r-Fredholm theory in Banach algebras

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ABSTRACT

Harte [10] initiated the study of Fredholm theory relative to a unital homomorphism $T : A \to B$ between unital Banach algebras A and B based on the following notions: an element $a \in A$ is called *Fredholm* if 0 is not in the spectrum of Ta, while a is *Weyl (Browder)* if there exist (commuting) elements b and c in A with $a = b + c$ such that 0 is not in the spectrum of b and c is in the null space of T. We introduce and investigate the concepts of r-Fredholm, r-Weyl and r-Browder elements, where 0 in these definitions is replaced by the spectral radii of a and b, respectively.

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1. Introduction

In the early 1980's Harte [10] introduced Fredholm, Weyl and Browder theory relative to a unital homomorphism $T : A \to B$ between general unital Banach algebras A and B. Several authors have continued this investigation (see [11], [12], [18], [19], [15], [17], [4] and [5]).

In certain applications it is necessary to study the elements $a \in A$ with the property that the spectral radius $r(a)$ of a is an isolated point of the spectrum $\sigma(a)$ of a but outside the spectrum $\sigma(Ta)$ of $Ta \in B$

(see [1], [2] and [5]). In symbols, this property can be expressed as follows:

$$0 \in (\mathrm{iso}\ \sigma(a - r(a)\mathbf{1}))\backslash\sigma(T(a - r(a)\mathbf{1})),$$

which, in the language of Fredholm theory, says that $a - r(a)\mathbf{1}$ is *almost invertible Fredholm* (excluding, here, the trivial case where $r(a) \notin \sigma(a)$). Introducing the notions "a is *almost r-invertible* if $r(a)$ is not an accumulation point of $\sigma(a)$" and "a is *r-Fredholm* if $r(a) \notin \sigma(Ta)$", this becomes (the non-trivial case of) "a is *almost r-invertible r-Fredholm*".

In studying elements b with the property that $0 \in (\mathrm{iso}\ \sigma(b))\backslash\sigma(Tb)$, the spectral idempotent $p(b, 0)$ associated with b and 0 plays an important role (see [19]). However, information about this spectral idempotent is scarce, and, especially in the context of ordered Banach algebras, more information is available about $p(a, r(a))$ (if $r(a) \in \mathrm{iso}\ \sigma(a)$) — see, for instance, [20]. It therefore seems useful to study the related concepts of "r-Fredholm theory". However, it is not obvious if, and when, the concept "$a - r(a)\mathbf{1}$ is Weyl (Browder)" coincides with a being "r-Weyl (r-Browder)", where an element a is *r-Weyl (r-Browder)* if there exist (commuting) elements b and c in A with $a = b + c$ such that $r(b) \notin \sigma(b)$ and $Tc = 0$. (We recall that this is the definition of "a is *Weyl (Browder)*" where the condition $0 \notin \sigma(b)$ is replaced by the condition $r(b) \notin \sigma(b)$.)

As a result, in this note we investigate what happens to Harte's Fredholm theory if we replace the group A^{-1} of invertible elements of a unital Banach algebra A with the set $A^r = \{a \in A : a - r(a)\mathbf{1} \in A^{-1}\}$. We say that the elements of A^r have *the spectral radius invertibility property*, or simply that they are *r-invertible*. We shall study r-Fredholm, r-Weyl and r-Browder elements, prove analogues of important results of Harte [10] concerning Browder elements, and show that if the relevant homomorphism has the Riesz property, then $a - r(a)\mathbf{1}$ is Browder if and only if a belongs to a particular subset of the r-Browder elements (see Theorems 7.2 and 7.4).

Our paper is organised as follows: In Section 2 the necessary background is given, including important results and examples regarding Fredholm, Weyl and Browder elements. Section 3 introduces r-Fredholm, r-Weyl and r-Browder elements formally and provides some basic properties of these elements, while in Section 4 a number of illustrative examples are given. Some additional properties are obtained if the relevant homomorphism is spectral radius preserving. This is the topic of Section 5, and an important observation in Proposition 5.2 leads to the definition and investigation of certain special sets of r-Weyl and r-Browder elements in Section 6. Finally, the main results and some applications are presented in Section 7.

2. Preliminaries

All our Banach algebras will be complex and unital (with unit $\mathbf{1}$). We denote the set of all invertible elements of a Banach algebra A by A^{-1} and the radical of A by $\mathrm{Rad}(A)$. By "ideal" we will always mean "proper two-sided ideal". If A and B are Banach algebras, then a linear operator $T : A \to B$ (not necessarily continuous) is called a *(unital algebra) homomorphism* if $T(ab) = TaTb$ $(a, b \in A)$ and T maps the unit of A onto the unit of B. Clearly, $TA^{-1} \subseteq B^{-1}$. The null space (kernel) of T will be indicated by $\mathrm{N}(T)$.

The spectrum of an element a in a Banach algebra A will be denoted by $\sigma(a)$, the sets of isolated and accumulation points of $\sigma(a)$ by $\mathrm{iso}\,\sigma(a)$ and $\mathrm{acc}\,\sigma(a)$, respectively, and the spectral radius of a by $r(a)$. In addition, we will denote the *peripheral spectrum* $\{\lambda \in \sigma(a) : |\lambda| = r(a)\}$

of a by $\sigma_{\mathrm{per}}(a)$. Note that this is a non-empty, closed subset of $\sigma(a)$. For $\lambda \in \mathbb{C}\backslash \mathrm{acc}\ \sigma(a)$, we write $p(a,\lambda)$ for the spectral idempotent of $a \in A$ relative to λ. An element $a \in A$ is said to be *almost invertible* if $0 \notin \mathrm{acc}\ \sigma(a)$, and the set of all almost invertible elements of A is denoted by A^D. In the literature almost invertible elements are also called *generalised Drazin invertible elements* (see [13], Theorem 4.2), motivating this notation. An ideal I in A will be called *inessential* whenever $\mathrm{acc}\ \sigma(a) \subseteq \{0\}$ for all $a \in I$. If $T : A \to B$ is a homomorphism and $\mathrm{N}(T)$ is an inessential ideal of A, then T is said to have the *Riesz property*. Clearly, $\sigma(Ta) \subseteq \sigma(a)$ for all $a \in A$. In addition, T is said to be *spectrum preserving* if $\sigma(Ta) = \sigma(a)$ for all $a \in A$, and T is *spectral radius preserving* if $r(Ta) = r(a)$ for all $a \in A$.

