_1+2]}, \sigma_j^{(5)} \rangle$ is contained in state, send $(\textsc{Verify}, \mathsf{sid}, \mathsf{ssid}, (\overline{m}_{i,\ell}^{(j)}, \pi_{i,j,\ell}^{(6)})_{i \in [n'], \ell \in [\lambda_1+2]})_{i \in [n'], \ell \in [\lambda_1+2]}, \sigma_j^{(5)})$ to $\widehat{\mathcal{F}}_{\textsc{Cert}}$, and receive $(\textsc{Verified}, \mathsf{sid}, \mathsf{ssid}, (\overline{m}_{i,\ell}^{(j)}, \pi_{i,j,\ell}^{(6)})_{i \in [n'], \ell \in [\lambda_1+2]}, b_j^{(5)})$ from $\widehat{\mathcal{F}}_{\textsc{Cert}}$. If $\prod_{j=1}^{k} b_j^{(5)} = 1$, for all $j \in [k], i \in [n'], \ell \in [\lambda_1+2]$, check $\mathsf{NIZK}_{\mathcal{R}_8}.\mathsf{Verify}((e_{i,\ell}^{(k)}, \overline{m}_{i,\ell}^{(j)}, \overline{\mathtt{pk}}_i), \pi_{i,j,\ell}^{(6)}) = 1$. If any of the above checks is invalid, return $(\textsc{Error}, \mathsf{sid})$ to the environment $\mathcal{Z}$ and halt.
- For $i \in [n']$, $\ell \in [\lambda_1 + 2]$: compute $m_{i,\ell} \leftarrow \mathsf{TRE.ShareCombine}(e_{i,\ell}^{(k)}, \{\overline{m}_{i,\ell}^{(j)}\}_{j=1}^{k})$, $\ell \in [\lambda_1 + 2]$; define $B_i := (m_{i,\ell})_{\ell \in [\lambda_1+2]}$.
- Calculate election result $\mathit{result} \leftarrow \mathsf{TallyProcess}(\{B_i\}_{i \in [n']})$, and return $(\textsc{ReadResultReturn}, \mathsf{sid}, \mathit{result})$ to $\mathcal{Z}$.

Figure 15: Mix-net based statement voting scheme $\Pi_{\text{MIX-SV}}$ in the $\{\bar{\mathcal{G}}_{\text{BB}}, \widehat{\mathcal{F}}_{\textsc{Cert}}\}$-hybrid world (Part IV).

7 Application to Liquid Democracy

As mentioned before, *liquid democracy* is an emerging type of voting system that receives high attentions since the spread of its concept; however, there is no satisfactory solution in the form of either paper-voting or e-voting yet. We now show that how to define a simple statement to enable liquid democracy. We are particularly interested in the mix-net based scheme due to its efficiency. In the following, we will realize a liquid democracy voting scheme on top of the scheme presented in Section 6.

The preparation phase of the liquid democracy scheme is identical to Fig. 12 In the ballot casting phase, in round 1, the voter V_i, $i \in [n]$ picks a random temporal ID, and submits its encryption to $\bar{\mathcal{G}}_{\mathrm{BB}}$ as described in Fig. 13. In round 2, the liquid democracy statement consists of two ciphertexts (U, S); if the voter V_i wants to delegate her vote to voter V_j, she sets $U \leftarrow \mathsf{TRE.ReRand}(\mathtt{pk}, W_j)$ and $S \leftarrow \mathsf{TRE.Enc}(\perp)$; if the voter V_i wants to directly cast her vote x_i, she sets $U \leftarrow \mathsf{TRE.ReRand}(\mathtt{pk}, W_0)$ and $S \leftarrow \mathsf{TRE.Enc}(x_i)$. The tally phase is also identical to the one depicted in Fig. 14 and Fig. 15.

The statement interpretation step in the $\mathsf{TallyProcess}$ is defined as follows. Each ballots is in form of either $B_i = (w_i, u_i, \perp)$ or $B_i = (w_i, \perp, x_i)$, where w_i and u_i are temporal ID's, and x_i is a vote. To resolve the delegation, the algorithm needs to follow the "chain of delegation", i.e., for each ballot B_i:

- If B_i is in form of $(w_i, u_i, \perp)$, try to locate a ballot B_j in form of (u_i, X, Y). If founded, replace $B_i := (w_i, X, Y)$.
- Repeat the above step, until B_i is in form of $(w_i, \perp, Z)$. If there is a delegation loop, define $B_i := (w_i, \perp, \perp)$.

Namely, in case of delegation loop, we set the ballot to blank ballot. Of course, we can enrich the statement by adding another variable to indicate whether a voter wants to be delegated. When the "chain of delegation" breaks by V_i wants to delegate his vote to V_j, while V_j does not want to be delegated. In this case, V_i's ballot will be re-set to a blank ballot. The most preferable statement for liquid democracy in practice shall be determined by computational social choice theory, which is outside the scope of this paper.

Acknowledgement: We thank Zengpeng Li for his helpful discussions about TFHE instantiation.

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A Supplementary material for Section 4

A.1 Proof for Theorem 4.2

Before providing the proof details, we first present a simplified version of our statement voting functionality as in Figure 16. We note that, this functionality achieves strong privacy.

Functionality $\mathcal{F}_{\text{SV}}$

The functionality $\mathcal{F}_{\text{SV}}$ interacts with a set of voters $\mathbb{V} := \{\mathsf{V}_1, \ldots, \mathsf{V}_n\}$, a set of trustees $\mathbb{T} := \{\mathsf{T}_1, \ldots, \mathsf{T}_k\}$, and the adversary $\mathcal{S}$. Let $\mathbb{V}_{\mathsf{honest}}$, $\mathbb{V}_{\mathsf{corrupt}}$ and $\mathbb{T}_{\mathsf{honest}}$, $\mathbb{T}_{\mathsf{corrupt}}$ denote the set of honest/corrupt voters and trustees, respectively.
Functionality $\mathcal{F}_{\text{SV}}$ is parameterized by an algorithm $\mathsf{TallyProcess}$, and variables $result$, T_1, T_2, and B_i for all $i \in [n]$.

Initially, set $result := \emptyset$, $T_1 := \emptyset$, $T_2 := \emptyset$; for $i \in [n]$, set $B_i := \emptyset$.

Preparation:

1. Upon receiving input $(\textsc{InitialTrustee}, \mathsf{sid})$ from the trustee $\mathsf{T}_j \in \mathbb{T}$, set $T_1 := T_1 \cup \{\mathsf{T}_j\}$, and send a notification message $(\textsc{InitialTrusteeNotify}, \mathsf{sid}, \mathsf{T}_j)$ to the adversary $\mathcal{S}$.

Ballot Casting:

1. Upon receiving input $(\textsc{Cast}, \mathsf{sid}, s_i)$ from the voter $\mathsf{V}_i \in \mathbb{V}$, if $|T_1| < k$, ignore the input. Otherwise, record $B_i := (\mathsf{V}_i, s_i)$; send a message $(\textsc{CastNotify}, \mathsf{sid}, \mathsf{V}_i)$ to the adversary $\mathcal{S}$. If $|\mathbb{T}_{\mathsf{corrupt}}| = k$, then additionally send a message $(\textsc{Leak}, \mathsf{sid}, \mathsf{V}_i, B_i)$ to the adversary $\mathcal{S}$.

Tally:

1. Upon receiving input $(\textsc{Tally}, \mathsf{sid})$ from the trustee $\mathsf{T}_j \in \mathbb{T}$, if $\mathsf{T}_j \notin T_2$ then set $T_2 := T_2 \cup \{\mathsf{T}_j\}$.
   Send a notification message $(\textsc{TallyNotify}, \mathsf{sid}, \mathsf{T}_j)$ to $\mathcal{S}$.
   If $|T_2 \cap \mathbb{T}_{\mathsf{honest}}| + |\mathbb{T}_{\mathsf{corrupt}}| = k$, send $(\textsc{Leak}, \mathsf{sid}, \mathsf{TallyProcess}(B_1, \ldots, B_n))$ to $\mathcal{S}$.
   If $|T_2| = k$, compute $result \leftarrow \mathsf{TallyProcess}(B_1, \ldots, B_n)$.
2. Upon receiving input $(\textsc{ReadResult}, \mathsf{sid})$ from a voter $\mathsf{V}_i \in \mathbb{V}$, if $result = \emptyset$, ignore the input.
   Else, return $(\textsc{ResultReturn}, \mathsf{sid}, result)$ to V_i.

Figure 16: The statement voting functionality.

Proof. To prove the theorem, we construct a simulator $\mathcal{S}$ such that no non-uniform PPT environment $\mathcal{Z}$ can distinguish between (i) the real execution $\mathsf{EXEC}^{\bar{\mathcal{G}}_{\text{BB}}, \widehat{\mathcal{F}}_{\text{CERT}}}_{\Pi_{\text{FHE-SV}}, \mathcal{A}, \mathcal{Z}}$ where the parties $\mathbb{V} := \{\mathsf{V}_1, \ldots, \mathsf{V}_n\}$ and $\mathbb{T} := \{\mathsf{T}_1, \ldots, \mathsf{T}_k\}$ run protocol $\Pi_{\text{FHE-SV}}$ in the $\{\bar{\mathcal{G}}_{\text{BB}}, \widehat{\mathcal{F}}_{\text{CERT}}\}$-hybrid world and the corrupted parties are controlled by a dummy adversary $\mathcal{A}$ who simply forwards messages from/to $\mathcal{Z}$, and (ii) the ideal execution $\mathsf{EXEC}^{\bar{\mathcal{G}}_{\text{BB}}}_{\mathcal{F}_{\text{SV}}, \mathcal{S}, \mathcal{Z}}$ where the parties interact with functionality $\mathcal{F}_{\text{SV}}$ in the $\bar{\mathcal{G}}_{\text{BB}}$-hybrid model and corrupted parties are controlled by the simulator $\mathcal{S}$. Let $\mathbb{V}_{\mathsf{corrupt}} \subseteq \mathbb{V}$ and $\mathbb{T}_{\mathsf{corrupt}} \subseteq \mathbb{T}$ be the set of corrupted voters and trustees, respectively. We consider following cases.

Case 1: $0 \leq |\mathbb{V}_{\mathsf{corrupt}}| < n \ \wedge \ 0 \leq |\mathbb{T}_{\mathsf{corrupt}}| < k$.

Simulator. The simulator $\mathcal{S}$ internally runs $\mathcal{A}$, forwarding messages to/from the environment $\mathcal{Z}$. The simulator $\mathcal{S}$ simulates honest voters $\mathsf{V}_i \in \mathbb{V} \setminus \mathbb{V}_{\mathsf{corrupt}}$, honest trustees $\mathsf{T}_j \in \mathbb{T} \setminus \mathbb{T}_{\mathsf{corrupt}}$ and functionalities $\widehat{\mathcal{F}}_{\text{CERT}}$. In addition, the simulator $\mathcal{S}$ simulates the following interactions with $\mathcal{A}$.

- In the preparation phase:
  - Upon receiving $(\text{INITIALTRUSTEENOTIFY}, \mathsf{sid}, \mathsf{T}_j)$ from the external $\mathcal{F}_{\text{SV}}$ for an honest trustee $\mathsf{T}_j \in \mathbb{T} \setminus \mathbb{T}_{\mathsf{corrupt}}$, the simulator $\mathcal{S}$ acts as T_j, following the protocol $\Pi_{\text{FHE-SV}}$ as if T_j receives $(\text{INITIALTRUSTEE}, \mathsf{sid})$ from $\mathcal{Z}$.
  - Monitoring $\bar{\mathcal{G}}_{\text{BB}}$, when a valid $(\overline{\mathtt{pk}}_j, \pi_j^{(1)})$ is posted on $\bar{\mathcal{G}}_{\text{BB}}$ from a corrupted trustee $\mathsf{T}_j \in \mathbb{T}_{\mathsf{corrupt}}$, use $\mathsf{NIZK}_{\mathcal{R}_1}.\mathsf{Ext}^{\text{RO}}(\overline{\mathtt{pk}}_j, \pi_j^{(1)})$ to extract the corresponding secret key $\overline{\mathtt{sk}}_j$.
- In the ballot casting phase:
  - Upon receiving $(\text{CASTNOTIFY}, \mathsf{sid}, \mathsf{V}_i)$ from the external $\mathcal{F}_{\text{SV}}$ for an honest voter $\mathsf{V}_i \in \mathbb{V} \setminus \mathbb{V}_{\mathsf{corrupt}}$, the simulator $\mathcal{S}$ creates $c_i \leftarrow \mathsf{TFHE.Enc}(\mathtt{pk}, 0)$. It then uses $\mathsf{NIZK}_{\mathcal{R}_2}.\mathsf{Sim}$ to simulate the corresponding proofs $\pi_i^{(2)}$. The simulator $\mathcal{S}$ then follows the protocol to post $(c_i, \pi_i^{(2)})$ to $\bar{\mathcal{G}}_{\text{BB}}$.
  - Monitoring $\bar{\mathcal{G}}_{\text{BB}}$, when a valid $(c_i, \pi_i^{(2)})$ is posted on $\bar{\mathcal{G}}_{\text{BB}}$ from a corrupted voter $\mathsf{V}_i \in \mathbb{V}_{\mathsf{corrupt}}$, uses the extracted $\{\overline{\mathtt{sk}}_j\}_{j\in[k]}$ to decrypt c_i to (V_i, s_i). The simulator $\mathcal{S}$ then acts as V_i to send $(\text{CAST}, \mathsf{sid}, s_i)$ to $\mathcal{F}_{\text{SV}}$.
- In the tally phase:
  - Upon receiving $(\text{TALLYNOTIFY}, \mathsf{sid}, \mathsf{T}_j)$ from the external $\mathcal{F}_{\text{SV}}$ for an honest trustee $\mathsf{T}_j \in \mathbb{T}\setminus\mathbb{T}_{\mathsf{corrupt}}$, if $\overline{\tau}_j$ are not defined yet, the $\mathcal{S}$ acts as T_j, following the protocol $\Pi_{\text{FHE-SV}}$ as if T_j receives $(\text{TALLY}, \mathsf{sid})$ from $\mathcal{Z}$. $\mathcal{S}$ then adds j to $\mathcal{J}$, where $\mathcal{J}$ is initially empty. If $\overline{\tau}_j$ is defined, $\mathcal{S}$ uses $\mathsf{NIZK}_{\mathcal{R}_3}.\mathsf{Sim}$ to simulate the corresponding proof $\pi_j^{(3)}$. It then follows the protocol to post $(\overline{\tau}_j, \pi_j^{(3)})$ on the $\bar{\mathcal{G}}_{\text{BB}}$.
  - Upon receiving $(\text{LEAK}, \mathsf{sid}, \tau)$ from the external $\mathcal{F}_{\text{SV}}$, the simulator $\mathcal{S}$ uses the extracted secret key $\overline{\mathtt{sk}}_j$ to compute $\overline{\tau}_j \leftarrow \mathsf{TFHE.ShareDec}(\overline{\mathtt{sk}}_j, c)$ for all the corrupted trustees $\mathsf{T}_j \in \mathbb{T}_{\mathsf{corrupt}}$. It then adds all the indices of the corrupted trustees to $\mathcal{J}$. The simulator $\mathcal{S}$ computes $\{\overline{\tau}_j\}_{j\in[k]\setminus\mathcal{J}} \leftarrow \mathsf{SimShareDec}(c, \tau, \{\overline{\tau}_i\}_{i\in\mathcal{J}})$.

