Mobile vs. point guards

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Abstract

We study the problem of guarding orthogonal art galleries with horizontal mobile guards (alternatively, vertical) and point guards, using "rectangular vision". We prove a sharp bound on the minimum number of point guards required to cover the gallery in terms of the minimum number of vertical mobile guards and the minimum number of horizontal mobile guards required to cover the gallery. Furthermore, we show that the latter two numbers can be computed in linear time.

1 Introduction

The number of mobile and point guards required to control the interior of a general or an orthogonal polygon (without holes) has been well-studied as a function of the number of vertices of the polygon (in the introduction we assume the reader is familiar with the concept of mobile guards, point guards, etc., but all of these notion are defined precisely in Section 2). Kahn, Klawe, and Kleitman in 1980 [13], and a few years later Győri [10], and O'Rourke [18] proved that $\lfloor n/4 \rfloor$ point guards are sufficient and sometimes necessary to cover the interior of an orthogonal polygon of n vertices. Aggarwal proved in his thesis [1] that any n-vertex orthogonal polygon can be covered by at most $\lfloor \frac{3n+4}{16} \rfloor$ mobile guards, and a strengthening of this result has been shown in [12]. These estimates are also shown to be sharp as extremal results. These theorems imply that — from an extremal point of view — only 4/3 times as many point guards as mobile guards are needed. However, the ratio of these optima has not been studied.

The main goal of this paper is to explore the ratio between the numbers of mobile guards and points guards required to control an orthogonal polygon without holes. At first, this appears to be hopeless, as Figure 1 shows a comb, which can be guarded by one mobile guard (whose patrol is shown by a dotted horizontal line). However, to cover the comb using point guards, one has to be placed for each tooth, so ten point guards

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are needed (marked by solid disks). Combs with arbitrarily high number of teeth clearly demonstrate that the minimum number of points guards required to control an orthogonal polygon cannot be bounded by the minimum size of a mobile guard system covering the comb.

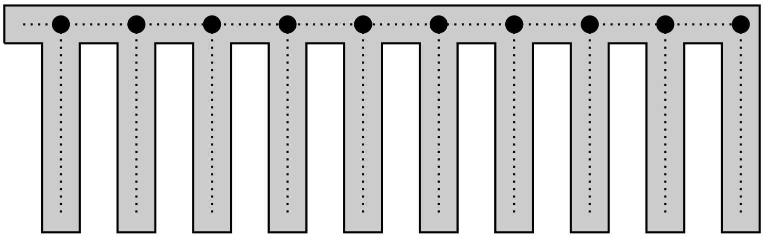


Figure 1: A comb with 10 teeth

In this paper, we study point and mobile guards that are equipped with rectangular vision, or r-vision for short: two points are visible to each other if their axis-parallel bounding rectangle is contained in the gallery. The results of [10, 18, 12] show that the worst case bounds on the number of point- and mobile guard required to control an n-vertex orthogonal polygon do not increase if line of sight vision is restricted to r-vision.

Even though the point guard problem in orthogonal polygons is NP-hard for line of sight vision [19], the problem becomes polynomial for r-vision [20]. The $\tilde{\mathcal{O}}(n^{17})$ time complexity is brought down by Biedl and Mehrabi [3] to a linear running time for thin orthogonal polygons. (An orthogonal polygon is thin if for any point x in the gallery there exists a vertex v on the orthogonal polygon to which everything seen by x via r-vision is r-visible.) Furthermore, a linear time 3-approximation algorithm for the point guard problem with r-vision in orthogonal polygons has been developed by Lingas, Wasylewicz, and Żyliński [16].

Katz and Morgenstern [14]) defined and studied the notion of "horizontal sliding cameras", which is a horizontal line segment $h \subset D$ inside the gallery, which sees a point $x \in D$ in the gallery if there is a point $y \in h$ on the line segment such that $\overline{xy} \perp h$. For a maximal horizontal line segment, the area covered by h as a horizontal mobile r-guard (guard with rectangular vision) and as a horizontal sliding camera are identical up to a 0-measure subset (see Lemma 1).

The main result of our paper, Theorem 2, shows that a constant factor times the sum of the minimum sizes of a horizontal and a vertical mobile r-guard system can be used to estimate the minimum size of a point r-guard system. It is surprising to have such a result given that this ratio cannot be bounded if the region may contain holes.

Take, for example, Figure 2, which generally contains $3k^2 + 4k + 1$ square holes (in the figure $k = 4$). The regions covered by line of sight vision by the black dots are pairwise disjoint, because the distance between adjacent square holes is less than half of the length of a square hole's side. Therefore no two of the black dots can be covered by one point guard, so at least k^2 point guards are necessary to control gallery. However, $2k + 2$ horizontal mobile guards can easily cover the polygon, and the same holds for vertical mobile guards.

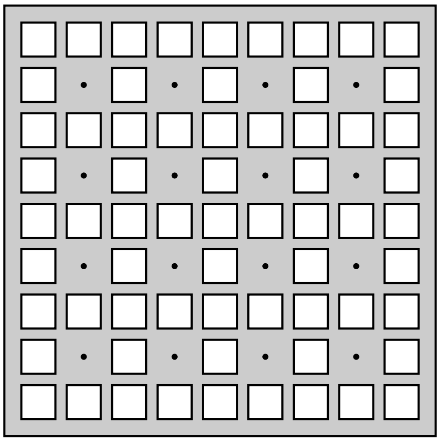


Figure 2: A polygon with holes — unlimited ratio.

In the last section of the paper, we show that a minimum size horizontal mobile r-guard system can be found in linear time (Theorem 22). This improves the result in [14], where it is shown that this problem can be solved in polynomial time.

2 Definitions and preliminaries

Our universe for the study of art galleries is the plane $\mathbb{R}^2$. A **polygon** is defined by a cyclically ordered list of pairwise distinct vertices in the plane. It is drawn by joining each successive pair of vertices on the list by line segments, that only intersect in vertices of the polygon. The last requirement ensures that the closed domain bounded by the polygon is simply connected (to emphasize this, such polygons are often referred to as simple polygons in the literature). An **orthogonal polygon** is a polygon such that its line segments are alternatingly parallel to one of the axes of $\mathbb{R}^2$. Consequently, it is simply connected, and its angles are $\frac{1}{2}\pi$ (convex) or $\frac{3}{2}\pi$ (reflex).

A **rectilinear domain** is a closed region of the plane ($\mathbb{R}^2$) whose boundary is an orthogonal polygon, i.e., a closed polygon without self-intersection, so that each segment is parallel to one of the two axes. A **rectilinear domain with holes** is a rectilinear domain with pairwise disjoint simple rectilinear domain holes. Its boundary is referred to as an **orthogonal polygon with holes**.

The definitions imply that number of vertices of an orthogonal polygon (even with holes) is even. We denote the number of vertices of the polygon by $n(P)$, and define $n(D) = n(P)$, where D is the domain bounded by P. Conversely, we write $P = \partial D$. We want to emphasize that in our problems not just the walls, but also the interior of the gallery must be covered. In the proofs of the theorems, therefore, we are working on rectilinear domains, not orthogonal polygons, even though one defines the other uniquely, and vice versa.

Whenever results about objects that are allowed to have holes are mentioned, it is explicitly stated.

Name	Notation	Meaning
Orthogonal polygon	P	A simple polygon made up of horiz. and vert. segments
Rectilinear domain	D	A bounded region of $\mathbb{R}^2$ s.t. ∂D is an orthogonal polygon
Side		A maximal horizontal or vertical segment of P or ∂D
Vertex		A non-empty intersection of two distinct sides
Convex hull	$\mathrm{Conv}(X)$	The smallest convex set containing $X \subset \mathbb{R}^2$
Pixel	$\cap e$	The intersection of the elements of e
Centroid	$c(X)$	The arithmetic mean position of $X \subset \mathbb{R}^2$

Table 1: Notation used in the paper

To avoid confusion, we state that throughout this part, **vertices** and **sides** refer to subsets of an orthogonal polygon or a rectilinear domain; whereas any **graph** will be defined on a set of **nodes**, of which some pairs are joined by some **edges**. Given a graph G, the edge set $E(G)$ is a subset of the 2-element subsets of the vertices $V(G)$.

Unless otherwise noted, we adhere to the same terminology in the subject of art galleries as O'Rourke [18]. However, for technical reasons, sometimes we need to assume extra conditions over what is traditionally assumed. In Lemma 1, we prove that we may, without restricting the problem, require the assumptions typeset in *italics* in the following definitions.

Two points x, y in a domain D have **line of sight vision**, **unrestricted vision**, or simply just **vision** of each other if the line segment spanned by x and y is contained in D.

A **point guard** in an art gallery D is a point $y \in D$. It has vision of a point $x \in D$ if the line segment $\overline{xy}$ is a subset of D. The term "stationary guard" refers to the same meaning, and is used mostly in contrast with "mobile guards".

A **mobile guard** is a line segment $L \subset D$. A point $x \in D$ is seen by the guard if there is a point $y \in L$ which has vision of x. Intuitively, a mobile guard is a point guard patrolling the line segment L.

The points **covered by a guard** is just another name for the set of points of D that are seen by the guard. A **system of guards** is a set of guards in D which cover D, i.e., for any point $x \in D$, there is a guard in the system covering x.

Two points x, y in a rectilinear domain D have ***r*-vision** of each other (alternatively, x is r-visible from y) if there exists an axis-aligned *non-degenerate* rectangle in D which contains both x and y. This vision is natural to use in orthogonal art galleries instead of the more powerful line of sight vision. For example, r-vision is invariant on the transformation depicted on Figure 3.

A **point *r*-guard** is a point $y \in D$, *such that the two maximal axis-parallel line segments in D containing y do not intersect vertices of D.* A set of point guards ***r*-cover** D if any point $x \in D$ is r-visible from a member of the set. Such a set is called a **point *r*-guard system**.

A **vertical mobile *r*-guard** is a vertical line segment in D, *such that the maximal line segment in D containing it does not intersect vertices of D.* **Horizontal** mobile guards are defined analogously. A **mobile *r*-guard** is either a vertical or a horizontal mobile r-guard. A mobile r-guard ***r*-covers** any point $x \in D$ for which there exists a point y on its line segment such that x is r-visible from y.

Lemma 1. *Any rectilinear domain D can be transformed into another rectilinear domain D' so that the point guard r-cover, and the vertical/horizontal mobile guard r-cover problems in D, without the restrictions typeset in italics, are equivalent to the respective problems, as per our definitions (i.e., with the restrictions), in D'.*

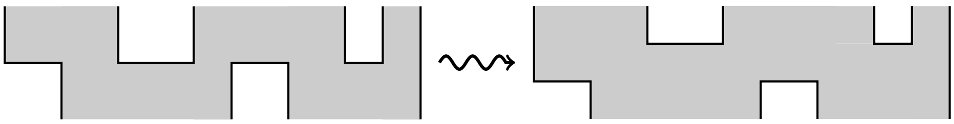


Figure 3: After this transformation, those mobile guards whose maximal containing line segment does not intersect vertices of the rectilinear domain, are just as powerful as mobile guards that are not restricted in such a way.

Proof. Let ε be the minimal distance between any two horizontal line segments of ∂D. The transformation depicted in Figure 3 in D takes a maximal horizontal line segment L in D which is touched from both above and below by the exterior of D, and maps D to

$$D' = D \bigcup \left(L + \overline{(0, -\varepsilon/4)(0, \varepsilon/4)} \right),$$

where addition is taken in the Minkowski sense. There is a trivial correspondence between the point and mobile guards of D and D' such that taking this correspondence guard-wise transforms a guarding system of D (guards without the restrictions) into a guarding system of D' (guards with the restrictions), and vice versa.

