

ON CHAIN DOMAINS, PRIME RINGS AND TORSION PRERADICALS

by

JOHN ERIC VAN DEN BERG

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To my wife Mandy and sons
Nicholas and Christopher

Preface

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The thesis represents original work by the author and has not been submitted in any form to another university. Where use was made of the work of others, it has been duly acknowledged in the text.

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Abstract

The main aims of this dissertation are to investigate the notion of an \mathfrak{m} -Jansian preradical on the module category of a ring, to study rings which are characterized by the behaviour of their \mathfrak{m} -Jansian torsion preradicals, and to examine (ring and radical theoretic aspects of) a resultant classification of prime rings. To facilitate this, a study of right chain domains is also undertaken.

The preliminary chapter (Chapter 0) establishes notational conventions, introduces terminology and presents general algebraic results that will be required in the sequel.

Chapter I develops techniques for the construction of noncommutative right chain domains (with a view to application in Chapter II). The main result (Theorem I.5.1) asserts that every algebraic chain is isomorphic to the chain of all proper (two-sided) ideals of a right chain domain T with identity. (The full ideal lattice may be realized if the requirement that T have identity is dropped.) Furthermore, T may be chosen such that the chains of right ideals strictly between covering pairs in the proper ideal lattice of T contain dually cofinal copies of further preassigned chains; these latter chains must be unbounded above, but may otherwise be quite arbitrary (Theorem I.5.3). The ordinal $\omega \oplus 3$, ordered naturally, is an algebraic chain that is not isomorphic to the ideal lattice of any commutative (or even right duo) ring (Proposition I.5.5). The universal algebraic significance of these results is discussed.

Chapter II concerns torsion theoretic aspects of module categories and is pitched at a slightly more general level than the traditional setting. The primitive notion is that of a preradical on a module subcategory. (A class of (not necessarily unital) modules \mathcal{C} over some fixed ring (possibly without identity) is called a module subcategory if \mathcal{C} is closed under homomorphic images, submodules, direct products and essential extensions.) The lattice of all torsion preradicals on a module subcategory is modular and algebraic (Proposition III.1.6). Notions of \mathfrak{m} -Jansian preradicals and \mathfrak{m} -Jansian topologizing filters (\mathfrak{m} a regular cardinal) are introduced and studied. The former are preradicals whose pretorsion classes are closed under direct products of fewer than \mathfrak{m} modules, and the latter are topologizing filters closed under intersections of fewer than \mathfrak{m} right ideals. A ring R for which every right topologizing filter is \mathfrak{m} -Jansian is said to be right \mathfrak{m} -closed. For $\mathfrak{m} \geq \aleph_1$ every right \mathfrak{m} -closed ring R satisfies the descending chain condition on (two-sided) ideals (Corollary II.4.2) and has only finitely many maximal proper ideals (Proposition II.5.8); also, $R/J(R)$ is either zero or isomorphic to a finite direct product of finite dimensional matrix rings over division rings (Corollary II.4.8). If R is a right \mathfrak{m} -closed ring ($\mathfrak{m} \geq \aleph_1$) then the conditions: (1) R_R has Gabriel dimension; (2) $J(R)$ is nilpotent; (3) R_R is artinian, are equivalent (Theorem II.4.15). A ring R (possibly without identity) is artinian if and only if every right topologizing filter on R is closed under arbitrary intersections (Theorem II.4.16). For every regular cardinal \mathfrak{m} , there exists a ring which is right \mathfrak{m} -closed but not right \mathfrak{n} -closed for any $\mathfrak{n} > \mathfrak{m}$ (Theorem II.4.20). This result applies the

techniques developed in Chapter I. Two rings R and S (possibly without identity) are said to be Morita $*$ -equivalent if their respective Dorroh Extensions are Morita equivalent. If R and S are Morita $*$ -equivalent rings and \mathfrak{m} is a regular cardinal then R is right \mathfrak{m} -closed if and only if S is right \mathfrak{m} -closed (Theorem II.5.10). Consequently, for rings with identity, \mathfrak{m} -closure is a Morita invariant property (in the classical sense).

For a ring R and a regular cardinal \mathfrak{m} the following conditions are equivalent: (1) For every nonzero $a \in R$ there exists a subset X of R such that $|X| < \mathfrak{m} + 1$ (here '+' is cardinal addition) and aX has trivial right annihilator; (2) $\sigma(R_R) = 0$ for all proper \mathfrak{m} -Jansian torsion preradicals $\sigma \geq \tau_{\text{zero}}$ on the category of right R -modules (Theorem II.6.3). (The torsion preradical τ_{zero} is defined by $\tau_{\text{zero}}(M) = \{x \in M : xR = 0\}$ for all modules M .) For an arbitrary nonzero cardinal \mathfrak{m} (possibly finite), a ring R is said to be right prime of bound \mathfrak{m} if \mathfrak{m} is the least cardinal for which condition (1) above is satisfied. We use $P_r(\mathfrak{m})$ to denote the class of all such rings and define $\bar{P}_r(\mathfrak{m}) = \bigcup_{\mathfrak{k} \leq \mathfrak{m}} P_r(\mathfrak{k})$. The left analogues of these classes are denoted by $P_l(\mathfrak{m})$ and $\bar{P}_l(\mathfrak{m})$. Several categorical characterizations of the members of $\bar{P}_r(\mathfrak{m})$ are presented for infinite \mathfrak{m} .

Chapter III is devoted primarily to an investigation of the classes $P_r(\mathfrak{m})$. We study the right bound of primeness of the (row-finite) matrix ring $\mathbb{M}_{\mathfrak{m}}(R)$ where \mathfrak{m} is a nonzero cardinal (possibly infinite) and R a prime ring. If D is a division ring then $\mathbb{M}_{\mathfrak{m}}(D) \in P_r(\mathfrak{m})$ if $\mathfrak{m} < \aleph_0$ and $\mathbb{M}_{\mathfrak{m}}(D) \in P_r(\mathfrak{m}^+)$ if $\mathfrak{m} \geq \aleph_0$ (Proposition III.1.4). (Here, \mathfrak{m}^+ denotes the cardinal successor of \mathfrak{m} .) Using various matrix ring constructions it is proved that the class $P_r(\mathfrak{m})$ is nonempty for all $\mathfrak{m} > 0$ (Proposition III.1.5). In general, the left bound of primeness of $\mathbb{M}_{\mathfrak{n}}(R)$ (R a prime ring) is not determined simply by \mathfrak{n} and the bound of primeness of R . For example, if D is a division ring and \mathfrak{n} is infinite then $\mathbb{M}_{\mathfrak{n}}(D) \in P_l(\mathfrak{k}^+)$ where \mathfrak{k} is the smallest infinite cardinal such that $|D|^{\mathfrak{k}} \geq \mathfrak{n}$ (Proposition III.1.11). We consider the question: for what values of \mathfrak{m} and \mathfrak{n} is the class $P_r(\mathfrak{m}) \cap P_l(\mathfrak{n})$ nonempty? The classes $P_r(\mathfrak{m}) \cap P_l(\mathfrak{m})$, $P_r(1) \cap P_l(\mathfrak{n})$ and $P_r(\mathfrak{m}) \cap P_l(1)$ are nonempty for all $\mathfrak{m}, \mathfrak{n} > 0$ and $P_r(\mathfrak{m}) \cap P_l(\mathfrak{n})$ is nonempty for all $\mathfrak{m}, \mathfrak{n} > \aleph_0$ (Theorem III.1.13). If $\mathfrak{m} > 1$ and $R \in P_r(\mathfrak{m})$ then $R \in P_l(\mathfrak{m})$ or $R \in P_l(1)$ (Proposition III.1.7). The only obstacle to a complete answer to the above question is a longstanding open problem of Goodearl, Handelman and Lawrence: is $P_r(\aleph_0) \subseteq P_l(\aleph_0) \cup P_l(1)$? For finite (resp. infinite) \mathfrak{m} , a ring S with identity is isomorphic to the ring of all linear transformations, written on the right, of an \mathfrak{m} -dimensional left vector space if and only if S is von Neumann regular, left self-injective with nonzero socle and $S \in P_r(\mathfrak{m})$ (resp. $S \in P_r(\mathfrak{m}^+)$) (Proposition III.1.14). For rings with identity and infinite \mathfrak{m} , membership of $\bar{P}_r(\mathfrak{m})$ is Morita invariant (Theorem III.1.6). If R and S are Morita $*$ -equivalent prime rings and \mathfrak{m} is infinite, then $R \in \bar{P}_r(\mathfrak{m})$ if and only if $S \in \bar{P}_r(\mathfrak{m})$ (Theorem III.1.18). The stipulation that R and S are prime rings cannot be dropped.

Closure properties of the class $\bar{P}_r(\mathfrak{m})$ under various standard overring and subring constructions are investigated. Particular attention is given to semigroup, monoid and polynomial rings as well as monomial algebras. It is known that a right order in a prime ring need not be prime.

If \mathbf{k} and \mathbf{m} are cardinals with $\mathbf{m} \geq \mathbf{k}, \aleph_0$ and R is a right \mathbf{m} -order in S (this means that for every subset X of S with $|X| < \mathbf{m}$, there is a unit u of S , contained in R , such that $Xu \subseteq R$) then $R \in P_r(\mathbf{k})$ if and only if $S \in P_r(\mathbf{k})$ (Theorem III.2.3). If $\mathbf{m} > 1$ and H is an arbitrary semigroup then $R \in \bar{P}_r(\mathbf{m})$ if the semigroup ring RH is in $P_r(\mathbf{m})$ (Proposition III.3.1). Conversely, for $\mathbf{m} > 0$ we have: (1) if $\mathbf{m} \geq \aleph_0$, H is a unique product semigroup and $R \in \bar{P}_r(\mathbf{m})$ then $RH \in \bar{P}_r(\mathbf{m})$; (2) if H is a cancellative strictly right ordered semigroup and $R \in \bar{P}_r(\mathbf{m})$ then $RH \in \bar{P}_r(\mathbf{m})$; (3) if H is the free product of two nontrivial monoids and $R \in \bar{P}_r(\mathbf{m})$ then $RH \in \bar{P}_r(2\mathbf{m})$ (Proposition III.3.3).

A ring R is said to be uniformly prime of bound \mathbf{m} if \mathbf{m} is the least cardinal for which R contains a subset X with $|X| < \mathbf{m} + 1$, such that $aXb \neq 0$ whenever $0 \neq a, b \in R$. We use $UP(\mathbf{m})$ to denote the class of all such rings. Rings belonging to $UP(\mathbf{n})$ ($1 < \mathbf{n} < \aleph_0$) are necessarily prime right or left Goldie (Theorem III.4.4). If \mathbf{n} is a finite nonzero cardinal, D a division ring and R a right order in $\mathbb{M}_{\mathbf{n}}(D)$ then $R \in UP(\mathbf{m})$ for some \mathbf{m} such that $\mathbf{n} \leq \mathbf{m} \leq 2\mathbf{n} - 1$ (Theorem III.4.3). This inequality cannot be sharpened. If F is an algebraically closed field, \mathbf{n} a finite nonzero cardinal and R a (right) order in $\mathbb{M}_{\mathbf{n}}(F)$ then $R \in UP(2\mathbf{n} - 1)$ (Theorem III.4.11). Hilbert's Nullstellensatz is used in the proof of this theorem. If D is a commutative noetherian domain with identity which contains a nonzero prime element, Q the field of quotients of D and R a (right) order in $\mathbb{M}_{\mathbf{n}}(Q)$ for some finite $\mathbf{n} > 0$, then $R \in UP(\mathbf{n})$ (Theorem III.4.14). Consequently, the class $UP(\mathbf{n})$ is nonempty for all finite nonzero cardinals \mathbf{n} (Corollary III.4.15).

