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# On the hereditary character of new strong variations of Weyl type Theorems

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**Abstract** Berkani and Kachad [18], [19], and Sanabria et al. [32], introduced and studied strong variations of Weyl type Theorems. In this paper, we study the behavior of these strong variations of Weyl type theorems for an operator $T$ on a proper closed and $T$-invariant subspace $W \subseteq X$ such that $T^n(X) \subseteq W$ for some $n \geq 1$, where $T \in L(X)$ and $X$ is an infinite-dimensional complex Banach space. The main purpose of this paper is to prove that for these subspaces (which generalize the case $T^n(X)$ closed for some $n \geq 0$), these strong variations of Weyl type theorems are preserved from $T$ to its restriction on $W$ and vice-versa. As consequence of our results, we give sufficient conditions for which these strong variations of Weyl type Theorems are equivalent for two given operators. Also, some applications to multiplication operators acting on the boundary variation space $BV[0,1]$ are given.

**Keywords** new Weyl-type theorems · strong variations of Weyl type theorems · restrictions of operators · spectral properties · multiplication operators

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## 1 Introduction

In 1909, H. Weyl [35] studied the spectra of compact perturbations for a Hermitian operator and showed that a point belong to spectra of all compact perturbations of the operator if and only if this point is not an isolated point of finite multiplicity in the spectrum of the operator. L. Coburn [21] was one of the first to make a systematic investigation about this result and introduced in abstract form the Weyl's theorem for operators acting on a Banach space. Later, W. Rakočević [29] introduce a stronger property, called $a$-Weyl's theorem. Berkani and Koliha [15] introduced generalized versions for the Weyl's theorems by using some spectra of a new theory of semi B-Fredholm operators given in [13]. After them many authors have introduced and studied a large number of spectral properties associated to an operator by using spectra derived from either Fredholm operators theory or B-Fredholm operators theory (see [2], [4], [6], [10], [14], [16], [17], [25], [31], [33] and [34]). Today all these results are known as Weyl type theorems or Weyl type properties and over the last years there has been a considerable interest to study these properties in operator theory. Recently, Berkani and Kachad [18], [19], and Sanabria et al. [32], introduced and studied new spectral properties related with Weyl type theorems. On the other hand B. Barnes [8](resp. [9]) studied the relationship between some properties of an operator and its extensions (resp. restrictions) on certain superspaces (resp. subspaces) and showed that some Fredholm properties (resp. closed range and generalized inverses) are transmitted from the operator to its extensions (resp. restrictions). In this paper, using the framework offered by Barnes [9] (which extends the context treated by Berkani [11]), we study the behavior of new strong variations of Weyl type theorems for an operator $T$ on a proper closed and $T$-invariant subspace $W \subseteq X$ such that $T^n(X) \subseteq W$ for some $n \geq 1$, where $T$ is a bounded linear operators acting on an infinite-dimensional complex Banach space $X$. The main purpose of this paper is to prove that for these subspaces (which generalize the case $T^n(X)$ closed for some $n \geq 0$, [22], [24]), these strong variations of Weyl type theorems are preserved from $T$ to its restriction on $W$ and vice-versa. As consequence of our results, we obtain sufficient conditions for which these strong variations of Weyl type theorems are equivalent for two given operators. Also, some applications to multiplication operators acting on the boundary variation space $BV[0,1]$ are given.

## 2 Preliminaries

Throughout this paper $L(X)$ denotes the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space $X$. The classes of operators studied in the classical Fredholm theory generate several spectra associated with an operator $T \in L(X)$. The *Fredholm spectrum* is defined by

$$\sigma_{\mathrm{f}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Fredholm}\},$$

and the *upper semi-Fredholm spectrum* is defined by

$$\sigma_{\mathrm{uf}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Fredholm}\}.$$

The *Browder spectrum* and the *Weyl spectrum* are defined, respectively, by

$$\sigma_{\mathrm{b}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Browder}\},$$

and

$$\sigma_{\mathrm{w}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl}\}.$$

Since every Browder operator is Weyl, $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$. Analogously, the *upper semi-Browder spectrum* and the *upper semi-Weyl spectrum* are defined by

$$\sigma_{\mathrm{ub}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Browder}\},$$

and

$$\sigma_{\mathrm{uw}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi-Weyl}\}.$$

For further information on Fredholm operators theory, we refer to [1] and [26]. Another important class of operators is the quasi-Fredholm operators defined in the sequel. Previously, we consider the following set.

$$\Delta(T) = \{n \in \mathbb{N} : m \geq n, m \in \mathbb{N} \Rightarrow T^n(X) \cap N(T) \subseteq T^m(X) \cap N(T)\}.$$

The *degree of stable iteration* is defined as $\mathrm{dis}(T) = \inf \Delta(T)$ if $\Delta(T) \neq \emptyset$, while $\mathrm{dis}(T) = \infty$ if $\Delta(T) = \emptyset$.

**Definition 2.1.** *$T \in L(X)$ is said to be* quasi-Fredholm of degree $d$, *if there exists $d \in \mathbb{N}$ such that:*
(a) *$dis(T) = d$,*
(b) *$T^n(X)$ is a closed subspace of $X$ for each $n \geq d$,*
(c) *$T(X) + N(T^d)$ is a closed subspace of $X$.*

For further information on quasi-Fredholm operators, we refer to [3], [5], [12] and [13].

An operator $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated, SVEP at $\lambda_0$)[23], if for every open disc $\mathbb{D}_{\lambda_0} \subseteq \mathbb{C}$ centered at $\lambda_0$ the only analytic function $f : \mathbb{D}_{\lambda_0} \to X$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0 \quad \text{for all } \lambda \in \mathbb{D}_{\lambda_0},$$

is the function $f \equiv 0$ on $\mathbb{D}_{\lambda_0}$. The operator $T$ is said to have SVEP if $T$ has the SVEP at every point $\lambda \in \mathbb{C}$. Evidently, $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) = \mathbb{C} \setminus \sigma(T)$. Also, the single valued extension property is inherited by restrictions on invariant closed subspaces. Moreover, from the identity theorem for analytic functions it is easily seen that $T$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, $T$ has SVEP at every isolated point of the spectrum. Note that (see [1, Theorem 3.8])

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{2.1}$$

and dually

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda, \tag{2.2}$$

where $p(T)$ is the smallest non-negative integer $p = p(T)$ such that $N(T^p) = N(T^{p+1})$ ( if such an integer does not exist, we put $p(T) = \infty$), and $q(T)$ is the smallest non-negative integer $q = q(T)$ such that $R(T^q) = R(T^{q+1})$ (if such an integer does not exist, we put $q(T) = \infty$).