If $X \neq \{0\}$ is a Banach space, then $\mathcal{L}(X)$ will denote the Banach algebra of bounded linear operators on X with unit the identity operator I on X. If $\dim X = \infty$, then $\mathcal{K}(X)$ will denote the closed inessential ideal in $\mathcal{L}(X)$ of all compact operators on X. The Banach algebra of all upper triangular 2×2 matrices with complex entries will be denoted by $M_2^u(\mathbb{C})$, while $C(K)$ will be the Banach algebra of all continuous complex-valued functions on a compact Hausdorff space K, with unit the constant function $\mathbf{1}(x) = 1$ for all $x \in K$. If $\mathbb{D}$ is the open unit disc in the complex plane, then $\mathcal{A}(\mathbb{D})$ denotes the *disc algebra*, i.e. the closed subalgebra of $C(\overline{\mathbb{D}})$ consisting of the functions which are analytic on $\mathbb{D}$.

The following result (which is obvious if T is bounded) holds for any homomorphism:

Proposition 2.1. *(*[8]*, Proposition 2.1) Let $T : A \to B$ be a homomorphism. If $a \in A$ and Γ is a contour in $\mathbb{C}\backslash\sigma(a)$ having winding* *number* 0 *or* 1 *around each point of* $\sigma(a)$*, then*

$$T\left(\frac{1}{2\pi i}\int_\Gamma(\lambda\mathbf{1}-a)^{-1}d\lambda\right)=\frac{1}{2\pi i}\int_\Gamma(\lambda\mathbf{1}-Ta)^{-1}d\lambda.$$

We now recall Harte's definitions of Fredholm, Weyl and Browder elements in Banach algebras:

Definition 2.2. ([10], p.431–432) Let $T : A \to B$ be a homomorphism. An element $a \in A$ is called:

- *Fredholm* if $Ta \in B^{-1}$,
- *Weyl* if there exist elements $b \in A^{-1}$ and $c \in \mathrm{N}(T)$ such that $a = b + c$,
- *Browder* if there exist commuting elements $b \in A^{-1}$ and $c \in \mathrm{N}(T)$ such that $a = b + c$,
- *almost invertible Fredholm* if it is Fredholm and almost invertible.

Denote by $\mathcal{F}_T$, $\mathcal{W}_T$, $\mathcal{B}_T$ and $A^D \cap \mathcal{F}_T$ the sets of Fredholm, Weyl, Browder and almost invertible Fredholm elements of A relative to T, respectively. If A is commutative then, obviously, $\mathcal{W}_T = \mathcal{B}_T$.

The above definitions are motivated by the following classical results in operator theory (see, e.g., [6]):

Example 2.3. Let X be an infinite-dimensional Banach space and $\pi : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{K}(X)$ the canonical homomorphism. If $U \in \mathcal{L}(X)$, then:

(1) $U \in \mathcal{F}_\pi$ if and only if $\dim \mathrm{N}(U) < \infty$, $U(X)$ is closed and $\dim(X/U(X)) < \infty$.

(2) $U \in \mathcal{W}_\pi$ if and only if $U \in \mathcal{F}_\pi$ with index zero (i.e. $\dim \mathrm{N}(U) = \dim\,(X/U(X))$), if and only if there exist operators $V \in \mathcal{L}(X)^{-1}$ and $W \in \mathcal{K}(X)$ such that $U = V + W$.

(3) $U \in \mathcal{B}_\pi$ if and only if $U \in \mathcal{F}_\pi$ with finite ascent and descent (in the operator theoretic sense), if and only if there exist commuting operators $V \in \mathcal{L}(X)^{-1}$ and $W \in \mathcal{K}(X)$ such that $U = V + W$.

The following example is due to Harte:

Example 2.4. ([10], p.432) Let K and L be compact Hausdorff spaces and $A := C(K)$ and $B := C(L)$ the Banach algebras of continuous complex-valued functions on K and L, respectively. Consider the homomorphism $T : A \to B$ defined by $Tf = f \circ \theta$, where $\theta : L \to K$ is a continuous map. Then:

(1) $\mathcal{F}_T = \{f \in A : 0 \notin f(\theta(L))\}$.
(2) $f \in \mathcal{W}_T$ if and only if its restriction to $\theta(L)$ has an invertible extension to K.

The non-trivial parts of the following result were proven by Harte for bounded homomorphisms, and generalised to unbounded homomorphisms in [19]:

Theorem 2.5. *([10], (1.4), Theorem 1, [19], p.19) Let $T : A \to B$ be a homomorphism. Then*

$$A^{-1} \subseteq A^D \cap \mathcal{F}_T \subseteq \mathcal{B}_T \subseteq \mathcal{W}_T \subseteq \mathcal{F}_T.$$

Moreover, $A^D \cap \mathcal{F}_T = \mathcal{B}_T$ if and only if T has the Riesz property.

Clearly, if T is spectrum preserving, then all the above inclusions are equalities. If T is only spectral radius preserving, then $\mathrm{N}(T) \subseteq \mathrm{Rad}(A)$, and hence

$$A^{-1} = A^D \cap \mathcal{F}_T = \mathcal{B}_T = \mathcal{W}_T,$$

since $\mathrm{Rad}(A) = \{a \in A : a + A^{-1} \subseteq A^{-1}\}$ (see [14], Theorem 2.5). In this case, the inclusion $\mathcal{W}_T \subseteq \mathcal{F}_T$ is, however, in general strict. This can be seen by considering the homomorphism $T : \mathcal{A}(\mathbb{D}) \to C(\mathbb{T})$ from the disc algebra $\mathcal{A}(\mathbb{D})$ into the Banach algebra $C(\mathbb{T})$ of continuous complex-valued functions on the unit circle $\mathbb{T}$, where $Tf = f_{|\mathbb{T}}$. This homomorphism is spectral radius preserving but not spectrum preserving, and if $f(z) = z$ for all $z \in \overline{\mathbb{D}}$, then $f \in \mathcal{A}(\mathbb{D})$ is Fredholm, but not invertible, so that $f \in \mathcal{F}_T \backslash \mathcal{W}_T$. (See ([9], Example 3).)

Finally, we record the following fact:

Proposition 2.6. *(*[19]*, Theorem 2.4) Let $T : A \to B$ be a homomorphism. If $a \in A^D \cap \mathcal{F}_T$, then $p(a, 0) \in \mathrm{N}(T)$.*

3. Basic r-Fredholm theory

An element a in a Banach algebra A has *the spectral radius invertibility property* if $r(a) \notin \sigma(a)$. We abbreviate this by simply saying that a is *r-invertible*. Analogously, a is said to be *almost r-invertible* if $r(a) \notin \mathrm{acc}\ \sigma(a)$. We shall denote by A^r and A^{D^r} the sets of r-invertible and almost r-invertible elements of A, respectively. Consequently, $A^r \subseteq A^{D^r}$.

