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# ON A HIGHER DIMENSIONAL VERSION OF THE BENJAMIN–ONO EQUATION

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ABSTRACT. We consider a higher dimensional version of the Benjamin–Ono equation,  $\partial_t u - \mathcal{R}_1 \Delta u + u \partial_{x_1} u = 0$ , where  $\mathcal{R}_1$  denotes the Riesz transform with respect to the first coordinate. We first establish space–time estimates for the associated linear equation, many of which are sharp. These estimates enable us to show that the initial value problem for the nonlinear equation is locally well-posed in  $L^2$ -Sobolev spaces  $H^s(\mathbb{R}^d)$ , with  $s > 5/3$  if  $d = 2$  and  $s > d/2 + 1/2$  if  $d \geq 3$ . We also provide ill-posedness results.

## 1. INTRODUCTION

With  $d \geq 2$ , we consider the initial value problem for a higher dimensional version of the Benjamin–Ono equation;

$$(HBO) \quad \begin{cases} \partial_t u - \mathcal{R}_1 \Delta u + u \partial_{x_1} u = 0, & x \in \mathbb{R}^d, t \in \mathbb{R}, \\ u(x, 0) = u_0. \end{cases}$$

Here  $\mathcal{R}_1$  denotes the Riesz transform with respect to the first coordinate  $x_1$  and  $\Delta$  denotes the Laplacian with respect to the full spatial variable  $x \in \mathbb{R}^d$ . If  $u(\cdot, t)$  solves (HBO) at a certain time  $t$ , then  $-u(\cdot, -t)$  solves the analogous problem with  $-\mathcal{R}_1$  replaced by  $\mathcal{R}_1$ , and so the sign in the equation is not important.

Taking  $d = 1$ , the Riesz transform coincides with the Hilbert transform, and so we recover the extensively studied Benjamin–Ono equation; see [3, 4, 20] and the references therein. The equation maintains its physical interest with  $d = 2$ ; see for example [1, 19, 27] and the references therein. Indeed, Mariş [15] found that solitary wave solutions can still propagate; of the form  $u(x_1, x_2, t) = \varphi(x_1 - ct, x_2)$ . Both the cases  $d = 1$  and 2 have been used to model one dimensional internal waves in stratified fluids of  $\mathbb{R}^3$  (with a vertical discontinuity in the density of the fluid).

With  $d = 1$ , the available local well-posedness theory has been based on compactness methods. Indeed, Molinet, Saut and Tzvetkov [17] proved that the problem cannot be solved in  $L^2$ -Sobolev spaces  $H^s$  by Picard iteration. We will show that this remains true with  $d \geq 2$ , and so compactness methods will also play a role here.

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In higher dimensions, the  $d = 2$  case presents the most mathematical difficulties, at least with the techniques that we will employ.

Combining the Kato–Ponce commutator estimate [10] with Gronwall’s inequality, one can show that smooth solutions of (HBO) satisfy

$$(1.1) \quad \sup_{[0,T]} \|u(t)\|_{H^s} \leq \|u(0)\|_{H^s} \exp\left(c \int_0^T \|\nabla u(t)\|_{L^\infty} dt\right)$$

for all  $T > 0$ . Thus, if we could control the argument of the exponential function by the  $H^s$ -norm, we could argue by compactness in order to establish the existence of solutions with less regularity. If this were to be done using Sobolev embedding, the required order of regularity would be  $s > d/2 + 1$ . However this would fail to take advantage of the additional *dispersion* of the higher dimensional equation.

Instead we follow the idea introduced by Koch and Tzvetkov [13] to study the local well-posedness of the one dimensional Benjamin–Ono equation. Roughly this consists of using Strichartz estimates rather than Sobolev embeddings. Extensions of this method were given by Kenig and König [11]. This will be the starting point in our analysis and so we first establish Strichartz estimates for the linear problem.

**1.1. Main Results.** Our first result is a sharp *Strichartz estimate* with  $d \geq 3$ . More precisely, we consider the linear equation

$$(LHBO) \quad \begin{cases} \partial_t v = \mathcal{R}_1 \Delta v, & x \in \mathbb{R}^d, t \in \mathbb{R}, \\ v(x, 0) = f(x), \end{cases}$$

where we can write  $\mathcal{R}_1 = -(-\Delta)^{-1/2} \partial_{x_1}$  with the fractional Laplacian defined as usual in terms of the Fourier transform  $(-\Delta)^s f := (|\cdot|^s \widehat{f})^\vee$ . Smooth solutions to (LHBO) can be similarly written as

$$v(x, t) = e^{t\mathcal{R}_1 \Delta} f(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi \cdot x} e^{i\xi_1 |\xi| t} \widehat{f}(\xi) d\xi.$$

We will prove the following theorem for data in homogeneous Sobolev spaces, with norm given by  $\|f\|_{\dot{H}^s} := \|(-\Delta)^{s/2} f\|_{L^2}$ . The estimate is sharp with respect to the regularity and the Lebesgue exponents.

**Theorem 1.1.** *Let  $d \geq 3$  and  $q < \infty$ . Then there is a constant  $C \equiv C(d, q, r)$  such that*

$$\|e^{t\mathcal{R}_1 \Delta} f\|_{L_t^r(\mathbb{R}, L_x^q(\mathbb{R}^d))} \leq C \|f\|_{\dot{H}^s(\mathbb{R}^d)}$$

*holds for all  $f \in \dot{H}^s(\mathbb{R}^d)$  if and only if  $\frac{2}{q} + \frac{2}{r} \leq 1$  and  $s = d(\frac{1}{2} - \frac{1}{q}) - \frac{2}{r}$ .*

With  $d = 2$ , we prove the sharp *dispersive estimate* (there is slightly less decay than in the higher dimensional cases), however we do not know if the Strichartz estimate yielded by this could hold in a larger range or not. Kenig, Ponce and Vega [12, Theorem 2.4] proved the analogous estimate for  $d = 1$  with data in  $L^2(\mathbb{R})$ .

**Theorem 1.2.** *Let  $d = 2$  and  $q < \infty$ . Then there is a constant  $C \equiv C(q, r)$  such that*

$$\|e^{t\mathcal{R}_1\Delta}f\|_{L_t^r(L_x^q(\mathbb{R}^2))} \leq C\|f\|_{\dot{H}^s(\mathbb{R}^2)}$$

*holds for all  $f \in \dot{H}^s(\mathbb{R}^2)$  if  $\frac{10}{q} + \frac{12}{r} \leq 5$  and  $s = 1 - \frac{2}{q} - \frac{2}{r}$ .*

We will also prove a *local smoothing* estimate. This kind of estimate was first proved by Kato [9] for the Korteweg–de Vries equation, and by Vega [26] and Constantin–Saut [5] for the Schrödinger equation. With  $d = 1$ , the following estimate follows from an identity first observed by Kenig, Ponce and Vega [12, Lemma 2.1].

**Theorem 1.3.** *Let  $\alpha > 1/2$ . Then*

$$\int_{\mathbb{R}^{d+1}} |e^{t\mathcal{R}_1\Delta}f(x)|^2 \frac{dxdt}{(1+|x|^2)^\alpha} \leq C_\alpha \|f\|_{\dot{H}^{-1/2}(\mathbb{R}^d)}.$$

As remarked in the introduction, one can prove that (HBO) is locally well-posed in inhomogeneous Sobolev spaces  $H^s(\mathbb{R}^d)$  for  $s > d/2 + 1$  using standard compactness methods. These spaces are defined in the same way as the homogeneous spaces, with  $(-\Delta)^{s/2}$  replaced by  $J^s := (I - \Delta)^{s/2}$ . Using the Strichartz estimates we make the following improvement of the standard result. The Sobolev space  $W^{1,\infty}$  is defined as usual with norm  $\|f\|_{W^{1,\infty}} := \|f\|_{L^\infty} + \|\nabla f\|_{L^\infty}$ .

**Theorem 1.4.** *Let  $s > s_d$  where  $s_d = d/2 + 1/2$  for  $d \geq 3$  and  $s_2 = 5/3$ . Then, for any  $u_0 \in H^s(\mathbb{R}^d)$ , there exist a time  $T = T(\|u_0\|_{H^s})$  and a unique solution  $u$  to (HBO) that belongs to*

$$C([0, T]; H^s(\mathbb{R}^d)) \cap L^1([0, T]; W^{1,\infty}(\mathbb{R}^d)).$$

*Moreover, the flow map  $u_0 \mapsto u(t)$  is continuous from  $H^s(\mathbb{R}^d)$  to  $H^s(\mathbb{R}^d)$ .*

As mentioned above, we will show that the flow map  $u_0 \mapsto u(t)$  is not of class  $C^2$  for any  $s \in \mathbb{R}$ . In particular, this implies that (HBO) cannot be solved using the Duhamel formulation and the contraction mapping principle.

**Theorem 1.5.** *Let  $s \in \mathbb{R}$ . Then (HBO) does not admit a solution  $u$  such that the flow map  $u_0 \mapsto u(t)$  is  $C^2$ -differentiable from  $H^s(\mathbb{R}^d)$  to  $H^s(\mathbb{R}^d)$ .*

With  $d = 2$ , we use the existence of solitary wave solutions [15] to show that the flow map would not be uniformly continuous in  $L^2(\mathbb{R}^2)$ .

**Proposition 1.6.** *Let  $d = 2$ . Then (HBO) does not admit a solution  $u$  such that the flow map  $u_0 \mapsto u(t)$  is uniformly continuous from  $L^2(\mathbb{R}^2)$  to  $L^2(\mathbb{R}^2)$ .*

Some remarks are in order:

- (1) If  $u$  solves (HBO), then so does the scaled version  $u_\lambda$  defined by

$$u_\lambda(x, t) := \lambda u(\lambda x, \lambda^2 t),$$

for any positive  $\lambda$ . On the other hand, one can calculate that

$$\|u_\lambda(\cdot, t)\|_{\dot{H}^s} = \lambda^{1-d/2+s} \|u(\cdot, \lambda^2 t)\|_{\dot{H}^s}.$$

As a consequence, the scale-invariant regularity for (HBO) is  $s = d/2 - 1$ . In particular, the  $d = 2$  problem is  $L^2$ -critical. Thus our results are far from reaching the regularity suggested by scaling.

- (2) The higher dimensional Benjamin–Ono equation has a Hamiltonian structure. Formally, there are at least three quantities conserved by the flow;

$$\begin{aligned} I(u) &= \int u(x, t) dx, \\ M(u) &= \int u^2(x, t) dx, \\ H(u) &= \int \left| (-\Delta)^{1/4} u(x, t) \right|^2 - \frac{1}{3} u^3(x, t) dx. \end{aligned}$$

Note that  $H^{1/2}(\mathbb{R}^d) \hookrightarrow L^3(\mathbb{R}^d)$  by Sobolev embedding when  $d \leq 3$ , so that  $H(u)$  is well-defined in those cases. Unfortunately our local well-posedness results require too much regularity to be able to take advantage of this.

- (3) For the one dimensional Benjamin–Ono equation, Tao [23] introduced a gauge transformation which allowed him to establish local and global results in  $H^1(\mathbb{R})$ . In the end, it was possible to go all the way to  $L^2(\mathbb{R})$  using this gauge transformation; see [7] and [16]. We do not know if there is such a gauge transformation for the higher dimensional Benjamin–Ono equation.

We will begin with the linear equation (LHBO). In the following section, we prove the local smoothing estimate of Theorem 1.3. In the third section, we prove sharp decay rates for certain relevant oscillatory integrals, and in the fourth section we use a well-known variant of an argument due to Tomas [25] to establish the Strichartz estimates of Theorems 1.1 and 1.2. We then proceed to consider the initial value problem for the nonlinear equation (HBO). The fifth section is devoted to proving Theorem 1.4. We conclude the paper with an appendix where we show the ill-posedness results stated in Theorem 1.5 and Proposition 1.6.

## 2. PROOF OF THEOREM 1.3

We will require the following trace estimate for domains with boundaries that can be written as graphs of measurable functions.

**Theorem 2.1.** *Let  $\alpha > 1/2$  and  $\gamma : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  be any measurable function. Then the trace operator  $T : f \mapsto f(\cdot, \gamma(\cdot))$  satisfies*

$$\|Tf\|_{L^2(\mathbb{R}^{d-1})} \leq C_\alpha \|f\|_{H^\alpha(\mathbb{R}^d)},$$

where  $C_\alpha$  is a constant independent of  $\gamma$ .