It is known that for each nonzero cardinal \mathbf{m} the classes $P_r(\mathbf{m})$ and $\bar{P}_r(\mathbf{m})$ are special classes (in the sense of Andrunakievič) and so determine upper (special) radicals in the category of rings. We denote these radicals by $\mathfrak{U}P_r(\mathbf{m})$ and $\mathfrak{U}\bar{P}_r(\mathbf{m})$ respectively. If \mathbf{m} is a limit cardinal we denote by $B_r(\mathbf{m})$ the upper (special) radical determined by the class $\bigcup_{0 < \mathbf{k} < \mathbf{m}} P_r(\mathbf{k})$. Most of Chapter IV is devoted to a study of the above radicals. The radical $\mathfrak{U}\bar{P}_r(\mathbf{m})$ is right strong for all $\mathbf{m} > 0$ and $B_r(\mathbf{m})$ is right strong for all limit cardinals \mathbf{m} (Proposition IV.2.14). Examples are presented which show that the radical $\mathfrak{U}\bar{P}_r(\mathbf{m})$ (resp. $B_r(\mathbf{m})$) is not left strong and is neither right nor left hereditary for any nonzero cardinal (resp. limit cardinal) \mathbf{m} . The radical $\mathfrak{U}P_r(\mathbf{m})$ (resp. $B_r(\mathbf{m})$) is, moreover, neither right nor left stable for any nonzero cardinal (resp. limit cardinal) \mathbf{m} . An internal characterization of the elements of $\mathfrak{U}\bar{P}_r(\mathbf{m})(R)$, for an arbitrary ring R , is given (Proposition IV.2.16). If \mathbf{n} is a finite nonzero cardinal then $\mathfrak{U}\bar{P}_r(\mathbf{m})(\mathbb{M}_{\mathbf{n}}(R)) = \mathbb{M}_{\mathbf{n}}(\mathfrak{U}\bar{P}_r(\mathbf{m})(R))$ for all rings R and $\mathbf{m} \geq \aleph_0$, while $B_r(\mathbf{m})(\mathbb{M}_{\mathbf{n}}(R)) = \mathbb{M}_{\mathbf{n}}(B_r(\mathbf{m})(R))$ for all rings R and limit cardinals \mathbf{m} (Proposition IV.2.21). If R is an arbitrary ring and H an arbitrary monoid then: (1) $\mathfrak{U}\bar{P}_r(\mathbf{m})(R) \supseteq R \cap \mathfrak{U}\bar{P}_r(\mathbf{m})(RH)$ for all $\mathbf{m} > 0$ and $B_r(\mathbf{m})(R) \supseteq R \cap B_r(\mathbf{m})(RH)$ for all limit cardinals \mathbf{m} ; (2) $\mathfrak{U}\bar{P}_r(\mathbf{m})(R) = R \cap \mathfrak{U}\bar{P}_r(\mathbf{m})(RH)$ for all $\mathbf{m} \geq \aleph_0$ and $B_r(\mathbf{m})(R) = R \cap B_r(\mathbf{m})(RH)$ for all limit cardinals $\mathbf{m} > \aleph_0$ (Proposition IV.3.4). The restrictions on \mathbf{m} in condition (2) above cannot be dispensed with. If R is an arbitrary ring and H a unique product monoid, a cancellative strictly right ordered monoid or the free product of two nontrivial monoids, then $[\mathfrak{U}\bar{P}_r(\mathbf{m})(R)]H = \mathfrak{U}\bar{P}_r(\mathbf{m})(RH)$ for all $\mathbf{m} \geq \aleph_0$ and $[B_r(\mathbf{m})(R)]H = B_r(\mathbf{m})(RH)$ for all limit cardinals $\mathbf{m} > \aleph_0$ (Corollary IV.3.6). For a nonempty class \mathcal{A} of prime rings, a ring R is said to be \mathcal{A} -Jacobson if

every prime ideal of R is an intersection of ideals A of R such that $R/A \in \mathcal{A}$. Theorem IV.3.12 asserts that if $\mathfrak{m} \geq \aleph_0$ and R is $\bar{P}_r(\mathfrak{m})$ -Jacobson then so is the polynomial ring $R[x]$. The requirement that \mathfrak{m} be infinite cannot be dropped.

We locate the radicals $\mathfrak{U}P_r(\mathfrak{m})$, $\mathfrak{U}\bar{P}_r(\mathfrak{m})$ and $B_r(\mathfrak{m})$ in the lattice of radicals by comparing them with, inter alia, the following: the Prime, Levitzki, Nil, Jacobson, Behrens, Brown-McCoy, Generalized Nil and Antisimple radicals as well as the upper (special) radicals determined by the special classes of uniformly strongly prime rings, and right superprime rings. The radicals $\mathfrak{U}\bar{P}_r(\mathfrak{m})$ ($\mathfrak{m} > 0$) form a strictly descending chain (bounded below by the Prime radical and above by the Generalized Nil radical), while if \mathfrak{m} is a limit cardinal then $\mathfrak{U}\bar{P}_r(\mathfrak{m}) \subsetneq B_r(\mathfrak{m}) \subsetneq \mathfrak{U}\bar{P}_r(\mathfrak{n})$ (Proposition IV.4.6). The radicals $\mathfrak{U}P_r(\mathfrak{m})$ are incomparable with most other radicals of interest to us (Proposition IV.4.9). The radicals $\mathfrak{U}\bar{P}_r(\mathfrak{m})$ and $\mathfrak{U}\bar{P}_l(\mathfrak{m})$ are incomparable for all $\mathfrak{m}, \mathfrak{n} > 0$, as are $\mathfrak{U}P_r(\mathfrak{m})$ and $\mathfrak{U}P_l(\mathfrak{m})$ (Proposition IV.4.11). In response to an open question of Olson and Veldsman, it is shown that the class of rings which are uniformly prime of bound at most \mathfrak{m} is special for all $\mathfrak{m} > 0$ (Theorem IV.5.1). The upper (special) radical determined by this class is denoted by $\tau_{\mathfrak{m}}$. We determine the positions of the $\tau_{\mathfrak{m}}$'s (for finite \mathfrak{m}) relative to some of the radicals described above.

CONVENTION ON NUMBERING OF RESULTS

Example: Theorem II.3.4 refers to Theorem 4 of §3 of Chapter II. Within §3 of Chapter II, this result will be referred to as Theorem 4; within Chapter II, it will be called Theorem 3.4.

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Introduction

The subject matter of this thesis concerns two strongly related subfields of the theory of rings and modules, namely the study of prime rings and the torsion theoretic study of a ring's module category.

Primeness is a rather universal notion in mathematics; its motivations are usually self-evident. In ring theory, for example, the fact that any ring, modulo the intersection of its prime ideals, is a subdirect product of prime rings suggests that an understanding of primeness is at the heart of a better understanding of rings in general. The class of all prime (not necessarily commutative) rings is vast. One possible view of its spectrum is that at one "extreme" lie right chain domains of arbitrary "height" while at the other, there are full linear rings of arbitrary "width". By the former, we mean rings without zero divisors whose right ideals are linearly ordered by set inclusion; by the latter, we mean the rings of linear transformations of vector spaces of arbitrary dimension (which are littered with zero divisors and may have very large sets of mutually incomparable right ideals).

Torsion preradicals on the module category of a ring are a newer focus of research; they are functors that associate with a module a "torsion submodule" (originally envisaged as an analogue of the notion of "torsion subgroup") and hence associate with the ring the class of modules equal to their torsion submodules, which is called the "hereditary pretorsion class" of the torsion preradical. (A topological approach to such functors is also possible.) Our interest lies primarily in the question of what information about a ring is encoded in the behaviour of its set of torsion preradicals.

It is not immediately clear from the above that primeness and torsion preradicals are related topics. An early indication of this fact was the discovery, in the mid 1970s that the classes of "strongly prime" and "absolutely torsion-free" rings with identity coincide. A spiritually similar result, going beyond prime rings, is the fact that the rings with identity all of whose hereditary pretorsion classes are closed under arbitrary direct products are just the artinian rings. In this thesis, we offer a perspective (with a set theoretic flavour) from which these and certain other classical results emerge as special cases. It is easiest to describe the coverage of the thesis by referring first to Chapter II, in which this perspective first emerges and undergoes considerable development. (We shall return shortly to the content of Chapter I.)

Our perspective has the following starting point. The hereditary pretorsion class of a torsion preradical on the module category of a ring is closed under the formation of direct sums (in particular, finite direct products) but not, in general, under arbitrary direct products. A focal point of our study (introduced in Chapter II) will be the intermediate notion of an " \mathfrak{m} -Jansian" torsion preradical, i.e., one whose hereditary pretorsion class is closed under the formation of products of fewer than \mathfrak{m} modules, where \mathfrak{m} is a given infinite cardinal. We are interested in classes of rings that are determined by the behaviour of the sets of \mathfrak{m} -Jansian torsion preradicals of their members.

For the most part, these will be classes of prime rings. One notable exception is the class of “ \mathfrak{m} -closed” rings (also introduced and studied in Chapter II). The general definition of an \mathfrak{m} -closed ring requires that “almost all” torsion preradicals on the ring’s module category be \mathfrak{m} -Jansian. (The meaning of “almost all” will be made precise in the text but for rings with identity, “almost all” may be taken to mean all torsion preradicals on the category of *unital* modules.) All rings are \aleph_0 -closed, while the intersection (over all infinite \mathfrak{m}) of the classes of \mathfrak{m} -closed rings turns out to be the class of all artinian rings. No generality is lost here if we confine our attention to regular cardinals \mathfrak{m} . We show that distinct classes of \mathfrak{m} -closed rings arise for distinct regular values of \mathfrak{m} . (Historically, it was the proof of this fact that gave rise to the material in Chapter I.) The classes of \mathfrak{m} -closed rings turn out to be rather interesting because they all enjoy a number of surprisingly strong (“artinian-like”) properties which may be stated *without* reference to the cardinal \mathfrak{m} . For example, they all turn out to satisfy the descending chain condition on two-sided ideals and their factor rings modulo the Jacobson radical are artinian.

The light thrown on rings by the study of \mathfrak{m} -Jansian torsion preradicals does not end here. Within the class of prime rings, we find (in the latter part of Chapter II) that the behaviour of a certain interval of these preradicals (in the lattice of all torsion preradicals) determines a sort of “degree of primeness” of the ring. For simplicity, we shall assume in describing this result here that all rings have identity and that torsion preradicals act on the category of unital modules. (By not imposing these restrictions in the thesis, we obtain stronger results whose statements are necessarily more technical.) A module may be considered “torsion-free” with respect to a torsion preradical if its corresponding torsion submodule is zero. Consider the class $\bar{P}(\mathfrak{m})$ of rings which, as modules over themselves, are torsion-free with respect to all \mathfrak{m} -Jansian torsion preradicals. All members of this class turn out to be prime in the strong sense that each of their nonzero elements a is “insulated” from annihilation on the right by a suitable subset X_a of the ring of cardinality less than \mathfrak{m} , by which we mean that $aX_a b$ is never zero, unless b is. Conversely, $\bar{P}(\mathfrak{m})$ includes all rings that have this property.

Of course, the primeness of a prime ring amounts to precisely the fact that the ring itself functions as just such an “insulator” for all of its nonzero elements – but its cardinality may be wastefully large for this purpose. (For example, singleton insulators are always available in a domain.) Put differently, the union over \mathfrak{m} of all $\bar{P}(\mathfrak{m})$ is the class of all prime rings. We therefore see the least infinite \mathfrak{m} for which a prime ring R lies in $\bar{P}(\mathfrak{m})$ as a “bound of primeness” of R . (Of course, it would be terminologically more accurate to refer to the cardinal \mathfrak{m} as the “right bound of primeness” of R . For the sake of simplicity we have chosen to omit the prefix “right” in this introduction.) Consequently, we see \mathfrak{m} -Jansian torsion preradicals as inducing a classification of prime rings. From this perspective, Handelman, Lawrence and Viola-Prioli’s result that the “strongly prime” rings are just the “absolutely torsion-free” rings is the case $\mathfrak{m} = \aleph_0$. (The essential role of closure of hereditary pretorsion classes under finite products is quite hidden in this special case as it holds universally.) This observation concludes Chapter II.

Since “ \mathfrak{m} -closed” rings and “ \mathfrak{m} -prime” rings (i.e., rings belonging to $\bar{P}(\mathfrak{m})$) are both characterizable in terms of torsion preradicals on their module categories, it is an unsurprising fact that “ \mathfrak{m} -closure” and “ \mathfrak{m} -primeness” are Morita invariant properties. There is, of course, an obvious limitation here; the classical notion of Morita equivalence is defined only for rings with identity, so the above observations yield results pertaining to rings with identity only. In an earlier paragraph, however, we alluded to the fact that many of our ring and module theoretic investigations are conducted at the more general level of arbitrary (not necessarily unital) modules over an arbitrary associative ring (possibly without identity). In keeping with this spirit, it is natural to ask first whether a more general theory of Morita equivalence exists for rings without identity and secondly, whether the aforementioned properties are invariant with respect to any such general notion of Morita equivalence.

We point out, in response to the first question, that although several fairly new generalized notions of Morita equivalence exist in the literature, we have chosen (for reasons that will be explained later) to formulate our own definition of Morita equivalence for rings without identity. This we shall call Morita \ast -equivalence. Briefly, arbitrary rings R and S are Morita \ast -equivalent if and only if their respective Dorroh Extensions, R^\ast and S^\ast , are Morita equivalent in the classical sense. In response to the second question raised above, it turns out that “ \mathfrak{m} -closure” is a Morita \ast -invariant property but “ \mathfrak{m} -primeness” is Morita \ast -invariant only within the class of all prime rings.

