

# 1 Diagnosing and Correcting the Effects of Multicollinearity: Bayesian 2 Implications of Ridge Regression 3 4

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## 28 **Abstract**

29  
30 When faced with the problem of multicollinearity most tourism researchers recommend mean-  
31 centering the variables. This procedure however does not work. It is actually one of the biggest  
32 misconceptions we have in the field. We propose instead using Bayesian ridge regression and treat  
33 the biasing constant as a parameter about which inferences are to be made. It is well known that  
34 many estimates of the biasing constant have been proposed in the literature. When the coefficients  
35 in ridge regression have a conjugate prior distribution, formal selection can be based on the marginal  
36 likelihood. In the non-conjugate case, we propose a conditionally conjugate prior for the biasing  
37 constant, and show that Gibbs sampling can be employed to make inferences about ridge regression  
38 parameters as well as the biasing constant itself. We examine posterior sensitivity and apply the  
39 techniques to a tourism data set.

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42 *Keywords:* Multicollinearity; Bayesian analysis; ridge regression; Gibbs sampling.  
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44

## 45 Introduction

46

47 The problem of multicollinearity is highly common in tourism research. One particular example is  
48 the regression model with moderators. Such model is usually highly prone to having collinearity  
49 problems because the interaction term is created by multiplying two exogenous variables to create  
50 another exogenous variable. To “alleviate” the potential problems of collinearity, tourism  
51 researchers routinely mean center the variables by subtracting the item value from the mean value of  
52 the item. This simply does not fix the problem. Mean centering does not really help or harm  
53 (Echambadi and Hess, 2007; and Dalal and Zickar, 2011). While the mean-centered coefficients  
54 have different interpretations than the original coefficients, we rarely see them being compared  
55 against each other in the tourism literature. In fact, anytime an interaction is included in the model,  
56 the original coefficients should not be used directly to assess the impact of X on Y. Instead, one  
57 needs to use the marginal effect which is actually what we obtain when we mean center the variables.

58

59 Assuming that data for the dependent variable are arranged in the  $n \times 1$  vector  $\mathbf{y}$  and the data for  
60 the explanatory variables are in the  $n \times p$  matrix  $\mathbf{X}$ , so that we have  $n$  observations and  $p$   
61 regressors, it is well established that the least squares (LS) estimator  $\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{y}$ , under the  
62 stated assumptions about the error term is the best linear unbiased estimator (BLUE). However,  
63 multicollinearity can result in ill conditioning of the matrix  $\mathbf{X}'\mathbf{X}$  rendering the LS estimator  
64 undesirable. For example when this matrix is nearly non-invertible, the covariance matrix will have  
65 large elements in the diagonal, implying that standard errors of LS estimators will be quite large.  
66 Effectively, in specific samples, it is quite likely that we may end up with LS coefficients having the  
67 wrong sign, being non-significant, *etc.*

68

69 A regularization method that has been proposed is the use of the ridge regression estimator (Hoerl  
70 and Kennard, 1970), with a biasing constant  $k$ , usually small. Effectively, “the procedure can be  
71 used to portray the sensitivity of the estimates to the particular set of data being used, and it can be  
72 used to obtain a point estimate with a smaller mean square error” (Hoerl and Kennard, 1970, p.55).  
73 As a matter of fact, Hoerl and Kennard (1970) discussed the Bayesian foundation of their approach  
74 (p.64) and also proposed a more general ridge regression.

75

76 A main challenge in the literature has been finding the appropriate value of  $k$ , as different  
77 procedures (Dorugade and Kashid, 2010; Uslu, Egrioglu and Bas, 2014) have been used for that  
78 purpose. Hoerl and Kennard (1970) suggested using the ridge trace to find the appropriate value of  
79  $k$ , for which the regression coefficients have been stabilized. Hoerl and Kennard (1976) proposed  
80 an iterative approach for selecting  $k$ . However, their procedure does not necessarily converge. As  
81 there is no consensus on what is a reasonable procedure to select the value of  $k$ , we propose here a  
82 Bayesian approach to address this issue. Our aim is to provide tourism researchers with more  
83 flexibility in estimating ridge regressions. The Bayesian approach is appealing because it treats  $k$  as a  
84 parameter which is to be selected in light of the data. In fact, we do not select a single value of  $k$ ,  
85 but we produce the whole marginal posterior of this parameter given the data. This, in turn, is one  
86 attractive way to address the uncertainty about  $k$ .

87  
88 The push for Bayesian estimation is taking place across several disciplines such as management  
89 (Zyphur & Oswald, 2015; Cabantous and Gond, 2015; McKee and Miller, 2015), marketing (Rossi  
90 and Allenby, 2003; Rossi et al. 2012), psychology (Van De Schoot, et al., 2017) and tourism (Assaf  
91 and Tsionas, 2018 a, b). Over the last decade, we have seen a strong increase in the use of the  
92 Bayesian methodology in tourism and other related fields (Wong et al. 2006; Wang et al. 2011; Assaf,  
93 2012; Barros, 2014; Assaf et al. 2017; Assaf et al., 2018a). A recent special issue in the Journal of  
94 Management is a clear indication on the growing popularity of this method (Zyphur & Oswald,  
95 2015). Across several research areas in tourism, recent studies have demonstrated the effectiveness  
96 of the Bayesian approach. For instance, Wong et al. (2006) have shown that a Bayesian vector  
97 autoregressive model resulted in better forecasting accuracy than traditional non-Bayesian models.  
98 Assaf et al. (2018b) have also recently shown that the Bayesian global vector autoregressive  
99 (BGVAR) consistently outperforms other non-Bayesian models. Moreover, in related areas, such as  
100 tourism performance, recent studies have also demonstrated how the Bayesian approach can handle  
101 more complicated models than traditional estimation techniques (Assaf and Tsionas, 2018 a, b).

102  
103 Recent papers has provided comprehensive introductions on the advantages of the Bayesian  
104 approach (Muthen, 2010, Zyphur and Oswald, 2015). The Bayesian approach is not simply about  
105 fitting more advanced models with MCMC (Markov chain Monte Carlo) but is a completely  
106 different paradigm and philosophy in statistics. It offers several advantages in the estimation of  
107 regression models including “ rich diagnostic information about parameters and models; controlling  
108 for multiple comparisons as a function of the data; handling low-frequency, unbalanced, missing  
109 data; and exploration of prior assumptions about model parameters” (Zyphur and Oswald, 2013,  
110 p.7). Probably, on the most known advantages of the method is its ability to incorporate prior  
111 information about a parameter and form a prior distribution. For instance, the Bayes’ theorem can  
112 be expressed as:  $p(\theta | y) \propto p(y | \theta)p(\theta)$ , where  $\propto$  is the proportionality symbol. Here,  $p(\theta | y)$  is  
113 the posterior distribution which is used to carry out all inferences, and is proportional to the product  
114 of the prior  $p(\theta)$  and the likelihood function  $p(y | \theta)$ <sup>1</sup>. Different choices of priors can be used  
115 such as conjugate vs. non-conjugate priors. The prior is said to be conjugate if it belongs to the  
116 family of distribution as the posterior distribution (i.e. the posterior has posterior has the same  
117 distributional form as the prior distribution). For example, in the context when the likelihood  
118 function is binomial  $y \sim Bin(n, \theta)$  , a conjugate prior in the form of a beta distribution on  $\theta$  will  
119 also lead to a posterior distribution that follows a beta distribution. A prior distribution which is not  
120 conjugate is called a non-conjugate prior.