*Proof.* Writing  $x = (\bar{x}, x_d)$ , by duality, it will suffice to prove

$$(2.1) \quad \int_{\mathbb{R}^d} |\widehat{g d\sigma}(\xi)|^2 \frac{d\xi}{(1 + |\xi|^2)^\alpha} \leq C_\alpha \|g\|_{L^2(d\sigma)}^2,$$

where  $d\sigma(\bar{x}, x_d) = \delta(x_d - \gamma(\bar{x}))$ . By squaring out and Fubini's theorem, it would suffice to prove

$$\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} g(\bar{x}) \overline{g(\bar{y})} \int_{\mathbb{R}^d} e^{-i(\bar{\xi} \cdot (\bar{x} - \bar{y}) + \xi_d(\gamma(\bar{x}) - \gamma(\bar{y})))} \frac{d\xi}{(1 + |\xi|^2)^\alpha} d\bar{x} d\bar{y} \leq C_s \|g\|_{L^2(d\sigma)}^2.$$

The inner integral is a Fourier transform, and so the left-hand side of this can be written as

$$(2\pi)^d \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} g(\bar{x}) \overline{g(\bar{y})} J^{-2\alpha}((\bar{y} - \bar{x}, \gamma(\bar{y}) - \gamma(\bar{x}))) d\bar{x} d\bar{y},$$

where the Bessel potential  $J^{-2\alpha}$  is well-known to satisfy

$$|J^{-2\alpha}(x)| \leq C_{d,\alpha} \begin{cases} |x|^{-(d-2\alpha)}, & \text{when } |x| \leq 1 \\ |x|^{-d}, & \text{when } |x| > 1. \end{cases}$$

In particular, by simply ignoring the  $x_d$ -variable, we have that

$$|J^{-2\alpha}((\bar{y} - \bar{x}, \gamma(\bar{y}) - \gamma(\bar{x})))| \leq \phi_s(\bar{x} - \bar{y})$$

where  $\phi_s : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  is an integrable function as long as  $\alpha > 1/2$ . Thus it remains to prove

$$\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} |g(\bar{x}) \overline{g(\bar{y})}| \phi_s(\bar{x} - \bar{y}) d\bar{x} d\bar{y} \leq C_\sigma \|g\|_{L^2(d\sigma)}^2,$$

which follows by the Cauchy–Schwarz inequality and then Young's inequality.  $\square$

Now, by Plancherel's theorem,

$$(2\pi)^{2d+1} \int |e^{t\mathcal{R}_1 \Delta} f(x)|^2 dt = \int \left| \int_{\mathbb{R}^d} \widehat{f}(\xi) \delta(\xi_1 |\xi| - \tau) e^{ix \cdot \xi} d\xi \right|^2 d\tau,$$

so by changing variables  $u = \xi_1 |\xi|$ , we see that

$$\begin{aligned} (2\pi)^{2d+1} \int |e^{t\mathcal{R}_1 \Delta} f(x)|^2 dt &= \int \left| \int_{\mathbb{R}^d} \widehat{f}(\xi) \delta(u - \tau) e^{ix \cdot \xi} \frac{|\xi| du d\bar{\xi}}{2\xi_1^2 + |\bar{\xi}|^2} \right|^2 d\tau \\ &= \int \left| \int_{\mathbb{R}^{d-1}} \widehat{f}(\gamma_\tau(\bar{\xi}), \bar{\xi}) e^{i[x_1 \gamma_\tau(\bar{\xi}) + \bar{x} \cdot \bar{\xi}]} \frac{\sqrt{\gamma_\tau^2(\bar{\xi}) + |\bar{\xi}|^2}}{2\gamma_\tau^2(\bar{\xi}) + |\bar{\xi}|^2} d\bar{\xi} \right|^2 d\tau, \end{aligned}$$

where  $\gamma_\tau^2(\bar{\xi}) = \sqrt{\tau^2 + (\frac{1}{2}|\bar{\xi}|^2)^2 - \frac{1}{2}|\bar{\xi}|^2}$ . Now by integrating both sides with respect to  $(1 + |x|^2)^{-\alpha} dx$ , applying Fubini's theorem and the trace theorem in dual form (2.1), we obtain

$$\int_{\mathbb{R}^{d+1}} |e^{t\mathcal{R}_1 \Delta} f(x)|^2 \frac{dx dt}{(1 + |x|^2)^\alpha} \leq C_\alpha \int \int_{\mathbb{R}^{d-1}} \left| \widehat{f}(\gamma_\tau(\bar{\xi}), \bar{\xi}) \frac{\sqrt{\gamma_\tau^2(\bar{\xi}) + |\bar{\xi}|^2}}{2\gamma_\tau^2(\bar{\xi}) + |\bar{\xi}|^2} \right|^2 d\bar{\xi} d\tau.$$

Finally, we reverse the change of variables so that

$$\int_{\mathbb{R}^{d+1}} |e^{t\mathcal{R}_1 \Delta} f(x)|^2 \frac{dx dt}{(1 + |x|^2)^\alpha} \leq C_\alpha \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 \frac{|\xi| d\xi}{2\xi_1^2 + |\bar{\xi}|^2},$$

which is slightly better than the desired bound.

## 3. THE OSCILLATORY INTEGRAL

An analogous estimate to the following with  $d = 1$  was proven by Ponce and Vega [21, Corollary 2.3] with decay of order  $|t|^{-1/2}$ .

**Proposition 3.1.** *If  $d \geq 3$ , then there is a constant  $C$ , depending only on the Schwartz function  $\psi$  supported in  $[1/2, 2]$ , such that*

$$\left| \int_{\mathbb{R}^d} \psi(|\xi|) e^{i[t\xi_1|\xi|+x\cdot\xi]} d\xi \right| \leq C|t|^{-1}, \quad x \in \mathbb{R}^d.$$

Moreover, with  $d = 2$ ,

$$\left| \int_{\mathbb{R}^2} \psi(|\xi|) e^{i[t\xi_1|\xi|+x\cdot\xi]} d\xi \right| \leq C|t|^{-5/6}, \quad x \in \mathbb{R}^2.$$

Both rates of decay are optimal.

*Proof.* We write the integral as a two-fold iterated integral in radial and spherical coordinates;

$$\begin{aligned} (3.1) \quad I(x, t) &:= \int_{\mathbb{R}^d} \psi(|\xi|) e^{i[t\xi_1|\xi|+x\cdot\xi]} d\xi = \int_0^\infty \rho(r) \left[ \int_{\mathbb{S}^{d-1}} e^{i[tr^2\omega_1+x\cdot r\omega]} d\sigma(\omega) \right] dr \\ &= \int_0^\infty \rho(r) \widehat{\sigma}(y(r)) dr \end{aligned}$$

where  $\rho(r) = \psi(r)r^{d-1}$  and  $y(r) = y_{x,t}(r) = (tr^2 + x_1r, x_2r, \dots, x_dr)$ .

We consider first the easier higher dimensional cases, when  $d \geq 4$ , then  $d = 3$ , finally treating the harder  $d = 2$  case.

We first remark that the decay rate  $|t|^{-1}$  when  $d \geq 3$  is best possible since a better decay rate would lead to an improvement of Theorem 1.1 (and also a better decay in Proposition 4.1 when  $d \geq 3$ ) which is not possible.

**Decay rate dimensions  $d \geq 4$ .** Considering the second iterated integral in (3.1) where the integral over the sphere  $\mathbb{S}^{d-1}$  is performed first and computed as a Fourier transform  $\widehat{\sigma}(y(r))$ , we use the stationary phase estimate

$$|\widehat{\sigma}(y(r))| \leq C_d \min(1, \|y(r)\|^{-(d-1)/2}) \leq C \min(1, |tr^2 + x_1r|^{-(d-1)/2})$$

to give a quick proof of the desired uniform estimate for  $I(x, t)$  for  $d \geq 4$ . We split the radial integration via  $\text{supp}(\rho) = E^1 \cup E^2$  where

$$E^1 = \{r \in \text{supp}(\rho) : |tr^2 + rx_1| \leq 1\}$$

and decompose  $I(x, t)$  accordingly. On the one hand, we have the uniform estimate  $|E^1| \leq C|t|^{-1}$  for the measure of  $E^1$ . On the other hand,

$$\int_{E^2} |tr^2 + x_1r|^{-(d-1)/2} dr \leq C_d |t|^{-(d-1)/2} \int_{E^2} |r + (x_1/t)|^{-(d-1)/2} dr \leq C|t|^{-1}$$

when  $d \geq 4$ , giving us the uniform estimate  $|I(x, t)| \leq C|t|^{-1}$  in that case.

For the lower dimensional cases, we adjust the argument in the following way. We proceed as before, splitting the radial integral into two parts via a decomposition  $F^1$  and  $F^2$  where  $F^1$  contains  $E^1$  (and hence  $F^2$  will be contained in  $E^2$ ) however will still satisfy the uniform bound  $|F^1| \leq C|t|^{-1}$ . Thus we can estimate the part of  $I(x, t)$  defined over  $F^1$  by a constant multiple of  $|t|^{-1}$  as before – this only uses the trivial bound  $|\widehat{\sigma}(y(r))| \lesssim 1$ <sup>1</sup> and the estimate  $|F^1| \lesssim |t|^{-1}$ . For the part over  $F^2$ , we need the more refined asymptotic expansion

$$\widehat{\sigma}(y(r)) = c_1 \frac{e^{i\|y(r)\|}}{\|y(r)\|^{(d-1)/2}} + c_2 \frac{e^{-i\|y(r)\|}}{\|y(r)\|^{(d-1)/2}} + \mathcal{E}_{x,t}(r),$$

valid for large  $\|y(r)\|$ , where  $c_1$  and  $c_2$  are constants depending only on  $d$ , and the uniform estimate  $|\mathcal{E}_{x,t}(r)| \leq C_d \|y(r)\|^{-(d+1)/2}$  holds. The argument above works as before for the integral of  $\mathcal{E}_{x,t}(r)$  over  $F^2$  (since  $F^2$  is contained in  $E^2$ ) but is now valid for  $d = 2$  and  $d = 3$ .

Hence matters are reduced to bounding integrals of the form

$$(3.2) \quad \int_{F^2} e^{\pm i\phi(r)} \rho(r) \phi(r)^{-(d-1)/2} dr$$

where  $\phi(r) = \|y(r)\|$ . We will need to examine the phase  $\phi(r)$  more closely; write  $\phi(r) = \sqrt{f(r)}$  where  $f(r) = (tr^2 + x_1 r)^2 + r^2 \|x'\|^2$  and  $x' = (x_2, \dots, x_d)$  so that  $x = (x_1, x')$ .

**Decay rate dimension  $d = 3$ .** We may assume that  $|x_1| \sim t$  and  $\|x'\|/t \leq c_0$  for any small absolute constant  $c_0$  (to be determined later); otherwise,  $\phi(r) \gtrsim |t|$  on the support of  $\rho$ . Then, no matter how we define  $F^1$  and  $F^2$  (in this case, we can take  $F^1 = E^1$  and  $F^2 = E^2$ ), we obtain

$$\left| \int_{F^2} e^{\pm i\phi(r)} \rho(r) \phi(r)^{-1} dr \right| \lesssim \int_{F^2} |\rho(r)| |\phi(r)|^{-1} dr \lesssim |t|^{-1}.$$

Now, when  $d = 3$ , we can write the integral in (3.2) as

$$(3.3) \quad \mp 2i \int_{F^2} \frac{d}{dr} [e^{\pm i\phi(r)}] \rho(r) f'(r)^{-1} dr.$$

Although the derivative  $f'$  can vanish to order two on the support of  $\rho$ , this cannot happen when  $|x_1| \sim t$  and  $\|x'\|/t$  is small. In this case, we enlarge the set  $E^1$  to  $F^1 := E^1 \cup \{r \in \text{supp}(\rho) : |f'(r)| \leq t\}$  and so

$$F^2 = \{r \in \text{supp}(\rho) : |r + (x_1/t)| \geq 1/t \text{ and } |f'(r)| \geq t\}$$

which is contained in  $E^2$ . We will see that the uniform estimate  $|F^1| \lesssim |t|^{-1}$  still holds. On the other hand, the set  $F^2$  can be written as a finite union of intervals such that  $\rho/f'$  is monotone on each subinterval. Taking one such interval  $(a, b)$ , we

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<sup>1</sup>We write  $A \lesssim B$  if there exists a constant  $c > 0$  such that  $A \leq cB$ . Also we write  $A \sim B$  if both  $A \lesssim B$  and  $B \lesssim A$  hold.



then integrate by parts;

$$\begin{aligned}
(3.4) \quad & \left| \int_a^b \frac{d}{dr} [e^{\pm i\phi(r)}] \rho(r) f'(r)^{-1} dr \right| \\
& \leq \left| \left[ e^{\pm i\phi(r)} \rho(r) f'(r)^{-1} \right]_a^b \right| + \int_a^b \left| e^{\pm i\phi(r)} \left( \rho(r) f'(r)^{-1} \right)' \right| dr \\
& \lesssim \left[ \min_{r \in (a,b)} |f'(r)| \right]^{-1} + \left| \int_a^b \left( \rho(r) f'(r)^{-1} \right)' dr \right| \\
& \lesssim \left[ \min_{r \in (a,b)} |f'(r)| \right]^{-1} \lesssim |t|^{-1},
\end{aligned}$$

which yields the desired bound. To prove the bound on  $F^1$ , a computation shows that  $f'$  has two real roots away from  $r = 0$ ;  $f'(t) = 4rt^2(r - r_+)(r - r_-)$  where

$$r_{\pm} = -\frac{3x_1}{4t} \pm \frac{1}{4} \sqrt{\frac{x_1^2}{t^2} - 8 \frac{\|x'\|^2}{t^2}}.$$

The constant  $c_0 > 0$  above is chosen small enough to guarantee that  $r_{\pm}$  are real and  $|r_+ - r_-| \sim 1$ . From these observations, it is easy to verify the uniform bound  $|F^1| \lesssim |t|^{-1}$  (if the roots  $r_+$  and  $r_-$  were not separated, we would be stuck with the bad bound  $|F^1| \lesssim t^{-1/2}$ ). This finishes the case  $d = 3$ .

