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A Hierarchy of Subgraph Projection-Based Semidefinite Relaxations for some NP-Hard Graph Optimization Problems *

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Abstract

Many important NP-hard combinatorial problems can be efficiently approximated using semidefinite programming relaxations. We propose a new hierarchy of semidefinite relaxations for classes of such problems that are based on graphs and for which the projection of the problem onto a subgraph shares the same structure as the original problem. This includes the well-studied max-cut and stable-set problems. Each level k of the proposed hierarchy consists of the basic semidefinite relaxation of the problem augmented by the constraints enforcing the structural projection condition on every k-node subgraph of the problem. This hierarchy has the distinguishing feature that all the relaxations are formulated in the space of the original semidefinite relaxation. Because the size of the relaxations increases rapidly with the number of subgraphs, we explore the possibility of adding the projection constraints only for selected subgraphs. Preliminary computational results show that the proposed hierarchy yields improved bounds when compared to the initial relaxation for benchmark instances of the max-cut and stable-set problems, and that the improved bounds result in significantly smaller enumeration trees when the relaxation is used in a branch-and-bound scheme.

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1 Introduction

It is well known that many important combinatorial problems are NP-hard in general. The inherent difficulty of these problems makes finding global optimal solutions hard, and therefore there is great interest in finding tighter convex relaxations that are tractable. For this reason semidefinite programming (SDP) relaxations that often produce strong bounds for these combinatorial optimization problems are of great interest. Surveys by Goemans [17] and Lovász [24] outline the connection between SDP and NP-hard problems. SDP relaxations exist for a variety of such problems, and many ways to tighten them have been proposed, see e.g. [1].

In particular, several hierarchies of relaxations have been proposed that provide increasingly tight bounds. Well-known hierarchies include the Sherali-Adams reformulation-linearization technique (RLT) [28], the Lovász-Schrijver lift-and-project [25], and the Lasserre relaxations [22]. For combinatorial problems, these hierarchies have the property that they converge to the integral hull in a finite number of steps. On the other hand, their size grows exponentially in terms of the numbers of variables in the combinatorial problem.

This paper introduces a new hierarchy of SDP relaxations for the classes of NP-hard graph problems that satisfy a certain projection property. Specifically we are interested in problems for which the projection of the graph problem onto a subgraph shares the same structure as the original problem. This includes the well-studied max-cut and stable-set problems. For max-cut for example, the projection of any cut of the original graph onto a subgraph induces a cut in every subgraph. In the same way, a stable set in the original graph induces a stable set in every subgraph.

The level k of the proposed hierarchy consists of the basic SDP relaxation for the problem at hand augmented by the constraints that the solution projected onto every k-node subgraph should satisfy the structure of the problem on that subgraph. This hierarchy has the distinguishing feature that all the SDP relaxations are formulated in the space of the original SDP relaxation. In this paper we focus on cases where the projected problem has a "simple" description but the hierarchy we propose applies to any class of problems that shares the required projection property.

The size of the relaxations increases rapidly with k because of the large number of subgraphs. For this reason we also explore the possibility of adding the projection constraints only for selected subgraphs. Such a selective addition of constraints provides flexibility in the construction of the SDP relaxation and results in more efficient computation of improved bounds.

We provide computational results showing that the proposed hierarchy yields improved bounds when compared to the basic SDP relaxations for benchmark instances of the max-cut and stable-set problems. We also report results showing that the improved bounds result in significantly smaller enumeration trees when the SDP relaxation is used in a branch-and-bound scheme to solve the problems to optimality or near-optimality.

The paper is organized as follows: Section 2 gives a general description of our hierarchy of relaxations. The hierarchy is further examined within the context of the max-cut and stable-set problems in Sections 3 and 4 respectively. Preliminary computational results for both small and large examples are reported for both problems. Section 5 contains concluding remarks.

2 A Hierarchy of Relaxations Based on k-Projections

For a general description of our hierarchy of SDP relaxations, let us assume that $N := \{1, \ldots, n\}$ denotes the vertex set of the graph in the combinatorial problem at hand, and that the combinatorial optimization problem under consideration is given through its set of feasible solutions $\{X_1, \ldots\} \subseteq \mathcal{S}_n$, where $\mathcal{S}_n$ denotes the set of symmetric matrices of order n. We further denote by $\mathcal{P}$ the convex hull of all feasible points, i.e.,

$$\mathcal{P} = \mathrm{conv}\{X_1, \ldots\}.$$

Given a cost matrix C, our goal is to find the solution X_i that maximizes $\langle C, X_i \rangle$:

$$z_{\mathcal{P}} := \max_i \langle C, X_i \rangle = \max\{\langle C, X \rangle \colon X \in \mathcal{P}\}.$$

For a set $I \subseteq N$, let

$$\pi_I(X) = X_I$$

denote the (orthogonal) projection mapping X to X_I, the principal submatrix of X indexed by I. Similarly

$$\pi_I(\mathcal{P}) = \mathrm{conv}\{\pi_I(X_1), \ldots\}$$

denotes the projection of $\mathcal{P}$ onto I. We are particularly interested in problems for which $\pi_I(\mathcal{P})$ has a "simple" description in the sense described below.