Indistinguishability. The indistinguishability is proven through a series of hybrid worlds $\mathcal{H}_0, \ldots, \mathcal{H}_4$.

Hybrid $\mathcal{H}_0$: It is the real protocol execution $\mathsf{EXEC}^{\bar{\mathcal{G}}_{\text{BB}}, \bar{\mathcal{F}}_{\text{CERT}}}_{\Pi_{\text{FHE-SV}}, \mathcal{A}, \mathcal{Z}}$.

Hybrid $\mathcal{H}_1$: $\mathcal{H}_1$ is the same as $\mathcal{H}_0$ except that $\mathcal{H}_1$ runs $\mathsf{NIZK}_{\mathcal{R}_1}.\mathsf{Ext}^{\text{RO}}(\overline{\mathtt{pk}}_j, \pi_j^{(1)})$ to extract the corrupted trustee's secret key $\overline{\mathtt{sk}}_j$. $\mathcal{H}_1$ halt if the extraction fails.

Claim A.1. *$\mathcal{H}_1$ and $\mathcal{H}_0$ are indistinguishable.*

Proof. According to Def. 2.3, the probability $\mathsf{NIZK}_{\mathcal{R}_1}.\mathsf{Ext}^{\text{RO}}$ extraction fails (a.k.a. knowledge error) is negligible, so the probability that any adversary $\mathcal{A}$ and the environment $\mathcal{Z}$ can distinguish $\mathcal{H}_1$ from $\mathcal{H}_0$ is $\mathsf{negl}(\lambda)$. □

Hybrid $\mathcal{H}_2$: $\mathcal{H}_2$ is the same as $\mathcal{H}_1$ except the following: During the tally phase, uses the extracted $\mathtt{sk}_j$ from Hybrid $\mathcal{H}_1$ to decrypt each ciphertext, and the last honest trustee's message shares of each ciphertext are calculated by $\mathsf{TFHE.SimShareDec}$ instead of using $\mathsf{TFHE.ShareDec}$.

Claim A.2. *$\mathcal{H}_2$ and $\mathcal{H}_1$ are indistinguishable.*

Proof. By the share-simulation indistinguishability of the underlying TFHE scheme, the distribution of the simulated decryption share(s) are computationally indistinguishable to the real ones. Moreover, by soundness of

$$\mathsf{NIZK}_{\mathcal{R}_3}\left\{\begin{array}{l}(c, \overline{\tau}_j, \overline{\mathtt{pk}}_j), (\overline{\mathtt{sk}}_j, \alpha_j) : \\ (\overline{\mathtt{pk}}_j, \overline{\mathtt{sk}}_j) = \mathsf{TFHE.Keygen}(\mathsf{param}; \alpha_j) \ \wedge \ \overline{\tau}_j = \mathsf{TFHE.ShareDec}(\overline{\mathtt{sk}}_j, c)\end{array}\right\}$$

the corrupted trustees have negligible probability to post an invalid decryption share that is different from $\overline{\tau}_j \leftarrow \mathsf{TFHE.ShareDec}(\overline{\mathtt{sk}}_j, c)$. Therefore, the adversary's advantage of distinguishing $\mathcal{H}_2$ from $\mathcal{H}_1$ is $\mathsf{negl}(\lambda)$. □

Hybrid $\mathcal{H}_3$: $\mathcal{H}_3$ is the same as $\mathcal{H}_2$ except the followings. During the vote phase, $\mathcal{H}_3$ uses $\mathsf{NIZK}_{\mathcal{R}_2}.\mathsf{Sim}$ to simulate $\pi_i^{(2)}$ for all the honest voter $\mathsf{V}_i \in \mathbb{V}$.

Claim A.3. *$\mathcal{H}_3$ and $\mathcal{H}_2$ are indistinguishable.*

Proof. The advantage of the adversary is bounded by the ZK property of $\mathsf{NIZK}_{\mathcal{R}_2}$ as defined by Def. 2.2. □

Hybrid $\mathcal{H}_4$: $\mathcal{H}_4$ is the same as $\mathcal{H}_3$ except the followings. During the vote phase, the simulator posts $c_i \leftarrow \mathsf{TFHE.Enc}(\mathtt{pk}, 0)$ for all the honest voter $\mathsf{V}_i \in \mathbb{V}$.

Claim A.4. *$\mathcal{H}_4$ and $\mathcal{H}_3$ are indistinguishable.*

Proof. The probability that any adversary $\mathcal{A}$ can distinguish $\mathcal{H}_4$ from $\mathcal{H}_3$ is bounded by $\mathsf{AdvCPA}_{\mathcal{A}}(1^\lambda)$ and ciphertext transformative indistinguishability. More specifically, we now show the if there exists an adversary $\mathcal{A}$ who can distinguish $\mathcal{H}_4$ from $\mathcal{H}_3$, then we can construction an adversary $\mathcal{B}$ that can break the IND-CPA game of the underlying TFHE by reduction. During the IND-CPA game, $\mathcal{B}$ receives a public key $\mathtt{pk}^*$ from the challenger. There must be at least one honest trustee in this case, and with our loss of generality, assume T_x is honest. During the preparation phase, $\mathcal{B}$ posts $\mathtt{pk}^*$ as T_x's public key together with simulated proof. During the ballot casting phase, for each honest voter V_i, $i \in [n]$, $\mathcal{B}$ sends $m_0 := 0$ and $m_1 := s_i$ to the IND-CPA challenger, and receives c^*. $\mathcal{B}$ then computes $c' \leftarrow \mathsf{TFHE.Trans}(c^*, \{\mathtt{sk}_i\}_{i\in[k]\setminus\{x\}})$. It posts c' as the honest voter's encrypted ballot. It is easy to see that, when c^* encrypts m_0, the adversary's view is indistinguishable from $\mathcal{H}_4$; when c^* encrypts m_1, the adversary's view is indistinguishable from $\mathcal{H}_3$. Hence, if $\mathcal{A}$ can distinguish $\mathcal{H}_4$ from $\mathcal{H}_3$ with non-negligible probability, then $\mathcal{B}$ can break the IND-CPA game with the same probability. □

The adversary's view of $\mathcal{H}_4$ is identical to the simulated view $\mathsf{EXEC}^{\bar{\mathcal{G}}_{\mathrm{BB}}}_{\mathcal{F}_{\mathrm{SV}},\mathcal{S},\mathcal{Z}}$. Therefore, no PPT $\mathcal{Z}$ can distinguish the view of the ideal execution from the view of the real execution with more than negligible probability.

Case 2: $0 \le |\mathbb{V}_{\mathsf{corrupt}}| < n \ \wedge \ |\mathbb{T}_{\mathsf{corrupt}}| = k$.

Simulator. Similar as Case 1, the simulator $\mathcal{S}$ internally runs $\mathcal{A}$, forwarding messages to/from the environment $\mathcal{Z}$. The simulator $\mathcal{S}$ simulates honest voters $\mathsf{V}_i \in \mathbb{V} \setminus \mathbb{V}_{\mathsf{corrupt}}$, honest trustees $\mathsf{T}_j \in \mathbb{T} \setminus \mathbb{T}_{\mathsf{corrupt}}$ and functionalities $\widehat{\mathcal{F}}_{\textsc{Cert}}$. In addition, the simulator $\mathcal{S}$ simulates the following interactions with $\mathcal{A}$.

- In the preparation phase:
  - Monitoring $\bar{\mathcal{G}}_{\mathrm{BB}}$, when a valid $(\overline{\mathtt{pk}}_j, \pi_j^{(1)})$ is posted on $\bar{\mathcal{G}}_{\mathrm{BB}}$ from a corrupted trustee $\mathsf{T}_j \in \mathbb{T}_{\mathsf{corrupt}}$, use $\mathsf{NIZK}_{\mathcal{R}_1}.\mathsf{Ext}^{\mathrm{RO}}(\overline{\mathtt{pk}}_j, \pi_j^{(1)})$ to extract the corresponding secret key $\overline{\mathtt{sk}}_j$.
- In the ballot casting phase:
  - Upon receiving $(\textsc{Leak}, \mathsf{sid}, \mathsf{V}_i, s_i)$ from the external $\mathcal{F}_{\mathrm{SV}}$ for an honest voter $\mathsf{V}_i \in \mathbb{V} \setminus \mathbb{V}_{\mathsf{corrupt}}$, the simulator $\mathcal{S}$ acts as V_i, following the protocol $\Pi_{\text{FHE-SV}}$ as if V_i receives $(\textsc{Cast}, \mathsf{sid}, s_i)$ from $\mathcal{Z}$.
  - Monitoring $\bar{\mathcal{G}}_{\mathrm{BB}}$, when a valid $(c_i, \pi_i^{(2)})$ is posted on $\bar{\mathcal{G}}_{\mathrm{BB}}$ from a corrupted voter $\mathsf{V}_i \in \mathbb{V}_{\mathsf{corrupt}}$, uses the extracted $\{\overline{\mathtt{sk}}_j\}_{j\in[k]}$ to decrypt c_i to (V_i, s_i). The simulator $\mathcal{S}$ then acts as V_i to send $(\textsc{Cast}, \mathsf{sid}, s_i)$ to $\mathcal{F}_{\mathrm{SV}}$.
- In the tally phase:
  - The simulator $\mathcal{S}$ monitoring $\bar{\mathcal{G}}_{\mathrm{BB}}$; once a $\overline{\tau}_j, \pi_j^{(3)}$ is posted from a corrupted trustee $\mathsf{T}_j \in \mathbb{T}_{\mathsf{corrupt}}$, the simulator $\mathcal{S}$ acts as T_j to send $(\textsc{Tally}, \mathsf{sid})$ to $\mathcal{F}_{\mathrm{SV}}$.

Indistinguishability. The indistinguishability in this case is straightforward, as $\mathcal{S}$ never simulate a single message to either any corrupted parties or the external $\bar{\mathcal{G}}_{\text{BB}}$. The simulator $\mathcal{S}$ knows all the honest voters' ballot from the external $\mathcal{F}_{\text{SV}}$, it simply acts as the honest voters according to the protocol $\Pi_{\text{FHE-SV}}$. Meanwhile, it also extracts the ballot of the malicious voters by using the extracted trustees' secret keys. Hence, the simulator $\mathcal{S}$ can submit the extracted ballot to the external $\mathcal{F}_{\text{SV}}$ on the malicious voters' behave. Therefore, when NIZK extraction for trustees' secret keys are successful, the view of $\mathcal{Z}$ in the ideal execution has identical distribution to the view of $\mathcal{Z}$ in the real execution.

Case 3: $|\mathbb{V}_{\mathsf{corrupt}}| = n \ \wedge \ 0 \leq |\mathbb{T}_{\mathsf{corrupt}}| \leq k$.

Simulator. Trivial case. There is nothing needs to extract, as the trustees do not have input. The simulator $\mathcal{S}$ just run trustee according to protocol $\Pi_{\text{FHE-SV}}$.

Indistinguishability. The view of $\mathcal{Z}$ in the ideal execution has identical distribution to the view of of $\mathcal{Z}$ in the real execution.

□

A.2 Instantiation of TFHE via GSW

In this subsection, we present our construction TFHE. Assume there exist N players in our system, and each player has a $(\mathtt{pk}, \mathtt{sk})$ pair. Without lose of generality, for player i with $(\mathtt{pk}_i, \mathtt{sk}_i)$ generated from Keygen for $i \in [N]$.

- $\mathsf{param} \leftarrow \mathsf{TFHE.Setup}(1^\lambda)$: The algorithm takes as input the security parameter λ then outputs $\mathsf{param} := (n, m, q, \chi)$ as public parameter;
- $(\mathtt{pk}_i, \mathtt{sk}_i) \leftarrow \mathsf{TFHE.Keygen}(1^\lambda)$: The algorithm first samples a public matrix $\mathbf{B} \leftarrow \mathbb{Z}_q^{n \times m}$, a secret vector $\mathbf{s}_i \leftarrow \mathbb{Z}_q^{1 \times n}$, and an error vector $\mathbf{e}_i \leftarrow \chi^{1 \times m}$; Then, the algorithm computes $\mathbf{b}_i := \mathbf{s}_i \cdot \mathbf{B} + \mathbf{e}_i \pmod q \in \mathbb{Z}_q^{1 \times m}$; The algorithm constructs and broadcasts the public key

  $$\mathtt{pk} := \mathbf{A} = \left(\frac{\mathbf{B}}{\mathbf{b}}\right) \in \mathbb{Z}_q^{(n+1) \times m}$$

  and keeps secret key $\mathtt{sk} := \mathbf{t} := [-\mathbf{s}, 1] \in \mathbb{Z}_q^{1 \times (n+1)}$ privately; Observe that

  $$[-\mathbf{s}|1] \cdot \left(\frac{\mathbf{B}}{\mathbf{b}}\right) = \mathbf{e} \pmod q;$$

- $\mathtt{pk} := \mathsf{TFHE.CombinePK}(\mathtt{pk}_1, \ldots, \mathtt{pk}_N)$: It takes input as a set of public keys $(\mathtt{pk}_1, \ldots, \mathtt{pk}_N)$, and outputs a combined public key $\mathtt{pk}$. i.e., $\mathtt{pk} = \sum_{i=1}^N \mathtt{pk}_i$
- $c \leftarrow \mathsf{TFHE.Enc}(\mathtt{pk}, m)$: In this setting, each player uses the combine public key to encrypt his message m, in more detail:

  1. Most importantly, each player will broadcasts their public key $\mathtt{pk}_i$, and receives the other player's public keys, then generates a combine public key via $\mathsf{CombinePK}(\mathtt{pk}_1, \ldots, \mathtt{pk}_N)$:

     $$\mathtt{pk} := \sum_i^N \mathtt{pk}_i = \left(\frac{\mathbf{B}^*}{\mathbf{b}^*}\right) \in \mathbb{Z}_q^{(n+1) \times m}$$

     where $\mathbf{b}^* := \sum_i^N \mathbf{b}_i$ and $\mathbf{B}^* := \sum_i^N \mathbf{B}_i$;

2. Samples a random matrix $\mathbf{R}_i \leftarrow \{0,1\}^{m\times(n+1)\ell}$, then, computes and broadcasts

$$\mathbf{C} := \left(\frac{\mathbf{B}^*}{\mathbf{b}^*}\right) \cdot \mathbf{R}_i + m_i \cdot \mathbf{G} \pmod q \in \mathbb{Z}_q^{(n+1)\times(n+1)\ell}$$

Here, we stress that, in general encryption scheme, each player encrypts $\mathsf{msg} \in \{0,1\}$ under his public key $\mathtt{pk}$ by using $\mathbf{C}_i \leftarrow \mathsf{Enc}(\mathtt{pk}_i, m_i) := \mathtt{pk}_i\mathbf{R}_i + m_i\mathbf{G}$ where the $\mathtt{pk}_i$ is not the common public key.