For the case  $d = 2$ , observe that the above arguments break down and in particular the argument allowing the restriction to  $\|x'\|/t$  being small is not valid when  $d = 2$ . We first study the sharpness of the decay rate.

**Sharpness of the decay rate dimension  $d = 2$ .** We begin by showing that the decay bound  $|t|^{-1}$  is not possible. We examine the integral  $I(x, t)$  for  $t > 0$  large and  $x = (-(4/3)t, (\sqrt{2}/3)t)$ . The reason for this particular choice of  $x$  is that

$$f(r) = t^2[(r + 1/3)(r - 1)^3 + 1/3] \quad \text{and} \quad f'(r) = 4t^2r(r - 1)^2.$$

We choose  $\psi$  so that its support is a small neighborhood of  $r = 1$  and so that  $\psi = 1$  near  $r = 1$  (hence  $\rho$  is supported in a small neighborhood of  $r = 1$ ). We set  $g(r) = f(r)/t^2 = [\phi(r)/t]^2$  and note that we may also write  $g(r) = r^2[(r - 4/3)^2 + 2/9]$ , showing in particular that  $g$  is smooth and uniformly bounded away from zero on the support of  $\rho$ .

When  $d = 2$ ,  $\widehat{\sigma}(y(r)) = cJ_0(\|y(r)\|)$  where  $J_0$  is the Bessel function of order zero which has the asymptotic expansion  $J_0(s) = bs^{-1/2} \cos(s - \pi/4) + O(s^{-3/2})$ , and so we see that for  $x = x(t)$  given above,

$$I(x(t), t) = c \int \rho(r) J_0(\|y(r)\|) dr = \frac{a}{\sqrt{t}} \int \frac{\cos(t\sqrt{g(r)} - \pi/4)}{g(r)^{1/4}} \rho(r) dr + O(t^{-3/2})$$

for some  $a \neq 0$ . Let  $\mathcal{I}$  denote the integral on the right so that

$$(3.5) \quad I(x(t), t) = at^{-1/2} \mathcal{I} + O(t^{-3/2}).$$

We will attempt to evaluate  $\mathcal{I}$  *asymptotically* by making a series of changes of variables. First we make the change of variables  $s = (r + 1/3)^{1/3}(r - 1)$  so that

$$\mathcal{I} = \int \rho_1(s) \cos(t\sqrt{s^3 + 1/3} - \pi/4) ds$$

where  $\rho_1(s) = \rho(r(s))g(r(s))^{-1/4}s'(r(s))^{-1}$ . Note that  $s = s(r) = (r+1/3)^{1/3}(r-1)$  is a nice smooth change of variables with a smooth inverse  $r(s)$  on the support of  $\rho$ . We note that  $\rho_1$  is supported in a small neighborhood of 0, smooth and  $\rho_1(0) = \rho(1)3^{1/4}[3/4]^{1/3} \neq 0$ .

We make a further change of variables  $u = [\sqrt{s^3 + 1/3} - 1/\sqrt{3}]^{1/3}$  which leads to

$$\mathcal{I} = \int \rho_2(u) \cos(tu^3 + t/\sqrt{3} - \pi/4) du$$

where  $\rho_2$  is smooth, supported in a small neighborhood of 0, and  $\rho_2(0) \neq 0$ . Finally the scaling  $v = t^{1/3}u$  change of variables leads to

$$\mathcal{I} = \frac{1}{t^{1/3}} \int \rho_2(v/t^{1/3}) \cos(v^3 + t/\sqrt{3} - \pi/4) dv.$$

It is easy to see that the oscillatory integral  $A = A(t) := \int \cos(v^3 + t/\sqrt{3} - \pi/4) dv$  exists in the sense that

$$\lim_{R \rightarrow \infty} \int_{|v| \leq R} \cos(v^3 + t/\sqrt{3} - \pi/4) dv$$

exists for every  $t > 0$ . We claim that

$$(3.6) \quad \mathcal{I} = \frac{\rho_2(0)}{t^{1/3}} A + O(t^{-2/3}).$$

To see that the bound (3.6) is sufficient for our purposes, note first that it allows us to write (3.5) as  $I(x(t), t) = bt^{-5/6}A + O(t^{-7/6})$  for some  $b \neq 0$ . But writing

$$\cos(v^3 + t/\sqrt{3} - \pi/4) = \cos(t/\sqrt{3} - \pi/4) \cos(v^3) - \sin(t/\sqrt{3} - \pi/4) \sin(v^3)$$

and noting that  $\sin(v^3)$  is an odd function, we have

$$A = \lim_{R \rightarrow \infty} \int_{|v| \leq R} \cos(v^3 + t/\sqrt{3} - \pi/4) dv = \cos(t/\sqrt{3} - \pi/4) \int_{\mathbb{R}} \cos(v^3) dv$$

which show that  $I(x(t), t) = b \cos(t/\sqrt{3} - \pi/4) t^{-5/6} + O(t^{-7/6})$  for  $t > 0$  large. Thus we arrive at the desired conclusion that the best possible uniform estimate for  $I(x, t)$  is  $O(t^{-5/6})$  when  $d = 2$ .

Now to establish the claim (3.6), we write

$$\rho_2(vt^{-1/3}) - \rho_2(0) = vt^{-1/3} \int_0^1 \rho_2'(\sigma vt^{-1/3}) d\sigma$$

and hence

$$t^{1/3} \mathcal{I} - \rho_2(0) A = \lim_{R \rightarrow \infty} \int_{|v| \leq R} vt^{-1/3} \left[ \int_0^1 \rho_2'(\sigma vt^{-1/3}) d\sigma \right] \cos(v^3 - t/\sqrt{3} - \pi/4) dv.$$

We fix  $R > 1$  and interchange the order of integration so that the above iterated integral is equal to

$$t^{-1/3} \int_0^1 \left[ \int_{|v| \leq R} \cos(v^3 + t/\sqrt{3} - \pi/4) \rho_2'(\sigma vt^{-1/3}) v dv \right] d\sigma =: t^{-1/3} \int_0^1 H_R(\sigma) d\sigma.$$

Of course  $H_R(\sigma)$  also depends on  $t$  but if we can show (a)  $H_R = O(1)$ , uniformly in  $\sigma, t$  and  $R$  and (b)  $\lim_{R \rightarrow \infty} H_R(\sigma)$  exists pointwise for every  $\sigma \in (0, 1)$ ,

then Lebesgue's Dominated Convergence Theorem implies that  $t^{1/3}\mathcal{I} - \rho_2(0)A = O(t^{-1/3})$ , establishing the claim.

First we establish (a). We write  $H_R(\sigma) = H_{R,1}(\sigma) + H_{R,2}(\sigma)$ , where

$$H_{R,1}(\sigma) = \int_{|v| \leq 1} \cos(v^3 + t/\sqrt{3} - \pi/3) \rho_2'(\sigma vt^{-1/3}) v \, dv$$

$$H_{R,2}(\sigma) = \int_{1 \leq |v| \leq R} \frac{d}{dv} \left[ \sin(v^3 + t/\sqrt{3} - \pi/4) \right] \rho_3(v) \, dv$$

and  $\rho_3(v) = \rho_2'(\sigma vt^{-1/3})/3v$ . Clearly  $H_{R,1} = O(1)$ , uniformly in  $\sigma$  and  $t$ , and is independent of  $R$ . Carrying out the integration by parts in  $H_{R,2}$ , we see that

$$H_{R,2} = B_1(\sigma, t) + B_2(\sigma, t, R) - \int_{1 \leq |v| \leq R} \sin(v^3 + t/\sqrt{3} - \pi/4) \rho_3'(v) \, dv$$

where  $B_1 = O(1)$  and  $B_2 = O(R^{-1})$ , uniformly in  $\sigma$  and  $t$ . Labelling the integral above as  $B_3(\sigma, t, R)$ , we divide up the intervals  $[1, R]$  and  $[-R, -1]$  of integration into subintervals  $\cup J$  so that  $\rho_3$  is monotone on each interval  $J$ . Note that the specific subintervals will depend on  $\sigma$  and  $t$  but the number of subintervals will not. With respect to the subintervals  $\{J\}$ , we decompose  $B_3 = \sum_J B_{3,J}(\sigma, t)$  where

$$B_{3,J} = \int_J \sin(v^3 + t/\sqrt{3} - \pi/4) \rho_3'(v) \, dv$$

If  $J = [a, b] \subset \mathbb{R}$  with  $a > 1$  say, we have

$$(3.7) \quad |B_{3,J}| \leq \int_J |\rho_3'(v)| \, dv = \left| \int_J \rho_3'(v) \, dv \right| = |\rho_3(b) - \rho_3(a)| = O(1/a),$$

uniformly in  $\sigma, t$  and  $R$ , since  $\rho_3(v) = O(1/v)$  uniformly in  $\sigma$  and  $t$ . This establishes (a) since  $O(1/a) = O(1)$  when  $a > 1$  and the number of intervals  $\{J\}$  is uniformly  $O(1)$ , even if the intervals  $J$  themselves depend on  $\sigma, t$  and  $R$ .

For (b), it suffices to show that

$$\int_{R \leq |v| \leq R'} \frac{d}{dv} \left[ \sin(v^3 + t/\sqrt{3} - \pi/4) \right] \rho_3(v) \, dv \longrightarrow 0$$

as  $R, R' \rightarrow \infty$ , for each fixed  $\sigma$  and  $t$ . This follows exactly in the same way as (a), breaking up the above integral into subintervals on which  $\rho_3$  is monotone and using (3.7) which is  $O(1/R)$  in this context.

**Decay rate dimension  $d = 2$ .** It remains to establish the uniform bound  $|I(x, t)| \lesssim |t|^{-5/6}$ . To do this, we go back to (3.2) with the choice  $F^1 = E^1$  and  $F^2 = E^2$  and write the integral appearing there as

$$(3.8) \quad \int_{E^2} e^{\pm i\phi(r)} \rho(r) \phi(r)^{-(d-1)/2} \, dr = \frac{1}{\sqrt{t}} \int_{E^2} e^{\pm it\sqrt{g(r)}} \rho(r) g(r)^{-1/4} \, dr$$

where  $g(r) = [\phi(r)/t]^2$ . Hence matters are reduced to showing that

$$(3.9) \quad I := \int_{E^2} e^{\pm it\sqrt{g(r)}} \rho(r) g(r)^{-1/4} \, dr = O(t^{-1/3})$$

holds uniformly in  $x$ . We begin by reducing the parameters to the region  $|x_1| \sim t$  and  $|x_2| \lesssim t$ . Recall that  $g(r)/r^2 = [r + x_1/t]^2 + x_2^2/t^2$  and so

$$(3.10) \quad \frac{g'(r)}{2r} = \left(r + \frac{x_1}{t}\right)\left(2r + \frac{x_1}{t}\right) + \frac{x_2^2}{t^2}.$$

Hence if  $|x_1| \ll t$ , we have  $g'(r) \sim 2r^2 + x_2^2/t^2 \gtrsim 1$  and  $g(r) \sim r^2 + x_2^2/t^2$  on the support of  $\rho$ . We write

$$(3.11) \quad I = \pm \frac{2}{it} \int_{E^2} \frac{d}{dr} [e^{\pm it\sqrt{g(r)}}] \rho(r) \frac{g(r)^{1/4}}{g'(r)} dr,$$

and breaking up  $E^2$  into intervals where  $g^{1/4}/g'$  is monotonic as before, this yields the bound  $I = O(t^{-1})$  via integration by parts, noting that  $|g^{1/4}/g'| \lesssim 1$ .

Next, if  $|x_1| \gg t$ , we have  $g'(r), g(r) \sim (x_1^2 + x_2^2)/t^2 \gtrsim 1$  which again leads to the bound  $I = O(t^{-1})$ . Finally if  $|x_1| \sim t$  and  $|x_2| \gg t$ , we have  $g'(r), g(r) \sim x_2^2/t^2 \gtrsim 1$ , again leading to the bound  $I = O(t^{-1})$ , all of which are better than the claimed bound for  $I$  in (3.9).