121  
122 We illustrate below the flexibility of the Bayesian approach and prior information within the context  
123 of ridge regression. In particular, we introduce a Bayesian ridge estimator for both conjugate and  
124 non-conjugate priors though we rely more on the non-conjugate prior as the conjugate priors are  
125 restrictive and have certain problems, for example they have the same tails with the likelihood and

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<sup>1</sup> The likelihood function summarizes the information from the data.

126 they are rarely used in practice. A singular advantage of the Bayesian approach is that ridge  
 127 regression can be interpreted as Bayes posterior mean when the prior on the regression parameters  
 128 is multivariate normal with zero mean and covariance matrix a diagonal matrix whose diagonal  
 129 elements have the same variance / precision. Moreover, the significance of the Bayesian approach to  
 130 regression is that the celebrated James-Stein estimator has a direct empirical Bayes estimator. The  
 131 James-Stein estimator is well-known to improve on maximum likelihood / OLS estimator in terms  
 132 of risk and MSE across all values of the parameter space.

133  
 134 In this paper we proceed as follows: In section 2 we provide an introduction to ridge regression.  
 135 Sections 3 and 4 present the Bayesian ridge regression approach with conjugate and non-conjugate  
 136 setting in comparison with the diffuse prior assumptions. We conduct a Monte Carlo study in  
 137 section 5 to illustrate the issue diagnosing and correcting the effect multicollinearity. We then  
 138 present illustration on the Bayesian ridge regression using a tourism application.  
 139

140  
 141 **2. How to Proceed?**

142 So, if mean centering does not work, how to proceed from here? One of the most common  
 143 approaches is to use ridge regression to analyze regression data that is subject to multicollinearity. As  
 144 mentioned, with OLS the regression parameters can be estimated using the following formula:  
 145

146  $\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{y}$

147  
 148 The ridge regression differentiates by adding a biased constant  $k > 0$  to the diagonal elements of the  
 149 correlations matrix:  
 150

151  $\mathbf{b}_k = (\mathbf{X}'\mathbf{X} + k\mathbf{I}_p)^{-1} \mathbf{X}'\mathbf{y},$

152  
 153 This is where the term “ridge regression” comes from as the diagonal of one in the correlation  
 154 matrix are thought of as a ridge). What we know from Hoerl and Kennard (1970) is that there is  
 155 always a  $k \in (0, \bar{k})$  for which ridge regression dominates OLS in terms of mean squared error  
 156 (MSE), and  $\bar{k} = \frac{\sigma^2}{\alpha_{\max}^2}$ , where  $\mathbf{X}'\mathbf{X} = \mathbf{P}'\mathbf{\Lambda}\mathbf{P}$ , and  $\boldsymbol{\alpha} = \mathbf{P}\boldsymbol{\beta}$ . Here,  $\mathbf{P}$  is the orthonormal matrix of  
 157 eigenvectors of  $\mathbf{X}'\mathbf{X}$ , and  $\mathbf{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_p)$ , where  $\lambda_1, \dots, \lambda_p$  represent the distinct eigenvalues  
 158 of  $\mathbf{X}'\mathbf{X}$ . Another result of Hoerl and Kennard (1970) was that the total MSE of the ridge estimator  
 159 is<sup>2</sup>:  
 160

161 
$$MSE(\mathbf{b}_k) = \sigma^2 \sum_{i=1}^p \frac{\lambda_i}{(\lambda_i + k)^2} + k^2 \boldsymbol{\beta}' (\mathbf{X}'\mathbf{X} + k\mathbf{I}_p)^{-2} \boldsymbol{\beta}$$

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<sup>2</sup> The notation  $A^{-2}$  for a matrix  $A$ , means  $A^{-2} = A^{-1}A^{-1}$ .

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163  
164  
165 Minimizing the MSE, unfortunately, depends on the ratio of  $\beta/\sigma$ . Depending on this result  
166 several settings for the parameter  $k$  have been proposed. See for example Khalaf and Shukur  
167 (2005), Lawless and Wang (1976), Nomura (1988) and Maruyama and Strawderman (2005). A similar  
168 idea is the Bayesian lasso regression (Park and Casella, 2008, Hans, 2009).

169  
170 The goal of this paper is to propose a more flexible way to estimate  $k$  using the Bayesian approach.  
171 As mentioned, one of the advantages is that with the Bayesian approach we do not (necessarily)  
172 select a single value of  $k$  but we produce the whole marginal posterior of this parameter given the  
173 data. We aim here to diagnose and correct the effects of multicollinearity through a full non-  
174 conjugate Bayesian approach to Ridge Regression. In particular we take up Bayesian inference in  
175 conjugate and non-conjugate ridge regression models by using the fact that a prior can be placed on  
176 the ridge parameter(s)  $k$  and proceed with posterior analysis on all parameters using MCMC  
177 techniques. We run different simulations to illustrate the performance of the method. We also  
178 provided evidence based on a real dataset from the hotel industry. Our goal is to show that  
179 collinearity can be simultaneously diagnosed and corrected using priors on all parameters. Our  
180 techniques detect and correct the adverse effects of collinearity in a transparent way.

181  
182 Specifically, given the general regression model, we consider first ridge regression from the Bayesian  
183 point of view treating the biasing constant ( $k$ ) as a parameter about which inferences are to be  
184 made to avoid selecting a particular value of  $k$ . For the conjugate case we have derived the marginal  
185 likelihoods and showed how selection of the  $k$  parameter can be performed to choose the  
186 appropriate value. It is important to notice that the original ridge regression estimators depend  
187 crucially on a conjugacy assumption, namely that the regression coefficients,  $\beta | \sigma, k \sim N_p\left(0, \frac{\sigma^2}{k} \mathbf{I}_p\right)$ .  
188 Conjugate priors have certain problems, for example they have the same tails with the likelihood and  
189 they are rarely used in practice.