Recall that $T \in L(X)$ is said to be *bounded below* if $T$ is injective and has closed range. Denote by $\sigma_{\mathrm{ap}}(T)$ the classical *approximate point spectrum* defined by

$$\sigma_{\mathrm{ap}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\}.$$

Note that if $\sigma_{\mathrm{su}}(T)$ denotes the *surjectivity spectrum*

$$\sigma_{\mathrm{su}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not onto}\},$$

then $\sigma_{\mathrm{ap}}(T)=\sigma_{\mathrm{su}}(T^*)$, $\sigma_{\mathrm{su}}(T)=\sigma_{\mathrm{ap}}(T^*)$ and $\sigma(T)=\sigma_{\mathrm{ap}}(T)\cup\sigma_{\mathrm{su}}(T)$.

It is easily seen from definition of localized SVEP, that

$$\lambda\notin \operatorname{acc}\sigma_{\mathrm{ap}}(T)\Rightarrow T \text{ has SVEP at } \lambda, \tag{2.3}$$

and

$$\lambda\notin \operatorname{acc}\sigma_{\mathrm{su}}(T)\Rightarrow T^* \text{ has SVEP at } \lambda, \tag{2.4}$$

where $\operatorname{acc} K$ means the set of all accumulation points of a subset $K\subseteq\mathbb{C}$.

**Remark 2.2.** The implications (2.1), (2.2), (2.3) and (2.4) are actually equivalences, if $T\in L(X)$ is semi-Fredholm (see [1, Chapter 3]). More generally, if $T\in L(X)$ is quasi-Fredholm (see [3]). On the other hand $\sigma_{\mathrm{b}}(T)=\sigma_{\mathrm{w}}(T)\cup\operatorname{acc}\sigma(T)$, $\sigma_{\mathrm{ub}}(T)=\sigma_{\mathrm{uw}}(T)\cup\operatorname{acc}\sigma_{ap}(T)$ and $\sigma(T)=\sigma_{\mathrm{ap}}(T)\cup\Xi(T)$, where $\Xi(T)$ denote the set $\{\lambda\in\mathbb{C}: T \text{ does not have SVEP at } \lambda\}$ (see [1, Chapter 3]).

We denote by $\operatorname{iso} K$, the set of all isolated points of $K\subseteq\mathbb{C}$, by $\alpha(T)=\dim N(T)$ and by $\beta(T)=\operatorname{codim} R(T)$. Let $T\in L(X)$, we define

$$\begin{aligned}
E(T)&=\{\lambda\in\operatorname{iso}\sigma(T): 0<\alpha(\lambda I-T)\},\\
E_a(T)&=\{\lambda\in\operatorname{iso}\sigma_{\mathrm{ap}}(T): 0<\alpha(\lambda I-T)\},\\
\Pi(T)&=\{\lambda\in\mathbb{C}: 0<p(\lambda I-T)=q(\lambda I-T)<\infty\},\\
\Pi_a(T)&=\{\lambda\in\sigma_{\mathrm{ap}}(T): p(\lambda I-T)<\infty \text{ and } R((\lambda I-T)^{p(\lambda I-T)+1}) \text{ is closed}\}.
\end{aligned}$$

$E(T)$ is the set of isolated eigenvalues in $\sigma(T)$, $E_a(T)$ is the set of isolated eigenvalues in $\sigma_{\mathrm{ap}}(T)$, while $\Pi(T)$ is the set of poles of $T$ and $\Pi_a(T)$ is the set of left poles of $T$ (see [19]).

The following definition describes new strong variations of Weyl type theorems introduced recently in [18], [19] and [32].

**Definition 2.3.** *An operator $T\in L(X)$ is said to satisfy property:*

*(i)* $(W_E)$, *if* $\sigma(T)=\sigma_{\mathrm{w}}(T)\cup E(T)$ *([18]);*
*(ii)* $(\bigcup W_{E_a})$, *if* $\sigma_{\mathrm{ap}}(T)=\sigma_{\mathrm{uw}}(T)\cup E_a(T)$ *([18]);*
*(iii)* $(W_\Pi)$, *if* $\sigma(T)=\sigma_{\mathrm{w}}(T)\cup \Pi(T)$ *([19]);*
*(iv)* $(\bigcup W_{\Pi_a})$, *if* $\sigma_{\mathrm{ap}}(T)=\sigma_{\mathrm{uw}}(T)\cup \Pi_a(T)$ *([19]);*
*(v)* $(\bigcup W_{\Pi})$ *if* $\sigma_{\mathrm{ap}}(T)=\sigma_{\mathrm{uw}}(T)\cup \Pi(T)$ *([19]);*
*(vi)* $(\bigcup W_{E})$, *if* $\sigma_{\mathrm{ap}}(T)=\sigma_{\mathrm{uw}}(T)\cup E(T)$ *([19]);*
*(vii)* $(V_E)$, *if* $\sigma(T)\setminus\sigma_{\mathrm{uw}}(T)=E(T)$ *([32]);*
*(viii)* $(V_{E_a})$, *if* $\sigma(T)\setminus\sigma_{\mathrm{uw}}(T)=E_a(T)$ *([32]);*

As in Barnes [9], in the sequel of this paper we always assume that $W$ is a proper closed subspace of a Banach space $X$. Also, we denote

$$\mathcal{P}(X,W)=\{T\in L(X): T(W)\subseteq W \text{ and for some integer } n\geq 1, T^n(X)\subseteq W\}.$$

For each $T\in\mathcal{P}(X,W)$, $T_W$ denote the restriction of $T$ on the subspace $T$-invariant $W$ of $X$. Observe that $0\in\sigma_{\mathrm{su}}(T)$ for all $T\in\mathcal{P}(X,W)$. Later we shall see that $\sigma_{\mathrm{su}}(T)$ and $\sigma_{\mathrm{su}}(T_W)$ may differ only in 0.

**Remark 2.4.** *Observe that an operator $F \in L(W)$ with $n$-dimensional range has the form $F(w) = \sum_{k=1}^{n} f_k(w)F(w_k)$, where $F(w_k) \in W$ and $f_k \in W^*$ ($W^*$ denotes the dual space of $W$) for $k = 1, ..., n$. By the Hahn-Banach Theorem, each $f_k \in W^*$ has an extension $\widehat{f_k} \in X^*$ ($X^*$ denotes the dual space of $X$), then $F$ has an extension $\widehat{F} \in L(X)$, given by $\widehat{F}(x) = \sum_{k=1}^{n} \widehat{f}_k(x)F(w_k)$ for all $x \in X$. Also, $\widehat{F} \in \mathcal{P}(X, W)$ and $\widehat{F}_W = F$.*

We end this section by stating the following lemmas which were proved in [9] and [5], respectively.