Therefore we may assume that  $|x_1| \sim t$  and  $|x_2| \lesssim t$  (recall that when  $d \geq 3$ , the reduction to  $\|x'\| \gtrsim t$  was straightforward – this is not the case when  $d = 2$ ). Recall that  $E^2 = \{r \sim 1 : |r + x_1/t| \geq 1/t\}$ . If  $\Phi(r) = \sqrt{g(r)}$ , then  $\Phi'(r) = g'(r)/2\sqrt{g(r)}$  and

$$2\Phi''(r) = \frac{g''(r)g(r) + [g'(r)]^2/2}{g(r)^{3/2}}$$

where  $g'(r)$  is given in (3.10) and

$$(3.12) \quad g''(r)/2 = r^2 + x_2^2/t^2 + 5(r + x_1/t)(r + x_1/5t) = 6(r - r_+)(r - r_-).$$

Here the roots  $r_{\pm}$  may be complex but when  $|x_2| \ll t$ , the roots are real and  $r_{\pm} \sim -x_1/2t(1 \pm 1/\sqrt{3})$ .

We split  $E^2 = E_1^2 \cup E_2^2$  where  $E_1^2 = \{r \sim 1 : 1/t \leq |r + x_1/t| \leq \epsilon_0\}$  for some small  $\epsilon_0 > 0$  and decompose  $I = I_1 + I_2$  accordingly. Hence on  $E_1^2$ ,  $g''(r) \sim 1$  if  $\epsilon_0$  is sufficiently small (less than  $1/100$  is enough). To analyse  $I_1$  we consider two cases:  $|x_2| \sim t$  and  $|x_2| \ll t$ . In the first case, both  $g(r), g'(r) \sim 1$  and so an integration by parts argument, as shown in (3.11), shows  $|I_1| \lesssim t^{-1}$  as before. In the second case, fix a small positive (absolute)  $\delta > 0$  such that  $|x_2| \leq \delta t$ . In this case, we split  $E_1^2 = E_{1,1}^2 \cup E_{1,2}^2$  further where  $E_{1,1}^2 = \{r \in E_1^2 : |r + x_1/t| \leq \epsilon x_2^2/t^2\}$  for some small  $\epsilon > 0$  and write  $I_1 = I_{1,1} + I_{1,2}$  accordingly.

On  $E_{1,1}^2$ ,  $g(r) \sim x_2^2/t^2$  and  $|g'(r)| \leq 10|r + x_1/t| + x_2^2/t^2$  implying  $g'(r)^2 \leq 10^{-2}x_2^2/t^2$  if  $\epsilon > 0$  is small enough. Therefore

$$|\Phi''(r)| \gtrsim 1/\sqrt{g(r)} \sim t/|x_2|$$

on  $E_{1,1}^2$ . Since the amplitude  $\rho(r)/g(r)^{1/4}$  in (3.8) is  $\sim t^{1/2}/|x_2|^{1/2}$  on  $E_{1,1}^2$ , an application of van der Corput's lemma, together with an integration by parts argument (see for example [22, pp. 234]), gives the bound  $|I_{1,1}| \lesssim t^{-1/2}$ .

On  $E_{1,2}^2$ ,  $g(r) \sim |r + x_1/t|^2$  and

$$|g'(r)| \gtrsim |r + x_1/t| - 10\epsilon^{-1}|r + x_1/t|^2 \gtrsim |r + x_1/t|$$

if we choose  $\epsilon_0 \leq 10^{-2}\epsilon$ . Hence by integrating by parts, as in (3.11), we have

$$|I_{1,2}| \lesssim t^{-1} \max_{r \in E_2^2} |r + x_1/t|^{-1/2} \lesssim t^{-1/2}$$

which, together with our bound on  $I_{1,1}$ , implies  $|I_1| \lesssim t^{-1/2}$  in this second case. Hence in either case, we have  $I_1 = O(t^{-1/2})$ .

To estimate  $I_2$ , we note that  $g(r) \sim 1$  on  $E_2^2$ . By splitting the set  $E_2^2$  into two parts where  $g'(r)$  is respectively small and large, one can easily take care of the part where  $g'(r)$  is bounded away from zero by integrating by parts (recall now  $g(r) \sim 1$ ) as in (3.11), obtaining a contribution of  $O(t^{-1})$  for this part of the integral  $I_2$ . Hence we may assume that on  $E_2^2$ , we have the further restriction that  $|g'(r)| \leq \eta$  for some small absolute constant  $\eta > 0$ .

To understand the size of  $g''(r)$ , we split  $E_2^2 = E_{2,1}^2 \cup E_{2,2}^2$  where

$$E_{2,1}^2 = \{r \in E_2^2 : |r + x_1/2t| \leq \epsilon_0\}$$

and write  $I_2 = I_{2,1} + I_{2,2}$  accordingly. On  $E_{2,1}^2$ , since  $|g'(r)| \leq \eta$ , we see from (3.10) that  $|x_2|^2/t^2 \leq 10[\epsilon_0 + \eta]$ . Hence by (3.12), we see that  $|g''(r)| \sim 1$  on  $E_{2,1}^2$ . Therefore, since  $g \sim 1$  on  $E_2^2$ , we see that  $|\Phi''(r)| \gtrsim 1$  on  $E_{2,1}^2$  and now an application of van der Corput's lemma as before shows that  $|I_{2,1}| \lesssim t^{-1/2}$ .

Finally we turn to  $I_{2,2}$  where we are integrating over  $E_{2,2}^2$  and in particular we have  $|g'''(r)| \geq 24\epsilon_0$  since  $g'''(r) = 24[r + x_1/2t]$ . We also have  $|g'(r)| \leq \eta$  and  $g(r) \sim 1$  for  $r \in E_2^2$  and so we compute

$$\frac{d^3}{dr^3} \sqrt{g(r)} = \frac{g'''(r)g(r)^2 - (3/2)g'(r)[g''(r)g(r) - (1/2)g'(r)^2]}{2g(r)^{5/2}}$$

and deduce the bound  $|d^3/dr^3 \sqrt{g(r)}| \gtrsim 1$  on  $E_{2,2}^2$  if  $\epsilon_0$  and  $\eta$  are chosen so that  $\eta \ll \epsilon_0$ . Hence a final application of van der Corput's lemma, together with an integration by parts argument, gives the bound  $|I_{2,2}| \lesssim t^{-1/3}$  which implies that  $|I_2| \lesssim t^{-1/3}$  and this completes the proof of the claim  $I = O(t^{-1/3})$ .

This finishes the proof of the proposition.  $\square$

#### 4. PROOF OF THEOREMS 1.1 AND 1.2

A dispersive estimate of the following type with  $d = 1$  was proven by Ponce and Vega [21, Lemma 2.4] with decay of the order  $|t|^{-(1/2-1/p)}$ . Thus one could argue that we have more dispersion in the higher dimensional cases, however the decay rate does not continue to increase with the dimension. This is somehow consistent with the notion that the higher dimensional equation continues to model unidimensional behaviour.

**Proposition 4.1.** *Let  $d \geq 3$  and  $2 \leq p < \infty$ . Then*

$$\|e^{t\mathcal{R}_1\Delta} f\|_{L^p(\mathbb{R}^d)} \leq C_p |t|^{-2(\frac{1}{2}-\frac{1}{p})} \|(-\Delta)^{(d-2)(\frac{1}{2}-\frac{1}{p})} f\|_{L^{p'}(\mathbb{R}^d)},$$

and the rate of decay is optimal. Moreover, with  $d = 2$ ,

$$\|e^{t\mathcal{R}_1\Delta}f\|_{L^p(\mathbb{R}^2)} \leq C_p |t|^{-\frac{5}{3}(\frac{1}{2}-\frac{1}{p})} \|(-\Delta)^{(d-\frac{5}{3})(\frac{1}{2}-\frac{1}{p})}f\|_{L^{p'}(\mathbb{R}^2)}.$$

*Proof.* First we calculate the necessary condition. Letting  $\phi$  be a smooth Schwartz function, supported on  $(-\pi/2, \pi/2)$  and equal to one on  $(-1, 1)$ , we define  $f$  by

$$f(x) = R^{-1}\phi^\vee(R^{-1}x_1)\phi^\vee(x_2)\dots\phi^\vee(x_d),$$

so that, for all  $\beta > 0$ ,

$$(4.1) \quad \|(-\Delta)^\beta f\|_{p'} \leq cR^{-1+1/p'} = cR^{-1/p}.$$

On the other hand,

$$e^{t\mathcal{R}_1\Delta}f(x) = \frac{1}{(2\pi)^d} \int \dots \int \phi(R\xi_1)\phi(\xi_2)\dots\phi(\xi_d)e^{it|\xi|\xi_1}e^{i[x_1\xi_1+\dots+x_d\xi_d]}d\xi_1\dots d\xi_d,$$

so we see that as long as  $|t| \leq \frac{R}{2}$ ,  $|x_1| \leq \frac{R}{2d}$  and  $|x_2|, \dots, |x_d| \leq \frac{1}{2d}$ , then

$$|e^{t\mathcal{R}_1\Delta}f(x)| \geq \frac{1}{(2\pi)^d} \int_0^{R^{-1}} \int_0^1 \dots \int_0^1 \cos(1) d\xi_1 \dots d\xi_d \geq cR^{-1}.$$

Thus, for  $|t| = \frac{R}{4}$ , we have that

$$(4.2) \quad \|e^{t\mathcal{R}_1\Delta}f\|_{L^p(\mathbb{R}^d)} \geq cR^{-1+1/p}.$$

Comparing (4.1) and (4.2) and letting  $R$  tend to infinity, we see that the rate of decay cannot exceed  $1 - 2/p$ .

Then, by the change of variables  $\xi \rightarrow R\xi$ ,  $x \rightarrow R^{-1}x$  and  $t \rightarrow R^{-2}t$ , and letting  $R$  tend to zero and infinity, one can calculate that, given the decay rates, the loss of regularity is necessary.

Now to prove the estimates, we consider  $\psi$  as defined in the previous section, and the smooth partition of unity  $\sum_{k \in \mathbb{Z}} \psi(2^{-k}|\cdot|) = 1$ . Defining the projection operators  $P_k$  by

$$(4.3) \quad P_k f = (\psi(2^{-k}|\cdot|)\widehat{f})^\vee,$$

the Littlewood–Paley inequality yields

$$\begin{aligned} \|e^{t\mathcal{R}_1\Delta}f\|_{L^p(\mathbb{R}^d)} &\leq C \left\| \left( \sum_{k \in \mathbb{Z}} |e^{t\mathcal{R}_1\Delta}P_k f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \\ &\leq C \left( \sum_{k \in \mathbb{Z}} \|e^{t\mathcal{R}_1\Delta}P_k f\|_{L^p(\mathbb{R}^d)}^2 \right)^{1/2}, \end{aligned}$$

where in the final estimate we use the triangle inequality in  $L^{p/2}$ . Thus, with  $d \geq 3$ , it will suffice to prove that, for all  $k \in \mathbb{Z}$ ,

$$(4.4) \quad \|e^{t\mathcal{R}_1\Delta}P_k f\|_{L^p(\mathbb{R}^d)} \leq C |t|^{-2(\frac{1}{2}-\frac{1}{p})} 2^{(d-2)(1-\frac{2}{p})k} \|f\|_{L^{p'}(\mathbb{R}^d)}.$$

Then, with  $\tilde{P}_k$  defined in the same way as  $P_k$  but with cut-off function equal to one on the support of  $\psi$ , we would obtain

$$\begin{aligned} \|e^{t\mathcal{R}_1\Delta}f\|_{L^p(\mathbb{R}^d)} &\leq C|t|^{-2(\frac{1}{2}-\frac{1}{p})} \left( \sum_{k\in\mathbb{Z}} 2^{(d-2)(1-\frac{2}{p})k} \|\tilde{P}_k f\|_{L^{p'}(\mathbb{R}^d)}^2 \right)^{1/2} \\ &\leq C|t|^{-2(\frac{1}{2}-\frac{1}{p})} \left\| \left( \sum_{k\in\mathbb{Z}} 2^{(d-2)(1-\frac{2}{p})k} |\tilde{P}_k f|^2 \right)^{1/2} \right\|_{L^{p'}(\mathbb{R}^d)}, \end{aligned}$$

and the desired inequality would follow by a further Littlewood–Paley application.