As a first example we consider the max-cut problem on a graph with n nodes. The feasible solutions X are cut matrices of the form $X = cc^T$ with $c \in \{-1, 1\}^n$. We denote the convex hull of cut matrices by CUT_n. This is generally known as the cut polytope. It follows from the definition that if $|I| = k$ then

$$\pi_I(\mathrm{CUT}_n) = \mathrm{conv}\{cc^T \colon c \in \{-1, 1\}^k\} = \mathrm{CUT}_k. \tag{1}$$

A similar situation occurs for the stable-set problem. Let G be a given graph on n nodes. We consider

$$\mathrm{STAB}(G) := \mathrm{conv}\{s_i \ : \ s_i \text{ is the incidence vector of a stable set in } G\}. \tag{2}$$

Note that $\mathrm{STAB}(G) \subseteq \mathbb{R}^n$ and that

$$\pi_I(\mathrm{STAB}(G)) = \mathrm{STAB}(G_I), \tag{3}$$

where G_I denotes the subgraph induced by I. In a slight abuse of notation we also denote by $\pi_I(x)$ the projection of the vector x onto the coordinates in I.

The important observation is that the projection properties in (1) and (3) are a consequence of the fact that cuts and stable sets, when restricted to G_I, induce cuts and stable sets (in G_I).

This property does not hold for combinatorial problems in general. In contrast, the convex hull HAM(G) of Hamiltonian cycles in a graph G does not have this property. In particular, the restriction of a Hamiltonian cycle to a proper subgraph will not be a cycle.

From now on we consider optimization problems where the convex hull of feasible solutions, defined on some graph G and denoted by $\mathcal{P}(G)$, satisfies the following projection property:

$$\pi_I(\mathcal{P}(G)) = \mathcal{P}(G_I). \tag{4}$$

To simplify notation we write in the following $\mathcal{P}$ instead of $\mathcal{P}(G)$ and $\mathcal{P}_I$ instead of $\mathcal{P}(G_I)$.

A formal description of our new hierarchy of relaxations starts with a superset $\mathcal{R}$ of $\mathcal{P}$, $\mathcal{R} \supseteq \mathcal{P}$, that is tractable in the sense that

$$z_{\mathcal{R}} := \max\{\langle C, X\rangle \colon X \in \mathcal{R}\} \tag{5}$$

can be solved efficiently. We are particularly interested in cases where $\mathcal{R}$ is a spectrahedron, i.e., the intersection of the cone of semidefinite matrices $\mathcal{S}_n^+$ with an affine linear space.

For $k \in \mathbb{N}$ fixed, we tighten the relaxation (5) by adding the *k-projection constraints*:

$$\pi_I(X) \in \pi_I(\mathcal{P}) \ \forall I \subseteq N, \ |I| = k.$$

Under our assumption (4), this simplifies to

$$X_I \in \mathcal{P}_I.$$

For small values of k, we can express this condition in a more convenient way by exploiting the fact that the vertices v_i^I of $\mathcal{P}_I$ can be enumerated explicitly, and requiring that X_I lay in the convex hull of the vertices of $\mathcal{P}_I$:

$$X_I = \sum_i \lambda_i^I v_i^I \text{ with } \lambda_i^I \geq 0, \ \sum_i \lambda_i^I = 1. \tag{6}$$

Thus level k of our hierarchy reads

$$\begin{aligned} z_{\mathcal{R},k} := \max\big\{\langle C, X\rangle \quad : \quad & X \in \mathcal{R}, \ X_I = \sum_i \lambda_i^I v_i^I \text{ with} \\ & \lambda_i^I \geq 0, \ \sum_i \lambda_i^I = 1 \ \forall I \subseteq N, \ |I| = k\big\}. \end{aligned} \tag{7}$$

It is clear from the definitions that

$$z_{\mathcal{R}} \geq z_{\mathcal{R},1} \geq \ldots \geq z_{\mathcal{R},n} = z_{\mathcal{P}}.$$