190  
191  
192 The reader can refer to Leamer (1969) and Judge et al (1985) regarding this point. As they mention,  
193 despite the fact that the natural conjugate setting is a convenient approach (since it provides an  
194 analytical solution to the integrations involved), it has been criticized because it employs the prior  
195 information as a previous imaginary sample from the same process: When we set the degrees of  
196 freedom and the precision matrix equal to zero to obtain the limiting distribution of the normal-  
197 (inverse) gamma prior, the resulting ignorant prior is different to the usual diffuse prior and the  
198 posterior distribution has different degrees of freedom. Therefore, a non - conjugate prior can be  
199 adopted instead, and numerical posterior inference can rely on the Gibbs sampler. In the Gibbs  
200 sampler  $k$  is treated as a parameter and, therefore, formal statistical inferences can be made about  
201 this parameter thus solving a long-standing problem in the literature. Moreover, a formal test for  
202 collinearity can be developed if we compare the marginal posterior of  $k$  with its value at  $k=0$   
203 (corresponding to OLS or Bayes with diffuse prior). An equivalent test is to compare the marginal  
204 likelihood at the optimal  $k$  with its value when  $k=0$ .

205

206 We discuss below a Bayesian ridge estimator for both conjugate and non-conjugate priors though we  
 207 rely more on the non-conjugate prior as the conjugate priors are restrictive and have certain  
 208 problems, for example they have the same tails with the likelihood and they are rarely used in  
 209 practice.

210  
 211 **3. Bayesian ridge regression**

212  
 213 For the Bayesian interpretation of the ridge regression estimator, the model is given by:

214  
 215 
$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}, \mathbf{u} \sim N_n(\mathbf{0}, \sigma^2 \mathbf{I}_n)$$

216 The prior on the unknown parameters is

217 
$$\boldsymbol{\beta} | \sigma \sim N_p\left(\mathbf{0}, \frac{\sigma^2}{k} \mathbf{I}_p\right), p(\sigma) \propto \sigma^{-1},$$

218 where  $k > 0$  is prior precision *relative* to the error variance,  $\sigma^2$ . It is not difficult to show that,  
 219 under these conditions, the posterior mean is given by:  $E(\boldsymbol{\beta} | \mathbf{y}, \mathbf{X}) = \mathbf{b}_k$ , that is the ridge regression  
 220 estimator. As mentioned, since the choice of parameter  $k$  has been an active area of research for  
 221 many years and, many choices have been proposed, it is natural to investigate the implications of a  
 222 fully Bayesian approach to the problem. To this effect, we consider both a conjugate and non-  
 223 conjugate prior on the regression parameters,  $\boldsymbol{\beta}$

224  
 225 The conjugate prior provides explicitly in analytical form the ridge regression estimator so there is  
 226 much in favor of it. However, the non-conjugate case is also interesting and can be considered as an  
 227 alternative.

228  
 229 **3.1 Optimal biasing parameter through conjugacy**

230  
 231 Suppose  $\boldsymbol{\beta} | \sigma \sim N_p\left(\mathbf{0}, \frac{\sigma^2}{k} \mathbf{I}_p\right)$ , and  $\frac{\underline{\nu} \underline{s}^2}{\sigma^2} \sim \chi^2(\underline{\nu})$ , where  $\underline{\nu}, \underline{s}^2$  are prior hyperparameters. This prior  
 232 is conjugate because it depends on  $\sigma$  and matches exactly the likelihood to provide as posterior mean  
 233  $\mathbf{b}_k$  below.

234  
 235 The marginal likelihood (“or evidence”), for a given value of  $k$ , can be derived analytically in this  
 236 case<sup>3</sup>:

237  
 238 
$$p_k(\mathbf{y}) \propto \left(\frac{|\mathbf{V}|}{|\mathbf{V}|}\right)^{1/2} (\bar{\nu} \bar{s}^2)^{-\bar{\nu}/2}, \text{ where}$$

$$\mathbf{V} = (\mathbf{X}'\mathbf{X} + k\mathbf{I}_p)^{-1}, \underline{\mathbf{V}} = k^{-1}\mathbf{I}_p,$$

$$\bar{\nu} \bar{s}^2 = \underline{\nu} \underline{s}^2 + \mathbf{y}' \left(\mathbf{I}_p - \mathbf{X}(\mathbf{X}'\mathbf{X} + k\mathbf{I}_p)^{-1} \mathbf{X}'\right) \mathbf{y}$$

$$\mathbf{b} = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{y} \text{ and } \bar{\nu} = \underline{\nu} + n.$$

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<sup>3</sup>See Zellner (1971), p.309.

239 Moreover,  $\frac{|V|}{|Y|} = \frac{k^p}{\prod_{i=1}^p (k + \lambda_i)}$ , where  $\lambda_1, \dots, \lambda_p$  are the eigenvalues of  $\mathbf{X}'\mathbf{X}$ , and

240  $\bar{v}s^2 = \underline{v}s^2 + RSS + \mathbf{b}'(\mathbf{X}'\mathbf{X})(\mathbf{b} - \mathbf{b}_k)$ ,  $RSS = (\mathbf{y} - \mathbf{X}\mathbf{b})'(\mathbf{y} - \mathbf{X}\mathbf{b})$ . By (2.3) in Hoerl and Kennard

241 (1970), we have  $\left[ \mathbf{I}_p + k(\mathbf{X}'\mathbf{X})^{-1} \right]^{-1} \mathbf{b} = \mathbf{b}_k$ , where  $\mathbf{b}_k = (\mathbf{X}'\mathbf{X} + k\mathbf{I}_p)^{-1} \mathbf{X}'\mathbf{y}$  is the ridge estimate.

242 So the log marginal likelihood simplifies to the expression:

243

244 
$$\log p_k(\mathbf{y}) = 0.5p \log k - 0.5 \sum_{i=1}^p \log(\lambda_i + k) - 0.5\bar{v} \log(\underline{v}s^2 + RSS + \mathbf{b}'(\mathbf{X}'\mathbf{X})(\mathbf{b} - \mathbf{b}_k)) \quad (1)$$

245  
246 This expression involves only the eigenvalues of  $\mathbf{X}'\mathbf{X}$ , standard LS quantities and the ridge  
247 estimates.