After performing this operation at every vertical and horizontal occurrence, we get a rectilinear domain D'', in which any vertical or horizontal line segment is contained in a non-degenerate rectangle in D''. Therefore, degenerate vision between any two points implies non-degenerate vision between the pair. Furthermore, the line segment of any mobile guard can be translated slightly along its normal (at least in one direction) while staying inside D'', and this clearly does not change the set of points r-covered by the guard. Similarly, we can perturb the position of a point guard without changing the set of points of D'' it r-covers. $\square$

Theorem 2. *Given a rectilinear domain D let m_V be the minimum size of a vertical mobile r-guard system of D, let m_H be defined analogously for horizontal mobile r-guard systems, and finally let p be the minimum size of a point r-guard system of D. Then*

$$\left\lfloor \frac{4(m_V + m_H - 1)}{3} \right\rfloor \geq p.$$

Observe, that the magical $4 : 3$ ratio highlighted by O'Rourke [18, Section 3.1] appears between the minimum number of (horizontal plus vertical) mobile and point guards required to control the gallery, even though the theorem does not use the number of vertices of the gallery as a parameter. Before moving onto the proof of Theorem 2, we discuss the aspects of its sharpness.

For $m_V + m_H \le 6$, sharpness of the theorem is shown by the examples in Figure 4. The polygon in Figure 4f can be easily generalized to one satisfying $m_V + m_H = 3k + 1$ and $p = 4k$. For $m_V + m_H = 3k + 2$ and $m_V + m_H = 3k + 3$, we can attach 1 or 2 plus signs to the previously constructed polygons, as shown in Figure 4d and 4e. Thus Theorem 2 is sharp for any fixed value of $m_V + m_H$.

By stringing together a number of copies of the polygons in Figure 4a and 4c in an L-shape (Figure 4f is a special case of this), we can construct rectilinear domains for any (m_H, m_V) pair satisfying $m_V \le 2(m_H - 1)$ and $m_H \le 2(m_V - 1)$, such that the polygon satisfies Theorem 2 sharply. The analysis in Section 3 immediately yields that if $m_V = 1$ or $m_H = 1$, then $m_V + m_H - 1$ is an upper bound for the minimum size of a point guard system (see Proposition 8), whose sharpness is shown by combs (Figure 1).

3 Translating the problem into the language of graphs

For graph theoretical notation and theorems used in this chapter (say, the block decomposition of graphs), the reader is referred to [5].

Definition 3 (Chordal bipartite or bichordal graph, [9])**.** A graph G is chordal bipartite iff any cycle C of ≥ 6 vertices of G has a chord (that is $E(G[C]) \supsetneq E(C)$).

Let S_V be the set of internally disjoint rectangles we obtain by cutting vertically at each reflex vertex of a rectilinear domain D. Similarly, let S_H be defined analogously for horizontal cuts of D. We may refer to the elements of these sets as **vertical and horizontal slices**, respectively.

The horizontal R-tree T_H of D is equal to

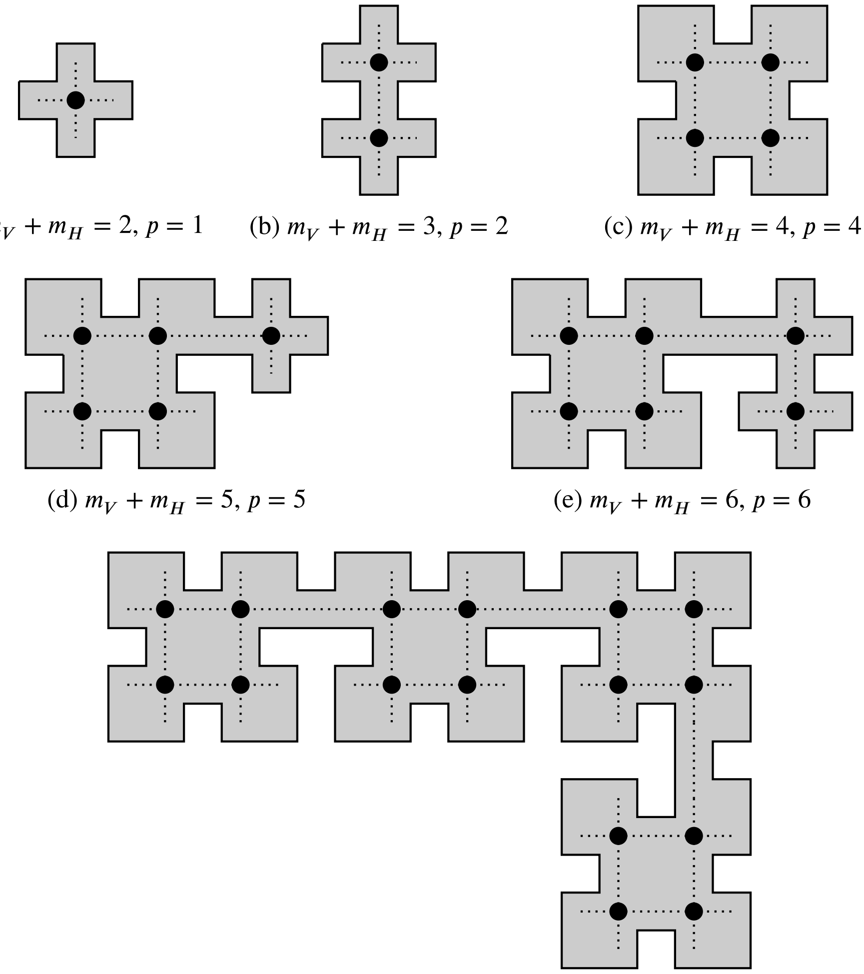
$$T_H = \left(S_H, \left\{ \{h_1, h_2\} \subseteq S_H \ : \ h_1 \neq h_2,\ h_1 \cap h_2 \neq \varnothing \right\} \right),$$

i.e., T_H is the intersection graph of the horizontal slices of D. The graph T_H is indeed a tree as its connectedness is trivial, and since any cut creates two internally disjoint rectilinear domains, T_H is also cycle-free. We can think of T_H as a sort of dual of the planar graph determined by the union of ∂D and its horizontal cuts. Similarly, T_V is the intersection graph of the vertical slices of D.

Let G be the intersection graph of S_H and S_V, i.e.,

$$G = \left(S_H \cup S_V, \left\{ \{h, v\} \ : \ h \in S_H,\ v \in S_V,\ \mathrm{int}(h) \cap \mathrm{int}(v) \neq \varnothing \right\} \right).$$

In other words, a horizontal and a vertical slice are joined by an edge iff their interiors intersect; see Figure 5. We may also refer to G as the **pixelation graph** of D. Clearly, the **set of pixels** $\{\cap e \mid e \in E(G)\}$ is a cover of D. Let us define $c(e)$ as the centroid of $\cap e$ (the pixel determined by e).



(a) $m_V + m_H = 2$, $p = 1$ (b) $m_V + m_H = 3$, $p = 2$ (c) $m_V + m_H = 4$, $p = 4$

(d) $m_V + m_H = 5$, $p = 5$ (e) $m_V + m_H = 6$, $p = 6$

(f) $m_V + m_H = 13$, $p = 16$

Figure 4: Vertical dotted lines: a minimum size vertical mobile guard system;
Horizontal dotted lines: a minimum size horizontal mobile guard system;
Solid disks: a minimum size point guard system.

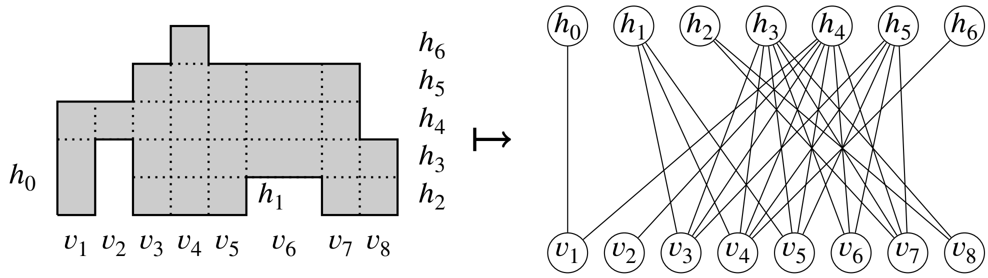


Figure 5: A rectilinear domain and its associated pixelation graph

Lemma 4. *G is a connected chordal bipartite graph.*

Proof. Connectedness of D immediately yields that G is connected too. Suppose C is a cycle of ≥ 6 vertices in G. For each node of the cycle C, connect the centroids of the pixels of its two incident edges with a line segment. This way we get a (not necessarily simple) orthogonal polygon P in D.

If P is self-intersecting, then the vertices which are represented by the two intersecting line segments are intersecting. This clearly corresponds to a chord of C in G.

If P is simple, then the number of its vertices is $|V(C)|$, thus one of them is a reflex vertex, say $c(v_1 \cap h_1)$ is one. As P lives in D, its interior is a subset of D as well (here we use that D is simply connected). The simpleness of P also implies that the vertical line segment intersecting $c(v_1 \cap h_1)$, after entering the interior of P at $c(v_1 \cap h_1)$, intersects P at least once more when it emerges, say at $c(v_1 \cap h_2)$. As this is not an intersection of the line segments corresponding to two vertices of D, the edge $\{v_1, h_2\}$ is a chord of C. $\square$

It is worth mentioning that even if D is a rectilinear domain with rectilinear hole(s), G may still be chordal bipartite. Take, for example, $[0,3]^2 \setminus (1,2)^2$; the graph associated to it has only one cycle, which is of length 4.

We will use the following technical claim to translate r-vision of points of D into relations in G.

Claim 5. *Let $e_1, e_2 \in E(G)$, where $e_1 = \{v_1, h_1\}$, $e_2 = \{v_2, h_2\}$, $v_1, v_2 \in S_V$, and $h_1, h_2 \in S_H$. The points $p_1 \in \mathrm{int}(\cap e_1)$ and $p_2 \in \mathrm{int}(\cap e_2)$ have r-vision of each other in D iff $e_1 \cap e_2 \neq \emptyset$ or $e_1 \cup e_2$ induces a 4-cycle in G.*

Proof. If $v_1 \in e_1 \cap e_2$, then $p_1, p_2 \in v_1$, therefore p_1 and p_2 have r-vision of each other. If $h_1 \in e_1 \cap e_2$, the same holds. If $\{v_1, h_1, v_2, h_2\}$ induces a 4-cycle, then

$$\mathrm{Conv}((v_1 \cap h_1) \cup (v_1 \cap h_2)) \subseteq v_1 \subseteq D$$

by v_1's convexity. Moreover,

$$\begin{aligned} B = &\mathrm{Conv}((v_1 \cap h_1) \cup (v_1 \cap h_2)) \cup \mathrm{Conv}((v_1 \cap h_2) \cup (v_2 \cap h_2)) \cup \\ &\cup \mathrm{Conv}((v_2 \cap h_2) \cup (v_2 \cap h_1)) \cup \mathrm{Conv}((v_2 \cap h_1) \cup (v_1 \cap h_1)) \end{aligned}$$

is contained in D. Since D is simply connected, we have $\mathrm{Conv}(B) \subseteq D$, which is a rectangle containing both p_1 and p_2.

In the other direction, suppose $e_1 \cap e_2 = \emptyset$. If R is an axis-aligned rectangle which contains both p_1 and p_2, then R clearly intersects the interiors of each element of $e_1 \cup e_2$, which implies that $\mathrm{int}(v_2) \cap \mathrm{int}(h_1) \neq \emptyset$ and $\mathrm{int}(v_1) \cap \mathrm{int}(h_2) \neq \emptyset$. Thus $e_1 \cup e_2$ induces a cycle in G. $\square$

This easily implies the following claim.

Claim 6. *Two points $p_1, p_2 \in D$ have r-vision of each other iff $\exists e_1, e_2 \in E(G)$ such that $p_1 \in \cap e_1$, $p_2 \in \cap e_2$, and either $e_1 \cap e_2 \neq \emptyset$ or $e_1 \cup e_2$ induces a 4-cycle in G.*

These claims motivate the following definition.

Definition 7 (r-vision of edges)**.** For any $e_1, e_2 \in E(G)$ we say that e_1 and e_2 have r-vision of each other iff $e_1 \cap e_2 \neq \emptyset$ or there exists a C_4 in G which contains both e_1 and e_2.

