Iterative algorithms for approximating solutions of variational inequality problems and monotone inclusion problems

by

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Dissertation submitted in fulfilment of the requirements for the degree of
Master of Science

The University of KwaZulu-Natal

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March, 2017.

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As the candidate's supervisor, I have approved this dissertation for submission.

Dr. O. T. Mewomo

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Abstract

In this work, we introduce and study an iterative algorithm independent of the operator norm for approximating a common solution of split equality variational inequality problem and split equality fixed point problem. Using our algorithm, we state and prove a strong convergence theorem for approximating an element in the intersection of the set of solutions of a split equality variational inequality problem and the set of solutions of a split equality fixed point problem for demicontractive mappings in real Hilbert spaces. We then considered finite families of split equality variational inequality problems and proposed an iterative algorithm for approximating a common solution of this problem and the multiple-sets split equality fixed point problem for countable families of multivalued type-one demicontractive-type mappings in real Hilbert spaces. A strong convergence result of the sequence generated by our proposed algorithm to a solution of this problem was also established. We further extend our study from the frame work of real Hilbert spaces to more general p-uniformly convex Banach spaces which are also uniformly smooth. In this space, we introduce an iterative algorithm and prove a strong convergence theorem for approximating a common solution of split equality monotone inclusion problem and split equality fixed point problem for right Bregman strongly nonexpansive mappings. Finally, we presented numerical examples of our theorems and applied our results to study the convex minimization problems and equilibrium problems.

Acknowledgements

I am highly grateful to God Almighty who has always crowned my efforts with success and out of his infinite mercy, has provided me with life, good health and strength to pursue this program to the end.

My most profound gratitude goes to my mother for her financial support and encouragement throughout my academic life. I also thank other members of my family for their endless supports and encouragement.

I am heartily thankful to my supervisor Dr. O. T. Mewomo, who gave me this great work and whose guidance and fatherly directions helped me to get through it. His resilience in resolving challenging research problems were motivational and contagious for me throughout my program.

I am highly indebted to Mr. F. U. Ogbuisi for devoting his time, energy and intelligence to criticize, guide and make constructive contributions to this work and to Mr. C. C. Okeke for his endless supports, love and care throughout my program.

Special thanks to the University of KwaZulu Natal (UKZN), for her financial support and hospitality during my stay in UKZN. My thanks goes to Princess Bavuyile Nhlangulela, the administrator, Howard College, School of Mathematics, Statistics and Computer Science, UKZN and the entire staff of the school for their kindness and hospitality as well.

I wish to acknowledge the National Mathematical Center (NMC), Abuja for funding my flight ticket during my travel to UKZN. I also acknowledge the financial support, encouragement and guidance from Professor U. A. Osisiogu and Dr. J. N. Ezeora right from my undergraduate up to M.Sc level. Their fatherly supports and contributions have been the driving force that has sustained me even during tough times of my M.Sc pursuit. My appreciation also goes to Prof. E. U. Ofuedu, Dr. M. S. Minjibir, Mrs. T. E. Efor, Mr. F. L. Adum, Mr. L. A. Ogbonna, Mr. E. E. Otubo and the entire staff in the Department of Mathematics and Computer Science, Ebonyi State University, Nigeria.

My wholehearted gratitude goes to my colleagues: Mebawondu Akindele Adebawo, Olanipekun Peter, Abass Ahmed, Jolaoso Lateef, Adiele Ugochukwu Oliver and Abinu Matthew. Special thanks to my loved ones; Nnakwe Ezinne Vivian, Eze Francisco Chinedu, Eze Henry, Akanimo David, Amino David, Ogudu Harrison, Nnakwe Monday, Ogonnaya Michael Ogbonna, Ann Oliver, Otu Jane, Okafor Chuma, Okafor Ann, Okafor Paul, Okafor Chichi and Igwe Obinna. Thank you all for each moment of support and encouragement. Your love and care will remain indelible in my memory.

Dedication

This dissertation is dedicated to God Almighty and to my beloved mother.

Contents

Declaration

This dissertation has not been submitted to this or any other institution in support of an application for the award of a degree. It represents the author's own work and where the work of others has been used, proper reference has been made.

Izuchukwu Chinedu

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Chapter 1

Introduction

1.1 Background of study

Let C be a nonempty subset of a real Hilbert space H and $A : H \to H$ be a nonlinear operator, then the Variational Inequality Problem (VIP) is the problem of finding $x \in C$ such that

$$\langle A(x), y - x \rangle \geq 0 \ \forall y \in C. \tag{1.1.1}$$

The first problem in the form of a variational inequality problem was the Signorini problem which was posed in 1959 by Signorini (see [9], [53]) and was solved in 1963 by Fichera [54]. In order to study the regularity problem for partial differential equations, Stampacchia [111] studied a generalization of the Lax-Milgram theorem and called all problems involving inequalities of such kind, the VIPs. The VIP was later found to have applications in many fields such as mechanics, optimization, nonlinear programming, economics, finance, applied sciences, among others. As a result of this, the theory of variational inequalities became an area of great research interest, thus great deal of research efforts were invested to study the VIPs and their generalizations in both finite and infinite dimensional spaces by numerous authors.

A useful and important generalization of the VIP is the Monotone Inclusion Problem (MIP) which is the problem of finding a point $x \in H$ such that

$$0 \in B(x), \tag{1.1.2}$$

where $B : H \to 2^H$ is a maximal monotone operator.

Various methods for solving VIPs and MIPs have been developed and studied by numerous authors, these methods includes fixed point methods, proximal-like methods, auxiliary principles, decomposition techniques, extra-gradient methods and normal map equations (for example, see [5, 7, 66, 76, 77, 78, 121]). In recent years, these two problems have been studied in diverse directions by using these methods. The fixed point methods are known to be one of the most effective methods for finding solutions of VIPs and MIPs. As a result of this, intensive research efforts have been devoted in developing different techniques for finding solutions of VIPs and MIPs using the fixed point methods.

Let H be a real Hilbert space, a point $x \in H$ is called a fixed point of a nonlinear operator $T : H \to H$ if

$$Tx = x. \tag{1.1.3}$$

If T is a multivalued mapping, then $x \in H$ is called a fixed point of T if $x \in Tx$. The fixed point theory is known to be one of the most flourishing areas of research in nonlinear analysis and can be considered the kernel of the modern nonlinear analysis due to the role it plays in diverse mathematical models arising from optimization problems and differential equations. Recent development of efficient techniques for computing fixed points has enormously increased the usefulness of the theory of fixed points for applications. Thus, fixed point theory is increasingly becoming a powerful and effective tool in applied mathematics. In a wide range of mathematical problems, the existence of a solution is equivalent to the existence of a fixed point for a suitable map. The existence of a fixed point is therefore of paramount importance in several areas of mathematics and other sciences. For example, consider the following nonlinear ordinary differential equation

$$x'(t) = f(t, x(t)), \qquad\qquad x(t_0) = x_0. \tag{1.1.4}$$

Finding a solution of (1.1.4) is equivalent to finding a solution of the equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds. \tag{1.1.5}$$

To establish the existence of solution of problem (1.1.4), we consider the operator $T : C([a, b]) \to C([a, b])$ defined by

$$Tx = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

Then we have that any x which solves (1.1.5) is a fixed point of the operator T. Thus finding a solution of (1.1.4) is equivalent to finding a fixed point of T. However, the existence theorems only involves the establishment of sufficient conditions under which a given problem has a solution, but does not tell us how to find such solution (see [67, 71, 74] and the references therein).

On the other hand, the iterative method is concerned with the approximation or computation of sequences which converges to fixed points of nonlinear operators and solutions of (1.1.1) and (1.1.2). There are several of such approximations in literature, feasible iterative algorithms for approximating solutions of (1.1.1), (1.1.2) and (1.1.3) have been studied by many researchers. This method is our major concern in this dissertation and it will be well treated in subsequent chapters.

1.2 Research motivation

Ansari *et al.* [6] introduced and studied the Split Hierarchical Variational Inequality Problem (SHVIP) for two strongly nonexpansive operators in real Hilbert spaces, for which one of the operators is in addition cutter. They proposed an algorithm for finding a solution of the SHVIP and proved that the sequence generated by their algorithm converges weakly to a solution of the SHVIP.

Censor *et al.* [33] introduced the general Common Solutions to Variational Inequality Problem (CSVIP) which consists of finidng common solutions to unrealated variational inequalities for finite number of sets in real Hilbert spaces.

Shehu [104] studied the Multiple-Set Split Equality Fixed Point Problem (MSSEFPP) for infinite families of multivalued quasi-nonexpansive mappings in real Hilbert spaces and established strong convergence result for the MSSEFPP using his proposed algorithm.

Chidume *et al.* [40] studied the Split Equality Fixed Point Problem (SEFPP) for demi-contractive mappings in real Hilbert spaces. Using their proposed algorithm, they established weak convergence result for the SEFPP and obtain strong convergence result by imposing the demi-compactness condition on the demi-contractive mappings considered by them. Chidume *et al.* [41] also studied the MSSEFPP for countable families of multivalued demi-contractive mappings which are more general than mutivalued quasi-nonexpansive mappings considered by Shehu [104].

Motivated by the works of Ansari *et al.* [6] and Chidume *et al.* [40], we study a split-type problem by combining a Split Equality Variational Inequality Problem (SEVIP) and a SEFPP for demi-contractive mappings. Also motivated by Zhoa [125], we introduce an iterative algorithm independent of the operator norm to approximate a common solution of the combined problem and obtained strong convergence result without imposing the demi-compactness condition on the demi-contractive mappings.

Further motivated by the works of Censor *et al.* [33], Shehu [104] and Chidume *et al.* [41], we extend our result to finite families of SEVIPs and to MSSEFPP for countable families of multi-valued type-one demicontractive-type mappings introduced by Isiogugu *et al.* [65].

The idea of accretive operators introduced by Browder [19] in 1967 has proved to be very useful in partial differential equations. Consider for example, an initial value problem of the form

$$\frac{dx}{dt} + Ax(t) = 0, \quad x(0) = x_0, \tag{1.2.1}$$

which describes an evolution system where A is an accretive map from a Banach space E into itself. At equilibrium state, $\frac{dx}{dt} = 0$ and a solution of

$$Ax = 0 \tag{1.2.2}$$

describes the equilibrium state of the system. Since generally A is nonlinear, there is no closed form solution of equation (1.2.2). Thus to approximate a solution of (1.2.2), Browder [19] converted (1.2.2) to a Fixed Point Problem (FPP). He called an operator

$T = I - A$ pseudo-contractive, where A is accretive and I is the identity map defined on E. We then see that any zero of A is a fixed point of T. Thus, finding a solution of (1.2.2) amounts to finding a fixed point of T (see [37]). Riech and Sabach [96] studied problem (1.2.2) in a case where A is a multivalued maximal monotone operator defined on a reflexive real Banach space E and obtained strong convergence result. These motivates our study on split equality monotone inclusion problems and SEFPPs in real Banach spaces.

1.3 Statement of problem

In this dissertation, we studied the following problems in a unified manner:

- Let H_1, H_2, H_3 be real Hilbert spaces and $C_1,\ C_2$ be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $T : C_1 \to C_1$ and $S : C_2 \to C_2$ be demicontractive mappings with $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$. Let $f_i : C_i \to C_i$ $(i = 1, 2)$ be ρ-inverse strongly monotone operators $(\rho > 0)$ and $A : H_1 \to H_3,\ B : H_2 \to H_3$ be bounded linear operators. Find $(\bar{x}, \bar{y}) \in F(T) \times F(S)$ such that

$$\langle f_1(\bar{x}),\ x - \bar{x}\rangle \geq 0, \quad \forall x \in C_1,$$

$$\langle f_2(\bar{y}),\ y - \bar{y}\rangle \geq 0, \quad \forall y \in C_2$$

  and

$$A\bar{x} = B\bar{y}.$$

- Let H_1, H_2, H_3 be real Hilbert spaces and for $l = 1, 2, \ldots, N,\ r = 1, 2, \ldots, m$, let C_l and Q_r be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $T_i : H_1 \to CB(H_1),\ i = 1, 2, \ldots$ and $S_j : H_2 \to CB(H_2),\ j = 1, 2, \ldots$ be two countable families of multi-valued type-one demicontractive-type mappings with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $\cap_{j=1}^{\infty} F(S_j) \neq \emptyset$. Let $f_l : C_l \to C_l,\ h_r : Q_r \to Q_r$ be α_l, (respectively, μ_r)-inverse strongly monotone operators and $A : H_1 \to H_3,\ B : H_2 \to H_3$ be bounded linear operators. Find $(\bar{x}, \bar{y}) \in \cap_{i=1}^{\infty} F(T_i) \times \cap_{j=1}^{\infty} F(S_j)$ such that

$$\langle f_l(\bar{x}),\ x - \bar{x}\rangle \geq 0,\ \forall\ x \in C_l,\ l = 1, 2, \ldots, N,$$

$$\langle h_r(\bar{y}),\ y - \bar{y}\rangle \geq 0,\ \forall\ y \in Q_r,\ r = 1, 2, \ldots, m, \text{ and } A\bar{x} = B\bar{y}.$$

- Let E_1, E_2, E_3 be three real Banach spaces and $A : E_1 \to E_3,\ B : E_2 \to E_3$ be bounded linear operators. Let $M_1 : E_1 \to 2^{E_1^*},\ M_2 : E_2 \to 2^{E_2^*}$ be multivalued maximal monotone mappings and $T : E_1 \to E_1,\ S : E_2 \to E_2$ be right Bregman strongly nonexpansive mappings: Find $(\bar{x}, \bar{y}) \in F(T) \times F(S)$ such that

$$0 \in M_1(\bar{x}),$$

$$0 \in M_2(\bar{y}) \text{ and } A\bar{x} = B\bar{y}.$$

1.4 Objectives

The main objectives of this work are to:

(i) review some known and useful results on VIPs, MIPs and SEFPPs,

(ii) introduce and study iterative algorithms for approximating solutions of the problems stated in (i) above,

(iii) establish strong convergence results for these problems using our proposed algorithms,

(iv) apply our results to some optimization problems and present numerical examples of our results.

The rest of this dissertation is organized as follows: In Chapter 2, we recall some basic definitions, concepts, theorems and propositions that will be useful throughout this work and that will serve as the bedrock for the formulation of our main results. Detailed literature review on VIPs, MIPs and SEFPPs were also presented. We also gave some non-trivial examples for better understanding of these concepts. In Chapter 3, a common solution of SEVIP and SEFPP for demicontractive mappings with applications were studied in real Hilbert spaces. Chapter 4 deals with an extension to finite families of SEVIP and to MSSEFPP for countable families of multivalued demicontractive-type mappings in real Hilbert spaces. Chapter 5 is concerned with the approximation of common solution of SEMIP and SEFPP for right Bregman strongly nonexpansive mappings with applications in real Banach spaces. We present in Chapter 6, our conclusion, contribution to knowledge and possible future research.

Chapter 2

Literature Review

In this chapter, we provide definitions of basic terms and concepts that will be useful throughout our work. We also present some useful results and give detailed literature review of concepts that are relevant to our work.

2.1 Preliminaries and definitions

Throughout this section, we shall denote the real Hilbert space by H, the norm and inner product in H by $||.||$ and $\langle . \, , \, . \rangle$ respectively.

2.1.1 Nonlinear single-valued mappings

Definition 2.1.1. *A mapping $T : H \to H$ is said to be*

- *L-Lipschitzian if there exists $L > 0$ such that*
$$||Tx - Ty|| \leq L||x - y|| \ \forall x, y \in H, \tag{2.1.1}$$
- *nonexpansive if*
$$||Tx - Ty|| \leq ||x - y|| \ \forall x, y \in H, \tag{2.1.2}$$
- *quasi-nonexpansive if $F(T) \neq \emptyset$ and*
$$||Tx - y|| \leq ||x - y|| \ \forall x \in H, y \in F(T), \tag{2.1.3}$$
- *firmly quasi-nonexpansive if $F(T) \neq \emptyset$ and*
$$||Tx - y||^2 \leq ||x - y||^2 - ||x - Tx||^2 \ \forall x \in H, y \in F(T), \tag{2.1.4}$$
- *directed (or cutter) if $F(T) \neq \emptyset$ and*
$$\langle Tx - y, Tx - x \rangle \leq 0 \ \forall x \in H, \ y \in F(T), \tag{2.1.5}$$

- *k-strictly pseudo-contractive if there exists a constant $k \in (0,1)$ such that*

$$||Tx - Ty||^2 \le ||x-y||^2 + k||(I-T)x - (I-T)y||^2 \ \forall x, y \in H. \tag{2.1.6}$$

If $k = 1$ in (2.1.6), then T is called a pseudocontractive mapping.

Remark 2.1.2. *It is known that (2.1.5) is equivalent to*

$$||Tx - y||^2 \le ||x-y||^2 - ||Tx - x||^2 \ \forall \ x \in H, \ y \in F(T). \tag{2.1.7}$$

Thus, directed (cutter) mappings and firmly quasi-nonexpansive mappings coincide (see [11]).

Definition 2.1.3. *A mapping $T : H \to H$ is said to be strongly nonexpansive (SNE) if T is nonexpansive and for all bounded sequences $\{x_n\}$ and $\{y_n\}$ in H, $\lim_{n\to\infty}(||x_n - y_n|| - ||Tx_n - Ty_n||) = 0$ implies $\lim_{n\to\infty} ||(x_n - y_n) - (Tx_n - Ty_n)|| = 0$.*

Definition 2.1.4. *A mapping $T : H \to H$ is called α-inverse strongly monotone (or α-cocoercive), if there exists $\alpha > 0$ such that*

$$\langle Tx - Ty, x - y\rangle \ge \alpha||Tx - Ty||^2 \ \forall x, y \in H. \tag{2.1.8}$$

If $\alpha = 1$, then T is called a firmly nonexpansive (FNE) mapping, while T is called a monotone operator if $\alpha = 0$.

Definition 2.1.5. *Let C be a nonempty subset of H and $T : C \to C$ be an operator, T is called hemicontinuous if it is continuous along each line segment in C.*

Remark 2.1.6. *(i) It is obvious that any α-inverse strongly monotone operator is monotone and $\frac{1}{\alpha}$-Lipschitz continuous.*

(ii) Every L-Lipschitz operator is $\frac{2}{L}$-inverse strongly monotone operator.

Definition 2.1.7. *A mapping $T : H \to H$ is said to be averaged nonexpansive (ANE) if $\forall x, y \in H$, $T = (1-\alpha)I + \alpha S$ holds for a nonexpansive operator $S : H \to H$ and $\alpha \in (0,1)$.*

Remark 2.1.8. *(i) In a real Hilbert space, T is firmly nonexpansive if and only if it is averaged with $\alpha = \frac{1}{2}$ (see [91]).*

(ii) FNE $\subset$ ANE $\subset$ SNE $\subset$ QNE, where QNE means quasi-nonexpansive.

Definition 2.1.9. *A mapping $T : H \to H$ is called a demicontractive mapping if $F(T) \neq \emptyset$ and there exists a constant $k \in (0,1)$ such that*

$$\|Tx - y\|^2 \le \|x - y\|^2 + k||x - Tx||^2 \quad \forall x \in H, \ y \in F(T). \tag{2.1.9}$$

It is known that (2.1.9) is equivalent to

$$\langle Tx - y, x - y\rangle \le \|x - y\|^2 - \frac{1-k}{2}\|x - Tx\|^2. \tag{2.1.10}$$

Remark 2.1.10. *If inequality (2.1.6) holds only for $y \in F(T)$, then T is called a quasi-strictly pseudocontractive mapping. Therefore, the notion of quasi-strictly pseudocontractivity coincides with the notion of demicontractivity.*

We now give some examples of demicontractive mappings.

Example 2.1.11. *Let $H = \mathbb{R}$ be endowed with the usual metric and $T : \mathbb{R} \to \mathbb{R}$ be defined by*

$$T(x) = -\left(\frac{2\alpha+1}{2}\right)x, \quad \forall \alpha > \frac{1}{2}.$$

Then T is a demicontractive mapping with constant $k = \frac{4\alpha^2+4\alpha-3}{(2\alpha+3)^2}$.

Example 2.1.12. *Let $H = l_2(\mathbb{R})$ and $T : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ be defined by*

$$T(x_1, x_2, x_3, \dots) = -\left(\frac{2\alpha+1}{2}\right)(x_1, x_2, x_3, \dots), \forall \alpha > \frac{1}{2}.$$

Then T is a demicontractive mapping.

To see this, recall that $l_2(\mathbb{R}) := \{\bar{x} = (x_1, x_2, x_3, \dots),\ x_i \in \mathbb{R} : \sum_{i=1}^{\infty} x_i^2 < \infty\}$.
With this definition, we first show that T is well defined (that is, we show that for each $x \in l_2(\mathbb{R}),\ T(x) \in l_2(\mathbb{R})$).
Now, for each $x \in l_2(\mathbb{R})$, we have that

$$\sum_{i=1}^{\infty}\left[-\left(\frac{2\alpha+1}{2}\right)x_i\right]^2 = \left(\frac{2\alpha+1}{2}\right)^2\sum_{i=1}^{\infty}x_i^2 < \infty.$$

Hence, $T(x_1, x_2, x_3, \dots) = -\left(\frac{2\alpha+1}{2}\right)(x_1, x_2, x_3, \dots) \in l_2(\mathbb{R})$ and thus T is well defined.

Observe that $F(T) = \{0\}$ and for $x \in l_2(\mathbb{R})$, we have

$$\begin{aligned} ||Tx - 0||^2 &= ||-\left(\frac{2\alpha+1}{2}\right)x - 0||^2 \\ &= \left(\frac{2\alpha+1}{2}\right)^2 ||x-0||^2 \\ &= ||x-0||^2 + \left(\frac{4\alpha^2+4\alpha-3}{4}\right)||x-0||^2. \end{aligned} \tag{2.1.11}$$

Also,

$$\begin{aligned} ||x - Tx||^2 &= ||x + \frac{2\alpha+1}{2}x||^2 \\ &= \left(\frac{2\alpha+3}{2}\right)^2 ||x-0||^2. \end{aligned} \tag{2.1.12}$$

Using (2.1.11) *and* (2.1.12), *we have*

$$||Tx - 0||^2 = ||x - 0||^2 + \left(\frac{4\alpha^2 + 4\alpha - 3}{(2\alpha + 3)^2}\right) ||x - Tx||^2,$$

which implies that T is demicontractive with constant $k = \frac{4\alpha^2+4\alpha-3}{(2\alpha+3)^2}$.

Remark 2.1.13. *By restricting $l_2(\mathbb{R})$ to $\mathbb{R}$, one can use the same argument as in Example 2.1.12 to show that T defined in Example 2.1.11 is a demicontractive mapping.*

We now state the relationships that exists between demicontractive mappings and some other nonlinear mappings in Hilbert spaces:
Firmly nonexpansive mappings (with nonempty fixed points set) $\subset$ nonexpansive mappings (with nonempty fixed points set) $\subset$ quasi-nonexpansive mappings $\subset$ demicontractive mappings.

Nonexpansive mappings (with nonempty fixed points set) $\subset$ k-strictly pseudocontractive mappings (with nonempty fixed points set) $\subset$ Lipschiptz pseudocontractive mappings (with nonempty fixed points set) $\subset$ demicontractive mappings.
We give some examples to show that the above inclusions are proper.

Example 2.1.14. *[85]. Let $H = \mathbb{R}^2$ with the usual norm and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a function defined by*

$$T(x, y) = (-y, x), \ \forall (x, y) \in \mathbb{R}^2. \tag{2.1.13}$$

Then T is a nonexpansive mapping, but T is not firmly nonexpansive. To see that T is not firmly nonexpansive, take $x = (1, 2)$ and $y = (4, 6)$.

Example 2.1.15. *Let $H = \mathbb{R}$ with the usual norm and $T : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$T(x) = \begin{cases} \frac{x}{2}\cos(\frac{1}{x}), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} \tag{2.1.14}$$

then T is a quasi-nonexpansive mapping but not nonexpansive.

Observe that $F(T) = \{0\}$. Then, for all $x \in (-\infty, 0) \cup (0, \infty)$, we have

$|Tx - 0| = |\frac{x}{2}\cos(\frac{1}{x}) - 0| = \frac{1}{2}|x - 0| < |x - 0|$. Hence, T is quasi-nonexpansive.

However, if we take $x = \frac{1}{\pi}$ and $y = \frac{1}{2\pi}$, we have that

$|Tx - Ty| = |\frac{1}{2\pi}\cos(\pi) - \frac{1}{4\pi}\cos(2\pi)| = \frac{3}{4\pi}$.

Also, $|x - y| = |\frac{1}{\pi} - \frac{1}{2\pi}| = \frac{1}{2\pi} < \frac{3}{4\pi}$. Hence, T is not nonexpansive.

Example 2.1.16. *Let $H = l_2(\mathbb{R})$ and $T : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ be defined by*

$$T(x_1, x_2, x_3, \dots) = -(\alpha + 1)(x_1, x_2, x_3, \dots), \ \forall \alpha > 0.$$

Then T is a demicontractive mapping but not quasi-nonexpansive mapping.

Observe that $F(T) = \{0\}$, thus we can follow the argument in Example 2.1.12 to show that T is a demicontractive mapping for any $\alpha > 0$.
However, for arbitrary $x \in l_2(\mathbb{R})$, we have

$$||Tx - 0||^2 = || - (\alpha + 1)x - 0||^2 = (\alpha + 1)^2||x - 0||^2,$$

which implies that T is not quasi-nonexpansive for any $\alpha > 0$.

Example 2.1.17. *[60]. Let $H = \mathbb{R}$ with the usual norm and $C = [-2, 0]$. Let $T : C \to C$ be defined by*

$$Tx = \begin{cases} x^2 - 2, & \text{if } x \in [-1, 0], \\ -\frac{1}{8}, & \text{if } x = -\frac{3}{2}, \\ -1, & x \in [-2, -\frac{3}{2}) \cup (-\frac{3}{2}, -1]. \end{cases} \tag{2.1.15}$$

Then T is a demicontractive mapping with $k = \frac{3}{4}$. However, T is neither pseudocontractive nor quasi-nonexpansive.

2.1.2 Nonlinear multivalued mappings

Definition 2.1.18. *Let (X, d) be a metric space and 2^X be the family of all subsets of X. Let $\mathcal{H}$ denote the Hausdorff metric induced by the metric d, then for all $A, B \in 2^X$,*

$$\mathcal{H}(A, B) = \max\{\sup_{a \in A} d(a, B), \ \sup_{b \in B} d(b, A)\}, \tag{2.1.16}$$

where $d(a, B) := \inf_{b \in B} d(a, b)$.

Definition 2.1.19. *Let $T : H \to 2^H$ be a multi-valued mapping, then $P_T x := \{u \in Tx : ||x - u|| = d(x, Tx)\}$.*

Example 2.1.20. *Let $H = \mathbb{R}$ (endowed with the usual metric) and $T : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by*

$$Tx = \begin{cases} \left[-5x, \ -\frac{9}{2}x\right], & x \in [0, \infty), \\ \\ \left[-\frac{9}{2}x, \ -5x\right], & x \in (-\infty, 0), \end{cases}$$

then $P_T x = \{-\frac{9}{2}x\}$. To see this, observe that $d(x, Tx) = |x - \left(-\frac{9}{2}\right) x| = |x - u|$, where $u = -\frac{9}{2}x \in Tx$.

Definition 2.1.21. *A multi-valued mapping $T : H \to 2^H$ is said to be L-Lipschitzian if there exists $L > 0$ such that*

$$\mathcal{H}(Tx, Ty) \leq L||x - y||, \ \forall x, y \in H. \tag{2.1.17}$$

In (2.1.17), if $L \in (0, 1)$, then T is called a contraction while T is called nonexapansive if $L = 1$.

Definition 2.1.22. *A multivalued mapping $T : H \to 2^H$ is said to be*

- *of type-one if*

$$||u - v|| \leq \mathcal{H}(Tx, Ty) \ \forall x, y \in H, \ u \in P_T x, \ v \in P_T y, \tag{2.1.18}$$

- *quasi-nonexpansive if $F(T) \neq \emptyset$ and*

$$\mathcal{H}(Tx, Ty) \leq ||x - y||, \ \forall x \in H, y \in F(T), \tag{2.1.19}$$

- *k-strictly pseudocontractive in the sense of Browder and Petryshyn[20] if there exists $k \in [0, 1)$ such that $\forall x, y \in H$ and $u \in Tx$, there exists $v \in Ty$ satisfying $||u - v|| \leq \mathcal{H}(Tx, Ty)$ and*

$$\mathcal{H}^2(Tx, Ty) \leq ||x - y||^2 + k||x - u - (y - v)||^2. \tag{2.1.20}$$

If $k = 1$ in (2.1.20), then T is called a pseudocontractive-type mapping while T is called a nonexpansive-type mapping if $k = 0$.

Remark 2.1.23. *[64].*

(i) Every multivalued nonexpansive mapping is a nonexpansive-type mapping.

(ii) Every multivalued nonexpansive-type mapping is k-strictly psuedocontractive-type mapping. However, the converse of this is not always true.

(iii) Every k-strictly pseudocontractive-type mapping is pseudocontractive-type mapping. The converse of this statement is not true always.

Definition 2.1.24. *Let $T : H \to CB(H)$ be a multivalued mapping, then T is called k-strictly pseudocontractive mapping in the sense of [38] if there exists $k \in (0, 1)$ such that $\forall x, y \in H$ one has*

$$\mathcal{H}^2(Tx, Ty) \leq ||x - y||^2 + k||x - u - (y - v)||^2, \ \forall u \in Tx, \ v \in Ty. \tag{2.1.21}$$

If $k = 1$, then T is called a pseudocontractive mapping.

We now give the definition of multivalued demicontractive-type mappings which are more general than the multi-valued quasi-nonexpansive mappings and are also related to the multivalued k-strictly pseudocontractive and pseudocontractive-type mappings (see [39] for more information).

Definition 2.1.25. *Let $T : H \to 2^H$ be a multivalued mapping, then T is called demicontractive in the sense of Hicks and Kubices [61] if $F(T) \neq \emptyset$ and for all $y \in F(T)$, $x \in H$, there exists $k \in [0, 1)$ such that*

$$\mathcal{H}^2(Tx, Ty) \leq ||x - y||^2 + kd^2(x, Tx), \tag{2.1.22}$$

where $\mathcal{H}^2(Tx, Ty) = [\mathcal{H}(Tx, Ty)]^2$ and $d^2(x, y) = [d(x, y)]^2$.

Clearly, every multivalued quasi-nonexpansive mapping is a multivalued demicontractive mapping. We give the following example to show that the converse of this statement is not always true.

Example 2.1.26. *Let $H = \mathbb{R}$ (endowed with the usual metric) and $T : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by*

$$Tx = \begin{cases} \left[-(\alpha+1)x, \ -\frac{2\alpha+1}{2}x\right], & x \in [0, \infty), \\ \\ \left[-\frac{2\alpha+1}{2}x, \ -(\alpha+1)x\right], & x \in (-\infty, 0), \ \forall \ \alpha > 0. \end{cases}$$

Then $F(T) = \{0\}$. For each $x \in (-\infty, 0) \cup (0, \infty)$, we have

$$\begin{aligned} \mathcal{H}^2(Tx, T0) = |-(\alpha+1)x - 0|^2 &= (\alpha+1)^2|x-0|^2 \\ &= |x-0|^2 + (\alpha^2+2\alpha)|x-0|^2. \end{aligned} \tag{2.1.23}$$

Also,

$$d^2(x, Tx) = \left|x + \frac{2\alpha+1}{2}x\right|^2 = \left(\frac{2\alpha+3}{2}\right)^2 |x-0|^2,$$

which implies

$$|x-0|^2 = \frac{4}{(2\alpha+3)^2} d^2(x, Tx). \tag{2.1.24}$$

Substituting (2.1.24) into (2.1.23), we obtain

$$\mathcal{H}^2(Tx, T0) = |x-0|^2 + \frac{4(\alpha^2+2\alpha)}{(2\alpha+3)^2} d^2(x, Tx),$$

which implies that T is a demicontractive multivalued mapping with $k = \frac{4(\alpha^2+2\alpha)}{(2\alpha+3)^2} \in (0, 1)$, $\forall \ \alpha > 0$. However, (2.1.23) implies that T is not a quasi-nonexpansive multivalued mapping. Hence, the class of quasi-nonexpansive multivalued mappings is properly contained in the class of demicontractive multivalued mappings.

Remark 2.1.27. *[61]. Let $T : H \to 2^H$ be any multivalued mapping such that $F(T) \neq \emptyset$. If P_T is k-strictly pseudocontractive-type mapping, then P_T is a demicontractive-type mapping. However, the following example shows that the converse of this statement is not always true.*

Example 2.1.28. *Let $H = \mathbb{R}$ be endowed with the usual metric. Define $T : [-1, 1] \to 2^{[-1,1]}$ by*

$$Tx = \begin{cases} [-1, \ \frac{2}{3}x \sin\frac{1}{2}], & x \in (0, 1], \\ \{0\}, & x = 0, \\ [\frac{2}{3}x \sin\frac{1}{x}, 1], & x \in [-1, 0). \end{cases} \tag{2.1.25}$$

Then $F(T) = \{0\}$. For each $x \in [-1, 1]$, we have that

$$P_T x = \begin{cases} \{\frac{2}{3}x \sin\frac{1}{x}\}, & x \neq 0, \\ \{0\}, & x = 0, \end{cases} \tag{2.1.26}$$

is a demicontractive-type multivalued mapping but not k-strictly pseudocontractive-type multivalued mapping (see, for example [61]).

2.1.3 The metric projection on Hilbert spaces

Definition 2.1.29. *Let C be a nonempty, closed and convex subset of H. The metric (or nearest point) projection onto C is the mapping $P_C : H \to C$ which assigns to each $x \in H$ the unique point $P_C x$ in C such that*

$$||x - P_C x|| = \inf\{||x - y|| \ : \ y \in C\}. \tag{2.1.27}$$

Example 2.1.30. *Let $x \in \mathbb{R}$ and $C = [-\alpha, \ \alpha]$ for $\alpha > 0$, we define a map $P_C : \mathbb{R} \to C$ by*

$$P_C(x) = \begin{cases} x, & \text{if } x \in C, \\ \frac{\alpha x}{|x|}, & \text{otherwise.} \end{cases}$$

Then P_C is the metric projection onto C.