**Lemma 2.5.** *[9, Proposition 3] Let $T \in \mathcal{P}(X, W)$. Then $(\lambda I - T)^{-1}(W) = W$, for all $\lambda \neq 0$.* □

**Lemma 2.6.** *[9, Theorem 6(1)] Let $T \in \mathcal{P}(X, W)$. Then for all $\lambda \neq 0$, we have*

$$R(\lambda I - T) \text{ is closed in } X \text{ if and only if } R(\lambda I - T_W) \text{ is closed in } W.$$

□

**Lemma 2.7.** *[5, Lemma 2.3] If $T \in L(X)$ and $p = p(T) < \infty$, then the following statements are equivalent:*

*(i) There exists $n \geq p + 1$ such that $T^n(X)$ is closed;*
*(ii) $T^n(X)$ is closed for all $n \geq p$.*

□

## 3 Relations between the spectra of $T$ and $T_W$

In this section, we establish several lemmas and theorems that we need in the sequel.

**Lemma 3.1.** *Let $T \in L(X)$. Then*

$$(T^n)^{-1}(R(T^{n+m})) = R(T^m) + N(T^n),$$

*for any non-negative integers $n$, $m$.*

*Proof.* It is clear. □

**Lemma 3.2.** *Let $T \in L(X)$. Then $N((\lambda I - T)^m) = T^n(N((\lambda I - T)^m))$ for all $\lambda \neq 0$ and any $n, m \in \mathbb{N}$.*

*Proof.* It follows by mathematical induction. □

The following is a generalization of Lemma 2.6.

**Theorem 3.3.** *Let $T \in \mathcal{P}(X, W)$. Then for all $\lambda \neq 0$, we have*

$$R((\lambda I - T)^m) \text{ is closed in } X \text{ if and only if } R((\lambda I - T_W)^m) \text{ is closed in } W$$

*for any integer $m \geq 1$.*

*Proof.* Observe that if $T \in \mathcal{P}(X, W)$, for any integer $m \geq 1$, we have

$$\begin{aligned}(\lambda I - T)^m &= \sum_{k=0}^{m} \binom{m}{k} (-1)^k \lambda^{m-k} T^k \\ &= \lambda^m I - \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} \lambda^{m-k} T^k. \\ &= \mu I - S,\end{aligned}$$

where $S = \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} \lambda^{m-k} T^k \in \mathcal{P}(X, W)$ and $\mu = \lambda^m \neq 0$.

Similarly,

$$\begin{aligned}(\lambda I - T_W)^m &= \sum_{k=0}^{m} \binom{m}{k} (-1)^k \lambda^{m-k} (T_W)^k \\ &= \lambda^m I - \sum_{k=1}^{m} \binom{m}{k} (-1)^{k+1} \lambda^{m-k} (T_W)^k. \\ &= \mu I - S_W.\end{aligned}$$

From the above equalities and by Lemma 2.6, we obtain

$$R(\mu I - S) \text{ is closed in } X \text{ if and only if } R(\mu I - S_W) \text{ is closed in } W,$$

or equivalently,

$$R((\lambda I - T)^m) \text{ is closed in } X \text{ if and only if } R((\lambda I - T_W)^m) \text{ is closed in } W.$$

□

The next lemma is a generalization of [20, Lemma 2.1], but in the framework dealt by Barnes in [9].

**Lemma 3.4.** *If $T \in \mathcal{P}(X, W)$, then for all $\lambda \neq 0$:*

*(i)* $N((\lambda I - T_W)^m) = N((\lambda I - T)^m)$, *for any $m$;*
*(ii)* $R((\lambda I - T_W)^m) = R((\lambda I - T)^m) \cap W$, *for any $m$;*
*(iii)* $\alpha(\lambda I - T_W) = \alpha(\lambda I - T)$;
*(iv)* $p(\lambda I - T_W) = p(\lambda I - T)$;
*(v)* $\beta(\lambda I - T_W) = \beta(\lambda I - T)$.

*Proof.* The proof is similar to that of [20, Lemma 2.1], making use of Lemma 3.2 in part (i) and Lemma 2.5 in part (ii). □

In the same style as in Lemma 2.6, the following result treats the relationship between the SVEP of an operator $T \in \mathcal{P}(X, W)$ and its restriction $T_W$.

**Lemma 3.5.** *If $T \in \mathcal{P}(X, W)$, then $T$ has SVEP at $\lambda$ if and only if $T_W$ has SVEP at $\lambda$.*

*Proof.* It is easy to see that $T$ (resp. $T_W$) has the SVEP at $\lambda$ if and only if $\lambda I - T$ (resp. $\lambda I - T_W$) has the SVEP at 0. Thus, we may assume without loss of generality $\lambda = 0$. Since the SVEP is inherited by restrictions on invariant closed subspaces, if $T$ has the SVEP at 0 then $T_W$ has the SVEP at 0. Reciprocally, suppose that $T_W$ has the SVEP at 0 and let us consider an open disc $\mathbb{D}_0 \subseteq \mathbb{C}$ centered at 0 and an analytic function $f : \mathbb{D}_0 \to X$ such that $(\mu I - T) f(\mu) = 0$, for all $\mu \in \mathbb{D}_0$. From this it follows that $\mu^k f(\mu) = T^k f(\mu)$, for all $k \in \mathbb{N}$. Consequently, since $T \in \mathcal{P}(X, W)$, there exists $n \geq 1$ such that $T^n(X) \subseteq W$ and so $f(\mu) = \mu^{-n} T^n f(\mu) \in T^n(X) \subseteq W$, for all $\mu \in \mathbb{D}_0 \setminus \{0\}$. On the other hand, if $\mu = 0$ there exists a sequence $(\lambda_k)_{k=1}^{\infty} \subseteq \mathbb{D}_0$, such that $\lambda_k \neq 0$ and $\lambda_k \to 0$. Hence, $(f(\lambda_k))_{k=1}^{\infty} \subseteq W$ and $f(\lambda_k) \to f(0)$. Being $W$ a closed subspace, we conclude that $f(0) \in W$. Therefore $f : \mathbb{D}_0 \to W$ is an analytic function such that $(\mu I - T_W) f(\mu) = 0$ for every $\mu \in \mathbb{D}_0$. From this, by the assumption that $T_W$ has the SVEP at 0, we deduce that $f \equiv 0$ on $\mathbb{D}_0$ and therefore $T$ has the SVEP at 0.