248

249

### 250 3.2. Bayesian ridge regression in the non-conjugate case

251

252 In the Bayesian context it is reasonable to treat  $k$  as unknown parameter whose prior is  $p(k)$   
253 *independently* of  $\boldsymbol{\beta}$  and  $\sigma$ . Therefore, it is useful to depart from the conjugate case which involves the  
254 unpleasant feature that the tails of the posterior and the prior are the same. Then we have:

255

256 
$$\boldsymbol{\beta} | \sigma, k \sim N_p\left(\mathbf{0}, \frac{1}{k}\mathbf{I}_p\right), p(\sigma | k) \propto \sigma^{-1},$$

257 and the prior of  $k$  is proportional to  $p(k)$ . It is not necessary for this prior to be proper. The  
258 joint posterior is as follows:

259

260 
$$p(\boldsymbol{\beta}, \sigma, k | \mathbf{y}, \mathbf{X}) \propto \sigma^{-(n+1)} k^{p/2} p(k) \exp\left[-\frac{(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) + k\sigma^2 \boldsymbol{\beta}'\boldsymbol{\beta}}{2\sigma^2}\right] \quad (2)$$

261 Completing the square  $Q = (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) + k\sigma^2 \boldsymbol{\beta}'\boldsymbol{\beta}$ , we obtain the expression:

262 
$$Q = (\boldsymbol{\beta} - \mathbf{b}_k)'(\mathbf{X}'\mathbf{X} + k\sigma^2\mathbf{I}_p)(\boldsymbol{\beta} - \mathbf{b}_k) + \mathbf{y}'\mathbf{M}_k\mathbf{y},$$

263

264 where  $\mathbf{M}_k = \mathbf{I}_p - \mathbf{X}\mathbf{V}_k\mathbf{X}'$  and  $\mathbf{V}_k = (\mathbf{X}'\mathbf{X} + k\sigma^2\mathbf{I}_p)^{-1}$ . Therefore, the posterior distribution is:

265

266 
$$p(\boldsymbol{\beta}, \sigma, k | \mathbf{y}, \mathbf{X}) \propto \sigma^{-(n+1)} k^{p/2} p(k) \exp\left[-\frac{(\boldsymbol{\beta} - \mathbf{b}_k)'(\mathbf{X}'\mathbf{X} + k\sigma^2\mathbf{I}_p)(\boldsymbol{\beta} - \mathbf{b}_k) + \mathbf{y}'\mathbf{M}_k\mathbf{y}}{2\sigma^2}\right] \quad (3)$$

267 In this expression,  $\mathbf{b}_k = (\mathbf{X}'\mathbf{X} + k\sigma^2\mathbf{I}_p)^{-1} \mathbf{X}'\mathbf{y}$  is the ridge regression estimate. From the expression  
268 in (3), we can extract the following posterior conditional distributions:

269

270 
$$\boldsymbol{\beta} | \sigma, k, \mathbf{y}, \mathbf{X} \sim N_p \left( \mathbf{b}_k, \sigma^2 (\mathbf{X}'\mathbf{X} + k\sigma^2 \mathbf{I}_p)^{-1} \right),$$

271 
$$\frac{(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})' (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})}{\sigma^2} \Big| \boldsymbol{\beta}, k, \mathbf{y}, \mathbf{X} \sim \chi^2(n),$$

272 where  $\chi^2(n)$  denotes the chi-square distribution with  $n$  degrees of freedom. Finally, the posterior  
273 conditional distribution of the biasing parameter can be derived from (2) as:

274 
$$p(k | \boldsymbol{\beta}, \sigma, \mathbf{y}, \mathbf{X}) \propto k^{p/2} p(k) \exp\left(-\frac{k\boldsymbol{\beta}'\boldsymbol{\beta}}{2}\right) \quad (4)$$

275 The conditionally conjugate prior for the biasing parameter,  $k$ , is clearly a  $Gamma(\bar{A}/2, \bar{B}/2)$ <sup>4</sup>  
276 distribution whose density is of the following form:  $p(k) = \frac{(\bar{B}/2)^{\bar{A}/2}}{\Gamma(\bar{A}/2)} k^{\bar{A}/2-1} \exp(-\frac{\bar{B}}{2}k)$ , where  $\bar{A} \geq 0$   
277 and  $\bar{B} \geq 0$  are hyperparameters. Then we obtain:

278 
$$p(k | \boldsymbol{\beta}, \sigma, \mathbf{y}, \mathbf{X}) \propto k^{(p+\bar{A})/2-1} \exp\left(-\frac{\bar{B} + \boldsymbol{\beta}'\boldsymbol{\beta}}{2}k\right)$$

279 Therefore, the posterior conditional distribution of the biasing parameter is:

280 
$$k | \boldsymbol{\beta}, \sigma, \mathbf{y}, \mathbf{X} \sim Gamma\left(\frac{n + \bar{A}}{2}, \frac{\bar{B} + \boldsymbol{\beta}'\boldsymbol{\beta}}{2}\right) \quad (5)$$

282  
283 Prior elicitation of the hyperparameters  $\bar{A}$  and  $\bar{B}$  is facilitated by the fact that, in the prior,  
284  $E(k) = \frac{\bar{A}}{\bar{B}}$  and  $Var(k) = 2\frac{\bar{A}}{\bar{B}^2}$ . If we believe that  $E(k) = 0.1$  and the standard deviation of the  
285 biasing parameter is  $\sigma_k$ , then  $\bar{B} = \frac{0.2}{\sigma_k^2}$ . If  $\sigma_k^2$  is 0.1, 0.5, 1 or 5 then we obtain respectively that  $\bar{B}$  is  
286 2.0, 0.40, 0.20 or 0.04. Therefore  $\bar{A}$  must be, respectively, 0.2, 0.004, 0.02 or 0.004. In what follows  
287 we adopt the reference prior  $p(\sigma) \propto \sigma^{-1}$ .

288

289

290

## 291 4. Sampling properties of diagnosing and correcting multicollinearity

292

293

### 294 4.1. Diagnosing Collinearity

295

296 Two reasonable questions: how we diagnose for collinearity using the Bayesian approach, and how  
297 the Bayesian ridge model in section 3.2<sup>5</sup> behaves compared to ordinary least square (OLS)<sup>6</sup>.

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<sup>4</sup>Notice that  $Gamma(\bar{A}/2, \bar{B}/2)$  reduce to an exponential prior for  $k$ , by setting  $\bar{A} = 2$ .

<sup>5</sup> As mentioned, we recommend relying on the non-conjugate prior for the reasons mentioned in Section 4.2.