Let $Z \subseteq E(G)$ be such that for any $e_0 \in E(G)$ there exists an $e_1 \in Z$ so that e_1 has r-vision of e_0. According to Claim 6, if we choose a point from $\mathrm{int}(\cap e_1)$ for each $e_1 \in Z$, then we get a point r-guard system of D.

Observe that any vertical mobile r-guard is contained in $\mathrm{int}(v)$ for some $v \in S_V$ (except ≤ 2 points of the patrol). Extending the line segment the mobile guard patrols increases the area that it covers, therefore we may assume that this line segment intersects each element of $\{\mathrm{int}(\cap e) \mid v \in e \in E(G)\}$, which only depends on some $v \in S_V$. Using Claim 6, we conclude that the set which such a mobile guard covers with r-vision is exactly $\cup\{h \in S_H \mid \{h, v\} \in E(G)\}$. The analogous statement holds for horizontal mobile guards as well.

Thus, a set of vertical mobile guards of D can be represented by a set $M_V \subseteq S_V$. Clearly, M_V covers D if and only if

$$D = \bigcup_{v \in M_V} \left(\bigcup N_G(v) \right), \text{ which holds iff } S_H = \bigcup_{v \in M_V} N_G(v),$$

or in other words, M_V dominates each element of S_H in G. Similarly, a horizontal mobile guard system has a representative set $M_H \subseteq S_H$, which dominates S_V in G. Equivalently, $M_H \cup M_V$ is a totally dominating set of G, i.e., a subset of $V(G)$ that dominates every node of G (even the nodes of $M_H \cup M_V$).

Kosowski and Małafiejski [15] studies weakly cooperative mobile guards in grids. A grid is the connected union of vertical and horizontal segments in the plane, and a mobile guard is a maximal horizontal or vertical line segment of the grid. A set of mobile guards is called weakly cooperative, if the segment of each mobile guard intersects another guard's segment. An important observation of [15] is that the weakly cooperative mobile guard set problem in grids reduces to the total dominating set problem in the intersection graph of the grid. In Section 5, we discuss their complexity results as well.

The observations about G can be extended to a mixed set of vertical and horizontal mobile r-guards, which is represented by a set of vertices of $S \subseteq V(G)$. The set of

Orthogonal polygon	**Pixelation graph**
Mobile guard	Vertex
Point guard	Edge
Simply connected	Chordal bipartite ($\Rightarrow$, but $\not\Leftarrow$)
r-vision of two points	$e_1 \cap e_2 \neq \emptyset$ or $G[e_1 \cup e_2] \cong C_4$
Horiz. mobile guard cover	$M_H \subseteq S_H$ dominating S_V
Covering system of mobile guards	Dominating set

Table 2: Translating the orthogonal art gallery problem to the pixelation graph

guards is a covering system of guards of D if and only if every node $V(G) \setminus S$ has neighbor in S, i.e., S is a dominating set in G. Table 2 is the dictionary that lists the main notions of the original problem and their corresponding phrasing in the pixelation graph.

As promised, the following claim has a very short proof using the definitions and claims of this section.

Proposition 8. *If $m_V = 1$ or $m_H = 1$, then $p \leq m_V + m_H - 1$.*

Proof. Let Z be the set of edges of G induced by $M_H \cup M_V$. Clearly, $G[M_H \cup M_V]$ is a star, thus $|Z| = |M_H| + |M_V| - 1$.

We claim that Z covers $E(G)$. There exist two slices, $h_1 \in M_H$ and $v_1 \in M_V$, which are joined by an edge to v_0 and h_0, respectively. Since $G[M_H \cup M_V]$ is a star, $\{v_1, h_1\} \in Z$. This edge has r-vision of e_0, as either $\{v_1, h_1\}$ intersects e_0, or $\{v_0, h_0, v_1, h_1\}$ induces a C_4 in Z. □

Finally, we can state Theorem 2 in a stronger form, conveniently via graph theoretic concepts.

Theorem 2′. *Let A_V be a set of internally disjoint axis-parallel rectangles of a rectilinear domain D, called the vertical slices. Similarly, let A_H be another set with the same property, whose elements we call the horizontal slices. Also, suppose that for any $v \in A_V$, its top and bottom sides are a subset of ∂D, and for any $h \in A_H$, its left and right sides are a subset of ∂D. Furthermore, suppose that their intersection graph*

$$G = \left(A_H \cup A_V, \left\{\{h, v\} \subseteq A_V \cup A_H \ : \ \mathrm{int}(v) \cap \mathrm{int}(h) \neq \emptyset\right\}\right)$$

is connected.

If $M_V \subseteq A_V$ dominates A_H in G, and $M_H \subseteq A_H$ dominates A_V in G, then there exists a set of edges $Z \subseteq E(G)$ such that any element of $E(G)$ is r-visible from some element of Z, and

$$|Z| \leq \frac{4}{3} \cdot (|M_V| + |M_H| - 1).$$

Now we are ready to prove the main theorem of this paper.

4 Proof of Theorem $2'$

The set A_H can be extended to a set S_H of internally disjoint axis-parallel rectangles which completely cover D, and whose left and right sides are subsets of ∂D. Similarly, extend A_V to a complete partition S_V of D. By Lemma 4, G is a subgraph induced by $A_H \cup A_V$ in a chordal bipartite graph, thus G is chordal bipartite as well. Let $M = G[M_V \cup M_H]$ be the subgraph induced by the dominating sets. Notice, that the bichordality of G is inherited by M.

Given a pair of subsets $A_H \subseteq S_H$ and $A_V \subseteq S_V$ such that their intersection graph G is connected, join two slices $h_1, h_2 \in A_H$ by an edge if there exists a $v \in A_V$ such that $\{h_1, v\}, \{h_2, v\} \in E(G)$ and there does not exist $h_3 \in A_H$ which is between h_1 and h_2 in the path induced by $N_G(v)$ in T_H. We call the constructed graph the R-tree on A_H. The definition for A_V goes analogously.

Claim 10. *For any $h_1, h_2 \in A_H$ the following statements hold:*

- *$N_G(h_1)$ is the vertex set of a path in the R-tree on A_V, or in other words $N_G(h_1)$ induces a path in the R-tree on A_V.*
- *$N_G(h_1) \bigcap N_G(h_2)$ is either empty, contains exactly one slice, or induces a path in the R-tree on A_V.*
- *If G is 2-connected and h_1 is a neighbor of h_2 in the R-tree on A_H, then*

$$\left| N_G(h_1) \bigcap N_G(h_2) \right| \geq 2.$$

Proof. The first two statements are trivial. Suppose that G is 2-connected, h_1 is joined to h_2 in the R-tree on A_H. There is a path connecting h_1 to h_2 in G. Every second node of this path is a vertical slice, and the neighborhoods of two vertical slices distance two apart have a common neighbor. The neighborhood of a vertical slice is path in the R-tree on A_H, so there exists a vertical slice v_1 such that $h_1, h_2 \in N_G(v_1)$. Moreover, $G - v$ is still connected, so in the same manner we can find another vertical slice v_2 which is also joined to both h_1 and h_2 in G. □

Claim 11. *If M is connected, then any edge $e_0 = \{h_0, v_0\} \in E(G)$ is r-visible from some edge of M.*

Proof. As $N_G(M_V \cup M_H) = V(G)$, there exists two vertices, $v_1 \in M_V$ and $h_1 \in M_H$, such that $\{v_1, h_0\}, \{v_0, h_1\} \in E(G)$.

If $v_0 \in M_V$ or $h_0 \in M_H$, then $\{v_0, h_1\}$ or $\{v_1, h_0\}$ is in $E(M)$.

Otherwise, there exists a path in M, whose endpoints are v_1 and h_1, and this path and the edges $\{v_1, h_0\}$,$\{h_0, v_0\}$,$\{v_0, h_1\}$ form a cycle in G. By the bichordality of G, there exists a C_4 in G which contains an edge of M and e_0. □

Claim 11 implies that for connected M, we can select a subset of edges of M that guard every edge of G. Section 4.3 shows that if Theorem $2'$ holds for connected M, then it also holds when M has multiple connected components. Furthermore, if Theorem $2'$ holds when M is 2-connected, Section 4.2 shows that theorem also holds when M is connected.

Based on the level connectivity of M, we distinguish three cases: M is 2-connected, M is connected, and M has multiple connected components. These cases and their proofs are quite different. When M is connected or has multiple connected components, the proofs are relatively short and simple, and more importantly, only rely on elementary graph theory.

The spirit of the proof dwells in Section 4.1, which holds the deepest insight into the problem and is vastly longer and more complex than the other two cases following it. A few geometric arguments are present, but the overwhelming majority of reasoning in the 2-connected case is graph theoretic. Although this means that the proof is somewhat technical, we believe it is also quite robust, being built on the abstraction provided by R-trees and the pixelation graph.

4.1 M is 2-connected

The $\frac{4}{3}$ constant in the statement of Theorem $2'$ is determined by this case. Let us first present an outline of this case.

First, we describe two fundamental properties of M in Claim 12 and 13. Then, some of the horizontal slices of M are refined into two thinner slices each, so as to avoid technical difficulties later in the proof. From then on, we work in this refined structure, denoted by M' and G'. Claim 14 provides the link between point guards of G' and G. Next, we establish a relation between edges of M' (Definition 15), which describes when an edge can be replaced by another one, so that the replacement edge r-covers any edge seen exclusively by the replaced edge. This leads to the definition of "hyperguards" of M' (Definition 16), which are proven to be point guard systems of G' in Lemma 17.

After the lengthy preparation, we are finally ready to construct a hyperguard of M' in Section 4.1.2. In the following Section 4.1.3, the size of the constructed hyperguard is estimated, finishing the proof of this case.

If $E(M)$ consists of a single edge e, then $Z = \{e\}$ is clearly a point guard system of G by Claim 11.

Suppose now, that M has more than two vertices. Any edge of M is contained in a cycle of M, and by the bichordality property, there is such a cycle of length 4. It is easy to see that the convex hull of the pixels determined by the edges of a C_4 is a rectangle. Define

$$D_M = \bigcup_{\{e_1,e_2,e_3,e_4\} \text{ is a } C_4 \text{ in } M} \operatorname{Conv}\left(\bigcup_{i=1}^{4} \cap e_i\right).$$

The simply connectedness of D implies that $D_M \subseteq D$.

Claim 12. *For any slice $s \in V(M)$ the intersection of s and D_M is connected.*

Proof. Suppose that $e_1, e_2 \in E(M)$ are such that $\cap e_1$ and $\cap e_2$ are in two different components of $s \cap D_M$. Since M is 2-connected, there is a path connecting $e_1 \setminus \{s\}$ and $e_2 \setminus \{s\}$ in $M - s$.

Take the shortest cycle in M containing e_1 and e_2. If this cycle contains 4 edges, then the convex hull of their pixels is in D_M, which is a contradiction. Similarly, if the cycle contains more than 4 edges, the bichordality of M implies that s is joined to every second node of the cycle, which contradicts our assumption that $s \cap D_M$ is disconnected. □

Claim 13. *D_M is simply connected.*

Proof. Connectedness of D_M follows from the connectedness of M and Claim 12. Suppose there is a hole in D_M. If the hole is a rectangle, the four slices of M bounding it induce a C_4, which contradicts the definition of D_M.

If the hole has more than 6 vertices, take a reflex vertex x of it, and let $e \in E(M)$ be such that x is a vertex of $\cap e$. Since $D_M \subseteq D$ and D is simply connected, the horizontal slice of e crosses the hole, and intersects another vertical slice of M. This contradicts Claim 12. □

Let $B_H \subset M_H$ be the set of those slices whose top and bottom sides both intersect ∂D_M in an uncountable number of points of $\mathbb{R}^2$.