Remark 1 *In our applications we focus mostly on relaxations where $\mathcal{R}$ is some spectrahedron. It is a nontrivial task to actually identify subsets I so that the current iterate x violates $x \in \mathcal{P}_I$ by a substantial amount. We select the cardinality of I in such a way that $\mathcal{P}_I$ has a relatively small number of vertices. In this case we prefer maintaining the vertex-based description (6) of $\pi_I(\mathcal{P})$, which imposes much more structure than adding a single cutting plane.*

Remark 2 *An important distinguishing feature of our hierarchy, as compared to other generic hierarchies or SDP relaxations such as the Anjos-Wolkowicz [2] relaxation, lies in the crucial fact that all our relaxations are formulated in the original space $\mathcal{S}_n$, and only the number of constraints in the hierarchies increases exponentially. In the other constructions mentioned previously, the dimension of the matrix space also grows exponentially, hence even the smallest levels in these hierarchies are computationally challenging.*

This informal description of the new hierarchies leaves open several issues which are relevant in a practical implementation. It may for instance not be a good idea to include all projection constraints $\pi_I(X) \in \pi_I(\mathcal{P})$ at once, as there are $\binom{n}{k}$ of them altogether. After surveying some related work in the literature in Section 2.1, we provide in the rest of the paper some very preliminary computational experience applied to max-cut and stable-set problems.

2.1 Related work

The idea of generating cutting planes for $\mathcal{P}$ from smaller polytopes $\mathcal{P}_I$ has a long history in polyhedral combinatorics. In the context of graph-based optimization problems, these smaller polytopes have been obtained either by shrinking the graph (as done in Applegate et al. [3]) or by considering subgraphs of the graph. Our interest here is in the latter approach.

Given some polyhedral relaxation $\mathcal{R}$ it seems natural to consider the following scheme. Suppose we have solved the relaxation over $\mathcal{R}$ with optimal solution x. One way to further tighten this relaxation would be to check whether $\pi_I(x) \in \pi_I(\mathcal{P})$ for some set I of small cardinality. If it turns out that $\pi_I(x) \notin \pi_I(\mathcal{P})$, then one of the facets of $\pi_I(\mathcal{P})$ defines a "local" cutting plane, which separates $\pi_I(x)$ from the small polytope $\pi_I(\mathcal{P})$. Lifting it back to $\mathbb{R}^n$ yields a linear constraint that is violated by the current iterate x but is valid for $\mathcal{P}$. Christof and Reinelt [9] apply the same idea to the linear ordering and the betweenness problem. The relaxation, obtained from lifting facets of $\mathcal{P}_I$ to facets of $\mathcal{P}$ is called 'small instance relaxation' by Christof and Reinelt. They also address computational issues like heuristics for the separation of facets of $\mathcal{P}_I$ and parallel implementation, see also [8, 10] by the same authors. More recently Buchheim, Liers and Oswald [7] introduce *target cuts* to improve polyhedral relaxations. These correspond to liftings of facets of polytopes $\mathcal{P}_I$. Finally, Bonato, Jünger, Reinelt and Rinaldi [5] apply this approach to the cut polytope. Boros, Crama and Hammer [11] introduce a hierarchy of polyhedral relaxations for max-cut which agrees with our hierarchy when started from the metric polytope relaxation. More recently Boros and Lari [6] study polyhedral hierarchies for max-cut and compare hierarchies based on functions of k variables, degree k posiforms and lift-and-project hierarchies.

From a worst-case point of view it is known that the metric polytope relaxation of max-cut has an integrality gap of $2-\epsilon$, see Poljak and Tuza [26]. Even worse, de la Vega and Kenyon-Mathieu [12] show that for any fixed k, the level k hierarchy of the metric polytope relaxation obtained by including all valid inequalities for the cut polytope on at most k vertices still has integrality gap of $2-\epsilon$.

Cutting planes generated from small polytopes have also been used for non-polyhedral relaxations $\mathcal{R}$. Helmberg and Rendl [20] consider the semidefinite relaxation (9) for the max-cut problem and combine it with clique inequalities and general hypermetric inequalities from small subgraphs.

While we also generate cutting planes for $\mathcal{P}$ from smaller polytopes, we do not use cutting planes expressed via valid inequalities. Instead we use the complete inner (vertex) description of $\mathcal{P}_I$ for the smaller problem to tighten the relaxation of the larger (original) problem. It is important to note that our approach is only applied to problems satisfying the projection property (4); for example, the travelling salesman problem does not fit in this category.