- $c^* \leftarrow \mathsf{TFHE.Eval}(C, \mathtt{pk}, c_1, \cdots, c_N)$: **Homomorphic evaluation algorithm,** upon receiving all the encrypted data from other players, each player invokes the Eval algorithm (e.g., addition or multiplication) to generate the evaluation ciphertext. We stress that the Eval algorithm is the same as the evaluation algorithm of Gentry et al. [GSW13].

- $\mathtt{sk} \leftarrow \mathsf{CombineSK}(\mathtt{sk}_1, \ldots, \mathtt{sk}_N)$. It takes input as a set of secret key $(\mathtt{sk}_1, \ldots, \mathtt{sk}_k)$, and outputs combined secret key $\mathtt{sk}$. More concretely, each player uses the secret key share functional polynomial f_i to share the secret key share, e.g., $f(\mathbf{x}) = \mathtt{sk} + r_1\mathbf{x}^1 + r_2\mathbf{x}^2 + \cdots + r_N\mathbf{x}^N$ and sends each f_j to the player j for $j \in [N]$. Then, the player i re-constructs these shares to generate $\mathtt{sk}_i := \sum_{j\in S} f_j(i)$. For example, we parse $\mathtt{sk}_i$ into k pieces, $\mathtt{sk}_i := (f_i(1), \cdots, f_i(N))$, at the end of secret share, we set $\mathtt{sk}_i := (f_1(i), \cdots, f_N(i))$.

- $y \leftarrow \mathsf{TFHE.Dec}(c^*, \mathtt{sk}_1, \cdots, \mathtt{sk}_N)$:

  1. Upon receiving the shares from other players, the player i combines his shares of secret key by computing $\mathtt{sk}_i := \sum_{j\in S} f_j(i)$ and broadcasts $\mu_i := (\sum_{j\in S} \mathtt{sk}_j \cdot c^*) \cdot \mathbf{G}^{-1}(\mathbf{w}^T) + smdg_i$;
  2. Upon receiving all the partial messages $\{\mu_i\}_{i\in T}$, each player picks an arbitrary subset $T \subseteq S \subseteq [N]$ such that $|T| = [N/2]+1$. Then, they use the "*Lagrange interpolation*" polynomial to compute $result = \sum_{k\in T} \delta_k(0) \cdot \mu_k = \lfloor \frac{q}{2} \rfloor \cdot m + noise$ for $k \in T$;
  3. Finally, they output m.

- $\mu_i \leftarrow \mathsf{TFHE.ShareDec}(\mathtt{sk}_i, c^*)$. It takes the secret key of player i and the evaluated ciphertext as input. Upon receiving the shares from other players, the player i combines his shares of secret key by computing $\mathtt{sk}_i := \sum_{j\in S} f_j(i)$ and broadcasts the partial message $\mu_i := (\sum_{j\in S} \mathtt{sk}_j \cdot c^*) \cdot \mathbf{G}^{-1}(\mathbf{w}^T) + \mathsf{smdg}_i$;

- $m \leftarrow \mathsf{TFHE.ShareCombine}(c, \mu_1, \ldots, \mu_k)$. It takes input as a ciphertext c and k decryption shares $(\mu_1, \ldots, \mu_k)$, and outputs a plaintext m. More concretely, upon receiving all the partial messages $\{\mu_i\}_{i\in T}$, each player picks an arbitrary subset $T \subseteq S \subseteq [N]$ such that $|T| = [N/2] + 1$. Then, they use the "*Lagrange interpolation*" polynomial to compute $result = \sum_{k\in T} \delta_k(0) \cdot \mu_k = \lfloor \frac{q}{2} \rfloor \cdot m + noise$ for $k \in T$; Lastly, outputs m.

Theorem A.5. *The construction* TFHE *above is a secure publicly evaluable key-homomorphic threshold FHE under the LWE assumption.*

Proof. To prove the above theorem, we need to show

1). Correct Key Combination: Consider the combination of keys, it is easily seen that

$$\begin{aligned}\mathtt{pk}^* &= \mathtt{pk}_1 + \mathtt{pk}_2 + \cdots + \mathtt{pk}_N = \sum_{i=1}^{N} \mathbf{b}_i = \sum_{i=1}^{N} \left(\mathbf{s}_i \cdot \mathbf{B} + \mathbf{e}_i \pmod q \in \mathbb{Z}_q^{1\times m}\mathbf{s}_i\right) \cdot \mathbf{B} \\ &= (\sum_{i=1}^{N} \mathbf{s}_i)\mathbf{B} + (\sum_{i=1}^{N} \mathbf{e}_i) \pmod q.\end{aligned}$$

Obliviously, then $(\mathtt{pk}^*, \mathtt{sk}^*)$ are valid key tuples.

2). Ciphertext transformative indistinguishability: We note that, the PPT algorithm Trans takes input as the current ciphertext c under the set $\{\mathtt{sk}_i\}$ for $i \in [k]$, and outputs the transformed ciphertext $c' \approx c$. As mentioned earlier, in our setting, we obtain that, $\mathbf{C}' := \mathsf{Trans}(\mathbf{C}, \{\mathtt{sk}_i\}_{i\in[k]}) = \left(\frac{\mathbf{B}}{\sum_{i\in j}\mathbf{b}_i}\right) \cdot \bar{\mathbf{R}} + m \cdot \mathbf{G}$, where we recall the original ciphertext as follows $\mathbf{C} = \left(\frac{\mathbf{B}}{\mathbf{b}}\right) \cdot \mathbf{R} + m \cdot \mathbf{G}$ under the secret key $\mathtt{sk} := \mathbf{t} := [-\mathbf{s}, 1] \in \mathbb{Z}_q^{1\times(n+1)}$. Notably $\mathbf{b} = \mathbf{s} \cdot \mathbf{B} + \mathbf{e} \pmod q$. In order to prove the $\mathbf{C}'$ and $\mathbf{C}$ indistinguishability, we only consider $\left(\frac{\mathbf{B}}{\sum_{i\in j}\mathbf{b}_i}\right) \cdot \bar{\mathbf{R}}$ and $\left(\frac{\mathbf{B}}{\mathbf{b}}\right) \cdot \mathbf{R}$ indistinguishability. In a simple, the simulator can easily obtain the original ciphertext and the public keys which from the parties. Once the simulator obtain the randomness from one of the parties, he could create a matrix $\bar{\mathbf{R}} = \mathbf{R} + \mathbf{Y} \in \{0,1\}^{m\times(n+1)\ell}$ for $\mathbf{z} = \sum_{i\in[k]\setminus[j]}\mathbf{b}_i\mathbf{R} + (\sum_{i\in[k]\setminus[j]}\mathbf{s}_i \cdot \mathbf{B} + \sum_{i\in[k]\setminus[j]}\mathbf{e}_i)\mathbf{Y} \in \mathbb{Z}_q^{1\times m}$. Hence, they are identical and there is no PPT adversary can distinguish them.

3). Share-simulation indistinguishability:

We first define the $\mathsf{SimShareDec}(c, m, \{\mu_i\}_{i\in\mathcal{I}})$, then fix $j^* \in \bar{T} = [N] \setminus T$. Sample the partial message μ_j uniformly and let $\mu_{j^*} := \left(\frac{\mathbf{B}^*}{\mathbf{b}^*}\right) \cdot \mathbf{R}_i - \sum_{i\in N, i\neq j^*}\mu_i$ and output $\{\mu_j\}_{j\in\bar{T}}$. By correct share decryption, we know that regardless of how $\{\mu_j\}_{j\in\bar{T}}$ were created, $\mu_{j^*} := \left(\frac{\mathbf{B}^*}{\mathbf{b}^*}\right) \cdot \mathbf{R}_i - \sum_{i\in N, i\neq j^*}\mu_i$. Since this is a deterministic function of the rest of the variables, we simply need to prove that $\{\mu_j := (\mathtt{sk}_j^* \cdot c^*) \cdot \mathbf{G}^{-1}(\mathbf{w}^T) + \mathsf{smdg}_j\}_{\{j\in\bar{T}, j\neq j^*\}} \approx_c \{\mu_j \leftarrow \mathbb{Z}_q\}_{\{j\in\bar{T}, j\neq j^*\}}$. Obliviously, inspired by the security of TFHE, utilizing the leftover hash lemma and the LWE assumption, we can prove the above equation satisfy the property of computational indistinguishability.

□

A.3 Fully homomorphic encryption

A fully homomorphic encryption scheme FHE consists of a tuple of algorithms: (Setup, Keygen, Enc, Eval, Dec) as follows.

- $\mathsf{param} \leftarrow \mathsf{Setup}(1^\lambda)$. The algorithm Setup takes input as the security parameter λ, and outputs public parameters param. All the other algorithms implicitly take param as input.
- $(\mathtt{pk}, \mathtt{sk}) \leftarrow \mathsf{Keygen}(\mathsf{param})$. The algorithm Keygen takes input as the public parameter param, and outputs a public key pk, a secret key sk.
- $c \leftarrow \mathsf{Enc}(\mathtt{pk}, m)$. The algorithm Enc takes input as the public key pk and the message m, and outputs the ciphertext c.
- $c' := \mathsf{Eval}(\mathtt{pk}, \mathcal{F}, c_1, \ldots, c_n)$. The algorithm Eval takes input as the public (a.k.a., evaluation) key pk, the description of the evaluation function (circuit) $\mathcal{F}$, and a set of ciphertexts $c_1, \ldots, c_n$, and outputs the result ciphertext c'.
- $m \leftarrow \mathsf{Dec}(\mathtt{sk}, c)$. The algorithm Dec takes input as the secret key sk and a ciphertext c, and outputs the decrypted plaintext m.

Definition A.6. *We say* $\mathsf{FHE} = \{\mathsf{Setup}, \mathsf{Keygen}, \mathsf{Enc}, \mathsf{Eval}, \mathsf{Dec}\}$ *is a* secure *fully homomorphic encryption if the following properties hold:*

- Correctness: *The correctness properties are required as follows:*
  - *For any* λ, $m \in \{0,1\}^*$, *and* $(\mathtt{pk}, \mathtt{sk})$ *output by* $\mathsf{Keygen}(1^\lambda)$, *we have that*

$$m = \mathsf{Dec}\Big(\mathtt{sk}, \big(\mathsf{Enc}(\mathtt{pk}, m)\big)\Big);$$

  - *For any* λ, *any* $m_1, \cdots, m_l \in \{0,1\}^*$, *and* $C \in \mathcal{C}_\lambda$, *we have that*

$$\mathcal{C}(m_1, \cdots, m_\ell) \quad = \quad \mathsf{Dec}\Big(\mathtt{sk}, \big(\mathsf{Eval}\big(\mathtt{pk}, (C, \mathsf{Enc}(\mathtt{pk}, m_1), \cdots, \mathsf{Enc}(\mathtt{pk}, m_\ell))\big)\big)\Big).$$

- IND-CPA security: *We say that a* TFHE *scheme achieves* indistinguishability under plaintext attacks (IND-CPA) *if for any* PPT *adversary* $\mathcal{A}$ *the following advantage* AdvCPA *is negligible.*

$\underline{\textit{Experiment}^{\mathsf{CPA}}(1^\lambda)}$

1. *Run* $\mathsf{param} \leftarrow \mathsf{TFHE.Setup}(1^\lambda)$.
2. *Run* $(\mathtt{pk}, \mathtt{sk}) \leftarrow \mathsf{TFHE.Keygen}(\mathsf{param})$;
4. $\mathcal{A}(\mathtt{pk})$ *outputs* m_0, m_1 *of equal length;*
5. *Pick* $b \leftarrow \{0,1\}$; *Run* $c \leftarrow \mathsf{TFHE.Enc}(\mathtt{pk}, m_b)$;
6. $\mathcal{A}(c)$ *outputs* b^*; *It returns* 1 *if* $b = b^*$; *else, returns* 0.

We define the advantage of $\mathcal{A}$ *as*

$$\mathsf{AdvCPA}_{\mathcal{A}}(1^\lambda) = \left|\Pr[\textit{Experiment}^{\mathsf{CPA}}(1^\lambda) = 1] - \frac{1}{2}\right| \ .$$

A.4 Gentry-Sahai-Waters (GSW) construction

Let k be a security parameter and let L be the number of levels for the somewhat homomorphic scheme. We describe the algorithms that form the GSW scheme [GSW13]. The algorithm is originally defined in terms of the functions
$\mathsf{BitDecomp}, \mathsf{BitDecomp}^{-1}$ and $\mathsf{Flatten}$, but we tend to follow the formulation in [AP14, MW16] and so use the matrix $\mathbf{G}$.