For technical reasons, we split each element of $h \in B_H$ horizontally through $c(h)$ to get two isometric rectangles in $\mathbb{R}^2$; let the set of the resulting refined horizontal slices be B'_H. Replace the elements of A_H and M_H contained in B_H with their corresponding two halves in B'_H to get

$$A'_H = B'_H \bigcup A_H \setminus B_H \quad \text{and} \quad M'_H = B'_H \bigcup M_H \setminus B_H,$$

respectively. Let $A'_V = A_V$, $M'_V = M_V$. Let the R-tree on A'_H be T'_H. Let τ be the function which maps $h \in B'_H$ to the $\tau(h) \in A_H$ for which $h \subseteq \tau(h)$ holds, and let τ be the identity function on $A'_V \cup A'_H \setminus B'_H$.

Let G' be the intersection graph of A'_H and A'_V (as in the statement of Theorem 2′). Also, let $M' = G'[M'_H \cup M'_V] = \tau^{-1}(M)$. Observe that τ naturally defines a graph homomorphism $\tau : G' \to G$ (edges are mapped vertex-wise).

Claim 14. *In G', the set M'_H dominates A'_V, and M'_V dominates A'_H. Furthermore, if $Z' \subseteq E(M')$ is a point guard system of G', then $Z = \tau(Z') \subseteq E(M)$ is a point guard system of G.*

Proof. The first statement of this claim holds, since τ maps non-edges to non-edges, and both $M'_H = \tau^{-1}(M_H)$ and $M'_V = \tau^{-1}(M_V)$ by definition. As τ is a graph homomorphism, it preserves r-visibility, which implies the second statement of this claim. □

Notice, that M' is 2-connected and $D_M = D_{M'}$. It is straightforward to verify that an edge $e \in E(M')$ falls into one of the following 4 categories:

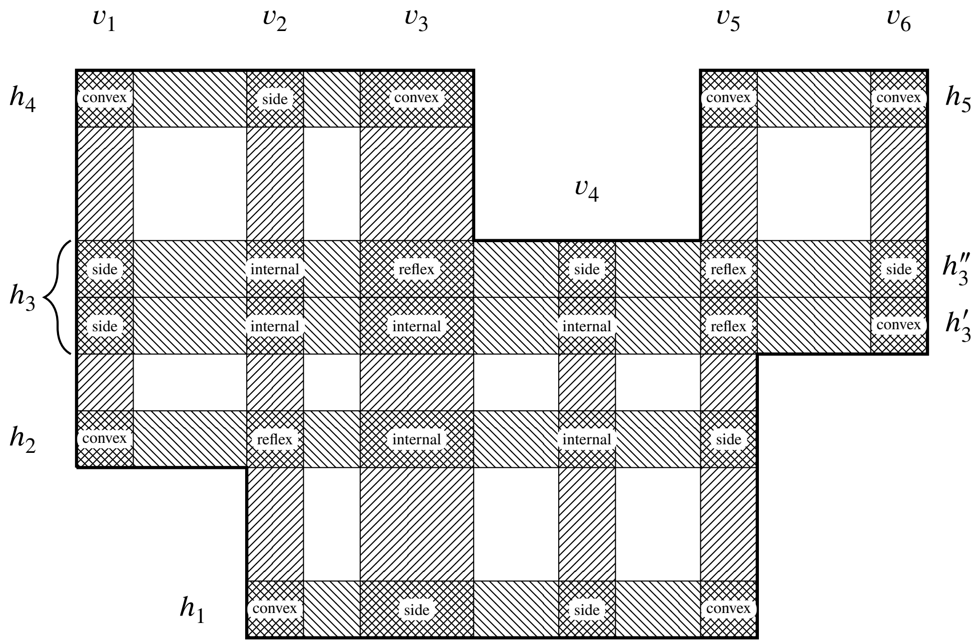


Figure 6: We have $M_H = \{h_1, h_2, h_3, h_4, h_5\}$, $M'_H = M_H - h_3 + h'_3 + h''_3$, and $M_V = M'_V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. The thick line is the boundary of D_M. Each rectangle pixel is labeled according to the type of its corresponding edge of M'.

Convex edge: 3 vertices of $\cap e$ fall on ∂D_M, e.g., the edge $\{h_2, v_1\}$ on Figure 6;

Reflex edge: exactly 1 vertex of $\cap e$ falls on ∂D_M, e.g., $\{h''_3, v_3\}$ on Figure 6;

Side edge: two neighboring vertices of $\cap e$ fall on ∂D_M, e.g., $\{h_1, v_4\}$ on Figure 6;

Internal edge: zero vertices of $\cap e$ fall on D_M, e.g., $\{h_2, v_3\}$ on Figure 6.

Notice that on Figure 6, the edge $\{h_3, v_5\}$ falls into neither of the previous categories, as two non-neighboring (diagonally opposite) vertices of pixel $h_3 \cap v_5$ fall on D_M. This clearly cannot happen with edges of G', but G may contain edges of this type. The τ preimage of such edges are two reflex edges of M'.

The preimages of a convex edge are a convex edge and a side edge (M' is 2-connected), the preimages of a side edge are two side edges, and the preimages of a reflex edge are a reflex edge and an internal edge. In the other direction, τ maps convex edges to convex edges, and side edges to convex or side edges.

The following definition allow us to break our proof into smaller, transparent parts, which ultimately boils down to presenting a precise proof. It captures a condition which in certain circumstances allows us to conclude that a guard e_1 can be replaced by e_2 such that we still have complete coverage of G'.

Definition 15. We say that a slice s_0 is **between** slices s_1 and s_2 (all vertical or horizontal), if in the corresponding R-tree s_0 is on the path between s_1 and s_2. For any two edges $e_1, e_2 \in E(M')$, where $e_1 = \{v_1, h_1\}$ and $e_2 = \{v_2, h_2\}$, we write $e_2 \to e_1$ (e_2 dominates e_1) iff either

- $h_1 = h_2$, and $\exists h_3, h_4 \in M'_H$ such that $\{v_1, v_2, h_3, h_4\}$ induces a C_4 in M', and $h_1 = h_2$ is between h_3 and h_4; or
- $v_1 = v_2$, and $\exists v_3, v_4 \in M'_V$ such that $\{v_3, v_4, h_1, h_2\}$ induces a C_4 in M', and $v_1 = v_2$ is between v_3 and v_4; or
- $e_1 \cap e_2 = \emptyset$, and $\exists v_3 \in M'_V$ and $h_3 \in M'_H$ such that both $\{v_1, h_2, v_2, h_3\}$ and $\{h_1, v_3, h_2, v_2\}$ induces a C_4 in M'; furthermore, v_1 is between v_2 and v_3, and h_1 is between h_2 and h_3.

We write $e_2 \leftrightarrow e_1$ iff both $e_2 \rightarrow e_1$ and $e_1 \rightarrow e_2$ hold. Note that $\leftrightarrow$ is a symmetric, but generally intransitive relation. For convenience, we define both relations to be reflexive.

For example, on Figure 6, $\{h_1, v_3\} \leftrightarrow \{h''_3, v_3\}$, and $\{h_1, v_2\} \rightarrow \{h''_3, v_3\}$. Also, $\{h''_3, v_3\} \leftrightarrow \{h''_3, v_1\}$, but $\{h''_3, v_3\} \nrightarrow \{h'_3, v_1\}$. This is a technicality which makes the proofs easier, but does not cause any issues in the end, as $\tau(\{h''_3, v_1\}) = \tau(\{h'_3, v_1\})$. The fact that $\{h''_3, v_3\} \rightarrow \{h'_3, v_5\}$ and $\{h'_3, v_5\} \nrightarrow \{h''_3, v_3\}$ shows that $\rightarrow$ is not symmetric.

4.1.1 Hyperguards

We will search for a point guard system of M' with very specific properties, which are described by the following definition.

Definition 16. Suppose $Z' \subseteq E(M')$ is such, that

1. Z' contains every convex edge of M',
2. for any non-internal edge $e_1 \in E(M') \setminus Z'$, there exists some $e_2 \in Z'$ for which $e_2 \rightarrow e_1$, and
3. if $h_3, h_4 \in M'_H$ are neighboring slices in the R-tree on M'_H, and v_3, v_4 are the end-nodes of the path induced by $N_{M'}(h_3) \bigcap N_{M'}(h_4)$ in the R-tree on M'_V, and $\{v_3, h_3, v_4, h_4\}$ induces a C_4 in M', then there exists $e_2 \in Z'$ such that $e_2 \rightarrow \{v_3, h_3\}, \{v_4, h_3\}$ or $e_2 \rightarrow \{v_3, h_4\}, \{v_4, h_4\}$ holds.

If these three properties hold, we call Z' a **hyperguard** of M'.

The 3$^{\text{rd}}$ property of hyperguards corresponds to the configuration in D_M shown on Figure 7.

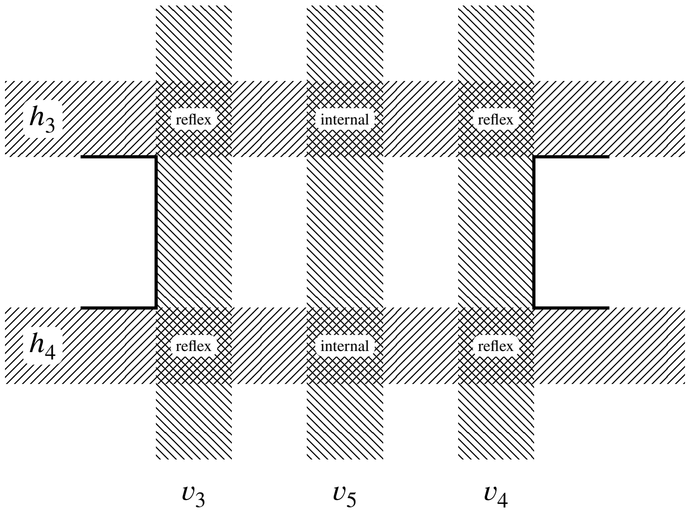


Figure 7: A neck in D_M. There are no horizontal slices of M'_H between h_3 and h_4, but there can be vertical slices between v_3 and v_4.

Lemma 17. *Any hyperguard Z' of M' is a point guard system of G', i.e., any edge of G' is r-visible from some element of Z'.*

Proof. Let $e_0 = \{v_0, h_0\} \in E(G')$ be an arbitrary edge. By Claim 11, there exists an edge $e_1 = \{v_1, h_1\} \in E(M')$ which has r-vision of e_0, and we also suppose that e_1 is chosen so that $\mathrm{dist}_{T'_H}(h_0, h_1) + \mathrm{dist}_{T'_V}(v_0, v_1)$ is minimal.

Trivially, if $e_1 \in Z'$ (for example, if e_1 is a **convex edge** of M'), then e_0 is r-visible from e_1. Assume now, that $e_1 \notin Z'$.

- If e_1 is a **reflex or side edge** of M', then $\exists e_2 = \{v_2, h_2\} \in Z'$ so that $e_2 \to e_1$. We claim that e_2 has r-vision of e_0 in G' (this is the main motivation for Definition 15).

  1. If $h_1 = h_2$: by the choice of e_1 and e_2, v_1 is joined to h_0, h_3, h_4 in G'. Since $N_{G'}(v_1)$ is the vertex set of a path in T'_H, the choice of e_1 guarantees that h_0 is between h_3 and h_4, which are neighbors of v_2 in G'. Therefore $\{v_2, h_0\} \in E(G')$, so $\{v_0, h_0, v_2, h_1(= h_2)\}$ induces a C_4 in G'.
  2. If $v_1 = v_2$: the proof proceeds analogously to the previous case.
  3. If $e_1 \cap e_2 = \emptyset$: by the choice of e_1 and e_2, v_1 is joined to h_0, h_3, h_2 in G', and v_1 is joined to v_0, v_3, v_2 in G'. The choice of e_1 guarantees that h_0 is between h_3 and h_2, and that v_0 is between v_3 and v_2. Therefore $\{v_2 \cap h_0\}, \{v_0 \cap h_2\} \in E(G')$, so $\{v_0, h_0, v_2, h_2\}$ induces a C_4 in G'.

  In any of the three cases, e_0 is r-visible from e_2 in G'.