To see this, observe that if $x \in C$, then $|x - P_C x| = 0 \leq |x - y| \ \forall y \in C$.

Also observe that if $x \notin C$, it implies that $|x| > \alpha$, i.e., $x < -\alpha$ or $x > \alpha$.
Now, for $x < -\alpha$, we have
$|x - P_C x| = \left|x - \frac{\alpha x}{|x|}\right| = |x - (-\alpha)|$.
Observe that for any $y \in [-\alpha, \alpha]$, $x - (-\alpha) \geq x - y$. Also, since $x - (-\alpha) < 0$, we have that $x - y < 0$. Hence,
$|x - (-\alpha)| \leq |x - y|$, which implies that $|x - P_C x| \leq |x - y| \ \forall y \in [-\alpha, \ \alpha]$.

Similarly, for $x > \alpha$, we have that
$|x - P_C x| = \left|x - \frac{\alpha x}{|x|}\right| = |x - \alpha| \leq |x - y| \ \forall y \in [-\alpha, \alpha]$.

Hence, we obtain that $|x - P_C x| \leq |x - y| \ \forall y \in C, \ x \in \mathbb{R}$. Thus, P_C is the metric projection onto C.

Example 2.1.31. *We now list some examples of metric projections in Hilbert spaces.*

1. *Let $C = \{x \in H : ||x - x_0|| \leq r\}$, that is C is a closed ball centered at $x_0 \in H$ with radius $r > 0$, then*

$$P_C x = \begin{cases} x_0 + r\frac{(x - x_0)}{||x - x_0||}, & \text{if } \ ||x - x_0|| > r; \\ \\ x, & \text{otherwise} \end{cases} \tag{2.1.28}$$

   is the metric projection onto C.

   We note that if C is a closed ball in $\mathbb{R}$ centered at the origin and $r = \alpha$, then the example above reduces to Example 2.1.30.

2. *Let* $C = [a, b]$ *be a closed rectangle in* $\mathbb{R}^n$*, where* $a = (a_1, a_2, \ldots, a_n)^T$ *and* $b = (b_1, b_2, \ldots, b_n)^T$*, then for* $1 \leq i \leq n$*, then*

$$(P_C x)_i = \begin{cases} a_i, & x_i < a_i, \\ x_i, & x_i \in [a_i, b_i], \\ b_i, & x_i > b_i \end{cases} \tag{2.1.29}$$

is the metric projection with the i^{th} *coordinate.*

3. *Let* $C = \{y \in H : \langle a, y \rangle \leq \alpha\}$ *be a closed halfspace, with* $a \neq 0$ *and* $\alpha \in \mathbb{R}$*, then*

$$P_C x = \begin{cases} x - \frac{\langle a,x \rangle - \alpha}{||a||^2} a, & \text{if } \langle a, x \rangle > \alpha, \\ x, & \text{if } \langle a, x \rangle \leq \alpha \end{cases} \tag{2.1.30}$$

is the metric projection onto C*.*

We note that if "$\leq$*" is replaced with "*$=$*" in the definition of* C *above, then* C *becomes a hyperplane and we have that*

$$P_C x = x - \frac{\langle a, x \rangle - \alpha}{||a||^2} a \tag{2.1.31}$$

is the metric projection onto C*.*

4. *Let* C *be the range of an* $m \times n$ *matrix* A *with full column rank and* A^* *be the adjoint of* A*, then*

$$P_C x = A(A^* A)^{-1} A^* x \tag{2.1.32}$$

is the metric projection P_C *onto* C*.*

Proposition 2.1.32. *(Characterization of metric projections, see [10]). Let* $x \in H$*, then*

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \ \forall y \in C. \tag{2.1.33}$$

The following are consequences of Proposition 2.1.32:

(i) The metric projection is firmly nonexpansive, that is $||P_C x - P_C y||^2 \leq \langle x - y, P_C x - P_C y \rangle \ \forall x, y \in H$,

(ii) $||x - P_C x||^2 \leq ||x - y||^2 - ||y - P_C x||^2 \ \forall x \in H$ and $y \in C$,

(iii) if C is a closed subspace of H, then P_C coincides with the orthogonal projection from H onto C; that is, $\forall x \in H$, $x - P_C x$ is orthogonal to C.

Proposition 2.1.33. *[33]. Let* C *be a nonempty, closed and convex subset of* H *and* $f : H \to H$ *be an* α*-inverse strongly monotone operator on* H*. If* $\lambda \in (0, 2\alpha)$*, then the operator* $P_C(I - \lambda f)$ *is averaged.*

Remark 2.1.34. *It follows from Remark 2.1.8 and Proposition 2.1.33 that* $P_C(I - \lambda f)$ *is firmly nonexpansive with* $\alpha = \frac{1}{2}$*.*

We shall see in Chapter 3 and Chapter 4 that the operator $P_C(I - \lambda f)$ plays crucial role in establishing the proofs of our main theorems. Thus Remark 2.1.34 is very important to our study.

2.1.4 The resolvent of maximal monotone operators

Definition 2.1.35. *A set-valued operator $B : H \to 2^H$ is called maximal monotone if B is monotone, i.e.,*

$$\langle u - v, x - y\rangle \geq 0 \ \forall x, y \in H, \ u \in B(x) \ \text{ and } v \in B(y),$$

and the graph $G(B)$ defined by

$$G(B) := \{(x, y) \in H \times H : \ y \in B(x)\}$$

is not properly contained in the graph of any other monotone operator.

Definition 2.1.36. *Let $B : H \to 2^H$ be a maximal monotone operator. The resolvent of B with parameter $\lambda > 0$ is denoted and defined by $J_\lambda^B := (I + \lambda B)^{-1}$, where I is the identity operator.*

The resolvent operator is known to be very useful in the study of MIPs. We shall see in Section 2.4 that the resolvent of a maximal monotone operator plays crucial role in approximating solutions of MIPs, thus the algorithms proposed by the authors mentioned in Section 2.4 depends heavily on resolvent operators. The following proposition brings out the relationship between the fixed point of a resolvent operator and the solution set of MIP (1.1.2).

Proposition 2.1.37. *[25]. Let $B : H \to 2^H$ be any set-valued operator and J_λ^B be the resolvent of B with parameter $\lambda > 0$. Then we have the following.*

(i) If B is a maximal monotone operator, then a point $\bar{x} \in H$ is a fixed point of J_λ^B if and only if $\bar{x} \in B^{-1}(0) := \{x \in H : 0 \in B(x)\}$.

(ii) B is monotone if and only if the resolvent J_λ^B is single-valued and firmly nonexpansive.

(iii) B is maximal monotone if and only if J_λ^B is single-valued, firmly nonexpansive and $dom(J_\lambda^B) = H$.

2.1.5 Fejér monotone sequences

Definition 2.1.38. *Let C be a nonempty, closed and convex subset of H. The sequence $\{x_n\}_{n\geq 1}$ in H is said to be Fejér monotone with respect to C if*

$$||x_{n+1} - x|| \leq ||x_n - x|| \ \forall n \in \mathbb{N}, \ x \in C.$$

Example 2.1.39. *[12]. Let $H = \mathbb{R}^2$, $C = \{0\} \times \mathbb{R}$ and $x_n = ((-1)^n, 0)\,, \ \forall n \geq 1$. Then $\{x_n\}_{n\geq 1}$ is a Fejér monotone sequence with respect to C.*

In this case, we have that

$$||x_n - x|| = ||x_{n+1} - x|| \ \forall x \in C.$$

Example 2.1.40. *[12]. Let $H = \mathbb{R}^2$, $C = \{(0,0)\}$, $y_n = \sum_{k=1}^{n} \frac{1}{k}$ and $x_n = \cos(y_n)(1,0) + \sin(y_n)(0,1)$. Then $\{x_n\}_{n\geq 1}$ is a Fejér monotone sequence with respect to C.*

We now give some basic properties of Fejér monotone sequences which are widely used in establishing convergence results in fixed point theory.

Proposition 2.1.41. *[12]. Let $\{x_n\}_{n\geq 1}$ be a Fejér monotone sequence in H with respect to $C \subset H$ and P_C be the metric projection onto C, then the following holds;*

(i) the sequence $\{x_n\}_{n\geq 1}$ is bounded,

(ii) for each $x \in C$, the sequence $\{||x_n - x||\}_{n\geq 1}$ converges,

(iii) the sequence $\{P_C x_n\}_{n\geq 1}$ converges strongly to a point in C,

(iv) if int $C \neq \emptyset$, then $\{x_n\}_{n\geq 1}$ converges strongly to a point in H,

(v) every weak cluster point of $\{x_n\}_{n\geq 1}$ that belongs to C must be $\lim_{n\to\infty} P_C x_n$,

(vi) the sequence $\{x_n\}_{n\geq 1}$ converges weakly to some point in C if and only if all weak cluster points of $\{x_n\}_{n\geq 1}$ lie in C,

(vii) if all weakly cluster points of $\{x_n\}_{n\geq 1}$ lie in C, then the $\{x_n\}_{n\geq 1}$ converges weakly to $\lim_{n\to\infty} P_C x_n$.

Remark 2.1.42. *We shall see in subsequent chapters that the sequences generated by our algorithms are Fejér monotone sequences. Consequently, we apply Proposition 2.1.41(i) to obtain the boundedness of these sequences.*

2.2 Geometric properties of Banach spaces

The Hilbert space is known to have the most simplest and clearly discernible geometric structure among all Banach spaces. Some of the geometric properties that characterizes Hilbert spaces and makes problems in Hilbert spaces more manageable than those in general Banach spaces includes (see [37]); the availability of the inner product, the nonexpansivity property of the nearest point map defined on a real Hilbert space H onto a closed convex subset C of H and the following two identities which holds for all $x, y \in H$, $\lambda \in (0,1)$,

$$||x + y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2 \tag{2.2.1}$$

and

$$||\lambda x + (1-\lambda)y||^2 = \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)||x-y||^2. \tag{2.2.2}$$

It follows from (2.2.1) that

$$2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x-y||^2 = ||x+y||^2 - ||x||^2 - ||y||^2, \quad \forall\, x, y \in H. \tag{2.2.3}$$

We note that the parallelogram identity $||x+y||^2 + ||x-y||^2 = 2\left(||x||^2 + ||y||^2\right)$ is equivalent to inequality (2.2.1). Also, we shall see in chapter 3 and chapter 4 that inequalities (2.2.2) and (2.2.3) play important role in the proofs of our main results.

Observe that (2.2.1) does not make sense in a general Banach space due to lack of inner product, this makes working in general Banach space more difficult than in Hilbert spaces. However, most real life problems do not occur in Hilbert spaces. Therefore, to overcome these challenges, researchers use the concept of the duality mappings which are considered as one of the most important canonical operators in Banach spaces. The duality mapping can be seen as a suitable analogue of the inner product in Hilbert spaces. Beside the introduction of the duality mappings, the distance function $\Delta(.,.)$ (called the Bregman distance, which was introduced by Bregman [18]) were used instead of the operator norm to make computations in Banach spaces less difficult to handle. In Chapter 5, we shall use the Bregman distance for our computations and we shall also see that the duality mapping plays central role in establishing our result. In what follows, we give definitions of some spaces more general than Hilbert spaces and the connections between them.

2.2.1 Uniformly convex spaces

Definition 2.2.1. *A normed linear space E is said to be uniformly convex if for each $\epsilon \in (0,2]$ there exists a $\delta_\epsilon > 0$ such that, for all $x, y \in E$ with $||x|| = 1 = ||y||$ and $||x-y|| \geq \epsilon$, then $||\frac{x+y}{2}|| \leq 1 - \delta_\epsilon$.*

Equivalently, E is uniformly convex if for any $\epsilon \in (0,2]$ there exists $\delta = \delta_\epsilon > 0$ such that if $x, y \in E$ with $||x|| \leq 1$, $||y|| \leq 1$ and $||x-y|| \geq \epsilon$, then $||\frac{1}{2}(x+y)|| \leq 1-\delta$.

Example 2.2.2. *[37]. The L_p spaces, $1 < p < \infty$ are uniformly convex.*

Definition 2.2.3. *Let dim $E \geq 2$, then the modulus of convexity of a normed linear space E is the function*

$$\delta_E : (0,2] \to [0,1]$$

defined by

$$\delta_E(\epsilon) := \inf\{1 - ||\frac{x+y}{2}|| : ||x|| = ||y|| = 1, \epsilon = ||x-y||\}.$$

Definition 2.2.4. *Let $p > 1$ be a real number. A normed linear space E is said to be p-uniformly convex if there exists $C_p > 0$ such that*

$$\delta_E(\epsilon) \geq C_p \epsilon^p, \text{ for any } \epsilon \in (0,2].$$

Definition 2.2.5. *A normed linear space E is called strictly convex if for all $x, y \in E$, $x \neq y$, $||x|| = ||y|| = 1$, we have $||\lambda x + (1-\lambda)y|| < 1 \ \forall \lambda \in (0,1)$.*

Theorem 2.2.6. *[37]. A normed linear space E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$.*

2.2.2 Smooth spaces

Throughout this work, we denote the paring $\langle \xi, x\rangle$ by the action of $\xi \in E^*$ at $x \in E$, that is, $\langle \xi, x\rangle := \xi(x)$.

Definition 2.2.7. *A normed linear space E is called smooth if for every $x \in E$ with $||x|| = 1$, there exists a unique $x^* \in E^*$ such that $||x^*|| = 1$ and $\langle x^*, x\rangle = ||x||$.*

Definition 2.2.8. *Let E be a real Banach space and $S(E) = \{x \in E : ||x|| = 1\}$. Then the norm of E is said to be Gâteaux differentiable if*

$$\lim_{t\to 0} \frac{||x + ty|| - ||x||}{t}$$

exists for each $x, y \in S(E)$. In this case E is called smooth.

Definition 2.2.9. *A normed linear space E is said to be uniformly smooth if for any given $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E$ with $||x|| = 1$ and $||y|| \leq \delta$, then*

$$||x + y|| + ||x - y|| < 2 + \epsilon||y||.$$

Definition 2.2.10. *Let E be a normed linear space with dim $E \geq 2$, then the modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by*

$$\rho_E(t) := \sup \Big\{\frac{1}{2}(||x + y|| + ||x - y||) - 1 : ||x|| \leq 1, ||y|| \leq t\Big\}. \tag{2.2.4}$$

Proposition 2.2.11. *[37]. A normed linear space E is uniformly smooth if and only if*

$$\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0.$$

Definition 2.2.12. *For $q > 1$, a Banach space E is said to be q-uniformly smooth if there exists $C_q > 0$ such that*

$$\rho_E(t) \leq C_q t^q \text{ for any } t > 0.$$

2.2.3 Reflexive Banach spaces

Definition 2.2.13. *Let E^* and E^{**} be the dual and the bidual of a Banach space E respectively. Then there exists a canonical (or canonical embedding) mapping $\mathbf{J} : E \to E^{**}$ defined, for each $x \in E$ by*

$$\mathbf{J}(x) = \Phi_x \in E^{**},$$

where

$$\Phi_x : E^* \to \mathbb{R} \text{ is defined by}$$

$$\langle \Phi_x, f\rangle = \langle f, x\rangle, \text{ for each } f \in E^*.$$

Thus, $\langle \mathbf{J}(x), f\rangle \equiv \langle f, x\rangle$ for each $f \in E^*$. If the canonical mapping $\mathbf{J}$ is an onto mapping, then E is called reflexive. Thus, a reflexive Banach space is a Banach space in which the canonical embedding is onto.

Remark 2.2.14. *[36]. The canonical mapping* **J** *defined above have the following properties.*

(i) **J** *is linear,*

(ii) **J** *is an isometry, i.e.,* $||\mathbf{J}x|| = ||x|| \; \forall x \in E$*.*

Remark 2.2.15. *[37].*

(i) Every uniformly convex space is strictly convex.

(ii) Every uniformly convex space is reflexive.

(iii) Every uniformly smooth space is smooth.

(iv) E *is uniformly smooth if and only if* E^* *is uniformly convex.*

(v) If the dual space E^* *is reflexive, then* E *is reflexive.*

From Remarks 2.2.15 (iv), 2.2.15(ii) and 2.2.15(v), we have the following important remark.

Remark 2.2.16. *Every uniformly smooth space is reflexive.*

We now study the notion of the duality mapping and some of it properties in the following subsection.

2.2.4 The duality mapping

Definition 2.2.17. *Let* E *be a real Banach space, then for each* $p > 1$*, the duality mapping* $J_p : E \to 2^{E^*}$ *is defined by*

$$J_p(x) = \{x^* \in E^* : \langle x, x^* \rangle = ||x||.||x^*||, ||x^*|| = ||x||^{p-1}\}.$$

If $p = 2$*, then* $J_p = J_2 = J$ *and this is called the normalized duality mapping on* E*.*

Definition 2.2.18. *The duality mapping* J_p *is said to be weak-to-weak continuous if* $x_n \rightharpoonup x \implies \langle J_p x_n, y\rangle \to \langle J_p x, y\rangle$ *holds for any* $y \in E$*.*

We note that l_p $(p > 1)$ spaces has this property, but L_p $(p > 2)$ does not posses this property. For $1 < q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have the following important remark.

Remark 2.2.19. *[4] . It is known that* E *is* p*-uniformly convex and uniformly smooth if and only if* E^* *is* q*-uniformly smooth and uniformly convex. In this case, the duality mapping* J_p *is one-to-one, single valued and satisfies* $J_p = (J_q^*)^{-1}$*, where* J_q^* *is the duality mapping of* E^**.*

Proposition 2.2.20. *[37]. Let* E *be a normed linear space and* J *be the normalized duality mapping on* E*, then the following holds.*

(i) *$J(x)$ is a nonempty, closed, convex and bounded subset of E^*.*

(ii) $J(\lambda x) = \lambda J(x), \ \forall x \in E, \ \lambda \in \mathbb{R}$.

(iii) *If E is a real uniformly smooth Banach space, then J is norm-to-norm uniformly continuous on bounded subsets of E.*

(iv) *J is the identity map on E if E is a real Hilbert space.*

In what follows, we discuss some notions in Banach spaces that are useful to our study. For the rest of this section, we shall denote the real Banach space by E.

2.2.5 Some notions in Banach spaces

Definition 2.2.21. *Let C be a nonempty, closed and convex subset of E. Let $f : E \to (-\infty, +\infty]$ be any function, then the domain of f is defined as*

$$dom\ f := \{x \in E : f(x) < +\infty\}.$$

The function f is called proper if dom $f \neq \emptyset$.

Definition 2.2.22. *Let $f : D \subset E \to (-\infty, +\infty]$ be any mapping. Then f is said to be convex if D is a convex set and for each $\alpha \in [0, 1]$, $x_1, x_2 \in D$, we have*

$$f(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha f(x_1) + (1 - \alpha)f(x_2). \tag{2.2.5}$$

Definition 2.2.23. *Let $f : D \subset E \to (-\infty, +\infty]$ be any mapping and for arbitrary $x_0 \in E$, let $U(x_0)$ be the set of all neighbourhoods of x_0. Then f is called a lower semi-continuous function at x_0 if and only if*

$$\forall \lambda \in \mathbb{R} \text{ suth that } \lambda < f(x_0), \ \exists V \in U(x_0) : f(x) > \lambda \ \forall x \in V.$$

Proposition 2.2.24. *A function $f : E \to (-\infty, +\infty]$ is lower semi-continuous at $x_0 \in X$ if whenever $\{x_n\}$ is a sequence in E such that $x_n \to x_0$, as $n \to \infty$, then*

$$f(x_0) \leq \limsup_{n\to\infty} f(x_n).$$

Definition 2.2.25. *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable convex function, then the Bregman distance with respect to f is defined as*

$$\Delta_f(x, y) = f(y) - f(x) - \langle f'(x), y - x\rangle, \ \forall x, y \in E,$$

where f' is the Gâteaux derivative of f.

Remark 2.2.26. *[106]. The duality mapping J_p is the derivative of the function $f_p(x) = (\frac{1}{p})||x||^p$. Given that $f = f_p$ in the definition above, the Bregman distance with respect to f_p now becomes*

$$\begin{aligned} \Delta_p(x, y) &= \frac{1}{q}||x||^p - \langle J_p x, y\rangle + \frac{1}{p}||y||^p \\ &= \frac{1}{p}\left(||y||^p - ||x||^p\right) + \langle J_p x, x - y\rangle \\ &= \frac{1}{q}\left(||x||^p - ||y||^p\right) - \langle J_p x - J_p y, y\rangle. \end{aligned} \tag{2.2.6}$$

We note that the Bregman distance is not symmetric, therefore it is not a metric but it has the following important properties for all $x, y, z \in E$.

(i) $\Delta_p(x, x) = 0$,

(ii) $\Delta_p(x, y) \geq 0$,

(iii) $\Delta_p(x, y) = \Delta_p(x, z) + \Delta_p(z, y) + \langle z - y, J_p x - J_p y\rangle$,

(iv) $\Delta_p(x, y) + \Delta_p(y, x) = \langle x - y, J_p x - J_p y\rangle$.

For any p-uniformly convex Banach space E, the metric and Bregman distance have the following relation:

$$k||x - y||^p \leq \Delta_p(x, y) \leq \langle x - y, J_p x - J_p y\rangle,$$

where $k > 0$ is a fixed number.

Definition 2.2.27. *Let C be a nonempty, closed and convex subset of int domf, where int domf means interior domain of f. Let $T : C \to C$ be any mapping, a point $p \in C$ is called a fixed point of T if $Tp = p$. While $p \in C$ is called an asymptotic fixed point of T if C contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges weakly to p and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. The set of fixed points of T and asymptotic fixed points of T are denoted by $F(T)$ and $\hat{F}(T)$ respectively.*

$T : C \to C$ is said to be

(i) right Bregman firmly nonexpansive if

$$\langle J_p(Tx) - J_p(Ty), Tx - Ty\rangle \leq \langle J_p(Tx) - J_p(Ty), x - y\rangle, \ \forall x, y \in C,$$

equivalently,

$$\Delta_p(Tx, Ty) + \Delta_p(Ty, Tx) + \Delta_p(x, Tx) + \Delta_p(y, Ty) \leq \Delta_p(x, Ty) + \Delta_p(y, Tx),$$

(ii) right Bregman strongly nonexpansive (see [81]) with respect to a nonempty $\hat{F}(T)$ if

$$\Delta_p(Tx, y) \leq \Delta_p(x, y), \ \forall x \in C, \ y \in \hat{F}(T)$$

and if whenever $\{x_n\} \subset C$ is bounded, $y \in \hat{F}(T)$ and

$$\lim_{n\to\infty} (\Delta_p(x_n, y) - \Delta_p(Tx_n, y)) = 0,$$

it follows that

$$\lim_{n\to\infty} \Delta_p(x_n, Tx_n) = 0.$$

Remark 2.2.28. *[81]. Every right Bregman firmly nonexpansive mapping is right Bregman strongly nonexpansive mapping with respect to $F(T) = \hat{F}(T)$.*

Definition 2.2.29. *A mapping $B : E \to 2^{E^*}$ is called monotone if*

$$\langle \xi - \eta, x - y \rangle \geq 0 \ \forall x, y \in E, \ \xi \in B(x), \eta \in B(y). \tag{2.2.7}$$

B is said to be maximal if the graph of B denoted by $G(B)$ is not properly contained in the graph of any other monotone mapping. It is generally known that a monotone mapping B is maximal if and only if $\langle x - y, u - v \rangle \geq 0$, for all $(x, u) \in E \times E$, $(y, v) \in G(B)$ implies $u \in Bx$.

The following are examples of monotone mappings.

Example 2.2.30. *Let E be a real Banach space and $f : E \to (-\infty, +\infty]$ be a proper, convex and lower semi-continuous function. The subdifferential ∂f of f defined by*

$$\partial f(x) = \{\xi \in E^* : \langle \xi, y - x \rangle \leq f(y) - f(x) \ \forall x \in E\},$$

is maximal monotone (see [100]).

Example 2.2.31. *Let E be a real Banach space, then the duality mapping J is monotone. Indeed for any $x, y \in E$, $u \in J(x)$, $v \in J(y)$, we have*

$$\begin{aligned} \langle x - y, u - v \rangle &= ||x||^2 + ||y||^2 - \langle x, v \rangle - \langle y, u \rangle \\ &\geq ||x||^2 + ||y||^2 - ||x||||v|| - ||y||||u|| \\ &= ||x||^2 + ||y||^2 - 2||x||||y|| \\ &= (||x|| - ||y||)^2 \geq 0. \end{aligned}$$

Example 2.2.32. *Let A be an $n \times n$ matrix with real entries. Consider the operator $f : \mathbb{R}^n \to \mathbb{R}^n$ defined by $f(x) = Ax$. Then f is maximal monotone if f is a positive linear operator (see [56]).*

Definition 2.2.33. *Let E be a $p-$uniformly convex Banach space and J_p be the duality mapping of E. The resolvent of a maximal monotone mapping B is the operator $Res_p^{\lambda B} : E \to 2^E$ defined by*

$$Res_p^{\lambda B} := (J_p + \lambda B)^{-1} \circ J_p, \quad \lambda > 0. \tag{2.2.8}$$

Remark 2.2.34. *The resolvent operator $Res_p^{\lambda B}$ is a Bregman firmly nonexpansive operator. Furthermore, $0 \in B(x)$ if and only if $x = Res_p^{\lambda B}(x)$ (see [96], for more details).*

Definition 2.2.35. *Let C be a nonempty, closed and convex subset of E. The Bregman projection $\prod_C$ is defined by*

$$\prod_C x = arg \min_{y \in C} \Delta_p(x, y), \ \forall \ x \in E, \tag{2.2.9}$$

which is a unique minimizer of the Bregman distance.

Definition 2.2.36. *[106]. Let E be a $p-$uniformly convex Banach space. The function $V_p : E^* \times E \to [0, +\infty)$ is defined by*

$$V_p(x, y) := \frac{1}{q}||x||^q - \langle x, y \rangle + \frac{1}{p}||y||^p, \ \forall x \in E^*, y \in E. \tag{2.2.10}$$

V_p is nonnegative and $V_p(x,y) = \Delta_p(J_q^*(x), y)$ for all $x \in E^*$ and $y \in E$. Also, by the subdifferential inequality, we have

$$V_p(x^*, x) + \langle y^*, J_q^*(x^*) - x\rangle \leq V_p(x^* + y^*, x), \ \forall x \in E, \ x^*, y^* \in E^* \text{ (see, [106])}. \quad (2.2.11)$$

Furthermore, V_p is convex in the second variable. Thus for all $z \in E$, we have

$$\Delta_p\left(J_q^*\left(\sum_{i=1}^{N} t_i J_p(x_i)\right), z\right) \leq \sum_{i=1}^{N} t_i \Delta_p(x_i, z), \quad (2.2.12)$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$ (see [4, 35, 105, 110, 106] for more details).

2.3 Variational inequality problems

The theory of VIP is known to be very useful in solving diverse mathematical problems which includes optimization problems, equilibrium problems, boundary valued problems, among others. It is known that many mathematical problems can be posed as a VIP. In particular, VIPs are known to be natural generalization of the theory of boundary value problems and are considered in optimization theory as natural extension of minimization problems (see [43]). We now give some examples of a VIP in the following subsection. To do this, we first recall the definition of a VIP in a real Hilbert space H. Let $A : H \to H$ be an operator and C be a nonempty, closed and convex subset of H. The VIP is the problem of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0 \ \forall x \in C. \quad (2.3.1)$$

We denote by $VIP(C, A)$ the solution set of the VIP (2.3.1). It is generally known that $VIP(C, A)$ is a closed and convex subset of C (for example, see [112]).

2.3.1 Examples of variational inequality problems

1. Consider the following VIP. The mapping A and the set C are defined by

   $$A(x) = \begin{bmatrix} 3x_1 - \frac{1}{x_1} + 3x_2 - 2 \\ 3x_1 + 3x_2 \\ 4x_3 + 4x_4 \\ 4x_3 + 4x_4 - \frac{1}{x_4} - 3 \end{bmatrix} \text{ and}$$

   $C = \{x \in \mathbb{R}^n \mid x_1 + x_2 = 1, \ x_3 + x_4 \geq 0, \ l \leq x \leq h\}$, where $l = (0.1, 0, 0, 1)^T$ and $h = (10, 10, 10, 10)^T$. Then the above nonlinear VIP has a unique solution $x^* = (1, 0, 0, 1)^T$ (see [124]).

2. Let A and C associated with the VIP (2.3.1) be defined by

$$A(x) = \begin{bmatrix} 22x_1 - 2x_2 + 6x_3 - 4 \\ 2x_2 - 2x_1 \\ 2x_3 + 6x_1 \end{bmatrix} \text{ and}$$

$C = \{x \in \mathbb{R}^3 \mid x_1 - x_2 \geq 1, \ -3x_1 - x_3 \geq -4, \ 2x_1 + 2x_2 + x_3 = 0, \ l \leq x \leq h\}$, where $l = (-6, -6, -6)^T$ and $h = (6, 6, 6)^T$. The above VIP is linear and has only one solution $x^* = (2, 1, -6)^T$ (see also [124]).

3. Consider the following problem of finding the minimal value of a differentiable function f over a closed interval $I = [a, b]$. For $x^* \in I$, we have the following three possible cases,

   (a) if $a < x^* < b$ then $f'(x^*) = 0$,

   (b) if $x^* = a$ then $f'(x^*) \geq 0$,

   (c) if $x^* = b$ then $f'(x^*) \leq 0$.

   The above cases can be summarized as a VIP of finding $x^* \in I$ such that $f'(x^*)(x - x^*) \geq 0 \ \forall x \in I$ (see [9] for more details).

2.3.2 Previous works on variational inequality problems

VIPs have been extensively studied in both finite and infinite dimensional spaces by numerous authors. The break through in the study of VIPs in finite dimensional spaces happened in 1980 when Dafermos [47] identified that a certain traffic network equilibrium conditions had a structure of variational inequalities under the monotonicity assumption. Dafermos [47] used the techniques of the theory of variational inequalities to establish existence of a traffic equilibrium pattern for which he developed an algorithm for the construction of the pattern and derived estimates on the speed of convergence of the algorithm. He also used his algorithm to estimate the user-optimized equilibrium pattern for a simple network with two-way streets. His work attracted the interest of numerous researchers, as a result of this, a lot of research efforts were devoted to the study of VIPs in finite dimensional spaces (for example, see [48], [49], [75], [90]).

The study of VIPs was further extended to infinite dimensional spaces. There are several monographs on VIPs in infinite dimensional spaces, however, we shall mention here a few. Stampacchia [111] established the existence and uniqueness of the solution of problem (2.3.1) under the assumption that A is a coercive and linear operator from a Hilbert space H to its dual space H^*. Lions and Stampacchia [79] further considered the case where A is positive or semicoercive. Hatman and Stampacchia [59] worked on partial differential equations using the VIP (2.3.1) as a tool, with applications to problems arising from mechanics. They proved the existence and uniqueness theorem of the solution of problem (2.3.1) in a reflexive real Banach space when A is assumed to be a monotone hemicontinuous operator. In fact they proved the following theorem.

Theorem 2.3.1. *Let X be a Banach space and X^* be its dual. Let $A : X \to X^*$ be a monotone hemicontinuous operator and K be a bounded convex subset of X. Then there exists at least one solution of problem* (2.3.1).

As we mentioned earlier, there are different methods or ways of obtaining solutions of problem (2.3.1) in infinite dimensional spaces. The methods used by the authors mentioned above has to do with the existence and uniqueness of solutions of VIPs (see [112] for detailed information on different approaches for solving problem (2.3.1)). Unlike the existence and uniqueness theorems which is only concerned with establishing conditions under which problem (2.3.1) has solution, the iterative methods of finding solutions of problem (2.3.1) is concerned with the actual computation or approximation of sequences to a solution of problem (2.3.1). As we have stated earlier, this will be our focus in this dissertation.

Let C and Q be nonempty, closed and convex subsets of the real Hilbert spaces H_1 and H_2 respectively. Let $f : H_1 \to H_1$, $g : H_2 \to H_2$ be inverse strongly monotone operators and $A : H_1 \to H_2$ be bounded linear operator. Consider the following problem which is called the Split Variational Inequality Problem (SVIP): Find $x^* \in C$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0 \ \forall x \in C \tag{2.3.2}$$

and such that $y^* = Ax^* \in Q$ solves

$$\langle g(y^*), y - y^* \rangle \geq 0 \ \forall y \in Q. \tag{2.3.3}$$

If (2.3.2) and (2.3.3) are considered separately, we have that (2.3.2) is a VIP with its solution set $VIP(C, f)$ and (2.3.3) is a VIP with its solution set $VIP(Q, g)$. The SVIP was introduced and studied by Censor *et al.* [32]. They studied this problem as a pair of VIPs in which they obtained a solution of one VIP in H_1 whose image under a given bounded linear operator A is a solution of the second VIP in the second space H_2. They considered two approaches for establishing the solution of the SVIP (2.3.2)-(2.3.3). In each of these approaches they proposed an iterative algorithm and using their algorithms they obtained strong convergence results of the SVIP (2.3.2)-(2.3.3).