- $\mathsf{GSW.Setup}(1^k, 1^L)$:
  1. Choose a modulus q of $\kappa = \kappa(k, L)$ bits, parameter $n = n(k, L) \in \mathbb{N}$, and error distribution $\chi = \chi(k, L)$ on $\mathcal{Z}$ so that the (n, q, m, χ)-LWE problem achieves at least 2^k security against known attacks.
     Choose a parameter $m = m(k, L) = O(n \log(q))$;
  2. Output: $\mathsf{param} = (n, q, m, \chi)$.
     We also use the notation $\ell = \lfloor \log(q) \rfloor + 1$ and $N = (n+1) \cdot \ell$.
- $\mathsf{GSW.Keygen}(\mathsf{param})$:
  1. Sample uniformly $\mathbf{t} = (t_1, \ldots, t_n)^T \leftarrow \mathbb{Z}_q^n$ and compute

$$\mathbf{s} \leftarrow (1, -\mathbf{t}^T)^T = (1, -t_1, \cdots, -t_n)^T \in \mathbb{Z}_q^{(n+1)\times 1};$$

  2. Generate a matrix $\mathbf{B} \leftarrow \mathbb{Z}_q^{m \times n}$ uniformly and a vector $\mathbf{e} \leftarrow \chi^m$;

3. Compute $\mathbf{b} = \mathbf{Bt} + \mathbf{e} \in \mathbb{Z}_q^m$ and construct the matrix $\mathbf{A} = (\mathbf{b}|\mathbf{B}) \in \mathbb{Z}_q^{m\times(n+1)}$ as the vector $\mathbf{b}$ followed by the n columns of $\mathbf{B}$.
   Observe that
$$\mathbf{As} = (\mathbf{b}|\mathbf{B})\mathbf{s} = (\mathbf{Bt} + \mathbf{e}|\mathbf{B}) \begin{pmatrix} 1 \\ -\mathbf{t} \end{pmatrix} = \mathbf{Bt} + \mathbf{e} - \mathbf{Bt} = \mathbf{e}.$$
4. Return $\mathtt{sk} \leftarrow \mathbf{s}$ and $\mathtt{pk} \leftarrow \mathbf{A}$.

- $\mathbf{C} \leftarrow \mathsf{GSW.Enc}(\mathsf{param}, \mathtt{pk}, \mu)$: In order to encrypt one-bit messages $\mu \in \{0, 1\}$:

1. Let $\mathbf{G}$ be the $(n + 1) \times N$ gadget matrix as above;
2. Sample uniformly a matrix $\mathbf{R} \leftarrow \{0, 1\}^{m\times N}$;
3. Compute $\mathbf{C} = \mu\mathbf{G} + \mathbf{A}^T\mathbf{R} \pmod q \in \mathbb{Z}_q^{(n+1)\times N}$.

- $\mu' \leftarrow \mathsf{GSW.Dec}(\mathsf{param}, \mathtt{sk}, \mathbf{C})$:

1. We have $\mathtt{sk} = \mathbf{s} \in \mathbb{Z}_q^{n+1}$;
2. Define a vector $\mathbf{w} = [\lceil q/2 \rceil | 0, \cdots, 0] \in \mathbb{Z}_q^{1\times(n+1)}$
3. Compute $v = \mathbf{s}^T\mathbf{G} \cdot \mathbf{G}^{-1}(\mathbf{w}^T) \in \mathbb{Z}_q$ and output $\mu = |\lfloor \frac{v}{q/2} \rceil|$ as the decrypted message. So if $|\mu| \leq q/4$ then return 0 otherwise return 1.

- $\mathsf{GSW.Eval}(\mathsf{param}, \mathbf{C}_1, \cdots, \mathbf{C}_\ell)$:

  - $\mathsf{GSW.Add}(\mathbf{C}_1, \mathbf{C}_2)$: output
$$\mathbf{C}_1 + \mathbf{C}_2 = (\mu_1 + \mu_2)\mathbf{G} + \mathbf{A}^T(\mathbf{R}_1 + \mathbf{R}_2) \in \mathbb{Z}_q^{(n+1)\times N};$$
  - $\mathsf{GSW.Mult}(\mathbf{C}_1, \mathbf{C}_2)$: Compute $\mathbf{G}^{-1}(\mathbf{C}_2) \in \{0, 1\}^{N\times N}$ and output $\mathbf{C}_1\mathbf{G}^{-1}(\mathbf{C}_2)$.
    Note that
$$\begin{aligned} \mathbf{C}_1\mathbf{G}^{-1}(\mathbf{C}_2) &= (\mu_1\mathbf{G} + \mathbf{A}^T\mathbf{R}_1)\mathbf{G}^{-1}(\mathbf{C}_2) \\ &= \mu_1\mathbf{C}_2 + \mathbf{A}^T\mathbf{R}_1\mathbf{G}^{-1}(\mathbf{C}_2) \\ &= \mu_1\mu_2\mathbf{G} + \mathbf{A}^T\mathbf{R}_1\mathbf{G}^{-1}(\mathbf{C}_2) + \mu_1\mathbf{A}^T\mathbf{R}_2 \\ &= \mu_1\mu_2\mathbf{G} + \mathbf{A}^T\big(\mathbf{R}_1\mathbf{G}^{-1}(\mathbf{C}_2) + \mu_1\mathbf{R}_2\big) \in \mathbb{Z}_q^{(n+1)\times N}. \end{aligned}$$

  One may also compute a homomorphic NAND gate by outputting $\mathbf{G} - \mathbf{C}_1\mathbf{G}^{-1}(\mathbf{C}_2)$.

Remark A.7. *Note that the formulation of the decryption algorithm in [MW16] is to choose an appropriate vector* $\mathbf{w}$ *and compute* $\mathbf{sCG}^{-1}(\mathbf{w}^T)$. *This is considerably less efficient than the original GSW decryption algorithm (both in terms of computation time and also the size of the error term). Hence we employ the original GSW decryption algorithm for our scheme.*

There is also a variant of the scheme that handles messages in $\mathbb{Z}_q$ when q is a power of two. We refer to [GSW13] for the details.

A.4.1 Security

A sketch proof is given in [GSW13] of the following theorem.

Theorem A.8. *Let (n, q, m, χ) be such that the $\mathsf{LWE}_{(n,q,m,\chi)}$ assumption holds and let $m = O(n \log(q))$. Then the GSW scheme is IND-CPA secure.*

The main step in the proof is showing that $(\mathbf{A}, \mathbf{RA})$ is computationally indistinguishable from uniform.

Definition A.9 ([GSW13, AP14, BV14, BP16]). *If the ciphertexts $\mathbf{C} = \mu\mathbf{G} + \mathbf{A} \cdot \mathbf{R}$, along with the secret key $\mathbf{s} = (-\mathbf{t}, 1)$, then the noise of $\mathbf{C}$ is the infinity norm of the noise vector: $\mathsf{noise}_{(\mathbf{s},\mu)}(\mathbf{C}) = \|\mathbf{C} - \mu\mathbf{G}\|_\infty$, i.e., $\mathsf{noise}_{(\mathbf{s},\mu)}(\mathbf{C}) = \|\mathbf{tA} \cdot \mathbf{R}\| = \|\mathbf{e} \cdot \mathbf{R}\| \leq mB \cdot m \leq E$.*

Lemma A.10 ([GSW13, AP14, BV14, BP16]). *For the ciphertexts $\mathbf{C} = \mu\mathbf{G} + \mathbf{A} \cdot \mathbf{R} \in \mathbb{Z}_q^{n \times m}$, along with the secret key $\mathbf{s} = (-\mathbf{t}, 1)$ and $\mathbf{G} \in \mathbb{Z}_q^{n \times m}$, then the noise in negation, addition and multiplication is bounded as follows:*

- ***Addition:*** *for all messages $\mu_1, \mu_2 \in \{0, 1\}$, it holds that $\mathsf{noise}_{(\mathbf{s},\mu_1+\mu_2)}(\mathbf{C}_1 + \mathbf{C}_2) \leq \mathsf{noise}_{(\mathbf{s},\mu_1)}(\mathbf{C}_1) + \mathsf{noise}_{(\mathbf{s},\mu_2)}(\mathbf{C}_2)$;*
- ***Multiplication:*** *for all messages $\mu_1, \mu_2 \in \{0, 1\}$, it holds that $\mathsf{noise}_{(\mathbf{s},\mu_1\mu_2)}(\mathbf{C}_1\mathbf{G}^{-1}(\mathbf{C}_2)) \leq \mu_1 \cdot \mathsf{noise}_{(\mathbf{s},\mu_2)}(\mathbf{C}_2) + m \cdot \mathsf{noise}_{(\mathbf{s},\mu_1)}(\mathbf{C}_1)$ for an efficiently computable function $\mathbf{G}^{-1} : \mathbb{Z}_q^n \to \mathbb{Z}_q^m$. i.e., $\mathsf{noise}_{(\mathbf{s},\mu_1\mu_2)}(\mathbf{C}_1\mathbf{G}^{-1}(\mathbf{C}_2)) \leq \|\mu_1 \cdot (\mathbf{e}_2\mathbf{R}_2) + (\mathbf{e}_1\mathbf{R}_1) \cdot \mathbf{G}^{-1}(\mathbf{C}_2)\|$.*
- ***Negation:*** *for all message $\mu \in \{0, 1\}$, it holds that $\mathsf{noise}_{(\mathbf{s},1-\mu)}(\mathbf{G} - \mathbf{C}) = \mathsf{noise}_{(\mathbf{s},\mu)}(\mathbf{C})$.*

A.5 LWE assumption

Definition A.11 ([Bra12] Def2.1). *A distribution ensemble $\chi = \chi(\lambda)$ over the integers is called B-bounded (denoted $|\chi| \leq B$) if there exists:*

$$\Pr_{x \xleftarrow{\$} \chi}[|x| \geq B] \leq 2^{-\tilde{\Omega}(n)}$$

Definition A.12 (LWE Distribution). *For the security parameter λ, let $n = n(\lambda)$ and $m = m(\lambda)$ be integers, let $\chi = \chi(\lambda)$ be error distribution over $\mathcal{Z}$ bounded by $B = B(\lambda)$, and let $q = q(\lambda) \geq 2$ be an integer modulus for any polynomial $p = p(\lambda)$ such that $q \geq 2^p \cdot B$. Then, sample a vector $\mathbf{s} \in \mathbb{Z}_q^{n \times 1}$ called the secret, the LWE distribution $\mathcal{A}_{\mathbf{s},\chi}$ over $\mathbb{Z}_q^n \times \mathbb{Z}_q$ is sampled by choosing $\mathbf{A} \in \mathbb{Z}_q^{m \times n}$ uniformly at random, choosing $\mathbf{e} \leftarrow \chi^{m \times 1}$, and outputting $\big(\mathbf{A}, \mathbf{b} = \mathbf{A} \cdot \mathbf{s} + \mathbf{e} \pmod q\big)$.*

We define the decisional version as follows,

Definition A.13 (Decision-$\mathsf{LWE}_{n,q,\chi,m}$). *Assume given an independent sample $(\mathbf{A}, \mathbf{b}) \in \mathbb{Z}_q^{m \times n} \times \mathbb{Z}_q^{m \times 1}$, where the sample is distributed according to either: (1) $\mathcal{A}_{\mathbf{s},\chi}$ for a uniformly random $\mathbf{s} \in \mathbb{Z}_q^n$ (i.e., $\{(\mathbf{A}, \mathbf{b}) : \mathbf{A} \leftarrow \mathbb{Z}_q^{m \times n}, \mathbf{s} \leftarrow \mathbb{Z}_q^{n \times 1}, \mathbf{e} \leftarrow \chi^{m \times 1}, \mathbf{b} = \mathbf{A} \cdot \mathbf{s} + \mathbf{e} \pmod q\}$), or (2) the uniform distribution (i.e., $\{(\mathbf{A}, \mathbf{b}) : \mathbf{A} \leftarrow \mathbb{Z}_q^{m \times n}, \mathbf{b} \leftarrow \mathbb{Z}_q^{m \times 1}\}$). Then, the above two distributions are computationally indistinguishable.*

Remark A.14. *Regev and others [Reg05, Pei09, PW08, MW16] show that reductions between the LWE assumption and approximating the shortest vector problem in lattices (for appropriate parameters). We omit the corollary of these schemes' results. More details will be find [Reg05, Pei09, PW08, MW16].*

Lemma A.15 (Smudging Lemma). *Let $B_1 = B_1(\lambda)$, and $B_2 = B_2(\lambda)$ be positive integers and let $v^{\mathsf{sm}} \in [-B_1, B_1]$ be a fixed integer. Let $v_2^{\mathsf{sm}} \leftarrow [-B_2, B_2]$ be chosen uniformly at random. Then the distribution of v_2^{sm} is statistically indistinguishable from that of $v_2^{\mathsf{sm}} + v_1^{\mathsf{sm}}$ as long as $B_1/B_2 = \mathsf{negl}(\lambda)$.*

Lemma A.16 ([MP12]). *For any $N \geq m\lceil \log q \rceil$ there exists a computable gadget matrix $\mathbf{G} \in \mathbb{Z}_q^{m \times N}$ and an efficiently computable deterministic inverse (a.k.a., "short preimage") function $\mathbf{G}^{-1}(\cdot)$. The inverse function $\mathbf{G}^{-1}(\mathbf{M})$ takes as input a matrix $\mathbf{M} \in \mathbb{Z}_q^{m \times m'}$ for any m' and outputs a matrix $\mathbf{G}^{-1}(\mathbf{M}) \in \{0, 1\}^{N \times m'}$ such that $\mathbf{G}\mathbf{G}^{-1}(\mathbf{M}) = \mathbf{M}$.*

B Supplementary material for Section 5

B.1 Proof for Theorem 5.2

Proof. To prove the theorem, we construct a simulator $\mathcal{S}$ such that no non-uniform PPT environment $\mathcal{Z}$ can distinguish between (i) the real execution $\mathsf{EXEC}^{\bar{\mathcal{G}}_{\mathrm{BB}},\widehat{\mathcal{F}}_{\mathrm{CERT}},\mathcal{F}^{\mathcal{C}}_{\mathrm{MPC}}}_{\Pi_{\text{MPC-SV}},\mathcal{A},\mathcal{Z}}$ where the parties $\mathbb{V} := \{\mathsf{V}_1,\ldots,\mathsf{V}_n\}$ and $\mathbb{T} := \{\mathsf{T}_1,\ldots,\mathsf{T}_k\}$ run protocol $\Pi_{\text{MPC-SV}}$ in the $\{\bar{\mathcal{G}}_{\mathrm{BB}},\widehat{\mathcal{F}}_{\mathrm{CERT}},\mathcal{F}^{\mathcal{C}}_{\mathrm{MPC}}\}$-hybrid world and the corrupted parties are controlled by a dummy adversary $\mathcal{A}$ who simply forwards messages from/to $\mathcal{Z}$, and (ii) the ideal execution $\mathsf{EXEC}^{\bar{\mathcal{G}}_{\mathrm{BB}}}_{\mathcal{F}_{\mathrm{SV}},\mathcal{S},\mathcal{Z}}$ where the parties interact with functionality $\mathcal{F}_{\mathrm{SV}}$ in the $\bar{\mathcal{G}}_{\mathrm{BB}}$-hybrid model and corrupted parties are controlled by the simulator $\mathcal{S}$. Let $\mathbb{V}_{\mathsf{corrupt}} \subseteq \mathbb{V}$ and $\mathbb{T}_{\mathsf{corrupt}} \subseteq \mathbb{T}$ be the set of corrupted voters and trustees, respectively. We consider following cases.