- If e_1 is an **internal edge** of M', then $\cap e_0 \subset D_M$, so $\cap e_0$ is in a rectangle corresponding to a C_4 of M'. Thus there are two elements $h_3, h_4 \in M'_H \cap N_{G'}(v_0)$ such that there does not exist an element of M'_H which is between h_3 and h_4, but h_0 is between h_3 and h_4 (or is equal to one of them). Because M' is 2-connected, Claim 10 applies. Let the end-points of the path induced by $N_{M'}(h_3) \bigcap N_{M'}(h_4)$ be v_3 and v_4.

We claim that the edges of the C_4 induced by $\{v_3, h_3, v_4, h_4\}$ are non-internal edges. Take $\{v_3, h_3\}$, for example.

- If v_3 is an end-point of the path induced by $N_{M'}(h_3)$, then Claim 12 implies that one of the sides of the pixel $v_3 \cap h_3$ is a subset of ∂D_M. In other words, $\{v_3, h_3\}$ is a side or a convex edge of M'.
- Otherwise, there is a neighbor v_5 of v_3 in the path induced by $N_{M'}(h_3)$ in the R-tree on M'_H, such that $v_5 \notin N_{M'}(h_4)$. If $\{v_3, h_3\}$ is an internal edge, then $\{v_5, h_4\} \in E(M')$, so $\{v_3, h_3\}$ can only be a reflex edge.

The same reasoning holds for the other three edges induced by $\{v_3, h_3, v_4, h_4\}$. Clearly, e_0 is r-visible to all four edges; if any of them is a convex edge, we are done.

If, say, $\{v_3, h_3\}$ is a side edge, then $\exists e_2 = \{v_2, h_2\} \in Z'$ such that $e_2 \to \{v_3, h_3\}$. Because v_3 is an end-point of the path induced by $N_{M'}(h_3)$ in the R-tree on M'_H, we must have $h_2 = h_3$. There are two horizontal slices $h_5, h_6 \in M'_H$ which intersect both v_2 and v_3, and h_3 is between them. Both $N_{M'}(v_2)$ and $N_{M'}(v_3)$ are the vertex set of a path in the R-tree on M'_H, and so is their intersection $N_{M'}(v_2) \cap N_{M'}(v_3)$. It contains the vertices of the path from h_5 to h_6 through h_3, therefore it contains h_4 (there is no slice of M'_H between h_3 and h_4). Thus e_2 has r-vision of the four induced edges of $\{v_3, h_3, v_4, h_4\}$, and consequently, of e_0.

If each of the four induced edges of $\{v_3, h_3, v_4, h_4\}$ are reflex edges, then without loss of generality, we may assume that there $\exists e_2 = \{v_2, h_2\} \in Z'$ such that $e_2 \to \{v_3, h_3\}, \{v_4, h_3\}$. This implies that $v_2, v_3, v_4 \in N_{G'}(h_3)$. If v_2 is between v_3 and v_4 (or is equal to one of them), then $v_2 \in N_{G'}(h_4)$, so e_2 has r-vision of each of the four induced edges of $\{v_3, h_3, v_4, h_4\}$ and of e_0.

Suppose now, that v_2 is not between v_3 and v_4, i.e., $v_2 \notin N_{M'}(h_3) \cap N_{M'}(h_4)$. Thus h_2 is not equal to either h_3 or h_4, and so cannot be between them. Because $e_2 \to \{v_3, h_3\}$, there is an h_5 such that $\{v_3, h_5\} \in E(M')$, and h_3 is between h_2 and h_5 (all of which are joined to v_3 in M'). By construction, h_4 is between h_2 and h_5. Since v_2 is joined to both h_2 and h_5, it should be joined to h_4, a contradiction.

We have verified the statement in every case, so the proof of this lemma is complete. □

Observe that if D_M does not contain a "neck" (see Figure 7), even the first two properties of a hyperguard are sufficient to prove Lemma 17.

Notice, that the set of all convex, reflex, and side edges of $E(M')$ form a hyperguard of M'. By Lemma 17, this set is a point guard system of G', and Claim 14 implies that its τ-image is a point guard system of G. The cardinality of the τ-image of this hyperguard is bounded by $2|V(M)| - 4$ (we will see this shortly), which is already a magnitude lower than what the trivial choice of $E(M)$ would give (generally, $|E(M)|$ can be equal to $\Omega(|V(M)|^2)$).

Let the number of convex, side, and reflex edges in M' be c', s', and r', respectively. Claim 12 and Claim 13 allow us to count these objects.

1. The number of reflex vertices of D_M is equal to r': any reflex vertex is a vertex of a reflex edge, and the way M' and D_M is constructed guarantees that exactly one vertex of the pixel of a reflex edge is a reflex vertex of D_M.

2. The number of convex vertices of D_M is equal to c': any convex vertex is a vertex of the pixel of a convex edge, and the way D_M is constructed guarantees that exactly one vertex of the pixel of a convex edge is a convex vertex.

3. The cardinality of $V(M')$ is $c' + \frac{1}{2}s'$: the first and last edge incident to any element of $V(M')$ ordered from left-to-right (for elements of M'_H) or from top-to-bottom (for elements of M'_V) is a convex or a side edge. Conversely, any convex edge is the first or last incident edge of exactly one element of M'_H and one element of M'_V. A side edge is the first or last incident edge of exactly one element of $V(M')$.

4. For any reflex edge $e_1 = \{v_1, h_1\} \in E(M')$, there is exactly one reflex or side edge in $E(M')$ which contains v_1 and is in the $\leftrightarrow$ relation with e_1, and the same can be said about h_1.

5. Any side edge $e_1 \in E(M')$ is in $\leftrightarrow$ relation with exactly one reflex or side edge which it intersects. The intersection is the slice in $V(M')$ on which e_1 is a boundary edge.

We can now compute the size of the set of all convex, reflex, and side edges of M':

$$c' + r' + s' = 2c' - 4 + s' = 2|V(M')| - 4.$$

Furthermore, it is clear that taking the τ-image of this set decreases its cardinality by $2|B_H|$ (new reflex and side edges are created at both ends of slices in B_H when splitting them). Thus the cardinality of the τ-image of all convex, reflex, and side edges of M' is

$$2|V(M')| - 4 - 2|B_H| = 2|V(M)| - 4,$$

proving the claim from the previous page. Readers who are only interested in a result which is sharp up to a constant factor, may skip to Section 4.2. Further analysis of M' allows us to lower the coefficient 2 to $\frac{4}{3}$.

Define the **auxiliary graph X** as follows: let $V(X)$ be the set of reflex and side edges of M', and let

$$E(X) = \Big\{ \{e, f\} \,:\, e \neq f,\ e \cap f \neq \emptyset,\ e \leftrightarrow f \Big\}.$$

By our observations, X is the disjoint union of some cycles and $\frac{1}{2}s'$ paths. This structure allows us to select a hyperguard which contains a subset of the reflex and side edges of M', instead of the whole set.

In the next section, we use the following trivial fact several times.

Claim 18. *A path on k nodes has a dominating set of size*

$$\left\lceil \frac{k}{3} \right\rceil = \left\lfloor \frac{k+2}{3} \right\rfloor.$$

4.1.2 Constructing a hyperguard Z' of M'.

We will define $(Z'_j)_{j=0}^{\infty}$, a sequence of (set theoretically) increasing sequence of subsets of $E(M')$, and $(X_j)_{j=0}^{\infty}$, a decreasing sequence of induced subgraphs of X.

Additionally, we will define a function $w_j : V(X) \to \{0,1,2\}$, and extend its domain to any subgraph $H \subseteq X$ by defining $w_j(H) = \sum_{e \in V(H)} w_j(e)$. The purpose of w_j, very vaguely, is that as Z' will contain every third node of X, we need to keep count of the modulo 3 remainders. Furthermore, w_j serves as buffer in a(n implicitly defined) weight function (see inequality (2)).

For a set $E_0 \subseteq E(X)$, let the indicator function of E_0 be

$$\mathbb{1}_{E_0}(e) = \begin{cases} 1, & \text{if } e \in E_0, \\ 0, & \text{if } e \in E(X) \setminus E_0. \end{cases}$$

Let $Z'_0 = \varnothing$ and $X_0 = X$. By our previous observations, X does not contain isolated nodes. Define $w_0 : V(X) \to \{0,1,2\}$ such that

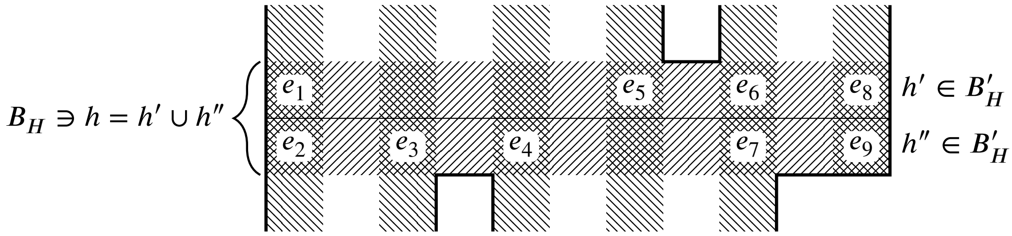
$$w_0(e) = \begin{cases} 1, & \text{if } d_{X_0}(e) = 1, \\ 0, & \text{if } d_{X_0}(e) = 0 \text{ or } 2. \end{cases}$$

In the j^{th} step, we will define Z'_j, X_j, and w_j so that

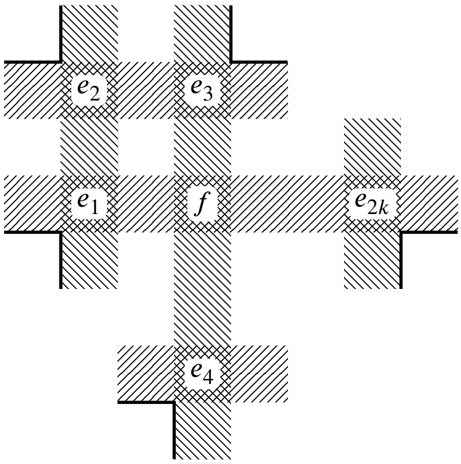
- $Z'_{j-1} \subseteq Z'_j$, $X_j \subseteq X_{j-1}$,
- $\{e \in V(X_j) \mid d_{X_j}(e) = 1\} \subseteq w_j^{-1}(1)$,
- $\{e \in V(X_j) \mid d_{X_j}(e) = 0\} = w_j^{-1}(2)$, and
- $\forall e_0 \in V(X) \setminus V(X_j)$, either $e_0 \in Z'_j$, or $\exists e_1 \in Z'_j$ so that $e_1 \to e_0$.

If these hold, then for any path component P_j in X_j, we have $w_j(P_j) \geq 2$.

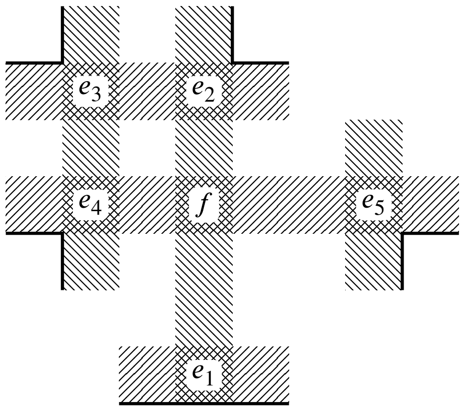
As j increases, the construction goes through 5 phases. In each of Phase 2-4, j is incremented for multiple iterations, until X_j satisfies some predefined condition. The different phases and the relevant parts of D_M are depicted on Figure 8.