□

As in the above lemma, the following result treats spectral relationships between the operator $T \in \mathcal{P}(X, W)$ and its restriction $T_W$ for several spectra derived from the classical Fredholm theory.

**Theorem 3.6.** *If $T \in \mathcal{P}(X, W)$ and $q(T) = \infty$, or $p(T) = \infty$, then the following equalities are true:*

*(i)* $\sigma_{\mathrm{su}}(T) = \sigma_{\mathrm{su}}(T_W)$;
*(ii)* $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{ap}}(T_W)$;
*(iii)* $\sigma(T) = \sigma(T_W)$;
*(iv)* $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{w}}(T_W)$;
*(v)* $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{uw}}(T_W)$;
*(vi)* $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{b}}(T_W)$;
*(vii)* $\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{ub}}(T_W)$.

*Proof.* (i) Observe first that $\lambda I - T$ (resp. $\lambda I - T_W$) is onto if and only if $\beta(\lambda I - T) = 0$ (resp. $\beta(\lambda I - T_W) = 0$). Now, by Lemma 3.4, $\beta(\lambda I - T) = \beta(\lambda I - T_W)$ for all $\lambda \neq 0$, then $\sigma_{\mathrm{su}}(T) \setminus \{0\} = \sigma_{\mathrm{su}}(T_W) \setminus \{0\}$. To show the equality $\sigma_{\mathrm{su}}(T) = \sigma_{\mathrm{su}}(T_W)$ we need only to prove that $0 \in \sigma_{\mathrm{su}}(T)$ and $0 \in \sigma_{\mathrm{su}}(T_W)$. Since $T \in \mathcal{P}(X, W)$, $0 \in \sigma_{\mathrm{su}}(T)$. We claim that $0 \in \sigma_{\mathrm{su}}(T_W)$. To see this, suppose that $0 \notin \sigma_{\mathrm{su}}(T_W)$. Then $T_W$ is onto, thus $W = (T_W)^k(W) = T^k(W)$ for $k = 0, 1, 2, \ldots$. Being $T \in \mathcal{P}(X, W)$, there exist $n \geq 1$ such that $T^n(X) \subseteq W$, then $W = T^m(W) \subseteq T^m(X) \subseteq T^n(X) \subseteq W$

for all $m \geq n$. Therefore $T^m(X) = T^n(X) = T^m(W) = W$ for all $m \geq n$, which implies that $q(T) < \infty$, contradicting our assumption that $q(T) = \infty$. On the other hand, $T_W$ onto implies that $q(T_W) = 0$, and so $(T_W)^*$ has the SVEP at 0. Hence, $0 \notin \Xi((T_W)^*)$. From this,

$$0 \notin \sigma_{\mathrm{su}}(T_W) \cup \Xi((T_W)^*) = \sigma((T_W)^*) = \sigma(T_W) = \sigma_{\mathrm{ap}}(T_W) \cup \Xi(T_W).$$

Consequently $0 \notin \Xi(T_W)$, that is, $T_W$ has the SVEP at 0. Since $T \in \mathcal{P}(X, W)$, by Lemma 3.5, $T$ has the SVEP at 0. But, as it was observed above, $T \in \mathcal{P}(X, W)$ implies that there exists $n \geq 1$ such that $T^m(X) = T^n(W) = W$ for all $m \geq n$. Then, by the isomorphism $\frac{T^k(X)}{T^{k+1}(X)} \cong \frac{X}{N(T^k)+T(X)}$ ($\forall k \in \mathbb{N}$), given by $T^k x + T^{k+1}(X) \to x + (N(T^k) + T(X))$, we conclude that $X = N(T^m) + T(X)$ for all $m \geq n$. Also $\mathrm{dis}(T) = \inf \Delta(T) \leq n$, because $T^m(X) \cap N(T) = T^n(X) \cap N(T)$ for all $m \geq n$. Thus, $T$ is a quasi-Fredholm operator and $T$ has the SVEP at 0. By [3, Theorem. 2.7], $p(T) < \infty$, contradicting our assumption that $p(T) = \infty$.

(ii) Note first that for each $\lambda \in \sigma_{\mathrm{ap}}(T) \setminus \{0\}$, $\lambda I - T$ is not bounded below and $\lambda \neq 0$. Therefore, we have the following possibilities: $p(\lambda I - T) > 0$ or $R(\lambda I - T)$ is not closed in $X$. But, by Lemmas 3.4 and 2.6, these possibilities are equivalent to $p(\lambda I - T_W) > 0$ or $R(\lambda I - T_W)$ is not closed in $W$. Hence $\sigma_{\mathrm{ap}}(T) \setminus \{0\} = \sigma_{\mathrm{ap}}(T_W) \setminus \{0\}$. As in the part (i), for the equality $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{ap}}(T_W)$, it suffices to show that $0 \in \sigma_{\mathrm{ap}}(T)$ and $0 \in \sigma_{\mathrm{ap}}(T_W)$. Suppose that $0 \notin \sigma_{\mathrm{ap}}(T)$ then $T$ is injective. Consequently $T$ has SVEP at 0, then $0 \notin \Xi(T)$. But, since $\sigma_{\mathrm{ap}}(T) \cup \Xi(T) = \sigma(T) = \sigma_{\mathrm{ap}}(T) \cup \sigma_{\mathrm{su}}(T)$, we have that $0 \notin \sigma_{\mathrm{su}}(T)$, a contradiction. Therefore $0 \in \sigma_{\mathrm{ap}}(T)$. Similarly, $0 \notin \sigma_{\mathrm{ap}}(T_W)$ implies $T_W$ injective. Thus, $T_W$ has SVEP at 0 and $0 \notin \Xi(T_W)$. Again, since $\sigma_{\mathrm{ap}}(T_W) \cup \Xi(T_W) = \sigma(T_W) = \sigma_{\mathrm{ap}}(T_W) \cup \sigma_{\mathrm{su}}(T_W)$, we have that $0 \notin \sigma_{\mathrm{su}}(T_W)$. By part (i), $0 \notin \sigma_{\mathrm{su}}(T)$, and as it has been observed above this is impossible. Then $0 \in \sigma_{\mathrm{ap}}(T_W)$, so the equality $\sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{ap}}(T_W)$ holds.

(iii) To show the equality $\sigma(T) = \sigma(T_W)$, observe that $\sigma(T) = \sigma_{\mathrm{ap}}(T) \cup \sigma_{\mathrm{su}}(T)$ (resp. $\sigma(T_W) = \sigma_{\mathrm{ap}}(T_W) \cup \sigma_{\mathrm{su}}(T_W)$). Hence, combining these equalities with (i) and (ii), we obtain that $\sigma(T) = \sigma(T_W)$.