A general methodological aim of this thesis has been to illustrate the theory presented as richly as possible with examples. This was particularly necessary in the study of \mathfrak{m} -closed rings, since it is not obvious from the definition that one obtains distinct classes of such rings for distinct values of \mathfrak{m} . As it happens, even the fact that an \aleph_1 -closed ring need not be artinian turns out to be quite difficult to prove. In seeking examples to establish this, we find ourselves gravitating naturally towards that end of the prime ring spectrum occupied by right chain domains. For reasons that will become clear in the text of Chapter II, we seek, for an arbitrary infinite \mathfrak{m} , a right chain domain with a unique nontrivial two-sided ideal, into the dual of whose chain of proper right ideals the cardinal \mathfrak{m} can be cofinally embedded (as a well ordered set). No such rings seemed to be to hand in the literature (despite vigorous recent developments in the theory of chain rings). The construction techniques that appear in Chapter I resulted from this need, but their development takes on a theoretical (rather than merely illustrative) character there because they lead to quite general representation theorems, which we shall now describe. (This fact and their independence of subsequent results accounts for their location at the beginning of the thesis, but in this more historically minded introduction, we do not see them as our point of departure.)

The (two-sided) ideal lattice (i.e., the congruence lattice) of a ring is algebraic and modular. It is known to lattice theorists and universal algebraists that not all algebraic modular lattices are realizable as ideal lattices of rings (nor indeed, as congruence lattices of algebras of any fixed “type”). Nothing is known about the possibility of realizing all algebraic *distributive* lattices as the congruence

lattices of algebras of any fixed type. (A longstanding conjecture asks, in particular, whether the algebras may be chosen to be lattices themselves.) Certainly, it is not known whether all algebraic distributive lattices are the ideal lattices of (necessarily arithmetical) rings, but a partial positive result in this direction (proved by Kim and Roush) asserts that every *finite* distributive lattice is the ideal lattice of a (von Neumann regular, not necessarily finite) ring. In the light of the question just posed, this result seems to have a natural (but previously unaddressed) complement. Just as finite implies algebraic, linearly ordered implies distributive. Thus, a second obvious “test” which the general question should pass is that every algebraic *chain* should be realizable as the ideal lattice of a ring.

Most of Chapter I is devoted to proving precisely this result, its first main theorem. We cannot insist that the ring have identity, but every algebraic chain is also realizable as the lattice of *proper* (two-sided) ideals of a right chain domain with identity, which may be chosen such that all of its (two-sided) ideals are idempotent. The second main theorem of the chapter refines the first by showing that in such a realization of a given algebraic chain, we may also require the intervals of right ideals strictly between covering pairs of proper ideals of the ring to contain dually cofinal copies of other preassigned chains. These latter chains must be unbounded above, but otherwise, they may be specified quite arbitrarily. (A special case of this theorem, presented in Chapter II, settles the question of distinctness of the notions “ \mathfrak{m} -closed” for different \mathfrak{m} .) In Chapter I, we also show that, as far as the representation of algebraic chains by rings is concerned, the first main theorem cannot be improved in any of the seemingly “obvious” ways: for example, certain algebraic chains are *not* realizable as the ideal lattices of *commutative* rings. The chapter closes with a discussion of the universal algebraic significance of its first main result.

The classification of prime rings that emerged from the torsion theoretic discourse of Chapter II undergoes further investigation in Chapter III. At this point, however, it is desirable to refine our definitions slightly. While the torsion theoretic definition of being “prime of bound \mathfrak{m} ” given above makes no sense for finite cardinals \mathfrak{m} , the more elementary characterization in terms of insulators is meaningful for all \mathfrak{m} , and allows us to consider finite bounds of primeness as well. (For the sake of unification, we need to alter the requirement $|X_a| < \mathfrak{m}$ to $|X_a| < \mathfrak{m} + 1$ at this point; $+$ denotes cardinal addition. Of course, this makes no difference if \mathfrak{m} is infinite.) In other words, we further divide the class of strongly prime rings, $\bar{P}(\aleph_0)$, into a nested (denumerable) sequence of subclasses. This has the effect that our classification incorporates the existing notion of a “bounded strongly prime” ring, which was introduced to the literature by Handelman and Lawrence and which proved useful in Goodearl and Handelman’s complete classification of simple self-injective rings. The study of the classes $\bar{P}(\mathfrak{m})$ therefore unifies purely ring-theoretic and torsion theoretic earlier investigations. We see the inclusion of the former as advantageous at this point, in view of the useful role played by strongly prime rings in the historical development of ring theory, which takes over, to some extent, from torsion theory in influencing the spirit of Chapters III and IV. Thus Chapter III marks the beginning of a purer ring theoretic exploration.

In order that the classification of prime rings introduced in Chapter II be considered significant, it is obviously necessary that examples of rings which are “prime of bound \mathfrak{m} ” be produced, if possible, for all values of \mathfrak{m} . In the early part of Chapter III, matrix rings are put to use in constructing such examples. Indeed, we show that given any nonzero cardinal \mathfrak{m} , there are rings which are prime of bound precisely \mathfrak{m} . We supplement this investigation later with an “almost complete” answer to the question: for what values of \mathfrak{m} and \mathfrak{n} do there exist rings which are prime of bound \mathfrak{m} on the right and of bound \mathfrak{n} on the left?

In earlier ring theoretic investigations, there has been much study of the “survival” of prime ideals in the passage between rings and overrings. This research has its origins in the study of prime ideals in polynomial rings and, in particular, the correspondence between prime ideals of a ring R and those of the polynomial (over) ring $R[x]$. While the spirit of these investigations has changed very little, many of the early results for polynomial rings have been surpassed by subtler results about more general types of overring such as “centralizing extensions” or “normalizing extensions”. Many of these results have been recast further in terms of graded rings. A study such as this must be based on a knowledge of prime rings and a familiarity with conditions under which primeness conditions are transferred from a ring to a subring, or to an overring. In Chapter III and, to a lesser extent, in Chapter IV, we contribute to this foundation by examining preservation of the property “prime of bound \mathfrak{m} ” under various standard subring and overring constructions such as right orders, matrix rings, monoid rings, group rings and polynomial rings.

In their pioneering work [HL75] Handelman and Lawrence remark that the set of all matrix units of a finite dimensional matrix ring over a division ring acts as a kind of “uniform insulator” in that it “insulates” *all* nonzero ring elements from annihilation on the right. More generally, they define a ring R to be uniformly strongly prime if R contains a finite subset X (called a uniform insulator) such that for all nonzero elements a in R , aXb is never zero unless b is. If, in the above definition, the finiteness condition on X is dropped in favour of assigning to R a cardinal bound on the size of the uniform insulator X , we obtain a classification of prime rings which is entirely analogous to the one based on one-sided bounds of primeness. To be precise, for each nonzero cardinal \mathfrak{m} we define $\overline{UP}(\mathfrak{m})$ to be the class of all rings which contain a uniform insulator of cardinality less than $\mathfrak{m} + 1$. The union over \mathfrak{m} of all $\overline{UP}(\mathfrak{m})$ is the class of all prime rings, and we regard the least cardinal \mathfrak{m} for which a prime ring R lies in $\overline{UP}(\mathfrak{m})$ as the “uniform bound of primeness” of R . The final section of Chapter III is devoted to a study of these “uniform” notions of primeness. One important respect in which the notions “uniformly prime of bound \mathfrak{m} ” and “right prime of bound \mathfrak{m} ” differ is that the former is left-right symmetric while the latter is not. It is also worth pointing out that many of the results on uniform bounds of primeness are not predictable analogues of any results on right bounds of primeness. For example, the uniform bound of a finite dimensional $\mathfrak{n} \times \mathfrak{n}$ matrix ring over a field F is not determined solely by \mathfrak{n} (as is the case with the right bound) but depends also on some quite interesting properties of the field F . Finally, it turns out that our study of (finite) uniform bounds of primeness throws light on some very concrete facts about fields whose statements

have no apparent connection with uniform bounds of primeness.

The final chapter of this thesis (Chapter IV) is devoted to a study of certain radicals (arising from primeness conditions) in the category of rings. Radicals can be useful tools in analysing the structure of rings: in order to gain an insight into the structure of a ring R we often analyse first the “simpler” factor ring $R/\mathfrak{R}(R)$, where \mathfrak{R} is some carefully selected radical. An example which illustrates the usefulness of this strategy is the famous Artin-Wedderburn Theorem, which provides almost complete information about the structure of every artinian ring, modulo its Jacobson radical.

Subsequent to the discovery of the classical radicals of Jacobson, Baer, Köthe, Levitzki and others, Amitsur and Kuroš formulated in the early 1950’s an axiomatic base for the development of a general theory of radicals. An important notion emerging from Kuroš’ work is that of the “upper radical” determined by a class of rings: if \mathcal{M} is a class of rings with the property that every nonzero ideal of a ring in \mathcal{M} has a nonzero homomorphic image in \mathcal{M} , then the upper radical determined by \mathcal{M} , denoted $\mathfrak{U}\mathcal{M}$, is the class of all rings which have no nonzero homomorphic image in \mathcal{M} . (It is also the largest radical with respect to which all rings in \mathcal{M} are semisimple. These notions force us to consider rings without identity, which accounts, to an extent, for the level of generality of the thesis as a whole.) The class \mathcal{M} is quite arbitrary in this definition. In 1958, Andrunakievič observed that if certain more stringent conditions are imposed on the class \mathcal{M} then $\mathfrak{U}\mathcal{M}$ enjoys a number of nice properties, one of these being the so-called intersection property: if R is any ring then the $\mathfrak{U}\mathcal{M}$ radical of R is equal to the intersection of all ideals of R modulo which R lies in \mathcal{M} . Andrunakievič called such classes \mathcal{M} “special classes” and the upper radicals determined by them “special radicals”. He showed further that many of the classical radicals (e.g., the Jacobson, Prime and Brown-McCoy radicals) are the upper (special) radicals determined by special classes of prime rings.

Recently, Raftery showed that each of the classes $\bar{P}(\mathbf{m})$ is special in Andrunakievič’s sense. This observation extended the independent discoveries by Desale and Varadarajan, and Groenewald and Heyman in the early 1980’s, that the class of all strongly prime rings is special. The fact that $\bar{P}(\mathbf{m})$ is special for both finite and infinite values of \mathbf{m} added to our interest in the classification of prime rings by bounds of primeness. Chapter IV is devoted almost entirely to a detailed study of the special radicals $\mathfrak{U}\bar{P}(\mathbf{m})$. One of the topics of this study is an examination of the relationship between the $\mathfrak{U}\bar{P}(\mathbf{m})$ radical of a ring R and the $\mathfrak{U}\bar{P}(\mathbf{m})$ radical of several types of monoid ring over R . Many of the results in Chapter III on the transfer of primeness conditions from rings to overrings are exploited here. Particular attention is also given to the positions of the $\mathfrak{U}\bar{P}(\mathbf{m})$ radicals in the lattice of all radicals. A Hasse diagram in the lattice of radicals is sketched which locates the $\mathfrak{U}\bar{P}(\mathbf{m})$ ’s relative to various classical radicals.

Chapter IV concludes with a brief examination of radicals associated with uniform bounds of primeness. We demonstrate that each of the classes $\overline{UP}(\mathbf{m})$ is special, thus answering in the affirmative a published question of Olson and Veldsman.

Chapter 0

Preliminaries

The aim of this chapter is twofold, being in the first place to describe the notational conventions to be used throughout this dissertation, and secondly, to assemble in readily usable form a selection of mostly standard results from the theories of ordered sets and rings and modules, which will be required in the sequel. We have chosen to provide proofs only in the few cases where the result is less well known and we have been unable to locate a suitable reference. Otherwise, and wherever possible, we provide a reference from such standard texts as [AF74], [Fai73], [Fai76], [Hun74], [Lam66], and [Row88].

Although drawing on results and definitions from a broad spectrum of mathematical fields, this dissertation is concerned principally with rings. We have therefore assumed that the reader is more familiar with algebra, and in particular ring and module theory, than with other mathematical theories. Nevertheless, the study of any algebraic structure, and rings are no exception, inevitably demands that the reader have some knowledge of sets and ordered structures. In addition to this, there is a particularly set theoretic flavour to this dissertation. For this reason, we have chosen to devote the first section of this chapter entirely to a discussion of sets and ordered structures. Here, we shall assume a familiarity with no more than the very rudimentary concepts such as the notions of well ordered set, ordinal and cardinal. We require also that the reader be acquainted with the basics of ordinal and cardinal arithmetic and with the statement of the Generalized Continuum Hypothesis, and have a working familiarity with Zorn's Lemma and transfinite induction.