In 2012, Censor *et al.* [33] introduced the general Common Solutions to Variational Inequality Problem (CSVIP), which consist of finding common solutions to unrelated variational inequalities for finite number of sets. That is, find $x^* \in \cap_{i=1}^N C_i$ such that for each $i = 1, 2, \ldots, N$,

$$\langle A_i(x^*), x - x^* \rangle \geq 0, \text{ for all } x \in C_i, \ i = 1, 2, \ldots, N, \tag{2.3.4}$$

where $A_i : H \to H$ is a nonlinear operator for each $i = 1, 2, \ldots, N$ and C_i is a nonempty, closed and convex subset of H. They obtained the solution of problem (2.3.4) by considering first, a case where $i = 1, 2$ and later obtained the result of the problem for $i = 1, 2, \ldots, N$. They proposed the following algorithm and proved the corresponding theorem.

$$\begin{cases} x^0 \in H, \\ x^{k+1} = \prod_{i=1}^N (P_{C_i}(I - \lambda A_i))(x^k). \end{cases} \tag{2.3.5}$$

Theorem 2.3.2. *Let H be a real Hilbert space and C_i be nonempty, closed and convex subsets of H for each $i = 1, 2, \ldots, N$. Let $A_i : H \to H$ be α_i-inverse strongly* *monotone operators with* $\lambda \in (0, 2\alpha)$ *and* $\alpha := \min_i\{\alpha_i\}$*. Assume that* $\cap_{i=1}^{\infty} C_i \neq \emptyset$ *and* $\Gamma := \cap_{i=1}^{N} SOL(C_i, A_i) \neq \emptyset$*. Then any sequence* $\{x^k\}_{k=0}^{\infty}$ *generated by Algorithm* (2.3.5) *converges weakly to a point* $x^* \in \Gamma$ *and furthermore,*

$$x^* = \lim_{k\to\infty} P_\Gamma(x^k). \tag{2.3.6}$$

Censor *et al.* [34] also considered problem (2.3.4) in the case where the operator A is a multivalued mapping. More precisely, they studied the following problem which they called Common Solutions to Variational Inequalities Problem (CSVIP):
Let H be a real Hilbert space and C_i be a nonempty, closed and convex subset of H with $\cap_{i=1}^{N} C_i \neq \emptyset$. Let $A_i : H \to 2^H$ be a multivalued mapping for each $i = 1, 2, \ldots, N$. The CSVIP is the problem of finding a point $x \in \cap_{i=1}^{N} C_i$ such that for each $i = 1, 2, \ldots, N$, there exists $u_i \in A_i(x)$ satisfying

$$\langle u_i, y - x\rangle \geq 0 \ \forall y \in C_i, \ i = 1, 2, \ldots, N. \tag{2.3.7}$$

Their motivation stems from the observation that if $A_i = 0$, then the CSVIP (2.3.7) reduces to the Convex Feasibility Problem (CFP) of finding a point $x \in \cap_{i=1}^{N} C_i$. Also observe that if C_i are the fixed point sets of a family of nonlinear operators defined on H, then the CFP becomes the Common Fixed Point Problem (CFPP) (for example, see [33] and [34]).

2.3.3 Variational inequality problems and fixed point problems

Fixed point problems are closely related to VIPs. As a result of this relationship, iterative methods for finding common solutions to both problems have been widely studied by numerous researchers. In 2005, Iiduka and Takahashi [62] established a strong convergence result for approximating an element in the intersection of the set of solutions of a VIP and the set of solutions of a FPP for nonexpansive mappings. They proposed the following iterative algorithm:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) T P_C (I - \lambda_n A) x_n, \ n \geq 1, \end{cases} \tag{2.3.8}$$

where C is a nonempty, closed and convex subset of a real Hilbert space H, $P_C : H \to C$ is a metric projection, $T : C \to C$ is a nonexpansive mapping, $A : C \to H$ is an α-inverse strongly monotone mapping, $\{\alpha_n\} \subset (0, 1)$ and $\{\lambda_n\} \in (0, 2\alpha)$.

A year later, Takahashi and Toyoda [113], introduced the following iterative algorithm for approximating a common solution of VIP and fixed point problem:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) T P_C (I - \lambda_n A) x_n, \ n \geq 0, \end{cases} \tag{2.3.9}$$

where C is a nonempty, closed and convex subset of a real Hilbert space H, $P_C : H \to C$ is a metric projection, $A : C \to H$ is an α-inverse strongly monotone mapping and $T : C \to C$

is a nonexpansive mapping. They proved that if $\Gamma := F(T) \cap VI(C, A) \neq \emptyset$, then the sequence generated by Algorithm (2.3.9) converges weakly to an element of Γ.

The common solution to VIPs and FPPs for nonexpansive mappings were also studied by Yao *et al.* [123]. In his study, he assumed that the operator associated with the VIP is a monotone k-Lipschitzian continuous mapping. Under some suitable conditions, he obtained strong convergence result using the following iterative scheme: For a fixed $u \in H$ and arbitrary $x_0 \in H$,

$$\begin{cases} x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(x_n - \lambda_n y_n), \\ y_n = P_C(I - \lambda_n) x_n, \ n \geq 0. \end{cases} \tag{2.3.10}$$

Let $T : H \to H$ be an operator such that $F(T) \neq \emptyset$ and $f : H \to H$ be an operator, then the hierarchical variational inequality problem is to find $x^* \in F(T)$ such that

$$\langle f(x^*), \ x - x^* \rangle \geq 0, \quad \forall x \in F(T). \tag{2.3.11}$$

Ansari *et al.* [6] introduced a split-type problem by combining a split fixed point problem and a hierarchical variational inequality problem; thus, presenting the split hierarchical variational inequality problem which is to find $x^* \in F(T)$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in F(T) \tag{2.3.12}$$

and such that $Ax^* \in F(S)$ satisfies

$$\langle h(Ax^*), y - Ax^* \rangle \geq 0 \ \ \forall y \in F(S), \tag{2.3.13}$$

where H_1, H_2 are two real Hilbert spaces, $T : H_1 \to H_1$ is a strongly nonexpansive operator such that $F(T) \neq \emptyset$, $S : H_2 \to H_2$ is a strongly nonexpansive cutter operator such that $F(S) \neq \emptyset$, $A : H_1 \to H_2$ is a bounded linear operator with $R(A) \cap F(S) \neq \emptyset$, f(respectively h) is a monotone and continuous operator on H_1(respectively H_2). With these assumptions, they proposed the following iterative scheme for finding a solution of problem (2.3.12)-(2.3.13):

$$\begin{cases} \forall x_1 \in H_1, \\ y_n := x_n - \gamma A^*(I - S(I - \beta_k h)) A x_n, \\ x_{n+1} := T(I - \alpha_n f) y_n, \end{cases} \tag{2.3.14}$$

where $\gamma \in (0, \frac{2}{||A||^2})$, $\{\alpha_n\}$, $\{\beta\} \subset (0, +\infty)$. They proved that the sequence generated by Algorithm (2.3.14) converges weakly to a solution of (2.3.12)-(2.3.13). See [68] and the references therein for the works of other authors done in this direction.

Observe that the operator $P_C(I - \lambda A)$ appears in most of the algorithms stated above. Hence, we present the connection between the fixed points set of the operator $P_C(I - \lambda A)$ and the solution set of VIP:

Proposition 2.3.3. *[33]. Let C be a nonempty, closed and convex subset of a real Hilbert space H and $A : C \to C$ be an inverse strongly monotone operator. If $VIP(C, A)$ is the solution set of the VIP (2.3.1), then for any $\lambda > 0$, we have that $F\left(P_C(I - \lambda A)\right) = VIP(C, A)$.*

Proof. Let $\lambda > 0$ and $y \in C$, then from the characterization of the metric projection (see Proposition (2.1.32)), we have that

$$\begin{aligned} x \in F(P_C(I-\lambda A)) &\iff x = P_C(I-\lambda A)x, \\ &\iff \langle x-(I-\lambda A)x, y-x\rangle \geq 0, \\ &\iff \langle \lambda A(x), y-x\rangle \geq 0, \\ &\iff \langle A(x), y-x\rangle \geq 0, \\ &\iff x \in \text{VIP}(C,A). \end{aligned}$$

Hence, $F(P_C(I-\lambda A)) = \text{VIP}(C,A)$. □

2.3.4 Variational inequality problems and related problems

The VIPs are deeply related to various mathematical problems, such as complementarity problems, minimization problems, Minty variational inequality problems, inclusion problems, among others. We shall state some of these relationships below.

Complementarity problems:

Definition 2.3.4. *Let H be a real Hilbert space and K be a nonempty subset of H. We say that K is a convex cone if the following properties holds:*

P1 $K + K \subseteq K$,

P2 $\lambda k \subseteq K$, *for* $\lambda \geq 0$.

If in addition, K satisfies $K \cap (-K) = \{0\}$, then K is called a pointed convex cone.

Definition 2.3.5. *Let K be a pointed convex cone in H. Then the dual of K is defined by*

$$K^* = \{y \in H : \langle x, y\rangle \geq 0, \ \forall x \in K\}. \tag{2.3.15}$$

Remark 2.3.6. *It is well known that for any pointed convex cone K, the dual K^* of K is always a closed and convex cone.*

Definition 2.3.7. *Let H be a real Hilbert space and $K \subseteq H$ be a nonempty, closed and pointed convex cone. Let $f : H \to H$ be any nonlinear mapping, then the Complementarity Problem (CP) is to find $x^* \in K$ such that*

$$f(x^*) \in K^* \quad \textit{and} \quad \langle f(x^*), x^*\rangle = 0. \tag{2.3.16}$$

We denote the solution set of problem (2.3.16), by $CP(f, K)$. The following result brings out the connection between VIPs and CPs.

Theorem 2.3.8. *[63]. Let H be a Hilbert space and C be a nonempty, closed and convex subset of H. Let $K \subseteq H$ be a nonempty, closed and convex cone. Then $VI(C, f)$ and $CP(K, f)$ are equivalent if and only if $C = K$.*

Minty variational inequality problems:

Let C be a nonempty, closed and convex subset of a real Hilbert space H and $A : H \to H$ be a nonlinear operator. The Minty Variational Inequality Problem (MVIP) is the problem of finding $x \in C$ such that

$$\langle Ay, y - x\rangle \geq 0 \ \forall y \in C. \tag{2.3.17}$$

We denote the solution set of problem (2.3.17) by $MVIP(C, A)$. The minty variational inequality problem was introduced and studied by Minty in 1967. Minty, in his study states the relationship between VIPs and MVIPs under the continuity and monotonicity assumption of the associated operator A.

Theorem 2.3.9. *[84]. Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. Then $MVIP(C, A)$ and $VIP(C, A)$ are equivalent if and only if $A : H \to H$ is a monotone and continuous operator.*

Minimization problems:

Let H be a real Hilbert space and K be a nonempty, closed and convex subset of H. Let $f : H \to \mathbb{R}$ be differentiable on an open set containing K. Then the Minimization Problem (MP) is the problem of finding a point $x^* \in K$ such that

$$f(x^*) = \min_{x\in K} f(x). \tag{2.3.18}$$

Consider the following VIP: Find $x^* \in K$ such that

$$\langle f'(x^*), x - x^*\rangle \geq 0 \ \forall x \in K, \tag{2.3.19}$$

where f' is the gradient of f, then we have the following proposition.

Proposition 2.3.10. *[46]. Let K be a nonempty, closed and convex subset of a real Hilbert space H and $f : H \to \mathbb{R}$ be a differentiable function on an open set containing K, then*

(i) if $x^* \in K$ is a solution of MP (2.3.18), then x^* is also a solution of VIP (2.3.19),

(ii) if f is convex and $x^* \in K$ is a solution of VIP (2.3.19), then x^* is also a solution of MP (2.3.18).

2.4 Monotone inclusion problems

In the previous section, we reviewed some of the important works done on VIPs and we also discussed their relationships with some related mathematical problems. In this section however, we shall study a useful and important generalization of the VIPs, namely the MIP. We begin by discussing some of the important works done in this area.

2.4.1 Previous works on monotone inclusion problems

Like the VIPs, many mathematical problems such as optimization problems, equilibrium problems, VIPs, saddle point problems, among others, can be modelled as a MIP. The MIP has been widely studied by numerous researchers in both finite and infinite dimensional spaces. Rockafellar [98] was the first to introduce and study the MIP, which he defined as a problem of finding a point $x \in H$ such that

$$0 \in B(x), \tag{2.4.1}$$

where H is a real Hilbert space and $B : H \to 2^H$ is a maximal monotone operator. The solution set of problem (2.4.1) is denoted by $B^{-1}(0)$, which is known to be closed and convex, see [98].

Byrne *et al.* [25] introduced and studied the following Split Monotone Inclusion Problem (SMIP): Let $B_i : H_1 \to 2^{H_1},\ 1 \le i \le p$ and $F_j : H_2 \to 2^{H_2},\ 1 \le j \le r$ be maximal monotone mappings. Let $A_j : H_1 \to H_2,\ 1 \le j \le r$ be bounded linear operators,

$$\text{find } \bar{x} \in H_1 \text{ such that } 0 \in \cap_{i=1}^{p} B_i(\bar{x}) \tag{2.4.2}$$

and

$$\bar{y}_j = A_j(\bar{x}) \in H_2 \text{ such that } 0 \in \cap_{j=1}^{r} F_j(\bar{y}). \tag{2.4.3}$$

They denote problem (2.4.2)-(2.4.3) by SCNPP(p, r) in order to emphasize the multiplicity of the mappings B_i and F_j. In [25], Byrne observed that the SVIP introduced and studied by Censor *et al.* [32] can be structurally considered as a special case of SCNPP(p, r), for $p = r = 1$. Byrne *et al.* [25] proved strong convergence of their proposed algorithms to a solution of problem (2.4.2)-(2.4.3) by first considering a case where $p = r = 1$. Then for $1 \le i \le p$ and $1 \le j \le r$, they employed a product space formulation in order to transform SCNPP(p, r) into SCNPP$(1, 1)$ and obtained a strong convergence result.

Inspired by the work of Byrne *et al.* [25], Kazmi and Rizvi [70] introduced the following algorithm for finding a common solution of the SMIP and the FPP of a nonexpansive mapping: Let $x_0 \in H_1$ and the sequences $\{u_n\}$ and $\{x_n\}$ be generated by

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n,\ n \neq 0, \end{cases} \tag{2.4.4}$$

where $S : H_1 \to H_1$ is a nonexpansive mapping. They proved that the sequences $\{u_n\}$ and $\{x_n\}$ converges strongly to $z \in F(S) \cap \Omega$, where Ω is the solution set of SMIP (2.4.2)-(2.4.3).

Guo *et al.* [56] introduced and studied the Split Equality Monotone Inclusion Problem (SEMIP) in real Hilbert spaces. They stated the problem as follows: Find

$$x \in U^{-1}(0) = F(J_{u_n}^U),\ y \in V^{-1}(0) = F(J_{u_n}^V) \text{ such that } Ax = By, \tag{2.4.5}$$

where H_1, H_2, H_3 are real Hilbert spaces, $U : H_1 \to 2^{H_1},\ V : H_2 \to 2^{H_2}$ are maximal monotone mappings and $A : H_1 \to H_3,\ B : H_2 \to H_3$ are two bounded linear operators. They proved the following result.

Theorem 2.4.1. *[56]. Let H_1, H_2, H_3 be real Hilbert spaces and $U : H_1 \to 2^{H_1}$, $V : H_2 \to 2^{H_2}$ be maximal monotone mappings. Let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators and A^*, B^* be the adjoints of A and B respectively. Let $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, where f_i, $i = 1, 2$ are contraction mappings on H_i with constant $k \in (0, 1)$ and $\{S_n\}$ a sequence of nonexpansive mappings on H_1, D a strongly positive bounded linear operator with coefficient $\gamma > 0$. Assume that the solution set SEMIP (2.4.5) is nonempty*

$$J_{u_n}^{(U,V)} = \begin{bmatrix} J_{u_n}^U \\ J_{u_n}^V \end{bmatrix}, \quad G = [A, \quad -B], \quad G^*G = \begin{bmatrix} A^*A & -A^*B \\ -B^*A & B^*B \end{bmatrix}.$$

Let w_n be generated by

$$\begin{cases} u_n = J_{u_n}^{(U,V)}(I - \gamma G^*G)w_n, \\ w_{n+1} = \alpha_n \sigma f(w_n) + (1 - \alpha_n D)S_n V_n. \end{cases} \tag{2.4.6}$$

Let $F(S) = \cap_{i=1}^{\infty} F(S_n)$ and S_n satisfy the AKTT condition:

$$\sum_{n=1}^{\infty} \sup\{||S_{n+1}v - S_n v|| : v \in C\} < \infty.$$

If $F(S) \cap \Gamma$ is nonempty (where Γ is the solution set of problem (2.4.5)) and the following conditions are satisfied

(i) $\alpha_n \in (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

(iii) $\sum_{n=0}^{\infty} |u_{n+1} - u_n| < \infty$,

(iv) $0 < \gamma < \frac{1}{\alpha_n}$, $0 < \sigma < \frac{\gamma}{k}$.

Then, the sequence $\{w_n\}$ convergences strongly to a point w^*, where $w^* = P_{F(S)\cap\Gamma}(I - D - \sigma f)(w^*)$ is a unique solution of the variational inequalities

$$\langle (D - \sigma f)w^*, w^* - z \rangle \leq 0, \ z \in F(S) \cap \Gamma.$$

Riech and Sabach [96] used the following algorithm to obtain strong convergence result for problem (2.4.1) in a more general reflexive Banach space:

$$\begin{cases} x_0 \in X, y_n^i = \mathrm{Res}_{\lambda_n^i B_i}^f (x_n + e_n^i), \\ C_n^i = \{z \in X : D_f(z, y_n^i) \leq D_f(z, x_n + e_n^i)\}, \\ C_n := \cap_{i=1}^N C_n^i, \\ Q_n = \{z \in X : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0\}, \\ x_{n+1} = \mathrm{Proj}_{C_{n+1}}^f (x_0), \ n \geq 0. \end{cases} \tag{2.4.7}$$

They proved the following theorem using Algorithm (2.4.7).

Theorem 2.4.2. *[96]. Let X be a reflexive real Banach space and X^* be its dual space. Let $B_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$ be N maximal monotone operators such that $Z := \cap_{i=1}^{N} B_i^{-1}(0^*) \neq \emptyset$. Let $f : X \to \mathbb{R}$ be a Legendre function which is bounded, uniform Frétchet differentiable and totally convex on bounded subsets of X. Suppose that ∇f^* is bounded on bounded subsets of X^*. Then, for each $x_0 \in X$, there are sequences $\{x_n\}_{n\in\mathbb{N}}$ which satisfy Algorithm (2.4.7). If for each $i = 1, 2, \ldots, N$, $\liminf_{n\to\infty} \lambda_n^i > 0$ and the sequence of errors $\{e_n^i\}_{n\in\mathbb{N}} \subset X$ satisfies $\lim_{n\to\infty} e_n^i = 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly as $n \to \infty$ to $Proj_Z^f(x_0)$.*

2.4.2 Relationships between monotone inclusion problems and variational inequality problems

Definition 2.4.3. *Let C be a nonempty, closed and convex subset of H. The normal cone of C, $N_C : H \to 2^H$ is defined as*

$$N_C z = \begin{cases} \emptyset, & \text{if } z \notin C, \\ \{d \in H : \langle d, y - z\rangle \leq 0, \quad \forall\, y \in C\}, & \text{if } z \in C. \end{cases} \tag{2.4.8}$$

Observe that if $H = \mathbb{R}^n$ and $B(x) := T(x) + N_C(x) \quad \forall x \in \mathbb{R}^n$, where T is a maximal monotone mapping and N_C is a normal cone of C. Then we have that dom $B = C\cap$ dom T and int $C \neq \emptyset$ (where int means interior). Furthermore, we assume that dom $T\cap$ int $C \neq \emptyset$, so that $B = T + N_C$ is maximal monotone (see [17]). In this case, we have that the MIP (2.4.1) is equivalent to the following generalized VIP of finding $x \in C$ and $v \in T(x)$ such that

$$\langle v, y - x\rangle \geq 0, \ \forall y \in C. \tag{2.4.9}$$

The solution set of problem (2.4.9) is denoted by $GVIP(T, C)$. It has been shown that if $C = \mathbb{R}^n$, then the generalized VIP (2.4.9) reduces to the MIP (2.4.1) (for more information, see [17]).

We also note that if $H = \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is a single valued, continuous and monotone operator, then $B = F + N_C$ is a maximal monotone operator, since dom$B = C$ and dom$F = \mathbb{R}^n$. Thus, the MIP (2.4.9) reduces to the following VIP; find $x \in C$ such that

$$\langle F(x), y - x\rangle \geq 0 \ \forall y \in C. \tag{2.4.10}$$

For an inverse strongly monotone operator A associated with a given VIP, we have the following result.

Proposition 2.4.4. *[99]. Let C be a nonempty, closed and convex subset of H and A be an inverse strongly monotone operator on C. Let $N_C z$ be the normal cone of C at the point $z \in C$ and $B : H \to 2^H$ be a set-valued operator defined by*

$$Bz = \begin{cases} Az + N_C z, & z \in C, \\ \emptyset, & \text{otherwise}. \end{cases} \tag{2.4.11}$$

Then B is a maximal monotone operator. Furthermore, $B^{-1}(0) = VIP(C, A)$.

2.5 On split equality fixed point problems

The SEFPP introduced and studied by Moudafi and Al-Shemas [88] is stated as follows:

$$\text{Find } \ x \in C := F(T), \ \ y \in Q := F(S) \ \text{ such that } \ Ax = By, \tag{2.5.1}$$

where $C \subset H_1$, $Q \subset H_2$ are two nonempty, closed and convex sets, $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded linear operators, $F(T)$ and $F(S)$ denotes the sets of fixed points of operators T and S defined on H_1 and H_2 respectively. Moudafi and Al-Shemas presented the following algorithm for solving the SEFPP:

$$\begin{cases} x_{n+1} = T(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = S(y_n + \gamma_n B^*(Ax_n - By_n)), \quad \forall n \geq 1, \end{cases} \tag{2.5.2}$$

where $T : H_1 \to H_1$, $S : H_2 \to H_2$ are two firmly quasi-nonexpansive mappings, $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded linear operators, A^* and B^* are the adjoints of A and B respectively, $\{\gamma_n\} \subset \left(\epsilon, \frac{2}{\lambda_{A^*A} + \lambda_{B^*B}} - \epsilon\right)$, λ_{A^*A} and λ_{B^*B} denote the spectral radii of A^*A and B^*B respectively. They established the weak convergence result for problem (2.5.1) using Algorithm (2.5.2).

The SEFPP is generally known to be a useful generalization of the Split Feasibility Problem (SFP), which was introduced in 1994 by Censor and Elfving [28] and which has wide applications in many fields, such as phase retrieval, medical image reconstruction, signal processing and radiation therapy treatment planning (for example, see [23, 24, 27, 28, 30, 83, 86, 106, 107, 115, 119, 117, 122, 126] and the references therein). Observe that if $H_2 = H_3$ and $B = I$ (where I is the identity map on H_2) in problem (2.5.1), then the SEFPP (2.5.1) reduces to the following SFP of Censor and Elfving [28]: Find a point

$$x \in C, \quad \text{such that} \quad Ax \in Q, \tag{2.5.3}$$

where C and Q are nonempty, closed and convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and A is an $m \times n$ real matrix. The SFP was later extended to Multiple-Sets Split Feasibility Problem (MSSFP) by Censor *et al.* [29] which he defined as the problem of finding

$$x^* \in C = \cap_{i=1}^N C_i \text{ such that } Ax^* \in Q = \cap_{j=1}^M Q_j,$$

where N and M are positive integers, $\{C_1, \ldots, C_N\}$ and $\{Q_1, \ldots, Q_N\}$ are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively and $A : H_1 \to H_2$ is a bounded linear map. Recently, researchers have started approximating solutions of SFP in Banach spaces (for example, see [103, 114, 106] and the references therein).

Yaun-Fang *et al.* [52] presented the following algorithm (which is an extension of Algorithm (2.5.2)) for solving SEFPP (2.5.1):

$$\begin{cases} \forall x_1 \in H_1, \ y_1 \in H_2, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n S(y_n + \gamma_n B^*(Ax_n - By_n)), \quad \forall n \geq 1, \end{cases} \tag{2.5.4}$$

where $T : H_1 \to H_1$, $S : H_2 \to H_2$ are two firmly quasi-nonexpansive mappings, $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded linear operators, A^* and B^* are the adjoints of A and B respectively. The sequences $\{\gamma_n\}$ and $\{\alpha_n\}$ are in $\left(\epsilon, \frac{2}{\lambda_{A^*A}+\lambda_{B^*B}} - \epsilon\right)$ and $[\alpha, 1]$ respectively, for $\alpha > 0$ and for ϵ small enough. λ_{A^*A} and λ_{B^*B} denote the spectral radii of A^*A and B^*B respectively. They established a strong and weak convergence results using Algorithm (2.5.4).

Based on the work of Moudafi and Al-Shemas [88], Chidume *et al.* [40] proposed the following algorithm for solving the SEFPP for demi-contractive mappings:

$$\begin{cases} \forall x_1 \in H_1, \quad \forall y_1 \in H_2, \\ x_{n+1} = (1-\alpha)\left(x_n - \gamma A^*(Ax_n - By_n)\right) + \alpha T\left(x_n - \gamma A^*(Ax_n - By_n)\right), \\ y_{n+1} = (1-\alpha)\left(y_n + \gamma B^*(Ax_n - By_n)\right) + \alpha S\left(y_n + \gamma B^*(Ax_n - By_n)\right), \quad \forall n \geq 1, \end{cases} \tag{2.5.5}$$

where $T : H_1 \to H_1$, $S : H_2 \to H_2$ are two demi-contractive mappings. Chidume *et al.* [40] proved weak and strong convergence theorems of the iterative scheme (2.5.5) to a solution of the SEFPP in real Hilbert spaces.

The approximation of fixed point of multi-valued mappings with respect to the Hausdorrf metric has been an area of great research interest due to its numerous applications in various fields such as game theory, mathematical economics, optimization theory, among others. Thus, it is ideal to extend the known results on SEFPP for single-valued mappings to multi-valued mappings. Wu *et al.* [116] studied the Multiple-Set Split Equality Fixed Point Problem (MSSEFPP) for finite families of multi-valued mappings. They stated the problem as follows: Find

$$x \in C = \cap_{j=1}^{N} F(R_1^j) \text{ and } y \in Q = \cap_{j=1}^{N} F(R_2^j) \text{ such that } Ax = By, \tag{2.5.6}$$

where N is a positive integer, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators, $R_i^j : H \to CB(H_i)$, $i = 1, 2$, $j = 1, 2, \ldots, N$ is a family of multi-valued quasi-nonexpansive mappings. They established strong convergence result to a solution of problem (2.5.6).

Shehu [104] also studied the MSSEFPP for infinite families of multi-valued quasi-nonexpansive mappings: Find

$$x \in \cap_{i=1}^{\infty} F(S_i) \text{ and } y \in \cap_{i=1}^{\infty} F(T_i) \text{ such that } Ax = By, \tag{2.5.7}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators, $S_i : H_1 \to CB(H_1)$ and $T_i : H_2 \to CB(H_2)$, $i = 1, 2, \ldots$ are two infinite families of multi-valued quasi-nonexpansive mappings. With these assumptions, he proposed the following algorithm for finding a solution of problem (2.5.7):

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = t_n u + (a_{0,n} - t_n)u_n + \sum_{i=1}^{\infty} \alpha_{i,n} w_{i,n}, \quad w_{i,n} \in S_i u_n, \\ v_n = y_n - \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = t_n v + (a_{0,n} - t_n)v_n + \sum_{i=1}^{\infty} \alpha_{i,n} z_{i,n}, \quad z_{i,n} \in T_i u_n, \end{cases} \tag{2.5.8}$$

where $\{\gamma_n\} \in \left(\epsilon, \frac{2||Ax_n - By_n||^2}{||A^*(Ax_n - By_n)||^2 + ||B^*(Ax_n - By_n)||^2} - \epsilon\right), n \in \Omega$, otherwise $\gamma_n = \gamma$ (γ being any nonegative value), where the set of indexes $\Omega = \{n : Ax_n - By_n \neq 0\}$. Shehu [104] established a strong convergence result for problem (2.5.7) using Algorithm (2.5.8).

Based on the works of Chidume *et al.* [40] and Wu *et al.* [116], Chidume *et al.* [41] introduced the following algorithm for solving the MSSEFPP for countable families of multi-valued demi-contractive mappings which are more general than the mappings considered by the authors above:

$$\begin{cases} x_{n+1} = a_0 \left(x_n - \gamma A^*(Ax_n - By_n)\right) + \sum_{i=1}^{\infty} \alpha_i z_n^i, \\ y_{n+1} = a_0 \left(y_n - \gamma B^*(Ax_n - By_n)\right) + \sum_{i=1}^{\infty} \alpha_j w_n^j, \forall n \geq 1, \end{cases} \tag{2.5.9}$$

where $z_n^i \in S_i\left(x_n - \gamma A^*(Ax_n - By_n)\right),\ w_n^j \in T_i\left(y_n - \gamma B^*(Ax_n - By_n)\right),\ A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear maps, $S_i : H_1 \to CB(H_1),\ i = 1, 2\ldots$ and $T_j : H_2 \to CB(H_2),\ j = 1, 2, \ldots$ are two families of multi-valued demi-contractive mappings. Chidume *et al.* [41] proved weak and strong convergence results for problem (2.5.7) using Algorithm (2.5.9).

Chapter 3

Split Equality Variational Inequality Problem and Split Equality Fixed Point Problem

3.1 Introduction

In this chapter, we study the following problem which we call SEVIP and SEFPP: Let H_1, H_2, H_3 be real Hilbert spaces and $C_1,\ C_2$ be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $T : C_1 \to C_1$ and $S : C_2 \to C_2$ be mappings with $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$. Let $f_i : C_i \to C_i,\ (i = 1, 2)$ be ρ-inverse strongly monotone operators ($\rho > 0$) and $A : H_1 \to H_3,\ B : H_2 \to H_3$ be bounded linear operators. Find $(\bar{x}, \bar{y}) \in F(T) \times F(S)$ such that

$$\langle f_1(\bar{x}),\ x - \bar{x}\rangle \geq 0 \ \ \forall x \in C_1, \tag{3.1.1}$$

$$\langle f_2(\bar{y}),\ y - \bar{y}\rangle \geq 0 \ \ \forall y \in C_2 \tag{3.1.2}$$

and such that

$$A\bar{x} = B\bar{y}. \tag{3.1.3}$$

Furthermore, we propose an iterative scheme and using the iterative scheme, we state and prove a strong convergence result for the approximation of a solution of (3.1.1)-(3.1.3). We apply our result to study other related problems. In what follows, we give the following definitions which will be very useful throughout this chapter.

Definition 3.1.1. *Let H be a real Hilbert space and $T : H \to H$ be a nonlinear mapping. Then $(I - T)$ is said to be demi-closed at 0 if for any sequence $\{x_n\} \subset H$ such that $x_n \rightharpoonup x^*$ and $(I - T)x_n \to 0$, we have that $x^* = Tx^*$.*

Definition 3.1.2. *A mapping $T : C \to C$ is said to be demi-contractive if $F(T) \neq \emptyset$ and there exists a constant $k \in (0,1)$ such that*

$$\|Tx - y\|^2 \leq \|x - y\|^2 + k||x - Tx||^2 \quad \forall x \in C, \; y \in F(T). \tag{3.1.4}$$

In a real Hilbert space H, it is known that (3.1.4) is equivalent to

$$\langle Tx - y, x - y\rangle \leq \|x - y\|^2 - \frac{1-k}{2}\|x - Tx\|^2. \tag{3.1.5}$$

Example 3.1.3. *[60]. Let $H = \mathbb{R}$ with the usual norm and $C = [-2, 0]$. Let $T : C \to C$ be defined by*

$$Tx = \begin{cases} x^2 - 2, & \text{if } x \in [-1, 0], \\ -\frac{1}{8}, & \text{if } x = -\frac{3}{2}, \\ -1, & x \in [-2, -\frac{3}{2}) \cup (-\frac{3}{2}, -1]. \end{cases} \tag{3.1.6}$$

Then T is a demicontractive mapping with $(I - T)$ been demiclosed at 0.

3.2 Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. We denote the strong and weak convergence by "$\longrightarrow$" and "$\rightharpoonup$" respectively and the solution set of (3.1.1)-(3.1.3) by Γ defined by

$$\Gamma := \{(\bar{x}, \bar{y}) \in F(T) \times F(S) : \langle f_1(\bar{x}), x - \bar{x}\rangle \geq 0, \forall x \in C_1, \; \langle f_2(\bar{y}), y - \bar{y}\rangle \geq 0, \forall y \in C_2 \text{ and } A\bar{x} = B\bar{y}\}.$$

Lemma 3.2.1. *Let H be a real Hilbert space, then*

$$2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x - y||^2 = ||x + y||^2 - ||x||^2 - ||y||^2, \quad \forall\; x, y \in H.$$

Proof.

$$\begin{aligned} ||x + y||^2 = \langle x + y, x + y\rangle &= \langle x, x + y\rangle + \langle y, x + y\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= ||x||^2 + \langle x, y\rangle + \langle x, y\rangle + ||y||^2 \\ &= ||x||^2 + 2\langle x, y\rangle + ||y||^2. \end{aligned} \tag{3.2.1}$$

Similarly, we have

$$||x - y||^2 = ||x||^2 - 2\langle x, y\rangle + ||y||^2. \tag{3.2.2}$$

From (3.2.1) and (3.2.2), we obtain

$$2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x - y||^2 = ||x + y||^2 - ||x||^2 - ||y||^2, \quad \forall\; x, y \in H.$$

□

Lemma 3.2.2. *[37]. Let H be a real Hilbert space, then $\forall\ x, y \in H$ and $\alpha \in (0,1)$, we have*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2.$$

Lemma 3.2.3. *[119]. Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n, \quad n \ge 0, \tag{3.2.3}$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
(i) $\Sigma_{n=0}^{\infty}\gamma_n = \infty$, or equivalently, $\prod_{n=1}^{\infty}(1-\gamma_n) = 0$,
(ii) $\limsup_{n\to\infty}\delta_n \le 0$ or $\Sigma_{n=0}^{\infty}|\delta_n\gamma_n| < \infty$,
Then $\lim_{n\to\infty} a_n = 0$.