Case 1: $0 \leq |\mathbb{V}_{\mathsf{corrupt}}| < n \ \wedge \ 0 \leq |\mathbb{T}_{\mathsf{corrupt}}| < k$.

Simulator. The simulator $\mathcal{S}$ internally runs $\mathcal{A}$, forwarding messages to/from the environment $\mathcal{Z}$. The simulator $\mathcal{S}$ simulates honest voters $\mathsf{V}_i \in \mathbb{V} \setminus \mathbb{V}_{\mathsf{corrupt}}$, honest trustees $\mathsf{T}_j \in \mathbb{T} \setminus \mathbb{T}_{\mathsf{corrupt}}$ and functionalities $\widehat{\mathcal{F}}_{\mathrm{CERT}}$ and $\mathcal{F}^{\mathcal{C}}_{\mathrm{MPC}}$. In addition, the simulator $\mathcal{S}$ simulates the following interactions with $\mathcal{A}$.

- In the preparation phase:
  - Upon receiving $(\textsc{InitialTrusteeNotify}, \mathsf{sid}, \mathsf{T}_j)$ from the external $\mathcal{F}_{\mathrm{SV}}$ for an honest trustee $\mathsf{T}_j \in \mathbb{T} \setminus \mathbb{T}_{\mathsf{corrupt}}$, the simulator $\mathcal{S}$ acts as T_j, following the protocol $\Pi_{\text{MPC-SV}}$ as if T_j receives $(\textsc{InitialTrustee}, \mathsf{sid})$ from $\mathcal{Z}$.
  - Monitoring $\bar{\mathcal{G}}_{\mathrm{BB}}$, when a valid $(\overline{\mathtt{pk}}_j, \pi_j^{(1)})$ is posted on $\bar{\mathcal{G}}_{\mathrm{BB}}$ from a corrupted trustee $\mathsf{T}_j \in \mathbb{T}_{\mathsf{corrupt}}$, use $\mathsf{NIZK}_{\mathcal{R}_{11}}.\mathsf{Ext}^{\mathrm{RO}}(\overline{\mathtt{pk}}_j, \pi_j^{(1)})$ to extract the corresponding secret key $\overline{\mathtt{sk}}_j$ and random coin α_j.
- In the ballot casting phase:
  - Upon receiving $(\textsc{CastNotify}, \mathsf{sid}, \mathsf{V}_i)$ from the external $\mathcal{F}_{\mathrm{SV}}$ for an honest voter $\mathsf{V}_i \in \mathbb{V} \setminus \mathbb{V}_{\mathsf{corrupt}}$, the simulator $\mathcal{S}$ creates $c_i \leftarrow \mathsf{TE.Enc}(\mathtt{pk}, 0)$. It then uses $\mathsf{NIZK}_{\mathcal{R}_{12}}.\mathsf{Sim}$ to simulate the corresponding proofs $\pi_i^{(2)}$. The simulator $\mathcal{S}$ then follows the protocol to post $(c_i, \pi_i^{(2)})$ to $\bar{\mathcal{G}}_{\mathrm{BB}}$.
  - Monitoring $\bar{\mathcal{G}}_{\mathrm{BB}}$, when a valid $(c_i, \pi_i^{(2)})$ is posted on $\bar{\mathcal{G}}_{\mathrm{BB}}$ from a corrupted voter $\mathsf{V}_i \in \mathbb{V}_{\mathsf{corrupt}}$, uses the extracted $\{\overline{\mathtt{sk}}_j\}_{j\in[k]}$ to decrypt c_i to (V_i, s_i). The simulator $\mathcal{S}$ then acts as V_i to send $(\textsc{Cast}, \mathsf{sid}, s_i)$ to $\mathcal{F}_{\mathrm{SV}}$.
- In the tally phase:
  - Upon receiving $(\textsc{Leak}, \mathsf{sid}, \tau)$ from the external $\mathcal{F}_{\mathrm{SV}}$, the simulator $\mathcal{S}$ acts as the simulated $\mathcal{F}^{\mathcal{C}}_{\mathrm{MPC}}$ to send τ to each of the trustees $\mathsf{T}_j \in \mathbb{T}$. For any honest trustee $\mathsf{T}_j \in \mathbb{T} \setminus \mathbb{T}_{\mathsf{corrupt}}$, the simulator $\mathcal{S}$ acts as T_j to post τ on the $\bar{\mathcal{G}}_{\mathrm{BB}}$.

Indistinguishability. The indistinguishability is proven through a series of hybrid worlds $\mathcal{H}_0,\ldots,\mathcal{H}_3$.

Hybrid $\mathcal{H}_0$: It is the real protocol execution $\mathsf{EXEC}^{\bar{\mathcal{G}}_{\mathrm{BB}},\widehat{\mathcal{F}}_{\mathrm{CERT}},\mathcal{F}^{\mathcal{C}}_{\mathrm{MPC}}}_{\Pi_{\text{MPC-SV}},\mathcal{A},\mathcal{Z}}$.

Hybrid $\mathcal{H}_1$: $\mathcal{H}_1$ is the same as $\mathcal{H}_0$ except that $\mathcal{H}_1$ runs $\mathsf{NIZK}_{\mathcal{R}_{11}}.\mathsf{Ext}^{\mathrm{RO}}(\overline{\mathtt{pk}}_j, \pi_j^{(1)})$ to extract the corrupted trustee's secret key $\overline{\mathtt{sk}}_j$. $\mathcal{H}_1$ halt if the extraction fails.

Claim B.1. *$\mathcal{H}_1$ and $\mathcal{H}_0$ are indistinguishable.*

Proof. According to Def. 2.3, the probability $\mathsf{NIZK}_{\mathcal{R}_{11}}.\mathsf{Ext}^{\mathrm{RO}}$ extraction fails (a.k.a. knowledge error) is negligible, so the probability that any adversary $\mathcal{A}$ and the environment $\mathcal{Z}$ can distinguish $\mathcal{H}_1$ from $\mathcal{H}_0$ is $\mathsf{negl}(\lambda)$. □

Hybrid $\mathcal{H}_2$: $\mathcal{H}_2$ is the same as $\mathcal{H}_1$ except the followings. During the vote phase, $\mathcal{H}_3$ uses $\mathsf{NIZK}_{\mathcal{R}_{12}}.\mathsf{Sim}$ to simulate $\pi_i^{(2)}$ for all the honest voter $\mathsf{V}_i \in \mathbb{V}$.

Claim B.2. *$\mathcal{H}_2$ and $\mathcal{H}_1$ are indistinguishable.*

Proof. The advantage of the adversary is bounded by the ZK property of $\mathsf{NIZK}_{\mathcal{R}_{12}}$ as defined by Def. 2.2. □

Hybrid $\mathcal{H}_3$: $\mathcal{H}_3$ is the same as $\mathcal{H}_2$ except the followings. During the vote phase, the simulator posts $c_i \leftarrow \mathsf{TE.Enc}(\mathtt{pk}, 0)$ for all the honest voter $\mathsf{V}_i \in \mathbb{V}$.

Claim B.3. *$\mathcal{H}_3$ and $\mathcal{H}_2$ are indistinguishable.*

Proof. The probability that any adversary $\mathcal{A}$ can distinguish $\mathcal{H}_4$ from $\mathcal{H}_3$ is bounded by $\mathsf{AdvCPA}_{\mathcal{A}}(1^\lambda)$ and ciphertext transformative indistinguishability. More specifically, we now show the if there exists an adversary $\mathcal{A}$ who can distinguish $\mathcal{H}_4$ from $\mathcal{H}_3$, then we can construction an adversary $\mathcal{B}$ that can break the IND-CPA game of the underlying TE by reduction. During the IND-CPA game, $\mathcal{B}$ receives a public key $\mathtt{pk}^*$ from the challenger. There must be at least one honest trustee in this case, and with our loss of generality, assume T_x is honest. During the preparation phase, $\mathcal{B}$ posts $\mathtt{pk}^*$ as T_x's public key together with simulated proof. During the ballot casting phase, for each honest voter V_i, $i \in [n]$, $\mathcal{B}$ sends $m_0 := 0$ and $m_1 := s_i$ to the IND-CPA challenger, and receives c^*. $\mathcal{B}$ then computes $c' \leftarrow \mathsf{TE.Trans}(c^*, \{\mathtt{sk}_i\}_{i\in[k]\setminus\{x\}})$. It posts c' as the honest voter's encrypted ballot. It is easy to see that, when c^* encrypts m_0, the adversary's view is indistinguishable from $\mathcal{H}_3$; when c^* encrypts m_1, the adversary's view is indistinguishable from $\mathcal{H}_2$. Hence, if $\mathcal{A}$ can distinguish $\mathcal{H}_3$ from $\mathcal{H}_2$ with non-negligible probability, then $\mathcal{B}$ can break the IND-CPA game with the same probability. □

The adversary's view of $\mathcal{H}_3$ is identical to the simulated view $\mathsf{EXEC}^{\bar{\mathcal{G}}_{\mathrm{BB}}}_{\mathcal{F}_{\mathrm{SV}},\mathcal{S},\mathcal{Z}}$. Therefore, no PPT $\mathcal{Z}$ can distinguish the view of the ideal execution from the view of the real execution with more than negligible probability.

Case 2: $0 \le |\mathbb{V}_{\mathsf{corrupt}}| < n \ \wedge \ |\mathbb{T}_{\mathsf{corrupt}}| = k$.

Simulator. Similar as Case 1, the simulator $\mathcal{S}$ internally runs $\mathcal{A}$, forwarding messages to/from the environment $\mathcal{Z}$. The simulator $\mathcal{S}$ simulates honest voters $\mathsf{V}_i \in \mathbb{V} \setminus \mathbb{V}_{\mathsf{corrupt}}$, honest trustees $\mathsf{T}_j \in \mathbb{T} \setminus \mathbb{T}_{\mathsf{corrupt}}$ and functionalities $\widehat{\mathcal{F}}_{\textsc{Cert}}$. In addition, the simulator $\mathcal{S}$ simulates the following interactions with $\mathcal{A}$.

- In the preparation phase:
  - Monitoring $\bar{\mathcal{G}}_{\mathrm{BB}}$, when a valid $(\overline{\mathtt{pk}}_j, \pi_j^{(1)})$ is posted on $\bar{\mathcal{G}}_{\mathrm{BB}}$ from a corrupted trustee $\mathsf{T}_j \in \mathbb{T}_{\mathsf{corrupt}}$, use $\mathsf{NIZK}_{\mathcal{R}_{11}}.\mathsf{Ext}^{\mathrm{RO}}(\overline{\mathtt{pk}}_j, \pi_j^{(1)})$ to extract the corresponding secret key $\overline{\mathtt{sk}}_j$.
- In the ballot casting phase:
  - Upon receiving $(\textsc{Leak}, \mathsf{sid}, \mathsf{V}_i, s_i)$ from the external $\mathcal{F}_{\mathrm{SV}}$ for an honest voter $\mathsf{V}_i \in \mathbb{V} \setminus \mathbb{V}_{\mathsf{corrupt}}$, the simulator $\mathcal{S}$ acts as V_i, following the protocol $\Pi_{\text{MPC-SV}}$ as if V_i receives $(\textsc{Cast}, \mathsf{sid}, s_i)$ from $\mathcal{Z}$.
  - Monitoring $\bar{\mathcal{G}}_{\mathrm{BB}}$, when a valid $(c_i, \pi_i^{(2)})$ is posted on $\bar{\mathcal{G}}_{\mathrm{BB}}$ from a corrupted voter $\mathsf{V}_i \in \mathbb{V}_{\mathsf{corrupt}}$, uses the extracted $\{\overline{\mathtt{sk}}_j\}_{j\in[k]}$ to decrypt c_i to (V_i, s_i). The simulator $\mathcal{S}$ then acts as V_i to send $(\textsc{Cast}, \mathsf{sid}, s_i)$ to $\mathcal{F}_{\mathrm{SV}}$.
- In the tally phase:
  - Once the simulated $\mathcal{F}^{\mathcal{C}}_{\mathrm{MPC}}$ receives $(\textsc{Input}, \mathsf{sid}, \alpha_j, \{\mathtt{pk}_\ell\}_{\ell\in[k]}, \{c_i\}_{i\in[n]})$ from a corrupted trustee $\mathsf{T}_j \in \mathbb{T}_{\mathsf{corrupt}}$, the simulator $\mathcal{S}$ acts as T_j to send $(\textsc{Tally}, \mathsf{sid})$ to $\mathcal{F}_{\mathrm{SV}}$.

Indistinguishability. The indistinguishability in this case is straightforward, as $\mathcal{S}$ never simulate a single message to either any corrupted parties or the external $\bar{\mathcal{G}}_{\text{BB}}$. The simulator $\mathcal{S}$ knows all the honest voters' ballot from the external $\mathcal{F}_{\text{SV}}$, it simply acts as the honest voters according to the protocol $\Pi_{\text{MPC-SV}}$. Meanwhile, it also extracts the ballot of the malicious voters by using the extracted trustees' secret keys. Hence, the simulator $\mathcal{S}$ can submit the extracted ballot to the external $\mathcal{F}_{\text{SV}}$ on the malicious voters' behave. Therefore, when NIZK extraction for trustees' secret keys are successful, the view of $\mathcal{Z}$ in the ideal execution has identical distribution to the view of $\mathcal{Z}$ in the real execution.

Case 3: $|\mathbb{V}_{\mathsf{corrupt}}| = n \ \wedge \ 0 \leq |\mathbb{T}_{\mathsf{corrupt}}| \leq k$.

Simulator. Trivial case. There is nothing needs to extract, as the trustees do not have input. The simulator $\mathcal{S}$ just run trustee according to protocol $\Pi_{\text{MPC-SV}}$.

Indistinguishability. The view of $\mathcal{Z}$ in the ideal execution has identical distribution to the view of of $\mathcal{Z}$ in the real execution.