Since ring theory is our primary domain of discourse, we shall adopt the ring theorist's convention of not distinguishing notationally between structures (i.e., nonempty sets with specified operations and/or relations) and their underlying universes (i.e., the sets themselves). For example, if $\langle R; +, \cdot, -, 0 \rangle$ is a ring and $\langle P; \leq \rangle$ a partially ordered set, we shall speak of the "ring R " and the "partially ordered set P ".

We shall, at times, need to appeal to elementary category theory. In some instances, this is done simply in order to achieve greater elegance and precision; in other instances, the need to refer to categories is unavoidable, for example when defining the notion of "Morita equivalence" for rings. Those elements of category theory that are needed may be found in §3 and §6. Again, we expect the reader to be familiar only with the very basic concepts of category theory, such as the notions of category, subcategory, covariant functor and contravariant functor.

The remainder of §2 to §10 is devoted entirely to algebraic structures and, in particular, to rings and modules. We assume that the reader is familiar with each of the following classes of algebraic objects: semigroups, monoids, groups, abelian groups (including torsion and torsion-free

abelian groups), rings, division rings, fields, vector spaces and modules. We also presuppose a knowledge of the various homomorphisms, substructures and quotient or factor structures associated with each of the above; these include: subgroups, factor groups, left ideals, right ideals, ideals (that is, two-sided ideals), subrings (and overrings), factor rings, submodules and factor modules.

It is important to note that *our rings will not necessarily possess an identity element*. Our motive for working at this level of generality is born of necessity: in the sequel, several concepts will be introduced which lead to the study of so-called “special radicals” in the category of rings, in which context it is unavoidable that we consider rings without identity. We have assumed, however, that the reader (like the author) is better acquainted with the theory of rings with identity and unital modules over such rings. For this reason we shall often discuss at length, in the wider context of arbitrary modules over rings (possibly without identity) some concepts which, for unital modules over rings with identity, are rather rudimentary. Results which are peculiar to this wider context and which have no obvious analogue in the theory of rings with identity have been highlighted.

We should warn the reader that many results in this chapter have been stated for arbitrary rings, but have as their given references sources which assume rings to have identity. This has been done, however, only in cases where the arguments provided in the reference may be used *mutatis mutandis* to establish the more general result pertaining to rings without identity.

§1. SETS AND ORDERED STRUCTURES.

The following standard notational conventions will be used for sets X and Y and functions $\varphi: X \rightarrow Y$.

$X \subseteq Y$ shall mean *only* that X is a subset of Y .

$Y \setminus X$ denotes the complement of X in Y .

$|X|$ denotes the cardinality of X .

$\varphi[X']$ denotes the image of X' in Y under φ , viz., $\{\varphi(x): x \in X'\}$, where $X' \subseteq X$.

$\varphi^{-1}[Y']$ denotes the preimage of Y' in X under φ , viz., $\{x \in X: \varphi(x) \in Y'\}$, where $Y' \subseteq Y$.

$\varphi|_{X'}$ denotes the restriction of φ to X' , where $X' \subseteq X$, so that $\varphi|_{X'}: X' \rightarrow Y$.

In almost all instances, we shall use 1_X to denote the identity map on X .

We cite [Dev79] as a set theoretic reference sufficient for our purposes in this dissertation. Throughout the dissertation, bold letters $\mathbf{m}, \mathbf{n}, \dots$ will be used to denote cardinals; \mathbf{m}^+ denotes the cardinal successor of \mathbf{m} , and if \mathbf{m} is a successor cardinal, we shall use \mathbf{m}^- occasionally to denote the cardinal predecessor¹ of \mathbf{m} . If \mathbf{m} and \mathbf{n} are cardinals then $\mathbf{m} + \mathbf{n}$ (resp. $\mathbf{m} \cdot \mathbf{n}$ or $\mathbf{m}\mathbf{n}$) denotes their cardinal sum (resp. their cardinal product) and $\mathbf{m}^{\mathbf{n}}$ denotes cardinal exponentiation. If $\{\mathbf{m}_i: i \in \Gamma\}$ is a family of cardinals, we use $\sum_{i \in \Gamma} \mathbf{m}_i$ to denote their cardinal sum. Recall that a cardinal \mathbf{m} is said

¹In cases where \mathbf{m} is finite and nonzero, we use the more natural $\mathbf{m}-1$ to denote the predecessor of \mathbf{m} .

to be *regular* if it is infinite and $\sum_{i \in \Gamma} m_i < m$ whenever $m_i < m$ for all $i \in \Gamma$ and $|\Gamma| < m$. Certainly, if m is an arbitrary infinite cardinal then m^+ is regular. An infinite cardinal is called *singular* if it is not regular.

We use lower case Greek letters $\alpha, \beta, \gamma, \dots$ to denote ordinals. We identify an ordinal α with the set of all ordinals β for which $\beta < \alpha$ and we identify cardinals with initial ordinals, i.e., an ordinal α is a cardinal if and only if for every ordinal $\beta < \alpha$, there is no bijection from α onto β . As usual, ω denotes the first infinite ordinal, i.e., the set of all nonnegative integers.

If α and β are ordinals then $\alpha \oplus \beta$ (resp. $\alpha \odot \beta$) denotes their ordinal sum (resp. their ordinal product). We remind the reader that these ordinal operations are different from their cardinal counterparts. We shall need the following ordinal analogue of the division algorithm for integers, which is a standard exercise in most introductory texts on axiomatic set theory, e.g., [Sup60, Ex 11, p223].

LEMMA 1. *Let α and β be ordinals with $\beta > 0$. Then α may be written uniquely in the form $(\beta \odot \gamma) \oplus \delta$ for some ordinals γ, δ with $\delta < \beta$. (Here, necessarily, $\gamma < \alpha$.)* \square

Recall that an infinite cardinal m is said to be a *limit cardinal* if it is of the form \aleph_α , where α is either 0 or a limit ordinal; m is called a *strong limit cardinal* if $2^n < m$ for all cardinals $n < m$, in which case m is, in particular, a limit cardinal. If we assume the Generalized Continuum Hypothesis then the limit cardinals are just the strong limit cardinals. An uncountable regular limit cardinal is called an *inaccessible cardinal*. (The existence of such cardinals is not provable in Zermelo-Fraenkel Set Theory with the Axiom of Choice.)

We remind the reader that the cartesian product $\prod_{i \in \Gamma} X_i$ is the set of all functions $\varphi: \Gamma \rightarrow \bigcup_{i \in \Gamma} X_i$ such that $\varphi(i) \in X_i$ for all $i \in \Gamma$. We shall use $\{x_i\}_{i \in \Gamma}$ to denote that element φ of $\prod_{i \in \Gamma} X_i$ for which $\varphi(i) = x_i \in X_i$ for all $i \in \Gamma$. If $X_i = X$ for all $i \in \Gamma$, we abbreviate $\prod_{i \in \Gamma} X_i$ by $\prod_\Gamma X$ or by X^Γ . If $\Gamma = \mathbf{n} := \{0, 1, \dots, \mathbf{n}-1\}$ is a finite cardinal, we write $\{x_\alpha\}_{\alpha \in \mathbf{n}} \in \prod_{\alpha \in \mathbf{n}} X_\alpha$ as $(x_0, x_1, \dots, x_{\mathbf{n}-1})$.

By a partially ordered set (briefly, a poset) $\langle P; \leq \rangle$, we mean a *nonempty* set P , endowed with a binary reflexive, antisymmetric and transitive relation \leq . We call \leq the *partial order* (or just the *order*) on this poset. For a poset $\langle P; \leq \rangle$, if $x, y \in P$ with $x \leq y$, we use the standard notation $[x, y]$ for the *interval* $\{p \in P: x \leq p \leq y\}$ from x to y in P . We shall denote the interval $\{a \in L: a > x\}$ by (x) and the interval $\{a \in L: a \geq x\}$ by $[x]$. (Clearly, if x is a maximal element of L then $(x) = \emptyset$.) If $x, y \in P$ with $x < y$ and $[x, y] = \{x, y\}$ (i.e., there is no $p \in P$ such that $x < p < y$), then we say that x is *covered by y in P* or that y *covers x in P* ; and we call the pair (x, y) a *covering pair* of P .

Recall that if $\langle P; \leq \rangle$ is a poset then the *dual order* of $\langle P; \leq \rangle$, denoted \leq^{-1} , is the relation on P defined by $x \leq^{-1} y$ if and only if $y \leq x$ ($x, y \in P$). The poset $\langle P; \leq^{-1} \rangle$ is called the *dual* of $\langle P; \leq \rangle$ and will be written as $\langle P; \leq \rangle^{\text{du}}$. An element x of a poset $\langle P; \leq \rangle$ is called a

bottom (resp. *top*) *element* of P if $x \leq p$ (resp. $p \leq x$) for all $p \in P$. Since bottom and top elements (if they exist) are obviously unique, we usually denote them by 0 and 1 respectively and we refer to a poset with a bottom (resp. top) element as a *poset with 0* (resp. *poset with 1*). A poset $\langle P; \leq \rangle$ is said to be a *chain* (or *linearly ordered*) if $x \leq y$ or $y \leq x$ whenever $x, y \in P$. If $\langle P; \leq \rangle$ is a poset then a subset X of P is called an (*upward*) *directed* subset of $\langle P; \leq \rangle$ provided that for any $x, y \in X$, there exists $z \in X$ such that $x, y \leq z$. A nonempty subset X of P is called a *hereditary* (resp. *dually hereditary*) subset of $\langle P; \leq \rangle$ provided that for any $x, y \in P$ with $x \in X$ and $y \leq x$ (resp. $y \geq x$), we have $y \in X$.

A nonempty subset X of a poset $\langle P; \leq \rangle$ is always a poset in its own right, endowed with the relation $\leq \cap (X \times X)$, but we denote this poset by $\langle X; \leq \rangle$ rather than $\langle X; \leq \cap (X \times X) \rangle$. A nonempty subset X of P is said to be *cofinal* in P if for every $p \in P$, there exists an $x \in X$ such that $p \leq x$. The *cofinality* of P (abbreviated *cof* P) is defined to be the least cardinal \mathfrak{m} such that P has a cofinal subset of cardinality \mathfrak{m} . Such a cardinal always exists because P is cofinal in itself, and so we always have $\text{cof } P \leq |P|$. It is easily checked that if X is a cofinal subset of P and Y is a cofinal subset of the poset $\langle X; \leq \rangle$ then Y is a cofinal subset of P . The following simple lemma will assist us in calculating the cofinalities of certain posets.

LEMMA 2. *If X is a cofinal subset of a poset $\langle P; \leq \rangle$ then $\text{cof } X = \text{cof } P$.*

Proof. Clearly, $\text{cof } X \geq \text{cof } P$. Let Y be any cofinal subset of P . For each $y \in Y$, choose $x_y \in X$ such that $x_y \geq y$. Certainly, $Z := \{x_y : y \in Y\}$ is cofinal in X , yet $|Z| \leq |Y|$, so $\text{cof } X \leq \text{cof } P$. Thus $\text{cof } X = \text{cof } P$. \square

Where order theoretic properties are attributed to an ordinal, it is assumed that the ordinal is being considered as a well ordered set (a poset in which every nonempty subset has a least element), endowed with the standard order ($\alpha \leq \beta$ if and only if $\alpha \in \beta$ or $\alpha = \beta$). It is well known that an infinite cardinal \mathfrak{m} is regular if and only if $\text{cof } \mathfrak{m} = \mathfrak{m}$ and that for every chain P with no top element, $\text{cof } P$ is a regular cardinal.

Let \mathfrak{m} be an infinite cardinal and $\langle P; \leq \rangle$ a poset. For any ordinal β , a family $\{p_\alpha : \alpha \in \beta\}$ of elements $p_\alpha \in P$ is called a *well ordered strictly ascending* (resp. *strictly descending*) *chain* in P if for any $\alpha, \gamma \in \beta$, we have $p_\alpha < p_\gamma$ (resp. $p_\alpha > p_\gamma$) whenever $\alpha < \gamma$. The poset P is said to satisfy the \mathfrak{m} -*ascending* (resp. \mathfrak{m} -*descending*) *chain condition*, briefly the \mathfrak{m} -*ACC* (resp. \mathfrak{m} -*DCC*) if, for every well ordered strictly ascending (resp. descending) chain $\{p_\alpha : \alpha \in \beta\}$ in P , we have $\beta < \mathfrak{m}$. We omit the prefix \mathfrak{m} - when $\mathfrak{m} = \aleph_0$.