(iv) Proceeding as in the first part of proofs (i) and (ii), by Lemmas 3.4 and 2.6, we see that $\sigma_{\mathrm{f}}(T) \setminus \{0\} = \sigma_{\mathrm{f}}(T_W) \setminus \{0\}$ and $\sigma_{\mathrm{w}}(T) \setminus \{0\} = \sigma_{\mathrm{w}}(T_W) \setminus \{0\}$. Again, as in the parts (i) and (ii), for the equality $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{w}}(T_W)$, it suffices to show that $0 \in \sigma_{\mathrm{w}}(T)$ and $0 \in \sigma_{\mathrm{w}}(T_W)$. Note first that, if $0 \notin \sigma_{\mathrm{w}}(T)$ then $T$ is a Weyl operator. That is, $T$ is a Fredholm operator with $\mathrm{ind}(T) = 0$. Being $T \in \mathcal{P}(X, W)$, there exists $n \geq 1$ such that $T^n(X) \subseteq W$. From which we obtain the inclusions

$$T^{n+m}(X) \subseteq T^m(W) \subseteq W \subseteq X \quad (\forall m \in \mathbb{N}),$$

and so the inequalities

$$\dim \frac{W}{T^{n+m}(X)} \geq \dim \frac{W}{T^m(W)} = \beta(T_W^m).$$

Since $\frac{W}{T^{n+m}(X)} \subseteq \frac{X}{T^{n+m}(X)}$, then

$$\beta(T^{n+m}) = \dim \frac{X}{T^{n+m}(X)} \geq \dim \frac{W}{T^{n+m}(X)} \geq \dim \frac{W}{T^m(W)} = \beta(T_W^m).$$

Thus, $\beta(T^{n+m}) \geq \beta(T_W^m)$ for any $m \in \mathbb{N}$. On the other hand, the inclusions $N(T_W^m) \subseteq N(T^m) \subseteq N(T^{n+m})$, implies $\alpha(T_W^m) \leq \alpha(T^{n+m})$. Then $T_W^m \in L(W)$ is a Fredholm operator, so $T_W$ is a Fredholm operator. Since $T \in L(X)$ is a Weyl operator, by [26, Proposición 26.2], there exists a bijective operator $R \in L(X)$ and a finite rank operator $K \in L(X)$ such that $T = R + K$. Therefore $T_W = R_W + K_W$, with $R_W$ injective and $K_W$ of finite rank. From this it follows that

$$\text{ind } (T_W) = \text{ind } (R_W + K_W) = \text{ind } (R_W) \leq 0$$

Thus, we conclude that $T_W \in L(W)$ is a upper semi-Weyl operator. Again, by [26, Proposición 26.2], there exists a injective operator $S \in L(W)$ and a finite rank operator $F \in L(W)$ such that $T_W = S + F$. From which $S = T_W - F$. But, since $T_W(W)$ is closed and $F(W)$ is a finite dimensional subspace of $W$, then $S(W)$ is closed in $W$. So $S \in L(W)$ is bounded below, and hence $0 \notin \sigma_{\text{ap}}(S) = \sigma_{\text{ap}}(T_W - F)$. By Remark 2.4, $F \in L(W)$ has an extension $\widehat{F} \in L(X)$ such that $\widehat{F} \in \mathcal{P}(X, W)$, then $T - \widehat{F} \in \mathcal{P}(X, W)$. Consequently, $(T - \widehat{F})_W = T_W - F$. Thus, by part (ii), $0 \in \sigma_{\text{ap}}(T - \widehat{F}) = \sigma_{\text{ap}}((T - \widehat{F})_W) = \sigma_{\text{ap}}(T_W - F)$. That is, $0 \in \sigma_{\text{ap}}(T_W - F)$ and $0 \notin \sigma_{\text{ap}}(T_W - F)$, a contradiction. Hence $0 \in \sigma_{\text{w}}(T)$. Now, we show that $0 \in \sigma_{\text{w}}(T_W)$. To see this, suppose that $0 \notin \sigma_{\text{w}}(T_W) = \sigma_{\text{uw}}(T_W) \cup \sigma_{\text{lw}}(T_W)$. From this it follows that $0 \notin \sigma_{\text{uw}}(T_W)$. That is, $T_W \in L(W)$ is an upper semi-Weyl operator. But, as it has been observed above this is impossible, then $0 \in \sigma_{\text{w}}(T_W)$. Consequently, we obtain the equality $\sigma_{\text{w}}(T) = \sigma_{\text{w}}(T_W)$.

(v) Again, as in the first part of proofs (i) y (ii), by Lemmas 3.4 and 2.6, we have that $\sigma_{\text{uf}}(T)\setminus\{0\} = \sigma_{\text{uf}}(T_W)\setminus\{0\}$ and $\sigma_{\text{uw}}(T)\setminus\{0\} = \sigma_{\text{uw}}(T_W)\setminus\{0\}$. As in the proof of part (iv), to show the equality $\sigma_{\text{uw}}(T) = \sigma_{\text{uw}}(T_W)$ we need only to prove that $0 \in \sigma_{\text{uw}}(T)$ and $0 \in \sigma_{\text{uw}}(T_W)$. By similar representation arguments for semi-Weyl operators as part (iv), we can prove that $0 \in \sigma_{\text{uw}}(T_W)$ and $0 \in \sigma_{\text{uw}}(T)$.

Finally to show parts (vi) and (vii). Observe that $\sigma_{\text{b}}(T) = \sigma_{\text{w}}(T) \cup \text{acc}\,\sigma(T)$ and $\sigma_{\text{b}}(T_W) = \sigma_{\text{w}}(T_W) \cup \text{acc}\,\sigma(T_W)$. Hence, combining these equalities with (iii) and (iv), we obtain that $\sigma_{\text{b}}(T) = \sigma_{\text{b}}(T_W)$. Similarly, combining the equalities $\sigma_{\text{ub}}(T) = \sigma_{\text{uw}}(T) \cup \text{acc}\,\sigma_{\text{ap}}(T)$ and $\sigma_{\text{ub}}(T_W) = \sigma_{\text{uw}}(T_W) \cup \text{acc}\,\sigma_{\text{ap}}(T_W)$ with (ii) and (v), $\sigma_{\text{ub}}(T) = \sigma_{\text{ub}}(T_W)$.