By scaling invariance, it will suffice to prove (4.4) with  $k = 0$ . That is to say, an estimate for the operator  $T$  defined by

$$Tf(x, t) = \int_{\mathbb{R}^d} \psi(|\xi|) \hat{f}(\xi) e^{i[t\xi_1|\xi|+x\cdot\xi]} d\xi$$

with data  $f \in L^{p'}(\mathbb{R}^d)$ . By Plancherel's theorem, we have that

$$(4.5) \quad \|Tf(\cdot, t)\|_2 \leq C\|f\|_2.$$

On the other hand, recalling that the Fourier transform is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(y) e^{-iy\cdot\xi} dy,$$

by Fubini's theorem, we can write

$$(4.6) \quad Tf(x, t) = \int_{\mathbb{R}^d} f(y) \int_{\mathbb{R}^d} \psi(|\xi|) e^{i[t\xi_1|\xi|+(x-y)\cdot\xi]} d\xi dy.$$

Applying Lemma 3.1, we obtain

$$\left| \int_{\mathbb{R}^d} \psi(|\xi|) e^{i[t\xi_1|\xi|+(x-y)\cdot\xi]} d\xi \right| \leq C|t|^{-1}.$$

Substituting into (4.6), this yields a dispersive estimate,

$$\|Tf(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \leq C|t|^{-1}\|f\|_{L^1(\mathbb{R}^d)},$$

so that by Riesz–Thorin interpolation with our  $L^2$  estimate (4.5), we obtain

$$(4.7) \quad \|Tf(\cdot, t)\|_{L^p(\mathbb{R}^d)} \leq \frac{C}{|t|^{1-2/p}} \|f\|_{L^{p'}(\mathbb{R}^d)}, \quad p \geq 2,$$

which is (4.4) with  $k = 0$ , and so we are done.

When  $d = 2$ , the same argument applies. In that case, the estimate (4.7) takes the form

$$\|Tf(\cdot, t)\|_{L^p(\mathbb{R}^2)} \leq \frac{C}{|t|^{\frac{5}{6}(1-2/p)}} \|f\|_{L^{p'}(\mathbb{R}^2)}, \quad p \geq 2,$$

and so (4.4) is changed accordingly.  $\square$

We now continue with the proof of the main Theorem 1.1.

*Proof.* Proof of Theorem 1.1 First we calculate the necessary condition in a similar fashion to the previous proposition. By the change of variables  $\xi \rightarrow R\xi$ ,  $x \rightarrow R^{-1}x$  and  $t \rightarrow R^{-2}t$ , then letting  $R$  tend to zero and infinity, one can calculate that it is necessary that  $s = d(\frac{1}{2} - \frac{1}{q}) - \frac{2}{r}$  for such an estimate to hold. Now define  $f$  by

$$\widehat{f}(\xi) = \chi_{[0, R^{-1}]}(\xi_1) \chi_{[0, 1]}(\xi_2) \cdots \chi_{[0, 1]}(\xi_d),$$

so that

$$(4.8) \quad \|f\|_{\dot{H}^s} \leq cR^{-1/2}.$$

On the other hand,

$$e^{t\mathcal{R}_1\Delta} f(x) = \frac{1}{(2\pi)^d} \int_0^{R^{-1}} \int_0^1 \cdots \int_0^1 e^{it|\xi|\xi_1} e^{i[x_1\xi_1 + \cdots + x_d\xi_d]} d\xi_1 \cdots d\xi_d.$$

We see that as long as  $|t|, |x_1| \leq \frac{R}{2d}$  and  $|x_2|, \dots, |x_d| \leq \frac{1}{2d}$ , then

$$|e^{t\mathcal{R}_1\Delta} f(x)| \geq \frac{1}{(2\pi)^d} \int_0^{R^{-1}} \int_0^1 \cdots \int_0^1 \cos(1) d\xi_1 \cdots d\xi_d \geq cR^{-1}.$$

Thus,

$$(4.9) \quad \|e^{t\mathcal{R}_1\Delta} f\|_{L_t^r(\mathbb{R}, L_x^q(\mathbb{R}^d))} \geq cR^{1/q+1/r-1}$$

and so by comparing (4.8) and (4.9) and letting  $R$  tend to infinity, we see that the condition  $\frac{2}{q} + \frac{2}{r} \leq 1$  is necessary.

For the positive part, the argument is completed using a variant of the  $TT^*$  argument of Tomas [25]. We consider first the estimates on the sharp line  $\frac{2}{q} + \frac{2}{r} = 1$ . Note that the  $r = \infty$  estimate is a consequence of (4.5), so we can suppose  $r$  is finite. By duality it will suffice to prove

$$\left\| (-\Delta)^{-s/2} \int e^{-t\mathcal{R}_1\Delta} F(\cdot, t) dt \right\|_{L_x^2(\mathbb{R}^d)} \leq C \|F\|_{L_t^{r'} L_x^{q'}},$$

where  $s = d(\frac{1}{2} - \frac{1}{q}) - \frac{2}{r}$ , which is equivalent to

$$\iint \left\langle (-\Delta)^{-s} e^{-t\mathcal{R}_1\Delta} F(\cdot, \tau), e^{-\tau\mathcal{R}_1\Delta} G(\cdot, t) \right\rangle_x dt d\tau \leq C \|F\|_{L_t^{r'} L_x^{q'}} \|G\|_{L_t^{r'} L_x^{q'}}.$$

This can be rewritten as

$$\int \left\langle \int (-\Delta)^{-s} e^{-(t-\tau)\mathcal{R}_1\Delta} F(\cdot, \tau) d\tau, G(\cdot, t) \right\rangle_x dt \leq C \|F\|_{L_t^{r'} L_x^{q'}} \|G\|_{L_t^{r'} L_x^{q'}},$$

which would follow, by Hölder's inequality, from

$$\left\| \int (-\Delta)^{-s} e^{-(t-\tau)\mathcal{R}_1\Delta} F(\cdot, \tau) d\tau \right\|_{L_t^{r'} L_x^{q'}} \leq C \|F\|_{L_t^{r'} L_x^{q'}}$$

This is a consequence of Minkowski's integral inequality, the decay estimate of Proposition 4.1, and the one dimensional Hardy–Littlewood–Sobolev inequality;



see for example [18, Proposition 7.8],

$$\begin{aligned} \left\| \int (-\Delta)^{-s} e^{-(t-\tau)\mathcal{R}_1\Delta} F(\cdot, \tau) d\tau \right\|_{L_t^r L_x^q} &\leq \left\| \int \|(-\Delta)^{-s} e^{-(t-\tau)\mathcal{R}_1\Delta} F(\cdot, \tau)\|_{L_x^q} d\tau \right\|_{L_t^r} \\ &\leq \left\| \int \frac{\|F(\cdot, \tau)\|_{L_x^{q'}}}{|t-\tau|^{1-\frac{2}{q}}} d\tau \right\|_{L_t^r} \\ &\leq C \|F\|_{L_t^{r'} L_x^{q'}}, \end{aligned}$$

noting that on the sharp line  $s = (d-2)(\frac{1}{2} - \frac{1}{q})$ . For the application of the Hardy–Littlewood–Sobolev inequality we require that  $0 < 1 - \frac{2}{q} = \frac{2}{r} < 1$ , which follows from the fact that  $q$  and  $r$  are finite.

Finally, in order to take  $q$  larger than those on the sharp line  $\frac{1}{2} = \frac{1}{q} + \frac{1}{r}$ , we can apply the Hardy–Littlewood–Sobolev inequality in the form; see [18, Corollary 7.9]

$$\|e^{t\mathcal{R}_1\Delta} f\|_{L^{\bar{q}}(\mathbb{R}^d)} \leq C \|(-\Delta)^{s/2} e^{t\mathcal{R}_1\Delta} f\|_{L^q(\mathbb{R}^d)}, \quad s = d\left(\frac{1}{q} - \frac{1}{\bar{q}}\right),$$

and so the proof for  $d \geq 3$  is complete.

When  $d = 2$ , the only thing that changes from above is that we initially consider estimates on the line  $\frac{10}{q} + \frac{12}{r} = 5$ , so that  $s = d(\frac{1}{2} - \frac{1}{q}) - \frac{2}{r}$  becomes

$$s = (d - 5/3)\left(\frac{1}{2} - \frac{1}{q}\right)$$

and the condition  $0 < 1 - \frac{2}{q} = \frac{2}{r} < 1$  becomes  $0 < \frac{5}{6} - \frac{10}{6q} = \frac{2}{r} < 1$ . This completes the proof.  $\square$

## 5. PROOF OF THEOREM 1.4

After using our linear estimates to prove some *a priori* nonlinear estimates, we present the proof of Theorem 1.4 in the final three subsections. In the first of these three parts we prove uniqueness, using the fact that  $u \in L^1([0, T], W^{1,\infty}(\mathbb{R}^d))$ . Then, we prove existence using a compactness argument. Finally we deduce continuous dependence.

**5.1. Linear estimates.** By an application of Sobolev embedding in the spatial variable and an application of Hölder’s inequality in the time variable, our Strichartz estimates of Theorems 1.1 and 1.2 yield the following corollary.

**Corollary 5.1.** *Let  $s > s_d - 3/2$  where  $s_d = d/2 + 1/2$  for  $d \geq 3$  and  $s_2 = 5/3$ . Then*

$$\left( \int_0^1 \|e^{t\mathcal{R}_1\Delta} f\|_{L^\infty}^2 dt \right)^{1/2} \leq C_s \|f\|_{H^s}.$$

This will be a key ingredient in the proof of the following inhomogeneous Strichartz estimate.

**Lemma 5.2.** *Let  $s > s_d - 1$  where  $s_d = d/2 + 1/2$  for  $d \geq 3$  and  $s_2 = 5/3$ . Then*

$$\int_0^T \|w(\cdot, t)\|_{L^\infty} dt \leq C_s T^{1/2} \left( \sup_{t \in [0, T]} \|w(\cdot, t)\|_{H^s} + \int_0^T \|F(\cdot, t)\|_{H^{s-1}} dt \right)$$

whenever  $T \leq 1$  and  $w$  is a solution to  $\partial_t w - \mathcal{R}_1 \Delta w = F$ .

*Proof.* We again consider the smooth partition of unity  $\sum_{k \in \mathbb{Z}} \psi(2^{-k}|\cdot|) = 1$  and the Littlewood–Paley projection operators  $P_k$  defined by

$$P_k f := (\psi(2^{-k}|\cdot|)\widehat{f})^\vee, \quad k \geq 2.$$

This time we consider the lower frequencies together;  $P_1 f = \sum_{k \leq 1} (\psi(2^{-k}|\cdot|)\widehat{f})^\vee$ . By the triangle inequality and summing a geometric series, it will suffice to prove that, for all  $k \geq 1$  and  $s > s_d - 1$ ,

$$\int_0^T \|P_k w(\cdot, t)\|_{L^\infty} dt \leq C_s T^{1/2} \left( \sup_{t \in [0, T]} \|P_k w(\cdot, t)\|_{H^s} + \int_0^T \|P_k F(\cdot, t)\|_{H^{s-1}} dt \right).$$

Following the arguments of [11], we split the interval  $[0, T] = \bigcup_m I_m$  into  $2^k$  intervals  $I_m = [a_m, b_m]$  with  $(b_m - a_m) \sim 2^{-k}T$  and use the Cauchy–Schwarz inequality to estimate

$$(5.1) \quad \int_0^T \|P_k w(\cdot, t)\|_{L^\infty} dt \lesssim (2^{-k}T)^{1/2} \sum_m \left( \int_{I_m} \|P_k w(\cdot, t)\|_{L^\infty}^2 dt \right)^{1/2}.$$

Now to estimate the right-hand side, we employ Duhamel’s principle on each  $I_m$ , so that

$$P_k w(\cdot, t) = e^{(t-a_m)\mathcal{R}_1 \Delta} P_k w(\cdot, a_m) + \int_{a_m}^t e^{(t-t')\mathcal{R}_1 \Delta} P_k F(\cdot, t') dt'$$

whenever  $t \in I_m$ . Then by an application of Minkowski’s integral inequality and applications of Corollary 5.1, we find that (5.1) can be bounded by

$$\begin{aligned} &\lesssim (2^{-k}T)^{1/2} \sum_m \left( \|P_k w(\cdot, a_m)\|_{H^{s-1/2}} + \int_{I_m} \|P_k F(\cdot, t')\|_{H^{s-1/2}} dt' \right) \\ &\lesssim T^{1/2} \left( \sup_{t \in [0, T]} \|P_k w(\cdot, t)\|_{H^s} + \int_0^T \|P_k F(\cdot, t')\|_{H^{s-1}} dt' \right), \end{aligned}$$

which completes the proof.  $\square$

**5.2. Energy estimates.** We will need the Kato–Ponce commutator estimates.

**Lemma 5.3.** [6, 10] *Let  $s > 0$  and write  $[J^s, f]g := J^s(fg) - fJ^s g$ . Then*

$$\|[J^s, f]g\|_{L^2} \lesssim \|\nabla f\|_{L^\infty} \|J^{s-1}g\|_{L^2} + \|J^s f\|_{L^2} \|g\|_{L^\infty}.$$

Moreover

$$\|J^s(fg)\|_{L^2} \lesssim C_s \|J^s f\|_{L^2} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|J^s g\|_{L^2}.$$

Using these we deduce the following *a priori* energy estimate for smooth solutions which we take in  $H^{d+1}(\mathbb{R}^d)$  from now on for definiteness.