As can be seen from Definition 2.2, the set of invertible elements plays a crucial role in Fredholm theory. Here we (analogously) introduce notions which make use of the set of r-invertible elements:

Definition 3.1. Let $T : A \to B$ be a homomorphism. An element $a \in A$ is called

- *r-Fredholm* if $r(a) \notin \sigma(Ta)$,
- *r-Weyl* if there exist elements $b \in A^r$ and $c \in \mathrm{N}(T)$ such that $a = b + c$,

- *r-Browder* if there exist commuting elements $b \in A^r$ and $c \in \mathrm{N}(T)$ such that $a = b + c$,
- *almost r-invertible r-Fredholm* if it is r-Fredholm and almost r-invertible.

Denote by $\mathcal{F}_T^r$, $\mathcal{W}_T^r$, $\mathcal{B}_T^r$ and $A^{D^r} \cap \mathcal{F}_T^r$ the sets of r-Fredholm, r-Weyl, r-Browder and almost r-invertible r-Fredholm elements of A relative to T, respectively. If A is commutative then, of course, $\mathcal{W}_T^r = \mathcal{B}_T^r$. We note that the sets of r-Weyl and r-Browder elements depend only on the null space of the homomorphism in the sense that if $S : A \to B$ and $T : A \to C$ are homomorphisms with $\mathrm{N}(S) = \mathrm{N}(T)$, then $\mathcal{W}_S^r = \mathcal{W}_T^r$ and $\mathcal{B}_S^r = \mathcal{B}_T^r$. (An analogous statement holds, of course, for the sets of Weyl and Browder elements.) The following inclusions are obvious:

Proposition 3.2. *Let $T : A \to B$ be a homomorphism. Then*

$$A^r \subseteq \mathcal{B}_T^r \subseteq \mathcal{W}_T^r \text{ and } A^r \cup T^{-1}(B^r) \subseteq \mathcal{F}_T^r.$$

However, in general no relation holds between A^r and $T^{-1}(B^r)$ — see Example 4.2.

We shall next describe the elementary connections between the above notions and their "classical" counterparts in Harte's Fredholm theory.

Lemma 3.3. *Let $T : A \to B$ be a homomorphism. Then:*

(1) *$a \in A^r$ if and only if $a - r(a)\mathbf{1} \in A^{-1}$.*
(2) *$a \in A^{D^r}$ if and only if $a - r(a)\mathbf{1} \in A^D$.*
(3) *If $a \in \mathcal{B}_T^r$, then $a - r(b)\mathbf{1} \in \mathcal{B}_T$, where b is the r-invertible component in a decomposition of a.*
(4) *If $a \in \mathcal{W}_T^r$, then $a - r(b)\mathbf{1} \in \mathcal{W}_T$, where b is the r-invertible component in a decomposition of a.*

(5) *$a \in \mathcal{F}_T^r$ if and only if $a - r(a)\mathbf{1} \in \mathcal{F}_T$, and hence $a \in A^{D^r} \cap \mathcal{F}_T^r$ if and only if $a - r(a)\mathbf{1} \in A^D \cap \mathcal{F}_T$.*

The proof of the above lemma follows easily from the definitions of the respective sets. We note that $r(b)$ in (3) and (4) of Lemma 3.3 cannot be replaced by $r(a)$ in general (see Example 4.2 below). Proposition 6.2 will show, however, that this can be done if $\mathcal{B}_T^r$ and $\mathcal{W}_T^r$ are replaced by certain special subsets. In Theorem 7.4 we will show that the converse of (3), with $r(b)$ replaced by $r(a)$, holds whenever T has the Riesz property.

4. Examples

The situation for function spaces corresponding to Example 2.4 is as follows:

Example 4.1. Let $T : A \to B$ be the homomorphism $Tf = f \circ \theta$, where K and L are compact Hausdorff spaces, $A := C(K)$, $B := C(L)$ and $\theta : L \to K$ is a continuous map, as in Example 2.4. Then:

(1) $A^r = \{f \in A : r(f) \notin f(K)\}$.

(2) $\mathcal{F}_T^r = \{f \in A : r(f) \notin f(\theta(L))\}$.

(3) $\mathcal{W}_T^r = \begin{cases} A^r & \text{if} \quad \theta(L) = K \\ A & \text{otherwise.} \end{cases}$

Proof. For (1) we only need to note that $\sigma(f) = f(K)$, while (2) follows directly from the definition of $\mathcal{F}_T^r$ (or from Example 2.4 and Lemma 3.3(5)).

Towards (3), we note that $\mathrm{N}(T) = \{f \in A : f(\theta(L)) = \{0\}\}$. Hence if $\theta(L) = K$, then $\mathrm{N}(T) = \{0\}$, so that $\mathcal{W}_T^r = A^r$. Otherwise, choose $x_0 \in K \backslash \theta(L)$. By Urysohn's Lemma ([21], Theorem 15.6) there exists $h \in A$ such that $h(\theta(L)) = \{0\}$, $h(x_0) = 1$ and $0 \leq h \leq 1$. Let

$f \in A$ and consider $g = (1-h)f + i(r(f)+1)h$. Then $g \in A$ and $(f-g)(\theta(L)) = \{0\}$. Therefore $f = g + u$ where $u = f - g \in \mathrm{N}(T)$.

Since $h(x_0) = 1$, we have that

$$r(g) = \max_{x \in K} |g(x)| \geq |g(x_0)| = |i(r(f)+1)| = r(f) + 1.$$

So if $g \notin A^r$, i.e. $r(g) = g(x_1)$ for some $x_1 \in K$, then since $0 \leq h \leq 1$,

$$r(g) = \mathrm{Re}\ g(x_1) = (1 - h(x_1))(\mathrm{Re}\ f(x_1)) \leq |f(x_1)| \leq \max_{x \in K} |f(x)| = r(f),$$

which contradicts the previous inequality. Therefore $g \in A^r$ and hence $f \in \mathcal{W}_T^r$. □

Example 4.2. In Example 4.1, let K be any 2-point space, L a 1-point subset of K and θ the inclusion map. Identifying $A = C(K)$ with $\mathbb{C}^2$ and $B = C(L)$ with $\mathbb{C}$ in the usual way, with $T : \mathbb{C}^2 \to \mathbb{C}$ becoming the homomorphism $T(z_1, z_2) = z_1$, there exists $a \in \mathcal{W}_T^r$ such that $a - r(a)\mathbf{1} \notin \mathcal{W}_T$. Also, neither of the inclusions $A^r \subseteq T^{-1}(B^r)$ or $T^{-1}(B^r) \subseteq A^r$ hold.