3 The New Hierarchy for Max-Cut

An instance of the max-cut problem is given through the weighted adjacency matrix A of the underlying graph G. It is assumed that $\mathrm{diag}(A)=0$ (no loops) and $A=A^T$ (G is undirected). The Laplacian associated to A is given by $L=\mathrm{Diag}(Ae)-A$ with e the all-ones vector. The *max-cut problem* is

$$z_{maxcut}=\max\{x^TLx:\ x\in\{-1,1\}^n\}=\max\{\langle L,X\rangle:\ X\in \mathrm{CUT}_n\}. \tag{8}$$

The cut polytope CUT_n is contained in the set $\mathcal{C}$ of correlation matrices:

$$\mathcal{C}:=\{X\in\mathcal{S}_n:\ \mathrm{diag}(X)=e,\ X\succeq 0\}.$$

Optimizing over $\mathcal{C}$ yields one of the most well-studied semidefinite optimization problems,

$$z_{\mathcal{C}}:=\max\{\langle L,X\rangle : X\in\mathcal{C}\}. \tag{9}$$

It was introduced (in dual form) by Delorme and Poljak [13]. An interior-point method for solving this relaxation is provided in Helmberg, Rendl, Vanderbei and Wolkowicz [21]. Goemans and Williamson [18] provide a theoretical error analysis showing that

$$z_{maxcut}\geq 0.878 z_{\mathcal{C}} \text{ for graphs with } A\geq 0.$$

The cut polytope is also contained in the *metric polytope* $\mathcal{M}$:

$$\mathcal{M}:=\{X\in\mathcal{S}_n:\ \mathrm{diag}(X)=e,\ f^TXf\geq 1\ \forall f\in\{-1,0,1\}^n,\ \mathrm{support}(f)=3\}.$$

The metric polytope therefore contains all matrices with main diagonal equal to the vector of all-ones that also satisfy the triangle inequalities:

$$x_{ij} + x_{ik} + x_{jk} \geq -1, \; x_{ij} - x_{ik} - x_{jk} \geq -1 \; \forall i, j, k.$$

The intersection $\mathcal{C} \cap \mathcal{M}$ is thus another relaxation of CUT_n and leads to the following SDP relaxation:

$$\max\{\langle L, X\rangle \colon X \in \mathcal{C} \cap \mathcal{M}\}.$$

This semidefinite optimization problem can be solved in polynomial time up to a fixed prescribed precision. However it contains $O(n^3)$ inequality constraints, and hence it is a challenge to standard SDP solvers. A computationally efficient way to deal with this relaxation was introduced by Fischer, Gruber, Rendl and Sotirov [16]. It combines interior-point methods with the bundle method to deal with the triangle inequalities. An exact method where this relaxation is used in a branch-and-bound setting was proposed by Rendl, Rinaldi and Wiegele [27].

Numerous strengthenings of these relaxations have been suggested in the literature. The cut polytope is for instance contained in the *hypermetric cone* investigated by Deza, Grishukhin and Laurent [14]. Hence hypermetric inequalities can be used to strengthen the relaxations.

Finally, the semidefinite relaxations have been refined by introducing hierarchies of relaxations of increasing matrix size. Anjos and Wolkowicz [2] introduced and investigated a *lifting* of the basic relaxation on $\mathcal{C}$. Later on, Lasserre [22] proposed another lifting procedure which yields the integer optimum after at most n lifting steps. These liftings have the computational drawback that their matrix dimensions increase in each step, and even the first nontrivial lifting step leads to matrices of order $\binom{n}{2}$ which is prohibitive even for very modest values of n such as $n \approx 50$.

To apply our new hierarchy, we take the SDP relaxation over the intersection $\mathcal{R} := \mathcal{C} \cap \mathcal{M}$ as our initial relaxation:

$$z_{\mathcal{R}} = \max\{\langle L, X\rangle \colon X \in \mathcal{R}\}.$$

The motivation for this choice is that this relaxation provides one of the most competitive bounds if both practical efficiency and strength of the relaxation are taken into account.

The new hierarchy applied to max-cut starting from (7) with $\mathcal{R} = \mathcal{C} \cap \mathcal{M}$ reads

$$z_{\mathcal{R},k} = \max\{\langle L, X\rangle \colon \; X \in \mathcal{R}, \; X_I \in \mathrm{CUT}_k \; \forall I \subseteq N, \; |I| = k\}. \tag{10}$$

In [4] it is shown that triangle inequalities give a complete description of the cut polytope for $n \leq 4$. Therefore the smallest interesting value for k in our hierarchy (10) is $k = 5$.