**Lemma 5.4.** *Let  $s \in (0, d + 1]$ . Then, there is a constant  $c_s > 0$  such that*

$$\sup_{t \in [0, T]} \|u(t)\|_{H^s}^2 \leq \|u_0\|_{H^s}^2 + c_s \sup_{t \in [0, T]} \|u(t)\|_{H^s}^2 \int_0^T \|\nabla u(t)\|_{L^\infty} dt$$

whenever  $u_0 \in H^{d+1}(\mathbb{R}^d)$  and  $u \in C([0, T]; H^{d+1}(\mathbb{R}^d))$  solves (HBO).

*Proof.* Applying  $J^s$  to the higher dimensional Benjamin–Ono equation, multiplying by  $J^s u$  and integrating in space yields

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} (J^s u)^2 dx = - \int_{\mathbb{R}^d} [J^s, u] \partial_{x_1} u J^s u dx - \int_{\mathbb{R}^d} u J^s \partial_{x_1} u J^s u dx.$$

The Riesz transform term is easily seen to be zero using the fact that it is skew-self-adjoint. The first term on the right-hand side is bounded by the Cauchy–Schwarz inequality and the commutator estimate of Lemma 5.3. Noting that  $J^s \partial_{x_1} u J^s u = \frac{1}{2} \partial_{x_1} (J^s u)^2$ , the second term is controlled by integrating by parts and using Hölder’s inequality. Together we deduce

$$\frac{d}{dt} \|J^s u\|_{L^2}^2 \lesssim \|\nabla u\|_{L^\infty} \|J^s u\|_{L^2}^2.$$

Integrating in time, the desired inequality follows.  $\square$

**5.3. Nonlinear estimates.** As in the previous section we will prove *a priori* estimates for smooth solutions with respect to norms of lower regularity. The following estimate for the Lipschitz norm is a consequence of the inhomogeneous Strichartz estimate of Lemma 5.2.

**Lemma 5.5.** *Let  $s \in (s_d, d + 1]$  where  $s_d = d/2 + 1/2$  for  $d \geq 3$  and  $s_2 = 5/3$ . For  $T \leq 1$ , let*

$$K(T) := \int_0^T \|u(t)\|_{L^\infty} + \|\nabla u(t)\|_{L^\infty} dt.$$

Then there is a constant  $C_s > 0$  such that

$$K(T) \leq C_s T^{1/2} \sup_{t \in [0, T]} \|u(t)\|_{H^s} (1 + K(T)),$$

whenever  $u_0 \in H^{d+1}(\mathbb{R}^d)$  and  $u \in C([0, T]; H^{d+1}(\mathbb{R}^d))$  solves (HBO).

*Proof.* Taking first  $w = u$  and  $F = -u \partial_{x_1} u$ , then  $w = \nabla u$  and  $F = -\nabla(u \partial_{x_1} u)$ , by Lemma 5.2 we have

$$(5.2) \quad K(T) \leq C_s T^{1/2} \left( \sup_{t \in [0, T]} \|u(t)\|_{H^s} + \int_0^T \|(u \partial_{x_1} u)(t')\|_{H^{s-1}} dt' \right)$$

Then using the second estimate of Lemma 5.3,

$$\begin{aligned} \|(u \partial_{x_1} u)(t)\|_{H^{s-1}} &\lesssim \|u(t)\|_{L^\infty} \|J^{s-1} \partial_{x_1} u(t)\|_{L^2} + \|J^{s-1} u(t)\|_{L^2} \|\partial_{x_1} u(t)\|_{L^\infty} \\ &\lesssim (\|u(t)\|_{L^\infty} + \|\nabla u(t)\|_{L^\infty}) \sup_{t \in [0, T]} \|u(t)\|_{H^s}, \end{aligned}$$

which can be plugged into (5.2) to yield the desired inequality.  $\square$

With  $s > d/2 + 1$ , Theorem 1.4 follows by a parabolic regularization of (HBO); an additional term  $-\mu\Delta u$  is added, after which the limit  $\mu \rightarrow 0$  is taken. The argument also yields a blow-up criteria. The result follows directly from the arguments of Iorio [8] and so is omitted.

**Lemma 5.6.** *Let  $s > d/2 + 1$ . Then, for any  $u_0 \in H^s(\mathbb{R}^d)$ , there exists a time  $T = T(\|u_0\|_{H^s})$  and a unique maximal solution  $u$  to (HBO) in  $C([0, T^*]; H^s(\mathbb{R}^d))$ . Moreover, if the maximal time of existence  $T^*$  is finite, then*

$$\lim_{t \rightarrow T^*} \|u(t)\|_{H^s} = \infty,$$

and the flow map  $u_0 \mapsto u(t)$  is continuous from  $H^s(\mathbb{R}^d)$  to  $H^s(\mathbb{R}^d)$ .

Given this lemma, we first prove that the smooth solutions exist long enough for our purposes, taking advantage of the blow-up criteria. We then provide another nonlinear estimate. The proof follows closely the arguments of [14].

**Lemma 5.7.** *Let  $s \in (s_d, d+1]$  where  $s_d = d/2 + 1/2$  for  $d \geq 3$  and  $s_2 = 5/3$ . Then there is a constant  $A_s > 0$  such that, for all  $u_0 \in H^{d+1}(\mathbb{R}^d)$ , there is a solution  $u \in C([0, T^*]; H^{d+1}(\mathbb{R}^d))$  of (HBO) with  $T^* \geq (1 + A_s \|u_0\|_{H^s})^{-2}$ . Moreover, there is a constant  $K_s > 0$  such that*

$$\sup_{t \in [0, T]} \|u(t)\|_{H^s} \leq 2 \|u_0\|_{H^s}, \quad \text{and} \quad K(T) \leq K_s$$

whenever  $T \leq (1 + A_s \|u_0\|_{H^s})^{-2}$ .

*Proof.* Set  $A_s = 8(1 + \max\{c_s, c_{d+1}\})C_s$  where  $c_s$  and  $c_{d+1}$  are the constants appearing in Lemma 5.4 and  $C_s$  is the constant of Lemma 5.5. We consider any time  $T \leq (1 + A_s \|u_0\|_{H^s})^{-2}$  for which

$$(5.3) \quad \sup_{t \in [0, T]} \|u(t)\|_{H^s} \leq 2 \|u_0\|_{H^s}$$

for all  $u_0 \in H^{d+1}$ . Then, by Lemma 5.5,

$$K(T) \leq 2C_s T^{1/2} \|u_0\|_{H^s} (1 + K(T)).$$

Calculating we find

$$K(T) \leq \frac{1}{3 \max\{c_s, c_{d+1}\}}.$$

From the energy estimates of Lemma 5.4 we deduce that both

$$\sup_{t \in [0, T]} \|u(t)\|_{H^s}^2 \leq \frac{3}{2} \|u_0\|_{H^s}^2 \quad \text{and} \quad \sup_{t \in [0, T]} \|u(t)\|_{H^{d+1}}^2 \leq \frac{3}{2} \|u_0\|_{H^{d+1}}^2$$

for all  $u_0 \in H^{d+1}$ . In view of the blow-up criteria of Lemma 5.6, the latter estimate implies that we can take  $T^* > T$ . On the other hand, the former estimate and continuity implies that  $T$  was not the largest time for which (5.3) holds. We conclude that the largest such  $T$  must be as least as large as  $(1 + A_s \|u_0\|_{H^s})^{-2}$  which completes the proof.  $\square$

5.4. **Uniqueness.** Let  $u_1$  and  $u_2$  be two solutions of (HBO) in our class

$$C([0, T]; H^s(\mathbb{R}^d)) \cap L^1([0, T]; W^{1, \infty}(\mathbb{R}^d)),$$

with respective initial data  $u_1(\cdot, 0) = \phi_1$  and  $u_2(\cdot, 0) = \phi_2$ . By setting  $v := u_1 - u_2$ , we find that

$$\partial_t v - \mathcal{R}_1 \Delta v + u_1 \partial_{x_1} u_1 - u_2 \partial_{x_1} u_2 = 0,$$

which can be rewritten as

$$\partial_t v - \mathcal{R}_1 \Delta v + \frac{1}{2} \partial_{x_1} ((u_1 + u_2)v) = 0.$$

Taking the inner product in  $L^2(\mathbb{R}^d)$  with  $v$ , we arrive at

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} v^2 dx = -\frac{1}{4} \int_{\mathbb{R}^d} \partial_{x_1} (u_1 + u_2) v^2 dx.$$

Again by skew-adjointness, the Riesz transform term is zero, and one can arrive to the form of the right-hand side by integrating by parts twice. Thus,

$$\frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 \lesssim (\|\nabla u_1\|_{L^\infty} + \|\nabla u_2\|_{L^\infty}) \|v\|_{L^2}^2.$$

An application of Gronwall's inequality (see for example [24, Theorem 1.12]) gives

$$\sup_{t \in [0, T]} \|u_1(t) - u_2(t)\|_{L^2} \leq \|\phi_1 - \phi_2\|_{L^2} \exp\left(c \int_0^T \|\nabla u_1\|_{L^\infty} + \|\nabla u_2\|_{L^\infty}\right)$$

from which uniqueness follows.

5.5. **Existence.** We mollify the initial datum as in the Bona-Smith argument [2]. Consider a radial and positive  $\rho \in C_0^\infty(\mathbb{R}^d)$  such that  $\rho(\xi) = 1$  for  $|\xi| \leq 1/2$  and  $\rho(\xi) = 0$  for  $|\xi| \geq 1$ , and define

$$u_{0,n} := (\rho(n^{-1} \cdot) \widehat{u}_0)^\vee$$

for any integer  $n \geq 1$ . First we state some properties of the regularized initial data.

**Lemma 5.8.** *Let  $m \geq n \geq 1$  and  $\alpha \leq s$ . Then*

$$\|u_{0,n} - u_{0,m}\|_{H^\alpha} \lesssim n^{-(s-\alpha)} \|u_0\|_{H^s}.$$

Moreover, with  $u_0 \in H^s$ ,

$$\|u_{0,n} - u_{0,m}\|_{H^s} \xrightarrow{n \rightarrow \infty} 0.$$

*Proof.* By support considerations, we observe

$$|\langle \xi \rangle^\alpha (\rho(\xi/m) - \rho(\xi/n)) \widehat{u}_0(\xi)|^2 \lesssim n^{-2(s-\alpha)} |(\rho(\xi/m) - \rho(\xi/n))|^2 |\langle \xi \rangle^s \widehat{u}_0(\xi)|^2$$

and so the first estimate follows by integrating and Plancherel's identity. With  $\alpha = s$ , the result follows by the Lebesgue dominated convergence theorem.  $\square$

Let  $s \in (s_d, d+1]$  where  $s_d = d/2 + 1/2$  for  $d \geq 3$  and  $s_2 = 5/3$  and take  $\alpha \in (s_d, s]$ . For integers  $n \geq 1$ , we consider solutions  $u_n \in C([0, T]; H^{d+1}(\mathbb{R}^d))$  of the higher dimensional Benjamin-Ono equation with mollified initial data  $u_{0,n}$ ;

$$(5.4) \quad \begin{cases} \partial_t u_n - \mathcal{R}_1 \Delta u_n + u_n \partial_{x_1} u_n = 0, & x \in \mathbb{R}^d, t \in \mathbb{R}, \\ u_n(x, 0) = u_{0,n}(x). \end{cases}$$

As  $\|u_{0,n}\|_{H^\alpha} \leq \|u_{0,n}\|_{H^s} \leq \|u_0\|_{H^s}$ , from Lemma 5.7, we find that

$$(5.5) \quad \sup_{t \in [0, T]} \|u_n(t)\|_{H^\alpha} \leq 2 \|u_0\|_{H^\alpha}, \quad \alpha \in (s_d, s]$$

whenever  $T \leq (1 + A_s \|u_0\|_{H^s})^{-2}$ , and

$$(5.6) \quad K := \sup_{n \geq 1} \int_0^T \|u_n(t)\|_{L^\infty} + \|\nabla u_n(t)\|_{L^\infty} dt < \infty.$$

Now we set  $v_{n,m} := u_n - u_m$ , so that  $v_{n,m}$  satisfies

$$(5.7) \quad \partial_t v_{n,m} - \mathcal{R}_1 \Delta v_{n,m} + u_n \partial_{x_1} u_n - u_m \partial_{x_1} u_m = 0,$$

with initial datum  $v_{n,m}(\cdot, 0) = u_{0,n} - u_{0,m}$ . Arguing as in the uniqueness Subsection 5.4 and using Lemma 5.8, we deduce

$$\sup_{t \in [0, T]} \|v_{n,m}(t)\|_{L^2} \leq e^{cK} \|u_{0,n} - u_{0,m}\|_{L^2} \lesssim n^{-s} \|u_0\|_{H^s}.$$

Interpolating with (5.5) we immediately get that

$$(5.8) \quad \sup_{t \in [0, T]} \|v_{n,m}(t)\|_{H^\alpha} \lesssim n^{-(s-\alpha)} \|u_0\|_{H^s}$$

so that  $\{v_{n,m}\}$  is a Cauchy sequence in  $C([0, T]; H^\alpha(\mathbb{R}^d))$  for all  $0 \leq \alpha < s$ .