298  
 299 Specifically, the question is whether the Bayesian approach can be useful in both diagnosing and  
 300 correcting the possibly harmful effects of multicollinearity in circumstances that are encountered in  
 301 practice. To illustrate this, we choose a design with  $n=500$  observations and  $p=10$  regressors.  
 302 The first regressor, say  $X_{i1}$ , is generated from a standard normal distribution. The remaining  
 303 regressors are  $X_{ij} = \alpha X_{i,j-1} + \omega Z_{ij}$ ,  $Z_{ij} \sim iidN(0,1)$ ,  $j=2, \dots, p$ , and  $\omega$  is set to  $1/500$  with  $\alpha=1$   
 304 for collinear data, and  $\alpha=0$ ,  $\omega=1$  for independent data. The data generating process is  
 305  $y_i = \sum_{j=1}^p \beta_j x_{ij} + u_i$ , where all regression coefficients are  $\beta_j=1$  ( $j=1, \dots, p$ ), and  $u_i \sim iidN(0, \sigma^2)$ ,  
 306 with  $\sigma=1$ <sup>7</sup>. Setting all coefficients equal to one is done only for simplicity and the results in no way  
 307 depend on the exact true values of the coefficients.  
 308

309 As our prior on the biasing constant,  $k$ , we choose an exponential with parameter  $\bar{B}=10^3$  implying  
 310 a prior average value of  $k$  equal to  $E(k)=10^{-3}$  which seems reasonable in view of experience with  
 311 collinear data. Holding the matrix of regressors,  $X$ , fixed we generate  $D=10,000$  different data  
 312 sets. For each data set, the Gibbs sampling technique presented in section 4.2 is applied using  
 313 11,000 draws, omitting the first 1,000 and taking only every other tenth draw (for a total for 1,000  
 314 draws). From the 1,000 available, approximately independent, draws we compute the posterior  
 315 means,  $\bar{\beta}^{(d)} = E(\beta | \mathbf{y})$ ,  $d=1, \dots, D$ . Our objective is to compare the sampling distribution resulting  
 316 from  $\bar{\beta}^{(d)}$ s (as an approximation to the actual sampling distribution) with the sampling distribution  
 317 of posterior means resulting from a diffuse prior, which is based on OLS quantities that are readily  
 318 available for each different data set.  
 319

320 With a diffuse prior (i.e. OLS), the sampling distribution of posterior means should be more  
 321 dispersed compared to the sampling distribution of  $E(\beta | \mathbf{y})$  under the stated prior on the biasing  
 322 constant,  $k$ . The sampling results are presented in the three panels of Figure 1. For orthogonal data  
 323 (i.e. no collinearity) ridge and diffuse posteriors are extremely close (Figure 1a), but this is not the  
 324 case when we have collinear data (Figures 1b and 1c). In other words, for collinear data, one would  
 325 observe significant difference between the diffuse posterior (i.e. OLS) and the ridge regression  
 326 results.  
 327

328 This, in turn, would provide a useful way to detect whether there is harmful multicollinearity and, at  
 329 the same time, correct it based on the Bayesian ridge estimator. The test can be made more formal  
 330 by using a Kolmogorov-Smirnov test for testing the equality of the two distributions. We believe,  
 331 however, that visual presentation is much more informative. We can also use a formal test for  
 332 collinearity in Bayesian analysis. In OLS settings such tests are not possible. For example, variance  
 333 inflation factors (VIF) commonly used are diagnostics of collinearity, not statistical tests.  
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<sup>6</sup> This is similar to Bayesian analysis using diffuse priors of the form  $p(\beta, \sigma) \propto \sigma^{-1}$ , from the sampling-theory point of view

<sup>7</sup> All computations were performed using the WinGauss software. The codes can be provided by the authors upon request.

335 In our case, one can formally test for collinearity using the Bayes factor. Given the marginal  
 336 posterior  $p(k|\mathbf{y})$  the Bayes factor in favor of ridge regression and against OLS can be  
 337 approximated using

$$338 \quad BF \simeq \frac{p(\hat{k}|\mathbf{y})}{p(k=0|\mathbf{y})} \quad (6)$$

339 where  $\hat{k}$  is the modal value of the marginal posterior. In our case the denominator is practically  
 340 zero, so the BF diverges to a very large value, indicating that the ridge regression model fits the data  
 341 best. For this approach see Berger (1980, p. 156).

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#### 344 4.2. Correcting for Collinearity

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346 To correct for collinearity we propose the Bayesian ridge estimator proposed in Section 4.2. While  
 347 we saw in Figure 1.c how the Bayesian diffuse prior (i.e. OLS) can seriously affect the regression  
 348 results in the presence of collinear data, the important issue that remains is how reliable our Bayesian  
 349 estimator is in the presence of collinearity.

350 We run a similar experiment to the above where we compared the performance of the Bayesian  
 351 diffuse prior against the Bayesian ridge regression. We also included in the comparison a traditional  
 352 ridge regression model (non-Bayesian) with a pre-specified value for  $k = 0.001^8$ . For the diffuse  
 353 prior context, we followed a common practice in the literature and tried to drop the collinear  
 354 variables from the model.

355 We use 10,000 Monte Carlo replications to compare the above models. We tried three versions of  
 356 the diffuse prior model by dropping one, two and three variables at a time. The true model for the  
 357 simulation is

358  $y_i = x_{i1} + x_{i2} + x_{i3} + x_{i4} + x_{i5} + 0.1u_i$  where  $u_i \sim N(0,1)$  and the regressors are generated as follows.

359  $x_{i1} \sim N(0,1)$

360  $x_{ij} = x_{i1} + 0.1v_{ij}, v_{ij} \sim N(0,1), j = 2, \dots, 5.$

361 For testing purposes we set all  $\beta_j$  ( $j = 1, \dots, p$ ) as equal to 1. Again, the results do not depend on  
 362 the exact values of these coefficients.

363 The results are presented in Figure 2. We can see that the Bayesian ridge regression based on the  
 364 optimal prior seems to performs best and is the one most centered around the true value of  $\beta$ .  
 365 Contrary to common belief, the practice of dropping variables from the models, on the other hand,

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<sup>8</sup> This is a standard value for  $k$  used in the literature. Of course, our proposed approach focuses on optimizing the value of  $k$  and not on pre-specifying the value of  $k$ , as discussed previously.

366 does not seem to be a good choice for correcting the results of the regression model. The closest to  
367 our model is the traditional ridge regression, but this has the problem of pre-specifying the value of  
368  $k$  in advance. Further illustration with real data is presented next.