To get a vertex-based inner description for $X_I \in \mathrm{CUT}_k$ we recall that CUT_k is the convex hull of 2^{k-1} cut matrices $C_r \in \mathcal{S}_k$ of the form $C_r = c_r c_r^T$ with

$c_r \in \{-1,1\}^k$. The standard simplex in $\mathbb{R}^m$ is denoted by

$$\Delta_m := \{\lambda \in \mathbb{R}^m : \ \lambda \geq 0, \ \sum_i \lambda_i = 1\}.$$

Thus k-projection constraint $X_I \in \mathrm{CUT}_k$ corresponding to the k-subset I can be conveniently expressed as

$$X_I = \sum_{r=1}^{2^{k-1}} \lambda_r^I C_r, \ \lambda^I \in \Delta_{2^{k-1}}.$$

(To improve readability, we write Δ for $\Delta_{2^{k-1}}$ if the dimension is clear from the context.)

The new hierarchy (10) therefore has the following form.

$$z_{\mathcal{R},k} = \max\{\langle L, X\rangle : \ X \in \mathcal{R}, \ X_I = \sum_{r=1}^{2^{k-1}} \lambda_r^I C_r, \ \lambda^I \in \Delta, \ \forall I \subseteq N, \ |I| = k\}. \tag{11}$$

Trivially, this hierarchy yields z_{maxcut} for $k = n$. Moreover, it can be computed in polynomial time for fixed k. Since there are $\binom{n}{k}$ distinct subsets I to be considered in (11) and k should be at least 5, it is impractical to work directly with this model. We use a simple enumeration approach to separate k-projection constraints for max-cut in Section 3.2 below. First we provide some insight into the behaviour of the new hierarchy with some small computational examples.

3.1 Small examples

In this subsection we illustrate the behaviour of the hierarchy (10) on selected small max-cut instances. Because these instances are small, all the relaxations in the hierarchy can be solved to optimality.

We first consider the 7×7 matrix

$$Q = -\frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 1 & -2 & -1 & 0 \\ 1 & 0 & 1 & 1 & -2 & 0 & -1 \\ 1 & 1 & 0 & 1 & -2 & -1 & 0 \\ 1 & 1 & 1 & 0 & -2 & 0 & -1 \\ -2 & -2 & -2 & -2 & 0 & 1 & 1 \\ -1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 1 & -1 & 0 \end{pmatrix}.$$

Grishukhin [19] showed that $\langle Q, X\rangle \geq 5$ is a facet of the cut polytope CUT_7. Hence maximizing $\langle Q, X\rangle$ over various supersets of CUT_7 shows how close we come to this facet using the respective relaxations. The results are reported in Table 1.

A similar distinction between the relaxations occurs in case of the clique web inequalities [15]. Recall that these inequalities are defined as follows: Let

Relaxation (bound)	Bound	Gap (%)
$\mathcal{C}_7$ ($z_{\mathcal{C}}$)	6.9518	39.04
$\mathcal{C}_7 \cap \mathcal{M}_7$ ($z_{\mathcal{R}}$)	6.0584	21.17
$\mathcal{C}_7 \cap \mathcal{M}_7$ and all CUT_5s ($z_{\mathcal{R},5}$)	5.8000	16.00
$\mathcal{C}_7 \cap \mathcal{M}_7$ and all CUT_6s ($z_{\mathcal{R},6}$)	5.6667	13.33
Anjos & Wolkowicz	5.7075	14.15
Lasserre level 2	5.6152	12.30
CUT_7 ($z_{\mathcal{P}}$)	5.0000	0.00

Table 1: Bounds and relative gaps to optimality (%) obtained from various relaxations for the Grishukhin inequality of CUT_7.

n, p, q, r be integers such that $n = p + q, p - q = 2r + 1, q \geq 2$ and let $b := (1, \dots, 1, -1, \dots, -1)^T$ be a vector of length n where the first p coefficients are equal to $+1$ and the last q coefficients are equal to -1. AW_p^r defines the antiweb as the graph with vertex set $V_p = \{1, 2, \dots, p\}$ and edge set defined by the pairs $(i, i+1), (i, i+2), \dots, (i, i+r)$, $\forall i \in V_p$. Then the clique web inequalities are

$$\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} - \sum_{ij \in \mathrm{AW}_p^r} x_{ij} \leq 0$$

We consider the cases $n = 9$ and $n = 11$ and compare again the various levels of our new hierarchy. These inequalities are parametrized by the integer r with $0 \leq r \leq \frac{n-5}{2}$. The results are reported in Tables 2 and 3.