Below we will find that  $\{v_{n,m}\}$  is also Cauchy in  $C([0, T]; H^s(\mathbb{R}^d))$ , but first we show that the sequence is Cauchy in  $L^1([0, T]; W^{1,\infty}(\mathbb{R}^d))$ . In fact we prove something stronger that will help later.

**Lemma 5.9.** *Let  $m \geq n \geq 1$ . Then*

$$\int_0^T n \|v_{n,m}(t)\|_{L^\infty} + \|\nabla v_{n,m}(t)\|_{L^\infty} dt \xrightarrow{n \rightarrow \infty} 0.$$

*Proof.* Let  $\alpha \in (s_d, s)$  and rewrite the nonlinear term in (5.7) as

$$u_n \partial_{x_1} u_n - u_m \partial_{x_1} u_m = \frac{1}{2} \partial_{x_1} ((u_n + u_m) v_{n,m}).$$

Taking  $w = \nabla v_{n,m}$  and  $F = -\frac{1}{2} \nabla \partial_{x_1} ((u_n + u_m) v_{n,m})$ , by Lemma 5.2,

$$\begin{aligned} & \int_0^T \|\nabla v_{n,m}(t)\|_{L^\infty} dt \\ & \lesssim T^{1/2} \left( \sup_{t \in [0, T]} \|v_{n,m}(\cdot, t)\|_{H^\alpha} + \int_0^T \|\partial_{x_1} ((u_n + u_m) v_{n,m})(\cdot, t)\|_{H^{\alpha-1}} dt \right) \\ & \lesssim T^{1/2} \left( \sup_{t \in [0, T]} \|v_{n,m}(\cdot, t)\|_{H^\alpha} + \int_0^T \|u_0\|_{H^\alpha} \|v_{n,m}(\cdot, t)\|_{H^\alpha} dt \right), \end{aligned}$$

where the second inequality follows from Lemma 5.3 and Sobolev embedding. The desired convergence for this part then follows by applying (5.8).

On the other hand, taking  $w = v_{n,m}$  and  $F = -\frac{1}{2}\partial_{x_1}((u_n + u_m)v_{n,m})$  in Lemma 5.2, we also have

$$\begin{aligned} & \int_0^T \|v_{n,m}(t)\|_{L^\infty} dt \\ & \lesssim T^{1/2} \left( \sup_{t \in [0, T]} \|v_{n,m}(\cdot, t)\|_{H^{\alpha-1}} + \int_0^T \|u_0\|_{H^{\alpha-1}} \|v_{n,m}(\cdot, t)\|_{H^{\alpha-1}} dt \right), \end{aligned}$$

and so we obtain an extra factor of  $n^{-1}$  when applying (5.8).  $\square$

**Proposition 5.10.** *Let  $m \geq n \geq 1$ . Then*

$$\sup_{t \in [0, T]} \|v_{n,m}(t)\|_{H^s} \xrightarrow{n \rightarrow \infty} 0.$$

*Proof.* Applying the operator  $J^s$  to (5.7), rewriting the nonlinearity as

$$u_n \partial_{x_1} u_n - u_m \partial_{x_1} u_m = v_{n,m} \partial_{x_1} u_n + u_m \partial_{x_1} v_{n,m},$$

and then multiplying the equation by  $J^s v_{n,m}$  and integrating in space, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|J^s v_{n,m}\|_{L^2}^2 &= - \int_{\mathbb{R}^d} J^s (u_m \partial_{x_1} v_{n,m}) J^s v_{n,m} - \int_{\mathbb{R}^d} J^s (v_{n,m} \partial_{x_1} u_n) J^s v_{n,m} \\ &=: -(A_1 + A_2). \end{aligned}$$

Now by integrating by parts, we can write

$$A_1 = \int_{\mathbb{R}^d} [J^s, u_m] \partial_{x_1} v_{n,m} J^s v_{n,m} - \frac{1}{2} \int_{\mathbb{R}^d} \partial_{x_1} u_m (J^s v_{n,m})^2,$$

from which it follows from the Cauchy–Schwarz inequality and the commutator estimate of Lemma 5.3 that

$$|A_1| \lesssim \|\nabla u_m\|_{L^\infty} \|J^s v_{n,m}\|_{L^2}^2.$$

On the other hand, we can write

$$A_2 = \int_{\mathbb{R}^d} [J^s, v_{n,m}] \partial_{x_1} u_n J^s v_{n,m} + \int_{\mathbb{R}^d} v_{n,m} (J^s \partial_{x_1} u_n) J^s v_{n,m}.$$

Again by the Cauchy–Schwarz inequality and the commutator estimate of Lemma 5.3,

$$\begin{aligned} |A_2| &\lesssim \|\nabla u_n\|_{L^\infty} \|J^s v_{n,m}\|_{L^2}^2 + \|\nabla v_{n,m}\|_{L^\infty} \|J^s u_n\|_{L^2} \|J^s v_{n,m}\|_{L^2} \\ &\quad + \|v_{n,m}\|_{L^\infty} \|J^{s+1} u_n\|_{L^2} \|J^s v_{n,m}\|_{L^2}. \end{aligned}$$

Now arguing as in the proof of Lemma 5.4, we have

$$\frac{d}{dt} \|J^{s+1} u_n\|_{L^2}^2 \lesssim \|\nabla u_n\|_{L^\infty} \|J^{s+1} u_n\|_{L^2}^2,$$

so, by Gronwall’s inequality,

$$\|J^{s+1} u_n\|_{L^2} \leq e^{cK} \|J^{s+1} u_{0,n}\|_{L^2} \lesssim n \|u_0\|_{H^s}.$$

where  $K$  is defined as in (5.6). In view of (5.5), we can bound both  $\|J^s u_n\|_{L^2}$  and  $\|J^s v_{n,m}\|_{L^2}$  by a constant multiple of  $\|u_0\|_{H^s}$ , so that

$$|A_2| \lesssim \|\nabla u_n\|_{L^\infty} \|J^s v_{n,m}\|_{L^2}^2 + (n \|v_{n,m}\|_{L^\infty} + \|\nabla v_{n,m}\|_{L^\infty}) \|u_0\|_{H^s}^2.$$

Summing up our estimates for  $A_1$  and  $A_2$ , we find that

$$(5.9) \quad \frac{d}{dt} \|J^s v_{n,m}(t)\|_{L^2}^2 \leq a(t) \|J^s v_{n,m}(t)\|_{L^2}^2 + b(t)$$

where

$$\begin{aligned} a(t) &= C_0 \left( \|\nabla u_n(t)\|_{L^\infty} + \|\nabla u_m(t)\|_{L^\infty} \right), \\ b(t) &= C_1 \left( n \|v_{m,m}(t)\|_{L^\infty} + \|\nabla v_{n,m}(t)\|_{L^\infty} \right) \|u_0\|_{H^s}^2. \end{aligned}$$

Now if  $g(t)$  solves

$$\begin{cases} \frac{d}{dt} g(t) = a(t)g(t) + b(t), \\ g(0) = \|u_{0,n} - u_{0,m}\|_{H^s}^2, \end{cases}$$

then

$$\frac{d}{dt} \left( \|J^s v_{n,m}(t)\|_{L^2}^2 - g(t) \right) \leq a(t) \left( \|J^s v_{n,m}(t)\|_{L^2}^2 - g(t) \right)$$

with initial condition,  $\|J^s v_{n,m}(0)\|_{L^2}^2 - g(0) = 0$ . Then by an application of Gronwall's inequality, we find that  $\|J^s v_{n,m}(t)\|_{L^2}^2 \leq g(t)$  for all  $t \geq 0$ . Now as  $g(t)$  has the explicit form

$$g(t) = g(0)e^{\int_0^t a(t') dt'} + \int_0^t b(\tau)e^{\int_\tau^t a(t') dt'} d\tau,$$

we find that  $\sup_{t \in [0, T]} \|J^s v_{n,m}(t)\|_{L^2}^2$  is bounded by

$$e^{cK} \left( \|u_{0,n} - u_{0,m}\|_{H^s}^2 + \|u_0\|_{H^s}^2 \int_0^T n \|v_{n,m}(t)\|_{L^\infty} + \|\nabla v_{n,m}(t)\|_{L^\infty} dt \right) \underset{n \rightarrow \infty}{=} 0,$$

where the convergence follows from Lemmas 5.8 and 5.9.  $\square$

We deduce from Proposition 5.10 and Lemma 5.9 that  $u_n$  has a limit  $u$  in

$$C([0, T]; H^s(\mathbb{R}^d)) \cap L^1([0, T]; W^{1, \infty}(\mathbb{R}^d)).$$

Now recalling that

$$(5.10) \quad u_n(t) = e^{t\mathcal{R}_1\Delta} u_{0,n} - \frac{1}{2} \int_0^t e^{(t-t')\mathcal{R}_1\Delta} \partial_{x_1} [u_n(t')]^2 dt',$$

and noting that

$$\begin{aligned} & \left\| \int_0^t e^{(t-t')\mathcal{R}_1\Delta} \partial_{x_1} [u_n(t')^2 - u(t')^2] dt' \right\|_{H^{s-1}} \\ & \leq \int_0^t \|u_n(t')^2 - u(t')^2\|_{H^s} dt' \\ & \lesssim \int_0^t \|u_n(t') + u(t')\|_{H^s} \|u_n(t') - u(t')\|_{H^s} dt', \end{aligned}$$

we see that  $u$  also solves the integral formulation of (HBO) in the  $C([0, T]; H^{s-1}(\mathbb{R}^d))$  sense.



**5.6. Continuity of the flow map.** Let  $s > s_d$  where  $s_d = d/2 + 1/2$  for  $d \geq 3$  and  $s_2 = 5/3$ . Fix  $u_0 \in H^s$  and  $t < T = T(\|u_0\|_{H^s})$ . We are required to prove that for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all data  $v_0$  such that  $\|u_0 - v_0\|_{H^s} < \delta$ , we have

$$(5.11) \quad \|u(t) - v(t)\|_{H^s} < \epsilon.$$

For  $n \geq 1$ , we regularize the initial data as in the previous section. Then from the triangle inequality we obtain

$$\|u(t) - v(t)\|_{H^s} \leq \|u(t) - u_n(t)\|_{H^s} + \|u_n(t) - v_n(t)\|_{H^s} + \|v_n(t) - v(t)\|_{H^s}.$$

By the definitions of  $u$  and  $v$ , we can take  $n$  sufficiently large so that

$$(5.12) \quad \|u(t) - u_n(t)\|_{H^s} + \|v_n(t) - v(t)\|_{H^s} < \epsilon/2.$$

On the other hand,

$$\|u_{0,n} - v_{0,n}\|_{H^{d+1}} \lesssim n^{d+1-s} \|u_0 - v_0\|_{H^s} \leq n^{d+1-s} \delta.$$

Then using the continuity of the flow map for smooth solutions, we can choose  $\delta > 0$  small enough to ensure

$$(5.13) \quad \|u_n(t) - v_n(t)\|_{H^s} \leq \|u_n(t) - v_n(t)\|_{H^{d+1}} \leq \epsilon/2.$$

Therefore estimate (5.11) follows by combining (5.12) and (5.13).

## 6. APPENDIX: ILL-POSEDNESS RESULTS

Here we prove that (HBO) cannot be solved in  $H^s(\mathbb{R}^d)$  by a Picard iterative scheme based on the Duhamel formula. This result can be viewed as an extension of [17], where the  $C^2$  ill-posedness in  $H^s(\mathbb{R})$  is established for the Benjamin–Ono equation.