Proof. Let $\epsilon > 0$ be given and let N be an integer big enough such that

$$\delta_n < \epsilon,\ \forall n \ge N.$$

Then, from (3.2.3) and by induction we obtain, for $n \ge N$ that

$$\begin{aligned} a_{n+1} &\le \left(\prod_{k=N}^{n}(1-\gamma_k)\right)a_N + \left(1 - \prod_{k=N}^{n}(1-\gamma_k)\right)\delta_k \\ &\le \left(\prod_{k=N}^{n}(1-\gamma_k)\right)a_N + \left(1 - \prod_{k=N}^{n}(1-\gamma_k)\right)\epsilon. \end{aligned} \tag{3.2.4}$$

Using condition (ii) in (3.2.4), we obtain

$$\limsup_{n\to\infty} a_n \le 2\epsilon,$$

which implies $\lim_{n\to\infty} a_n = 0$. □

3.3 Main result

Theorem 3.3.1. *Let H_1, H_2, H_3 be real Hilbert spaces and C_1, C_2 be nonempty, closed and convex subsets of H_1, H_2 respectively. Let $T : C_1 \to C_1$ and $S : C_2 \to C_2$ be demi-contractive mappings with constants k_1 and k_2 respectively such that $I - T$ and $I - S$ are demi-closed at 0. Let $f_i : C_i \to C_i$ be μ_i-inverse strongly monotone operators $(i = 1, 2)$ and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operators. Assume that the solution set $\Gamma \ne \emptyset$ and that the stepsize sequence γ_n is chosen in such a way that for some $\epsilon > 0$,*

$$\gamma_n \in \left(\epsilon,\ \frac{2||Aw_n - Bz_n||^2}{||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2} - \epsilon\right), n \in \Omega,$$

otherwise, $\gamma_n = \gamma$ (γ being any nonegative value), where the set of indexes $\Omega = \{n : Aw_n - Bz_n \ne 0\}$.

Let $u, x_0 \in C_1,\ v, y_0 \in C_2$ be arbitrary and the sequence $\{(x_n, y_n)\}$ be generated by

$$\begin{cases} w_n = (1-\alpha_n)x_n + \alpha_n u, \\ z_n = (1-\alpha_n)y_n + \alpha_n v; \\ u_n = P_{C_1}(I-\rho f_1)(w_n - \gamma_n A^*(Aw_n - Bz_n)), \\ v_n = P_{C_2}(I-\rho f_2)(z_n + \gamma_n B^*(Aw_n - Bz_n)), \\ x_{n+1} = (1-\beta_n)u_n + \beta_n T u_n, \\ y_{n+1} = (1-\lambda_n)v_n + \lambda_n S v_n,\ n \geq 0, \end{cases} \tag{3.3.1}$$

$0 < \rho < 2\mu_i$ *for each* $i = 1, 2,$ *with conditions*

(i) $\{\alpha_n\},\ \{\beta_n\}$ *and* $\{\lambda_n\}$ *are sequences in* $(0, 1)$*,*

(ii) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$

(iii) $\beta_n \in (a, 1-k_1) \subseteq (0, 1)$ for some $a > 0,$

(iv) $\lambda_n \in (b, 1-k_2) \subseteq (0, 1)$ for some $b > 0.$

Then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y})$ *in* Γ*.*

Proof. First we show that γ_n is well defined. For any $(x, y) \in \Gamma$, we have

$$\langle A^*(Aw_n - Bz_n), w_n - x\rangle = \langle Aw_n - Bz_n, Aw_n - Ax\rangle, \tag{3.3.2}$$

and

$$\langle B^*(Aw_n - Bz_n), y - z_n\rangle = \langle Aw_n - Bz_n, By - Bz_n\rangle. \tag{3.3.3}$$

Adding (3.3.2) and (3.3.3) and noting that $Ax = By$, we obtain $\forall n \in \Omega$,

$$\begin{aligned} ||Aw_n - Bz_n||^2 &= \langle A^*(Aw_n - Bz_n), w_n - x\rangle + \langle B^*(Aw_n - Bz_n), y - z_n\rangle \\ &\leq ||A^*(Aw_n - Bz_n)||||w_n - x|| + ||B^*(Aw_n - Bz_n)||||y - z_n||. \end{aligned}$$

Therefore, for $n \in \Omega$, that is, $||Aw_n - Bz_n|| > 0$, we have $||A^*(Aw_n - Bz_n)|| \neq 0$ or $||B^*(Aw_n - Bz_n)|| \neq 0$. Thus, γ_n is well defined.

Let $(x^*, y^*) \in \Gamma$, we have from (3.3.1) that

$$\begin{aligned} ||u_n - x^*||^2 &= ||P_{C_1}(I-\rho f_1)(w_n - \gamma_n A^*(Aw_n - Bz_n)) - x^*||^2 \\ &\leq ||w_n - \gamma_n A^*(Aw_n - Bz_n) - x^*||^2 \\ &= ||w_n - x^*||^2 - 2\gamma_n\langle w_n - x^*, A^*(Aw_n - Bz_n)\rangle \\ &\quad + \gamma_n^2||A^*(Aw_n - Bz_n)||^2. \end{aligned} \tag{3.3.4}$$

From Lemma 3.2.1 and noting that A^* is the adjoint of A, we have

$$\begin{aligned} -2\langle w_n - x^*, A^*(Aw_n - Bz_n)\rangle &= -2\langle Aw_n - Ax^*, Aw_n - Bz_n\rangle \\ &= -||Aw_n - Ax^*||^2 - ||Aw_n - Bz_n||^2 \\ &\quad + ||Bz_n - Ax^*||^2. \end{aligned} \tag{3.3.5}$$

Substituting (3.3.5) into (3.3.4), we have

$$\begin{aligned} ||u_n - x^*||^2 &\leq ||w_n - x^*||^2 - \gamma_n ||Aw_n - Ax^*||^2 - \gamma_n ||Aw_n - Bz_n||^2 \\ &+ \gamma_n ||Bz_n - Ax^*||^2 + \gamma_n^2 ||A^*(Aw_n - Bz_n)||^2. \end{aligned} \tag{3.3.6}$$

Similarly, from (3.3.1), we have

$$\begin{aligned} ||v_n - y^*||^2 &\leq ||z_n - y^*||^2 - \gamma_n ||Bz_n - By^*||^2 - \gamma_n ||Aw_n - Bz_n||^2 \\ &+ \gamma_n ||Aw_n - By^*||^2 + \gamma_n^2 ||B^*(Aw_n - Bz_n)||^2. \end{aligned} \tag{3.3.7}$$

Adding inequality (3.3.6) and (3.3.7), and using the fact that $Ax^* = By^*$, we obtain

$$\begin{aligned} ||u_n - x^*||^2 + ||v_n - y^*||^2 &\leq ||w_n - x^*||^2 + ||z_n - y^*||^2 - \gamma_n \big[2||Aw_n - Bz_n||^2 \\ &- \gamma_n \left(||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2 \right) \big] \\ &\leq ||w_n - x^*||^2 + ||z_n - y^*||^2. \end{aligned} \tag{3.3.8}$$

From (3.3.1) and the fact that T is demi-contractive, we obtain

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= ||(1-\beta_n)u_n + \beta_n T u_n - x^*||^2 \\ &= ||(1-\beta_n)(u_n - x^*) + \beta_n (Tu_n - x^*)||^2 \\ &= (1-\beta_n)^2 ||u_n - x^*||^2 + \beta_n^2 ||Tu_n - x^*||^2 + 2\beta_n (1-\beta_n) \langle u_n - x^*, Tu_n - x^* \rangle \\ &\leq (1-\beta_n)^2 ||u_n - x^*||^2 + \beta_n^2 \left[||u_n - x^*||^2 + k_1 ||u_n - Tu_n||^2 \right] \\ &\quad + 2\beta_n (1-\beta_n) \left[||u_n - x^*||^2 - \frac{1-k_1}{2} ||u_n - Tu_n||^2 \right] \\ &= (1 - 2\beta_n + \beta_n^2) ||u_n - x^*||^2 + \beta_n^2 \left[||u_n - x^*||^2 + k_1 ||u_n - Tu_n||^2 \right] \\ &\quad + 2\beta_n ||u_n - x^*||^2 - 2\beta_n^2 ||u_n - x^*||^2 - \beta_n (1-\beta_n)(1-k_1) ||u_n - Tu_n||^2 \\ &= ||u_n - x^*||^2 + \beta_n [k_1 + \beta_n - 1] ||u_n - Tu_n||^2 \\ &\leq ||u_n - x^*||^2. \end{aligned} \tag{3.3.9}$$

Similarly, we have that

$$||y_{n+1} - y^*||^2 \leq ||v_n - y^*||^2. \tag{3.3.10}$$

Adding (3.3.9) and (3.3.10), and using (3.3.8), we obtain

$$\begin{aligned} ||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2 &\leq ||u_n - x^*||^2 + ||v_n - y^*||^2 \\ &\leq ||w_n - x^*||^2 + ||z_n - y^*||^2. \end{aligned} \tag{3.3.11}$$

From (3.3.1), (3.3.11) and Lemma 3.2.2, we obtain

$$\begin{aligned}
||x_{n+1}-x^*||^2+||y_{n+1}-y^*||^2 &\leq ||(1-\alpha_n)x_n+\alpha_n u-x^*||^2+||(1-\alpha_n)y_n+\alpha_n v-y^*||^2\\
&= ||(1-\alpha_n)(x_n-x^*)+\alpha_n(u-x^*)||^2\\
&\quad +||(1-\alpha_n)(y_n-y^*)+\alpha_n(v-y^*)||^2\\
&\leq (1-\alpha_n)||x_n-x^*||^2+\alpha_n||u-x^*||^2\\
&\quad +(1-\alpha_n)||y_n-y^*||^2+\alpha_n||v-y^*||^2\\
&= (1-\alpha_n)\left[||x_n-x^*||^2+||y_n-y^*||^2\right]\\
&\quad +\alpha_n\left[||u-x^*||^2+||v-y^*||^2\right]\\
&\leq \max\left\{||x_n-x^*||^2+||y_n-y^*||^2, ||u-x^*||^2+||v-y^*||^2\right\}\\
&\ \ \vdots\\
&\leq \max\left\{||x_0-x^*||^2+||y_0-y^*||^2, ||u-x^*||^2+||v-y^*||^2\right\}.
\end{aligned}$$

Therefore, $\{||x_n-x^*||^2+||y_n-y^*||^2\}$ is bounded. Consequently $\{x_n\}$, $\{y_n\}$, $\{w_n\}$, $\{z_n\}$, $\{u_n\}$, $\{v_n\}$, $\{Ax_n\}$ and $\{By_n\}$ are bounded. From (3.3.8), (3.3.11) and Lemma 3.2.2, we obtain

$$\begin{aligned}
||x_{n+1}-x^*||^2+||y_{n+1}-y^*||^2 &\leq ||w_n-x^*||^2+||z_n-y^*||^2-\gamma_n\big[2||Aw_n-Bz_n||^2\\
&\quad -\gamma_n\left(||A^*(Aw_n-Bz_n)||^2+||B^*(Aw_n-Bz_n)||^2\right)\big]\\
&\leq (1-\alpha_n)\left[||x_n-x^*||^2+||y_n-y^*||^2\right]\\
&\quad +\alpha_n\left[||u-x^*||^2+||v-y^*||^2\right]-\gamma_n\big[2||Aw_n-Bz_n||^2\\
&\quad -\gamma_n\big(||A^*(Aw_n-Bz_n)||^2+||B^*(Aw_n-Bz_n)||^2\big)\big].
\end{aligned}$$

Let $P_n=||A^*(Aw_n-Bz_n)||^2+||B^*(Aw_n-Bz_n)||^2$, then we have (by the condition on γ_n)

$$\begin{aligned}
\epsilon P_n &\leq (1-\alpha_n)\left[||x_n-x^*||^2+||y_n-y^*||^2\right]+\alpha_n\left[||u-x^*||^2+||v-y^*||^2\right]\\
&\quad -\left[||x_{n+1}-x^*||^2+||y_{n+1}-y^*||^2\right].
\end{aligned} \tag{3.3.12}$$

We now consider two cases to establish the strong convergence of $\{(x_n,y_n)\}$ to $(\bar{x},\bar{y})$.
Case 1: Assume that $\{||x_n-x^*||^2+||y_n-y^*||^2\}$ is monotone decreasing, then $\{||x_n-x^*||^2+||y_n-y^*||^2\}$ is convergent, thus

$$\lim_{n\to\infty}\left[\left(||x_{n+1}-x^*||^2+||y_{n+1}-y^*||^2\right)-\left(||x_n-x^*||^2+||y_n-y^*||^2\right)\right]=0.$$

From (3.3.12), we have

$(||A^*(Aw_n-Bz_n)||^2+||B^*(Aw_n-Bz_n)||^2)\to 0$, as $n\to\infty$.
Since $Aw_n-Bz_n=0$, if $n\notin\Omega$, we have

$$\lim_{n\to\infty}||A^*(Aw_n-Bz_n)||^2=\lim_{n\to\infty}||B^*(Aw_n-B^*z_n)||^2=0. \tag{3.3.13}$$

From (3.3.1), we have

$$\begin{aligned}
||w_n-x_n||^2 &= ||(1-\alpha_n)x_n+\alpha_n u-x_n||^2\\
&= \alpha_n||u-x_n||^2\to 0,\ \text{as } n\to\infty,
\end{aligned}$$

which implies

$$\lim_{n\to\infty} ||w_n - x_n||^2 = 0. \tag{3.3.14}$$

Similarly, we have

$$\lim_{n\to\infty} ||z_n - y_n||^2 = 0. \tag{3.3.15}$$

Also, from (3.3.1), Lemma 3.2.1 and Lemma 3.2.2, we have that

$$\begin{aligned}
||u_n - x^*||^2 &= ||P_{C_1}(I - \rho f_1)(w_n - \gamma_n A^*(Aw_n - Bz_n)) - x^*||^2 \\
&\leq \langle u_n - x^*, w_n - \gamma_n A^*(Aw_n - Bz_n) - x^* \rangle \\
&= \frac{1}{2}[||u_n - x^*||^2 + ||w_n - \gamma_n A^*(Aw_n - Bz_n) - x^*||^2 \\
&\quad -||u_n - x^* - (w_n - \gamma_n A^*(Aw_n - Bz_n) - x^*)||^2] \\
&\leq \frac{1}{2}[||u_n - x^*||^2 + ||w_n - x^*||^2 + \gamma_n^2||A^*(Aw_n - Bz_n)||^2 \\
&\quad +2\gamma_n||w_n - x^*||||A^*(Aw_n - Bz_n)|| \\
&\quad -(||u_n - w_n||^2 + \gamma_n^2||A^*(Aw_n - Bz_n)||^2 - 2\gamma_n\langle u_n - w_n, A^*(Aw_n - Bz_n)\rangle]) \\
&= \frac{1}{2}[||u_n - x^*||^2 + ||w_n - x^*||^2 + 2\gamma_n||w_n - x^*||||A^*(Aw_n - Bz_n)|| \\
&\quad -||u_n - w_n||^2 + 2\gamma_n\langle u_n - w_n, A^*(Aw_n - Bz_n)\rangle] \\
&\leq \frac{1}{2}[||u_n - x^*||^2 + ||w_n - x^*||^2 + 2\gamma_n||w_n - x^*||||A^*(Aw_n - Bz_n)|| \\
&\quad -||u_n - w_n||^2 + 2\gamma_n||u_n - w_n||||A^*(Aw_n - Bz_n)||] \\
&\leq \frac{1}{2}[||u_n - x^*||^2 + (1-\alpha_n)||x_n - x^*||^2 + \alpha_n||u - x^*||^2 \\
&\quad +2\gamma_n||w_n - x^*||||A^*(Aw_n - Bz_n)|| - ||u_n - w_n||^2 \\
&\quad +2\gamma_n||u_n - w_n||||A^*(Aw_n - Bz_n)||] \\
&\leq \frac{1}{2}[||u_n - x^*||^2 + ||x_n - x^*||^2 + \alpha_n||u - x^*||^2 \\
&\quad +2\gamma_n||w_n - x^*||||A^*(Aw_n - Bz_n)|| \\
&\quad -||u_n - w_n||^2 + 2\gamma_n||u_n - w_n||||A^*(Aw_n - Bz_n)||], \qquad (3.3.16)
\end{aligned}$$

which implies

$$\begin{aligned}
||u_n - x^*||^2 &\leq ||x_n - x^*||^2 + \alpha_n||u - x^*||^2 + 2\gamma_n||w_n - x^*||||A^*(Aw_n - Bz_n)|| \\
&\quad -||u_n - w_n||^2 + 2\gamma_n||u_n - w_n||||A^*(Aw_n - Bz_n)||. \qquad (3.3.17)
\end{aligned}$$

From (3.3.9) and (3.3.17), we have

$$\begin{aligned}
||x_{n+1} - x^*||^2 &\leq ||x_n - x^*||^2 + \alpha_n||u - x^*||^2 + 2\gamma_n||w_n - x^*||||A^*(Aw_n - Bz_n)|| \\
&\quad -||u_n - w_n||^2 + 2\gamma_n||u_n - w_n||||A^*(Aw_n - Bz_n)||. \qquad (3.3.18)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
||y_{n+1} - y^*||^2 &\leq ||y_n - y^*||^2 + \alpha_n||v - y^*||^2 + 2\gamma_n||z_n - y^*||||B^*(Aw_n - Bz_n)|| \\
&\quad -||v_n - z_n||^2 + 2\gamma_n||v_n - z_n||||B^*(Aw_n - Bz_n)||. \qquad (3.3.19)
\end{aligned}$$

Adding (3.3.18) and (3.3.19), we have

$$\begin{aligned}
||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2 \;\; &\leq \;\; ||x_n - x^*||^2 + ||y_n - y^*||^2 + \alpha_n[||u - x^*||^2 + ||v - x^*||^2] \\
&+2\gamma_n[||w_n - x^*||||A^*(Aw_n - Bz_n)|| \\
&+||z_n - y^*||||B^*(Aw_n - Bz_n)||] \\
&-[||u_n - w_n||^2 + ||v_n - z_n||^2] \\
&+2\gamma_n[||u_n - w_n||||A^*(Aw_n - Bz_n)|| \\
&+||v_n - z_n||||B^*(Aw_n - Bz_n)||]. \qquad (3.3.20)
\end{aligned}$$

Using (3.3.13) together with the fact that $\alpha_n \to 0$, as $n \to \infty$ in (3.3.20), we have

$$\lim_{n\to\infty} \left[||u_n - w_n||^2 + ||v_n - z_n||^2\right] = 0,$$

which implies

$$\lim_{n\to\infty} ||u_n - w_n||^2 = 0 \qquad (3.3.21)$$

and

$$\lim_{n\to\infty} ||v_n - z_n||^2 = 0. \qquad (3.3.22)$$

Observe that since T is demicontractive and $x^* \in F(T)$, we have

$$\begin{aligned}
||Tx - x^*||^2 \;\; &\leq \;\; ||x - x^*||^2 + k_1||x - Tx||^2 \\
\implies \langle Tx - x^*, Tx - x^*\rangle \;\; &\leq \;\; \langle x - x^*, x - x^*\rangle + k_1||x - Tx||^2 \\
\implies \langle Tx - x^*, Tx - x\rangle + \langle Tx - x^*, x - x^*\rangle \;\; &\leq \;\; \langle x - x^*, x - x^*\rangle + k_1||x - Tx||^2 \\
\implies \langle Tx - x^*, Tx - x\rangle \;\; &\leq \;\; \langle x - Tx, x - x^*\rangle + k_1||x - Tx||^2 \\
\implies \langle Tx - x, Tx - x\rangle + \langle x - x^*, Tx - x\rangle \;\; &\leq \;\; \langle x - Tx, x - x^*\rangle + k_1||x - Tx||^2 \\
||Tx - x||^2 \;\; &\leq \;\; \langle x - x^*, x - Tx\rangle \\
&\quad -\langle x - x^*, Tx - x\rangle + k_1||x - Tx||^2 \\
\implies (1 - k_1)||Tx - x||^2 \;\; &\leq \;\; 2\langle x - x^*, x - Tx\rangle. \qquad (3.3.23)
\end{aligned}$$

From (3.3.1) and (3.3.23), we have

$$\begin{aligned}
||x_{n+1} - x^*||^2 \;\; &= \;\; ||(1 - \beta_n)u_n + \beta_n Tu_n - x^*||^2 \\
&= \;\; ||u_n - x^* + \beta_n(Tu_n - u_n)||^2 \\
&= \;\; ||u_n - x^*||^2 + \beta_n^2||Tu_n - u_n||^2 - 2\beta_n\langle u_n - x^*, u_n - Tu_n\rangle \\
&\leq \;\; ||u_n - x^*||^2 + \beta_n^2||Tu_n - u_n||^2 - (1 - k_1)\beta_n||Tu_n - u_n||^2 \\
&= \;\; ||u_n - x^*||^2 + \beta_n(\beta_n - (1 - k_1))||u_n - Tu_n||^2. \qquad (3.3.24)
\end{aligned}$$

Similarly, we have that

$$||y_{n+1} - y^*||^2 \leq ||v_n - y^*||^2 + \lambda_n(\lambda_n - (1 - k_2)||v_n - Sv_n||^2. \qquad (3.3.25)$$

Adding (3.3.24) and (3.3.25), we have

$$\begin{aligned}||x_{n+1}-x^*||^2+||y_{n+1}-y^*||^2 &\leq ||u_n-x^*||^2+||v_n-y^*||^2+\beta_n(\beta_n-(1-k_1))||Tu_n-u_n||^2\\ &\quad +\lambda_n(\lambda_n(1-k_2))||v_n-Sv_n||^2\\ &\leq ||w_n-x^*||+||z_n-y^*||^2+\beta_n(\beta_n-(1-k_1))||Tu_n-u_n||^2\\ &\quad +\lambda_n(\lambda_n-(1-k_2))||v_n-Sv_n||^2\\ &\leq (1-\alpha_n)\left[||x_n-x^*||^2+||y_n-y^*||^2\right]\\ &\quad +\alpha_n\left[||u-x^*||^2+||v-y^*||^2\right]\\ &\quad +\beta_n(\beta_n-(1-k_1))||u_n-Tu_n||^2\\ &\quad +\lambda_n(\lambda_n-(1-k_2))||v_n-Sv_n||^2. \end{aligned} \tag{3.3.26}$$

Let $K_n=\beta_n((1-k_1)-\beta_n)||u_n-Tu_n||^2+\lambda_n((1-k_2)-\lambda_n)||v_n-Sv_n||^2$, then

$$\begin{aligned}K_n &\leq (1-\alpha_n)\left[||x_n-x^*||^2+||y_n-y^*||^2\right]-\left[||x_{n+1}-x^*||^2+||y_{n+1}-y^*||^2\right]\\ &\quad +\alpha_n\left[||u||^2+||v||^2\right]\to 0,\ \text{ as } n\to\infty, \end{aligned} \tag{3.3.27}$$

which implies

$$||u_n-Tu_n||^2+||v_n-Sv_n||^2\to 0,\ \text{ as } n\to\infty.$$

Hence,

$$\lim_{n\to\infty}||u_n-Tu_n||^2=0, \tag{3.3.28}$$

and

$$\lim_{n\to\infty}||v_n-Sv_n||^2=0. \tag{3.3.29}$$

From (3.3.28), we have

$$\lim_{n\to\infty}||x_{n+1}-u_n|| = \lim_{n\to\infty}\beta_n||u_n-Tu_n||=0. \tag{3.3.30}$$

Similarly, from (3.3.29), we have

$$\lim_{n\to\infty}||y_{n+1}-v_n|| = \lim_{n\to\infty}\lambda_n||v_n-Sv_n||=0. \tag{3.3.31}$$

From (3.3.14) and (3.3.21), we have
$||x_n-u_n||\leq||x_n-w_n||+||w_n-u_n||\to 0$, as $n\to\infty$,
which implies that

$$\lim_{n\to\infty}||x_n-u_n||=0. \tag{3.3.32}$$

Similarly, from (3.3.15) and (3.3.22), we have

$$\lim_{n\to\infty}||y_n-v_n||=0. \tag{3.3.33}$$

Also, from (3.3.30) and (3.3.32), we have
$||x_{n+1} - x_n|| \leq ||x_{n+1} - u_n|| + ||u_n - x_n|| \to 0$, as $n \to \infty$,
which implies that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.3.34}$$

Similarly, from (3.3.31) and (3.3.33), we have

$$\lim_{n\to\infty} ||y_{n+1} - y_n|| = 0. \tag{3.3.35}$$

Since $\{x_n\}$ is bounded, there exist a subsequence of $\{x_n\}$ (without loss of generality, still denoted by $\{x_n\}$) such that $\{x_n\}$ converges weakly to $\bar{x} \in C_1$. By (3.3.32) and (3.3.14), we have that $\{u_n\}$ and $\{w_n\}$ converges weakly to $\bar{x}$ and by the demi-closeness of $I - T$ at 0 and (3.3.28), we have that $\bar{x} \in F(T)$. Since $\{y_n\}$ is bounded, there exist a subsequence of $\{y_n\}$ (without loss of generality, still denoted by $\{y_n\}$) such that $\{y_n\}$ converges weakly to $\bar{y} \in C_2$. By (3.3.33) and (3.3.15), we have that $\{v_n\}$ and $\{z_n\}$ converges weakly to $\bar{y}$ and by the demi-closeness of $I - S$ at 0 and (3.3.29), we have that $\bar{y} \in F(S)$.
Also, since A and B are bounded linear operators, we have that $\{Aw_n\}$ converges weakly to $A\bar{x}$ and $\{Bz_n\}$ converges weakly to $B\bar{y}$.

Next, we show that $A\bar{x} = B\bar{y}$.

$$\begin{aligned}
||A\bar{x} - B\bar{y}||^2 &= \langle A\bar{x} - B\bar{y}, A\bar{x} - B\bar{y}\rangle \\
&= \langle A\bar{x} - B\bar{y}, A\bar{x} - B\bar{y} + Aw_n - Aw_n + Bz_n - Bz_n\rangle \\
&= \langle A\bar{x} - B\bar{y}, A\bar{x} - Aw_n\rangle + \langle A\bar{x} - B\bar{y}, Aw_n - Bz_n\rangle \\
&\quad + \langle A\bar{x} - B\bar{y}, Bz_n - B\bar{y}\rangle \\
&= \langle A\bar{x} - B\bar{y}, A\bar{x} - Aw_n\rangle + \langle A\bar{x}, Aw_n - Bz_n\rangle \\
&\quad - \langle B\bar{y}, Aw_n - Bz_n\rangle + \langle A\bar{x} - B\bar{y}, Bz_n - B\bar{y}\rangle \\
&= \langle A\bar{x} - B\bar{y}, A\bar{x} - Aw_n\rangle + \langle \bar{x}, A^*(Aw_n - Bz_n)\rangle \\
&\quad - \langle \bar{y}, B^*(Aw_n - Bz_n)\rangle + \langle A\bar{x} - B\bar{y}, Bz_n - B\bar{y}\rangle \\
&\leq \langle A\bar{x} - B\bar{y}, A\bar{x} - Aw_n\rangle + ||\bar{x}||||A^*(Aw_n - Bz_n)|| \\
&\quad + ||\bar{y}||||B^*(Aw_n - Bz_n)|| \\
&+ \langle A\bar{x} - B\bar{y}, Bz_n - B\bar{y}\rangle \to 0, n \to \infty,
\end{aligned}$$

which implies that $||A\bar{x} - B\bar{y}|| = 0$. Hence $A\bar{x} = B\bar{y}$.

We now show that $\bar{x} \in VI(C_1, f_1)$, that is $\bar{x}$ satisfies $\langle f_1(\bar{x}), x - \bar{x}\rangle \geq 0 \ \forall x \in C_1$ and $\bar{y} \in VI(C_2, f_2)$, that is $\bar{y}$ satisfies $\langle f_2(\bar{y}), y - \bar{y}\rangle \geq 0 \ \forall y \in C_2$.
Let $N_{C_1}z$ be the normal cone of C_1 at a point $z \in C_1$, then we define the following set-valued operator $B : C_1 \to 2^{C_1}$ by

$$Mz = \begin{cases} f_1 z + N_{C_1} z, & z \in C_1 \\ \emptyset, & z \notin C_1. \end{cases}$$

Then, M is maximal monotone. Let $(z, w) \in G(M)$, then $w - f_1 z \in N_{C_1} z$. For $u_n \in C_1$, we have

$$\langle z - u_n, w - f_1 z\rangle \geq 0. \tag{3.3.36}$$

Let $a_n = w_n - \gamma_n A^*(Aw_n - Bz_n)$.
Then, $||a_n - w_n||^2 = \gamma_n^2 ||A^*(Aw_n - Bz_n)||^2 \to 0$, as $n \to \infty$.
and

$$||u_n - a_n|| \le ||u_n - w_n|| + ||w_n - a_n|| \to 0, \text{ as } n \to \infty. \tag{3.3.37}$$

From $u_n = P_{C_1}(a_n - \rho f_1 a_n)$, we have $\langle z - u_n, u_n - (a_n - \rho f_1 a_n)\rangle \ge 0$, which implies $\langle z - u_n, \frac{u_n - a_n}{\rho} + f_1 a_n\rangle \ge 0$.
From (3.3.36), we get

$$\begin{aligned} \langle z - u_n, w\rangle &\ge \langle z - u_n, f_1 z\rangle \\ &\ge \langle z - u_n, f_1 z\rangle - \langle z - u_n, \frac{u_n - a_n}{\rho} + f_1 a_n\rangle \\ &= \langle z - u_n, f_1 z - f_1 a_n - \frac{u_n - a_n}{\rho}\rangle \\ &= \langle z - u_n, f_1 z - f_1 u_n\rangle + \langle z - u_n, f_1 u_n - f_1 a_n\rangle - \langle z - u_n, \frac{u_n - a_n}{\rho}\rangle \\ &\ge \langle z - u_n, f_1 u_n - f_1 a_n\rangle - \langle z - u_n, \frac{u_n - a_n}{\rho}\rangle. \end{aligned} \tag{3.3.38}$$

Since f_1 is Lipschitz continuous (by Remark 2.1.6), we have from (3.3.37) that

$$\lim_{n\to\infty} ||f_1 u_n - f_1 a_n|| = 0. \tag{3.3.39}$$

Using (3.3.39) together with the fact that $\{u_n\}$ converges weakly to $\bar{x}$, we obtain from (3.3.38), that $\langle z - \bar{x}, w\rangle \ge 0$. Also, B is maximal monotone, this gives us that $\bar{x} \in M^{-1}(0)$ which implies $0 \in M(\bar{x})$. Hence, $\bar{x} \in VI(C_1, f_1)$, that is $\langle f_1(\bar{x}), z - \bar{x}\rangle \ge 0, \forall z \in C_1$. In the same manner, we obtain that $\langle f_2(\bar{y}), y - \bar{y}\rangle \ge 0, \forall y \in C_2$.

So far, we have succeeded in showing that $(\bar{x}, \bar{y}) \in \Gamma$. Next, we show that $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$.
From (3.3.11), we have

$$\begin{aligned} ||x_{n+1} - \bar{x}||^2 + ||y_{n+1} - \bar{y}||^2 &\le ||w_n - \bar{x}||^2 + ||z_n - \bar{y}||^2 \\ &= (1-\alpha_n)^2 ||x_n - \bar{x}||^2 + \alpha_n^2 ||u - \bar{x}||^2 + 2(1-\alpha_n)\alpha_n \langle x_n - \bar{x}, u - \bar{x}\rangle \\ &\quad + (1-\alpha_n)^2 ||y_n - \bar{y}||^2 + \alpha_n^2 ||v - \bar{y}||^2 + 2(1-\alpha_n)\alpha_n \langle y_n - \bar{y}, v - \bar{y}\rangle \\ &\le (1-\alpha_n)\left[||x_n - \bar{x}||^2 + ||y_n - \bar{y}||^2\right] + \alpha_n \big[\alpha_n ||u - \bar{x}||^2 \\ &\quad + 2(1-\alpha_n)\langle x_n - \bar{x}, u - \bar{x}\rangle + \alpha_n ||v - \bar{y}||^2 \\ &\quad + 2(1-\alpha_n)\langle y_n - \bar{y}, v - \bar{y}\rangle\big]. \end{aligned} \tag{3.3.40}$$

Since $x_n \rightharpoonup \bar{x}$ and $y_n \rightharpoonup \bar{y}$, then $\langle x_n - \bar{x}, u - \bar{x}\rangle \to 0$ and $\langle y_n - \bar{y}, v - \bar{y}\rangle \to 0$, as $n \to \infty$. Thus, applying Lemma 3.2.3 to (3.3.40), we have that $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$.