□

C Supplementary material for Section 6

C.1 Proof for Theorem 6.2

Functionality $\mathcal{F}_{\text{SV}}$

The functionality $\mathcal{F}_{\text{SV}}$ interacts with a set of voters $\mathbb{V} := \{\mathsf{V}_1, \ldots, \mathsf{V}_n\}$, a set of trustees $\mathbb{T} := \{\mathsf{T}_1, \ldots, \mathsf{T}_k\}$, and the adversary $\mathcal{S}$. Let $\mathbb{V}_{\mathsf{honest}}$, $\mathbb{V}_{\mathsf{corrupt}}$ and $\mathbb{T}_{\mathsf{honest}}$, $\mathbb{T}_{\mathsf{corrupt}}$ denote the set of honest/corrupt voters and trustees, respectively.
Functionality $\mathcal{F}_{\text{SV}}$ is parameterized by an algorithm TallyProcess (see Figure 4), a working table $\mathbb{W}$, and variables $result$, T_1, T_2, and B_i for all $i \in [n]$.

Initially, set $result := \emptyset$, $T_1 := \emptyset$, $T_2 := \emptyset$; for $i \in [n]$, set $B_i := \emptyset$.
Table $\mathbb{W}$ consists of n entries, and each entry consists of voter's real ID, voter's alternative ID, and the statement that the voter submitted; for all $i \in [n]$, the ith entry $\mathbb{W}[i] := (\mathsf{V}_i, w_i, statement_i)$, where $w_i \leftarrow \{0,1\}^\lambda$, $statement_i := \emptyset$.

Preparation:

1. Upon receiving input $(\textsc{InitialTrustee}, \mathsf{sid})$ from the trustee $\mathsf{T}_j \in \mathbb{T}$, set $T_1 := T_1 \cup \{\mathsf{T}_j\}$, and send a notification message $(\textsc{InitialTrusteeNotify}, \mathsf{sid}, \mathsf{T}_j)$ to the adversary $\mathcal{S}$.

Ballot Casting:

1. Upon receiving input $(\textsc{Cast}, \mathsf{sid}, (s_i, w_i^*))$ from the voter $\mathsf{V}_i \in \mathbb{V}$, if $|T_1| < k$, ignore the input.
   Otherwise,
   - if V_i is honest (now $w_i^* := \perp$), then update $\mathbb{W}[i] := (\mathsf{V}_i, w_i, s_i)$; send a message $(\textsc{CastNotify}, \mathsf{sid}, \mathsf{V}_i)$ to the adversary $\mathcal{S}$.
   - if V_i is corrupt, then update $\mathbb{W}[i] := (\mathsf{V}_i, w_i^*, s_i)$.

   If $|\mathbb{T}_{\mathsf{corrupt}}| = k$, then additionally send a message $(\textsc{Leak}, \mathsf{sid}, \mathbb{W}[i])$ to the adversary $\mathcal{S}$.

Tally:

1. Upon receiving input $(\textsc{Tally}, \mathsf{sid})$ from the trustee $\mathsf{T}_j \in \mathbb{T}$, set $T_2 := T_2 \cup \{\mathsf{T}_j\}$ and do the following:
   - set $\mathbb{U} := \mathbb{W}$; then eliminate all V_i's in $\mathbb{U}$; finally sort the entries in $\mathbb{U}$ lexicographically.

   Send a notification message $(\textsc{TallyNotify}, \mathsf{sid}, \mathsf{T}_j)$ to $\mathcal{S}$.
   If $|T_2 \cap \mathbb{T}_{\mathsf{honest}}| + |\mathbb{T}_{\mathsf{corrupt}}| = k$, send a leakage message $(\textsc{Leak}, \mathsf{sid}, \mathbb{U})$ to $\mathcal{S}$.
   If $|T_2| = k$, compute $result \leftarrow \mathsf{TallyProcess}(\mathbb{U})$.
2. Upon receiving input $(\textsc{ReadResult}, \mathsf{sid})$ from a voter $\mathsf{V}_i \in \mathbb{V}$, if $result = \emptyset$, ignore the input.
   Else, return $(\textsc{ResultReturn}, \mathsf{sid}, result)$ to V_i.

Figure 17: The voting functionality.

Proof. To prove the theorem, we construct a simulator $\mathcal{S}$ such that no non-uniform PPT environment $\mathcal{Z}$ can distinguish between (i) the real execution $\mathsf{EXEC}^{\bar{\mathcal{G}}_{\text{BB}}, \widehat{\mathcal{F}}_{\text{Cert}}}_{\Pi_{\text{MIX-SV}}, \mathcal{A}, \mathcal{Z}}$ where the parties $\mathbb{V} := \{\mathsf{V}_1, \ldots, \mathsf{V}_n\}$ and $\mathbb{T} := \{\mathsf{T}_1, \ldots, \mathsf{T}_k\}$ run protocol $\Pi_{\text{MIX-SV}}$ in the $\{\bar{\mathcal{G}}_{\text{BB}}, \widehat{\mathcal{F}}_{\text{Cert}}\}$-hybrid world and the corrupted parties are controlled by a dummy adversary $\mathcal{A}$ who simply forwards messages from/to $\mathcal{Z}$, and (ii) the ideal execution $\mathsf{EXEC}^{\bar{\mathcal{G}}_{\text{BB}}}_{\mathcal{F}_{\text{SV}}, \mathcal{S}, \mathcal{Z}}$ where the parties interact with functionality $\mathcal{F}_{\text{SV}}$ in the $\bar{\mathcal{G}}_{\text{BB}}$-hybrid model and corrupted parties are controlled by the simulator $\mathcal{S}$. Let $\mathbb{V}_{\mathsf{corrupt}} \subseteq \mathbb{V}$ and $\mathbb{T}_{\mathsf{corrupt}} \subseteq \mathbb{T}$ be the set of corrupted voters and trustees, respectively. We consider following cases.

Case 1: $0 \leq |\mathbb{V}_{\mathsf{corrupt}}| < n \ \wedge \ 0 \leq |\mathbb{T}_{\mathsf{corrupt}}| < k$.

Simulator. The simulator $\mathcal{S}$ internally runs $\mathcal{A}$, forwarding messages to/from the environment $\mathcal{Z}$. The simulator $\mathcal{S}$ simulates honest voters $\mathsf{V}_i \in \mathbb{V} \setminus \mathbb{V}_{\mathsf{corrupt}}$, honest trustees $\mathsf{T}_j \in \mathbb{T} \setminus \mathbb{T}_{\mathsf{corrupt}}$ and functionalities $\widehat{\mathcal{F}}_{\text{CERT}}$. In addition, the simulator $\mathcal{S}$ simulates the following interactions with $\mathcal{A}$.

- In the preparation phase:
  - Upon receiving $(\text{INITIALTRUSTEENOTIFY}, \mathsf{sid}, \mathsf{T}_j)$ from the external $\mathcal{F}_{\text{SV}}$ for an honest trustee $\mathsf{T}_j \in \mathbb{T} \setminus \mathbb{T}_{\mathsf{corrupt}}$, the simulator $\mathcal{S}$ acts as T_j, following the protocol $\Pi_{\text{MIX-SV}}$ as if T_j receives $(\text{INITIALTRUSTEE}, \mathsf{sid})$ from $\mathcal{Z}$.
  - Monitoring $\bar{\mathcal{G}}_{\text{BB}}$, when a valid $(\overline{\mathtt{pk}}_j, \pi_j^{(1)})$ is posted on $\bar{\mathcal{G}}_{\text{BB}}$ from a corrupted trustee $\mathsf{T}_j \in \mathbb{T}_{\mathsf{corrupt}}$, use $\mathsf{NIZK}_{\mathcal{R}_4}.\mathsf{Ext}^{\text{RO}}(\overline{\mathtt{pk}}_j, \pi_j^{(1)})$ to extract the corresponding secret key $\overline{\mathtt{sk}}_j$.
- In the ballot casting phase:
  - Upon receiving $(\text{CASTNOTIFY}, \mathsf{sid}, \mathsf{V}_i)$ from the external $\mathcal{F}_{\text{SV}}$ for an honest voter $\mathsf{V}_i \in \mathbb{V} \setminus \mathbb{V}_{\mathsf{corrupt}}$, the simulator $\mathcal{S}$ acts as V_i, following the protocol $\Pi_{\text{MIX-SV}}$ round 1 description as if V_i receives $(\text{CAST}, \mathsf{sid}, (\cdot, \perp))$ from $\mathcal{Z}$. In round 2, the simulator $\mathcal{S}$ creates $U_{i,\ell} \leftarrow \mathsf{TRE.Enc}(\mathtt{pk}, 0)$, $\ell \in [\lambda_1]$ and $S_i \leftarrow \mathsf{TRE.Enc}(\mathtt{pk}, 0)$. It then simulates the corresponding proofs $\pi_{i,\ell}^{(3)}$ and $\pi_i^{(4)}$. The simulator $\mathcal{S}$ then follows the protocol to post $(U_{i,\ell}, \pi_{i,\ell}^{(3)})_{\ell=1}^{\lambda_1}, S_i, \pi_i^{(4)})$ to $\bar{\mathcal{G}}_{\text{BB}}$.
  - The simulator $\mathcal{S}$ monitoring $\bar{\mathcal{G}}_{\text{BB}}$; once a $(W_i, \pi_i^{(2)})$ is posted from a corrupted voter $\mathsf{V}_i \in \mathbb{V}_{\mathsf{corrupt}}$, the simulator $\mathcal{S}$ uses the extracted $\{\overline{\mathtt{sk}}_j\}_{j\in[k]}$ to decrypt W_i to the temporal ID w_i. Record (V_i, w_i). When a valid $(U_{i,\ell}, \pi_{i,\ell}^{(3)})_{\ell=1}^{\lambda_1}, S_i, \pi_i^{(4)})$ is posted on $\bar{\mathcal{G}}_{\text{BB}}$ from a corrupted voter $\mathsf{V}_i \in \mathbb{V}_{\mathsf{corrupt}}$, uses the extracted $\{\overline{\mathtt{sk}}_j\}_{j\in[k]}$ to decrypt $U_{i,\ell}$ to $w_{i,\ell}$ and S_i to s_i. Replace the ID references in s_i to their actuarial voter ID's, and denoted the modified statement as s_i'. Record (V_i, s_i').
  - Upon receiving any $(\text{TALLYNOTIFY}, \mathsf{sid}, \mathsf{T}_j)$ from the external $\mathcal{F}_{\text{SV}}$ for an honest trustee $\mathsf{T}_j \in \mathbb{T} \setminus \mathbb{T}_{\mathsf{corrupt}}$ or any corrupted trustee has moved to the tally phase, the simulator $\mathcal{S}$ acts as each of the corrupted voters $\mathsf{V}_i \in \mathbb{V}_{\mathsf{corrupt}}$ to send $(\text{CAST}, \mathsf{sid}, (s_i', w_i))$ to $\mathcal{F}_{\text{SV}}$ if both (V_i, w_i) and (V_i, s_i') is recorded; otheriwse, it acts as V_i to send $(\text{CAST}, \mathsf{sid}, (s_i', \perp))$ to $\mathcal{F}_{\text{SV}}$ if only (V_i, s_i') is recorded.
- In the tally phase:
  - Upon receiving $(\text{TALLYNOTIFY}, \mathsf{sid}, \mathsf{T}_j)$ from the external $\mathcal{F}_{\text{SV}}$ for an honest trustee $\mathsf{T}_j \in \mathbb{T} \setminus \mathbb{T}_{\mathsf{corrupt}}$, if $\{\overline{m}_{i,\ell}^{(j)}\}_{i\in[n'],\ell\in[\lambda_1+2]}$ is not defined, the simulator $\mathcal{S}$ acts as T_j, following the protocol $\Pi_{\text{MIX-SV}}$ as if T_j receives $(\text{TALLY}, \mathsf{sid})$ from $\mathcal{Z}$. $\mathcal{S}$ then adds j to $\mathcal{J}$, where $\mathcal{J}$ is initially empty. If $\{\overline{m}_{i,\ell}^{(j)}\}_{i\in[n'],\ell\in[\lambda_1+2]}$ is defined, $\mathcal{S}$ uses $\mathsf{NIZK}_{\mathcal{R}_8}.\mathsf{Sim}$ to simulate the corresponding proof $\pi_{i,j,\ell}^{(6)}$. It then follows the protocol to post $(\overline{m}_{i,\ell}^{(j)}, \pi_{i,j,\ell}^{(6)})_{i\in[n'],\ell\in[\lambda_1+2]}$ on the $\bar{\mathcal{G}}_{\text{BB}}$.
  - The simulator $\mathcal{S}$ monitoring $\bar{\mathcal{G}}_{\text{BB}}$; once $(\overline{m}_{i,\ell}^{(j)}, \pi_{i,j,\ell}^{(6)})_{i\in[n'],\ell\in[\lambda_1+2]}$ is posted from a corrupted trustee $\mathsf{T}_j \in \mathbb{T}_{\mathsf{corrupt}}$, the simulator $\mathcal{S}$ acts as T_j to send $(\text{TALLY}, \mathsf{sid})$ to $\mathcal{F}_{\text{SV}}$.
  - Upon receiving $(\text{LEAK}, \mathsf{sid}, (\tilde{B}_1, \ldots, \tilde{B}_n))$ from the external $\mathcal{F}_{\text{SV}}$, the simulator $\mathcal{S}$ uses the extracted secret key $\overline{\mathtt{sk}}_j$ to compute $\overline{m}_{i,\ell}^{(j)} \leftarrow \mathsf{TRE.ShareDec}(\overline{\mathtt{sk}}_j, e_{i,\ell}^{(k)})$ for all the corrupted trustees $\mathsf{T}_j \in \mathbb{T}_{\mathsf{corrupt}}$. The simulator $\mathcal{S}$ then uses $\mathsf{TRE.SimShareDec.}$ to compute the message shares of the rest honest T_j's message shares $\overline{m}_{i,\ell}^{(j)}$ according to $(\tilde{B}_1, \ldots, \tilde{B}_n)$.

Indistinguishability. The indistinguishability is proven through a series of hybrid worlds $\mathcal{H}_0, \ldots, \mathcal{H}_4$.

Hybrid $\mathcal{H}_0$: It is the real protocol execution $\mathsf{EXEC}^{\bar{\mathcal{G}}_{\mathrm{BB}},\widehat{\mathcal{F}}_{\mathrm{CERT}}}_{\Pi_{\text{MIX-SV}},\mathcal{A},\mathcal{Z}}$.