*Proof of Theorem 1.5.* Suppose that there exists  $T > 0$  such that (HBO) is locally well-posed in  $H^s(\mathbb{R}^d)$  on the time interval  $[0, T)$  and such that the flow map

$$\Phi(t) : H^s(\mathbb{R}^d) \rightarrow H^s(\mathbb{R}^d), \quad u_0 \mapsto u(t)$$

is  $C^2$  differentiable at the origin. When  $\phi \in H^s(\mathbb{R}^d)$ , we have that  $\Phi(\cdot)\phi$  is a solution of (HBO) with initial data  $\phi$  so by Duhamel's principle  $\Phi(t)\phi$  must satisfy the integral equation

$$\Phi(t)\phi = e^{t\mathcal{R}_1\Delta}\phi - \frac{1}{2} \int_0^t e^{(t-t')\mathcal{R}_1\Delta} \partial_{x_1} [\Phi(t')\phi]^2 dt'.$$

We compute the Fréchet derivative of  $\Phi(t)$  at  $\psi$  with direction  $\phi_1$ ,

$$(6.1) \quad d_\psi \Phi(t)(\phi_1) = e^{t\mathcal{R}_1\Delta}\phi_1 - \int_0^t e^{(t-t')\mathcal{R}_1\Delta} \partial_{x_1} [\Phi(t')\psi d_\psi \Phi(t')(\phi_1)] dt'.$$

Supposing that (HBO) is well-posed, uniqueness implies that  $\Phi(t)(0) = 0$ , so that  $d_0\Phi(t)(\phi_1) = e^{t\mathcal{R}_1\Delta}\phi_1$ . Differentiating again we find that

$$\begin{aligned} d_0^2\Phi(t)(\phi_1, \phi_2) &= \frac{\partial}{\partial\gamma} \left( \gamma \mapsto d_{\gamma\phi_2}\Phi(t)(\phi_1) \right) \Big|_{\gamma=0} \\ &= - \int_0^t e^{(t-t')\mathcal{R}_1\Delta} \partial_{x_1} [d_{\gamma\phi_2}\Phi(t)(\phi_2) d_{\gamma\phi_2}\Phi(t)(\phi_1)] dt' \Big|_{\gamma=0} \\ &\quad - \int_0^t e^{(t-t')\mathcal{R}_1\Delta} \partial_{x_1} [\Phi(t)(\gamma\phi_2) d_{\gamma\phi_1}^2\Phi(t)(\phi_1, \phi_2)] dt' \Big|_{\gamma=0}, \end{aligned}$$

so that

$$d_0^2\Phi(t)(\phi_1, \phi_2) = - \int_0^t e^{(t-t')\mathcal{R}_1\Delta} \partial_{x_1} [(e^{t'\mathcal{R}_1\Delta}\phi_1)(e^{t'\mathcal{R}_1\Delta}\phi_2)] dt'.$$

Now, if the flow map were  $C^2$  then  $d_0^2\Phi(t)$  would be bounded from  $H^s \times H^s$  to  $H^s$ ;

$$\left\| \int_0^t e^{(t-t')\mathcal{R}_1\Delta} \partial_{x_1} [(e^{t'\mathcal{R}_1\Delta}\phi_1)(e^{t'\mathcal{R}_1\Delta}\phi_2)] dt' \right\|_{H^s} \lesssim \|\phi_1\|_{H^s} \|\phi_2\|_{H^s}.$$

We will prove that this does not hold in general, following the arguments in [17].

Indeed, we will construct two sequences of functions,  $\phi_{1,N}$  and  $\phi_{2,N}$ , such that

$$(6.2) \quad \|\phi_{1,N}\|_{H^s}, \|\phi_{2,N}\|_{H^s} \leq C$$

and

$$(6.3) \quad \lim_{N \rightarrow \infty} \left\| \int_0^t e^{(t-t')\mathcal{R}_1\Delta} \partial_{x_1} [(e^{t'\mathcal{R}_1\Delta}\phi_{1,N})(e^{t'\mathcal{R}_1\Delta}\phi_{2,N})] dt' \right\|_{H^s} = \infty.$$

We define  $\phi_{1,N}$  and  $\phi_{2,N}$  via their Fourier transforms as

$$\begin{cases} \widehat{\phi_{1,N}}(\xi) = \lambda^{\frac{1-2d}{2d}} N^{-s} \chi_{D_1}(\xi), & \text{with } D_1 = [N, N + \lambda] \times [\lambda^{1/d}/2, \lambda^{1/d}]^{d-1}, \\ \widehat{\phi_{2,N}}(\xi) = \lambda^{\frac{1-2d}{2d}} \chi_{D_2}(\xi), & \text{with } D_2 = [3\lambda, 4\lambda] \times [\lambda^{1/d}/2, \lambda^{1/d}]^{d-1} \end{cases}$$

where  $N \gg 1$ ,  $\lambda = N^{-(1+\epsilon)}$  and  $0 < \epsilon < 1/(2d-1)$ . First, we observe that  $\phi_{1,N}$  and  $\phi_{2,N}$  satisfy (6.2).

On the other hand, taking the Fourier transform with respect to the space variable,

$$\begin{aligned} (6.4) \quad \widehat{I}_N(\xi, t) &:= \left\{ \int_0^t e^{(t-t')\mathcal{R}_1\Delta} \partial_{x_1} [(e^{t'\mathcal{R}_1\Delta}\phi_{1,N})(e^{t'\mathcal{R}_1\Delta}\phi_{2,N})] dt' \right\}^\wedge(\xi) \\ &= \int_{K_\xi} \xi_1 e^{it\xi_1|\xi|} \frac{e^{i\sigma(\xi,\eta)t} - 1}{\sigma(\xi,\eta)} \widehat{\phi_{1,N}}(\eta) \widehat{\phi_{2,N}}(\xi - \eta) d\eta \end{aligned}$$

where the resonant function is given by

$$\sigma(\xi, \eta) = -\xi_1|\xi| + (\xi_1 - \eta_1)|\xi - \eta| + \eta_1|\eta|$$

and

$$K_\xi = \{\eta \in \mathbb{R}^d : \eta \in D_1, \xi - \eta \in D_2\}.$$

When  $\eta \in D_1$  and  $\xi - \eta \in D_2$ , we claim that

$$(6.5) \quad |\sigma(\xi, \eta)| \sim \lambda N.$$

Indeed, using that  $\widehat{I}_N(\xi)$  is supported on

$$D_3 = [N + 3\lambda, N + 5\lambda] \times [\lambda^{1/d}, 2\lambda^{1/d}]^{d-1}$$

we easily obtain

$$(6.6) \quad (\xi_1 - \eta_1)|\xi - \eta| \sim \lambda^{(d+1)/d}.$$

Moreover, from the inequality

$$|\xi| \leq \left( (N + 5\lambda)^2 + 4(d-1)\lambda^{2/d} \right)^{1/2} \leq N + 6\lambda$$

which holds for  $N$  large,  $\lambda = N^{-(1+\epsilon)}$  with  $0 < \epsilon < 1/(2d-1)$ , we have

$$(6.7) \quad (N + 3\lambda)^2 \leq \xi_1 |\xi| \leq (N + 6\lambda)^2.$$

Analogously, we get

$$(6.8) \quad N^2 \leq \eta_1 |\eta| \leq (N + 2\lambda)^2.$$

Then, (6.5) follows from (6.6), (6.7) and (6.8).

Now, since  $\lambda N = N^{-\epsilon}$  and  $|\sigma(\xi, \eta)| \sim \lambda N$  it follows

$$(6.9) \quad \left| \frac{e^{i\sigma(\xi, \eta)t} - 1}{\sigma(\xi, \eta)} \right| = |t| + O\left(\frac{1}{N^\epsilon}\right).$$

From (6.9) and  $|K_\xi| \sim \lambda^{(2d-1)/d}$ , we infer that

$$|\widehat{I}_N(\xi, t)| \chi_{D_3}(\xi) \gtrsim \frac{N\lambda^{(2d-1)/d}}{N^s \lambda^{(2d-1)/d}} |t| \chi_{D_3}(\xi).$$

Therefore we arrive at

$$\|I_N(t)\|_{H^s} \gtrsim N\lambda^{(2d-1)/2d} |t| = N^{1/2d - \epsilon((2d-1)/2d)} |t|.$$

Now as  $0 < \epsilon < 1/(2d-1)$ , from this we deduce (6.3), which completes the proof.  $\square$

The following corollary (of the proof) shows that it is not possible to solve (HBO) in  $H^s(\mathbb{R}^d)$  via the usual contraction argument.

**Corollary 6.1.** *Let  $s \in \mathbb{R}$  and  $T > 0$ . Then there does not exist a space  $X_T$  continuously embedded in  $C([0, T]; H^s(\mathbb{R}^d))$  such that*

$$(6.10) \quad \|e^{t\mathcal{R}_1 \Delta} \phi\|_{X_T} \leq C \|\phi\|_{H^s}$$

and

$$(6.11) \quad \left\| \int_0^t e^{(t-t')\mathcal{R}_1 \Delta} [F(\cdot, t') \partial_{x_1} F(\cdot, t')] dt' \right\|_{X_T} \leq C \|F(\cdot, t)\|_{X_T}^2.$$

*Proof.* We write  $F = F_1 + F_2$  and note that

$$\begin{aligned} & \left\| \int_0^t e^{(t-t')\mathcal{R}_1 \Delta} [F \partial_{x_1} F(\cdot, t')] dt' \right\|_{X_T} \geq \left\| \int_0^t e^{(t-t')\mathcal{R}_1 \Delta} \partial_{x_1} [F_1 F_2(\cdot, t')] dt' \right\|_{X_T} \\ & - \left\| \int_0^t e^{(t-t')\mathcal{R}_1 \Delta} [F_1 \partial_{x_1} F_1(\cdot, t')] dt' \right\|_{X_T} - \left\| \int_0^t e^{(t-t')\mathcal{R}_1 \Delta} [F_2 \partial_{x_1} F_2(\cdot, t')] dt' \right\|_{X_T}. \end{aligned}$$

Now taking  $F_1(\cdot, t') = e^{t'\mathcal{R}_1\Delta}\phi_{1,N}$  and  $F_2(\cdot, t') = e^{t'\mathcal{R}_1\Delta}\phi_{1,N}$ , by (6.10) and (6.2), we have

$$\|F\|_{X_T}, \quad \|F_1\|_{X_T}, \quad \|F_2\|_{X_T} \leq C.$$

Thus, if (6.11) held, we would find that

$$\left\| \int_0^t e^{(t-t')\mathcal{R}_1\Delta} \partial_{x_1} [(e^{t'\mathcal{R}_1\Delta}\phi_{1,N})(e^{t'\mathcal{R}_1\Delta}\phi_{2,N})] dt' \right\|_{X_T}$$

is uniformly bounded in  $N$ , contradicting (6.3).  $\square$

Next we prove that the flow map could not be uniformly continuous in  $L^2(\mathbb{R}^2)$ . We recall that Mariş [15] proved that there exists solitary wave solutions of the form  $u_c(x_1, x_2, t) = \varphi(x_1 - ct, x_2)$  with  $c > 0$ . That is to say,  $\varphi_c$  is a solution of the time independent equation

$$(6.12) \quad -c\partial_{x_1}\varphi - \mathcal{R}_1\Delta\varphi + \varphi\partial_{x_1}\varphi = 0$$

where  $\varphi_c \in H^s(\mathbb{R}^2)$  for all  $s \geq 0$ .

*Proof of Proposition 1.6.* Let  $\varphi_c(x_1, x_2) = c\varphi_1(cx_1, cx_2)$  where  $\varphi_1$  solves (6.12) with  $c = 1$ . Then  $\varphi_c$  solves (6.12) with  $c > 0$  and we consider solutions

$$u_c(x_1, x_2, t) = c\varphi_1(cx_1 - c^2t, cx_2)$$

to (HBO). In particular we will consider solutions  $u_{c_1}$  and  $u_{c_2}$  with  $c_1 \neq c_2$ .

By a change of variables it is easy to see that, for all  $t > 0$ ,

$$\|u_{c_1}(\cdot, t)\|_{L^2} = \|\varphi_1\|_{L^2} = \|u_{c_2}(\cdot, t)\|_{L^2},$$

so that

$$(6.13) \quad \|u_{c_1}(\cdot, t) - u_{c_2}(\cdot, t)\|_{L^2}^2 = 2\|\varphi_1\|_{L^2}^2 - 2\langle u_{c_1}(\cdot, t), u_{c_2}(\cdot, t) \rangle_{L^2}.$$

Changing variables by  $c_2x_1 - c_2^2t \rightarrow x_1$  and  $c_2x_2 \rightarrow x_2$ , we see that

$$\langle u_{c_1}(\cdot, t), u_{c_2}(\cdot, t) \rangle_{L^2} = \frac{c_1}{c_2} \int \varphi_1\left(\frac{c_1}{c_2}(x_1 - c_2(c_1 - c_2)t), \frac{c_1}{c_2}x_2\right) \overline{\varphi_1(x)} dx.$$

Therefore, taking  $c_1 = n + 1$ ,  $c_2 = n$ , from the Lebesgue dominated convergence theorem, it follows that, for all  $t > 0$ ,

$$\langle u_{c_1}(\cdot, t), u_{c_2}(\cdot, t) \rangle_{L^2} = \frac{c_1}{c_2} \int \varphi_1\left(\frac{c_1}{c_2}(x_1 - nt, x_2)\right) \overline{\varphi_1(x)} dx \rightarrow 0 \text{ as } n \rightarrow \infty,$$

while

$$\langle u_{c_1}(\cdot, 0), u_{c_2}(\cdot, 0) \rangle_{L^2} \rightarrow \|\varphi_1\|_{L^2}^2 \text{ as } n \rightarrow \infty.$$

Thus, in view of (6.13), we deduce

$$\|u_{c_1}(\cdot, 0) - u_{c_2}(\cdot, 0)\|_{L^2} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

while on the other hand, for all  $t > 0$ ,

$$\|u_{c_1}(\cdot, t) - u_{c_2}(\cdot, t)\|_{L^2} \rightarrow 2^{1/2} \|\varphi_1\|_{L^2} \text{ as } n \rightarrow \infty,$$

completing the proof.  $\square$

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