Proof. By Example 4.1(3) the element $a = (1, -1)$ is in $A = \mathcal{W}_T^r$, but since $a - r(a)\mathbf{1} = (0, -2) \in \mathrm{N}(T)$, it follows that $a - r(a)\mathbf{1} \notin \mathcal{W}_T$. In addition, for the element $a = (1, -2)$ we have that $a \in (\mathbb{C}^2)^r$, but $Ta = 1 \notin \mathbb{C}^r$, and if $a = (-1, 1)$, then $Ta = -1 \in \mathbb{C}^r$, while $a \notin (\mathbb{C}^2)^r$. □

For an elementary homomorphism from the algebra of upper triangular 2×2 matrices onto $\mathbb{C}$ we have:

Proposition 4.3. *Let $A = M_2^u(\mathbb{C})$ and $T : A \to \mathbb{C}$ the homomorphism $Ta = a_1$, where $a = \begin{pmatrix} a_1 & a_2 \\ 0 & a_4 \end{pmatrix}$. Then:*

(1) $\mathcal{B}_T = \mathcal{W}_T = A \backslash \mathrm{N}(T) = \left\{ \left(\begin{smallmatrix} a_1 & a_2 \\ 0 & a_4 \end{smallmatrix} \right) : a_1, a_2, a_4 \in \mathbb{C} \text{ and } a_1 \neq 0 \right\}$.

(2) $a - r(a)\mathbf{1} \in \mathcal{B}_T$ *if and only if* $r(a) \neq a_1$, *if and only if* $a \in \mathcal{F}_T^r$.

(3) $\mathcal{W}_T^r = A$.

(4) $\mathcal{B}_T^r = A \backslash \left\{ \left(\begin{smallmatrix} x & z \\ 0 & x \end{smallmatrix} \right) : x \geq 0 \ \mathit{and}\ z \neq 0 \right\}$.

Proof. (1) The final equality is obvious, since

$$\mathrm{N}(T) = \left\{ \left(\begin{smallmatrix} 0 & a_2 \\ 0 & a_4 \end{smallmatrix} \right) : a_2, a_4 \in \mathbb{C} \right\}.$$

Suppose that $a = b + c$, where $b \in A^{-1}$ and $c \in \mathrm{N}(T)$. Then $Ta = Tb \in B^{-1}$, so that $a \notin \mathrm{N}(T)$. Now let $a \in A \backslash \mathrm{N}(T)$. Then $a_1 \neq 0$. Therefore $b = a_1 \mathbf{1} = \left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_1 \end{smallmatrix} \right) \in A^{-1}$ and if $c = a - b$, then $c \in \mathrm{N}(T)$ and b commutes with c, so that $a \in \mathcal{B}_T$. We have proven that $\mathcal{W}_T \subseteq A \backslash \mathrm{N}(T) \subseteq \mathcal{B}_T$, and since $\mathcal{B}_T \subseteq \mathcal{W}_T$, the result follows.

(2) It follows from (1) that $a - r(a)\mathbf{1} = \left(\begin{smallmatrix} a_1 - r(a) & a_2 \\ 0 & a_4 - r(a) \end{smallmatrix} \right) \in \mathcal{B}_T$ if and only if $a_1 \neq r(a)$. In addition, $a_1 \neq r(a)$ if and only if $r(a) \notin \{a_1\} = \sigma(Ta)$, if and only if $a \in \mathcal{F}_T^r$.

(3) Let $a \in A$. If $b = \left(\begin{smallmatrix} a_1 & 0 \\ 0 & -|a_1|-1 \end{smallmatrix} \right)$ and $c = \left(\begin{smallmatrix} 0 & a_2 \\ 0 & a_4 + |a_1| + 1 \end{smallmatrix} \right)$, then $c \in \mathrm{N}(T)$ and $r(b) = |a_1| + 1 \notin \sigma(b)$, so that $b \in A^r$. Hence $a = b + c \in \mathcal{W}_T^r$.

(4) Let $a = \left(\begin{smallmatrix} a_1 & a_2 \\ 0 & a_4 \end{smallmatrix} \right)$. Then $a \neq \left(\begin{smallmatrix} x & z \\ 0 & x \end{smallmatrix} \right)$, where $x \geq 0$ and $z \neq 0$, if and only if

(i) $a_1 \notin [0, \infty)$, or

(ii) $a_4 \neq a_1 \geq 0$, or

(iii) $a_2 = 0$ and $a_1 = a_4 \geq 0$.

In case (i), let $b = a_1 \mathbf{1} = \left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_1 \end{smallmatrix} \right)$ and $c = \left(\begin{smallmatrix} 0 & a_2 \\ 0 & a_4 - a_1 \end{smallmatrix} \right)$. Then $r(b) = |a_1| \notin \{a_1\} = \sigma(b)$, so that $b \in A^r$. Clearly $c \in \mathrm{N}(T)$ and b commutes with c, so that $a = b + c \in \mathcal{B}_T^r$.

In case (ii), let $b = \left(\begin{smallmatrix} a_1 & b_2 \\ 0 & -a_1 - 1 \end{smallmatrix} \right)$ and $c = \left(\begin{smallmatrix} 0 & a_2 - b_2 \\ 0 & a_4 + a_1 + 1 \end{smallmatrix} \right)$, where $b_2 \in \mathbb{C}$ will be determined below. Then $c \in \mathrm{N}(T)$ and $r(b) = a_1 + 1 \notin \sigma(b)$, so that $b \in A^r$. Using $a_4 \neq a_1$ and comparing the entries of bc and cb in the upper right-hand corners, it can be seen that b commutes with c if (and only if) $b_2 = \frac{(2a_1+1)a_2}{a_1-a_4}$. Hence, choosing this value for b_2, we obtain that $a = b+c \in \mathcal{B}_T^r$.

Finally, in case (iii), let $b = \begin{pmatrix} a_1 & 0 \\ 0 & -a_1-1 \end{pmatrix}$ and $c = \begin{pmatrix} 0 & 0 \\ 0 & 2a_1+1 \end{pmatrix}$. As before, $b \in A^r$ and $c \in \mathrm{N}(T)$, and $bc = cb$ because they are diagonal. Hence $a = b + c \in \mathcal{B}_T^r$.