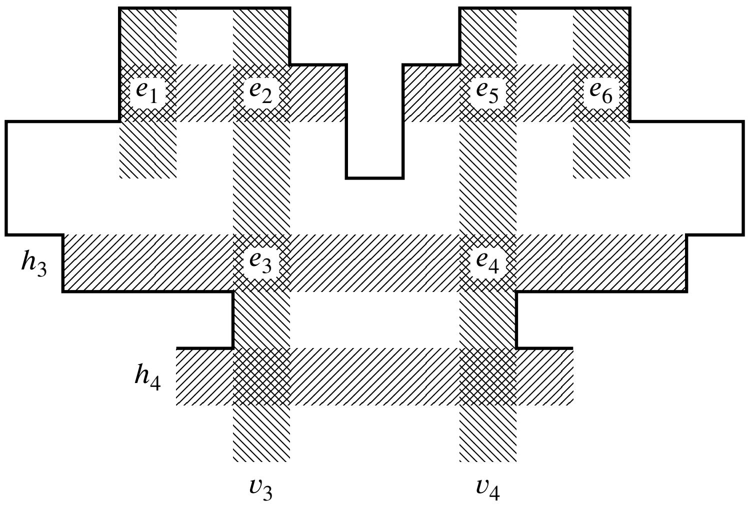
(a) Phase 1: handling the new reflex and side edges created on the refined slices. We have $e_1, e_2 \in S'$, $e_9 \in C'$, $e_8 \in U'$, and $e_3 \in T'$.



(b) Phase 2: cutting cycles



(c) Phase 3: cutting self-intersecting paths



(d) Phase 4: covering necks. Some slices are not shown or drawn completely to avoid clutter. The set $\{e_i \ : \ i = 1, \dots, 6\}$ induces a path in X, and $\{e_2, e_5\}$ is a minimum dominating set of it.

Figure 8: Demonstrating possible substructures of X which are handled in Phases 1-4.

Phase 1 Let the set of convex edges of M' be C'. Let

$$
\begin{aligned}
S' &= \left\{ e \in V(X) : \ \tau(e) \text{ is a side edge} \right\}, \\
T' &= \left\{ f \in V(X) : \ \exists e \in S' \ f \leftrightarrow e, \ \tau^{-1}(\tau(f)) \setminus \{f\} \to N_X(\tau^{-1}(\tau(e))) \setminus \{f\} \right\}, \\
U' &= \tau^{-1}(\tau(C')) \setminus C', \\
Q' &= \bigcup_{\substack{e_1, e_4 \in S' \cup U' \\ e_2, e_3 \in V(X) \\ e_1 \leftrightarrow e_2, e_2 \leftrightarrow e_3, e_3 \leftrightarrow e_4}} \{e_1, e_2, e_3, e_4\}.
\end{aligned}
$$

The reader is advised to look at Figure 8a to visualize the corresponding pixels. Take

$$
\begin{aligned}
Z_1' &= \tau^{-1}\left(\tau(C') \bigcup \tau(T')\right), \\
X_1 &= X - T' - N_X(T') - U' - N_X(U'), \\
w_1 &= w_0 - \mathbb{1}_{S'} - \mathbb{1}_{U'} + \sum_{f \in T'} \mathbb{1}_{N_X(N_X(f)) \setminus \{f\} \setminus Q'} + \sum_{e \in U'} \mathbb{1}_{N_X(N_X(e)) \setminus \{e\} \setminus Q'}.
\end{aligned}
$$

Phase 2 Take a cycle $e_1, e_2, \ldots, e_{2k_j}$ in X_j ($k_j \geq 2$, $j \geq 1$). This set of nodes of X_j is the edge set of a cycle of length $2k_j$ in M'.

- If $2k_j = 4$, observe that $e_1 \leftrightarrow e_2$, $e_1 \leftrightarrow e_4$, $e_2 \leftrightarrow e_3$, $e_4 \leftrightarrow e_3$ together imply that $e_1 \leftrightarrow e_3$. Take

$$
\begin{aligned}
Z_{j+1}' &= \{e_1\} \bigcup Z_j', \\
X_{j+1} &= X_j - \{e_1, e_2, e_3, e_4\}, \\
w_{j+1} &= w_j.
\end{aligned}
$$

- If $2k_j \geq 6$, the chordal bipartiteness of M' implies that without loss of generality there is a chord $f \in E(M')$ which forms a cycle with e_1, e_2, e_3 in M'. For example, see Figure 8b. Take

$$
\begin{aligned}
Z_{j+1}' &= \{f\} \bigcup Z_j', \\
X_{j+1} &= X_j - \{e_{2k_j}, e_1, e_2, e_3, e_4\}, \\
w_{j+1} &= w_j + \mathbb{1}_{e_5} + \mathbb{1}_{e_{2k_j - 1}}.
\end{aligned}
$$

By iterating the above operation, eventually we reach an index j_1 for which X_{j_1} is cycle-free ($|V(X_j)|$ decreases with every iteration).

Phase 3 Take a path $e_1, e_2, \ldots, e_k$ in X_j (for $j \geq j_1$), such that

$$E\left(M'\left[\bigcup_{i=2}^{k-1} e_i\right]\right) \setminus \{e_2, \ldots e_{k-1}\} \neq \emptyset.$$

Figure 8c shows such an example. Using the bichordality of M', there exists a chord $f \in E(M')$ which forms a C_4 with $\{e_{l-1}, e_l, e_{l+1}\}$, where $3 \leq l \leq k-2$. It is easy to see that $e_{l-2} \leftrightarrow e_{l-1}$ implies $f \to e_{l-2}$ and $f \to e_{l-1}$. Similarly, we have that $f \to e_{l+1}$ and $f \to e_{l+2}$. Also, $f \to e_{l-1}$ and $f \to e_{l+1}$ together imply $f \to e_l$. Therefore, we take

$$\begin{aligned}
Z'_{j+1} &= \{f\} \bigcup Z'_j, \\
X_{j+1} &= X_j - \{e_{l-2}, e_{l-1}, e_l, e_{l+1}, e_{l+2}\}, \\
w_{j+1} &= w_j + \mathbb{1}_{\{\mathrm{dist}_X(\bullet, e_l)=3\}}.
\end{aligned}$$

Since the number of nodes in X_j decreases with every iteration of this method, there is a j_2 for which X_{j_2} becomes free of the above defined paths.

Phase 4 The set M'_H is the subset of the nodes of a horizontal R-tree of D. Let $h_{\mathrm{root}} \in M'_H$ be a horizontal slice whose top side has maximal y-coordinate (so only convex and side edges are incident to it in M'). Process the elements of M'_H in decreasing distance (measured in the horizontal R-tree) from h_{root}.

Let $h_3 \in M'_H$ is the next horizontal slice to be processed. Let $h_4 \in M'_H$ be the neighbor of h_3 on the path between h_3 and h_{root}. Because M' is 2-connected, the path induced by $N_{M'}(h_3) \bigcap N_{M'}(h_4)$ contains at least two nodes; let the end-points of the path be v_3 and v_4. As it is shown in Lemma 17, in this case the edges of the cycle $\{v_3, h_3, v_4, h_4\}$ are non-internal edges of M'. If not each of them is a reflex edge, continue this phase with the next horizontal slice. Suppose now, that all four edges of the cycle are reflex edges of M'.

If $\{v_3, h_3\}$ and $\{v_4, h_3\}$, or $\{v_3, h_4\}$ and $\{v_4, h_4\}$ are removed in Phase 2 or Phase 3 in one iteration, then the edge by which Z' is extended in the same step satisfies the 3$^{\text{rd}}$ property of hyperguards for Z' and h_3, h_4, and we skip to the next horizontal slice to be processed. It is also quite possible, however, that $\{v_3, h_3\}$ and $\{v_4, h_3\}$ are removed in different iterations of the previous phases; this case, among others, is handled in the following paragraphs.

If $\{\{v_3, h_3\}, \{v_4, h_3\}\} \cap V(X_j)$ is non-empty, take the path component P_j of X_j containing this set; otherwise let P_j be the empty graph. Because of Phase 3, the path traced out by connecting the centroids of the pixels corresponding to the nodes of P_j is without self-intersection. This implies that for any node $e \in V(P)$, its horizontal slice $e \cap M_H$ is at least as far away from the root as h_3. See Figure 8d, for example.

Split the path P_j into two components $P_{j,1}$ and $P_{j,2}$ by deleting $\big\{\{v_3, h_3\}, \{v_4, h_3\}\big\}$ (if it it is not in $E(P_j)$, then one of the components is empty, and the other is P_j), so that $\{v_3, h_3\} \notin V(P_{j,2})$ and $\{v_4, h_3\} \notin V(P_{j,1})$.

- If $|V(P_{j,1})| \not\equiv 0 \pmod 3$ or $|V(P_{j,2})| \not\equiv 0 \pmod 3$, then let Y_j be a minimum size dominating set of P_j containing $\{v_3, h_3\}$ or $\{v_4, h_3\}$ (the size of Y_j is estimated in Section 4.1.3). Set

$$\begin{aligned} Z'_{j+1} &= Y_j \bigcup Z'_j, \\ X_{j+1} &= X_j - P_j, \\ w_{j+1}(e) &= \begin{cases} 0, & \text{if } e \in V(P_j), \\ w_j(e) & \text{if } e \notin V(P_j). \end{cases} \end{aligned}$$

  Clearly, one of $\{v_3, h_3\}$ and $\{v_4, h_3\}$ is contained in $Y_j \subset Z'_{j+1} \subseteq Z'$, and it satisfies the 3rd property of hyperguards for Z' and h_3, h_4.

- If $|V(P_{j,1})| \equiv |V(P_{j,2})| \equiv 0 \pmod 3$, then let Y_j be a minimal dominating set of P_j. Moreover, if $\{\{v_3, h_4\}, \{v_4, h_4\}\} \bigcap (V(X_j) \bigcup Z'_j)$ is non-empty, let f_j be an element of it, otherwise set $f_j = \{v_3, h_4\}$. Take

$$\begin{aligned} Z'_{j+1} &= Y_j \bigcup \{f_j\} \bigcup Z'_j, \\ X_{j+1} &= X_j - P_j - \{f_j\} - N_{X_j}(\{f_j\}), \\ w_{j+1}(e) &= \begin{cases} 0, & \text{if } e \in V(P_j) \bigcup \big\{\{v_3, h_4\}, \{v_4, h_4\}\big\}, \\ w_j(e) + 1, & \text{if } \mathrm{dist}_X(e, f_j) = 2, \\ w_j(e) & \text{otherwise.} \end{cases} \end{aligned}$$

  Observe, that f_j satisfies the 3rd property of hyperguards for Z' and h_3, h_4.

In any case, some element of $Z'_{j+1} \subseteq Z'$ satisfies the 3rd property of hyperguards for Z' and h_3, h_4.

Phase 5 Lastly, we get X_{j_3} which is the disjoint union of paths and isolated nodes (or it is an empty graph). Take a component P_j of X_j (for some $j \geq j_3$). Let Y_j be a dominating set of P_j (if $|V(P_j)| = 1$, then $Y_j = V(P_j)$). Take

$$\begin{aligned} Z'_{j+1} &= Y_j \bigcup Z'_j, \\ X_{j+1} &= X_j - P_j, \\ w_{j+1}(e) &= \begin{cases} 0, & \text{if } e \in V(P_j), \\ w_j(e) & \text{if } e \notin V(P_j). \end{cases} \end{aligned}$$

By repeating this procedure, eventually X_{j_4} is the empty graph for some $j_4 \geq j_3$.

Let $Z' = Z'_{j_4}$. This whole procedure is orchestrated in a way to guarantee that Z' is a hyperguard of M', so only an upper estimate on the cardinality of $\tau(Z')$ needs to be calculated to complete the proof of Section 4.1.

4.1.3 Estimating the size of $Z = \tau(Z')$.

We have

$$|V(X_0)| = r' + s', \ w_0(X) = s', \ |B'_H| = |T'| + |U'|.$$

By definition, $|Z'_1| = c' + |U'| + 2|T'|$ and $|\tau(Z'_1)| = |Z'_1| - |B'_H|$. It is easy to check that

$$|V(X_1)| + w_1(X) + 2|U'| + 5|T'| \leq |V(X_0)| + w_0(X).$$

Therefore, we have

$$\begin{aligned} |Z'_1| + \frac{|V(X_1)| + w_1(X)}{3} &\leq c' + |U'| + 2|T'| + \frac{|V(X_1)| + w_1(X)}{3} \leq \\ &\leq c' + |B'_H| + \frac{|V(X_0)| + w_0(X) - 2|U'| - 2|T'|}{3} \leq \\ &\leq c' + |B'_H| + \frac{r' + 2s' - 2|B'_H|}{3}. \end{aligned} \tag{1}$$

We now show that

$$|Z'_{j+1}| + \frac{|V(X_{j+1})| + w_{j+1}(X)}{3} \leq |Z'_j| + \frac{|V(X_j)| + w_j(X)}{3}. \tag{2}$$

holds for any $j \geq 1$.