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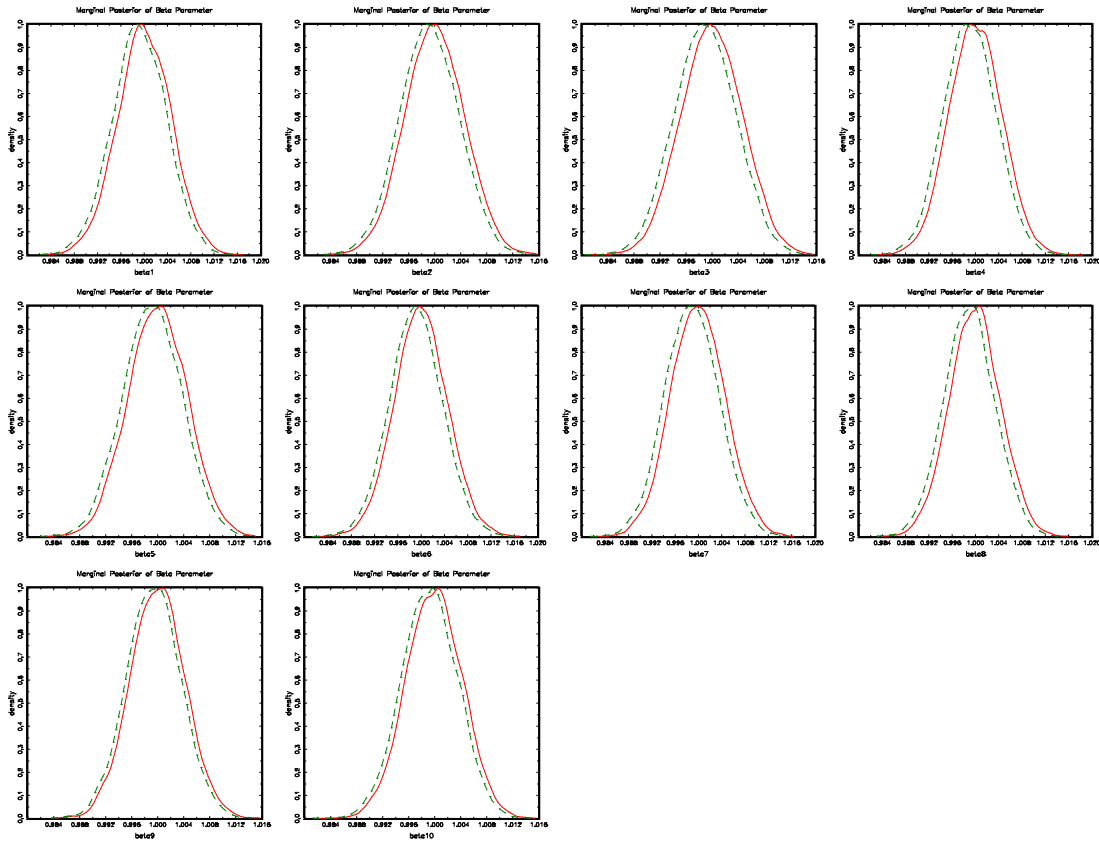
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395 Figure 1. Sampling distributions of different estimators of  $\beta_2$ .

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397 **Figure 1a. Sampling Distributions of Posterior Mean from Bayesian diffuse analysis and**  
398 **Bayesian ridge analysis ( $\bar{B} = 1$ ). Orthogonal data.**

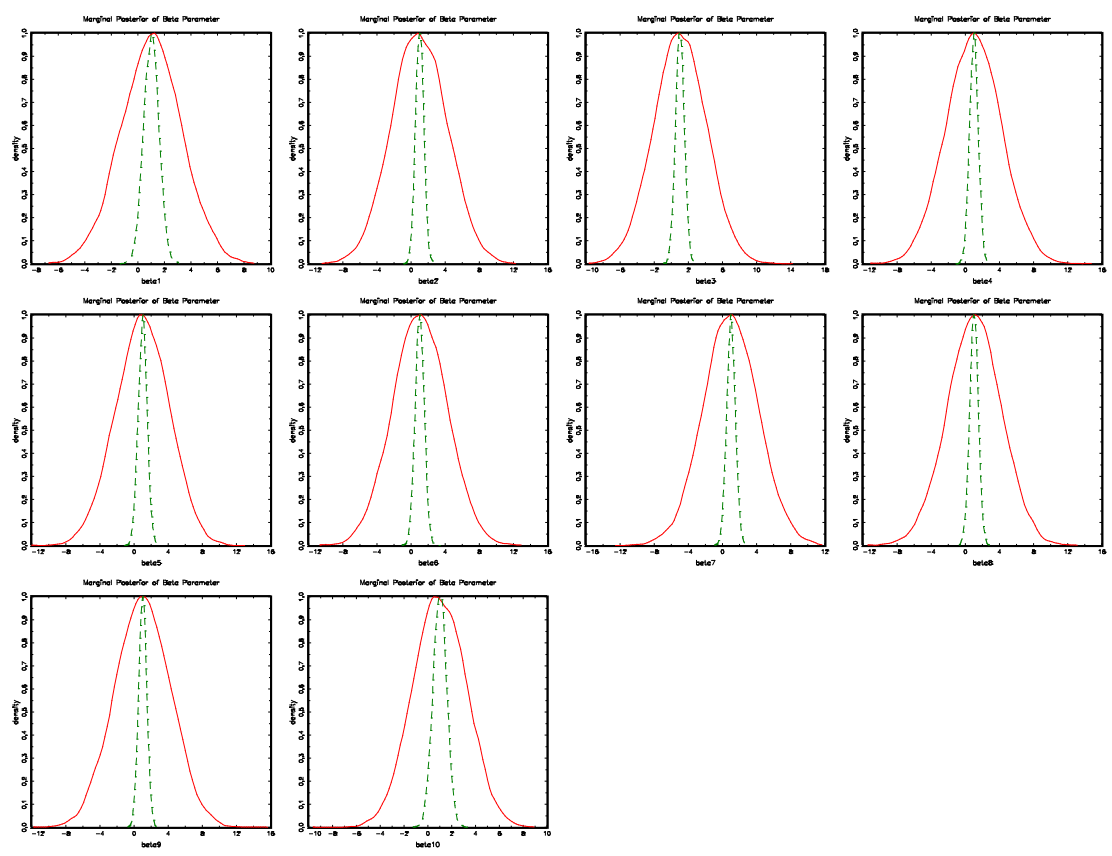
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417 **Figure 1b. Sampling Distributions of Posterior Mean from Bayesian diffuse analysis and**  
418 **Bayesian ridge analysis ( $\bar{B} = 10^3$ ). Collinear data.**