We conclude by proving that if $a = \begin{pmatrix} x & z \\ 0 & x \end{pmatrix}$ with $x \geq 0$ and $z \neq 0$, then $a \notin \mathcal{B}_T^r$. Indeed, assuming that $a \in \mathcal{B}_T^r$, we must have $a = b + c$ with $b = \begin{pmatrix} x & b_2 \\ 0 & b_4 \end{pmatrix} \in A^r$, i.e. $b_4 \neq |b_4| > x$, $c = \begin{pmatrix} 0 & z-b_2 \\ 0 & x-b_4 \end{pmatrix} \in \mathrm{N}(T)$ and $bc = cb$. However, the commutativity of b and c forces the equation $z(x - b_4) = 0$. Since $z \neq 0$, we obtain the contradiction $x = b_4$, and so $a \notin \mathcal{B}_T^r$. □

We note that (3) and (4) of Proposition 4.3 illustrate that, in general, the inclusion $\mathcal{B}_T^r \subseteq \mathcal{W}_T^r$ is proper.

5. Spectral radius preserving homomorphisms

We recall that, if $T : A \to B$ is a homomorphism, then the inclusion $TA^{-1} \subseteq B^{-1}$ holds. However, the inclusion $TA^r \subseteq B^r$ is not in general true (see Example 4.2). We have the following result.

Lemma 5.1. *Let $T : A \to B$ be a homomorphism. If $a \in A^r$ is such that $r(a) = r(Ta)$, then $Ta \in B^r$. Hence, if T is spectral radius preserving, then $TA^r \subseteq B^r$.*

Proof. Suppose that $a \in A^r$ satisfies $r(a) = r(Ta)$. Then $r(Ta) \notin \sigma(a)$, and hence $r(Ta) \notin \sigma(Ta)$ follows from the fact that $\sigma(Ta) \subseteq \sigma(a)$. This gives $Ta \in B^r$. □

Recall that the inclusion $\mathcal{W}_T \subseteq \mathcal{F}_T$ holds. An r-Weyl element is, however, not in general r-Fredholm — see Example 6.6. Our next result gives conditions under which r-Weyl elements are r-Fredholm.

Proposition 5.2. *Let $T : A \to B$ be a homomorphism and suppose that $a \in \mathcal{W}_T^r$, say $a = b + c$, where $b \in A^r$ and $c \in \mathrm{N}(T)$. If $r(b) \leq r(a)$, then $a \in \mathcal{F}_T^r$. In particular, if T is spectral radius preserving, then $\mathcal{W}_T^r \subseteq \mathcal{F}_T^r$.*

Proof. Suppose that $r(b) \leq r(a)$. Since $b \in A^r$, we have that $r(b) \notin \sigma(b)$, and hence $r(a) \notin \sigma(b)$, so that $r(a) \notin \sigma(Tb) = \sigma(Ta)$. This gives $a \in \mathcal{F}_T^r$.

Suppose now that T is spectral radius preserving. Then $r(a) = r(Ta) = r(Tb) = r(b)$, and hence the result follows from our previous reasoning. $\square$

It follows from Propositions 3.2 and 5.2 that if T is spectral radius preserving, then

$$A^r \subseteq \mathcal{B}_T^r \subseteq \mathcal{W}_T^r \subseteq \mathcal{F}_T^r.$$

In fact, in this case *all* these sets coincide — see Corollary 5.5. To prove this we need the following result of Mathieu and Schick:

Theorem 5.3. *([16], Proposition 4.7) Let A and B be Banach algebras, and let $T\colon A \to B$ be a unital, linear map which preserves the spectral radius. Then T preserves the peripheral spectrum, i.e.*

$$\sigma_{\mathrm{per}}(a) = \sigma_{\mathrm{per}}(Ta), \tag{5.4}$$

for all $a \in A$.

We remark that Mathieu and Schick stated the above result under the additional assumption that T is bijective. This assumption is, however, superfluous. Indeed, the proof of ([16], Proposition 4.7) establishes the inclusion $\subseteq$ in (5.4). Conversely, taking $\lambda \in \sigma_{\mathrm{per}}(Ta)$, we can argue as in the first display on p.198 of [16] to see that $r(a+\lambda) = 2r(a)$, so that there is $\alpha \in \sigma(a)$ such that $|\alpha + \lambda| = 2|\lambda|$. Now the rest of the argument carries over verbatim to show that $\lambda = \alpha \in \sigma_{\mathrm{per}}(a)$. We are grateful to Martin Mathieu for outlining this argument to us.

Corollary 5.5. *Let $T\colon A \to B$ be a spectral radius preserving homomorphism. Then $\mathcal{F}_T^r = A^r$. Hence*

$$A^r = \mathcal{B}_T^r = \mathcal{W}_T^r = \mathcal{F}_T^r.$$

Proof. We have that $r(a) \in \sigma(a)$ if and only if $r(a) \in \sigma_{\mathrm{per}}(a)$ if and only if $r(a) \in \sigma_{\mathrm{per}}(Ta)$ (using Theorem 5.3) if and only if $r(a) \in \sigma(Ta)$. $\square$

By Lemma 3.3 we have for any homomorphism T that $a \in \mathcal{F}_T^r$ if and only if $a - r(a)\mathbf{1} \in \mathcal{F}_T$ (and similarly for A^r and A^{D^r}). In addition, it now follows from Corollary 5.5, together with Lemma 3.3(1) and the remark following Theorem 2.5, that if T is spectral radius preserving, then $a \in \mathcal{W}_T^r$ if and only if $a - r(a)\mathbf{1} \in \mathcal{W}_T$ (and similarly for $\mathcal{B}_T^r$ and $\mathcal{B}_T$).

Recall that $T^{-1}(B^{-1}) = \mathcal{F}_T$ by definition. However, in general, $T^{-1}(B^r) \neq \mathcal{F}_T^r$ (see Example 4.2, where $\mathcal{F}_T^r \not\subseteq T^{-1}(B^r)$, since $A^r \not\subseteq T^{-1}(B^r)$). The following result gives a sufficient condition for equality to hold. It follows from Corollary 5.5 and Lemma 5.1, together with Proposition 3.2.

Corollary 5.6. *Let $T : A \to B$ be a spectral radius preserving homomorphism. Then $T^{-1}(B^r) = \mathcal{F}_T^r$.*

6. Special sets of r-Weyl and r-Browder elements

Motivated by the first part of Proposition 5.2, we make the following definitions:

Definition 6.1. Let $T : A \to B$ be a homomorphism. An element $a \in A$ is called

- *contractive* r*-Weyl* (or $(r,1)$*-Weyl* for short) if there exist elements $b \in A^r$ and $c \in \mathrm{N}(T)$ satisfying $r(b) \le r(a)$ and $a = b+c$,
- *contractive* r*-Browder* (or $(r,1)$*-Browder* for short) if there exist commuting elements $b \in A^r$ and $c \in \mathrm{N}(T)$ satisfying $r(b) \le r(a)$ and $a = b + c$.