Relaxation (bound)	$r = 1$	$r = 2$
$\mathcal{C}_9$ ($z_{\mathcal{C}}$)	8.40	8.99
$\mathcal{C}_9 \cap \mathcal{M}_9$ ($z_{\mathcal{R}}$)	7.12	7.12
$\mathcal{C}_9 \cap \mathcal{M}_9$ and all CUT_5s ($z_{\mathcal{R},5}$)	6.86	7.07
$\mathcal{C}_9 \cap \mathcal{M}_9$ and all CUT_6s ($z_{\mathcal{R},6}$)	6.86	7.07
$\mathcal{C}_9 \cap \mathcal{M}_9$ and all CUT_7s ($z_{\mathcal{R},7}$)	6.75	6.57
$\mathcal{C}_9 \cap \mathcal{M}_9$ and all CUT_8s ($z_{\mathcal{R},8}$)	6.64	6.56
Anjos & Wolkowicz	6.72	6.79
Lasserre level 2	6.59	6.55
CUT_9 ($z_{\mathcal{P}}$)	6.00	6.00

Table 2: Bounds for the clique web inequality with $n = 9$.

These first examples (Tables 1-3) show that our new hierarchy is competitive, even compared to the level 2 of the Lasserre hierarchy. Even though we have included the projection constraints for all subsets of cardinality k in these computations, a closer look at the computational results shows that in fact only a small fraction of these constraints are necessary to get the given bounds. It is also quite striking that going down to level $k = n - 1$ still leaves a rather

Relaxation (bound)	$r=1$	$r=2$	$r=3$
$\mathcal{C}_{11}$ $(z_{\mathcal{C}})$	10.83	13.31	12.10
$\mathcal{C}_{11} \cap \mathcal{M}_{11}$ $(z_{\mathcal{R}})$	9.28	10.62	9.28
$\mathcal{C}_{11} \cap \mathcal{M}_{11}$ and all CUT$_5$s $(z_{\mathcal{R},5})$	9.21	10.62	9.25
$\mathcal{C}_{11} \cap \mathcal{M}_{11}$ and all CUT$_6$s $(z_{\mathcal{R},6})$	9.21	10.62	9.25
$\mathcal{C}_{11} \cap \mathcal{M}_{11}$ and all CUT$_7$s $(z_{\mathcal{R},7})$	9.00	9.96	8.87
$\mathcal{C}_{11} \cap \mathcal{M}_{11}$ and all CUT$_8$s $(z_{\mathcal{R},8})$	8.86	9.96	8.87
$\mathcal{C}_{11} \cap \mathcal{M}_{11}$ and all CUT$_9$s $(z_{\mathcal{R},9})$	8.79	9.59	8.52
$\mathcal{C}_{11} \cap \mathcal{M}_{11}$ and all CUT$_{10}$s $(z_{\mathcal{R},10})$	8.56	9.50	8.44
Anjos & Wolkowicz	8.87	10.02	8.93
Lasserre level 2	8.72	9.62	8.59
CUT$_{11}$ $(z_{\mathcal{P}})$	8.00	9.00	8.00

Table 3: Bounds for the clique web inequality with $n = 11$.

large gap on these instances. Since the objective function corresponds to a facet of the cut polytope, this is an illustration of the worst case behaviour of our hierarchy.

We next turn to larger instances, address the practical issue of finding good projection constraints, and investigate the new hierarchy on some graphs from the literature.

3.2 Larger instances

This section reports the results of our computational experiments with the new hierarchy on larger instances of max-cut. The inclusion of all k-projection polytopes for some $k \geq 5$ is computationally prohibitive. Instead, we run through all 5-projection polytopes and include only the 100 most violated ones, iterating this process. To check whether or not $X_I \in \mathrm{CUT}_{|I|}$ we could compute the projection of X_I to $\mathrm{CUT}_{|I|}$. This requires in general the solution of a convex quadratic problem in 2^{k-1} variables if $|I| = k$.

For the case $k = 5$ we exploit the fact that the facets of CUT$_5$ are given by the triangle inequalities, which are always satisfied as we assume $X \in \mathcal{M}$, and the pentagonal inequalities $f^T X_I f \geq 1$ for all $f \in \{-1, 1\}^5$. We scan through all pentagonal inequalities and select the 100 subsets I corresponding to the largest violations. We add the corresponding projection constraints to the SDP relaxation, solve the resulting relaxation using SDPT3, and iterate this process. In the tables below, the number of these iterations is limited to 10. The final bound approximates $z_{\mathcal{C}\cap\mathcal{M},5}$ from above. The following tables contain representative results from our experiments.