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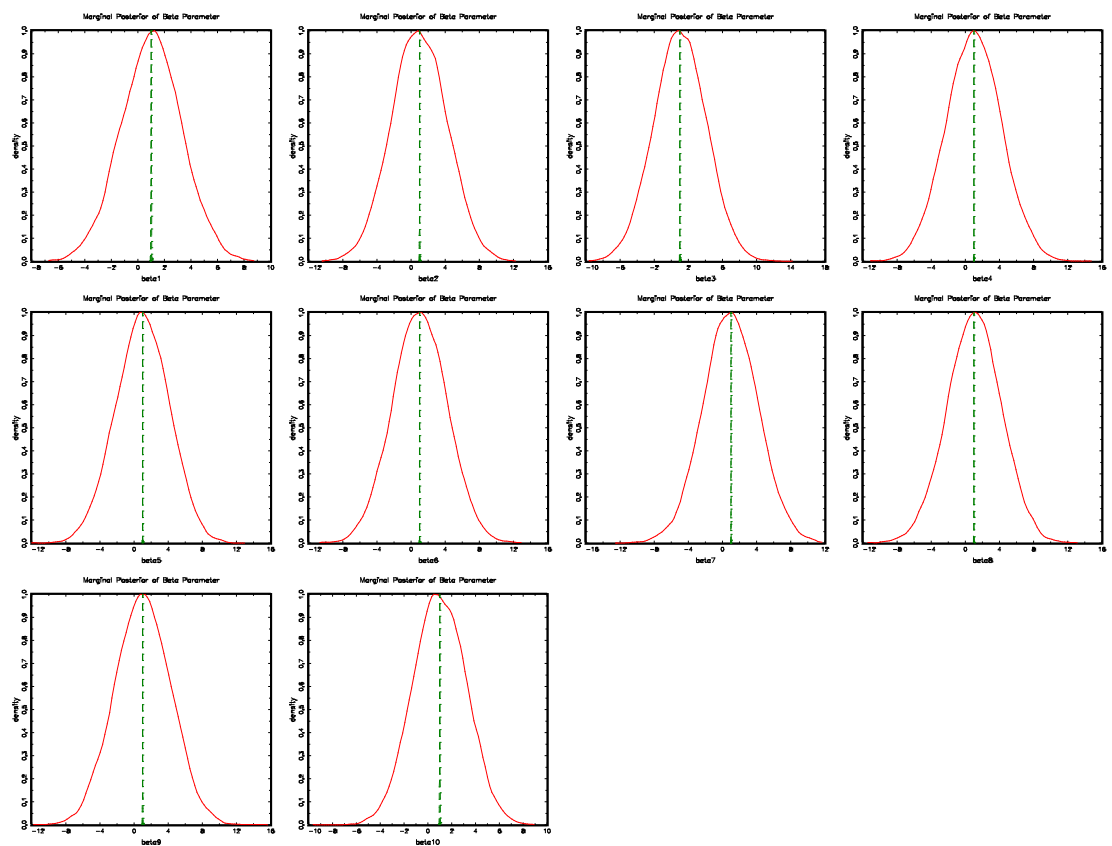
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434 **Figure 1c. Sampling Distributions of Posterior Mean from Bayesian diffuse analysis and**  
435 **Bayesian ridge analysis ( $\bar{B} = 1$ ). Collinear data.**  
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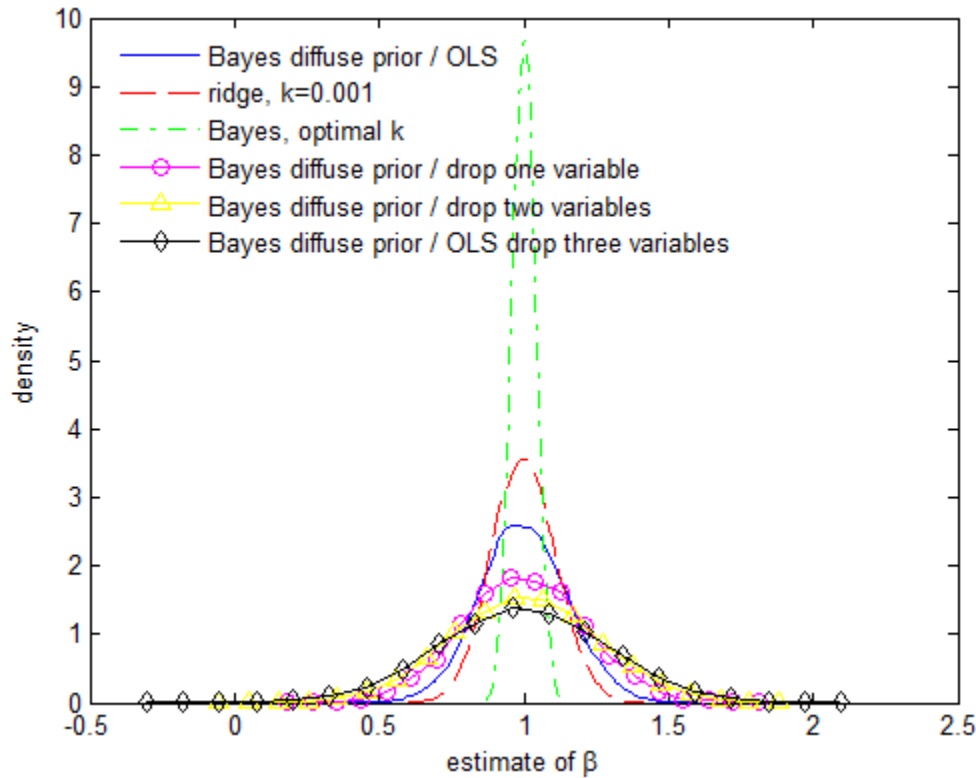


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451 **Figure 2. Performance of the Bayesian ridge regression against other alternatives**

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## 456 5. Illustration using Real Data

457 We also test our Bayesian ridge regression using a real application the hotel industry. The model  
458 focuses on the relationship between room revenue and the following covariates: room expenses,  
459 food and beverage (F&B) expenses, utility expenses, marketing expenses, property and maintenance  
460 (POM) expenses and number of rooms. All these variables are expected to be positively correlated  
461 with room revenue as higher revenue usually results in higher expenses in these categories.

462 The dataset for this study was obtained from Smith Travel Research, an independent company that  
463 tracks lodging supply and demand data for most major hotels in the US and internationally. The  
464 STR's data are highly comprehensive, reliable and mostly commonly by hotels to track their  
465 performance<sup>9</sup>.

466 We use here a unique panel sample of 78 US hotels (for the years 2012-2016). So, in total we have  
467 390 observations. The correlation matrix for all variables included in the model (Table 4) clearly

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<sup>9</sup> At least in the United States.

468 illustrates the high collinearity problem. Further evidence on the collinearity problem in this dataset  
469 is illustrated in Table 5 where we can see that the variance inflation factors (VIFs) for five of the six  
470 covariates are  $>10$ . Our Bayes factor (equation 6) also diverges to a very large value, indicating that  
471 the ridge regression model fits the data best.

472 We report in Table 6 the results from Bayesian ridge regression and linear regression (i.e. OLS). For  
473 the Bayesian estimation, we used the non-conjugate prior described in Section 4.2. As mentioned,  
474 one of the advantages is that with the Bayesian approach, we do not pre-set or select a single value  
475 of  $k$  but we produce the whole marginal posterior of this parameter given the data. For example, we  
476 report in Figure 3 the overall posterior density of  $k$ .