If $\langle P; \leq \rangle$ is a poset and $X \subseteq P$, the infimum or greatest lower bound or meet (resp. the supremum or least upper bound or join) of X in P , if it exists, is denoted by $\inf X$ or $\Lambda_P X$ (resp. by $\sup X$ or $\vee_P X$). The subscript is omitted if P is understood and $\Lambda_P \{x, y\}$ (resp. $\vee_P \{x, y\}$) is abbreviated as $x \wedge_P y$ or as $x \wedge y$ (resp. as $x \vee_P y$ or as $x \vee y$) if it exists. The poset P is called a

meet-semilattice (resp. a *join-semilattice*) if every pair of elements of P has a meet (resp. a join). In this case, the binary operation \wedge (resp. \vee) is associative, commutative and idempotent and we may write expressions like $x_0 \wedge x_1 \wedge \dots \wedge x_n$ (resp. $x_0 \vee x_1 \vee \dots \vee x_n$) unambiguously. A join- and meet-semilattice is called a *lattice*. A poset $\langle P; \leq \rangle$ is called a *complete lattice* if every subset of P has a meet. In this case, P is a poset with 0 and 1 (since $\emptyset \subseteq P$) and every subset of P has a join, viz., $\vee X = \wedge \{y \in P : x \leq y \text{ for all } x \in X\}$. In particular, $\langle P; \leq \rangle$ is necessarily a lattice.

For example, for any set S , the set $\exp S$ of all subsets of S is (the universe of) a complete lattice $\langle \exp S; \subseteq \rangle$, when ordered by set inclusion. Unless otherwise specified, whenever order theoretic properties are ascribed to a collection of subsets of a given set, the understood partial order is just set inclusion.

Given a poset $\langle P; \leq \rangle$, a nonempty subset X of P is called a *meet-subsemilattice* of $\langle P; \leq \rangle$ (resp. a *meet-complete subsemilattice* of $\langle P; \leq \rangle$) if $x \wedge_P y$ (resp. $\Lambda_P Y$) exists and $x \wedge_P y$ (resp. $\Lambda_P Y$) $\in X$ for any $x, y \in X$ (resp. any $Y \subseteq X$). In the latter case, since $\emptyset, X \subseteq X$, it follows that X must have a bottom element and that P must have a top element 1, with $1 \in X$. *Join-* and *join-complete subsemilattices* are defined dually. A *sublattice* (resp. a *complete sublattice*) of $\langle P; \leq \rangle$ is a nonempty subset of P that is both a meet- and a join-subsemilattice (resp. both a meet-complete and a join-complete subsemilattice) of $\langle P; \leq \rangle$. Notice that a meet-complete subsemilattice of $\langle P; \leq \rangle$ is a complete lattice in its own right but need not be a sublattice of $\langle P; \leq \rangle$. In order that $\langle P; \leq \rangle$ possess a complete sublattice X , it is necessary that P be a poset with 0 and 1 and that $0, 1 \in X$. If $\langle P; \leq \rangle$ is a meet-semilattice (resp. a lattice) and $x, y \in P$ with $x \leq y$ then the interval $[x, y]$ is always a meet-subsemilattice (resp. a sublattice) of $\langle P; \leq \rangle$, and is closed under existent binary joins (resp. contains the meets and joins of all of its *nonempty* subsets). Dual assertions may be made in the case where $\langle P; \leq \rangle$ is a join-semilattice.

Let $\langle P_1; \leq \rangle$ and $\langle P_2; \leq \rangle$ be posets. A map $\varphi: P_1 \rightarrow P_2$ is called *order preserving* (resp. *order reflecting*) if for any $x, y \in P_1$, $x \leq y$ implies (resp. is implied by) $\varphi(x) \leq \varphi(y)$. A bijection from P_1 to P_2 that is order preserving and order reflecting is called an *order isomorphism* from $\langle P_1; \leq \rangle$ to $\langle P_2; \leq \rangle$. The posets $\langle P_1; \leq \rangle$ and $\langle P_2; \leq \rangle$ are said to be *isomorphic* if there exists an order isomorphism from $\langle P_1; \leq \rangle$ to $\langle P_2; \leq \rangle$; we write $\langle P_1; \leq \rangle \cong \langle P_2; \leq \rangle$. We shall occasionally write $\langle P_1; \leq \rangle \lesssim \langle P_2; \leq \rangle$ if there exists a subset X of P_2 such that $\langle P_1; \leq \rangle \cong \langle X; \leq \rangle$. We say that P_2 contains a *cofinal copy* of P_1 if there exists a cofinal subset X of $\langle P_2; \leq \rangle$ such that $\langle P_1; \leq \rangle \cong \langle X; \leq \rangle$. We say that P_2 contains a *dually cofinal copy* of P_1 if there exists a cofinal subset X of $\langle P_2; \leq \rangle^{\text{du}}$ such that $\langle P_1; \leq \rangle \cong \langle X; \leq \rangle^{\text{du}}$. If $\langle P_1; \leq \rangle$ and $\langle P_2; \leq \rangle$ are meet-semilattices (resp. complete lattices) then a map $\varphi: P_1 \rightarrow P_2$ is called a *meet-semilattice homomorphism* (resp. a *meet-complete semilattice homomorphism*) if for any $x, y \in P_1$, we have $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$ (resp. for any subset X of P_1 , we have $\varphi(\Lambda X) = \Lambda \varphi[X]$). In this case, $\varphi[P_1]$ is a meet-subsemilattice (resp. a meet-complete subsemilattice) of P_2 . If in addition, φ is one-to-one and onto then φ is called a *meet-semilattice isomorphism*. It is easily checked that a meet-

semilattice isomorphism is just the same thing as an order isomorphism between meet-semilattices and that any meet-semilattice isomorphism between complete lattices is a meet-complete isomorphism. Dual definitions of join- and join-complete semilattice homomorphisms and isomorphisms are formulated in the obvious way. A join- and meet-semilattice homomorphism (resp. a join- and meet-complete semilattice homomorphism) between two lattices (resp. complete lattices) is called a *lattice homomorphism* (resp. a *complete lattice homomorphism*). We may replace “homomorphism” by *isomorphism* if the map is a bijection; however, any order isomorphism between lattices (resp. complete lattices) is already a lattice (resp. a complete lattice) homomorphism.

If $\langle P; \leq \rangle$ is a meet-semilattice then a nonempty subset F of P is called a *filter* of $\langle P; \leq \rangle$ if the following conditions hold:

- F1.** F is dually hereditary;
- F2.** $a, b \in F$ implies $a \wedge b \in F$.

In this case, if $\langle P; \leq \rangle$ has 1 then we must have $1 \in F$, by **F1** and the fact that F is nonempty. The set of all filters of $\langle P; \leq \rangle$ will be denoted by $\mathfrak{F}P$. It is easily checked that $\langle \mathfrak{F}P; \subseteq \rangle$ is a complete lattice. Dually, if $\langle P; \leq \rangle$ is a join-semilattice then a nonempty subset I of P is called an *ideal* of $\langle P; \leq \rangle$ if the following conditions are met:

- I1.** I is hereditary;
- I2.** $a, b \in I$ implies $a \vee b \in I$.

In this case, if $\langle P; \leq \rangle$ has 0 then we must have $0 \in I$, by **I1** and the fact that I is nonempty. The set of all ideals of $\langle P; \leq \rangle$ will be denoted by $\mathfrak{I}P$. Of course $\langle \mathfrak{I}P; \subseteq \rangle$ is also a complete lattice.

A lattice $\langle L; \leq \rangle$ is called *modular* if

$$(x \wedge y) \vee z = (x \vee z) \wedge y \text{ whenever } x, y, z \in L \text{ with } z \leq y.$$

(Notice that *any* lattice satisfies $(x \wedge y) \vee z \leq (x \vee z) \wedge y$ whenever $z \leq y$.) The following well known property of modular lattices will be useful to us. (See e.g., [CD73, 3.4, p19] for a proof.)

PROPOSITION 3. *Let $\langle L; \leq \rangle$ be a modular lattice. If $x, y \in L$ then the function $\varphi: [x \wedge y, x] \rightarrow [y, x \vee y]$ defined by $\varphi(z) = z \vee y$ for all $z \in L$, is a lattice isomorphism. \square*

A lattice $\langle L; \leq \rangle$ is said to be *distributive* if

$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z) \text{ for all } x, y, z \in L.$$

(Notice that *any* lattice satisfies $(x \wedge y) \vee z \leq (x \vee z) \wedge (y \vee z)$.) Clearly every distributive lattice is modular. A complete lattice $\langle L; \leq \rangle$ is called *Brouwerian* (resp. *continuous*) if it satisfies the join-infinite distributive identity

$$x \wedge (\vee Y) = \vee \{x \wedge y : y \in Y\}$$

(resp. the meet-infinite distributive identity

one. Also every closure (resp. algebraic closure) system X in $\langle L; \leq \rangle$ is the range $u[L]$ of a suitable closure (resp. algebraic closure) operator u on $\langle L; \leq \rangle$, viz., $u = u_X$, and the correspondence $u \mapsto u[L]$ is one-to-one. The aforementioned two correspondences are mutually inverse bijections between the set of closure (resp. algebraic closure) systems in $\langle L; \leq \rangle$ and the set of closure (resp. algebraic closure) operators on $\langle L; \leq \rangle$. The following results are to be found in most introductory lattice theory texts or, e.g., [Grä79, Theorem 5, p25].

PROPOSITION 4. *The following conditions on a lattice $\langle L; \leq \rangle$ are equivalent:*

- (i) $\langle L; \leq \rangle$ is an algebraic lattice;
- (ii) there exists a set S and an algebraic closure system X in the complete lattice $\langle \exp S; \subseteq \rangle$ of all subsets of S (ordered by set inclusion) such that $\langle L; \leq \rangle$ is isomorphic to the lattice $\langle X; \subseteq \rangle$. □

PROPOSITION 5. *Let $\langle L; \leq \rangle$ be an algebraic lattice and u an algebraic closure operator on $\langle L; \leq \rangle$. Then $y \in L$ is a compact element of $\langle u(L); \leq \rangle$ if and only if $y = u(x)$ for some compact element x of $\langle L; \leq \rangle$. □*

COROLLARY 6. *Let S be a set and $X \subseteq \exp S$. Then $\langle X; \subseteq \rangle$ is an algebraic lattice if and only if X is closed under arbitrary intersections and $\bigcup Y \in X$ for any directed subset Y of $\langle X; \subseteq \rangle$. In this case, the map $Z \mapsto u(Z) := \bigcap \{A \in X : A \supseteq Z\}$ ($Z \in \exp S$) is the algebraic closure operator corresponding to X and the compact elements of $\langle X; \subseteq \rangle$ are just the elements of the form $u(Z)$, where Z is any finite subset of S . □*

In Chapter I, we shall make essential use of the following proposition, which combines several classical results from lattice theory. Proofs of all statements in the proposition may be found in [Grä79, Chapter 0, §6].

PROPOSITION 7. (i) [Grä79, Theorem 2, p22] *The ideal lattice $\mathfrak{I}L$ of a join-semilattice L with 0 is an algebraic lattice. Dually, the filter lattice $\mathfrak{F}L$ of a meet-semilattice L with 1 is an algebraic lattice.*

(ii) [Grä79, Theorem 3, p22] *If L is an algebraic lattice and 0 is the bottom element of L then the set L^c of all compact elements of L is a join-subsemilattice of L (i.e., L^c is closed under finite joins), and $0 \in L^c$.*

(iii) [Grä79, Theorem 3, pp22-23] *If $\langle L; \leq \rangle$ is an algebraic lattice then the map $x \mapsto \{y \in L^c : y \leq x\}$ ($x \in L$) is a lattice isomorphism from L onto the ideal lattice $\mathfrak{I}L^c$ of the join-semilattice L^c (with 0) of all compact elements of L . Therefore:*

(vi) [Grä79, Theorem 5, pp25-26] *A lattice is algebraic if and only if it is isomorphic to*

§2. RINGS.