In Phase 2, we choose a node from each cycle of X_1. Inequality (2) is preserved, since

$$\begin{aligned} |Z'_{j+1}| &= |Z'_j| + 1, \\ |V(X_{j+1})| &= |V(X_j)| - 5 + \mathbb{1}_{\{4\}}(k_j), \\ w_{j+1}(X) &\leq w_j(X) + 2 - 2 \cdot \mathbb{1}_{\{4\}}(k_j). \end{aligned}$$

In Phase 3, for every $j_2 > j \geq j_1$, we have

$$\begin{aligned} |Z'_{j+1}| &= |Z'_j| + 1, \\ |V(X_{j+1})| &= |V(X_{j_1})| - 5, \\ w_{j+1}(X) &\leq w_j(X) + 2. \end{aligned}$$

Next, we analyze Phase 4. Let $j_3 > j \geq j_2$. If $|V(P_{j,1})| \not\equiv 0 \pmod 3$ and $|V(P_{j,2})| \not\equiv 2 \pmod 3$, then take a minimum size dominating set of P_j containing $\{v_1, h_1\}$. Using Claim 18, we have

$$\begin{aligned} |Y_j| &\leq 1 + \left\lceil \frac{|V(P_{j,1})| - 2}{3} \right\rceil + \left\lceil \frac{|V(P_{j,2})| - 1}{3} \right\rceil \leq \\ &\leq 1 + \frac{|V(P_{j,1})| - 1}{3} + \frac{|V(P_{j,2})|}{3} = \frac{|V(P_j)| + 2}{3}. \end{aligned}$$

Similarly, if $|V(P_{j,1})| \not\equiv 2 \pmod 3$ and $|V(P_{j,2})| \not\equiv 0 \pmod 3$, then there is a small dominating set of P_j containing $\{h_1, v_2\}$. Also, if both $|V(P_{j,1})| \equiv 2 \pmod 3$ and

$|V(P_{j,2})| \equiv 2 \pmod 3$ hold, then there is a small dominating set of P_j containing $\{h_1, v_2\}$. Thus, if $|V(P_{j,1})| \not\equiv 0 \pmod 3$ or $|V(P_{j,2})| \not\equiv 0 \pmod 3$, then

$$\begin{aligned} |Z'_{j+1}| &= |Z'_j| + |Y_j| \le |Z'_j| + \frac{|V(P_j)| + 2}{3}, \\ |V(X_{j+1})| &= |V(X_{j_1})| - |V(P_j)|, \\ w_{j+1}(X) &\le w_j(X) - 2. \end{aligned}$$

If both $|V(P_{j,1})| \equiv 0 \pmod 3$ and $|V(P_{j,2})| \equiv 0 \pmod 3$, then $|Y_j| = \frac{|V(P_j)|}{3}$. Observe, that

$$\{h_1, v_1\}, \{h_1, v_2\}, \{h_2, v_1\}, \{h_2, v_2\} \notin V(P_k) \text{ for any } k < j.$$

If both $\{h_1, v_1\} \notin Z'_j$ and $\{h_1, v_2\} \notin Z'_j$, but were removed in different iterations, then when $\{h_1, v_1\}$ is removed in iteration k we must have set $w_k(\{h_1, v_2\}) = 1$, which is the consequence of the previous observation. Thus, $w_j(\{h_1, v_2\}) = 1$. Similarly, we must have $w_j(\{h_1, v_1\}) = 1$. This reasoning holds for $\{h_2, v_1\}$ and $\{h_2, v_2\}$, as well.

If P_j is not the empty graph or $f_j \in Z(X_j)$, then inequality (2) trivially holds. If P_j is the empty graph, then $w_j(\{h_1, v_1\}) = w_j(\{h_1, v_2\}) = 1$. If $f_j \in V(X_j)$, these 2 extra weights can be used to compensate for the new degree 1 vertices of X_{j+1}. If $f_j \notin Z(X_j) \bigcup V(X_j)$, then even $w_j(\{h_2, v_1\}) = w_j(\{h_2, v_2\}) = 1$, and in total the 4 extra weights compensate for adding f_j to Z'_{j+1}.

In any case, inequality (2) holds for $j_3 > j \ge j_2$.

For any $j_4 > j \ge j_3$, we have

$$|Y_j| \le \left\lceil \frac{|V(P_j)|}{3} \right\rceil \le \frac{|V(P_j)| + 2}{3}$$

and $w_j(P_j) = 2$, so inequality (2) holds for j.

4.1.4 Summing it all up.

By definition, we have

$$|Z'| = |Z'_{j_4}|,\ X_{j_4} = \emptyset,\ 0 \le w_{j_4}(X).$$

Inequality (2) is preserved from Phase 2 up to Phase 5, therefore

$$|Z'| \le |Z'_{j_4}| + \frac{|V(X_{j_4})| + w_{j_4}(X)}{3} \le |Z'_1| + \frac{|V(X_1)| + w_1(X)}{3}.$$

Lastly, using inequality (1), we get

$$
\begin{aligned}
&|Z| = |\tau(Z')| = |\tau(Z' \setminus Z'_1)| + |\tau(Z'_1)| \leq |Z' \setminus Z'_1| + |Z'_1| - |B'_H| = \\
&= |Z'| - |B'_H| \leq c' + \frac{r' + 2s' - 2|B'_H|}{3} = c' + \frac{(c'-4) + 2s' - 2|B'_H|}{3} = \\
&= \frac{4\left(c' + \frac{1}{2}s'\right) - 4 - 2|B'_H|}{3} = \frac{4|V(M')| - 4 - 2|B'_H|}{3} = \\
&= \frac{4|M'_H| + 4|M'_V| - 4 - 2|B'_H|}{3} = \frac{4|M_H| + 4|B_H| + 4|M_V| - 4 - 2|B'_H|}{3} = \\
&= \frac{4(|M_H| + |M_V|) - 4}{3}, \text{ as desired.}
\end{aligned}
$$

4.2 M is connected, but not 2-connected

Let the 2-connected components (or blocks) of M be M_i for $i = 1, \dots, q$. Since induced graphs of G inherit the chordal bipartite property, by Section 4.1, there exists a subset $Z_i \subseteq E(M_i)$, such that for any edge $e_0 \in E(G[N_G(M_i)])$, there exists an edge $e_1 \in Z_i$ which has r-vision of e_0 in $G[N(M_i)]$, and $|Z_i| \leq \frac{4}{3}(|V(M_i)| - 1)$. Let $Z = \cup_{i=1}^{q} Z_i$.

Since the intersection graph of the vertex sets of the 2-connected components is a tree (and any two components intersect in zero or one elements), we have

$$
|Z| \leq \frac{4}{3}\left(-q + \sum_{i=1}^{q} |V(M_i)|\right) = \frac{4\,(-q + |V(M)| + (q-1))}{3} = \frac{4(|V(M)| - 1)}{3}.
$$

Furthermore, given an arbitrary $e_0 = \{v_0, h_0\} \in E(G)$, there exists a $v_1 \in M_V$ and an $h_1 \in M_H$ such that $\{v_1, h_0\}, \{v_0, h_1\} \in E(G)$.

- If $v_0 \in M_V$ or $h_0 \in M_H$, then $\{v_0, h_1\}$ or $\{v_1, h_0\}$ is in $E(M)$.
- Otherwise, there exists a path in M whose endpoints are v_1 and h_1, and this path and the edges $\{v_1, h_0\}$,$\{h_0, v_0\}$,$\{v_0, h_1\}$ form a cycle in G. By the bichordality of G, there exists a C_4 in G which contains an edge of M and e_0.

In any case, e_0 is r-visible from some $e_1 \in E(M)$. As e_1 is an edge of one of the 2-connected components M_i, we have $e_0 \subset N_G(M_i)$, therefore $e_0 \in E(G[N_G(M_i)])$. Thus, some $e_2 \in Z_i$ has r-vision of e_0.

4.3 M has more than one connected component.

Let us take a decomposition of M into connected components M_i for $i = 1, \dots, t$.

Let $N_i = N(M_i)$, so we have $M_i \subseteq N_i$ and $\cup_{i=1}^{t} N_i = V(G)$.

For all $i > 1$ let q_i be the number of components of $G[\cup_{k=1}^{i-1} N_k \setminus \cup_{k=i}^{t} N_k]$ to which $N_i \setminus \cup_{k=i+1}^{t} N_k$ is joined in $G[\cup_{k=1}^{i} N_k \setminus \cup_{k=i+1}^{t} N_k]$. Let $F_{i,j}$ be the set of edges joining $N_i \setminus \cup_{k=i+1}^{t} N_k$ to the j^{th} component of $G[\cup_{k=1}^{i-1} N_k \setminus \cup_{k=i}^{t} N_k]$. Furthermore, let $F_{i,j}^V = \{f \in F_{i,j} \mid f \cap A_V \cap N_i \neq \emptyset\}$ and $F_{i,j}^H = \{f \in F_{i,j} \mid f \cap A_H \cap N_i \neq \emptyset\}$.

Claim 19. *For any two edges $f_1, f_2 \in F^V_{i,j}$ either $f_1 \cap f_2 \neq \emptyset$ or $\exists f_3 \in F^V_{i,j}$ such that f_3 intersects both f_1 and f_2. The analogous statement holds for $F^H_{i,j}$.*

Proof. Suppose f_1 and f_2 are disjoint. Since M_i is connected, there is a path in G whose endpoints are $f_1 \cap N_i$ and $f_2 \cap N_i$, while its internal points are in $V(M_i)$; let the shortest such path be Q_1. There is also a path in the j^{th} component of $G[\cup_{k=1}^{i-1} N_k \setminus \cup_{k=i}^{t} N_k]$ whose endpoints are $f_1 \setminus N_1$ and $f_2 \setminus N_i$, let the shortest one be Q_2.

Now Q_1, f_1, Q_2, f_2 form a cycle in $G[\cup_{k=1}^{i} N_k \setminus \cup_{k=i+1}^{t} N_k]$, which is bipartite chordal. Since $V(Q_2) \cap N_i = \emptyset$, there cannot be a chord between $V(M_i) \cap V(Q_1)$ and $V(Q_2)$. This implies that $|V(Q_1)| = 3$ by its choice, and that either $(f_1 \cap N_i) \cup (f_2 \setminus N_i)$ or $(f_2 \cap N_i) \cup (f_1 \setminus N_i)$ is a chord. □

Claim 20. *For any two edges $f^V \in F^V_{i,j}$ and $f^H \in F^H_{i,j}$, the two-element set*

$$(f^V \cap N_i) \cup (f^H \cap N_i)$$

is an edge of $G[N_i]$.

Proof. Similar to the proof of Claim 19. □

Let $f^V_{i,j} \in F^V_{i,j}$ be the element which intersects the maximum number of edges from $F_{i,j}$, and choose $f^H_{i,j} \in F^H_{i,j}$ in the same way. If only one of these exist, let $w_{i,j}$ be the existing one, otherwise let $w_{i,j} = (f^V_{i,j} \cap N_i) \cup (f^H_{i,j} \cap N_i)$ (as in Claim 20). Let us finally define

$$W = \{w_{i,j} \mid i = 2, \dots, t \text{ and } j = 1, \dots, q_i\}.$$

Claim 21. $|W| = t - 1$.