Case 2. Assume that $\{||x_n - x^*||^2 + ||y_n - y^*||^2\}$ is not monotone decreasing. Set $\Gamma_n = ||x_n - x^*||^2 + ||y_n - y^*||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \ge n_0$ (for some large n_0) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \le n, \Gamma_k \le \Gamma_{k+1}\}.$$

Clearly, τ is a non-decreasing sequence such that $\tau(n) \to \infty$, as $n \to \infty$ and

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \forall n \geq n_0.$$

From (3.3.12), we have

$$\begin{aligned}\epsilon\left(||A^*(Aw_{\tau(n)} - Bz_{\tau(n)})||^2 + ||B^*(Aw_{\tau(n)} - Bz_{\tau(n)})||^2\right) &\leq [||x_{\tau(n)} - x^*||^2 + ||y_{\tau(n)} - y^*||^2] \\ &\quad -[||x_{\tau(n)+1} - x^*||^2 + ||y_{\tau(n)+1} - y^*||^2] \\ &\quad +\alpha_{\tau(n)}[||u - x^*||^2 + ||v - y^*||^2] \\ &\leq \alpha_{\tau(n)}[||u - x^*||^2 + ||v - y^*||^2]. \quad (3.3.41)\end{aligned}$$

Therefore,

$$\left(||A^*(Aw_{\tau(n)} - Bz_{\tau(n)})||^2 + ||B^*(Aw_{\tau(n)} - Bz_{\tau(n)})||^2\right) \to 0, \text{ as } n \to \infty.$$

Note that $Aw_{\tau(n)} - Bz_{\tau(n)} = 0$, if $\tau(n) \notin \Omega$.
Hence,

$$\lim_{n\to\infty} ||A^*(Aw_{\tau(n)} - Bz_{\tau(n)})||^2 = 0, \tag{3.3.42}$$

and

$$\lim_{n\to\infty} ||B^*(Aw_{\tau(n)} - Bz_{\tau(n)})||^2 = 0. \tag{3.3.43}$$

Using the same argument as in case 1, we have that $\{(x_{\tau(n)}, y_{\tau(n)})\}$ converges weakly to $(\bar{x}, \bar{y}) \in \Gamma$.
Now for all $n \geq n_0$,

$$\begin{aligned}0 &\leq [||x_{\tau(n)+1} - x^*||^2 + ||y_{\tau(n)+1} - y^*||^2] - [||x_{\tau(n)} - x^*||^2 + ||y_{\tau(n)} - y^*||^2] \\ &\leq (1 - \alpha_{\tau(n)})[||x_{\tau(n)} - \bar{x}||^2 + ||y_{\tau(n)} - \bar{y}||^2] - [||x_{\tau(n)} - x^*||^2 + ||y_{\tau(n)} - y^*||^2] \\ &\quad +\alpha_{\tau(n)}[\alpha_{\tau(n)}[||u - \bar{x}||^2 + ||v - \bar{y}||^2] + 2(1 - \alpha_{\tau(n)})(\langle x_{\tau(n)} - \bar{x}, u - \bar{x}\rangle + \langle y_{\tau(n)} - \bar{y}, v - \bar{y}\rangle)],\end{aligned}$$

which implies

$$\begin{aligned}||x_{\tau(n)} - \bar{x}||^2 + ||y_{\tau(n)} - \bar{y}||^2 &\leq \alpha_{\tau(n)}[||u - \bar{x}||^2 + ||v - \bar{y}||^2] \\ &\quad +2(1 - \alpha_{\tau(n)})(\langle x_{\tau(n)} - \bar{x}, u - \bar{x}\rangle + \langle y_{\tau(n)} - \bar{y}, v - \bar{y}\rangle) \to 0.\end{aligned}$$

Hence

$$\lim_{n\to\infty} (||x_{\tau(n)} - \bar{x}||^2 + ||y_{\tau(n)} - \bar{y}||^2) = 0.$$

Therefore,

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0.$$

Moreover, for $n \geq n_0$, it is clear that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$) because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$.
Consequently for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Thus, $\lim_{n\to\infty} \Gamma_n = 0$. That is $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$. □

Corollary 3.3.2. *Let H_1, H_2, H_3 be real Hilbert spaces and C_1, C_2 be nonempty, closed and convex subset of H_1 and H_2 respectively. Let $T : C_1 \to C_1$ and $S : C_2 \to C_2$ be k_1-strictly pseudocontractive and k_2-strictly pseudocontractive mappings respectively. Let $f_i : C_i \to C_i$ be μ_i-inverse strongly monotone operators $(i = 1, 2)$ and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operators. Assume that the solution set $\Gamma \neq \emptyset$ and that the stepsize sequence $\{\gamma_n\} \in \left(\epsilon, \frac{2||Aw_n - Bz_n||^2}{||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2} - \epsilon\right), n \in \Omega$,*
otherwise, $\gamma_n = \gamma$(γ being any nonegative value), where the set of indexes $\Omega = \{n : Aw_n - Bz_n \neq 0\}$.
Let $u, x_0 \in C_1$ and $v, y_0 \in C_2$ be arbitrary and the sequence $\{(x_n, y_n)\}$ be generated by Algorithm (3.3.1), with conditions

(i) $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\lambda_n\}$ *are sequences in* $(0, 1)$,

(ii) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\beta_n \in (a, 1 - k_1) \subseteq (0, 1)$ for some $a > 0$,

(iv) $\lambda_n \in (b, 1 - k_2) \subseteq (0, 1)$ for some $b > 0$.

Then $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$ in Γ.

Corollary 3.3.3. *Let H_1, H_2, H_3 be real Hilbert spaces and C_1, C_2 be nonempty, closed and convex subset of H_1 and H_2 respectively. Let $T : C_1 \to C_1$ and $S : C_2 \to C_2$ be quasi-nonexpansive mappings such that $(I - T)$ and $(I - S)$ are demiclosed at 0. Let $f_i : C_i \to C_i$ be μ_i-inverse strongly monotone operators $(i = 1, 2)$ and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operators. Assume that the solution set $\Gamma \neq \emptyset$ and that the stepsize sequence $\{\gamma_n\} \in \left(\epsilon, \frac{2||Aw_n - Bz_n||^2}{||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2} - \epsilon\right), n \in \Omega$,*
otherwise, $\gamma_n = \gamma$(γ being any nonegative value), where the set of indexes $\Omega = \{n : Aw_n - Bz_n \neq 0\}$.
Let $u, x_0 \in C_1$ and $v, y_0 \in C_2$ be arbitrary and the sequence $\{(x_n, y_n)\}$ be generated by Algorithm (3.3.1), with conditions

(i) $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\lambda_n\}$ *are sequences in* $(0, 1)$,

(ii) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\beta_n \in (a, 1) \subseteq (0, 1)$ for some $a > 0$,

(iv) $\lambda_n \in (b, 1) \subseteq (0, 1)$ for some $b > 0$.

Then $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$ in Γ.

Corollary 3.3.4. *Let H_1, H_2, H_3 be real Hilbert spaces and C_1, C_2 be nonempty, closed and convex subset of H_1 and H_2 respectively. Let $T : C_1 \to C_1$ and $S : C_2 \to C_2$ be directed mappings such that $(I - T)$ and $(I - S)$ are demiclosed at 0. Let $f_i : C_i \to C_i$ be μ_i-inverse strongly monotone operators $(i = 1, 2)$ and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operators. Assume that the solution set $\Gamma \neq \emptyset$ and that the stepsize*

sequence $\{\gamma_n\} \in \left(\epsilon, \frac{2||Aw_n - Bz_n||^2}{||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2} - \epsilon\right), n \in \Omega,$ *otherwise,* $\gamma_n = \gamma(\gamma$ *being any nonegative value), where the set of indexes* $\Omega = \{n : Aw_n - Bz_n \neq 0\}$.
Let $u, x_0 \in C_1$ *and* $v, y_0 \in C_2$ *be arbitrary and the sequence* $\{(x_n, y_n)\}$ *be generated by* (3.3.1)*, with conditions*

(i) $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\lambda_n\}$ *are sequences in* $(0, 1)$*,*

(ii) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$

(iii) $\beta_n \in (a, 1) \subseteq (0, 1)$ for some $a > 0$,

(iv) $\lambda_n \in (b, 1) \subseteq (0, 1)$ for some $b > 0$.

Then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y})$ *in* Γ*.*

3.4 Applications and numerical example

3.4.1 Split equality convex minimization problem

Let H_1, H_2, H_3 be real Hilbert spaces and $C_1,\ C_2$ be nonempty, closed and convex subset of H_1 and H_2 respectively. Let $T : C_1 \to C_1,\quad S : C_2 \to C_2$ be demi-contractive mappings and $A : H_1 \to H_3,\ B : H_2 \to H_3$ be bounded linear operators. Let $\phi_i : C_i \to \mathbb{R},\ (i = 1, 2)$ be convex and differentiable functions. Consider the following problem which we call the split equality fixed point convex minimizing problem (SEFPCMP): Find $x^* \in F(T)$ and $y^* \in F(S)$ such that

$$x^* = \arg\min_{x\in C_1} \phi_1(x), \tag{3.4.1}$$

$$y^* = arg \min_{y\in C_2} \phi_2(y) \text{ and } Ax^* = By^*. \tag{3.4.2}$$

We can formulate the SEFPCMP (3.4.1)-(3.4.2) as follows: Find $x^* \in F(T)$ and $y^* \in F(S)$ such that

$$\langle \nabla\phi_1(x^*), x - x^*\rangle \geq 0, \quad \forall x \in C_1, \tag{3.4.3}$$

$$\langle \nabla\phi_2(x^*), x - x^*\rangle \geq 0, \quad \forall y \in C_2 \text{ and } Ax^* = By^*, \tag{3.4.4}$$

where $\nabla\phi_1$ and $\nabla\phi_2$ are the gradient of ϕ_1 and ϕ_2 respectively (see for example [6]). If we assume that $\nabla\phi_1$ and $\nabla\phi_2$ are L_1 (respectively L_2)-Lipschitz continuous function, then we have from Remark 2.1.6 that $\nabla\phi_1$ and $\nabla\phi_2$ are $\frac{2}{L_1}$ (respectively $\frac{2}{L_2}$)-inverse strongly monotone operators. Therefore, if we let Γ to be the solution set of SEFPCMP (3.4.1)-(3.4.2), we have the following result.

Theorem 3.4.1. *Let H_1, H_2, H_3 be real Hilbert spaces and C_1, C_2 be nonempty, closed and convex subsets of H_1, H_2 respectively. Let $T : C_1 \to C_1$ and $S : C_2 \to C_2$ be demi-contractive mappings with constants k_1 and k_2 respectively such that $I - T$ and $I - S$ are demi-closed at 0. Let $\phi_i : C_i \to \mathbb{R}$, $(i = 1, 2)$ be convex and differentiable functions such that $\nabla\phi_i$, $(i = 1, 2)$ are L_i-Lipschitz continuous functions. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators. Assume that the solution set $\Gamma \neq \emptyset$ and that the stepsize sequence $\{\gamma_n\}$ is chosen in such a way that for some $\epsilon > 0$,*

$$\gamma_n \in \left(\epsilon,\ \frac{2||Aw_n - Bz_n||^2}{||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2} - \epsilon\right), n \in \Omega,$$

otherwise, $\gamma_n = \gamma$ (γ being any nonegative value), where the set of indexes $\Omega = \{n : Aw_n - Bz_n \neq 0\}$.
Let $u, x_0 \in C_1$, $v, y_0 \in C_2$ be arbitrary and the sequence $\{(x_n, y_n)\}$ be generated by

$$\begin{cases} w_n = (1-\alpha_n)x_n + \alpha_n u, \\ z_n = (1-\alpha_n)y_n + \alpha_n v, \\ u_n = P_{C_1}(I - \rho\nabla\phi_1)(w_n - \gamma_n A^*(Aw_n - Bz_n)), \\ v_n = P_{C_2}(I - \rho\nabla\phi_2)(z_n + \gamma_n B^*(Aw_n - Bz_n)), \\ x_{n+1} = (1-\beta_n)u_n + \beta_n T u_n, \\ y_{n+1} = (1-\lambda_n)v_n + \lambda_n S v_n, \end{cases} \tag{3.4.5}$$

with conditions

(i) $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\lambda_n\}$ *are sequences in* $(0, 1)$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\beta_n \in (a, 1 - k_1) \subseteq (0, 1)$ for some $a > 0$,

(iv) $\lambda_n \in (b, 1 - k_2) \subseteq (0, 1)$ for some $b > 0$.

Then $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$ in Γ.

3.4.2 Monotone inclusion problem and variational inequality problem

Let H be a real Hilbert space and $f : H \to H$ be an α-inverse strongly monotone mapping. Let $M : H \to 2^H$ be a maximal monotone operator. Consider the following monotone variational inclusion problem: Find $x^* \in H$ such that

$$0 \in f(x^*) + M(x^*). \tag{3.4.6}$$

Let SOL(f, M) be the solution set of problem (3.4.6), then for $\sigma > 0$ and $\lambda \in (0, 2\alpha)$, it is known that $F(J_\sigma^M(I - \lambda f))$ =SOL(f, M) and that the operator $J_\sigma^M(I - \lambda f)$ called the resolvent of M with parameter σ is a single valued and an averaged nonexpansive operator (see for example [6, 92]). It is also generally known that every averaged nonexpansive mapping with nonempty fixed point set is quasi-nonexpansive, which is in turn a demi-contractive mapping. Hence, we present the following result.

Theorem 3.4.2. *Let H_1, H_2, H_3 be real Hilbert spaces and C_1, C_2 be nonempty, closed and convex subset of H_1 and H_2 respectively. Let $M_i : H_i \to 2^{H_i}$, $(i = 1, 2)$ be maximal monotone operators and $f_i : H_i \to H_i$, $(i = 1, 2)$ be μ_i-inverse strongly monotone operators. Let $J_\sigma^{M_1}(I - \lambda f_1) : H_1 \to H_1$ and $J_\sigma^{M_2}(I - \lambda f_2) : H_2 \to H_2$ be two average nonexpansive mappings such that $(I - J_\sigma^{M_1}(I - \lambda f_1))$ and $(I - J_\sigma^{M_2}(I - \lambda f_2))$ are demiclosed at 0. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. Assume that the solution set $\Gamma \neq \emptyset$ and that the stepsize sequence $\{\gamma_n\}$ is chosen in such a way that for some $\epsilon > 0$, $\gamma_n \in \left(\epsilon, \frac{2||Aw_n - Bz_n||^2}{||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2} - \epsilon\right), n \in \Omega,$
otherwise, $\gamma_n = \gamma$(γ being any nonegative value), where the set of indexes $\Omega = \{n : Aw_n - Bz_n \neq 0\}$.
Let $u, x_0 \in H_1$ and $v, y_0 \in H_2$ be arbitrary and the sequence $\{(x_n, y_n)\}$ be generated by*

$$\begin{cases} w_n = (1 - \alpha_n)x_n + \alpha_n u, \\ z_n = (1 - \alpha_n)y_n + \alpha_n v, \\ u_n = P_{C_1}(I - \rho f_1)(w_n - \gamma_n A^*(Aw_n - Bz_n)), \\ v_n = P_{C_2}(I - \rho f_2)(z_n + \gamma_n B^*(Aw_n - Bz_n)), \\ x_{n+1} = (1 - \beta_n)u_n + \beta_n J_\sigma^{M_1}(I - \lambda f_1)u_n, \\ y_{n+1} = (1 - \lambda_n)v_n + \lambda_n J_\sigma^{M_2}(I - \lambda f_2)v_n, \end{cases} \tag{3.4.7}$$

$0 < \rho < 2\mu_i$ for each $i = 1, 2$, with conditions

(i) $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\lambda_n\}$ *are sequences in* $(0, 1)$,

(ii) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\beta_n \in (a, 1 - k_1) \subseteq (0, 1)$ for some $a > 0$,

(iv) $\lambda_n \in (b, 1 - k_2) \subseteq (0, 1)$ for some $b > 0$.

Then $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$ in Γ.

3.4.3 Equilibrium problem and variational inequality problem

Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \times C \to \mathbb{R}$ be a bifunction. The Equilibrium Problem (EP) is to find $x \in C$ such that

$$T(x, y) \geq 0 \ \forall y \in C. \tag{3.4.8}$$

We denote the solution set of EP (3.4.8) by $EP(C, T)$. It has been proved in [14] that T satisfies the following conditions:

C_1: $T(x,x) = 0\ \forall x \in C$,

C_2: T is monotone, i.e., $T(x,y) + T(y,x) \geq 0\ \forall x, y \in C$,

C_3: for each $x, y \in C$, $\lim_{t\to 0} T(tz + (1-t)x, y) \geq T(x,y)$,

C_4: for each $x \in C$, $y \mapsto T(x,y)$ is convex and lower semicontinuous

and for any $r > 0$, $z \in H$, we have that

$$J_r^T(z) := \{x \in C : T(x,y) + \frac{1}{r}\langle y - x, x - z\rangle \geq 0\ \forall y \in C\} \neq \emptyset,$$

where J_r^T is the resolvent operator of T with parameter r.

Furthermore, J_r^T is singled valued, firmly nonexpansive and $F(J_r^T) = EP(C,T)$ (see [45]). Using these facts, we can give the following theorem.

Theorem 3.4.3. *Let H_1, H_2, H_3 be real Hilbert spaces and C_1, C_2 be nonempty, closed and convex subset of H_1 and H_2 respectively. Let $T_1 : C_1 \times C_1 \to \mathbb{R}$, $T_2 : C_2 \times C_2 \to \mathbb{R}$ be two bifunctions and $f_i : C_i \to C_i$, $(i = 1,2)$ be μ_i-inverse strongly monotone operators. Let $J_{r_1}^{T_1} : H_1 \to C_1$, $J_{r_2}^{T_2} : H_2 \to C_2$ be two firmly nonexpansive mappings and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Assume that the solution set $\Gamma \neq \emptyset$ and that the stepsize sequence $\{\gamma_n\}$ is chosen in such a way that for some $\epsilon > 0$, $\gamma_n \in \left(\epsilon,\ \frac{2||Aw_n - Bz_n||^2}{||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2} - \epsilon\right), n \in \Omega$,*
otherwise, $\gamma_n = \gamma$(γ being any nonegative value), where the set of indexes $\Omega = \{n : Aw_n - Bz_n \neq 0\}$.
Let $u, x_0 \in C_1$ and $v, y_0 \in C_2$ be arbitrary and the sequence $\{(x_n, y_n)\}$ be generated by

$$\begin{cases} w_n = (1-\alpha_n)x_n + \alpha_n u, \\ z_n = (1-\alpha_n)y_n + \alpha_n v, \\ u_n = P_{C_1}(I - \rho f_1)(w_n - \gamma_n A^*(Aw_n - Bz_n)), \\ v_n = P_{C_2}(I - f_2)(z_n + \gamma_n B^*(Aw_n - Bz_n)), \\ x_{n+1} = (1-\beta_n)u_n + \beta_n J_{r_1}^{T_1} u_n, \\ y_{n+1} = (1-\lambda_n)v_n + \lambda_n J_{r_2}^{T_2} v_n, \end{cases} \tag{3.4.9}$$

$0 < \rho < \mu_i$, for each $i = 1, 2$, with conditions

(i) $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\lambda_n\}$ *are sequences in* $(0,1)$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\beta_n \in (a, 1-k_1) \subseteq (0,1)$ for some $a > 0$,

(iv) $\lambda_n \in (b, 1-k_2) \subseteq (0,1)$ for some $b > 0$.

Then $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$ in Γ.

3.4.4 Numerical example

We give a numerical example in $\mathbb{R}$ (with the usual metric) to support our main result. Let $H_1 = H_2 = H_3 = \mathbb{R}$. We define $T : \mathbb{R} \to \mathbb{R}$ and $S : \mathbb{R} \to \mathbb{R}$ as follows:

$$T(x) = -\frac{3x}{2}, \qquad S(x) = -\frac{7x}{2}.$$

Also, we define $A, B : \mathbb{R} \to \mathbb{R}$ as follows:
$A(x) = 2x, \quad B(x) = 6x$, So that $A^*(x) = 2x$ and $B^*(x) = 6x$.

Let $C_1 = [-1, 1]$ and $C_2 = [-\frac{1}{2}, \frac{1}{2}]$. Let $f_1 : C_1 \to C_1$ and $f_2 : C_2 \to C_2$ be given as
$f_1(x) = \frac{x}{3}, \quad f_2(x) = \frac{x}{30}$.

We define the metric projections P_{C_1} and P_{C_2} as

$$P_{C_1}(x) = \begin{cases} x, & \text{if } x \in C_1; \\ \frac{x}{|x|}, & \text{otherwise} \end{cases} \quad \text{and} \quad P_{C_2}(x) = \begin{cases} x, & \text{if } x \in C_2 \\ \frac{x}{2|x|}, & \text{otherwise} \end{cases} \quad \text{respectively.}$$

Take $\alpha_n = \frac{1}{2n+1}$, $\beta_n = \frac{1}{2+\frac{1}{n}}$ and $\lambda_n = \frac{1}{4+\frac{1}{n}}$. Let $\gamma_n \in \left(\epsilon, \frac{2||Aw_n - Bz_n||^2}{||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2} - \epsilon\right), n \in \Omega$, otherwise $\gamma_n = \gamma(\gamma$ being any nonegative value), where the set of indexes $\Omega = \{n : Aw_n - Bz_n \neq 0\}$.

It is not difficult to see that T and S are demicontractive mappings with constants $k_1 = \frac{1}{5}$ and $k_2 = \frac{5}{9}$ respectively, thus α_n, β_n and λ_n satisfies the conditions in Theorem (3.3.1).

Obviously, f_1, f_2 are inverse strongly monotone operators and A, B are bounded linear operators. Hence for x_0, $y_0 \in \mathbb{R}$, $u = 2$, $v = 3$, our Algorithm (3.3.1) becomes:

$$\begin{cases} w_n = (\frac{2n}{2n+1})x_n + \frac{2}{2n+1}, \\ z_n = (\frac{2n}{2n+1})y_n + \frac{3}{2n+1}, \\ u_n = P_{C_1}(I - \rho f_1)(w_n - \gamma_n(4w_n - 12z_n)), \\ v_n = P_{C_2}(I - \rho f_2)(z_n + \gamma_n(12w_n - 36z_n)), \\ x_{n+1} = \left(\frac{\frac{2}{n} - 1}{4 + \frac{2}{n}}\right) u_n, \\ y_{n+1} = \left(\frac{\frac{2}{n} - 1}{8 + \frac{2}{n}}\right) v_n; \ n \geq 0. \end{cases} \tag{3.4.10}$$

The following initial points can be used in Algorithm (3.4.10): $x_0 = \frac{1}{2}, \quad y_0 = 1$ and $\rho = 0.0001$; $x_0 = 0.1, \quad y_0 = 0.002$ and $\rho = 0.0001$ with appropriate tolerance levels.

Chapter 4

Systems of Split Equality Variational Inequalities and Multiple-sets Split Equality Fixed Point Problem

4.1 Introduction

In the previous chapter, we studied the SEVIP and SEFPP for single valued demicontractive mappings. In this chapter, we extend this study to systems of VIPs and SEFPP for countable families of multivalued type-one demicontractive-type mappings. Precisely, we study the following problem which we call SSEVIPs and MSSEFPP:

Let H_1, H_2, H_3 be real Hilbert spaces and for each $l = 1, 2, \dots, N,\ r = 1, 2, \dots, m$, let C_l and Q_r be nonempty, closed and convex subsets of $H_1,\ H_2$ respectively. Let $T_i : H_1 \to CB(H_1),\ i = 1, 2, \dots$ and $S_j : H_2 \to CB(H_2),\ j = 1, 2, \dots$ be two countable families of multi-valued mappings with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $\cap_{j=1}^{\infty} F(S_j) \neq \emptyset$. Let $f_l : C_l \to C_l,\ h_r : Q_r \to Q_r$ be α_l, (respectively, μ_r)-inverse strongly monotone operators, and $A : H_1 \to H_3,\ B : H_2 \to H_3$ be bounded linear operators: Find $(\bar{x}, \bar{y}) \in \cap_{i=1}^{\infty} F(T_i) \times \cap_{j=1}^{\infty} F(S_j)$ such that

$$\langle f_l(\bar{x}),\ x - \bar{x}\rangle \geq 0, \quad \forall\, x \in C_l,\ l = 1, 2, \dots, N, \tag{4.1.1}$$

$$\langle h_r(\bar{y}),\ y - \bar{y}\rangle \geq 0, \quad \forall\, y \in Q_r,\ r = 1, 2, \dots, m, \text{ and such that } A\bar{x} = B\bar{y}. \tag{4.1.2}$$

Problem (4.1.1)-(4.1.2) is equivalent to finding $(\bar{x}, \bar{y}) \in \cap_{i=1}^{\infty} F(T_i) \times \cap_{j=1}^{\infty} F(S_j)$ such that

$$(\bar{x}, \bar{y}) \in \cap_{l=1}^{N} VI(C_l, f_l) \times \cap_{r=1}^{m} VI(Q_r, h_r), \text{ and } A\bar{x} = B\bar{y}. \tag{4.1.3}$$

Furthermore, we propose an iterative scheme and using the iterative scheme, we state and prove a strong convergence result for the approximation of a solution of (4.1.3). Finally, we applied our result to study some related problems.

4.2 Preliminaries

We recall some definitions that are very important in this chapter.

Definition 4.2.1. *Let (X, d) be a metric space and 2^X be the family of all subsets of X. Let $\mathcal{H}$ denote the Hausdorff metric induced by the metric d, then for all $A, B \in 2^X$,*

$$\mathcal{H}(A, B) = \max\{\sup_{a\in A} d(a, B),\ \sup_{b\in B} d(b, A)\}, \tag{4.2.1}$$

where $d(a, B) := \inf_{b\in B} d(a, b)$.

Definition 4.2.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Assume that $T : C \to 2^C$ is a multi-valued mapping, then $P_T x := \{u \in Tx : ||x - u|| = d(x, Tx)\}$.*

Definition 4.2.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to 2^C$ be a multi-valued mapping, then T is said to be a demicontractive-type in the sense of [65] if $F(T) \neq \emptyset$ and*

$$\mathcal{H}^2(Tx, Ty) \leq ||x - y||^2 + kd^2(x, Tx),\ \forall x \in C, y \in F(T) \text{ and } k \in (0, 1).$$

Definition 4.2.4. *Let H be a real Hilbert space and $T : H \to 2^H$ be a multi-valued mapping. Then T is said to be demi-closed at 0 if for any sequence $\{x_n\} \subset H$ such that $x_n \rightharpoonup x^*$ and $d(x_n, Tx_n) \to 0$ as $n \to \infty$, we have that $x^* \in Tx^*$ (i.e. $x^* \in F(T)$).*

The following lemmas will be needed in the proof of our main result, thus we reproduce their proofs as follows.

Lemma 4.2.5. *[37]. Let H be a real Hilbert space, then $\forall\ x, y \in H$ and $\alpha \in (0, 1)$, we have*

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2.$$

Lemma 4.2.6. *[50]. Let H be a real Hilbert space and $\{x_i,\ i = 1, \ldots, m\} \subset H$. For $\alpha_i \in (0, 1),\ i = 1, \ldots, m$ such that $\sum_{i=1}^m \alpha_i = 1$, the following identity holds:*

$$||\sum_{i=1}^m \alpha_i x_i||^2 = \sum_{i=1}^m \alpha_i||x_i||^2 - \sum_{1\leq i<j\leq m} \alpha_i\alpha_j||x_i - x_j||^2. \tag{4.2.2}$$

Proof. (By mathematical induction).
Since $i < j$, then for $m = 2$ we have that

$$||\alpha_1 x_1 + \alpha_2 x_2||^2 = \alpha_1||x_1||^2 + \alpha_2||x_2||^2 - \alpha_1\alpha_2||x_1 - x_2||^2,$$

which holds by Lemma 4.2.5.
Assume that (4.2.2) is true for some $k \geq 2$, that is

$$||\sum_{i=1}^k \alpha_i x_i||^2 = \sum_{i=1}^k \alpha_i||x_i||^2 - \sum_{1\leq i<j\leq k} \alpha_i\alpha_j||x_i - x_j||^2. \tag{4.2.3}$$

Then, we show that (4.2.2) is true for $m = k+1$
Now,

$$\begin{aligned}
||\sum_{i=1}^{k+1}\alpha_i x_i||^2 &= ||\alpha_1 x_1 + \sum_{i=2}^{k+1}\alpha_i x_i||^2 \\
&= ||\alpha_1 x_1 + (1-\alpha_1)\sum_{i=2}^{k+1}\frac{\alpha_i}{(1-\alpha_1)}x_i||^2 \\
&= \alpha_1||x_1|| + (1-\alpha_1)||\sum_{i=2}^{k+1}\frac{\alpha_i}{(1-\alpha_1)}x_i||^2 - \alpha_1(1-\alpha_1)||\sum_{i=2}^{k+1}\frac{\alpha_i}{(1-\alpha_1)}(x_1 - x_i)||^2 \\
&= \alpha_1||x_1|| + (1-\alpha_1)||\sum_{i=1}^{k}\alpha'_{i+1}x_{i+1}||^2 \\
&\quad -\alpha_1(1-\alpha_1)||\sum_{i=1}^{k}\alpha'_{i+1}(x_1 - x_{i+1})||^2, \text{ where } \alpha'_i = \frac{\alpha_i}{(1-\alpha_1)}. \qquad (4.2.4)
\end{aligned}$$

Using (4.2.3) in (4.2.4), we have

$$\begin{aligned}
||\sum_{i=1}^{k+1}\alpha_i x_i||^2 &= \alpha_1||x_1||^2 + (1-\alpha_1)\left[\sum_{i=1}^{k}\alpha'_{i+1}||x_{i+1}||^2 - \sum_{1\leq i<j\leq k}\alpha'_{i+1}\alpha'_{j+1}||x_{i+1} - x_{j+1}||^2\right] \\
&\quad -\alpha_1(1-\alpha_1)\left[\sum_{i=1}^{k}\alpha'_{i+1}||x_1 - x_{i+1}||^2 - \sum_{1\leq i<j\leq k}\alpha'_{i+1}\alpha'_{j+1}||x_{i+1} - x_{j+1}||^2\right] \\
&= \sum_{i=1}^{k+1}\alpha_i||x_i||^2 - \sum_{i=2}^{k+1}\alpha_1\alpha_i||(x_1 - x_i)||^2 - \sum_{2\leq i<j\leq k+1}\alpha_i\alpha_j||x_i - x_j||^2 \\
&= \sum_{i=1}^{k+1}\alpha_i||x_i||^2 - \sum_{1\leq i<j\leq k+1}\alpha_i\alpha_j||x_i - x_j||^2.
\end{aligned}$$

Hence, by induction we have that (4.2.2) is true. □

Lemma 4.2.7. *[42]. Let H be a real Hilbert space and $\{x_i\}_{i\geq 1}$ be a bounded sequence in H. For $\alpha_i \in (0,1)$ such that $\sum_{i=1}^{\infty}\alpha_i = 1$, the following identity holds:*

$$||\sum_{i=1}^{\infty}\alpha_i x_i||^2 = \sum_{i=1}^{\infty}\alpha_i||x_i||^2 - \sum_{1\leq i<j<\infty}\alpha_i\alpha_j||x_i - x_j||^2.$$

Proof. Since $\{x_i\}_{i\geq 1}$ is a bounded sequence in H, then there exists $M > 0$ such that $||x_i|| \leq M$, $\forall i \geq 1$ and since $\sum_{i=1}^{\infty}\alpha_i = 1$, we have that $\sum_{i=1}^{\infty}\alpha_i||x_i||^2 < \infty$ and $\sum_{1\leq i<j<\infty}\alpha_i\alpha_j||x_i - x_j||^2 < \infty$.

Moreover,

$$\sum_{i=1}^{n}\alpha_i = 1 - \sum_{i=n+1}^{\infty}\alpha_i.$$

Thus setting

$$\alpha_i^n = \frac{\alpha_i}{1 - \sum_{i=n+1}^{\infty} \alpha_i}, \text{ then, we have that } \sum_{i=1}^{n} \alpha_i^n = 1. \tag{4.2.5}$$

Hence, from Lemma 4.2.6, we have

$$\begin{aligned}
||\sum_{i=1}^{\infty} \alpha_i x_i||^2 &= ||\lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i^n x_i||^2 \\
&= \lim_{n\to\infty} ||\sum_{i=1}^{n} \alpha_i^n x_i||^2 \\
&= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \alpha_i^n ||x_i||^2 - \sum_{1\le i<j\le n} \alpha_i^n \alpha_j^n ||x_i - x_j||^2 \right] \\
&= \sum_{i=1}^{\infty} \alpha_i ||x_i||^2 - \sum_{1\le i<j<\infty} \alpha_i \alpha_j ||x_i - x_j||^2. \qquad (4.2.6)
\end{aligned}$$

□

Lemma 4.2.8. *[41]. Let K be a nonempty subset of a real Hilbert space H and let $T : K \to CB(K)$ be a multi-valued k-demicontractive mapping. Assume that for every $p \in F(T)$, $Tp = \{p\}$. Then*

$$\mathcal{H}(Tx, Tp) \le \frac{1+\sqrt{k}}{1-\sqrt{k}} ||x - p||, \quad \forall x \in K, \ p \in F(T).$$

Proof.

$$\begin{aligned}
\mathcal{H}^2(Tx, Tp) &\le ||x - p||^2 + kd^2(x, Tx) \\
&\le ||x - p||^2 + k\mathcal{H}^2(\{x\}, Tx) \\
&\le \left(||x - p|| + \sqrt{k}\mathcal{H}(\{x\}, Tx)\right)^2,
\end{aligned}$$

which implies

$$\begin{aligned}
\mathcal{H}(Tx, Tp) &\le ||x - p|| + \sqrt{k}\,(||x - p|| + \mathcal{H}(Tx, Tp)) \\
&\le ||x - p|| + \sqrt{k}||x - p|| + \sqrt{k}\mathcal{H}(Tx, Tp). \qquad (4.2.7)
\end{aligned}$$

From (4.2.7), we have

$$\mathcal{H}(Tx, Tp) - \sqrt{k}\mathcal{H}(Tx, Tp) \le ||x - p||(1 + \sqrt{k}),$$

which implies

$$\mathcal{H}(Tx, Tp) \le \frac{1+\sqrt{k}}{1-\sqrt{k}} ||x - p||.$$

□

Lemma 4.2.9. *Let H be a real Hilbert space, then*

$$2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x-y||^2 = ||x+y||^2 - ||x||^2 - ||y||^2, \quad \forall\, x, y \in H.$$

Lemma 4.2.10. *[119]. Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n, \quad n \geq 0, \tag{4.2.8}$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
(i) $\Sigma_{n=0}^{\infty}\gamma_n = \infty$, or equivalently, $\prod_{n=1}^{\infty}(1-\gamma_n) = 0$,
(ii) $\limsup_{n\to\infty}\delta_n \leq 0$ or $\Sigma_{n=0}^{\infty}|\delta_n\gamma_n| < \infty$,
Then $\lim_{n\to\infty} a_n = 0$.