We shall use \mathbb{Z} to denote the ring of integers, \mathbb{N} to denote the subset of \mathbb{Z} consisting of all positive integers, \mathbb{Z}_n to denote the additive group of integers modulo n ($n \in \mathbb{N}$), and \mathbb{Q}, \mathbb{R} and \mathbb{C} to denote the fields of rational numbers, real numbers and complex numbers, respectively. We shall use 0 to denote the zero ring, the zero ideal and zero element of a ring, the zero module over any ring and the zero submodule and zero element of any module. The intended meaning of 0 will always be clear from the context in which it is used. All rings considered are associative but, unless otherwise stated, are not assumed to have an identity element. The identity element of a ring R , if one exists, will always be denoted by 1_R . We warn the reader that if R is a ring, with or without identity then by a *subring* of R we mean simply a multiplicatively closed subgroup of the additive group $\langle R; +, -, 0 \rangle$ of R ; we do not insist, therefore, that a subring of a ring with identity contain the identity element. Recall that if $\langle R; +, \cdot, -, 0 \rangle$ is a ring then R^{opp} denotes the *opposite ring* $\langle R; +, *, -, 0 \rangle$ of R , which has the same universe and the same addition and negation operations as R but in which a product $r * s$ of two elements r, s is equal to the product $s \cdot r$ (calculated in R). We remind the reader that the *centre* of a ring R , abbreviated $\text{cen } R$, is defined to be the set $\{c \in R : cr = rc \text{ for all } r \in R\}$. It is easily checked that $\text{cen } R$ is a subring of R . If there exists a least positive integer n such that $rn = 0$ for all $r \in R$, then R is said to have *characteristic* n ; we write $\text{char } R = n$. If no such n exists R is said to have *characteristic zero*; we write $\text{char } R = 0$.

A nonzero element r of a ring R is called a *divisor of zero* (or just a *zero divisor*) if $rs = 0$ or $sr = 0$ for some nonzero $s \in R$. A nonzero element of R which is not a zero divisor is called *regular*. A *domain* is a ring in which every nonzero element is regular. Recall also that if R is a ring with identity then an element $r \in R$ is said to be *right invertible* (resp. *left invertible*) if $rs = 1_R$ (resp. $sr = 1_R$) for some $s \in R$. An element of R which is both right and left invertible is called a *unit* of R . Recall that an element $r \in R$ is said to be *idempotent* if $r^2 = r$, and *nilpotent* if $r^n = 0$ for some $n \in \mathbb{N}$ and that a nonempty subset of R consisting entirely of nilpotent elements is called *nil*. If A_1, A_2, \dots, A_n ($n \in \mathbb{N}$) are subgroups of $\langle R; +, -, 0 \rangle$, we define $A_1 A_2 \dots A_n$ to be the subgroup of $\langle R; +, -, 0 \rangle$ generated by $\{a_1 a_2 \dots a_n : a_i \in A_i \text{ for } i = 1, \dots, n\}$. If $A_i = A$ for all $i \in \{1, \dots, n\}$, we write A^n instead of $AA \dots A$. (This notation coincides with that used for the cartesian product $A \times A \times \dots \times A$ but the intended meaning of A^n will always be clear from the context in which it is used.) We call a subgroup A of $\langle R; +, -, 0 \rangle$ *idempotent* if $A^2 = A$ and *nilpotent* if $A^n = 0$ for some $n \in \mathbb{N}$.

By an *ideal* of a ring, we shall always mean a two-sided ideal. We shall frequently use $\text{Id } R$ to denote the set of all ideals of a ring R . We shall also, occasionally, use $\text{Id } R_R$ (resp. $\text{Id}_R R$) to denote the set of all right (resp. left) ideals of a ring R . It is well known that $\langle \text{Id } R; \subseteq \rangle, \langle \text{Id } R_R; \subseteq \rangle$

and $\langle \text{Id}_R R; \subseteq \rangle$ are modular algebraic (in particular, complete) lattices. Recall that R is said to be a *simple ring* if it has no proper nonzero ideals, i.e., if $\text{Id } R = \{0, R\}$. If $X \subseteq R$, we denote by $\langle X \rangle$ the *ideal of R generated by X* , i.e., the intersection of all ideals of R containing X . If X is a nonempty subset of R then

$$\langle X \rangle = \left\{ \sum_{i=1}^n (r_i x_i r'_i + s_i x_i + x_i t_i + x_i m_i) : n \in \mathbf{N} \text{ and } x_i \in X, r_i, r'_i, s_i, t_i \in R, m_i \in \mathbb{Z} \right. \\ \left. \text{for } i = 1, \dots, n \right\}.$$

We warn the reader that the symbol $\langle X \rangle$ will not be used exclusively for the above purpose, however. (For example, we shall frequently use $\langle X \rangle$ to denote the free monoid on the set X : see §10.) For this reason, in instances where $\langle X \rangle$ is used, its meaning will always be stated explicitly.

We shall make frequent use of the following result of V.A. Andrunakievič, which is known as *Andrunakievič's Lemma*.

LEMMA 1. [And66, Lemma4, p102] *Let A be an ideal of a ring R and I an ideal of the ring A . If $\langle I \rangle$ denotes the ideal of R generated by I then $\langle I \rangle^3 \subseteq I$. \square*

§3. MODULES AND MODULE CATEGORIES.

We shall not distinguish notationally between a category and its class of objects. Recall that if \mathcal{C} is a category then a subcategory \mathcal{D} of \mathcal{C} is called a *full subcategory* of \mathcal{C} if every morphism of \mathcal{C} between two objects of \mathcal{D} is a morphism of \mathcal{D} (i.e., if $\text{Hom}_{\mathcal{C}}(A, B) = \text{Hom}_{\mathcal{D}}(A, B)$ for all objects A, B of \mathcal{D}). Trivially, every nonempty class \mathcal{A} of objects of \mathcal{C} gives rise to a full subcategory \mathcal{D} of \mathcal{C} whose class of objects is defined to be \mathcal{A} and whose class of morphisms is defined by $\text{Hom}_{\mathcal{D}}(A, B) = \text{Hom}_{\mathcal{C}}(A, B)$ for all objects A, B of \mathcal{D} . The subcategory \mathcal{D} defined in this manner is usually referred to as the *full subcategory of \mathcal{C} on \mathcal{A}* . We shall use $1_{\mathcal{C}}$ to denote the identity functor on \mathcal{C} . Recall that if \mathcal{C} and \mathcal{D} are categories then a covariant functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is called a *category isomorphism* if there exists a covariant functor $G: \mathcal{D} \rightarrow \mathcal{C}$ with the property that $GF = 1_{\mathcal{C}}$ and $FG = 1_{\mathcal{D}}$. The categories \mathcal{C} and \mathcal{D} are said to be *isomorphic* if there exists a category isomorphism $F: \mathcal{C} \rightarrow \mathcal{D}$.

Suppose that R is a ring and M is a right R -module (by which we mean that M is a right module over R). If A is a subgroup of $\langle R; +, -, 0 \rangle$ we define MA to be the subgroup of $\langle M; +, -, 0 \rangle$ generated by $\{xa: x \in M, a \in A\}$. We shall call M a *zero multiplication module* if $MR = 0$, and a *unital module* if $MR = M$. We warn the reader that, unless otherwise stated, *we do not assume our modules to be unital, even in situations where the ring in question has identity*. It is easily shown that if R is an arbitrary ring then every right R -module has a unique maximal zero multiplication submodule. We shall always use $\text{Mod-}R$ to represent the category of *all* right modules over the ring R . We use $\text{Mod-}R(\text{zero})$ (resp. $\text{Mod-}R(\text{unital})$) to denote the full subcategory of

$\text{Mod-}R$ on the class of all zero multiplication modules (resp. the class of all unital modules). The left analogues of $\text{Mod-}R$, $\text{Mod-}R(\text{zero})$ and $\text{Mod-}R(\text{unital})$ are denoted, respectively, by $R\text{-Mod}$, $R\text{-Mod}(\text{zero})$ and $R\text{-Mod}(\text{unital})$. We define $\mathbb{Z}_{\text{zero}} \in \text{Mod-}R(\text{zero})$ to be the abelian group $\langle \mathbb{Z}; +, -, 0 \rangle$ endowed with the zero multiplication. More generally, it is clear that every abelian group can be regarded as an element of $\text{Mod-}R(\text{zero})$ if it is endowed with the zero multiplication. It is also clear that every group homomorphism between zero multiplication right R -modules is automatically an R -module homomorphism. These observations are evidence of the easily verifiable fact that the category $\text{Mod-}R(\text{zero})$ is isomorphic to, and hence identifiable with, the category of all abelian groups.

In many situations, a structure M may be interpretable as a module in several different senses. For example, if I is an ideal of a ring R then I may be regarded as a right or left module over itself as well as a right or left module over R . If, in such situations, we wish to regard M specifically as a right (resp. left) module over a particular ring R , we indicate this by writing M_R (resp. ${}_R M$) rather than M . Suppose $M, N \in \text{Mod-}R$. If N is a submodule of M , we write $N \leq M$, and if N is isomorphic to a submodule of M , we write $N \lesssim M$. It is well known that the set of all submodules of M , ordered by set inclusion, is a modular algebraic (in particular, complete) lattice. We shall use $\text{Hom}_R(M, N)$ to denote the abelian group of all right R -module homomorphisms from M to N , and $\text{End}_R M$ to denote the ring of all right R -module endomorphisms of M . Unless stated otherwise, if M is a right (resp. left) R -module then R -module endomorphisms of M will be written on the left (resp. right) of their arguments. Also, $\text{Im } \varphi$ and $\text{Ker } \varphi$ are used as abbreviations for the image and kernel of $\varphi \in \text{Hom}_R(M, N)$. We shall often write $\varphi: M \cong N$ to indicate that $\varphi \in \text{Hom}_R(M, N)$ is an (R -module) isomorphism. We remind the reader that a sequence

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_n} M_n$$

of right R -modules and R -module homomorphisms is said to be *exact* if $\text{Im } \varphi_i = \text{Ker } \varphi_{i+1}$ for all $i \in \{1, 2, \dots, n-1\}$.

Let R be a ring and $\{M_i: i \in \Gamma\}$ a family of right R -modules. Suppose $x = \{x_i\}_{i \in \Gamma} \in \prod_{i \in \Gamma} M_i$. Recall that the *support* of x , abbreviated $\text{supp } x$, is defined to be $\{i \in \Gamma: x_i \neq 0\}$. The (external) direct sum of the family $\{M_i: i \in \Gamma\}$ will be denoted by $\bigoplus_{i \in \Gamma} M_i$. (Thus $\bigoplus_{i \in \Gamma} M_i = \{x \in \prod_{i \in \Gamma} M_i: |\text{supp } x| < \aleph_0\}$.) If $M_i = M$ for all $i \in \Gamma$, we abbreviate $\bigoplus_{i \in \Gamma} M_i$ by $\bigoplus_{\Gamma} M$ or by $M^{(\Gamma)}$. Obviously, if $\Gamma = \{0, 1, \dots, n-1\}$ is a finite cardinal then $\bigoplus_{i \in \Gamma} M_i$ and $\prod_{i \in \Gamma} M_i$ coincide; in such cases, we usually favour the direct sum notation and write $M_0 \oplus M_1 \oplus \dots \oplus M_{n-1}$ instead of $M_0 \times M_1 \times \dots \times M_{n-1}$. If \mathfrak{m} is an arbitrary cardinal, we define

$$\prod_{i \in \Gamma}^{(\mathfrak{m})} M_i = \{x \in \prod_{i \in \Gamma} M_i: |\text{supp } x| < \max\{\mathfrak{m}, \aleph_0\}\}.$$

It is easily checked that $\prod_{i \in \Gamma}^{(\mathfrak{m})} M_i$ is a submodule of $\prod_{i \in \Gamma} M_i$. Notice also that if $\mathfrak{m} \leq \aleph_0$ then $\prod_{i \in \Gamma}^{(\mathfrak{m})} M_i = \bigoplus_{i \in \Gamma} M_i$. If $M_i = M$ for all $i \in \Gamma$, we abbreviate $\prod_{i \in \Gamma}^{(\mathfrak{m})} M_i$ by $\prod_{\Gamma}^{(\mathfrak{m})} M$. The sum of a family $\{M_i: i \in \Gamma\}$ of submodules of $M \in \text{Mod-}R$ will be denoted by $\sum_{i \in \Gamma} M_i$; if this sum is

direct, we write $\bigoplus_{i \in \Gamma} M_i$ instead of $\sum_{i \in \Gamma} M_i$ and we refer to $\{M_i : i \in \Gamma\}$ as an *independent family* of submodules of M . (We therefore do not distinguish notationally between the external and internal direct sum of an independent family of submodules.)