Hybrid $\mathcal{H}_1$: $\mathcal{H}_1$ is the same as $\mathcal{H}_0$ except that $\mathcal{H}_1$ runs $\mathsf{NIZK}_{\mathcal{R}_4}.\mathsf{Ext}^{\mathrm{RO}}(\overline{\mathtt{pk}}_j, \pi_j^{(1)})$ to extract the corrupted trustee's secret key $\overline{\mathtt{sk}}_j$. $\mathcal{H}_1$ halt if the extraction fails.

Claim C.1. *$\mathcal{H}_1$ and $\mathcal{H}_0$ are indistinguishable.*

Proof. According to Def. 2.3, the probability $\mathsf{Ext}^{\mathrm{RO}}$ extraction fails (a.k.a. knowledge error) is negligible, so the probability that any adversary $\mathcal{A}$ and the environment $\mathcal{Z}$ can distinguish $\mathcal{H}_1$ from $\mathcal{H}_0$ is $\mathsf{negl}(\lambda)$. □

Hybrid $\mathcal{H}_2$: $\mathcal{H}_2$ is the same as $\mathcal{H}_1$ except the following: During the tally phase, uses the extracted $\mathtt{sk}_j$ from Hybrid $\mathcal{H}_1$ to decrypt each ciphertext, and the last honest trustee's message shares of each ciphertext are calculated by $\mathsf{TRE.SimShareDec}$ instead of using $\mathsf{TRE.ShareDec}$.

Claim C.2. *$\mathcal{H}_2$ and $\mathcal{H}_1$ are indistinguishable.*

Proof. By the share-simulation indistinguishability of the underlying TRE scheme, the distribution of the simulated decryption share(s) are computationally indistinguishable to the real ones. Moreover, by soundness of

$$\pi^{(6)}_{i,j,\ell} \leftarrow \mathsf{NIZK}_{\mathcal{R}_8}\left\{\begin{array}{l}(e^{(k)}_{i,\ell}, \overline{m}^{(j)}_{i,\ell}, \overline{\mathtt{pk}}_j), (\overline{\mathtt{sk}}_j, \alpha_j) : \\ (\overline{\mathtt{pk}}_j, \overline{\mathtt{sk}}_j) = \mathsf{TRE.Keygen}(\mathsf{param}; \alpha_j) \\ \wedge\, \overline{m}^{(j)}_{i,\ell} = \mathsf{TRE.ShareDec}(\overline{\mathtt{sk}}_j, e^{(k)}_{i,\ell})\end{array}\right\}$$

the corrupted trustees have negligible probability to post an invalid decryption share that is different from $\overline{m}^{(j)}_{i,\ell} \leftarrow \mathsf{TRE.ShareDec}(\overline{\mathtt{sk}}_j, e^{(k)}_{i,\ell})$. Therefore, the adversary's advantage of distinguishing $\mathcal{H}_2$ from $\mathcal{H}_1$ is $\mathsf{negl}(\lambda)$. □

Hybrid $\mathcal{H}_3$: $\mathcal{H}_3$ is the same as $\mathcal{H}_2$ except the followings. During the vote phase, $\mathcal{H}_3$ uses $\mathsf{NIZK}_{\mathcal{R}_6}.\mathsf{Sim}$ to simulate $\pi^{(3)}_{i,\ell}$, $\ell \in [\lambda_1]$ and uses $\mathsf{NIZK}_{\mathcal{R}_7}.\mathsf{Sim}$ to simulate $\pi^{(4)}_i$ for all the honest voter $\mathsf{V}_i \in \mathbb{V}$.

Claim C.3. *$\mathcal{H}_3$ and $\mathcal{H}_2$ are indistinguishable.*

Proof. The advantage of the adversary is bounded by the ZK property of NIZK as defined by Def. 2.2. □

Hybrid $\mathcal{H}_4$: $\mathcal{H}_4$ is the same as $\mathcal{H}_3$ except the followings. During the vote phase, the simulator posts $U_{i,\ell} \leftarrow \mathsf{TRE.Enc}(\mathtt{pk}, 0)$, $\ell \in [\lambda_1]$ and $S_i \leftarrow \mathsf{TRE.Enc}(\mathtt{pk}, 0)$ for all the honest voter $\mathsf{V}_i \in \mathbb{V}$.

Claim C.4. *$\mathcal{H}_4$ and $\mathcal{H}_3$ are indistinguishable.*

Proof. The probability that any adversary $\mathcal{A}$ can distinguish $\mathcal{H}_4$ from $\mathcal{H}_3$ is bounded by $\mathsf{AdvCPA}_{\mathcal{A}}(1^\lambda)$, $\mathsf{AdvUnlink}_{\mathcal{A}}(1^\lambda)$ and ciphertext transformative indistinguishability. More specifically, we now show the if there exists an adversary $\mathcal{A}$ who can distinguish $\mathcal{H}_4$ from $\mathcal{H}_3$, then we can construction an adversary $\mathcal{B}$ that can break the IND-CPA game of the underlying TRE by reduction. During the IND-CPA game, $\mathcal{B}$ receives a public key $\mathtt{pk}^*$ from the challenger. There must be at least one honest trustee in this case, and with our loss of generality, assume T_x is honest. During the preparation phase, $\mathcal{B}$ posts $\mathtt{pk}^*$ as T_x's public key together with simulated proof. During the ballot casting phase, for each honest voter V_i, $i \in [n]$, $\mathcal{B}$ sends $m_0 := (0, 0, \ldots, 0)$ and $m_1 := (w_{i,1}, \ldots, w_{i,\lambda_1}, s_i)$ to the IND-CPA challenger, and receives $\{c^*_\ell\}_{\ell\in[\lambda_1+1]}$. $\mathcal{B}$ then computes $c'_\ell \leftarrow \mathsf{TRE.Trans}(c^*_\ell, \{\mathtt{sk}_i\}_{i\in[k]\setminus\{x\}})$. It posts c' as the honest voter's encrypted ballot. It is easy to see that, due to $\mathsf{AdvUnlink}_{\mathcal{A}}(1^\lambda)$, when $\{c^*_\ell\}_{\ell\in[\lambda_1+1]}$ encrypts m_0, the adversary's view is indistinguishable from $\mathcal{H}_4$; when $\{c^*_\ell\}_{\ell\in[\lambda_1+1]}$ encrypts m_1, the adversary's view is indistinguishable from $\mathcal{H}_3$. Hence, if $\mathcal{A}$ can distinguish $\mathcal{H}_4$ from $\mathcal{H}_3$ with non-negligible probability, then $\mathcal{B}$ can break the IND-CPA game with the same probability.

□

The adversary's view of $\mathcal{H}_4$ is identical to the simulated view $\mathsf{EXEC}^{\bar{\mathcal{G}}_{\text{BB}}}_{\mathcal{F}_{\text{SV}},\mathcal{S},\mathcal{Z}}$. Therefore, no PPT $\mathcal{Z}$ can distinguish the view of the ideal execution from the view of the real execution with more than negligible probability.

Case 2: $0 \leq |\mathbb{V}_{\mathsf{corrupt}}| < n \ \wedge \ |\mathbb{T}_{\mathsf{corrupt}}| = k$.

Simulator. Similar as Case 1, the simulator $\mathcal{S}$ internally runs $\mathcal{A}$, forwarding messages to/from the environment $\mathcal{Z}$. The simulator $\mathcal{S}$ simulates honest voters $\mathsf{V}_i \in \mathbb{V} \setminus \mathbb{V}_{\mathsf{corrupt}}$, honest trustees $\mathsf{T}_j \in \mathbb{T} \setminus \mathbb{T}_{\mathsf{corrupt}}$ and functionalities $\widehat{\mathcal{F}}_{\text{CERT}}$. In addition, the simulator $\mathcal{S}$ simulates the following interactions with $\mathcal{A}$.

- In the preparation phase:
  - Monitoring $\bar{\mathcal{G}}_{\text{BB}}$, when a valid $(\overline{\mathtt{pk}}_j, \pi_j^{(1)})$ is posted on $\bar{\mathcal{G}}_{\text{BB}}$ from a corrupted trustee $\mathsf{T}_j \in \mathbb{T}_{\mathsf{corrupt}}$, use $\mathsf{NIZK}_{\mathcal{R}_4}.\mathsf{Ext}(\overline{\mathtt{pk}}_j, \pi_j^{(1)})$ to extract the corresponding secret key $\overline{\mathtt{sk}}_j$.
- In the ballot casting phase:
  - Upon receiving $(\textsc{Leak}, \mathsf{sid}, \mathsf{V}_i, B_i)$ from the external $\mathcal{F}_{\text{SV}}$ for an honest voter $\mathsf{V}_i \in \mathbb{V} \setminus \mathbb{V}_{\mathsf{corrupt}}$, the simulator $\mathcal{S}$ acts as V_i, following the protocol $\Pi_{\text{MIX-SV}}$ as if V_i receives $(\textsc{Cast}, \mathsf{sid}, B_i)$ from $\mathcal{Z}$.
  - The simulator $\mathcal{S}$ monitoring $\bar{\mathcal{G}}_{\text{BB}}$; once a $(W_i, \pi_i^{(2)})$ is posted from a corrupted voter $\mathsf{V}_i \in \mathbb{V}_{\mathsf{corrupt}}$, the simulator $\mathcal{S}$ uses the extracted $\left\{\overline{\mathtt{sk}}_j\right\}_{j\in[k]}$ to decrypt W_i to the temporal ID w_i. Record (V_i, w_i). When a valid $(U_{i,\ell}, \pi_{i,\ell}^{(3)})_{\ell=1}^{\lambda_1}, S_i, \pi_i^{(4)})$ is posted on $\bar{\mathcal{G}}_{\text{BB}}$ from a corrupted voter $\mathsf{V}_i \in \mathbb{V}_{\mathsf{corrupt}}$, uses the extracted $\left\{\overline{\mathtt{sk}}_j\right\}_{j\in[k]}$ to decrypt $U_{i,\ell}$ to $w_{i,\ell}$ and S_i to s_i. Replace the ID references in s_i to their actuarial voter ID's, and denoted the modified statement as s_i'. Record (V_i, s_i').
  - When any corrupted trustee has moved to the tally phase, the simulator $\mathcal{S}$ acts as each of the corrupted voters $\mathsf{V}_i \in \mathbb{V}_{\mathsf{corrupt}}$ to send $(\textsc{Cast}, \mathsf{sid}, (s_i', w_i))$ to $\mathcal{F}_{\text{SV}}$ if both (V_i, w_i) and (V_i, s_i') is recorded; otheriwse, it acts as V_i to send $(\textsc{Cast}, \mathsf{sid}, (s_i', \perp))$ to $\mathcal{F}_{\text{SV}}$ if only (V_i, s_i') is recorded.
- In the tally phase:
  - The simulator $\mathcal{S}$ monitoring $\bar{\mathcal{G}}_{\text{BB}}$; once $(\overline{m}_{i,\ell}^{(j)}, \pi_{i,j,\ell}^{(6)})_{i\in[n'],\ell\in[\lambda_1+2]}$ is posted from a corrupted trustee $\mathsf{T}_j \in \mathbb{T}_{\mathsf{corrupt}}$, the simulator $\mathcal{S}$ acts as T_j to send $(\textsc{Tally}, \mathsf{sid})$ to $\mathcal{F}_{\text{SV}}$.

Indistinguishability. The indistinguishability in this case is straightforward, as $\mathcal{S}$ never simulate a single message to either any corrupted parties or the external $\bar{\mathcal{G}}_{\text{BB}}$. The simulator $\mathcal{S}$ knows all the honest voters' ballot from the external $\mathcal{F}_{\text{SV}}$, it simply acts as the honest voters according to the protocol $\Pi_{\text{MIX-SV}}$. Meanwhile, it also extracts the ballot of the malicious voters by using the extracted trustees' secret keys. Hence, the simulator $\mathcal{S}$ can submit the extracted ballot to the external $\mathcal{F}_{\text{SV}}$ on the malicious voters' behave. Therefore, when NIZK extraction for trustees' secret keys are successful, the view of $\mathcal{Z}$ in the ideal execution has identical distribution to the view of $\mathcal{Z}$ in the real execution.

Case 3: $|\mathbb{V}_{\mathsf{corrupt}}| = n \ \wedge \ 0 \leq |\mathbb{T}_{\mathsf{corrupt}}| \leq k$.

Simulator. Trivial case. There is nothing needs to extract, as the trustees do not have input. The simulator $\mathcal{S}$ just run trustee according to protocol $\Pi_{\text{MIX-SV}}$.

Indistinguishability. The view of $\mathcal{Z}$ in the ideal execution has identical distribution to the view of of $\mathcal{Z}$ in the real execution.

□

C.2 Instantiation of TRE

We adopt threshold ElGamal encryption as a candidate for the threshold re-randomizable encryption (TRE) scheme. For any given security parameter λ, we pick a cyclic group $\langle g \rangle = \mathbb{G}$ with prime order q where the DDH assumption holds. The group information is denoted as param and is an implicit input of every algorithm.