Denote by $\mathcal{W}_T^{r,1}$ and $\mathcal{B}_T^{r,1}$ the sets of all $(r,1)$-Weyl elements and all $(r,1)$-Browder elements, respectively, of A relative to T.

Recalling (3) and (4) of Lemma 3.3 and Example 4.2, we now observe the following:

Proposition 6.2. *Let $T : A \to B$ be a homomorphism and let $a \in A$. If $a \in \mathcal{W}_T^{r,1}$, then $a - r(a)\mathbf{1} \in \mathcal{W}_T$ and if $a \in \mathcal{B}_T^{r,1}$, then $a - r(a)\mathbf{1} \in \mathcal{B}_T$.*

Proof. Suppose there exist (commuting) elements $b \in A^r$ and $c \in \mathrm{N}(T)$ such that $a = b + c$ and $r(b) \le r(a)$. If $r(b) = r(a)$, then $b - r(a)\mathbf{1} \in A^{-1}$ since $b \in A^r$. If $r(b) < r(a)$, then $r(a) \notin \sigma(b)$, so that, again, $b - r(a)\mathbf{1} \in A^{-1}$. Thus the result follows since we may write $a - r(a)\mathbf{1} = (b - r(a)\mathbf{1}) + c$. □

From the respective definitions and Propositions 3.2 and 5.2 we have the following inclusions:

Proposition 6.3. *Let $T : A \to B$ be a homomorphism. Then:*

$$\begin{array}{ccccccc}
A^r & \subseteq & \mathcal{B}_T^{r,1} & \subseteq & \mathcal{W}_T^{r,1} & \subseteq & \mathcal{F}_T^r \\
 & & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & \\
 & & & \mathcal{B}_T^r & \subseteq & \mathcal{W}_T^r. &
\end{array}$$

We note that, although $\mathcal{W}_T^r$ and $\mathcal{B}_T^r$ are not generally contained in $\mathcal{F}_T^r$ (see Example 6.6), $\mathcal{W}_T^{r,1}$ and $\mathcal{B}_T^{r,1}$ are always subsets of $\mathcal{F}_T^r$.

Example 6.4. Let $T : A \to B$ be the homomorphism $Tf = f \circ \theta$, where K and L are compact Hausdorff spaces, $A := C(K)$, $B := C(L)$ and $\theta : L \to K$ is a continuous map, as in Example 2.4. Then $\mathcal{W}_T^{r,1} = \mathcal{F}_T^r$.

Proof. By Proposition 6.3 the inclusion $\mathcal{W}_T^{r,1} \subseteq \mathcal{F}_T^r$ holds in general, so to prove the opposite inclusion, let $f \in \mathcal{F}_T^r$, i.e. $r(f) \notin f(\theta(L))$, by Example 4.1(2). If $r(f) \notin f(K)$, then $f \in A^r$, so that the result follows. Otherwise we have $r(f) \in f(K) \backslash f(\theta(L))$, so that

$$K_0 := \theta(L) \cup f^{-1}(\{-r(f)\}) \quad \text{and} \quad K_1 := f^{-1}(\{r(f)\})$$

are non-empty, closed sets. Since $f \in \mathcal{F}_T^r$ and $0 \notin \mathcal{F}_T^r$, it also follows that $K_0 \cap K_1 = \emptyset$. By applying Urysohn's Lemma we obtain $h \in A$ such that $h(K_0) = \{0\}$, $h(K_1) = \{1\}$ and $0 \leq h \leq 1$.

Let $g = (1 - 2h)f$. Then $g \in A$, $g_{|K_0} = f_{|K_0}$ and $g_{|K_1} = -f_{|K_1}$. Also, since $-1 \leq 1 - 2h \leq 1$, we have

$$|g(x)| \leq |f(x)| \tag{6.5}$$

for all $x \in K$, so that $r(g) \leq r(f)$. In fact, $r(g) = r(f)$, because if $x \in K_1$, then $g(x) = -f(x) = -r(f)$. Since $\theta(L) \subseteq K_0$, it is also clear that $g_{|\theta(L)} = f_{|\theta(L)}$, so that $f - g \in \mathrm{N}(T)$.

It remains to prove that $g \in A^r$, since then $f = g + (f - g) \in \mathcal{W}_T^{r,1}$. To this end, we prove that $r(f) \notin g(K)$, so suppose to the contrary that $r(f) = g(x_0)$ for some $x_0 \in K$. Then $f(x_0) \in \mathbb{R}$, and it follows from (6.5) that $r(f) = |f(x_0)|$, so that $f(x_0) = \pm r(f)$. However, if $f(x_0) = r(f)$, then $x_0 \in K_1$ so that $g(x_0) = -r(f)$, which contradicts $r(f) = g(x_0)$ since $f \neq 0$. Similarly, if $f(x_0) = -r(f)$, then $x_0 \in K_0$ so that $g(x_0) = -r(f)$, giving the same contradiction. Since $r(f) = r(g)$, it follows that $r(g) \notin g(K)$, i.e. $g \in A^r$, by Example 4.1(1). $\square$

It follows from Proposition 6.3, Example 6.4 and Example 4.1(3) that, if $\theta(L) \neq K$, then

$$C(K)^r \subseteq \mathcal{W}_T^{r,1} = \mathcal{F}_T^r \subseteq \mathcal{W}_T^r = C(K).$$

Our next example illustrates that the remaining inclusions are strict:

Example 6.6. If $\theta(L) \neq K$ in Example 6.4, then

$$C(K)^r \subsetneq \mathcal{W}_T^{r,1} = \mathcal{F}_T^r \subsetneq \mathcal{W}_T^r = C(K).$$

Proof. If $x_0 \in K \backslash \theta(L)$, then Urysohn's lemma provides a continuous function $f : K \to [0,1]$ such that $f(\theta(L)) = \{0\}$ and $f(x_0) = 1$. Then $r(f) = 1 \in f(K)$, so that $f \notin C(K)^r$, but $r(f) \notin \{0\} = f(\theta(L))$, so that $f \in \mathcal{F}_T^r$, by Example 4.1(2). This shows that $C(K)^r \subsetneq \mathcal{W}_T^{r,1}$.

To prove that $\mathcal{F}_T^r \subsetneq \mathcal{W}_T^r$, we simply note that for the contant function $\mathbf{1} \in C(K) = \mathcal{W}_T^r$ we have that $r(\mathbf{1}) = 1 \in \mathbf{1}(\theta(L))$, so that $\mathbf{1} \notin \mathcal{F}_T^r$. $\square$

Our final example illustrates that, in general, the inclusion $\mathcal{W}_T^{r,1} \subseteq \mathcal{F}_T^r$ is also strict.