We first look at max-cut for random unweighted graphs from the Erdős-Renyi model where each edge appears with probability p independent of the other edges. We consider graphs on $n = 80$ nodes with $p = \frac{1}{2}$. These instances can be found at the website `http://biqmac.uni-klu.ac.at/biqmaclib.html`.

For a comparison, we also provide the number of nodes used to prove optimality by the software package BiqMac [27]. The results are reported in Table 4.

We observe that our new bound is strong enough to solve most of the instances either at the root node or at the first two levels of the branching tree. In sharp contrast with the results for BiqMac, only two out of the ten instances could not be solved within the first two levels of a branch-and-bound procedure when using the new bound. (Note that for the instance g05_80.1, the relaxation $\mathcal{C} \cap \mathcal{M}$ already closes the gap.)

Graph name	Optimal cut value	Optimizing over $\mathcal{C}$	$\mathcal{C} \cap \mathcal{M}$	New bound	Nodes with BiqMac	Nodes with new bound
g05_80.0	929	950.92	934.24	931.01	59	5
g05_80.1	941	957.25	941.76	–	3	–
g05_80.2	934	955.55	937.24	934.52	17	1
g05_80.3	923	947.59	932.32	929.15	523	>7
g05_80.4	932	955.31	936.53	933.83	39	3
g05_80.5	926	947.51	931.42	928.41	65	7
g05_80.6	929	948.68	933.24	930.40	31	3
g05_80.7	929	949.86	932.63	929.58	23	1
g05_80.8	925	946.67	930.53	927.42	73	7
g05_80.9	923	943.66	929.95	926.67	157	>7

Table 4: Bounds and number of nodes in a branch-and-bound tree for unweighted graphs on $n = 80$ nodes.

We also look at larger instances of size $n = 100$. We consider graphs with both positive and negative edge weights and collect a sample of results in Table 5. Again these instances can be found in the BiqMac Library. Here we report the percentage gap between the optimal cut value and each of the bounds (with respect to the optimal). We again see that our rather simple-minded improvement strategy limited to $k = 5$ yields a significant improvement of the bounds.

It is far beyond the scope of this initial paper to provide an efficient implementation of the new SDP relaxations. We observe that the resulting SDP problems have a very special structure that should be exploited in a specialized implementation. Moreover, it may be worthwhile to include subsets of cardinalities larger than 5, and generally to vary the size of the subsets. The results of exploring these directions will be reported in a separate forthcoming paper.

To further emphasize the potential of our new bounding procedure, we include in the next section a short discussion of the new hierarchy applied to the stable-set problem.

Graph name	Optimal cut value	Optimizing over $\mathcal{C}$	$\mathcal{C} \cap \mathcal{M}$	New bound	Gap for $\mathcal{C} \cap \mathcal{M}$	Gap for new bound
w09_100.0	2121	2500.30	2234.39	2189.54	5.35	3.23
w09_100.1	2096	2522.03	2263.82	2218.30	8.01	5.83
w09_100.2	2738	3129.99	2880.60	2833.92	5.21	3.50
w09_100.3	1990	2333.05	2131.55	2084.76	7.11	4.76
w09_100.4	2033	2424.98	2154.71	2109.86	5.99	3.78
w09_100.5	2433	2733.64	2454.66	2433.08	0.89	0.00
w09_100.6	2220	2552.11	2281.17	2241.92	2.76	0.99
w09_100.7	2252	2639.73	2355.48	2312.90	4.60	2.70
w09_100.8	1843	2213.12	1924.37	1882.62	4.42	2.15
w09_100.9	2043	2409.78	2161.63	2116.84	5.81	3.61

Table 5: Bounds and relative gaps to optimality (%) for dense graphs with positive and negative weights on $n = 100$ nodes.

4 The Projection Bound for the Stable-Set Problem

We now take a closer look at the projection bound in the case of the stable-set problem. The stable-set polytope STAB(G) (see (2)) of a graph G with vertex set $V(G) = N$ is contained in $\mathbb{R}^n$. The stability number $\alpha(G)$ of a graph G, giving the cardinality of the largest stable set, is given by

$$\alpha(G) = \max\{\sum x_i : \; x \in \mathrm{STAB}(G)\}.$$

One of the most well-studied relaxations of STAB(G) is based on the theta body TH(G) introduced by Lovász [23]:

$$\mathrm{TH}(G) := \{x \in \mathbb{R}^n : \; \exists X \in \mathcal{S}_n, \; x = \mathrm{diag}(X), \; X - xx^T \succeq 0, \; x_{ij} = 0 \, \forall [i,j] \in E(G)\}.$$

Note that any characteristic vector $s \in \{0,1\}^n$ of a stable set in G yields a *stable-set matrix* $S := ss^T$ such that

$$s = \mathrm{diag}(S), \; S - ss^T \succeq 0, \; (S)_{ij} = s_i s_j = 0 \; \forall \; [i,j] \in E(G).$$

Hence STAB(G) $\subseteq$ TH(G).