□

**Remark 3.7.** Recall that for $T \in L(X)$, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ precisely when $\lambda$ is a pole of the resolvent of $T$ (see [26, Prop. 50.2]). Also, is well known that if $\lambda$ is a pole of the resolvent of $T$, then $\lambda \in \operatorname{iso}\sigma(T)$. Evidently, if $\lambda \in \operatorname{iso}\sigma(T)$ then $\lambda \in \partial\sigma(T)$. Thus, for $T \in \mathcal{P}(X, W)$, if $0 \notin \operatorname{iso}\sigma(T)$ (resp. $0 \notin \partial\sigma(T)$, $0 \in \Xi(T)$, $0 \in \Xi(T^*)$) then $q(T) = \infty$ or $p(T) = \infty$. Therefore, the conclusions of Theorem 3.6 remain true if the hypothesis $q(T) = \infty$ or $p(T) = \infty$ is replaced by one of the following hypothesis: $0 \notin \operatorname{iso}\sigma(T)$, $0 \notin \partial\sigma(T)$, $0 \in \Xi(T)$ or $0 \in \Xi(T^*)$.

We end this section by giving one illustrative example for the behavior of an operator $T$ and its restriction $T_W$, when $T$ does not satisfy the hypothesis of Theorem 3.6.

**Example 3.8.** *Let $X$ be a Banach space, and assume that $W$ and $Z$ are proper closed subspaces of $X$ with $X = W \oplus Z$. Let $T$ be the projection of $X$ on $W$ which is zero on $Z$. Since $T$ is a projection operator, i.e $T^2 = T$, then $\sigma(T) = \{0, 1\}$. Moreover, $\sigma_{\mathrm{su}}(T) = \sigma_{\mathrm{ap}}(T) = \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{ub}}(T) = \sigma(T)$. On the other hand, the operator $T_W = T|_{T(X)}$ is the identity operator on $W$, so $\sigma(T_W) = \{1\}$. Also, $\sigma_{\mathrm{su}}(T_W) = \sigma_{\mathrm{ap}}(T_W) = \sigma_{\mathrm{w}}(T_W) = \sigma_{\mathrm{uw}}(T_W) = \sigma_{\mathrm{b}}(T_W) = \sigma_{\mathrm{ub}}(T_W) = \sigma(T_W)$.*

## 4 New Strong Properties for $T$ and $T_W$

In this section we present the main applications of this paper. We show that, for all $T \in \mathcal{P}(X, W)$, the strong variations of Weyl type theorems studied in section one are preserved from $T$ to its restriction $T_W$ and vice-versa. Also, we give sufficient conditions for which these strong variations of Weyl type theorems are equivalent for two given operators. Additionally, some applications of our results to multiplication operators acting on the boundary variation space $BV[0, 1]$ are given.

We start with the following results which are crucial for our purposes.

**Lemma 4.1.** *Let $T \in \mathcal{P}(X, W)$. If $0 \notin \operatorname{iso}\sigma(T)$, then the following equalities are true:*

*(i)* $E(T) = E(T_W)$;
*(ii)* $E_a(T) = E_a(T_W)$.

*Proof.* (i) Suppose that $\lambda \in E(T)$, then $\lambda \in \operatorname{iso}\sigma(T)$ and $0 < \alpha(\lambda I - T)$. From the hypothesis $0 \notin \operatorname{iso}\sigma(T)$, by Remark 3.7, we have the equality $\sigma(T) = \sigma(T_W)$. Then $\lambda \in \operatorname{iso}\sigma(T) = \operatorname{iso}\sigma(T_W)$. Additionally, the hypothesis $0 \notin \operatorname{iso}\sigma(T)$ entails that $\lambda \neq 0$. Thus, by Lemma 3.4, $\alpha(\lambda I - T_W) = \alpha(\lambda I - T) > 0$. Therefore $\lambda \in \operatorname{iso}\sigma(T_W)$ and $\alpha(\lambda I - T_W) > 0$, which implies that $\lambda \in E(T_W)$. Consequently $E(T) \subseteq E(T_W)$. Reciprocally, if $\lambda \in E(T_W)$

then $\lambda \in \operatorname{iso}\sigma(T_W)$ and $0 < \alpha(\lambda I - T_W)$. As above, by the hypothesis $0 \notin \operatorname{iso}\sigma(T)$ and Remark 3.7, it then follows that $\sigma(T) = \sigma(T_W)$ and $\lambda \neq 0$. Then $\lambda \in \operatorname{iso}\sigma(T_W) = \operatorname{iso}\sigma(T)$. Again, by Lemma 3.4, $\alpha(\lambda I - T) = \alpha(\lambda I - T_W) > 0$. So $\lambda \in E(T)$, because $\lambda \in \operatorname{iso}\sigma(T)$ and $\alpha(\lambda I - T) > 0$. Hence, we have the inclusion $E(T_W) \subseteq E(T)$.

(ii) Suppose that $\lambda \in E_a(T)$, then $\lambda \in \operatorname{iso}\sigma_{\text{ap}}(T)$ and $0 < \alpha(\lambda I - T)$. In this case, by the hypothesis $0 \notin \operatorname{iso}\sigma(T)$ and Remark 3.7, $\sigma_{\text{ap}}(T) = \sigma_{\text{ap}}(T_W)$ and hence $\lambda \in \operatorname{iso}\sigma_{\text{ap}}(T) = \operatorname{iso}\sigma_{\text{ap}}(T_W)$. We claim that $\lambda \neq 0$. To see this, suppose that $\lambda = 0$. Then $0 \in \operatorname{iso}\sigma_{\text{ap}}(T)$, so there exists $\epsilon > 0$ such that $\mathbb{D}(0;\epsilon) \cap \sigma_{\text{ap}}(T) = \{0\}$. Since $\sigma(T) = \sigma_{\text{ap}}(T) \cup \sigma_{\text{su}}(T)$ and $0 \notin \operatorname{iso}\sigma(T)$, necessarily $0 \notin \operatorname{iso}\sigma_{\text{su}}(T)$. Thus $0 \in \operatorname{acc}\sigma_{\text{su}}(T)$, because $0 \in \sigma_{\text{su}}(T)$. Being $0 \in \operatorname{acc}\sigma_{\text{su}}(T)$, there exists an infinite sequence $(\mu_k)_{k=1}^{\infty} \subseteq \sigma_{\text{su}}(T)$ such that $\mu_k \neq 0$ and $\mu_k \to 0$. This implies that, there exists a non-negative integer $m$ such that $\mu_k \in \mathbb{D}(0;\epsilon)$ for all $k \geq m$. From this $\mu_k \notin \sigma_{\text{ap}}(T)$, then $\mu_k I - T$ is bounded below for $k \geq m$. Thus $\mu_k I - T$ has $SVEP$ at 0, for any $k \geq m$. Since $\mu_k I - T$ is bounded below we have that $\mu_k I - T$ is semi Fredholm which by Remark 2.2, implies that $0 \in \operatorname{iso}\sigma_{\text{ap}}(\mu_k I - T)$. Then $\mu_k \in \operatorname{iso}\sigma_{\text{ap}}(T)$ for all $k \geq m$, a contradiction. Therefore $0 \notin \operatorname{iso}\sigma_{\text{ap}}(T)$, and $\lambda \neq 0$. Consequently, by Lemma 3.4, we have $\alpha(\lambda I - T_W) = \alpha(\lambda I - T) > 0$, so $\lambda \in E_a(T_W)$. This proves the inclusion $E_a(T) \subseteq E_a(T_W)$. Reciprocally, if $\lambda \in E_a(T_W)$ then $\lambda \in \operatorname{iso}\sigma_{\text{ap}}(T_W)$ and $0 < \alpha(\lambda I - T_W)$. As above, by the hypothesis $0 \notin \operatorname{iso}\sigma(T)$ and Remark 3.7, it then follows that $\lambda \in \operatorname{iso}\sigma_{\text{ap}}(T_W) = \operatorname{iso}\sigma_{\text{ap}}(T)$ and $\lambda \neq 0$. Once again, by Lemma 3.4, $\alpha(\lambda I - T) = \alpha(\lambda I - T_W) > 0$. So $\lambda \in E_a(T)$, because $\lambda \in \operatorname{iso}\sigma_{\text{ap}}(T)$ and $\alpha(\lambda I - T) > 0$. Hence, we have the inclusion $E_a(T_W) \subseteq E_a(T)$.