4.3 Main result

Theorem 4.3.1. *Let H_1, H_2, H_3 be real Hilbert spaces and for each $l = 1, 2, \ldots, N$, $r = 1, 2, \ldots, m$, let C_l and Q_r be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $T_i : H_1 \to CB(H_1)$, $i = 1, 2, \ldots$ and $S_j : H_2 \to CB(H_2)$, $j = 1, 2, \ldots$ be two countable families of multi-valued type-one demicontractive-type mappings with constants k_i and k_j respectively such that T_i and S_j are demi-closed at 0. Let $f_l : C_l \to C_l$, $h_r : Q_r \to Q_r$ be μ_l (respectively ν_r)-inverse strongly monotone operators and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operators. Assume that the solution set $\Gamma \neq \emptyset$ and that the stepsize sequence $\{\gamma_n\}$ is chosen in such a way that for some $\epsilon > 0$,*

$$\gamma_n \in \left(\epsilon,\ \frac{2||Aw_n - Bz_n||^2}{||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2} - \epsilon\right), n \in \Omega,$$

otherwise $\gamma_n = \gamma$(γ being any nonegative value), where the set of indexes $\Omega = \{n : Aw_n - Bz_n \neq 0\}$.
Let $u, x_1 \in H_1$ and $v, y_1 \in H_2$ be arbitrary and the sequence $\{(x_n, y_n)\}$ be generated by

$$\begin{cases} w_n = (1-\alpha_n)x_n + \alpha_n u, \\ z_n = (1-\alpha_n)y_n + \alpha_n v, \\ u_n = P_{C_N}(I-\lambda f_N) \circ P_{C_{N-1}}(I-\lambda f_{N-1}) \circ \ldots \circ P_{C_1}(I-\lambda f_1)(w_n - \gamma_n A^*(Aw_n - Bz_n)), \\ v_n = P_{Q_m}(I-\lambda h_m) \circ P_{Q_{m-1}}(I-\lambda h_{m-1}) \circ \ldots \circ P_{Q_1}(I-\lambda h_1)(z_n + \gamma_n B^*(Aw_n - Bz_n)), \\ x_{n+1} = \beta_0 u_n + \sum_{i=1}^{\infty}\beta_i g_n^i, \\ y_{n+1} = \beta_0 v_n + \sum_{j=1}^{\infty}\beta_j h_n^j, \ \forall n \geq 1, \end{cases} \tag{4.3.1}$$

where $0 < \lambda < \min\{2\mu, 2\nu\}$, $\mu := \min\{\mu_l,\ l = 1, 2, \ldots, N\}$, $\nu := \min\{\nu_r,\ r = 1, 2, \ldots, m\}$ and A^*, B^* are the adjoint of A and B respectively. $g_n^i \in P_{T_i}u_n$, $z_n^j \in P_{S_j}v_n$ and $P_{T_i}u_n := \{g_n^i \in T_iu_n : ||g_n^i - u_n|| = d(u_n, T_iu_n)\}$,
with conditions

(i) $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(ii) $k \in (0,1)$, *where* $k := \max\{k_1, k_2\}$, $k_1 = \sup_{i\geq 1}\{k_i\}$, $k_2 = \sup_{j\geq 1}\{k_j\} \in (0,1)$,

(iii) $\beta_0 \in (k,1)$, $\beta_i, \beta_j \in (0,1)$, $i,j = 1,2,\dots$ *such that* $\sum_{i=0}^{\infty}\beta_i = 1$ *and* $\sum_{j=0}^{\infty}\beta_j = 1$.

(iv) for each $x^* \in \cap_{i=1}^{\infty}F(T_i)$; $T_i x^* = \{x^*\}$ *and for each* $y^* \in \cap_{j=1}^{\infty}F(S_j)$; $S_j y^* = \{y^*\}$.

Then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y})$ *in* Γ.

Proof. First, we show that for each $i = 1,2,\dots,$ $\{g_n^i\}$ is bounded. Using Lemma 4.2.8, we have

$$||g_n^i - x^*|| \leq \mathcal{H}(T_i u_n, T_i x^*) \leq \frac{1+\sqrt{k_1}}{1-\sqrt{k_1}}||u_n - x^*|| := P_n.$$

Hence $\{g_n^i\}_{i\geq 1}$ is bounded. Similarly, $\{h_n^j\}_{j\geq 1}$ is bounded.

Let $(x^*, y^*) \in \Gamma$, $\Phi^N = P_{C_N}(I-\lambda f_N) \circ P_{C_{N-1}}(I-\lambda f_{N-1}) \circ \dots \circ P_{C_1}(I-\lambda f_1)$, where $\Phi^0 = I$
and $\Psi^m = P_{Q_m}(I-\lambda h_m) \circ P_{Q_{m-1}}(I-\lambda h_{m-1}) \circ \dots \circ P_{Q_1}(I-\lambda h_1)$, where $\Psi^0 = I$, then we have from (4.3.1) that

$$\begin{aligned}
||u_n - x^*||^2 &= ||\Phi^N(w_n - \gamma_n A^*(Aw_n - Bz_n)) - x^*||^2 \\
&= ||P_{C_N}(I-\lambda f_N)\left(\Phi^{N-1}(w_n - \gamma_n A^*(Aw_n - Bz_n))\right) - x^*||^2 \\
&\leq ||\Phi^{N-1}(w_n - \gamma_n A^*(Aw_n - Bz_n)) - x^*||^2 \\
&\vdots \\
&\leq ||w_n - \gamma_n A^*(Aw_n - Bz_n) - x^*||^2 \\
&= ||w_n - x^*||^2 - 2\gamma_n\langle w_n - x^*, A^*(Aw_n - Bz_n)\rangle \\
&\quad + \gamma_n^2||A^*(Aw_n - By_n)||^2. \qquad (4.3.2)
\end{aligned}$$

From Lemma (4.2.9) and noting that A^* is the adjoint of A, we have

$$\begin{aligned}
-2\langle w_n - x^*, A^*(Aw_n - Bz_n)\rangle &= -2\langle Aw_n - Ax^*, Aw_n - Bz_n\rangle \\
&= -||Aw_n - Ax^*||^2 - ||Aw_n - Bz_n||^2 \\
&\quad + ||Bz_n - Ax^*||^2. \qquad (4.3.3)
\end{aligned}$$

Substituting (4.3.3) into (4.3.2), we have

$$\begin{aligned}
||u_n - x^*||^2 &\leq ||w_n - x^*||^2 - \gamma_n||Aw_n - Ax^*||^2 - \gamma_n||Aw_n - Bz_n||^2 \\
&\quad + \gamma_n||Bz_n - Ax^*||^2 + \gamma_n^2||A^*(Aw_n - Bz_n)||^2. \qquad (4.3.4)
\end{aligned}$$

Similarly, from (4.3.1), we have

$$\begin{aligned}
||v_n - y^*||^2 &\leq ||z_n - y^*||^2 - \gamma_n||Bz_n - By^*||^2 - \gamma_n||Aw_n - Bz_n||^2 \\
&\quad + \gamma_n||Aw_n - By^*||^2 + \gamma_n^2||B^*(Aw_n - Bz_n)||^2. \qquad (4.3.5)
\end{aligned}$$

From (4.3.1), Lemma 4.2.5 and adding inequality (4.3.4) and (4.3.5) together with the fact that $Ax^* = By^*$, we obtain

$$
\begin{aligned}
||u_n - x^*||^2 + ||v_n - y^*||^2 &\leq ||w_n - x^*||^2 + ||z_n - y^*||^2 - \gamma_n\big[2||Aw_n - Bz_n||^2 \\
&\quad -\gamma_n\left(||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2\right)\big] \\
&\leq ||w_n - x^*||^2 + ||z_n - y^*||^2 \\
&= ||(1-\alpha_n)x_n + \alpha_n u - x^*||^2 + ||(1-\alpha_n)y_n + \alpha_n v - y^*||^2 \\
&= ||(1-\alpha_n)(x_n - x^*) + \alpha_n(u - x^*)||^2 \\
&\quad +||(1-\alpha_n)(y_n - y^*) + \alpha_n(v - y^*)||^2 \\
&\leq (1-\alpha_n)||x_n - x^*||^2 + \alpha_n||u - x^*||^2 \\
&\quad +(1-\alpha_n)||y_n - y^*||^2 + \alpha_n||v - y^*||^2 \\
&= (1-\alpha_n)\left[||x_n - x^*||^2 + ||y_n - y^*||^2\right] \\
&\quad +\alpha_n\left[||u - x^*||^2 + ||v - y^*||^2\right]. \qquad (4.3.6)
\end{aligned}
$$

From (4.3.1), Lemma 4.2.7 and the fact that T is of type-one demi-contractive-type mapping, we obtain

$$
\begin{aligned}
||x_{n+1} - x^*||^2 &= ||\beta_0 u_n + \sum_{i=1}^{\infty}\beta_i g_n^i - x^*||^2 \\
&= ||\beta_0(u_n - x^*) + \sum_{i=1}^{\infty}\beta_i(g_n^i - x^*)||^2 \\
&= \beta_0||u_n - x^*||^2 + \sum_{i=1}^{\infty}\beta_i||g_n^i - x^*||^2 - \sum_{i=1}^{\infty}\beta_0\beta_i||u_n - g_n^i||^2 \\
&\quad - \sum_{1\leq i<j<\infty}\beta_i\beta_j||g_n^i - g_n^j||^2 \\
&\leq \beta_0||u_n - x^*||^2 + \sum_{i=1}^{\infty}\beta_i\mathcal{H}^2(T_iu_n, T_ix^*) - \sum_{i=1}^{\infty}\beta_0\beta_i||u_n - g_n^i||^2 \\
&\leq \beta_0||u_n - x^*||^2 + \sum_{i=1}^{\infty}\beta_i\left[||u_n - x^*||^2 + k_1 d^2(u_n, T_iu_n)\right] - \sum_{i=1}^{\infty}\beta_0\beta_i||u_n - g_n^i||^2 \\
&= \beta_0||u_n - x^*||^2 + \sum_{i=1}^{\infty}\beta_i\left[||u_n - x^*||^2 + k_1||u_n - g_n^i||^2\right] - \sum_{i=1}^{\infty}\beta_0\beta_i||u_n - g_n^i||^2 \\
&= ||u_n - x^*||^2 + (k_1 - \beta_0)\sum_{i=1}^{\infty}\beta_i||u_n - g_n^i||^2 \\
&\leq ||u_n - x^*||^2. \qquad (4.3.7)
\end{aligned}
$$

Similarly, we have that

$$||y_{n+1} - y^*||^2 \leq ||v_n - y^*||^2. \qquad (4.3.8)$$

Adding (4.3.7) and (4.3.8), and using (4.3.6), we obtain

$$\begin{aligned} ||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2 &\leq ||u_n - x^*||^2 + ||v_n - y^*||^2 \\ &\leq (1-\alpha_n)\left[||x_n - x^*||^2 + ||y_n - y^*||^2\right] \\ &\quad +\alpha_n\left[||u - x^*||^2 + ||v - y^*||^2\right] \\ &\leq \max\left\{||x_n - x^*||^2 + ||y_n - y^*||^2, ||u - x^*||^2 + ||v - y^*||^2\right\} \\ &\ \vdots \\ &\leq \max\left\{||x_0 - x^*||^2 + ||y_0 - y^*||^2, ||u - x^*||^2 + ||v - y^*||^2\right\}. \end{aligned}$$

Therefore, $\{||x_n - x^*||^2 + ||y_n - y^*||^2\}$ is bounded. Consequently $\{x_n\}$, $\{y_n\}$, $\{w_n\}$, $\{z_n\}$, $\{u_n\}, \{v_n\}$, $\{Ax_n\}$ and $\{By_n\}$ are all bounded.

We now consider two cases to establish the strong convergence of $\{(x_n, y_n)\}$ to $(\bar{x}, \bar{y}) \in \Gamma$.
Case 1: Assume that $\{||x_n - x^*||^2 + ||y_n - y^*||^2\}$ is monotone decreasing, then $\{||x_n - x^*||^2 + ||y_n - y^*||^2\}$ is convergent, thus

$$\lim_{n\to\infty}\left[\left(||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2\right) - \left(||x_n - x^*||^2 + ||y_n - y^*||^2\right)\right] = 0. \quad (4.3.9)$$

From (4.3.6) and (4.3.9), we obtain

$$\begin{aligned} ||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2 &\leq ||w_n - x^*||^2 + ||z_n - y^*||^2 - \gamma_n\big[2||Aw_n - Bz_n||^2 \\ &\quad -\gamma_n\left(||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2\right)\big] \\ &\leq (1-\alpha_n)\left[||x_n - x^*||^2 + ||y_n - y^*||^2\right] \quad (4.3.10) \\ &\quad +\alpha_n\left[||u - x^*||^2 + ||v - y^*||^2\right] - \gamma_n\big[2||Aw_n - Bz_n||^2 \\ &\quad -\gamma_n\left(||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2\right)\big] \end{aligned}$$

which implies (by the condition on γ_n)

$$\begin{aligned} \epsilon\left(||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2\right) &\leq (1-\alpha_n)\left[||x_n - x^*||^2 + ||y_n - y^*||^2\right] \\ &\quad +\alpha_n\left[||u - x^*||^2 + ||v - y^*||^2\right] \\ &\quad - \left[||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2\right]. \end{aligned}$$

From (4.3.9) and the fact that $\alpha_n \to 0$, as $n \to \infty$, we have

$$\lim_{n\to\infty}\left(||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2\right) = 0.$$

Since $Aw_n - Bz_n = 0$, if $n \notin \Omega$, we have

$$\lim_{n\to\infty} ||A^*(Aw_n - Bz_n)||^2 = \lim_{n\to\infty} ||B^*(Aw_n - Bz_n)||^2 = 0. \quad (4.3.11)$$

From (4.3.1), we have

$$\lim_{n\to\infty} ||w_n - x_n||^2 = \lim_{n\to\infty} \alpha_n^2 ||u - x_n||^2 = 0, \quad (4.3.12)$$

and

$$\lim_{n\to\infty} ||z_n - y_n||^2 = \lim_{n\to\infty} \alpha_n^2 ||v - y_n||^2 = 0. \tag{4.3.13}$$

Let $a_n = w_n - \gamma_n A^*(Aw_n - Bz_n)$ and $b_n = z_n + \gamma_n B^*(Aw_n - Bz_n)$. Then,

$$\lim_{n\to\infty} ||a_n - w_n||^2 = \lim_{n\to\infty} \gamma_n^2 ||A^*(Aw_n - Bz_n)||^2 = 0, \tag{4.3.14}$$

and

$$\lim_{n\to\infty} ||b_n - z_n||^2 = \lim_{n\to\infty} \gamma_n^2 ||B^*(Aw_n - Bz_n)||^2 = 0. \tag{4.3.15}$$

From (4.3.12) and (4.3.14), we have

$$\lim_{n\to\infty} ||a_n - x_n||^2 = 0. \tag{4.3.16}$$

Also, from (4.3.13) and (4.3.15), we obtain

$$\lim_{n\to\infty} ||b_n - y_n||^2 = 0. \tag{4.3.17}$$

From (4.3.1) and Lemma 4.2.9, we have that

$$\begin{aligned} ||u_n - x^*||^2 &= ||P_{C_N}(I - \lambda f_N)\Phi^{N-1}a_n - x^*||^2 \\ &\le \langle u_n - x^*, \Phi^{N-1}a_n - x^* \rangle \\ &= \frac{1}{2}\left[||u_n - x^*||^2 + ||\Phi^{N-1}a_n - x^*||^2 - ||u_n - \Phi^{N-1}a_n||^2\right], \end{aligned} \tag{4.3.18}$$

which implies

$$||u_n - \Phi^{N-1}a_n||^2 \le ||\Phi^{N-1}a_n - x^*||^2 - ||u_n - x^*||^2. \tag{4.3.19}$$

Similarly, we have that

$$||v_n - \Psi^{m-1}b_n||^2 \le ||\Psi^{m-1}b_n - y^*||^2 - ||v_n - y^*||^2. \tag{4.3.20}$$

Adding (4.3.19) and (4.3.20), we have

$$\begin{aligned} ||u_n - \Phi^{N-1}a_n||^2 + ||v_n - \Psi^{m-1}b_n||^2 &\le ||\Phi^{N-1}a_n - x^*||^2 + ||\Psi^{m-1}b_n - y^*||^2 \\ &\quad - \left(||u_n - x^*||^2 + ||v_n - y^*||^2\right) \\ &\le ||a_n - x^*||^2 + ||b_n - y^*||^2 \\ &\quad - \left(||u_n - x^*||^2 + ||v_n - y^*||^2\right) \\ &\le ||a_n - x^*||^2 + ||b_n - y^*||^2 \\ &\quad - \left(||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2\right) \\ &= ||a_n - x^*||^2 - ||x_n - x^*||^2 + ||b_n - y^*||^2 \\ &\quad - ||y_n - y^*||^2 + ||x_n - x^*||^2 + ||y_n - y^*||^2 \\ &\quad - \left(||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2\right) \to 0, \text{ as } n \to \infty, \end{aligned} \tag{4.3.21}$$

which implies

$$\lim_{n\to\infty} ||\Phi^N a_n - \Phi^{N-1} a_n|| = \lim_{n\to\infty} ||\Psi^m b_n - \Phi^{m-1} b_n|| = 0. \tag{4.3.22}$$

By the same argument as (4.3.18)-(4.3.21), we obtain that

$$\begin{aligned} ||\Phi^{N-1} a_n - \Phi^{N-2} a_n||^2 + ||\Psi^{m-1} b_n - \Psi^{m-2} b_n||^2 &\leq ||a_n - x^*||^2 + ||b_n - y^*||^2 \\ &\quad - \left(||\Phi^{N-1} a_n - x^*||^2 + ||\Psi^{m-1} b_n - y^*||^2\right) \\ &\leq ||a_n - x^*||^2 + ||b_n - y^*||^2 \\ &\quad - \left(||u_n - x^*||^2 + ||v_n - y^*||^2\right) \\ &\leq ||a_n - x^*||^2 + ||b_n - y^*||^2 \\ &\quad - \left(||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2\right) \to 0, \end{aligned} \tag{4.3.23}$$

which implies

$$\lim_{n\to\infty} ||\Phi^{N-1} a_n - \Phi^{N-2} a_n|| = \lim_{n\to\infty} ||\Psi^{m-1} b_n - \Psi^{m-2} b_n|| = 0. \tag{4.3.24}$$

Continuing in the same manner, we have that

$$\lim_{n\to\infty} ||\Phi^{N-2} a_n - \Phi^{N-3} a_n|| = \cdots = \lim_{n\to\infty} ||\Phi^2 a_n - \Phi^1 a_n|| = 0, \tag{4.3.25}$$

and

$$\lim_{n\to\infty} ||\Psi^{m-2} b_n - \Psi^{m-3} b_n|| = \cdots = \lim_{n\to\infty} ||\Psi^2 b_n - \Psi^1 b_n|| = 0. \tag{4.3.26}$$

From (4.3.22), (4.3.24), (4.3.25) and (4.3.26), we conclude that

$$\lim_{n\to\infty} ||\Phi^l a_n - \Phi^{l-1} a_n|| = 0,\ l = 1, 2, \ldots, N, \tag{4.3.27}$$

and

$$\lim_{n\to\infty} ||\Psi^r b_n - \Psi^{r-1} b_n|| = 0,\ r = 1, 2, \ldots, m. \tag{4.3.28}$$

Since f_l and h_r are Lipschitz continuous (by Remark (2.1.6)), we have from (4.3.27) and (4.3.28) that

$$\lim_{n\to\infty} ||f_l \Phi^l a_n - f_l \Phi^{l-1} a_n|| = 0, \tag{4.3.29}$$

and

$$\lim_{n\to\infty} ||h_r \Psi^r b_n - h_r \Psi^{r-1} b_n|| = 0. \tag{4.3.30}$$

Also,

$$\begin{aligned} ||u_n - a_n|| &\leq ||u_n - \Phi^{N-1} a_n|| + ||\Phi^{N-1} a_n - \Phi^{N-2} a_n|| + ||\Phi^{N-2} a_n - \Phi^{N-3} a_n|| \\ &\quad + \cdots + ||\Phi^1 a_n - a_n|| \to 0,\ \text{as } n \to \infty, \end{aligned}$$

which implies

$$\lim_{n\to\infty} ||u_n - a_n|| = 0. \tag{4.3.31}$$

Similarly, we have

$$\lim_{n\to\infty} ||v_n - b_n|| = 0. \tag{4.3.32}$$

From (4.3.14) and (4.3.31), we have

$$\lim_{n\to\infty} ||u_n - w_n| = 0. \tag{4.3.33}$$

Also, from (4.3.15) and (4.3.32), we have

$$\lim_{n\to\infty} ||v_n - z_n|| = 0. \tag{4.3.34}$$

From (4.3.12) and (4.3.33), we have

$$\lim_{n\to\infty} ||x_n - u_n|| \leq \lim_{n\to\infty} [||x_n - w_n|| + ||w_n - u_n||] = 0. \tag{4.3.35}$$

Similarly, from (4.3.13) and (4.3.34), we have

$$\lim_{n\to\infty} ||y_n - v_n|| = 0. \tag{4.3.36}$$

From (4.3.7), we have

$$\sum_{i=1}^{\infty} \beta_i(\beta_0 - k_1)||u_n - g_n^i||^2 \quad \leq \quad ||u_n - x^*||^2 - ||x_{n+1} - x^*||^2. \tag{4.3.37}$$

Similarly, we have

$$\sum_{j=1}^{\infty} \beta_j(\beta_0 - k_2)||v_n - h_n^j||^2 \quad \leq \quad ||v_n - y^*||^2 - ||y_{n+1} - y^*||^2. \tag{4.3.38}$$

Adding (4.3.37) and (4.3.38), and from (4.3.6) we have

$$\begin{aligned}
\sum_{i=1}^{\infty} \beta_i(\beta_0 - k_1)||u_n - g_n^i||^2 + \sum_{j=1}^{\infty} \beta_j(\beta_0 - k_2)||v_n - h_n^j||^2 \quad &\leq \quad ||u_n - x^*||^2 + ||v_n - y^*||^2 \\
&\quad - \left(||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2\right) \\
&\leq \quad (1 - \alpha_n)\left[||x_n - x^*||^2 + ||y_n - y^*||^2\right] \\
&\quad + \alpha_n\left[||u - x^*||^2 + ||v - y^*||^2\right] \\
&\quad - \left(||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2\right)
\end{aligned}$$

and for each $i, j = 1, 2, \ldots,$ we have

$$\begin{aligned}
\beta_i(\beta_0 - k_1)||u_n - g_n^i||^2 + \beta_j(\beta_0 - k_2)||v_n - h_n^j||^2 \quad &\leq \quad (1 - \alpha_n)\left[||x_n - x^*||^2 + ||y_n - y^*||^2\right] \\
&\quad + \alpha_n\left[||u - x^*||^2 + ||v - y^*||^2\right] \\
&\quad - \left(||x_{n+1} - x^*||^2 + ||y_{n+1} - y^*||^2\right) \to 0.
\end{aligned}$$

Hence,
$$\lim_{n\to\infty} \beta_i(\beta_0 - k_1)||u_n - g_n^i||^2 = \lim_{n\to\infty} \beta_j(\beta_0 - k_2)||v_n - h_n^j||^2 = 0.$$
By condition (iii), we have
$$\lim_{n\to\infty} ||u_n - g_n^i||^2 = \lim_{n\to\infty} ||v_n - h_n^j||^2 = 0.$$
Hence, we have
$$\lim_{n\to\infty} d(u_n, T_i u_n) = \lim_{n\to\infty} ||u_n - g_n^i|| = 0, \tag{4.3.39}$$
and
$$\lim_{n\to\infty} d(v_n, S_j v_n) = \lim_{n\to\infty} ||v_n - h_n^j|| = 0. \tag{4.3.40}$$
Since $\{x_n\}$ is bounded, there exist a subsequence of $\{x_n\}$ (without loss of generality, still denoted by $\{x_n\}$) such that $\{x_n\}$ converges weakly to $\bar{x} \in \cap_{l=1}^N C_l$. By (4.3.35) and (4.3.12), we have that $\{u_n\}$ and $\{w_n\}$ converges weakly to $\bar{x}$ and by the demi-closeness of T_i at 0 and (4.3.39), we have that $\bar{x} \in F(T_i)$ for each $i = 1, 2, \dots$. Similarly, since $\{y_n\}$ is bounded, there exist a subsequence of $\{y_n\}$ (without loss of generality, still denoted by $\{y_n\}$) such that $\{y_n\}$ converges weakly to $\bar{y} \in \cap_{r=1}^m Q_r$. By (4.3.36) and (4.3.13), we have that $\{v_n\}$ and $\{z_n\}$ converges weakly to $\bar{y}$ and by the demi-closeness of S_j at 0 and (4.3.40), we have that $\bar{y} \in F(S_j)$, for each $j = 1, 2, \dots$. Hence, $(\bar{x}, \bar{y}) \in \cap_{i=1}^\infty F(T_i) \times \cap_{j=1}^\infty F(S_j)$.

Next, we show that $A\bar{x} = B\bar{y}$.
Since A and B are bounded linear operators, we have that $Aw_n \rightharpoonup A\bar{x}$ and $Bz_n \rightharpoonup B\bar{y}$. Using the condition on $\{\gamma_n\}$ and (4.3.11) in (4.3.10), we obtain
$$\lim_{n\to\infty} ||Aw_n - Bz_n||^2 = 0. \tag{4.3.41}$$
By weakly semi continuity of the norm, we have
$$||A\bar{x} - B\bar{y}|| \le \liminf_{n\to\infty} ||Aw_n - Bz_n|| = 0. \tag{4.3.42}$$
That is,
$$A\bar{x} = B\bar{y}. \tag{4.3.43}$$

We now show that $(\bar{x}, \bar{y}) \in \cap_{l=1}^N VI(C_l, f_l) \times \cap_{r=1}^m VI(Q_r, h_r)$, that is $\bar{x}$ satisfies $\langle f_l(\bar{x}), x - \bar{x}\rangle \ge 0 \ \forall x \in \cap_{l=1}^N C_l$, and $\bar{y}$ satisfies $\langle h_r(\bar{y}), y - \bar{y}\rangle \ge 0 \ \forall y \in \cap_{r=1}^m Q_r$.
Let $N_{C_l} z$ be the normal cone of C_l at a point $z \in C_l$, $l = 1, 2 \dots, N$, we define the following set-valued operator $M_l : C_l \to 2^{C_l}$, for each $l = 1, 2, \dots, N$ by
$$M_l z = f_l z + N_{C_l} z.$$
Then, M_l is maximal monotone for each $l = 1, 2, \dots, N$. Let $(z, w) \in G(M_l)$, then $w - f_l z \in N_{C_l} z$. For $\Phi^l a_n \in C_l$, we have
$$\langle z - \Phi^l a_n, w - f_1 z\rangle \ge 0, \ l = 1, 2, \dots, N. \tag{4.3.44}$$

From $\Phi^l a_n = P_{C_l}(I - \lambda f_l)\Phi^{l-1} a_n$, we have $\langle z - \Phi^l a_n, \Phi^l a_n - (\Phi^{l-1} a_n - \lambda f_l \Phi^{l-1} a_n)\rangle \geq 0$, for each $l = 1, 2, \ldots, N$,
which implies $\langle z - \Phi^l a_n, \frac{\Phi^l a_n - \Phi^{l-1} a_n}{\lambda} + f_l \Phi^{l-1} a_n\rangle \geq 0$, for each $l = 1, 2, \ldots, N$.
From (4.3.44), we get

$$\begin{aligned}
\langle z - \Phi^l a_n, w\rangle &\geq \langle z - \Phi^l a_n, f_l z\rangle \\
&\geq \langle z - \Phi^l a_n, f_l z\rangle - \langle z - \Phi^l a_n, \frac{\Phi^l a_n - \Phi^{l-1} a_n}{\lambda} + f_l \Phi^{l-1} a_n\rangle \\
&= \langle z - \Phi^l a_n, f_l z - f_l \Phi^{l-1} a_n - \frac{\Phi^l a_n - \Phi^{l-1} a_n}{\lambda}\rangle \\
&= \langle z - \Phi^l a_n, f_l z - f_l \Phi^l a_n\rangle + \langle z - \Phi^l a_n, f_l \Phi^l a_n - f_l \Phi^{l-1} a_n\rangle \\
&\quad - \langle z - \Phi^l a_n, \frac{\Phi^l a_n - \Phi^{l-1} a_n}{\lambda}\rangle \\
&\geq \langle z - \Phi^l a_n, f_l \Phi^l a_n - f_l \Phi^{l-1} a_n\rangle \\
&\quad - \langle z - \Phi^l a_n, \frac{\Phi^l a_n - \Phi^{l-1} a_n}{\lambda}\rangle. \qquad (4.3.45)
\end{aligned}$$

Using (4.3.27) and (4.3.29) together with the fact that $\{u_n\} = \{\Phi^l a_n\}$ converges weakly to $\bar{x}$, we obtain from (4.3.45) that $\langle z - \bar{x}, w\rangle \geq 0$. Also, M_l is maximal monotone for each $l = 1, 2, \ldots, N$, this gives us that $\bar{x} \in M_l^{-1}(0)$, which implies that $0 \in M_l(\bar{x})$ for each $l = 1, 2, \ldots, N$. Hence, $\bar{x} \in \cap_{l=1}^N VI(C_l, f_l)$, that is $\langle f_l(\bar{x}), z - \bar{x}\rangle \geq 0, \forall z \in \cap_{l=1}^N C_l$. In the same manner, we obtain that $\langle h_r(\bar{y}), y - \bar{y}\rangle \geq 0, \forall y \in \cap_{r=1}^m Q_r$. Hence, we have that $(\bar{x}, \bar{y}) \in \Gamma$.

Next, we show that $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$.
From (4.3.6), we have

$$\begin{aligned}
||x_{n+1} - \bar{x}||^2 + ||y_{n+1} - \bar{y}||^2 &\leq ||w_n - \bar{x}||^2 + ||z_n - \bar{y}||^2 \\
&= (1 - \alpha_n)^2 ||x_n - \bar{x}||^2 + \alpha_n^2 ||u - \bar{x}||^2 + 2(1 - \alpha_n)\alpha_n \langle x_n - \bar{x}, u - \bar{x}\rangle \\
&\quad + (1 - \alpha_n)^2 ||y_n - \bar{y}||^2 + \alpha_n^2 ||v - \bar{y}||^2 + 2(1 - \alpha_n)\alpha_n \langle y_n - \bar{y}, v - \bar{y}\rangle \\
&\leq (1 - \alpha_n)\left[||x_n - \bar{x}||^2 + ||y_n - \bar{y}||^2\right] + \alpha_n\big[\alpha_n ||u - \bar{x}||^2 \\
&\quad + 2(1 - \alpha_n)\langle x_n - \bar{x}, u - \bar{x}\rangle + \alpha_n ||v - \bar{y}||^2 \\
&\quad + 2(1 - \alpha_n)\langle y_n - \bar{y}, v - \bar{y}\rangle\big]. \qquad (4.3.46)
\end{aligned}$$

Applying Lemma 4.2.10 to (4.3.46), we have that $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$.

Case 2. Assume that $\{||x_n - x^*||^2 + ||y_n - y^*||^2\}$ is not monotone decreasing. Set $\Gamma_n = ||x_n - x^*||^2 + ||y_n - y^*||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some large n_0) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, τ is a non-decreasing sequence such that $\tau(n) \to \infty$, as $n \to \infty$ and

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \forall n \geq n_0.$$

From (4.3.11), we have

$$\begin{aligned}\gamma_n^2\left(||A^*(Aw_{\tau(n)}-Bz_{\tau(n)})||^2+||B^*(Aw_{\tau(n)}-Bz_{\tau(n)})||^2\right) &\le [||x_{\tau(n)}-x^*||^2+||y_{\tau(n)}-y^*||^2]\\ &\quad -[||x_{\tau(n)+1}-x^*||^2+||y_{\tau(n)+1}-y^*||^2]\\ &\quad +\alpha_{\tau(n)}[||u-x^*||^2+||v-y^*||^2]\\ &\le \alpha_{\tau(n)}[||u-x^*||^2+||v-y^*||^2].\end{aligned}$$

Therefore,

$$\gamma_{\tau(n)}^2\left(||A^*(Aw_{\tau(n)}-Bz_{\tau(n)})||^2+||B^*(Aw_{\tau(n)}-Bz_{\tau(n)})||^2\right)\to 0,\ \text{as } n\to\infty.$$

By the condition on $\{\gamma_{\tau(n)}\}$, we have

$$\left(||A^*(Aw_{\tau(n)}-Bz_{\tau(n)})||^2+||B^*(Aw_{\tau(n)}-Bz_{\tau(n)})||^2\right)\to 0,\quad \text{as } n\to\infty.$$

Note that $Aw_{\tau(n)}-Bz_{\tau(n)}=0$, if $\tau(n)\notin\Omega$.
Hence,

$$\lim_{n\to\infty}||A^*(Aw_{\tau(n)}-Bz_{\tau(n)})||^2=0, \tag{4.3.47}$$

and

$$\lim_{n\to\infty}||B^*(Aw_{\tau(n)}-Bz_{\tau(n)})||^2=0. \tag{4.3.48}$$

Following the same line of argument as in case 1, we can show that

$$\lim_{n\to\infty}||\Phi^l a_{\tau(n)}-\Phi^{l-1}a_{\tau(n)}||=\lim_{n\to\infty}||\Psi^r b_{\tau(n)}-\Psi^{r-1}b_{\tau(n)}||=0,\ l=1,2,\dots,N,\ r=1,2,\dots,m,$$

$$\lim_{n\to\infty}d(u_{\tau(n)},T_iu_{\tau(n)})=\lim_{n\to\infty}d(v_{\tau(n)},S_jv_{\tau(n)})=0$$

and $\{(x_{\tau(n)},y_{\tau(n)})\}$ converges weakly to $(\bar{x},\bar{y})\in\Gamma$.