- $\mathsf{TRE.Keygen}(\mathsf{param})$: The algorithm randomly picks $\overline{\mathtt{sk}}_i \leftarrow \mathbb{Z}_q$ and outputs $(\overline{\mathtt{pk}}_i := g^{\overline{\mathtt{sk}}_i}, \overline{\mathtt{sk}}_i)$.
- $\mathsf{TRE.CombinePK}(\{\overline{\mathtt{pk}}_i\}_{i=1}^k)$: The algorithm sets $h := \prod_{i=1}^k \overline{\mathtt{pk}}_i$ and outputs $\mathtt{pk} := (h, \overline{\mathtt{pk}}_1, \ldots, \overline{\mathtt{pk}}_k)$.
- $\mathsf{TRE.CombineSK}(\overline{\mathtt{sk}}_1, \ldots, \overline{\mathtt{sk}}_k)$. The algorithm $\mathsf{CombineSK}$ takes input as a set of secret key $(\overline{\mathtt{sk}}_1, \ldots, \overline{\mathtt{sk}}_k)$, and outputs combined secret key $\mathtt{sk} := \sum_{i=1}^k \overline{\mathtt{sk}}_i$.
- $\mathsf{TRE.Enc}(\mathtt{pk}, m)$: The algorithm randomly picks $r \leftarrow \mathbb{Z}_q$ and outputs $e := (g^r, m \cdot h^r)$.
- $\mathsf{TRE.ReRand}(\mathtt{pk}, e)$: The algorithm first parses e into (e_1, e_2), then randomly picks $s \leftarrow \mathbb{Z}_q$ and outputs $e' := (g^s \cdot e_1, h^s \cdot e_2)$.
- $\mathsf{TRE.Dec}(\mathtt{sk}, e)$. The algorithm Dec first parses e into (e_1, e_2), and outputs the decrypted plaintext $m := e_2 / e_1^{\mathtt{sk}}$.
- $\mathsf{TRE.ShareDec}(\mathtt{pk}, \overline{\mathtt{sk}}_i, e)$: The algorithm first parses ciphertext e into (e_1, e_2); then it outputs $\overline{m}_i := e_1^{\overline{\mathtt{sk}}_i}$.
- $\mathsf{TRE.ShareCombine}(e, \{\overline{m}_i\}_{i=1}^k)$: The algorithm first parses ciphertext e into (e_1, e_2); then it outputs $m := e_2 / \prod_{i=1}^k \overline{m}_i$.
- $\mathsf{Trans}(e, \{\mathtt{sk}_i\}_{i \in [k] \setminus \{j\}})$. The algorithm first parses e into (e_1, e_2); then it outputs $(e_1, e_2 \cdot \prod_{i \in [k] \setminus \{j\}} e_1^{\mathtt{sk}_i})$.
- $\mathsf{SimShareDec}(e, m, \{\mu_i\}_{i \in \mathcal{I}})$. The algorithm first parses e into (e_1, e_2) and then generates random decryption shares $\{\mu_j\}_{j \in [k] \setminus \mathcal{I}}$ except for the last one, denoted as μ_x. It then set $\mu_x = \frac{e_2}{m \cdot \prod_{j \in [k] \setminus \{x\}} \mu_j}$ and outputs $\{\mu_j\}_{j \in [k] \setminus \mathcal{I}}$.

First of all, the correctness of the above scheme follows by inspection. Now let's examine the security properties. It is easy to see that $\mathsf{AdvCPA}_{\mathcal{A}}(1^\lambda) = \mathsf{negl}(\lambda)$ is guaranteed by the IND-CPA security of the underlying ElGamal encryption which is under the DDH assumption. Besides, $\mathsf{AdvUnlink}_{\mathcal{A}}(1^\lambda) = 0$, as each re-randomized ciphertext has the same distribution as a freshly encrypted ciphertext. In terms of the ciphertext transformative indistinguishability, it is perfectly indistinguishable as the resulting ciphertext has the same distribution as a freshly encrypted one. Finally, share-simulation indistinguishability is also straightforward and it is implied by IND-CPA security.

C.3 Instantiations of NIZKs

Several NIZK proofs are used in our construction. Hereby, we provide RO-based instantiation for these primitives.

NIZK for distributed key generation. In the preparation phase, we used a NIZK proof of knowledge for knowledge of the secret key and correctness of the distributed key generation, i.e.,

$$\mathsf{NIZK}_{\mathcal{R}_4}\left\{(\overline{\mathtt{pk}}), (\omega, \overline{\mathtt{sk}}) : (\overline{\mathtt{pk}}, \overline{\mathtt{sk}}) = \mathsf{TRE.Keygen}(\mathsf{param}; \omega)\right\}$$

In terms of ElGamal encryption, this NIZK can be realized by strong Fiat-Shamir heuristic of the Schnorr's proof [Sch91]. Schnorr's proof is Sigma proof of knowledge of discrete logarithm; however, its RO-NIZK version has a small caveat, i.e., the knowledge extraction is based on RO rewinding. Alternatively, to enable extractability, we propose to a NIZK in Fig. 18, where $H_1 : \{0,1\}^* \mapsto \mathbb{G}$ is a hash function. $\mathsf{NIZK}_{\mathcal{R}_9}$ allows the prover to show an ElGamal ciphertext is encryption of $0/1$ using a Sigma disjunction of Chaum-Pederden Sigma protocol. $\mathsf{NIZK}_{\mathcal{R}_{10}}$ is strong Fiat-Shamir heuristic of Chaum-Pederden Sigma protocol for DDH tuples.

NIZK for Discrete Logarithm

Statement: $h = g^s$
Witness: $s_1, \ldots, s_\kappa \in \{0,1\}$ s.t. $s = \sum_{i=1}^{\kappa} 2^{i-1} s_i$
Prove:

- Set $u := H_1(h)$ and pick $r_1, \ldots, r_\kappa \leftarrow \mathbb{Z}_q$.
- For $i \in [\kappa]$, compute $e_{i,1} := g^{r_1}$, $e_{i,2} := g^{s_i} u^{r_1}$, and prove

$$\pi_i \leftarrow \mathsf{NIZK}_{\mathcal{R}_9} \left\{ \begin{array}{l} (g, u, e_{i,1}, e_{i,2}), (s_i, r_i) : \\ (e_{i,1} = g^{r_i} \wedge e_{i,2} = u^{r_i}) \ \vee (e_{i,1} = g^{r_i} \wedge e_{i,2}/g = u^{r_i}) \end{array} \right\} .$$

- Compute $e_1 := \prod_{i=1}^{\kappa} (e_{i,1})^{2^{i-1}}$, $e_2 := \prod_{i=1}^{\kappa} (e_{i,2})^{2^{i-1}}$ and $r := \sum_{i=1}^{\kappa} 2^{i-1} r_i$. Prove

$$\phi \leftarrow \mathsf{NIZK}_{\mathcal{R}_{10}} \big\{ (g, u, e_1, e_2), (r) : (e_1 = g^r \wedge e_2/h = u^r) \big\} .$$

- Output $\pi := ((e_{i,1}, e_{i,2})_{i \in [\kappa]}, \pi_1, \ldots, \pi_\kappa, \phi)$.

Verify:

- Set $u := H_1(h)$, $e_1 := \prod_{i=1}^{\kappa} (e_{i,1})^{2^{i-1}}$, and $e_2 := \prod_{i=1}^{\kappa} (e_{i,2})^{2^{i-1}}$.
- For $i \in [\kappa]$, check $\mathsf{NIZK}_{\mathcal{R}_9}.\mathsf{Verify}\big\{(g, u, e_{i,1}, e_{i,2}), \pi_i\big\}$.
- Check $\mathsf{NIZK}_{\mathcal{R}_{10}}.\mathsf{Verify}\big\{(g, u, e_1, e_2), \phi\big\}$.

Figure 18: NIZK for Discrete Logarithm

Theorem C.5. *The* NIZK *described in Fig. 18 is an NIZK proof of knowledge of $s \in \mathbb{Z}_q$ for $h = g^s$ with extractability.*

Proof. The completeness and soundness follow directly by the completeness of the underlying $\mathsf{NIZK}_{\mathcal{R}_9}$ and $\mathsf{NIZK}_{\mathcal{R}_{10}}$. For ZK, the simulator generates $(e_{i,1}, e_{i,2})$ as encryption of 0 and computes $\mathsf{NIZK}_{\mathcal{R}_9}$ honestly. It then simulates ϕ using $\mathsf{NIZK}_{\mathcal{R}_{10}}.\mathsf{Sim}$. In terms of extractability, the knowledge extractor simulates the RO for H_1, and it outputs $u = g^x$ for a randomly chosen $x \in \mathbb{Z}_q$. Now the extractor can decrypt $(e_{i,1}, e_{i,2})$ and obtain s_i, for $i \in [\kappa]$; it then outputs $s = \sum_{i=1}^{\kappa} 2^{i-1} s_i$. □

Remark C.6. *We note that it is also possible to use Schnorr's proof (without extractability) for better computational efficiency, but at the cost of one more round. Namely, instead of directly posting the partial public keys on the bulletin board, we let the trustees first post a commitment of their partial public keys, and then decommit them. For instance, we can use simple hash based commitment. To commit m, pick a random $d \leftarrow \{0,1\}^\lambda$, output $c := H(m\|d)$. To verify a commitment, just check if $c = H(m\|d)$. Now the simulator can fix the combined public key to the one that the simulator knows its corresponding secrete key by equivocating the commitments. (cf. [BPW12] for more details of this technique.)*

NIZK for knowledge of plaintext. In our scheme, the voters post encryptions of their temporal ID on the BB. In order to prevent the adversary from copying and modifying their temporal ID, we use NIZK for the

correctness of TRE.Enc algorithm as the following.

$$\mathsf{NIZK}_{\mathcal{R}_5}\{(\mathtt{pk}, e), (\omega, m) : e = \mathsf{TRE.Enc}(\mathtt{pk}, m; \omega)\}$$

With regard to ElGamal encryption, the proof of knowledge of plaintext and randomness is the same as proof of knowledge of randomness, as given r, everyone can compute $m := e_2/\mathtt{pk}^r$. This can be done via strong Fiat-Shamir heuristic on Schnorr's proof [Sch91]. However, this NIZK assume the plaintext m is public. In practice, if the message space is small, we can use Sigma OR-composition to numerate each possible plaintext. However, this is not efficient. Alternatively, we we propose a Sigma protocol for knowledge of plaintext in Fig. 19.

Σ protocol for knowledge of plaintext

Statement: h and $(e_1, e_2) = (g^r, m \cdot h^r)$
Witness: $m \in \mathbb{G}$ and $r \in \mathbb{Z}_p$
Prover:

- Pick random $S \leftarrow \mathbb{G}$ and $t \in \mathbb{Z}_p$.
- Send $(c_1, c_2) = (g^t, S \cdot h^t)$ to the verifier.

Verifier:

- Send random challenge $\rho \leftarrow \{0,1\}^\lambda$ to the prover.

Prover:

- Send $u := r \cdot \rho + t$ and $W := m^\rho \cdot S$ to the verifier.

Verifier:

- Return valid if and only if $e_1^\rho \cdot c_1 = g^u$ and $e_2^\rho \cdot c_2 = W \cdot h^u$.

Figure 19: Σ protocol for knowledge of plaintext

Theorem C.7. *The* NIZK *described in Fig. 19 is a Sigma proof of knowledge of* $m \in \mathbb{G}$ *and* $r \in \mathbb{Z}_p$ *for* $(e_1, e_2) = (g^r, m \cdot h^r)$.

Proof. Perfect completeness follows by inspection. To special soundness, we can construct a knowledge extractor that takes in two set of valid transcripts $(c_1, c_2, \rho_1, u_1, W_1)$ and $(c_1, c_2, \rho_2, u_2, W_2)$ can output the witness. Indeed, we have $r := \frac{u_1 - u_2}{\rho_1 - \rho_2}$ and $m := (W_1/W_2)^{1/(\rho_1 - \rho_2)}$. Finally, for special honest verifier zero-knowledge, we will construct a PPT simulator Sim that given any challenge ρ^* can outputs a valid transcript that is indistinguishable from the real one. The simulator Sim first picks random $u \leftarrow \mathbb{Z}_p$ and $W \leftarrow \mathbb{G}$. It then computes $c_1 := g^u/e_1^{\rho^*}$ and $c_2 := W \cdot h^u/e_2^{\rho^*}$. It is easy to see that (c_1, c_2, ρ^*, u, W) has identical distribution as the real transcript. □

One-out-of-many NIZK. In our scheme, the voters need to use

$$\mathsf{NIZK}_{\mathcal{R}_6}\{(\mathtt{pk}, (e_1, \ldots, e_n), e'), (\omega, i) : e' = \mathsf{TRE.ReRand}(\mathtt{pk}, e_i; \omega)\}$$

to show that e' is re-randomized from one of a set of ciphertexts as follows. The statement can be re-stated as to show that one of the ciphertexts $(e_1/e', \ldots, e_n/e')$ is encryption of 0; namely, the prover knows i and r such that $e_i/e' := \mathsf{TRE.Enc}(\mathtt{pk}, 0; r)$. Groth and Kohlweiss [GK15] proposed an efficient one-out-of-many proof, whose proof size is $O(\log n)$. Their proof is a 3-move public coin special honest verifier zero-knowledge proof that allows the prover to convince the verifier that one out of a set of commitment commits to 0. Although they instantiate their proof to Pedersen commitment, their protocol is also compatible with ElGamal

commitment/encryption. Therefore, we can use strong Fiat-Shamir heuristic on their proof to instantiate our $\mathsf{NIZK}_{\mathcal{R}_6}$, and no knowledge extractor is needed. Due to space limitation, we refer interested readers to [GK15] for more details.

NIZK for shuffle correctness. Each trustee is shuffling the set of *triple ciphertext* (ballot) in turn. We need shuffle NIZK for the correctness of re-encryption mix-net, i.e.,

$$\mathsf{NIZK}_{\mathcal{R}_7}\left\{\begin{array}{c}(\mathtt{pk},(e_1,\ldots,e_n),(e'_1,\ldots,e'_n)),(\Pi,(\omega_1,\ldots,\omega_n)):\\ \forall i\in[n]: e'_i=\mathsf{TRE.ReRand}(\mathtt{pk},e_{\Pi(i)};\omega_i)\end{array}\right\}.$$

There are many ZK/NIZK of shuffling correctness for ElGamal re-encryption. To our best knowledge, the most efficient one is proposed by Bayer and Groth [BG12]. The proof size of their ZK is $O(\sqrt{n})$. Although the original proof is for shuffling single ElGamal ciphertexts rather than bundles of three ciphertexts, it is easy to modify their proof to meet our requirement. More concretely, the modified protocol consists of two sub-protocols. Let ρ be the permutation. The prover first uses generalized Pedersen commitment to commit $x^{\rho(1)},\ldots,x^{\rho(n)}$ and prove its correctness, where x is randomly chosen by the verifier; after that, the prover uses multi-exponentiation argument to show that $\prod_{i=1}^{n}(e_{i,j})^{x^i} = \mathsf{TRE.Enc}(\mathtt{pk},0;s)\cdot\prod_{i=1}^{n}(e'_{i,j})^{x^{\pi(i)}}$ for $j\in[3]$, where s is some randomness known to the prover. Their protocol is Fiat-Shamir friendly, and we refer interested readers to [BG12] for more details.

NIZK for share decryption correctness The NIZK proof of membership

$$\mathsf{NIZK}_{\mathcal{R}_8}\left\{(\overline{\mathtt{pk}}_i,e_1,\overline{m}_i),(\overline{\mathtt{sk}}_i):\overline{\mathtt{pk}}_i=g^{\overline{\mathtt{sk}}_i}\ \wedge\ \overline{m}_i=e_1^{\overline{\mathtt{sk}}_i}\right\}$$

invoked above can be instantiated by strong Fiat-Shamir heuristic on the well-known Chaum-Pedersen proof [CP93] for DDH tuples.