Proof. Observe that for every $i = 1, \dots, t$, the subgraph $G[N_i \setminus \cup_{k=i+1}^{t} N_k]$ is connected, since $M_i \subseteq N_i \setminus \cup_{k=i+1}^{t} N_k \subseteq N_i = N(M_i)$. Moreover, $G[\cup_{k=1}^{t} N_k] = G$ is connected, therefore $t - 1 = \sum_{i=2}^{t} q_i = |W|$. □

By Section 4.2, there exists a subset $Z_i \subseteq E(M_i)$, such that for any edge $e_0 \in E(G[N_i])$ there exists an edge $e_1 \in Z_i$ which has r-vision of e_0 in $G[N_i]$, and $|Z_i| \leq \frac{4}{3}(|V(M_i)| - 1)$.
Let $Z = W \cup \left(\cup_{i=1}^{t} Z_i\right)$. An easy calculation gives that

$$|Z| \leq (t-1) + \sum_{i=1}^{t} \frac{4|V(M_i)| - 4}{3} \leq \frac{4|V(M)| - 4t + 3(t-1)}{3} \leq$$
$$\leq \frac{4(|M_H| + |M_V| - 1)}{3}.$$

Take an arbitrary edge $e_0 = \{v_0, h_0\} \in E(G)$. We have three cases.

1. If $e_0 \in F^V_{i,j}$ for some i, j, then we claim that $f^V_{i,j}$ has r-vision of e_0. Suppose not; then $f^V_{i,j} \cap e_0 = \emptyset$, and $f_1 := \{v_0\} \cup (f^V_{i,j} \setminus N_i) \notin E(G)$ or $f_2 := \{h_0\} \cup (f^V_{i,j} \cap N_i) \notin E(G)$. By Claim 19 at least one of them is in $E(G)$. Suppose $f_1 \in E(G)$ and $f_2 \notin E(G)$. For any edge $e \in F^V_{i,j}$ intersecting $f^V_{i,j} \cap N_i$, there is

an edge $f(e) \in E(G)$ which intersects both e and e_0. As $f(e) \neq f_2$, we must have $f(e) = (f_{i,j}^V \cap N_i) \cup (e \setminus N_i)$. Furthermore, any edge $g \in F_{i,j}^V$ intersecting $f_{i,j}^V \setminus N_i$ is trivially intersected by f_1 also. Thus, f_1 intersects at least as many edges as $f_{i,j}^V$, and f_1 intersects e_0 too, which contradicts the choice of $f_{i,j}^V$. By symmetry, we are also done if $f_1 \notin E(G)$ and $f_2 \in E(G)$.

If $w_i = f_{i,j}^V$, then w_i trivially has r-vision of e_0. If both $f_{i,j}^V$ and $f_{i,j}^H$ exist, we have two cases.

- If $v_0 \in f_{i,j}^V$, then $v_0 \in w_i$ too, so w_i has r-vision of e_0.
- Otherwise, Claim 20 yields that $\{v_0\} \cup (f_{i,j}^H \cap N_i) \in E(G)$. Also, $f_{i,j}^V$ has r-vision of e_0, so $\left\{ f_{i,j}^H \cap N_i, v_0, h_0, f_{i,j}^V \cap N_i \right\}$ is the vertex set of a C_4 in G, so w_i has r-vision of e_0.

2. If $e_0 \in F_{i,j}^H$ for some i, j, the same argument as above gives that $w_{i,j}$ has r-vision of e_0.

3. If neither of the previous two cases holds, then $e_0 \in E(G[N_i])$ for some i, so some element of Z_i has r-vision of it.

Thus, Z satisfies Theorem 2′, and the proof is complete.

5 Algorithmic aspects

Finding a minimum cardinality horizontal mobile r-guard system, which is also known as the *Minimum cardinality Horizontal Sliding Cameras* or *MHSC* problem, is known to be polynomial [14] in orthogonal polygons without holes. In orthogonal polygons with holes, the problem is *NP*-hard as shown by Biedl, Chan, Lee, Mehrabi, Montecchiani, and Vosoughpour [2]. In their paper, a polynomial time constant factor approximation algorithm for the *MHSC* problem is described, too. As explained in Section 3, the *MHSC* problem translates to the *Total Dominating Set* problem in the pixelation graph (Section 3), which can be solved in polynomial time for chordal bipartite graphs [4].

The minimum cardinality weakly cooperative mobile guard set problem in two-dimensional grids (*MinWCMG* for short) is *NP*-complete [15]. However, Kosowski and Małafiejski also propose a quadratic time algorithm for *MinWCMG* in simple grids. This is exactly the same problem to which we reduce our problem in Section 3.

Finding a minimum cardinality mixed vertical and horizontal mobile r-guard system (also known as the *Minimum cardinality Sliding Cameras* or *MSC* problem) has been shown by Durocher and Mehrabi [6] to be *NP*-hard for orthogonal polygons with holes. For orthogonal polygons without holes, the problem translates to the *Dominating Set* problem in the pixelation graph. This reduction in itself has little use, as Müller and Brandstädt [17] have shown that *Dominating Set* is *NP*-complete even in chordal bipartite graphs. To our knowledge, the complexity of *MSC* is still an open question for orthogonal polygons. There is, however, a polynomial time 3-approximation algorithm by Katz and Morgenstern [14] for the *MSC* problem for x-monotone orthogonal polygons without holes. Also, for an orthogonal polygon of n vertices, a covering set of mobile guards of cardinality at most $\lfloor (3n+4)/16 \rfloor$ (which is the extremal bound shown by Aggarwal [1]) can be found in linear time [12]. In case holes are allowed, [2] give a polynomial time constant factor approximation algorithm.

The algorithm for the *MHSC* problem in [14] relies on a polynomial algorithm solving the *Clique Cover* problem in chordal graphs. Our analysis of the R-tree structures and the pixelation graph allows us to reduce the polynomial running time to linear.

Theorem 22. *The algorithm in Appendix A finds a solution to the MHSC problem in linear time for simple orthogonal polygons.*

Proof. Győri, Hoffmann, Kriegel, and Shermer [11, Section 5] showed that both the horizontal R-tree T_H and the vertical R-tree T_V of D can be constructed in linear time.

The main idea of the algorithm is to only sparsely construct the pixelation graph G of D. Observe, that the neighborhood of a vertical slice in G is a path in T_H, and vice versa. Label each horizontal edge of D by the horizontal slice that contains it. Furthermore, label each vertical edge of each horizontal slice by the edge of D containing it; do this for the horizontal edges of vertical slices as well. This step also takes linear time. The endpoints of a path induced by the neighborhood of any node in G (see Claim 10) can be identified via these labels in $O(1)$ time.

In Section 3, we showed that a horizontal guard system is a subset of $V(T_H)$ which intersects (covers) each element of $\mathscr{F}_H = \{N_G(v) \mid v \in V(T_V)\}$. Dirac's theorem [7, p. 10] states that ν, the maximum number of disjoint subtrees of the family, is equal to τ, the minimum number of nodes covering each subtree of the family. Obviously, $\nu \le \tau$. The other direction is proved using a greedy algorithm:

1. Choose an arbitrary node r of T_H to serve as its root. The distance of a vertical slice $v \in V(T_V)$ from r is $\mathrm{dist}_r(v) = \min_{h \in N_G(v)} \mathrm{dist}(h, r)$, and let $h_r(v) = \arg\min_{h \in N_G(v)} \mathrm{dist}(h, r)$.

2. Enumerate the elements of $V(T_V)$ in decreasing order of their distance from r, let $v_1, v_2, \dots, v_{|V(T_V)|}$ be such an indexing. Let $S_0 = \emptyset$.

3. If $N_G(v_i)$ is disjoint from the elements of $\{N_G(v) \mid v \in S_{i-1}\}$, let $S_i = S_{i-1} \cup \{v_i\}$; otherwise let $S_i = S_{i-1}$.

We claim that $\{h_r(v) \mid v \in S_{|V(T_V)|}\}$ is a cover of $\mathscr{F}_H$. Suppose there exists $v_j \in V(T_V)$ such that $N_G(v_j)$ is not covered. Let i be the smallest index such that $v_i \in S_i$ and $N_G(v_j) \cap N_G(v_i) \neq \emptyset$. Clearly, $i < j$, therefore $\mathrm{dist}_r(v_i) \ge \mathrm{dist}_r(v_j)$. However, this means that $h_r(v_i) \in N_G(v_j)$.

Now $\{h_r(v) \mid v \in S_{|V(T_V)|}\}$ is a cover of the same cardinality as the disjoint set system $\{N_G(v) \mid v \in S_{|V(T_V)|}\}$, proving that $\nu = \tau$.

Each neighborhood $N_G(v)$ for $v \in V(T_V)$ is the vertex set of a path in T_H. Therefore, the first part of the algorithm, including calculating $\mathrm{dist}_r(v)$ and $h_r(v)$ for each v, can be performed in $O(n)$ time, using the off-line lowest common ancestors algorithm of Gabow and Tarjan [8].

Calculating the distance decreasing order takes linear time via breadth-first search started from the root. In the i^{th} step of the third part of the algorithm, we maintain for each node in $V(T_H)$ whether it is under an element of $\{h_r(v) \mid v \in S_i\}$. Summed up for the $|V(T_H)|$ steps, this takes only linear time. $N_G(v_{i+1})$ is disjoint from the elements of $\{N_G(v) \mid v \in S_i\}$ if and only if one of the ends of the path induced by $N_G(v_{i+1})$ is under one of the elements of $\{h_r(v) \mid v \in S_i\}$, which now can be checked in constant time. Thus, the algorithm takes in total some constant factor times the size of the input time to run. □

If we replace the construction of R-trees with sweeps in Algorithm A, the modified algorithm solves the *MinWCMG* problem in simple grids. The two sweeps that recover the R-trees now dominate the increased time complexity of $O(n \log n)$.

The computational complexity of the *Point guard* problem in orthogonal polygons with or without holes has attracted significant interest since the inception of the problem. Schuchardt and Hecker [19] showed that even for orthogonal polygons (without holes), *Point guard* is *NP*-hard. However, a minimum cardinality *Point r-guard* system of an orthogonal polygon can be computed in $\tilde{O}(n^{17})$ time [20]. To our knowledge, the exponent of the running time is still in the double digits, which makes its use impractical. Therefore, approximate solutions to the problem are still relevant. A linear-time 3-approximation algorithm is described in [16].

Corollary 23. *An $\frac{8}{3}$-approximation of the minimum size of a point r-guard system of a simple orthogonal polygon can be computed in linear time.*

Proof. Compute m_V and m_H using the previous algorithm. By Theorem 2 and the trivial statement that both $m_H \leq p$ and $m_V \leq p$, we get that $\frac{4}{3} \cdot (m_H + m_V)$ is an $\frac{8}{3}$-approximation for p. □

Unfortunately, we can only compute the corresponding solution (guard system) in $O(n^2)$, because the pixelation graph may have $\Omega(n^2)$ edges. We consider it an interesting open problem to reduce this running time to linear as well, as such an algorithm would be comparable to the algorithm of Lingas, Wasylewicz, and Żyliński [16].

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Appendix A A linear time algorithm for *MHSC*

```
function SOLVE MHSC(P)
    T_H ← HORIZONTAL R-TREE(P)                          ▷ Algorithm of [11, Section 5]
    T_V ← VERTICAL R-TREE(P)
    for all vertical slice t ∈ T_V do
        a, b ← vertical sides of P bounding t
        h_a ← horizontal slice in V(T_H) containing a
        h_b ← horizontal slice in V(T_H) containing b
        N[t] ← {h_a, h_b}
    end for

    r ← arbitrary node of T_H to serve as root
    dist[] ← BREADTH FIRST SEARCH(T_H, r)                           ▷ distance from r

    LCA[] ← LOWEST COMMON ANCESTORS(T_H, r, N[])                ▷ Algorithm of [8]
            ▷ LCA[t] contains the lowest common ancestors of the elements of N[t]

    S ← ∅
    Set every node of T_H unmarked
    for all t ∈ V(T_V) so that dist[LCA[t]] is not increasing do ▷ reverse BFS-order
        if both elements of N[t] are unmarked then
            S ← S ∪ {LCA[t]}
            SET MARK(LCA[t])
        end if
    end for
    return S
end function

function SET MARK(u)
    mark u
    for all neighbor w of u in T_H do
        if dist[w] > dist[u] and w is unmarked then
            SET MARK(w)
        end if
    end for
end function
```