Now for all $n\ge n_0$, we have from (4.3.46) that

$$\begin{aligned}0 &\le [||x_{\tau(n)+1}-x^*||^2+||y_{\tau(n)+1}-y^*||^2]-[||x_{\tau(n)}-x^*||^2+||y_{\tau(n)}-y^*||^2]\\ &\le (1-\alpha_{\tau(n)})[||x_{\tau(n)}-\bar{x}||^2+||y_{\tau(n)}-\bar{y}||^2]-[||x_{\tau(n)}-x^*||^2+||y_{\tau(n)}-y^*||^2]\\ &\quad +\alpha_{\tau(n)}[\alpha_{\tau(n)}[||u-\bar{x}||^2+||v-\bar{y}||^2]+2(1-\alpha_{\tau(n)})(\langle x_{\tau(n)}-\bar{x},u-\bar{x}\rangle+\langle y_{\tau(n)}-\bar{y},v-\bar{y}\rangle)],\end{aligned}$$

which implies

$$\begin{aligned}||x_{\tau(n)}-\bar{x}||^2+||y_{\tau(n)}-\bar{y}||^2 &\le \alpha_{\tau(n)}[||u-\bar{x}||^2+||v-\bar{y}||^2]\\ &\quad +2(1-\alpha_{\tau(n)})(\langle x_{\tau(n)}-\bar{x},u-\bar{x}\rangle+\langle y_{\tau(n)}-\bar{y},v-\bar{y}\rangle)\to 0.\end{aligned}$$

Hence

$$\lim_{n\to\infty}(||x_{\tau(n)}-\bar{x}||^2+||y_{\tau(n)}-\bar{y}||^2)=0.$$

Therefore,

$$\lim_{n\to\infty}\Gamma_{\tau(n)}=\lim_{n\to\infty}\Gamma_{\tau(n)+1}=0.$$

Moreover, for $n\ge n_0$, it is clear that $\Gamma_{\tau(n)}\le\Gamma_{\tau(n)+1}$ if $n\ne\tau(n)$ (that is $\tau(n)<n$) because $\Gamma_j>\Gamma_{j+1}$ for $\tau(n)+1\le j\le n$.
Consequently for all $n\ge n_0$,

$$0\le\Gamma_n\le\max\{\Gamma_{\tau(n)},\Gamma_{\tau(n)+1}\}=\Gamma_{\tau(n)+1}.$$

Thus, $\lim_{n\to\infty}\Gamma_n=0$. That is $\{(x_n,y_n)\}$ converges strongly to $(\bar{x},\bar{y})$. □

Corollary 4.3.2. *Let H_1, H_2, H_3 be real Hilbert spaces and $C,\ Q$ be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $T_i : H_1 \to CB(H_1),\ i = 1, 2, \ldots$ and $S_j : H_2 \to CB(H_2),\ j = 1, 2, \ldots$ be two countable families of multi-valued type-one demicontractive-type mappings with constants k_i and k_j respectively, such that for $i, j = 1, 2, \ldots,$ T_i and S_j are demi-closed at 0. Let $f : C \to C,\ h : Q \to Q$ be μ (respectively, ν)-inverse strongly monotone operators and $A : H_1 \to H_3,\ B : H_2 \to H_3$ be bounded linear operators. Assume that the solution set $\Gamma^* := \{(\bar{x}, \bar{y}) \in \cap_{i=1}^{\infty} F(T_i) \times \cap_{i=1}^{\infty} F(S_j)\ :\ (\bar{x}, \bar{y}) \in VI(C, f) \times VI(Q, h)$ and $A\bar{x} = B\bar{y}\} \neq \emptyset$ and that the stepsize sequence $\{\gamma_n\}$ is chosen in such a way that for some $\epsilon > 0$,*
$\gamma_n \in \left(\epsilon,\ \frac{2||Aw_n - Bz_n||^2}{||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2} - \epsilon\right), n \in \Omega,$
otherwise $\gamma_n = \gamma$(γ being any nonegative value), where the set of indexes $\Omega = \{n : Aw_n - Bz_n \neq 0\}$.
Let $u, x_1 \in H_1$ and $v, y_1 \in H_2$ be arbitrary and the sequence $\{(x_n, y_n)\}$ be generated by

$$\begin{cases} w_n = (1 - \alpha_n)x_n + \alpha_n u, \\ z_n = (1 - \alpha_n)y_n + \alpha_n v; \\ u_n = P_C(I - \lambda f)(w_n - \gamma_n A^*(Aw_n - Bz_n)), \\ v_n = P_Q(I - \lambda h)(z_n + \gamma_n B^*(Aw_n - Bz_n)), \\ x_{n+1} = \beta_0 u_n + \sum_{i=1}^{\infty} \beta_i g_n^i, \\ y_{n+1} = \beta_0 v_n + \sum_{j=1}^{\infty} \beta_j h_n^j,\ \forall n \geq 1, \end{cases} \tag{4.3.49}$$

where $0 < \lambda < \min\{2\mu, 2\nu\}$ and A^*, B^* are the adjoint of A and B respectively. $g_n^i \in P_{T_i}u_n,\ z_n^j \in P_{S_j}v_n,\ P_{T_i}u_n := \{g_n^i \in T_i u_n : ||g_n^i - u_n|| = d(u_n, T_i u_n)\}$,
with conditions

- *(i) $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,*
- *(ii) $k \in (0, 1)$, where $k := \max\{k_1, k_2\},\ k_1 = \sup_{i\geq 1}\{k_i\},\ k_2 = \sup_{j\geq 1}\{k_j\} \in (0, 1)$,*
- *(iii) $\beta_0 \in (k, 1),\ \beta_i, \beta_j \in (0, 1),\ i, j = 1, 2, \ldots$ such that $\sum_{i=0}^{\infty} \beta_i = 1$ and $\sum_{j=0}^{\infty} \beta_j = 1$.*
- *(iv) for each $x^* \in \cap_{i=1}^{\infty} F(T_i);\ T_i x^* = \{x^*\}$ and for each $y^* \in \cap_{j=1}^{\infty} F(S_j);\ S_j y^* = \{y^*\}$.*

Then $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$ in Γ^*.

Corollary 4.3.3. *Let H_1, H_2, H_3 be real Hilbert spaces and for each $l = 1, 2, \ldots, N,\ r = 1, 2, \ldots, m$, let C_l and Q_r be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $T_i : H_1 \to CB(H_1),\ i = 1, 2, \ldots$ and $S_j : H_2 \to CB(H_2),\ j = 1, 2, \ldots$ be two countable families of multi-valued type-one quasi-nonexpansive mappings, such that T_i and S_j are demi-closed at 0. Let $f_l : C_l \to C_l,\ h_r : Q_r \to Q_r$ be μ_l (respectively, ν_r)-inverse strongly monotone operators and $A : H_1 \to H_3,\ B : H_2 \to H_3$ be bounded linear operators. Assume that the solution set $\Gamma \neq \emptyset$, and for each $x^* \in \cap_{i=1}^{\infty} F(T_i);\ T_i x^* = \{x^*\}$, for each $y^* \in \cap_{j=1}^{\infty} F(S_j);\ S_i y^* = \{y^*\}$. Let the stepsize sequence $\{\gamma_n\}$ be chosen in such a way that*
$\gamma_n \in \left(\epsilon,\ \frac{2||Aw_n - Bz_n||^2}{||A^*(Aw_n - Bz_n)||^2 + ||B^*(Aw_n - Bz_n)||^2} - \epsilon\right), n \in \Omega,$

otherwise $\gamma_n = \gamma$(γ *being any nonegative value), where the set of indexes* $\Omega = \{n : Aw_n - Bz_n \neq 0\}$.
Let $u, x_1 \in H_1$ *and* $v, y_1 \in H_2$ *be arbitrary and the sequence* $\{(x_n, y_n)\}$ *be generated by*

$$\begin{cases} w_n = (1-\alpha_n)x_n + \alpha_n u, \\ z_n = (1-\alpha_n)y_n + \alpha_n v; \\ u_n = P_{C_N}(I - \lambda f_N)\ o\ P_{C_{N-1}}(I - \lambda f_{N-1})\ o\ \ldots\ o\ P_{C_1}(I - \lambda f_1)(w_n - \gamma_n A^*(Aw_n - Bz_n)), \\ v_n = P_{Q_m}(I - \lambda h_m)\ o\ P_{Q_{m-1}}(I - \lambda h_{m-1})\ o\ \ldots\ o\ P_{Q_1}(I - \lambda h_1)(z_n + \gamma_n B^*(Aw_n - Bz_n)), \\ x_{n+1} = \beta_0 u_n + \sum_{i=1}^{\infty} \beta_i g_n^i, \\ y_{n+1} = \beta_0 v_n + \sum_{j=1}^{\infty} \beta_j h_n^j,\ \forall n \geq 1, \end{cases} \tag{4.3.50}$$

where $0 < \lambda < \min\{2\mu, 2\nu\}$, $\mu := \min\{\mu_l,\ l = 1, 2, \ldots, N\}$, $\nu := \min\{\nu_r,\ r = 1, 2, \ldots, m\}$ *and* A^*, B^* *are the adjoint of* A *and* B *respectively.* $g_n^i \in P_{T_i}u_n$, $z_n^j \in P_{S_j}v_n$ *and* $P_{T_i}u_n := \{g_n^i \in T_i u_n : ||g_n^i - u_n|| = d(u_n, T_i u_n)\}$. *Suppose* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Let* $\beta_i, \beta_j \in (0,1)$, $i, j = 0, 1, 2, \ldots$ *such that* $\sum_{i=0}^{\infty} \beta_i = 1$ *and* $\sum_{j=0}^{\infty} \beta_j = 1$.
Then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y})$ *in* Γ.

4.4 Applications

4.4.1 Multiple-sets split equality convex minimization problem.

Let H_1, H_2 and H_3 be real Hilbert spaces and for each $l = 1, 2, \ldots, N$, $r = 1, 2, \ldots, m$, let C_l and Q_r be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $T_i : H_1 \to CB(H_1)$, $i = 1, 2, \ldots$ and $S_j : H_2 \to CB(H_2)$, $j = 1, 2, \ldots$ be two countable families of multi-valued type-one demicontractive-type mappings with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $\cap_{j=1}^{\infty} F(S_j) \neq \emptyset$. Let $f_l : C_l \to C_l$, $h_r : Q_r \to Q_r$ be convex and differentiable functions and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear operators. Consider the following problem which we call the Multiple-Sets Split Equality Fixed Point Convex Minimization Problem (MSSEFPCMP): Find $(\bar{x}, \bar{y}) \in \cap_{i=1}^{\infty} F(T_i) \times \cap_{j=1}^{\infty} F(S_j)$ such that for each $l = 1, 2, \ldots, N$, and $r = 1, 2, \ldots, m$,

$$\bar{x} = \arg\min_{x \in C_l} f_l(x), \tag{4.4.1}$$

$$\bar{y} = \arg\min_{y \in Q_r} h_r(y) \text{ and } A\bar{x} = B\bar{y}. \tag{4.4.2}$$

We can formulate the MSSSEFPCMP (4.4.1)-(4.4.2) as follows: Find $(\bar{x}, \bar{y}) \in \cap_{i=1}^{\infty} F(T_i) \times \cap_{j=1}^{\infty} F(S_j)$ such that for each $l = 1, 2, \ldots, N$, and $r = 1, 2, \ldots, m$,

$$\langle \nabla f_l(\bar{x}), x - \bar{x} \rangle \geq 0, \quad \forall x \in C_l, \tag{4.4.3}$$

$$\langle \nabla h_r(\bar{y}), y - \bar{y} \rangle \geq 0, \quad \forall y \in Q_r \text{ and } A\bar{x} = B\bar{y}, \tag{4.4.4}$$

where ∇f_l and ∇h_r are the gradient of f_l and h_r respectively. If we assume that for each $l = 1, 2, \ldots, N$, $r = 1, 2, \ldots, m$, ∇f_l and ∇h_r are Lipschitz continuous functions, then by Remark 2.1.6, we can apply Algorithm (4.3.1) to obtain the solution of MSSSEF-PCMP (4.4.1)-(4.4.2). Furthermore, by applying Theorem 4.3.1, we have that the sequence $\{(x_n, y_n)\}$ converges to a solution of MSSSEFPCMP (4.4.1)-(4.4.2).

4.4.2 Systems of split equality variational inequality problem and monotone variational inclusion problem.

Let H be a real Hilbert space and $f_l : H \to H$ be α_l-inverse strongly monotone mappings. Let $M_l : H \to 2^H$ be maximal monotone mappings for $l = 1, 2, \ldots, N$. We consider the following System of Monotone Variational Inclusion Problem (SMVIP) which is to find $\bar{x} \in H$ such that for each $l = 1, 2, \ldots, N$,

$$0 \in f_l(\bar{x}) + M_l(\bar{x}). \tag{4.4.5}$$

Let SOL(f_l, M_l) be the solution set of SMVIP, then for $\sigma > 0$ and $\lambda \in (0, 2\alpha_l)$, we have that $F(J_\sigma^{M_l}(I - \lambda f_l))$ =SOL(f_l, M_l), $l = 1, 2, \ldots, N$ and $J_\sigma^{M_l}(I - \lambda f_l)$ are averaged nonexpansive operators, where $J_\sigma^{M_l}(I - \lambda f_l)$ is the resolvent of M_l for each $l = 1, 2, \ldots, N$, with parameter σ (see for example [6, 92]).

Let us consider the following Systems of Split Equality Variational Inequality Problem (SSEVIP) which is to find $(\bar{x}, \bar{y}) \in$ SOL$(f_l, M_l)\times$ SOL(h_r, K_r), $(l = 1, 2, \ldots, N,\ r = 1, 2, \ldots, m)$ such that

$$\langle f_l(\bar{x}), x - \bar{x} \rangle \geq 0,\ \forall\ x \in C_l, \tag{4.4.6}$$

$$\langle h_r(\bar{y}), y - \bar{y} \rangle \geq 0,\ \forall\ y \in Q_r \text{ and } A\bar{x} = B\bar{y}. \tag{4.4.7}$$

We know that every averaged nonexpansive mapping with nonempty fixed point set is quasi-nonexpansive. Thus by using this fact and adding the assumption that the resolvent operators are of type-one, we can apply Algorithm (4.3.50) and Corollary 4.3.3 to obtain a solution of problem (4.4.6)-(4.4.7).

Chapter 5

Split Equality Monotone Inclusion Problem and Split Equality Fixed Point Problem

5.1 Introduction

Our study in the last two chapters has been in the frame work of real Hilbert spaces. In this chapter however, we shall extend our study from real Hilbert spaces to more general $p-$uniformly convex Banach spaces which are also uniformly smooth. Precisely, we study the following problem which we call the SEMIP and SEFPP:

Let E_1, E_2 and E_3 be $p-$uniformly convex Banach spaces which are also uniformly smooth and $A : E_1 \to E_3$, $B : E_2 \to E_3$ be bounded linear operators. Let $M_1 : E_1 \to 2^{E_1^*}$, $M_2 : E_2 \to 2^{E_2^*}$ be multivalued maximal monotone mappings and $T : E_1 \to E_1$, $S : E_2 \to E_2$ be right Bregman strongly nonexpansive mappings: Find $(\bar{x}, \bar{y}) \in F(T) \times F(S)$ such that

$$0 \in M_1(\bar{x}), \tag{5.1.1}$$

$$0 \in M_2(\bar{y}) \quad \text{and} \quad A\bar{x} = B\bar{y}. \tag{5.1.2}$$

Furthermore, we propose an iterative algorithm and using the algorithm, we state and prove a strong convergence result for the approximation of a solution of problem (5.1.1)-(5.1.2). In what follows, we shall denote the solution set of problem (5.1.1)-(5.1.2) by Γ defined by
$\Gamma := \{(\bar{x}, \bar{y}) \in F(T) \times F(S)$ such that $0 \in M_1(\bar{x}),\ 0 \in M_2(\bar{y})$ and $A\bar{x} = B\bar{y}\}$.

5.2 Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem.

Lemma 5.2.1. *[106]. Let $x, y \in E$. If E is q-uniformly smooth, then there exists $C_q > 0$ such that*

$$||x - y||^q \leq ||x||^q - q\langle J_q^E(x), y\rangle + C_q||y||^q. \tag{5.2.1}$$

Lemma 5.2.2. *[119]. Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
(i) $\Sigma_{n=0}^{\infty}\gamma_n = \infty$,
(ii) $\limsup_{n\to\infty}\delta_n \leq 0$ or $\Sigma_{n=0}^{\infty}|\delta_n\gamma_n| < \infty$,
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 5.2.3. *[80]. Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_j\}$ of $\{n\}$ with $a_{n_j} < a_{n_j+1}$ $\forall j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{i \leq k : a_i < a_{i+1}\}$.

5.3 Main Result

Theorem 5.3.1. *Let E_1, E_2 and E_3 be three p-uniformly convex real Banach spaces which are also uniformly smooth and $A : E_1 \to E_3$, $B : E_2 \to E_3$ be two bounded linear operators. Let $M_1 : E_1 \to 2^{E_1^*}$, $M_2 : E_2 \to 2^{E_2^*}$ be multivalued maximal monotone mappings and $T : E_1 \to E_1$, $S : E_2 \to E_2$ be right Bregman strongly nonexpansive mappings such that $F(T) = \hat{F}(T)$ and $F(S) = \hat{F}(S)$. Suppose that $\Gamma \neq \emptyset$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $u, x_0 \in E_1$ and $v, y_0 \in E_2$ be arbitrary and the sequence $\{(x_n, y_n)\}$ be generated by*

$$\begin{cases} u_n = Res_p^{\lambda M_1} J_q^{E_1^*}\left[J_p^{E_1}(x_n) - t_nA^*J_p^{E_3}(Ax_n - By_n)\right], \\ v_n = Res_p^{\lambda M_2} J_q^{E_2^*}\left[J_p^{E_2}(y_n) + t_nB^*J_p^{E_3}(Ax_n - By_n)\right], \\ x_{n+1} = J_q^{E_1^*}\left[\alpha_nJ_p^{E_1}(u) + \beta_nJ_p^{E_1}(u_n) + \gamma_nJ_p^{E_1}(Tu_n)\right], \\ y_{n+1} = J_q^{E_2^*}\left[\alpha_nJ_p^{E_2}(v) + \beta_nJ_p^{E_2}(v_n) + \gamma_nJ_p^{E_2}(Sv_n)\right], \ n \geq 0, \end{cases} \tag{5.3.1}$$

with conditions

(i) $\lim_{n\to\infty}\alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(iii) $0 < t \leq t_n \leq k \leq \left(\frac{q}{2C_q||A||^q}\right)^{\frac{1}{q-1}}, \left(\frac{q}{2D_q||B||^q}\right)^{\frac{1}{q-1}}$, *(iv)* $(1-\alpha_n)a < \gamma_n,\ \alpha_n \le b < 1,\ a \in (0, \frac{1}{2})$.

Then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y}) \in \Gamma$.

Proof. Let $(x^*, y^*) \in \Gamma$, from (5.4.8) and Lemma 5.2.1, we have

$$\begin{aligned}
\Delta_p(u_n, x^*) &= \Delta_p(\text{Res}_p^{M_1} J_q^{E_1^*}\left[J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)\right], x^*) \\
&\le \Delta_p(J_q^{E_1^*}\left[J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)\right], x^*) \\
&= \frac{1}{q}||J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)||^q - \langle J_p^{E_1}(x_n), x^*\rangle \\
&\quad + t_n\langle Ax^*, J_p^{E_3}(Ax_n - By_n)\rangle + \frac{1}{p}||x^*||^p \\
&\le \frac{1}{q}||J_p^{E_1}(x_n)||^q - t_n\langle Ax_n, J_p^{E_3}(Ax_n - By_n)\rangle + \frac{C_q(t_n||A||)^q}{q}||J_p^{E_3}(Ax_n - By_n)||^q \\
&\quad - \langle J_p^{E_1}(x_n), x^*\rangle + \frac{1}{p}||x^*||^p + t_n\langle Ax^*, J_P^{E_3}(Ax_n - By_n)\rangle \\
&= \frac{1}{q}||x_n||^p - \langle J_p^{E_1}(x_n), x^*\rangle + \frac{1}{p}||x^*||^p + t_n\langle Ax^* - Ax_n, J_p^{E_3}(Ax_n - By_n)\rangle \\
&\quad + \frac{C_q(t_n||A||)^q}{q}||(Ax_n - By_n)||^p \\
&= \Delta_p(x_n, x^*) + t_n\langle J_p^{E_3}(Ax_n - By_n), Ax^* - Ax_n\rangle \\
&\quad + \frac{C_q(t_n||A||)^q}{q}||Ax_n - By_n||^p. \qquad (5.3.2)
\end{aligned}$$

Similarly, from (5.4.8) and Lemma 5.2.1, we have

$$\begin{aligned}
\Delta_p(v_n, y^*) &\le \Delta_p(y_n, y^*) - t_n\langle J_p^{E_3}(Ax_n - By_n), By^* - By_n\rangle \\
&\quad + \frac{D_q(t_n||B||)^q}{q}||Ax_n - By_n||^p. \qquad (5.3.3)
\end{aligned}$$

Adding (5.3.2) and (5.3.3) and using the fact that $Ax^* = By^*$, we have

$$\begin{aligned}
\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*) &\le \Delta_p(x_n, x^*) + \Delta_p(y_n, y^*) - t_n\langle J_p^{E_3}(Ax_n - By_n), Ax_n - By_n\rangle \\
&\quad + \frac{C_q(t_n||A||)^q}{q}||Ax_n - By_n||^p + \frac{D_q(t_n||B||)^q}{q}||Ax_n - By_n||^p \\
&= \Delta_p(x_n, x^*) + \Delta_p(y_n, y^*) \\
&\quad - \left[t_n - \left(\frac{C_q(t_n||A||)^q}{q} + \frac{D_q(t_n||B||)^q}{q}\right)\right]||Ax_n - By_n||^p \qquad (5.3.4)
\end{aligned}$$

Using condition (iii) in (5.3.4), we have

$$\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*) \le \Delta_p(x_n, x^*) + \Delta_p(y_n, y^*). \qquad (5.3.5)$$

From (5.4.8), we have

$$\begin{aligned}
\Delta_p(x_{n+1}, x^*) &= \Delta_p\left(J_q^{E_1^*}\left[\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(u_n) + \gamma_n J_p^{E_1}(Tu_n)\right], x^*\right) \\
&\le \alpha_n\Delta_p(u, x^*) + \beta_n\Delta_p(u_n, x^*) + \gamma_n\Delta_p(u_n, x^*) \\
&= \alpha_n\Delta_p(u, x^*) + (1-\alpha_n)\Delta_p(u_n, x^*). \qquad (5.3.6)
\end{aligned}$$

Similarly, from (5.4.8), we have

$$\Delta_p(y_{n+1}, y^*) \leq \alpha_n \Delta_p(v, y^*) + (1-\alpha_n)\Delta_p(v_n, y^*). \tag{5.3.7}$$

Adding (5.3.6) and (5.3.7) and using (5.3.5), we have

$$\begin{aligned}
\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*) &\leq \alpha_n \left[\Delta_p(u, x^*) + \Delta_p(v, y^*)\right] + (1-\alpha_n)\left[\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)\right] \\
&\leq \alpha_n \left[\Delta_p(u, x^*) + \Delta_p(v, y^*)\right] + (1-\alpha_n)\left[\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)\right] \\
&\leq \max\{\Delta_p(u, x^*) + \Delta_p(v, y^*),\ \Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)\} \\
&\ \vdots \\
&\leq \max\{\Delta_p(u, x^*) + \Delta_p(v, y^*),\ \Delta_p(x_0, x^*) + \Delta_p(y_0, y^*)\}. \qquad (5.3.8)
\end{aligned}$$

Therefore $\{\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)\}$ is bounded and consequently, $\{\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)\}$, $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{v_n\}$, $\{Ax_n\}$ and $\{By_n\}$ are all bounded.

Let $w_n = J_q^{E_1^*}\left(\frac{\beta_n}{1-\alpha_n} J_p^{E_1}(u_n) + \frac{\gamma_n}{1-\alpha_n} J_p^{E_1}(Tu_n)\right)$ and $z_n = J_q^{E_2^*}\left(\frac{\beta_n}{1-\alpha_n} J_p^{E_2}(v_n) + \frac{\gamma_n}{1-\alpha_n} J_p^{E_2}(Sv_n)\right)$, then

$$\begin{aligned}
\Delta_p(w_n, x^*) &= \Delta_p\left(J_q^{E_1^*}\left(\frac{\beta_n}{1-\alpha_n} J_p^{E_1} u_n + \frac{\gamma_n}{1-\alpha_n} J_p^{E_1} Tu_n\right), x^*\right) \\
&\leq \frac{\beta_n}{1-\alpha_n}\Delta_p(u_n, x^*) + \frac{\gamma_n}{1-\alpha_n}\Delta_p(Tu_n, x^*) \\
&\leq \frac{\beta_n}{1-\alpha_n}\Delta_p(u_n, x^*) + \frac{\gamma_n}{1-\alpha_n}\Delta_p(u_n, x^*) \\
&= \frac{\beta_n + \gamma_n}{1-\alpha_n}\Delta_p(u_n, x^*) \\
&= \Delta_p(u_n, x^*). \qquad (5.3.9)
\end{aligned}$$

Similarly, we have

$$\Delta_p(z_n, y^*) \leq \Delta_p(v_n, y^*). \tag{5.3.10}$$

Adding (5.3.9) and (5.3.10), we have

$$\Delta_p(w_n, x^*) + \Delta_p(z_n, y^*) \leq \Delta_p(u_n, x^*) + \Delta_p(v_n, y^*). \tag{5.3.11}$$

From the definition of w_n, we have

$$\begin{aligned}
\Delta_p(x_{n+1},x^*) &= \Delta_p\left(J_q^{E_1^*}\left[\alpha_n J_p^{E_1}(u)+\beta_n J_p^{E_1}(u_n)+\gamma_n J_p^{E_1}(Tu_n)\right],x^*\right)\\
&= \Delta_p\left(J_q^{E_1^*}\left[\alpha_n J_p^{E_1}(u)+(1-\alpha_n)J_p^{E_1}(w_n)\right],x^*\right)\\
&= V_p\left(\alpha_n J_p^{E_1}(u)+(1-\alpha_n)J_p^{E_1}(w_n),x^*\right)\\
&\leq V_p\left(\alpha_n J_p^{E_1}(u)+(1-\alpha_n)J_p^{E_1}(w_n)-\alpha_n(J_p^{E_1}(u)-J_p^{E_1}(x^*)),x^*\right)\\
&\quad -\langle-\alpha_n(J_p^{E_1}(u)-J_p^{E_1}(x^*)),J_q^{E_1^*}\left[\alpha_n J_p^{E_1}(u)+(1-\alpha_n)J_p^{E_1}(w_n)\right]-x^*\rangle\\
&= V_p\left(\alpha_n J_p^{E_1}(x^*)+(1-\alpha_n)J_p^{E_1}(w_n),x^*\right)\\
&\quad +\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle\\
&= \Delta_p\left(J_q^{E_1^*}\left[\alpha_n J_p^{E_1}(x^*)+(1-\alpha_n)J_p^{E_1}(w_n)\right],x^*\right)\\
&\quad +\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle\\
&\leq \alpha_n\Delta_p(x^*,x^*)+(1-\alpha_n)\Delta_p(w_n,x^*)\\
&\quad +\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle\\
&\leq (1-\alpha_n)\Delta_p(u_n,x^*)+\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle. \qquad (5.3.12)
\end{aligned}$$

Similarly, we have

$$\Delta_p(y_{n+1},y^*) \leq (1-\alpha_n)\Delta_p(v_n,y^*)+\alpha_n\langle J_p^{E_2}(v)-J_p^{E_2}(y^*),y_{n+1}-y^*\rangle. \qquad (5.3.13)$$

Adding (5.3.12) and (5.3.13), we have

$$\begin{aligned}
\Delta_p(x_{n+1},x^*)+\Delta_p(y_{n+1},y^*) &\leq (1-\alpha_n)\left[\Delta_p(u_n,x^*)+\Delta_p(v_n,y^*)\right]\\
&\quad +\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle\\
&\quad +\alpha_n\langle J_p^{E_2}(v)-J_p^{E_2}(y^*),y_{n+1}-y^*\rangle\\
&\leq (1-\alpha_n)\left[\Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)\right]\\
&\quad +\alpha_n\langle J_p^{E_1}(u)-J_p^{E_1}(x^*),x_{n+1}-x^*\rangle\\
&\quad +\alpha_n\langle J_p^{E_2}(v)-J_p^{E_2}(y^*),y_{n+1}-y^*\rangle. \qquad (5.3.14)
\end{aligned}$$

We now consider two cases to establish the strong convergence of $\{(x_n,y_n)\}$ to $(\bar{x},\bar{y})$.

Case 1. Suppose that $\{\Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)\}$ is monotone non-increasing, then $\{\Delta_p(x_n,x^*)+\Delta_p(y_n,y^*)\}$ is convergent. Thus,

$$\lim_{n\to\infty}\left[(\Delta_p(x_{n+1},x^*)+\Delta_p(y_{n+1},y^*))-(\Delta_p(x_n,x^*)+\Delta_p(y_n,y^*))\right]=0.$$

From (5.4.8), (5.3.5) and (5.3.11), we have

$$\begin{aligned}
0 &\leq (\Delta_p(u_n,x^*)+\Delta_p(v_n,y^*))-(\Delta_p(w_n,x^*)+\Delta_p(z_n,y^*))\\
&= (\Delta_p(u_n,x^*)+\Delta_p(v_n,y^*))-(\Delta_p(x_{n+1},x^*)+\Delta_p(y_{n+1},y^*))\\
&\quad +(\Delta_p(x_{n+1},x^*)+\Delta_p(y_{n+1},y^*))-(\Delta_p(w_n,x^*)+\Delta_p(z_n,y^*))\\
&\leq (\Delta_p(x_n,x^*)+\Delta_p(y_n,y^*))-(\Delta_p(x_{n+1},x^*)+\Delta_p(y_{n+1},y^*))\\
&\quad +\alpha_n\left(\Delta_p(u,x^*)+\Delta_p(v,y^*)\right)+(1-\alpha_n)\left(\Delta_p(w_n,x^*)+\Delta_p(z_n,y^*)\right)\\
&\quad -\left(\Delta_p(w_n,x^*)+\Delta_p(z_n,y^*)\right)\to 0, \text{ as } n\to\infty,
\end{aligned}$$

which implies

$$\lim_{n\to\infty}\left(\Delta_p(u_n,x^*)-\Delta_p(w_n,x^*)\right)=\lim_{n\to\infty}\left(\Delta_p(v_n,y^*)-\Delta_p(z_n,y^*)\right)=0. \tag{5.3.15}$$

Also, from the definition of w_n, we have

$$\begin{aligned}\Delta_p(w_n,x^*) &= \Delta_p\left(J_q^{E_1^*}\left(\frac{\beta_n}{1-\alpha_n}J_p^{E_1}(u_n)+\frac{\gamma_n}{1-\alpha_n}J_p^{E_1}(Tu_n)\right),x^*\right)\\ &\leq \frac{\beta_n}{1-\alpha_n}\Delta_p(u_n,x^*)+\frac{\gamma_n}{1-\alpha_n}\Delta_p(Tu_n,x^*)\\ &= \Delta_p(u_n,x^*)-\left(1-\frac{\beta_n}{1-\alpha_n}\right)\Delta_p(u_n,x^*)+\frac{\gamma_n}{1-\alpha_n}\left(\Delta_p(Tu_n,x^*)\right)\\ &= \Delta_p(u_n,x^*)+\frac{\gamma_n}{1-\alpha_n}\left(\Delta_p(Tu_n,x^*)-\Delta_p(u_n,x^*)\right), \end{aligned} \tag{5.3.16}$$

which implies

$$\frac{\gamma_n}{1-\alpha_n}\left(\Delta_p(u_n,x^*)-\Delta_p(Tu_n,x^*)\right)\leq \Delta_p(u_n,x^*)-\Delta_p(w_n,x^*)\to 0,\ \text{ as } n\to\infty. \tag{5.3.17}$$

By condition (iv), we have

$$\lim_{n\to\infty}\left(\Delta_p(u_n,x^*)-\Delta_p(Tu_n,x^*)\right)=0. \tag{5.3.18}$$

Similarly, we have

$$\lim_{n\to\infty}\left(\Delta_p(v_n,y^*)-\Delta_p(Sv_n,y^*)\right)=0. \tag{5.3.19}$$

Since T and S are right Bregman strongly nonexpansive mappings, then from (5.3.18) and (5.3.19), we have

$$\lim_{n\to\infty}\Delta_p(Tu_n,u_n)=0$$

and

$$\lim_{n\to\infty}\Delta_p(Sv_n,v_n)=0$$

respectively, which implies

$$\lim_{n\to\infty}||Tu_n-u_n||=0 \tag{5.3.20}$$

and

$$\lim_{n\to\infty}||Sv_n-v_n||=0. \tag{5.3.21}$$

From (5.3.4), we have

$$\begin{aligned}
&\left[t_n - \left(\frac{C_q(t_n||A||)^q}{q} + \frac{D_q(t_n||B||)^q}{q}\right)\right] ||Ax_n - By_n||^p \\
\leq\ & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\
=\ & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& + (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\
\leq\ & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& +(1-\alpha_n)\left(\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)\right) - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\
& +\alpha_n\left[\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle + \langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^*\rangle\right] \\
=\ & (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& +\alpha_n(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle - \Delta_p(u_n, x^*)) \\
& +\alpha_n(\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^*\rangle - \Delta_p(v_n, y^*)) \to 0, \text{ as } n \to \infty.
\end{aligned}$$

That is,

$$\lim_{n\to\infty}\left[t_n - \left(\frac{C_q(t_n||A||)^q}{q} + \frac{D_q(t_n||B||)^q}{q}\right)\right] ||Ax_n - By_n||^p = 0.$$

Since $0 < t\left(1 - \left(\frac{C_q k^{q-1}(||A||)^q}{q} + \frac{D_q k^{q-1}(||B||)^q}{q}\right)\right) \leq \left(t_n - \left(\frac{C_q(t_n||A||)^q}{q} + \frac{D_q(t_n||B||)^q}{q}\right)\right)$, we have

$$\lim_{n\to\infty} ||Ax_n - By_n||^p = 0. \tag{5.3.22}$$

Let $a_n = J_q^{E_1^*}\left[J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)\right]$ and $b_n = J_q^{E_2^*}\left[J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)\right]$. Then,

$$\begin{aligned}
||J_p^{E_1}a_n - J_p^{E_1}x_n|| &= ||J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n) - J_p^{E_1}(x_n)|| \\
&\leq t_n||A^*||||J_p^{E_3}(Ax_n - By_n)|| \\
&\leq \left(\frac{q}{C_q||A||^q}\right)^{\frac{1}{q-1}} ||A^*||||Ax_n - By_n|| \to 0, \ n \to \infty.
\end{aligned}$$

Since $J_p^{E_1^*}$ is norm to norm uniformly continuous on bounded subsets of E_1^*, we have

$$\lim_{n\to\infty} ||a_n - x_n|| = 0. \tag{5.3.23}$$

Similarly, we have

$$\lim_{n\to\infty} ||b_n - y_n|| = 0. \tag{5.3.24}$$

Since $\text{Res}_p^{\lambda M_1}$ is a Bregman firmly nonexpansive mapping (see Remark 2.2.34), we have

$$\begin{aligned}
&\Delta_p(x^*, \text{Res}_p^{\lambda M_1}a_n) + \Delta_p(\text{Res}_p^{\lambda M_1}a_n, x^*) + \Delta_p(x^*, x^*) + \Delta_p(a_n, \text{Res}_p^{\lambda M_1}a_n) \\
&\qquad\leq \Delta_p(x^*, \text{Res}_p^{\lambda M_1}a_n) + \Delta_p(a_n, x^*),
\end{aligned}$$

which implies

$$\Delta_p(a_n, \text{Res}_p^{\lambda M_1} a_n) \;\; \leq \;\; \Delta_p(a_n, x^*) - \Delta_p(\text{Res}_p^{\lambda M_1} a_n, x^*). \tag{5.3.25}$$

Similarly, we have

$$\Delta_p(b_n, \text{Res}_p^{\lambda M_2} b_n) \;\; \leq \;\; \Delta_p(b_n, y^*) - \Delta_p(\text{Res}_p^{\lambda M_2} b_n, y^*). \tag{5.3.26}$$

Adding (5.3.25) and (5.3.26), we have

$$\begin{aligned}
& \Delta_p(a_n, \text{Res}_p^{\lambda M_1} a_n) + \Delta_p(b_n, \text{Res}_p^{\lambda M_2} b_n) \\
\leq \;& (\Delta_p(a_n, x^*) + \Delta_p(b_n, y^*)) - \big(\Delta_p(\text{Res}_p^{\lambda M_1} a_n, x^*) + \Delta_p(\text{Res}_p^{\lambda M_2} b_n, y^*)\big) \\
\leq \;& (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - \big(\Delta_p(\text{Res}_p^{\lambda M_1} a_n, x^*) + \Delta_p(\text{Res}_p^{\lambda M_2} b_n, y^*)\big) \\
= \;& (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\
= \;& (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& + (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\
\leq \;& (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& +(1-\alpha_n)\, (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) - (\Delta_p(u_n, x^*) + \Delta_p(v_n, y^*)) \\
& +\alpha_n \left[\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle + \langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^*\rangle\right] \\
= \;& (\Delta_p(x_n, x^*) + \Delta_p(y_n, y^*)) - (\Delta_p(x_{n+1}, x^*) + \Delta_p(y_{n+1}, y^*)) \\
& +\alpha_n(\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n+1} - x^*\rangle - \Delta_p(u_n, x^*)) \\
& +\alpha_n(\langle J_p^{E_2}(v) - J_p^{E_2}(y^*), y_{n+1} - y^*\rangle - \Delta_p(v_n, y^*)) \to 0, \;\text{ as } n \to \infty,
\end{aligned} \tag{5.3.27}$$

which implies

$$\lim_{n\to\infty} ||a_n - \text{Res}_p^{\lambda M_1} a_n|| = \lim_{n\to\infty} ||b_n - \text{Res}_p^{\lambda M_2} b_n|| = 0. \tag{5.3.28}$$

That is,

$$\lim_{n\to\infty} ||a_n - u_n|| = 0 \tag{5.3.29}$$

and

$$\lim_{n\to\infty} ||b_n - v_n|| = 0. \tag{5.3.30}$$

Since $J_p^{E_1}$ and $J_p^{E_2}$ are uniformly continuous on bounded subsets of E_1 and E_2 respectively, we have

$$\lim_{n\to\infty} ||J_p^{E_1} a_n - J_p^{E_1} u_n|| = 0 \tag{5.3.31}$$

and

$$\lim_{n\to\infty} ||J_p^{E_2} b_n - J_p^{E_2} v_n|| = 0. \tag{5.3.32}$$

From (5.3.23) and (5.3.29), we have

$$\lim_{n\to\infty} ||x_n - u_n|| = 0. \tag{5.3.33}$$

Similarly, we have

$$\lim_{n\to\infty} ||y_n - v_n|| = 0. \tag{5.3.34}$$

Since $\{x_n\}$ is bounded in E_1 and E_1 is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to $\bar{x}$. By (5.3.20) and (5.3.33), we have that $\bar{x} \in F(T)$ since $F(T) = \hat{F}(T)$. Also since $\{y_n\}$ is bounded in E_2 and E_2 is reflexive, there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ that converges weakly to $\bar{y}$. By (5.3.21) and (5.3.34), we have that $\bar{y} \in F(S)$ since $F(S) = \hat{F}(S)$.