THEOREM 1. [Ker87, Proposition 15.2, p80] *Let R be a ring and for each right R -module M , define $M_0 = \{x \in M : xR = 0\}$. The following assertions are equivalent:*

- (i) R has identity;
- (ii) $M = M_0 \oplus MR$ for all (right) R -modules M . □

The reader will be aware that if R is a ring with identity, it is more common to define a right R -module M to be unital if $x \cdot 1_R = x$ for all $x \in M$. It is an easy consequence of Theorem 1, however, that if R is a ring with identity then this notion of “unital” is equivalent to the requirement that $MR = M$. Indeed, it is obvious that if $x \cdot 1_R = x$ for all $x \in M$ then $MR = M$ while conversely, if $MR = M$ then by Theorem 1, $\{x \cdot 1_R - x : x \in M\} \subseteq \{y \in M : yR = 0\} := M_0 = 0$.

Suppose $M \in \text{Mod-}R$ and $N \leq M$. We say that N is an *essential submodule* of M if $N \cap L \neq 0$ whenever $0 \neq L \leq M$. For $M, N \in \text{Mod-}R$, an R -module monomorphism $\iota : N \rightarrow M$ is said to be *essential* if $\iota[N]$ is an essential submodule of M , and the pair (ι, M) is then called an *essential extension* of N . If the essential monomorphism $\iota : N \rightarrow M$ is understood, it is customary to omit the first coordinate of the pair (ι, M) and to refer to M as an essential extension of N . If $N \leq M$ then Zorn’s Lemma may be used to establish the existence of a submodule L of M which is maximal with respect to the condition that $L \cap N = 0$. We shall call such a submodule L (which is not, in general, unique) an *orthogonal complement* for N in M . It is well known that in this situation, $N \oplus L$ is an essential submodule of M .

Let $M \in \text{Mod-}R$. If X and Y are nonempty subsets of M , we define $(X : Y) = \{r \in R : Yr \subseteq X\}$. Note that if $X = 0$ then $(X : Y) = (0 : Y)$ is just the *annihilator* of Y in R . For subsets X, Y of R , we define $(X : {}_r Y) = \{r \in R : Yr \subseteq X\}$ and $(X : {}_l Y) = \{r \in R : rY \subseteq X\}$. Thus $(0 : {}_r Y)$ (resp. $(0 : {}_l Y)$) denotes the *right annihilator* (resp. *left annihilator*) of Y in R . If Y is a singleton, say $Y = \{y\}$, then we write $(0 : y), (0 : {}_r y)$ and $(0 : {}_l y)$ instead of $(0 : \{y\}), (0 : {}_r \{y\})$ and $(0 : {}_l \{y\})$, respectively. Recall that M is said to be *faithful* if $(0 : M) = 0$, and *finitely annihilated* if there exists a finite subset X of M such that $(0 : X) = (0 : M)$. We call M *simple* if 0 and M are the only submodules of M . (We do not insist, as many authors do, that a simple module be unital.) Recall also that the *socle* of M , abbreviated $\text{soc } M$, is defined to be the sum of all simple submodules of M and that M is said to be *semisimple* if $M = \text{soc } M$.

PROPOSITION 2. [Row88, Theorem 2.4.7, p178] *If M is an arbitrary right R -module then $\text{soc } M = \bigcap \{N \leq M : N \text{ is essential in } M\}$.* □

A module is said to be *noetherian* (resp. *artinian*) if its lattice of submodules satisfies the

ACC (resp. the DCC). A ring R is called *right noetherian* (resp. *right artinian*) if the module R_R is noetherian (resp. artinian). It is well known that a right R -module M is noetherian if and only if every submodule of M is finitely generated. We point out that if X is a nonempty subset of M and N denotes the submodule of M generated by X then

$$N = \left\{ \sum_{i=1}^n (x_i r_i + x_i m_i) : n \in \mathbb{N}, x_i \in X, r_i \in R, m_i \in \mathbb{Z} \text{ for } i = 1, \dots, n \right\}.$$

THEOREM 3. [Lam66, Proposition 6, p22] *Let N be a submodule of a right R -module M . Then the following assertions are equivalent:*

- (i) M is noetherian (resp. artinian);
- (ii) N and M/N are noetherian (resp. artinian). □

For an infinite cardinal \mathfrak{m} and $M \in \text{Mod-}R$, we shall say that M satisfies the \mathfrak{m} -ACC (resp. the \mathfrak{m} -DCC) if the lattice of submodules of M satisfies the \mathfrak{m} -ACC (resp. the \mathfrak{m} -DCC).

§4. THE DORROH EXTENSION, PROJECTIVE AND INJECTIVE MODULES.

Let R be an arbitrary ring. We define R^* to be the ring $R \times \mathbb{Z}$ with addition and multiplication defined by:

$$\begin{aligned} (r_1, m_1) + (r_2, m_2) &= (r_1 + r_2, m_1 + m_2) \text{ and} \\ (r_1, m_1)(r_2, m_2) &= (r_1 r_2 + r_1 m_2 + r_2 m_1, m_1 m_2) \end{aligned}$$

for all $r_1, r_2 \in R$ and $m_1, m_2 \in \mathbb{Z}$. The ring R^* is usually called the *Dorroh Extension* of R and has its origin in [Dor32]. It is easily checked that $(0, 1) \in R^*$ is an identity element for R^* and that the map $\varphi: R \rightarrow R^*$ defined by $\varphi(r) = (r, 0)$ ($r \in R$) is a ring monomorphism whose image $(R, 0)$ ($:= R \times \{0\}$) is an ideal of R^* . If every $r \in R$ is identified with its image $(r, 0)$ in R^* , then R may be considered as an ideal of R^* and each $(r, m) \in R^*$ may be written as $(r, m) = (0, 1)(r, 0) + (0, 1)m = 1_{R^*} \cdot r + 1_{R^*} \cdot m$, whence $R^* = \{1_{R^*} \cdot r + 1_{R^*} \cdot m : r \in R, m \in \mathbb{Z}\}$. We point out that the Dorroh Extension R^* is not the only extension of R with identity; nor is it, in general, a minimal extension with an identity element, for R itself may possess an identity. It is an unsurprising and easily verifiable fact that the interval $[0, R]$ of the lattice $\langle \text{Id } R^*; \subseteq \rangle$ is isomorphic to the lattice $\langle \text{Id } R; \subseteq \rangle$. Also, since $R^*/R \cong \mathbb{Z}$, the interval $[R, R^*]$ of $\langle \text{Id } R^*; \subseteq \rangle$ is isomorphic to $\langle \text{Id } \mathbb{Z}; \subseteq \rangle$.

If $M \in \text{Mod-}R$ then M has the structure of a right R^* -module with

$$x \cdot (1_{R^*} \cdot r + 1_{R^*} \cdot m) := xr + xm \text{ for any element } 1_{R^*} \cdot r + 1_{R^*} \cdot m \text{ of } R^*.$$

Moreover, it is easily checked that if $M, N \in \text{Mod-}R$ then $\text{Hom}_{R^*}(M, N) = \text{Hom}_R(M, N)$ and the R^* -submodules of M are precisely the R -submodules of M . Conversely, since R is a subring (indeed, an ideal) of R^* , every unital right R^* -module has, automatically, the structure of a right R -module. Again, it is easily checked that if $M, N \in \text{Mod-}R^*$ (unital) then $\text{Hom}_{R^*}(M, N) = \text{Hom}_R(M, N)$ and the R^* -submodules of M are precisely the R -submodules of M . The above

observations can be phrased more categorically: the assignment of a unital right R^* -module structure to every right R -module defines, in essence, a covariant functor $F: \text{Mod-}R \rightarrow \text{Mod-}R^*$ (unital), while the reverse assignment defines a covariant functor $G: \text{Mod-}R^*$ (unital) $\rightarrow \text{Mod-}R$. A routine check shows that $FG = 1_{\text{Mod-}R^* \text{ (unital)}}$ and $GF = 1_{\text{Mod-}R}$. This proves the next result.

THEOREM 1. *If R is an arbitrary ring then the categories $\text{Mod-}R$ and $\text{Mod-}R^*$ (unital) are isomorphic. \square*

Although we seldom refer explicitly to Theorem 1, it is fundamentally important and underpins many of the assumptions made in the sequel. For example, making implicit use of Theorem 1, we shall systematically identify $\text{Mod-}R$ and $\text{Mod-}R^*$ (unital) henceforth.

Theorem 1 also reassures us that, in terms of its macroscopic features, the category $\text{Mod-}R$ is no different from the well-explored categories of unital modules over rings with identity. Many of the constructions available in the latter type of category are therefore also available in $\text{Mod-}R$. In particular, we shall make use of the fact that every right R -module has an “injective hull” in $\text{Mod-}R$. (In the sequel to Proposition 2 below, we have chosen, for the reader’s convenience, to recall some of the basic features of the injective hull.)

The identification of $\text{Mod-}R$ and $\text{Mod-}R^*$ (unital) requires that we refine our “annihilator” notation. Accordingly, if $M \in \text{Mod-}R$ and X, Y are nonempty subsets of M , we define

$$(X:Y)^* = \left\{ 1_{R^*} \cdot r + 1_{R^*} \cdot m \in R^* : Y(1_{R^*} \cdot r + 1_{R^*} \cdot m) = \{yr + ym : y \in Y\} \subseteq X \right\}.$$

Note that if X is a nonempty subset of M and N is the submodule of M generated by X then

$$\begin{aligned} N &= \left\{ \sum_{i=1}^n (x_i r_i + x_i m_i) : n \in \mathbf{N}, x_i \in X, r_i \in R, m_i \in \mathbb{Z} \text{ for } i = 1, \dots, n \right\} \\ &= \left\{ \sum_{i=1}^n x_i (1_{R^*} \cdot r_i + 1_{R^*} \cdot m_i) : n \in \mathbf{N}, x_i \in X, 1_{R^*} \cdot r_i + 1_{R^*} \cdot m_i \in R^* \text{ for } i = 1, \dots, n \right\} \\ &= XR^*. \end{aligned}$$

If, in particular, $X = \{x\}$ then $N = xR^* \cong R_R^* / (0:x)^*$. More generally, every cyclic right R -module is isomorphic to R_R^* / K for some right ideal K of R^* (or equivalently, some submodule K of R_R^*). This assertion follows from the remarks preceding Theorem 1 and the well known fact that if R is a ring with identity then every unital cyclic right R -module is isomorphic to R_R / K for some $K \leq R_R$. Note, in particular, that $\mathbb{Z}_{\text{zero}} \cong R_R^* / R_R$. We call a ring R a *principal right ideal ring* if every submodule of R_R is cyclic.

It follows from Theorem 3.1 that if R is a ring with identity and $R_0^* = \{x \in R_R^* : xR = 0\}$, then $R_R^* = R_0^* \oplus R_R$. It is easily shown that $R_0^* = \{1_{R^*} \cdot (-1_R m) + 1_{R^*} \cdot m : m \in \mathbb{Z}\} \cong \mathbb{Z}_{\text{zero}}$. Consequently, we have the following proposition.

PROPOSITION 2. *If R is a ring with identity then $R_R^* \cong \mathbb{Z}_{\text{zero}} \oplus R_R$.* □

It follows from Theorem 1 that R_R^* is a generator for the category $\text{Mod-}R$ (in the sense that every $M \in \text{Mod-}R$ is an epimorphic image of a direct sum of copies of R_R^*). More specifically, if R has identity, it is possible to deduce from Proposition 2 that \mathbb{Z}_{zero} is a generator for the category $\text{Mod-}R(\text{zero})$ and that R_R is a generator for the category $\text{Mod-}R(\text{unital})$.

Recall that a right R -module P is said to be *projective* if, given any diagram in $\text{Mod-}R$ of the form

$$\begin{array}{ccc}
 & P & \\
 & \downarrow \varphi & \searrow \bar{\varphi} \\
 0 & \longleftarrow N & \longleftarrow \pi M
 \end{array}$$

with exact row, there exists a homomorphism $\bar{\varphi} : P \rightarrow M$ making the diagram commute. Dually, a right R -module E is said to be *injective* if, given any diagram in $\text{Mod-}R$ of the form

$$\begin{array}{ccc}
 & E & \\
 & \uparrow \varphi & \swarrow \bar{\varphi} \\
 0 & \longrightarrow N & \xrightarrow{\iota} M
 \end{array}$$

with exact row, there exists a homomorphism $\bar{\varphi} : M \rightarrow E$ making the diagram commute. Recall also that if $M \in \text{Mod-}R$, there exists an injective right R -module E and an essential monomorphism $\iota : M \rightarrow E$. The pair (ι, E) is thus an essential extension of M which is, moreover, unique in the sense that if $E' \in \text{Mod-}R$ is injective and $\iota' : M \rightarrow E'$ is an essential monomorphism then there is an isomorphism $\varphi : E \rightarrow E'$ which makes the following diagram commute.

$$\begin{array}{ccc}
 & M & \\
 \iota \swarrow & & \searrow \iota' \\
 E & \xrightarrow{\varphi} & E'
 \end{array}$$

The pair (ι, E) is called an *injective hull* for M . For each $M \in \text{Mod-}R$, we choose a fixed injective hull for M and denote it by $(\iota_M, E(M))$. The above comments justify our calling $(\iota_M, E(M))$ the injective hull of M . If the essential monomorphism $\iota_M : M \rightarrow E(M)$ is understood, it is customary to refer to $E(M)$ as the injective hull of M .