□

**Lemma 4.2.** *Let $T \in \mathcal{P}(X, W)$. If $0 \notin \operatorname{iso}\sigma(T)$, then the following equalities are true:*

*(i)* $\Pi(T) = \Pi(T_W)$;
*(ii)* $\Pi_a(T) = \Pi_a(T_W)$.

*Proof.* (i) Suppose that $\lambda \in \Pi(T)$, then $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ and $\lambda \in \operatorname{iso}\sigma(T)$. Since $0 \notin \operatorname{iso}\sigma(T)$, then $\lambda \neq 0$, and so by Lemma 3.4, $p(\lambda I - T_W) = p(\lambda I - T)$ and $q(\lambda I - T_W) \leq q(\lambda I - T)$. Thus $0 < p(\lambda I - T_W) < \infty$ and $q(\lambda I - T_W) < \infty$, which implies that $0 < p(\lambda I - T_W) = q(\lambda I - T_W) < \infty$ by [26, Prop. 50.2]. That is, $\lambda$ is a pole of $T_W$. In consequence $\lambda \in \Pi(T_W)$, and we have the inclusion $\Pi(T) \subseteq \Pi(T_W)$. Reciprocally, $\lambda \in \Pi(T_W)$ implies that $0 < p(\lambda I - T_W) = q(\lambda I - T_W) < \infty$ and $\lambda \in \operatorname{iso}\sigma(T_W)$. By hypothesis $0 \notin \operatorname{iso}\sigma(T)$ and Remark 3.7, we have that $\sigma(T) = \sigma(T_W)$ and hence $\lambda \in \operatorname{iso}\sigma(T_W) = \operatorname{iso}\sigma(T)$. Thus $\lambda \neq 0$. Again, by Lemma 3.4, we obtain the equalities:

$$\begin{aligned} N(\lambda I - T_W) \cap R((\lambda I - T_W)^m) &= N(\lambda I - T_W) \cap R((\lambda I - T)^m) \cap W \\ &= (N(\lambda I - T_W) \cap W) \cap R((\lambda I - T)^m) \\ &= N(\lambda I - T_W) \cap R((\lambda I - T)^m) \\ &= N(\lambda I - T) \cap R((\lambda I - T)^m), \end{aligned}$$

for any $m$. From this, taking $r = q(\lambda I - T_W) = p(\lambda I - T_W)$ then

$$N(\lambda I - T) \cap R((\lambda I - T)^m) = N(\lambda I - T) \cap R((\lambda I - T)^{m+1}),$$

for all $m \geq r$. On the other hand, since $\lambda$ is a left pole of $T_W$ then $R((\lambda I - T_W)^{r+1})$ is closed in $W$. Moreover, by Lema 2.7, $R((\lambda I - T_W)^m)$ is closed in $W$ for all $m \geq r$. Thus, by Lemma 3.3, $R((\lambda I - T)^m)$ is closed in $X$ for all $m \geq r$. Also, by Lema 3.1, $R(\lambda I - T) + N((\lambda I - T)^{r+1})$ is closed in $X$. Hence $\lambda I - T$ is a quasi-Fredholm operator. But $0 \in \operatorname{iso}\sigma(\lambda I - T)$, because $\lambda \in \operatorname{iso}\sigma(T)$, then $\lambda I - T$ and $(\lambda I - T)^*$ have $SVEP$ at 0, from which both $p(\lambda I - T)$ and $q(\lambda I - T)$ are finite ([3, Theorem. 2.7]). Consequently $p(\lambda I - T) = q(\lambda I - T) < \infty$, but since $0 < p(\lambda I - T) = p(\lambda I - T_W)$, we obtain that $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$. Thus $\lambda$ is a pole of $T$, that is $\lambda \in \Pi(T)$. This proves the inclusion $\Pi(T_W) \subseteq \Pi(T)$.