Next, we show that $0 \in M_1(\bar{x})$ and $0 \in M_2(\bar{y})$.

Let $(z, \eta) \in G(M_1)$, then $\eta \in M_1 z$. From $u_n = \text{Res}_p^{\lambda M_1} a_n$, we have that

$$J_p^{E_1} a_n \in \left(J_p^{E_1} + \lambda M_1\right) u_n,$$

which implies

$$\frac{1}{\lambda}\left(J_p^{E_1} a_n - J_p^{E_1} u_n\right) \in M_1 u_n.$$

By the monotonicity of M_1, we have

$$\langle \eta - \frac{1}{\lambda}\left(J_p^{E_1} a_n - J_p^{E_1} u_n\right), z - u_n\rangle \geq 0.$$

This implies

$$\langle \eta, z - u_n\rangle \geq \left\langle \frac{1}{\lambda}\left(J_p^{E_1} a_n - J_p^{E_1} u_n\right), z - u_n \right\rangle.$$

Since $\{x_n\}$ converges weakly to $\bar{x}$, we have from (5.3.31) and (5.3.33) that

$$\langle \eta, z - \bar{x}\rangle \geq 0.$$

Hence, by the maximal monotonicity of M_1, we have that $0 \in M_1(\bar{x})$.

By similar argument, we obtain that $0 \in M_2(\bar{y})$.

We now show that $A\bar{x} = B\bar{y}$.
Since A and B are bounded linear operators, we have that $\{Ax_n\}$ and $\{By_n\}$ converge weakly to $\{A\bar{x}\}$ and $\{B\bar{y}\}$ respectively. Also, by weakly semi-continuity of the norm, we have

$$||A\bar{x} - B\bar{y}|| \leq \liminf_{n\to\infty} ||Ax_n - By_n|| = 0. \tag{5.3.35}$$

That is, $A\bar{x} = B\bar{y}$. Therefore $(\bar{x}, \bar{y}) \in \Gamma$.

We now show that $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$.

$$\begin{aligned}\Delta_p(x_{n+1}, u_n) &= \Delta_p(J_q^{E_1^*}\left[\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(u_n) + \gamma_n J_p^{E_1}(Tu_n)\right], u_n) \\ &\leq \alpha_n \Delta_p(u, u_n) + \beta_n \Delta_p(u_n, u_n) + \gamma_n \Delta_p(Tu_n, u_n) \to 0, \text{ as } n \to \infty,\end{aligned}$$

which implies

$$\lim_{n\to\infty} ||x_{n+1} - u_n|| = 0. \tag{5.3.36}$$

Similarly, we have

$$\lim_{n\to\infty} ||y_{n+1} - v_n|| = 0. \tag{5.3.37}$$

From (5.3.33) and (5.3.36), we have

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{5.3.38}$$

Similarly, we have

$$\lim_{n\to\infty} ||y_{n+1} - y_n|| = 0. \tag{5.3.39}$$

From (5.3.14), we have

$$\begin{aligned} & \Delta_p(x_{n+1}, \bar{x}) + \Delta_p(y_{n+1}, \bar{y}) \\ \leq \quad & (1-\alpha_n)\left[\Delta_p(x_n, \bar{x}) + \Delta_p(y_n, \bar{y})\right] \\ + \quad & \alpha_n \left[\langle J_p^{E_1}(u) - J_p^{E_1}(\bar{x}), x_{n+1} - \bar{x}\rangle + \langle J_p^{E_2}(v) - J_p^{E_2}(\bar{y}), y_{n+1} - \bar{y}\rangle\right]. \end{aligned} \tag{5.3.40}$$

Using Lemma 5.2.2 in (5.3.40), we conclude that $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y})$.

Case 2: Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\Delta_p(x_{n_i}, x^*) + \Delta_p(y_{n_i}, y^*) < \Delta_p(x_{n_i+1}, x^*) + \Delta_p(y_{n_i+1}, y^*) \ \forall \ i \in \mathbb{N}.$$

By Lemma 5.2.3, we can find a nondecreasing sequence $\{m_k\} \subseteq \mathbb{N}$ such that $m_k \to \infty$ and for all $k \in \mathbb{N}$, we have

$$\Delta_p(x_{m_k}, x^*) + \Delta_p(y_{m_k}, y^*) \leq \Delta_p(x_{m_k+1}, x^*) + \Delta_p(y_{m_k+1}, y^*)$$

and

$$\Delta_p(x_k, x^*) + \Delta_p(y_k, y^*) \leq \Delta_p(x_{m_k+1}, x^*) + \Delta_p(y_{m_k+1}, y^*). \tag{5.3.41}$$

Then, by the same arguments as in (5.3.11), (5.3.15), (5.3.16) and (5.3.17), we have that

$$\lim_{k\to\infty} ||Tu_{m_k} - u_{m_k}|| = 0 \tag{5.3.42}$$

and

$$\lim_{k\to\infty} ||Sv_{m_k} - v_{m_k}|| = 0. \tag{5.3.43}$$

From (5.3.14), we have

$$
\begin{aligned}
& \Delta_p(x_{m_k+1},\bar{x}) + \Delta_p(y_{m_k+1},\bar{y}) \\
\leq\ & (1-\alpha_{m_k})\left(\Delta_p(x_{m_k},\bar{x}) + \Delta_p(y_{m_k},\bar{y})\right) \\
+\ & \alpha_{m_k}\left(\langle J_p^{E_1}(u) - J_p^{E_1}(\bar{x}), x_{m_k+1}-\bar{x}\rangle + \langle J_p^{E_2}(v) - J_p^{E_2}(\bar{y}), y_{m_k+1}-\bar{y}\rangle\right), \quad (5.3.44)
\end{aligned}
$$

which implies

$$
\begin{aligned}
& \alpha_{m_k}\left(\Delta_p(x_{m_k},\bar{x}) + \Delta_p(y_{m_k},\bar{y})\right) \\
\leq\ & \left(\Delta_p(x_{m_k},\bar{x}) + \Delta_p(y_{m_k},\bar{y})\right) - \left(\Delta_p(x_{m_k+1},\bar{x}) + \Delta_p(y_{m_k+1},\bar{y})\right) \\
+\ & \alpha_{m_k}\left(\langle J_p^{E_1}(u) - J_p^{E_1}(\bar{x}), x_{m_k+1}-\bar{x}\rangle + \langle J_p^{E_2}(v) - J_p^{E_2}(\bar{y}), y_{m_k+1}-\bar{y}\rangle\right) \\
\leq\ & \alpha_{m_k}\left(\langle J_p^{E_1}(u) - J_p^{E_1}(\bar{x}), x_{m_k+1}-\bar{x}\rangle + \langle J_p^{E_2}(v) - J_p^{E_2}(\bar{y}), y_{m_k+1}-\bar{y}\rangle\right).
\end{aligned}
$$

That is

$$
\left(\Delta_p(x_{m_k},\bar{x}) + \Delta_p(y_{m_k},\bar{y})\right) \leq \left(\langle J_p^{E_1}(u) - J_p^{E_1}(\bar{x}), x_{m_k+1}-\bar{x}\rangle + \langle J_p^{E_2}(v) - J_p^{E_2}(\bar{y}), y_{m_k+1}-\bar{y}\rangle\right).
$$

Which implies

$$
\lim_{k\to\infty}\left(\Delta_p(x_{m_k},\bar{x}) + \Delta_p(y_{m_k},\bar{y})\right) = 0. \tag{5.3.45}
$$

From (5.3.41) and (5.3.45), we have

$$
\Delta_p(x_k,\bar{x}) + \Delta_p(y_k,\bar{y}) \leq \Delta_p(x_{m_k+1},\bar{x}) + \Delta_p(y_{m_k+1},\bar{y}) \to 0, \text{ as } k\to\infty,
$$

which implies that $\{(x_k, y_k)\}$ converges strongly to $(\bar{x},\bar{y})$. Thus, $\{(x_n, y_n)\}$ converges strongly to $(\bar{x},\bar{y}) \in \Gamma$. □

From Remark 2.2.28, we obtain the following corollary.

Corollary 5.3.2. *Let E_1, E_2 and E_3 be three p-uniformly convex real Banach spaces which are also uniformly smooth and $A : E_1 \to E_3$, $B : E_2 \to E_3$ be two bounded linear operators. Let $M_1 : E_1 \to 2^{E_1^*}$, $M_2 : E_2 \to 2^{E_2^*}$ be multivalued maximal monotone mappings and $T : E_1 \to E_1$, $S : E_2 \to E_2$ be right Bregman firmly nonexpansive mappings. Suppose that $\Gamma \neq \emptyset$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $u, x_0 \in E_1$ and $v, y_0 \in E_2$ be arbitrary and the sequence $\{(x_n, y_n)\}$ be generated by*

$$
\begin{cases}
u_n = Res_p^{\lambda M_1} J_q^{E_1^*}\left[J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)\right], \\
v_n = Res_p^{\lambda M_2} J_q^{E_2^*}\left[J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)\right], \\
x_{n+1} = J_q^{E_1^*}\left[\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(u_n) + \gamma_n J_p^{E_1}(Tu_n)\right], \\
y_{n+1} = J_q^{E_2^*}\left[\alpha_n J_p^{E_2}(v) + \beta_n J_p^{E_2}(v_n) + \gamma_n J_p^{E_2}(Sv_n)\right], \ n \geq 0,
\end{cases} \tag{5.3.46}
$$

with conditions

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < t \le t_n \le k \le \left(\frac{q}{2C_q||A||^q}\right)^{\frac{1}{q-1}}, \left(\frac{q}{2D_q||B||^q}\right)^{\frac{1}{q-1}}$,

(iv) $(1-\alpha_n)a < \gamma_n,\ \alpha_n \le b < 1,\ a \in (0, \frac{1}{2})$.

Then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y}) \in \Gamma$.

If we let $M_1 = M_2 = 0$ in Theorem 5.3.1, then from Remark 2.2.19 we have that $\text{Res}_p^{\lambda M_1} = I_1$ and $\text{Res}_p^{\lambda M_2} = I_2$ (where I_1 and I_2 are identity maps on E_1 and E_2 respectively). Thus, we obtain the following corollary.

Corollary 5.3.3. *Let* E_1*,* E_2 *and* E_3 *be three* p*-uniformly convex real Banach spaces which are also uniformly smooth and* $A : E_1 \to E_3$*,* $B : E_2 \to E_3$ *be two bounded linear operators. Let* $T : E_1 \to E_1$*,* $S : E_2 \to E_2$ *be right Bregman strongly nonexpansive mappings such that* $F(T) = \hat{F}(T)$ *and* $F(S) = \hat{F}(S)$*. Suppose that* $\Gamma^* := \{(\bar{x}, \bar{y}) \in F(T) \times F(S)$ *such that* $A\bar{x} = B\bar{y}\} \neq \emptyset$ *and* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$ *are sequences in* $(0, 1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$*. Let* $u, x_0 \in E_1$ *and* $v, y_0 \in E_2$ *be arbitrary and the sequence* $\{(x_n, y_n)\}$ *be generated by*

$$\begin{cases} u_n = J_q^{E_1^*}\left[J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)\right], \\ v_n = J_q^{E_2^*}\left[J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)\right], \\ x_{n+1} = J_q^{E_1^*}\left[\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(u_n) + \gamma_n J_p^{E_1}(Tu_n)\right], \\ y_{n+1} = J_q^{E_2^*}\left[\alpha_n J_p^{E_2}(v) + \beta_n J_p^{E_2}(v_n) + \gamma_n J_p^{E_2}(Sv_n)\right],\ n \ge 0, \end{cases} \tag{5.3.47}$$

with conditions

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < t \le t_n \le k \le \left(\frac{q}{2C_q||A||^q}\right)^{\frac{1}{q-1}}, \left(\frac{q}{2D_q||B||^q}\right)^{\frac{1}{q-1}}$,

(iv) $(1-\alpha_n)a < \gamma_n,\ \alpha_n \le b < 1,\ a \in (0, \frac{1}{2})$.

Then $\{(x_n, y_n)\}$ *converges strongly to* $(\bar{x}, \bar{y}) \in \Gamma^*$.

5.4 Applications and numerical example

5.4.1 Application to convex minimization problem

Let E_1, E_2 and E_3 be three p-uniformly convex real Banach spaces which are also uniformly smooth and $f : E_1 \to (-\infty, +\infty]$, $g : E_2 \to (-\infty, +\infty]$ be proper, convex and lower semi-continuous functions which attains their minimum over E_1 and E_2 respectively. Let $A : E_1 \to E_3$, $B : E_2 \to E_3$ be two bounded linear operators and $T : E_1 \to E_1$, $S : E_2 \to E_2$ be right Bregman strongly nonexpansive mappings such that $F(T) = \hat{F}(T)$ and $F(S) = \hat{F}(S)$. Consider the following problem which we call the Split Equality Fixed Point Convex Minimization Problem (SEFPCMP): Find $x^* \in F(T)$ and $y^* \in F(S)$ such that

$$f(x^*) = \min_{x \in E_1} f(x), \tag{5.4.1}$$

$$g(y^*) = \min_{y \in E_2} g(y), \quad \text{and} \quad Ax^* = By^*. \tag{5.4.2}$$

It is generally known that the above SEFPCMP can be formulated as follows: Find $x^* \in F(T)$ and $y^* \in F(S)$ such that

$$0 \in \partial f(x^*), \tag{5.4.3}$$

$$0 \in \partial g(y^*), \quad \text{and} \quad Ax^* = By^*, \tag{5.4.4}$$

where ∂f and ∂g are the subdifferentials of f and g respectively. We know that the subdifferentials ∂f and ∂g are maximal monotone operators whenever f and g are proper, convex and lower semicontinuous functions. Hence, by applying Algorithm (5.4.8), we obtain the solution of the SEFPCMP (5.4.1)-(5.4.2).

5.4.2 Application to a common solution of monotone inclusion and equilibrium problem

Let E be a p-uniformly convex real Banach space which is also uniformly smooth and C be nonempty, closed and convex subset of E. Let $f : C \times C \to \mathbb{R}$ be a bifunction, then the Equilibrium Problem (EP) is to find $x \in C$ such that

$$f(x, y) \geq 0 \ \forall y \in C. \tag{5.4.5}$$

We denote the solution set of EP (5.4.5) by $EP(f)$. Suppose f satisfies the following conditions:

C_1: $f(x, x) = 0 \ \forall x \in C$,

C_2: f is monotone, i.e., $f(x, y) + f(y, x) \geq 0 \ \forall x, y \in C$,

C_3: for each $x, y \in C$, $\lim_{t \to 0} f(tz + (1-t)x, y) \geq f(x, y)$,

C_4: for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semicontinuous. Then

$$F(\text{Res}_p^f) = EP(f).$$

We know that the resolvent operator Res_p^f is single-valued and Bregman firmly nonexpansive operator, hence a Bregman strongly nonexpansive operator with $F(\text{Res}_p^f) = \hat{F}(\text{Res}_p^f)$ (see for example [109]).

Consider the following problem, which we call Split Equality Monotone Inclusion and Equilibrium Problem (SEMIEP): Find $x \in F(\text{Res}_p^f)$ and $y \in F(\text{Res}_p^g)$ such that

$$0 \in M_1(x), \tag{5.4.6}$$

$$0 \in M_2(y) \quad \text{and} \quad Ax = By, \tag{5.4.7}$$

where $M_1 : E_1 \to 2^{E_1^*}$, $M_2 : E_2 \to 2^{E_2^*}$ are maximal monotone operators and $A : E_1 \to E_3$, $B : E_2 \to E_3$ are bounded linear operators. Let the set of solutions of problem (5.4.6)-(5.4.7) be Ω, then by setting $T = \text{Res}_p^f$ and $S = \text{Res}_p^g$ in Theorem 5.3.1, we obtain the following result.

Theorem 5.4.1. *Let E_1, E_2 and E_3 be three p-uniformly convex real Banach spaces which are also uniformly smooth and C, Q be nonempty, closed and convex subsets of E_1, E_2 respectively. Let $A : E_1 \to E_3$, $B : E_2 \to E_3$ be two bounded linear operators and $M_1 : E_1 \to 2^{E_1^*}$, $M_2 : E_2 \to 2^{E_2^*}$ be multivalued maximal monotone mappings. Let $f : C \times C \to \mathbb{R}$ and $g : Q \times Q \to \mathbb{R}$ be bifunctions satisfying conditions $(C_1) - (C_4)$. Suppose that $\Omega \neq \emptyset$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $u, x_0 \in E_1$ and $v, y_0 \in E_2$ be arbitrary and the sequence $\{(x_n, y_n)\}$ be generated by*

$$\begin{cases} u_n = Res_p^{\lambda M_1} J_q^{E_1^*}\left[J_p^{E_1}(x_n) - t_n A^* J_p^{E_3}(Ax_n - By_n)\right], \\ v_n = Res_p^{\lambda M_2} J_q^{E_2^*}\left[J_p^{E_2}(y_n) + t_n B^* J_p^{E_3}(Ax_n - By_n)\right], \\ x_{n+1} = J_q^{E_1^*}\left[\alpha_n J_p^{E_1}(u) + \beta_n J_p^{E_1}(u_n) + \gamma_n J_p^{E_1}\left(Res_p^f(u_n)\right)\right], \\ y_{n+1} = J_q^{E_2^*}\left[\alpha_n J_p^{E_2}(v) + \beta_n J_p^{E_2}(v_n) + \gamma_n J_p^{E_2}\left(Res_p^g(v_n)\right)\right], \ n \geq 0, \end{cases} \tag{5.4.8}$$

with conditions

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < t \leq t_n \leq k \leq \left(\frac{q}{2C_q||A||^q}\right)^{\frac{1}{q-1}}, \left(\frac{q}{2D_q||B||^q}\right)^{\frac{1}{q-1}}$,

(iv) $(1-\alpha_n)a < \gamma_n, \ \alpha_n \leq b < 1, \ a \in (0, \frac{1}{2})$.

Then $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y}) \in \Omega$.

5.4.3 Numerical example

We give a numerical example in $(\mathbb{R}^2, ||.||_2)$ to support our main result.
Let $E_1 = E_2 = E_3 = \mathbb{R}^2$. We define $A : \mathbb{R}^2 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$A(x) = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } B(x) = \begin{bmatrix} 5 & 1 \\ -8 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ respectively.}$$

Let $M_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $M_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$M_1(x) = (x_1 + x_2,\ x_2 - x_1) \text{ and } M_2(x) = (x_2,\ -x_1) \text{ respectively.}$$

Then, by Proposition 2.2.20 (iv), we have that

$$\text{Res}^{\lambda M_1}(x) = \left[(J_p + \lambda M_1)^{-1} \circ J_p\right](x) = (I + \lambda M_1)^{-1}(x) = T_\lambda^{M_1}.$$

So that

$$\begin{aligned} T_\lambda^{M_1}(x) &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda & \lambda \\ -\lambda & \lambda \end{bmatrix}\right)^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} 1+\lambda & \lambda \\ -\lambda & 1+\lambda \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \frac{1}{1+2\lambda+2\lambda^2} \begin{bmatrix} 1+\lambda & -\lambda \\ \lambda & 1+\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \left(\frac{(1+\lambda)x_1 - \lambda x_2}{1+2\lambda+2\lambda^2},\ \frac{\lambda x_1 + (1+\lambda)x_2}{1+2\lambda+2\lambda^2}\right). \end{aligned}$$

Similarly, we have that

$$T_\lambda^{M_2}(x) = \left(\tfrac{x_1 - \lambda x_2}{1+\lambda^2},\ \tfrac{x_2 + \lambda x_1}{1+\lambda^2}\right).$$

Take $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{2n}{3(n+1)}$ and $\gamma_n = \frac{n}{3(n+1)}$. Then α_n, β_n and γ_n satisfy the conditions in Theorem 5.3.1.

Since $\text{Res}^{\lambda M_1}$ and $\text{Res}^{\lambda M_2}$ are Bregman firmly nonexpansive mappings, then they are both right Bregman strongly nonexpansive mappings satisfying $F(\text{Res}^{\lambda M_1}) = \hat{F}(\text{Res}^{\lambda M_1})$ and $F(\text{Res}^{\lambda M_2}) = \hat{F}(\text{Res}^{\lambda M_2})$.

Therefore, we can take $T = \text{Res}^{\lambda M_1} = T_\lambda^{M_1}$ and $S = \text{Res}^{\lambda M_2} = T_\lambda^{M_2}$ as defined above.

Hence, for $x_0,\ y_0 \in \mathbb{R}^2$, our Algorithm (5.4.8) becomes:

$$\begin{cases} u_n = T_\lambda^{M_1}\left(x_n - t_n A^T(Ax_n - By_n)\right), \\ v_n = T_\lambda^{M_2}\left(y_n + t_n B^T(Ax_n - By_n)\right), \\ x_{n+1} = \frac{u}{n+1} + \frac{2n}{3(n+1)} u_n + \frac{n}{3(n+1)}\left(T_\lambda^{M_1} u_n\right), \\ y_{n+1} = \frac{v}{n+1} + \frac{2n}{3(n+1)} v_n + \frac{n}{3(n+1)}\left(T_\lambda^{M_2} v_n\right), n \geq 0. \end{cases} \tag{5.4.9}$$

We can make different choices of x_0, y_0, u, v and t_n with appropriate tolerance levels in Algorithm (5.4.9).

Chapter 6

Conclusion, Contribution to Knowledge and Future Research

6.1 Conclusion

This dissertation presented a systematic and comprehensive study of the approximation of common solutions of VIPs and FPPs in Hilbert spaces, and the approximation of solutions of MIPs and FPPs in $p-$uniformly convex Banach spaces which are also uniformly smooth. We have presented our study in a coherent manner, first by giving a brief background of our study for which we defined the subject matters and reviewed some of the important works done in this direction. We then recalled a number of theorems, propositions, lemmas and remarks that are very important to our study. As seen in chapter 3, chapter 4 and chapter 5, our main results extends many existing concepts and provides important insight of our contribution to existing ideas in this area. We also saw that chapter 3 and chapter 4 were devoted to the study of the approximation of common solutions of VIPs associated with inverse strongly monotone mappings and FPPs for both single-valued and multivalued demicontractive mappings in Hilbert spaces. The algorithms presented in both chapters are independent of the operator norm and were inspired by Zoa [125]. We ended chapter 3 by giving numerical example for the convergence speed of our algorithm. Chapter 5 extended the results in chapter 3 and chapter 4 to spaces more general than the Hilbert spaces in which we studied MIPs (which is a generalization of the VIP) and FPPs for right Bregman strongly nonexpansive mappings. Our computational technique makes use of the Bregman distance and were inspired by Bregman [18]. We also saw that strong convergence results were established in chapter 3, chapter 4 and chapter 5. These results are original results and they extend and complement some recent results in literature.

6.2 Contribution to knowledge

Our results generalizes and extends some recent results in literature (in particular, results that serves as motivation to our study) by making the following contributions, among others:

1. It is generally known that the class of demicontractive mappings is more general than the class of quasi-nonexpansive mappings. We saw that the example of the single-valued demicontractive mapping considered in Example 2.1.16 is not quasi-nonexpansive and the example of the multivalued demicontractive mapping considered in Example 2.1.26 is not multivalued quasi-nonexpansive. Hence, the class of quasi-nonexpansive mappings considered in [104] is a proper subclass of the class of demicontractive mappings considered in this work.

2. In [40], the author imposed the demi-compactness condition on the single-valued demicontractive mappings to obtain strong convergence result. Also, in [41], the author imposed the hemi-compactness condition on the multi-valued demicontractive mappings to obtain strong convergence result. However, we obtained strong convergence results without imposing these conditions on the mappings considered in our study. Hence, our results show that these conditions can be dispensed with.

3. In [6], the author proved weak convergence result for split hierarchical variational inequality problem, while in chapter 3 and chapter 4 of this dissertation, we obtained strong convergence results for both SEVIP and systems of SEVIP. Furthermore, the class of mappings considered in this work is more general than the class of mappings considered in [6].

4. In [33], the author obtained a general common solution to VIPs, while in chapter 4, we obtained a common solution to both MSSEFPP and systems of SEVIP. Hence, our result in chapter 4 extends the result in [33].

5. Our example of a multivalued demicontractive mapping given in chapter 2 of this dissertation (i.e., Example 2.1.26) generalizes the example of a multivalued demicontractive mapping given in [41]. In particular, if we take $\alpha = 2$ in Example 2.1.26, then Example 2.1.26 reduces to the example in [41].

6. Our result in Chapter 5 extends results for SEMIP and SEFPP from the frame work of Hilbert spaces to the more general p-uniformly convex Banach spaces which are also uniformly smooth.

The results obtained in chapters 3, 4 and 5 have been submitted for possible publications as follows:

(1) C. Izuchukwu, F. U. Obguisi and O. T. Mewomo, A solution to split equality variational inequality problem and split equality fixed point problem independent of operator norm. Submitted to Dynamics of Continuous Discrete and Impulsive Systems, Series B (Applications and Algorithm), (Scopos indexed Journal in Canada).

(2) C. Izuchukwu, C. C. Okeke and O. T. Mewomo, Systems of variational inequality problem and split equality fixed point problem. Submitted to Ukrainian Mathematical Journal (Scopos indexed Journal).

(3) C. Izuchukwu, F. U. Obguisi and O. T. Mewomo, A common solution of split equality monotone inclusion problem and split equality fixed point problem in real Banach spaces. Submitted to Acta Mathematica Vietnamica (Scopos indexed Journal).

6.3 Future research

As stated in the previous section, the results obtained in this work extends and generalizes some important results in this direction. However, there are lots of works to be done in this area and many researchers are developing new ideas for solving different problems in this direction. We are looking forward to study the problems considered in this dissertation and some other optimization problems in an interesting space, called the fuzzy normed space. Our future plan is to study some of the results we have in Hilbert spaces in the fuzzy normed spaces. To give the definition of a fuzzy normed space, we first define the following.

Definition 6.3.1. *[55]. A binary operation* $* : [0,1]\times[0,1] \to [0,1]$ *is a continuous* t*-norm if* $*$ *satisfies the following conditions:*

(i) $*$ *is commutative and associative,*

(ii) $*$ *is continuous,*

(iii) $a * 1 = a,\ \forall a \in [0,1]$,

(iv) $a * b \leq c * d$, *whenever* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [0,1]$. *For examples of continuous* t*-norms see [1].*

Definition 6.3.2. *[101]. A Fuzzy Normed Space (FNS) is a triple* $(X, N, *)$, *where* X *is a vector space over a scaler field* $\mathbb{F}$ *(or* $\mathbb{R}$*),* $*$ *is a continuous* t*-norm and* $N : X \times (0,\infty) \to [0,1]$ *is a fuzzy set (fuzzy norm) such that, for all* $x, y \in X$ *and* $t, s > 0$, *the following condition are satisfied:*

(i) $N(x,t) > 0$,

(ii) $N(x,t) = 1$ *if and only if* $x = 0$,

(iii) $N(cx,t) = N(x, \frac{t}{|c|})$ *for all* $c \neq 0$,

(iv) $N(x,s) * N(y,t) \leq N(x+y, s+t)$,

(v) $N(x,.)$ *is a continuous function of* $\mathbb{R}^+$ *and*

$$\lim_{t\to\infty} N(x,t) = 1, \quad \lim_{t\to 0} N(x,t) = 0.$$

The theory of fuzzy normed space is relatively recent in the field of fuzzy normed linear analysis, as a result of this, studies on fixed point theory and certain optimization problems in fuzzy normed spaces are still in the embryonic stage. Thus, there are ample scope of further works in this direction. For example, one may attempt to obtain the results in chapter 3 and chapter 4 of this dissertation in complete fuzzy normed spaces. For detailed information on fuzzy normed spaces, see [1, 2, 13, 55, 101] and the references therein.

We also intend to study the following monotone variational inclusion problem in a reflexive real Banach space X: Find $u \in X$ such that

$$0 \in A(u) + B(u), \tag{6.3.1}$$

where $A : X \to X^*$ is a Bregman inverse strongly monotone operator and $B : X \to 2^{X^*}$ is a maximal monotone operator. The resolvent operator $\mathrm{Res}^f_{\lambda B} : X \to 2^X$ with respect to a maximal monotone operator B and $\lambda > 0$ is defined by (see [69, 96])

$$\mathrm{Res}^f_{\lambda B} := (\nabla f + \lambda B)^{-1} \circ \nabla f,$$

where f is uniformly Frétchet differentiable and bounded on bounded subsets of X. Also, the anti-resolvent operator $A^f_\lambda : X \to X$ with respect to a Bregman inverse strongly monotone operator A is defined by (see [69])

$$A^f_\lambda := \nabla f^* \circ (\nabla f - \lambda A)\,.$$

We state here that a point $u \in X$ is a solution of problem (6.3.1) if and only if u is a fixed point of the composition $\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda$. Hence, the operator $\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda$ would be of paramount importance in the study of problem (6.3.1). Therefore, we shall continue our research in this direction to study those properties of this operator that will enable us obtain the solution of problem (6.3.1). In addition, we hope to extend other useful results from the frame work of real Hilbert spaces to more general Banach spaces.

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