Example 6.7. Consider the disc algebras $\mathcal{A}(\mathbb{D})$ and $\mathcal{A}\left(\frac{1}{2}\mathbb{D}\right)$ and let $T : \mathcal{A}(\mathbb{D}) \to \mathcal{A}\left(\frac{1}{2}\mathbb{D}\right)$ be defined by $Tf = f_{|\frac{1}{2}\mathbb{D}}$. Then $\mathcal{W}_T^{r,1} \subsetneq \mathcal{F}_T^r$.

Proof. Consider the function $f \in \mathcal{A}(\mathbb{D})$ defined by $f(z) = z$ for all $z \in \overline{\mathbb{D}}$. Then $\sigma(f) = \overline{\mathbb{D}}$ and $\sigma(Tf) = \frac{1}{2}\overline{\mathbb{D}}$. Since $r(f) = 1 \in \sigma(f) \backslash \sigma(Tf)$, we have that f is r-Fredholm but not r-invertible. By ([7], Theorem 3.7, p.78) $\mathrm{N}(T) = \{0\}$ and therefore $f \notin \mathcal{W}_T^r$, so that $f \notin \mathcal{W}_T^{r,1}$. $\square$

7. Main Results

In [10] and [19] Proposition 2.6 was used to show that every almost invertible Fredholm element is Browder. Here, we establish an analogue of this result for almost r-invertible r-Fredholm elements and use it to show that every almost r-invertible r-Fredholm element is r-Browder (see Theorem 7.2).

Lemma 7.1. *Let $T : A \to B$ be a homomorphism. If $a \in A^{D^r} \cap \mathcal{F}_T^r$, then $p(a, r(a)) \in \mathrm{N}(T)$.*

Proof. Let $a \in A^{D^r} \cap \mathcal{F}_T^r$. If $r(a) \notin \sigma(a)$, then $Tp(a, r(a)) = T0 = 0$. Hence, suppose that $r(a) \in \mathrm{iso}\, \sigma(a)$. By Proposition 2.1 we have that

$$Tp(a, r(a)) = T\left(\frac{1}{2\pi i}\int_\Gamma (\lambda \mathbf{1} - a)^{-1} d\lambda\right) = \frac{1}{2\pi i}\int_\Gamma (\lambda \mathbf{1} - Ta)^{-1} d\lambda,$$

where Γ is chosen to be a positively oriented circle centred at $r(a)$ and separating $r(a)$ from $\sigma(a)\backslash\{r(a)\}$, and hence, from $\sigma(Ta)\backslash\{r(a)\}$. Using the fact that $a \in \mathcal{F}_T^r$, that is, $r(a) \notin \sigma(Ta)$, it follows from Cauchy's theorem that $Tp(a, r(a)) = 0$. $\square$

We are now ready for our first main result:

Theorem 7.2. *Let $T : A \to B$ be a homomorphism. If $a \in A^{D^r} \cap \mathcal{F}_T^r$, then there exist commuting elements $b \in A^r$ and $c \in \mathrm{N}(T)$ such that $r(a) = r(b)$ and $a = b + c$. Therefore $A^{D^r} \cap \mathcal{F}_T^r \subseteq \mathcal{B}_T^{r,1}$.*

Proof. Let $a \in A^{D^r} \cap \mathcal{F}_T^r$. We may assume without loss of generality that $r(a) = 1$. If $1 \notin \sigma(a)$, then $a \in A^r$ so that the result follows with $b = a$ and $c = 0$. Hence suppose that $1 = r(a) \in \mathrm{iso}\, \sigma(a)$ and let p denote the spectral idempotent $p(a, 1)$ of a relative to 1. Let $b = a(\mathbf{1} - p) - p$ and $c = (a + \mathbf{1})p$. Then $a = b + c$, $bc = cb$ and it follows from Lemma 7.1 that $c \in \mathrm{N}(T)$.

Choose U_1 and U_0 to be disjoint open sets such that U_1 contains 1 and U_0 contains $\sigma(a)\backslash\{1\}$. If $f, g : U_0 \cup U_1 \to \mathbb{C}$ are defined by

$$f(\lambda) = \begin{cases} 0 & \text{if } \lambda \in U_0 \\ \\ 1 & \text{if } \lambda \in U_1 \end{cases} \quad \text{and} \quad g(\lambda) = \lambda(1 - f(\lambda)) - f(\lambda),$$

then, by the holomorphic functional calculus ([3], Theorem 3.3.4), $p = f(a)$ and $b = g(a)$. Using the spectral mapping theorem, we have that

$$\begin{aligned} \sigma(b) = \sigma(g(a)) &= g(\sigma(a)) \\ &= \{\lambda(1 - f(\lambda)) - f(\lambda) : \lambda \in \sigma(a)\} \\ &= \{-1\} \cup \sigma(a)\backslash\{1\}. \end{aligned}$$

Therefore $r(b) = 1 = r(a) \notin \sigma(b)$. This shows that $b \in A^r$ and finishes our proof. $\square$

It follows from the proof of Theorem 7.2 that if $a \in A$ is such that $r(a) \in (\text{iso}\, \sigma(a))\backslash\sigma(Ta)$ (with $r(a)$ now not necessarily equal to 1) and $p = p(a, r(a))$, then $a = b + c$, where $b = a(\mathbf{1} - p) - r(a)p \in A^r$ and $c = (a + r(a)\mathbf{1})p \in \mathrm{N}(T)$ commute, and $r(a) = r(b)$.

If T is a homomorphism satisfying the Riesz property, then Harte showed (see Theorem 2.5) that every Browder element relative to T is almost invertible Fredholm relative to T. This is, however, not the case for r-Browder and almost r-invertible r-Fredholm elements. We verify this in the next example.

Example 7.3. Let $A = M_2^u(\mathbb{C})$ and $T : A \to \mathbb{C}$ the homomorphism

$$T\begin{pmatrix} a_1 & a_2 \\ 0 & a_4 \end{pmatrix} = a_1,$$

which satisfies the Riesz property. Then $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathcal{B}_T^r\backslash(A^{D^r} \cap \mathcal{F}_T^r)$.