(ii) Suppose that $\lambda$ is a left pole of $T$, then $\lambda \in \sigma_{\mathrm{ap}}(T)$, $p = p(\lambda I - T) < \infty$ and $R((\lambda I - T)^{p+1})$ is closed in $X$. We claim that under the hypothesis $0 \notin \operatorname{iso}\sigma(T)$, necessarily $\lambda \neq 0$. To see this, suppose that $\lambda = 0$. That is, $p = p(T) < \infty$ and $R(T^{p+1})$ is closed in $X$. Since $p(T) < \infty$, by [26, Proposition 38.1], $T^m(X) \cap N(T) = \{0\} = T^p(X) \cap N(T)$ for all $m \geq p$. Thus $T^m(X) \cap N(T) = T^p(X) \cap N(T)$ for all $m \geq p$. Also, by Lemma 2.7, $R(T^m)$ is closed for any $m \geq p$. On the other hand, by Lemma 3.1, $R(T) + N(T^{p+1})$ is closed in $X$ which implies that $T$ is quasi-Fredholm and has $SVEP$ at 0, because $p(T) < \infty$. Consequently $0 \in \operatorname{iso}\sigma_{\mathrm{ap}}(T)$ ([3, Theorem. 2.7]), but as we proved in part (ii) of Lemma 4.1 this is impossible. Therefore $\lambda \neq 0$. Being $\lambda \in \sigma_{\mathrm{ap}}(T)$, $p = p(\lambda I - T) < \infty$, $R((\lambda I - T)^{p+1})$ closed in $X$ and $\lambda \neq 0$, by Remark 3.7, Lemma 3.4 and Lemma 3.3, we have that $\lambda \in \sigma_{\mathrm{ap}}(T_W)$, $p = p(\lambda I - T_W) < \infty$ and $R((\lambda I - T_W)^{p+1})$ is closed in $W$. That is, $\lambda$ is a left pole of $T_W$, thus $\Pi_a(T) \subseteq \Pi_a(T_W)$. Reciprocally, suppose that $\lambda \in \sigma_{\mathrm{ap}}(T_W)$, $p = p(\lambda I - T_W) < \infty$ and $R((\lambda I - T_W)^{p+1})$ is closed in $W$. If $\lambda = 0$, with analogous arguments as above, we obtain $0 \in \operatorname{iso}\sigma_{\mathrm{ap}}(T_W)$. But since $0 \notin \operatorname{iso}\sigma(T)$, by Remark 3.7, $\sigma_{\mathrm{ap}}(T_W) = \sigma_{\mathrm{ap}}(T)$. Then $0 \in \operatorname{iso}\sigma_{\mathrm{ap}}(T_W) = \operatorname{iso}\sigma_{\mathrm{ap}}(T)$ and as it has been proved before this is impossible. Therefore, $\lambda \neq 0$. Again as above, for $\lambda \neq 0$, by Remark 3.7, Lemma 3.4 and Lemma 3.3, it then follows that $\lambda \in \sigma_{\mathrm{ap}}(T)$, $p = p(\lambda I - T) < \infty$ and $R((\lambda I - T)^{p+1})$ is closed in $X$. Thus, $\Pi_a(T_W) \subseteq \Pi_a(T)$.

□

Now, we are ready to state and prove the main results.

**Theorem 4.3.** *If $T \in \mathcal{P}(X, W)$ and $0 \notin iso\,\sigma(T)$, then property (i) (resp.,(ii)-(viii)) in Definition 2.3 holds for $T$ if and only if property (i)(resp.,(ii)-(viii)) in Definition 2.3 holds for $T_W$.*

*Proof.* It follows by Remark 3.7, Lemma 4.1 and Lemma 4.2. □

Theorem 4.3, may be extended assuming weaker hypotheses as follows.

**Theorem 4.4.** *If $T \in \mathcal{P}(X, W)$ verifies one of the following conditions:*

*(i)* $0 \notin \partial\sigma(T)$,
*(ii)* $0 \in \Xi(T)$,
*(iii)* $0 \in \Xi(T^*)$,

*then property (i) (resp.,(ii)-(viii)) in Definition 2.3 holds for $T$ if and only if property (i) (resp.,(ii)-(viii)) in Definition 2.3 holds for $T_W$.*

*Proof.* It follows by Theorem 4.3 and Remark 3.7. □

As a immediate corollary of Theorem 4.4 and Remark 3.7, we obtain sufficient conditions for which new strong variations of Weyl type Theorems are equivalent for two given operators.

**Corollary 4.5.** *Suppose that $T$, $S \in \mathcal{P}(X, W)$ and $T$, $S$ coincide on $W$. If ones of the following conditions is valid*

*(i)* $0 \notin iso\,\sigma(T) \cup iso\,\sigma(S)$,
*(ii)* $0 \notin \partial\sigma(T) \cup \partial\sigma(S)$,
*(iii)* $0 \in \Xi(T) \cap \Xi(S)$,
*(iv)* $0 \in \Xi(T^*) \cap \Xi(S^*)$,

*then property (i) (resp.,(ii)-(xiii)) in Definition 2.3 holds for $T$ if and only if property (i)(resp.,(ii)-(xiii)) in Definition 2.3 holds for $S$.* □

Recently, Astudillo-Villalba and Ramos-Fernández [7] characterized invertibility, compactness and closedness of the range for multiplication operators acting on the space of functions of bounded variation $BV[0, 1]$. We give applications of our results for these class of operators.

**Corollary 4.6.** *Let $BV[0, 1]$ be the space of functions of bounded variation on $[0, 1]$. Suppose that $u \in BV[0, 1]$ and consider the multiplication operator induced by $u$, $M_u : BV[0, 1] \to BV[0, 1]$ given by $M_u(f) = u \cdot f$. If $Z_u = \{t \in [0, 1] : u(t) = 0\}$ is an infinite set and $X_{Z_u} = \{f \in BV[0, 1] : f(t) = 0, \forall t \in Z_u\} \neq \emptyset$, then property (i) (resp.,(ii)-(viii)) in Definition 2.3 holds for $M_u$ if and only if property (i)(resp.,(ii)-(viii)) in Definition 2.3 holds for its restriction on the subspace $X_{Z_u}$.*

*Proof.* Astudillo-Villalba and Ramos-Fernández proved that [7, Proposition 6], if $X_{Z_u} = \{f \in BV[0, 1] : f(t) = 0, \forall t \in Z_u\} \neq \emptyset$, then $X_{Z_u}$ is a proper closed $M_u$-invariant subspace of $BV[0,1]$ such that $M_u(BV[0,1]) \subset X_{Z_u}$. That is, $M_u \in \mathcal{P}(BV[0,1], X_{Z_u})$. Therefore, by Theorem 4.3, we can concluded that property (i) (resp.,(ii)-(viii)) in Definition 2.3 holds for $M_u$ if and only if property (i)(resp.,(ii)-(viii)) in Definition 2.3 holds for its restriction on the subspace $X_{Z_u}$.

□

**Corollary 4.7.** *If $u, v \in BV[0,1]$ are symbols such that $u \neq v$ but $Z_u = Z_v$, then property (i) (resp.,(ii)-(viii)) in Definition 2.3 holds for $M_u$ if and only if property (i)(resp.,(ii)-(viii)) in Definition 2.3 holds for $M_v$.*

*Proof.* If $u, v \in BV[0,1]$ are symbols such that $u \neq v$ but $Z_u = Z_v$, then $X_{Z_u} = X_{Z_v}$. Thus, taking $W = X_{Z_u} = X_{Z_v}$. By Corollary 4.5, we have that property (i) (resp.,(ii)-(viii)) in Definition 2.3 holds for $M_u$ if and only if property (i)(resp.,(ii)-(viii)) in Definition 2.3 holds for $M_v$.

□

**Remark 4.8.** Similar results as Corollaries 4.6 and 4.7, can be established for composition operators and integral operators by using our results.

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