A direct application of the projection approach would impose, for given $x \in \mathbb{R}^n$, the constraint $x_I \in \mathrm{STAB}(G_I)$ for subsets $I \subseteq N$. On the other hand, the set TH(G) can also be viewed as a matrix relaxation of the stable-set problem projected to the main diagonal. We define $\mathrm{STAB}^2(G)$ to be the convex hull of stable-set matrices:

$$\mathrm{STAB}^2(G) := \mathrm{conv}\{ss^T : s \text{ characteristic vector of stable set}\}.$$

Thus the projection of $\mathrm{STAB}^2(G)$ to the main diagonal gives STAB(G):

$$\mathrm{diag}(\mathrm{STAB}^2(G)) = \mathrm{STAB}(G).$$

We propose to apply the subgraph projection idea to $\mathrm{STAB}^2(G)$. Our starting point is the relaxation over $\mathrm{TH}(G)$. It is also called the Lovász theta function and can be formulated as

$$\theta(G) := \max\{\mathrm{tr}(X) \colon X \in \mathcal{S}_n, x_{ij} = 0 \,\forall [i,j] \in E(G),\ x = \mathrm{diag}(X),\ X - xx^T \succeq 0\}.$$

This relaxation is now strengthened by the k-projection polytopes

$$X_I \in \mathrm{STAB}^2(G_I) \ \forall I \subseteq N,\ |I| = k.$$

We emphasize the fact that it is possible to have $X_I \notin \mathrm{STAB}^2(G_I)$, but $\mathrm{diag}(X_I) \in \mathrm{STAB}(G_I)$. This could even happen for subsets $I = \{r, s\}$ if $x_{rs} < 0$.

As in the previous section, we close with some preliminary computational experiments. Here we iteratively include only the most violated k-projection polytopes for $k \leq 6$. We consider random graphs with edge density 25% (g60-25, g80-25) and a graph with density 10% (g100-10). We also consider a cubic graph with $n = 74$ (CubicVT74-9) available through the internet at `http://www.matapp.unimib.it/~spiga/census.html`, and finally a 3-dimensional grid graph (spin5). For these graphs there is a significant difference between θ and α.

In all cases the new bound (with 100 projection polytope constraints) provides a clear improvement over the theta number $\theta(G)$. This fact is particularly impressive for the cubic graph and the grid graph.

Graph	n	$\theta(G)$	New bound	$\alpha(G)$
g60-25	60	15.0058	14.71	14
cubic	74	34.8561	33.34	≥ 32
g80-25	80	17.1670	17.01	17
g100-10	100	32.1166	31.52	≥ 29
spin5	125	55.9017	51.61	≥ 50

Table 6: Results for instances of stable-set problems of various sizes and densities.

5 Conclusions

We have presented a hierarchical approach for tightening relaxations of NP-hard graph problems. The approach is based on projections to smaller polytopes corresponding to subgraphs of the original graph and has the distinguishing feature that all the resulting relaxations are formulated in the original matrix space.

The following observations can be made regarding our computational results for instances of max-cut and stable-set:

1. The hierarchy may not reach optimality until the final level and in the worst case situation the gap can still be quite large at the $k = n - 1$ level.

2. Significant improvements in the bound can be reached at the first level ($k = 5$) of the hierarchy. This strong bound can greatly reduce the number of nodes BiqMac requires.

3. Although there are $\binom{n}{k}$ k-projection polytopes at each level in the hierarchy, already after including a small fraction of these we attain a value close to the bound at that level. Therefore a good separation algorithm will be essential for the proposed approach to be efficient.

4. On the other hand, it is possible that including all k-projection polytope constraints from a level will not improve the bound from the previous level. However, this outcome seems to be atypical.

This paper does not provide an efficient implementation towards the new bound. A serious implementation will require exploiting the special structure of the relaxations. Future work will also examine the addition of k-projection polytopes with $k \geq 5$. This will require a more general separation algorithm to identify promising subgraphs.

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