Proof. By Proposition 4.3(4) $a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is r-Browder and since $r(a) = 1 = a_1$, it follows from Proposition 4.3(2) that a is not r-Fredholm. $\square$

In general, by Theorem 7.2 the inclusion $A^{D^r} \cap \mathcal{F}_T^r \subseteq \mathcal{B}_T^{r,1}$ holds. In our second main result we show that these two sets coincide whenever T satisfies the Riesz property:

Theorem 7.4. *Let $T : A \to B$ be a homomorphism which satisfies the Riesz property. Then $A^{D^r} \cap \mathcal{F}_T^r = \mathcal{B}_T^{r,1} = \{a \in A : a - r(a)\mathbf{1} \in \mathcal{B}_T\}$.*

Proof. Even if T does not have the Riesz property, we have from Theorem 7.2 that $A^{D^r} \cap \mathcal{F}_T^r \subseteq \mathcal{B}_T^{r,1}$, and from Proposition 6.2 that $\mathcal{B}_T^{r,1} \subseteq \{a \in A : a - r(a)\mathbf{1} \in \mathcal{B}_T\}$. Now, if T has the Riesz property and $a - r(a)\mathbf{1} \in \mathcal{B}_T$, then it follows from Theorem 2.5 that $a - r(a)\mathbf{1} \in A^D \cap \mathcal{F}_T$. Hence $a \in A^{D^r} \cap \mathcal{F}_T^r$ by Lemma 3.3(5). $\square$

We note that if T has the Riesz property, then we even have that a is $(r, 1)$-Browder if and only if $r(b) = r(a)$ holds for the r-invertible component b in a decomposition of a. This follows from the first equality in Theorem 7.4 and the first part of Theorem 7.2.

It was shown in Lemma 3.3(5) that, for any homomorphism T, $a \in \mathcal{F}_T^r$ if and only if $a - r(a)\mathbf{1} \in \mathcal{F}_T$. The second equality in Theorem 7.4 shows that if T has the Riesz property, then there is a similar relationship between $(r, 1)$-Browder and Browder elements. In particular, since $\mathcal{B}_T^{r,1} \subseteq \mathcal{B}_T^r$, the converse implication in Lemma 3.3(3) holds with $r(b)$ replaced by $r(a)$. For operators (see Example 2.3(3)), this means the following:

Corollary 7.5. *Let X be an infinite-dimensional Banach space. If $U \in \mathcal{L}(X)$ has the property that there exists $W \in \mathcal{L}(X)^{-1}$ commuting with U such that $U - r(U)I - W$ is a compact operator, then there* *exists* $V \in \mathcal{L}(X)$ *commuting with* U *such that* $V - r(V)I \in \mathcal{L}(X)^{-1}$ *and* $U - V$ *is a compact operator.*

The following characterisation of the set $\mathcal{B}_T^{r,1}$ in the matrix case follows from the second equality in Theorem 7.4 and Proposition 4.3(2):

Example 7.6. Let $A = M_2^u(\mathbb{C})$ and $T : A \to \mathbb{C}$ the homomorphism $Ta = a_1$, where $a = \left(\begin{smallmatrix} a_1 & a_2 \\ 0 & a_4 \end{smallmatrix}\right)$. Then

$$\mathcal{B}_T^{r,1} = \mathcal{F}_T^r = \left\{ a = \left(\begin{smallmatrix} a_1 & a_2 \\ 0 & a_4 \end{smallmatrix}\right) : r(a) \neq a_1 \right\}.$$

Our next example shows that, for the first equality in Theorem 7.4, the assumption "T satisfies the Riesz property" cannot in general be dropped.

Example 7.7. Consider the Banach algebra $A := C(K)$, where $K = [0,1]$, and let $T : A \to A$ be the homomorphism induced by composition with the zero function on K. If $f \in A$ is defined by $f(z) = z$, then $f \in \mathcal{B}_T^{r,1} \backslash (A^{D^r} \cap \mathcal{F}_T^r)$.

Proof. Since $r(f) = 1 \notin \{0\} = f(\theta(K))$, it follows from Example 4.1(2) that $f \in \mathcal{F}_T^r$. Therefore Example 6.4 implies that $f \in \mathcal{B}_T^{r,1}$. We have, however, that $r(f) = 1 \in \mathrm{acc}\ [0,1] = \mathrm{acc}\ \sigma(f)$, and so $f \notin A^{D^r}$. $\square$

Finally, we show that the second equality in Theorem 7.4 may also fail to hold if T does not have the Riesz property:

Example 7.8. Let $A = \mathcal{A}(\mathbb{D})$, define $f \in A$ by $f(z) = z$ for all $z \in \overline{\mathbb{D}}$ and let $T : A \to A/I$ be the canonical homomorphism where I is the closed ideal generated by f^2. Then:

(1) $h(0) = h'(0) = 0$ for all $h \in I$.

(2) $f - r(f)\mathbf{1} \in \mathcal{B}_T$ but $f \notin \mathcal{B}_T^{r,1}$.

Proof. (1) If $h = f^2 g$ for some $g \in A$, then clearly $h(0) = 0$, and since $h' = 2fg + f^2 g'$, it follows that $h'(0) = 0$ as well. Now let $h \in I$, i.e. there exists a sequence (g_n) in A such that the sequence (h_n), where $h_n = f^2 g_n$, converges uniformly to h on $\overline{\mathbb{D}}$. Then clearly $h(0) = 0$, and by ([7], Theorem 2.1, p.151) the sequence (h'_n) converges uniformly to h' on each compact subset of $\mathbb{D}$, so that $h'(0) = 0$ as well.

(2) We first note that $I = \mathrm{N}(T)$ and that $r(f) = 1 \in \overline{\mathbb{D}} = f(\overline{\mathbb{D}}) = \sigma(f)$. Let $h = \frac{1}{4}f^2$ and let $u = f - r(f)\mathbf{1} - h$. Then $f - r(f)\mathbf{1} = u + h$ with $h \in \mathrm{N}(T)$. Since $u(z) = z - 1 - \frac{1}{4}z^2$, we have that $u(z) = 0$ if and only if $z = 2$, and since $2 \notin \overline{\mathbb{D}}$, it follows that $0 \notin u(\overline{\mathbb{D}}) = \sigma(u)$, so that $u \in A^{-1}$. Since A is commutative, we have that $f - r(f)\mathbf{1}$ is Browder.

In order to show that f is not $(r, 1)$-Browder, suppose that $f = v + h$ for some $v \in A$ with $r(v) \leq r(f)$ and $h \in I$. Then $|v(z)| \leq ||v|| = r(v) \leq 1$ for all $z \in \overline{\mathbb{D}}$ and, by (1), $v(0) = f(0) - h(0) = 0$ and $v'(0) = f'(0) - h'(0) = 1$. It follows from Schwarz's Lemma ([7], Theorem 2.1, p.130) that $v(z) = kz$ for some constant k with $|k| = 1$. We see that $k = 1$ because $v'(0) = 1$. Hence $v = f$, but then $r(v) \in \sigma(v)$, so that v is not r-invertible. We have shown that f is not $(r, 1)$-Browder. □

We note that the homomorphism T in Example 7.8 does not have the Riesz property since the ideal I is not inessential.

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