477 The posterior mean of  $k$  is also included in Table 6. We can clearly the differences between the  
478 results obtained from the Bayesian ridge regression vs. OLS. For instance, despite the high positive  
479 correlation between the various covariates and the dependent variable, several coefficients from  
480 OLS have a negative sign and only three of them are significant. The Bayesian ridge regression  
481 however, indicates that all coefficients are positive and significant. This confirms our earlier results  
482 from the simulation that when collinearity exists, ridge and least square results can be very different.

483 Of course, we are not implying that collinearity is of less concern than is often implied in the  
484 literature. While our method seems to be more tolerant to collinearity, the results should not  
485 encourage tourism researchers to throw any variable into the model and expect the results to come  
486 out perfectly. The selection of variables should still be based on an educated theoretical approach.  
487 In contexts when collinearity cannot be avoided, the practices of mean centering, or dropping  
488 variables do not seem to be good choices for correcting the results of the regression model. Rather,  
489 the regression estimation should be conducted using more robust approaches such as the one we  
490 propose in this study.

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505 Table 4. Correlation Matrix

	1	2	3	4	5	6	7
Room revenue (1)	1						
Room Expenses (2)	0.9867	1					
F&B expenses (3)	0.9640	0.9744	1				
Utility Expenses (4)	0.9815	0.9741	0.9543	1			
Marketing Expenses(5)	0.9463	0.9279	0.9379	0.9523	1		
POM Expenses (6)	0.9615	0.9717	0.9741	0.9602	0.9514	1	
Number of Rooms (7)	0.8773	0.8839	0.8685	0.9075	0.9242	0.8967	1

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508 Table 5. Multicollinearity Diagnostic Criteria

	Eigen Values	VIF	1/VIF
Room Expenses	5.7033	34.3711	0.0291
F&B expenses	0.157	25.0432	0.0399
Utility Expenses	0.0566	28.6689	0.0349
Marketing Expenses	0.0423	19.6543	0.0509
POM Expenses	0.0242	33.7577	0.0296
Number of Rooms	0.0165	8.3613	0.1196

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511 Table 6. Bayesian Ridge Regression vs. OLS

Variable	Bayesian Ridge		OLS	
	Posterior Mean	Posterior t-stat	Estimate	t-stat
Room Expenses	5.871	11.068	10.442	16.202
F&B expenses	0.803	3.902	-0.504	-1.127
Utility Expenses	1.414	11.542	0.957	1.844
Marketing Expenses	3.913	38.391	4.628	12.172
POM Expenses	0.960	3.357	-1.378	-2.018
Number of Rooms	1.101	4.871	-0.911	-1.953
sigma	0.211	0.010		
$k$	0.129	0.075		

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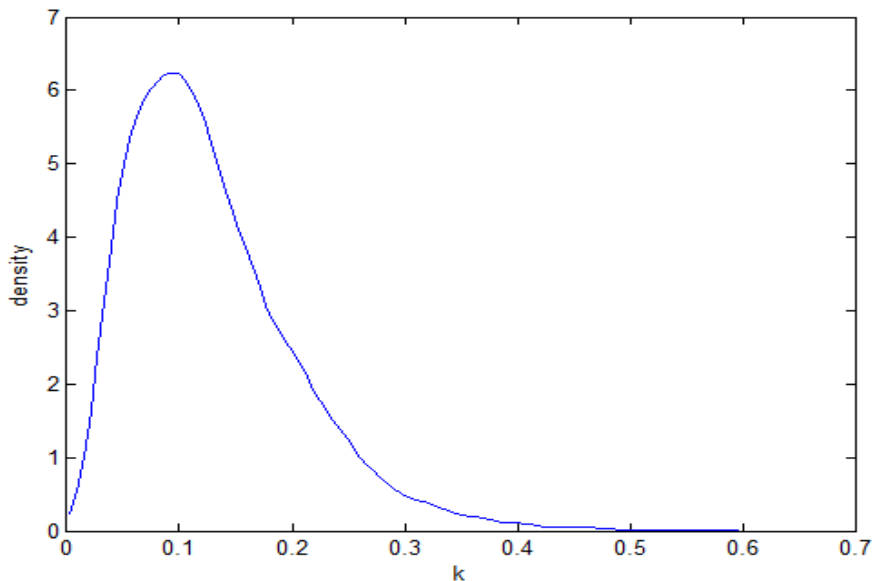
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518 **Figure 3. Posterior Distribution of the Bayesian ridge parameter ( $k$ )**



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## 524 **6. Concluding Remarks**

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526 In this paper, we have taken up Bayesian inference in conjugate and non-conjugate ridge regression  
527 models by using the fact that a prior can be placed on the ridge parameter(s)  $k$  proceed with  
528 posterior analysis on all parameters using standard MCMC techniques. For the conjugate case we  
529 have derived the marginal likelihoods and showed how selection of the  $k$  or  $g$  parameter can be  
530 based in an empirical Bayes context to choose the appropriate value. It is important to notice that  
531 the original ridge regression estimators depend crucially on a conjugacy assumption, namely that the  
532 regression coefficients,  $\boldsymbol{\beta} | \sigma, k \sim N_p\left(0, \frac{\sigma^2}{k} \mathbf{I}_p\right)$ . In the absence of  $\sigma$  the prior of  $\boldsymbol{\beta}$  is no longer in  
533 the normal-gamma prior form which is necessary for ordinary ridge regression to emerge. A non-  
534 conjugate prior of the form  $\boldsymbol{\beta} | k \sim N_p\left(0, \frac{1}{k} \mathbf{I}_p\right)$  can be adopted instead, and numerical posterior  
535 inference can rely on the Gibbs sampler. Conjugate priors have certain problems, for example they  
536 have the same tails with the likelihood and they are rarely used in practice.

537

538 We have applied these ideas to show that collinearity can be simultaneously diagnosed and corrected

539 using priors on all parameters. We also illustrated that the Bayesian ridge regression performs better  
540 than a Bayesian regression with diffuse prior (i.e. OLS). Contrary to common belief, the practice of  
541 dropping variables from the models, does not also seem to be a good choice for correcting the  
542 results of the regression model. Our method can handle collinearity more robustly when variables  
543 are collinear and should be theoretically left in the model. We showed through simulation and  
544 through marginal likelihood maximization that the approach we are proposing provide more robust  
545 findings. This should provide tourism researchers with more flexibility as they handle such type of  
546 data.

547  
548 One limitation of the paper is that we focus on a single  $k$ . Different  $k$ s can be used for each  
549 regressor easily, although at the cost of computing several values of the marginal likelihood  
550 depending on the value of such  $k$  coefficients. Our methods illustrate that biased estimators yielding  
551 lower mean squared error (MSE) are clearly desirable and, thus, future research could focus more on  
552 generalizations of our procedure. Another limitation is, of course, the assumption of normality of  
553 errors which, however, can be relaxed to consider more general models including other elliptical  
554 distributions, distributions with fat tails and / or asymmetry, etc.

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