It is well known that the injective (right) \mathbb{Z} -modules (i.e., abelian groups) are precisely the so-called divisible abelian groups. (An abelian group G is said to be *divisible* if, given any $g \in G$ and

any nonzero $n \in \mathbb{Z}$, we have $g'n = g$ for some $g' \in G$.) The groups $\langle \mathbb{Q}; +, -, 0 \rangle$ and \mathbb{Z}_{p^∞} (p a positive prime integer) are two prototypes of divisible abelian groups. We remind the reader that if p is a positive prime integer then

$$\mathbb{Z}_{p^\infty} := \left\{ \frac{m}{p^n} + \mathbb{Z} \in \mathbb{Q}_{\mathbb{Z}} / \mathbb{Z}_{\mathbb{Z}} : m \in \mathbb{Z}, n \in \mathbb{N} \right\}.$$

It is a well known and interesting fact that for each $k \in \mathbb{N}$, $H_k = \left\{ \frac{m}{p^k} + \mathbb{Z} \in \mathbb{Q}_{\mathbb{Z}} / \mathbb{Z}_{\mathbb{Z}} : m \in \mathbb{Z} \right\}$ is a subgroup of \mathbb{Z}_{p^∞} and every nonzero proper subgroup of \mathbb{Z}_{p^∞} is of this form. Inasmuch as every proper subgroup of \mathbb{Z}_{p^∞} is an element of the chain $0 \subsetneq H_1 \subsetneq H_2 \subsetneq \dots$, we may conclude that \mathbb{Z}_{p^∞} is an artinian but not a noetherian (right) \mathbb{Z} -module. It is easily checked that $E(\mathbb{Z}_{\mathbb{Z}}) \cong \mathbb{Q}_{\mathbb{Z}}$ and $E(\mathbb{Z}_p) \cong \mathbb{Z}_{p^\infty}$ for all positive prime integers p .

THEOREM 3. [Rot88, Theorem 10.14, p250] *The following assertions are equivalent for an abelian group G :*

- (i) G is divisible;
- (ii) $G_{\mathbb{Z}} \cong \mathbb{Q}_{\mathbb{Z}}^{(\Gamma)} \oplus (\oplus_{i \in \Gamma} \mathbb{Z}_{p_i^\infty})$ for some set Γ and some family $\{p_i : i \in \Gamma\}$ of (not necessarily distinct) positive prime integers. □

§5. THE JACOBSON AND PRIME RADICALS.

Recall that an ideal P of a ring R is called a *prime ideal* of R if $P \supseteq I$ or $P \supseteq J$ whenever I and J are ideals of R such that $P \supseteq IJ$. The set of all prime ideals of a ring R is often referred to as the *spectrum* of R , abbreviated $\text{Spec } R$. Recall that the *Prime* (or *Lower Baer*) *radical* of a ring R is defined to be the intersection of all prime ideals of R . We denote the Prime radical of R by $\beta(R)$.

PROPOSITION 1. [Lam66, Proposition 4, p54] *The following assertions are equivalent for a ring R :*

- (i) 0 is a prime ideal of R ;
- (ii) $IJ = 0$ implies $I = 0$ or $J = 0$, whenever I and J are ideals of R ;
- (iii) $aRb = 0$ implies $a = 0$ or $b = 0$, whenever $a, b \in R$. □

A ring R which satisfies the equivalent conditions of Proposition 1 is called *prime*.

PROPOSITION 2. [Lam66, Proposition 2, p56] *The following assertions are equivalent for a ring R :*

- (i) $\beta(R) = 0$;
- (ii) R has no nonzero nilpotent ideals;
- (iii) $aRa = 0$ implies $a = 0$, whenever $a \in R$. □

A ring R which satisfies the equivalent conditions of Proposition 2 is called *semiprime*.

Recall that a ring R is said to be *right primitive* if there exists a faithful simple right R -module. An ideal P of a ring R is called a *right primitive ideal* if R/P is a right primitive ring. Recall also that the *Jacobson radical*, $J(R)$, of a ring R is defined to be the intersection of all right primitive ideals of R . It is well known that the Jacobson radical is left-right symmetric in the sense that $J(R)$ is also equal to the intersection of all left primitive ideals of R . There are several standard characterizations of $J(R)$, some of which we list in Theorem 3 below. We remind the reader that a right (resp. left) ideal K of a ring R is said to be *regular* provided there exists $e \in R$ such that $r - er \in K$ (resp. $r - re \in K$) for all $r \in R$. Note that in a ring with identity, every right ideal and every left ideal is regular.

THEOREM 3. [Hun74, Theorem 2.3, p426] *If R is an arbitrary ring then:*

- (i) $J(R)$ is the intersection of all regular maximal right ideals of R ;
- (ii) $J(R)$ is the intersection of all annihilators of simple right R -modules.

Moreover, assertions (i) and (ii) remain valid if “right” is replaced by “left”. □

A ring R is said to be *semiprimitive* if $J(R) = 0$.

THEOREM 4. [Lam66, Proposition 1, p75] *The following assertions are equivalent for a ring R with identity:*

- (i) R has a unique maximal proper right ideal;
- (ii) $J(R)$ is the unique maximal proper right ideal of R ;
- (iii) $J(R)$ consists precisely of the nonunits of R ;
- (iv) the set of all nonunits of R is an ideal of R ;
- (v) $R/J(R)$ is a division ring.

Moreover, each of the above assertions is equivalent to each of assertions (i) and (ii) with “right” replaced by “left”. □

A ring R which satisfies the equivalent conditions of Theorem 4 is called *local*. A ring with identity is called *semilocal* if, for some finite positive cardinals $\mathbf{m}, \mathbf{n}_1, \dots, \mathbf{n}_\mathbf{m}$ and some division rings $D_1, \dots, D_\mathbf{m}$, the ring $R/J(R)$ is isomorphic to the direct product $\prod_{i=1}^{\mathbf{m}} M_{\mathbf{n}_i}(D_i)$, where $M_{\mathbf{n}_i}(D_i)$ denotes the ring of all $\mathbf{n}_i \times \mathbf{n}_i$ matrices over D_i .

PROPOSITION 5. [Ker87, Theorem 30.6, p144] *If R is a right artinian ring then $J(R)$ is nilpotent.* □

THEOREM 6. [Lam66, Corollary, p69] *Every right artinian ring with identity is right noetherian.* \square

The requirement that the ring in the above theorem possess an identity element cannot be dropped, as is witnessed by the abelian group \mathbb{Z}_p^∞ (p a positive prime integer) endowed with the zero multiplication and considered thus as a ring (see §4).

§6. MATRIX RINGS AND MORITA EQUIVALENCE.

Let \mathfrak{m} be a nonzero cardinal and R a ring. Extending the notation used in the previous section, we denote by $\mathbf{M}_{\mathfrak{m}}(R)$ (resp. $\mathbf{M}_{\mathfrak{m}}^*(R)$) the ring of all row-finite (resp. row- and column-finite) $\mathfrak{m} \times \mathfrak{m}$ matrices over R . Observe that the ring $\mathbf{M}_{\mathfrak{m}}^*(R)$ is left-right symmetric in the sense that $\mathbf{M}_{\mathfrak{m}}^*(R^{\text{opp}}) \cong (\mathbf{M}_{\mathfrak{m}}^*(R))^{\text{opp}}$. Note also that if \mathfrak{m} is finite then $\mathbf{M}_{\mathfrak{m}}(R) = \mathbf{M}_{\mathfrak{m}}^*(R)$; we shall occasionally refer to this ring as the *full $\mathfrak{m} \times \mathfrak{m}$ matrix ring over R* . If $A \in \mathbf{M}_{\mathfrak{m}}(R)$ (or $\mathbf{M}_{\mathfrak{m}}^*(R)$), we use $A_{(\alpha)}$ (resp. $A^{(\alpha)}$) to denote the α -th row (resp. the α -th column) of A , and $A_{\alpha\beta}$ to denote the (α, β) -th entry of A , where $\alpha, \beta \in \mathfrak{m}$. Recall that a matrix $A \in \mathbf{M}_{\mathfrak{m}}(R)$ is said to be *scalar* if $A_{\alpha\beta} = 0$ whenever $\alpha \neq \beta$, $\alpha, \beta \in \mathfrak{m}$, and $A_{\alpha\alpha} = A_{\beta\beta}$ for all $\alpha, \beta \in \mathfrak{m}$.

PROPOSITION 1. [Row88, Proposition 1.1.5, p31] *Let \mathfrak{n} be a finite nonzero cardinal and R a ring with identity. Then J is an ideal of $\mathbf{M}_{\mathfrak{n}}(R)$ if and only if $J = \mathbf{M}_{\mathfrak{n}}(I)$ for some ideal I of R .* \square

It is an obvious consequence of Proposition 1 that $\mathbf{M}_{\mathfrak{n}}(D)$ is simple whenever D is a division ring and $0 < \mathfrak{n} < \aleph_0$.

We remind the reader here of what it means to say that rings R and S are “Morita equivalent”. First, we need to recall some definitions from category theory. Suppose \mathcal{C} and \mathcal{D} are categories and that there are covariant functors $F_1: \mathcal{C} \rightarrow \mathcal{D}$ and $F_2: \mathcal{C} \rightarrow \mathcal{D}$. Let $\eta = \{\eta_A: A \in \mathcal{C}\}$ be an indexed class of morphisms in \mathcal{D} such that $\eta_A: F_1(A) \rightarrow F_2(A)$ for every $A \in \mathcal{C}$. Then η is called a *natural transformation* from F_1 to F_2 if for each pair of objects $A, B \in \mathcal{C}$ and each morphism $f: A \rightarrow B$, the diagram

$$\begin{array}{ccc}
 F_1(A) & \xrightarrow{F_1(f)} & F_1(B) \\
 \eta_A \downarrow & & \downarrow \eta_B \\
 F_2(A) & \xrightarrow{F_2(f)} & F_2(B)
 \end{array}$$

commutes. If each η_A is an isomorphism then η is called a *natural isomorphism*; in this case, we write $F_1 \cong F_2$. A covariant functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is said to be an *equivalence* if there exists a covariant functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that $GF \cong 1_{\mathcal{C}}$ and $FG \cong 1_{\mathcal{D}}$; we shall call such functors F and G *inverse category equivalences*. The categories \mathcal{C} and \mathcal{D} are said to be *equivalent* if there exists an equivalence $F: \mathcal{C} \rightarrow \mathcal{D}$.

A category \mathcal{C} is said to be *preadditive* if for each pair of objects $A, B \in \mathcal{C}$, the set $\text{Hom}_{\mathcal{C}}(A, B)$ is an abelian group and the composition map from $\text{Hom}_{\mathcal{C}}(B, C) \times \text{Hom}_{\mathcal{C}}(A, B)$ to $\text{Hom}_{\mathcal{C}}(A, C)$ is bilinear for all objects $A, B, C \in \mathcal{C}$. If \mathcal{C} and \mathcal{D} are preadditive categories, a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is said to be *additive* if for each pair of objects $A, B \in \mathcal{C}$, we have $F(\varphi + \psi) = F(\varphi) + F(\psi)$ for all $\varphi, \psi \in \text{Hom}_{\mathcal{C}}(A, B)$. Two rings R and S with identity are said to be *Morita equivalent* if there exists an additive equivalence $F: \text{Mod-}R(\text{unital}) \rightarrow \text{Mod-}S(\text{unital})$. It is a well known and interesting fact that the notion of Morita equivalence is left-right symmetric in the following sense: if R and S are rings with identity then the existence of an additive equivalence $F: \text{Mod-}R(\text{unital}) \rightarrow \text{Mod-}S(\text{unital})$ implies the existence of an additive equivalence $G: R\text{-Mod}(\text{unital}) \rightarrow S\text{-Mod}(\text{unital})$, and conversely.

THEOREM 2. [AF74, Corollary 22.7, p265] *Suppose R and S are Morita equivalent rings with identity. Then there exists a finite nonzero cardinal \mathfrak{n} and an idempotent element $e \in \mathbb{M}_{\mathfrak{n}}(R)$ such that $S \cong e\mathbb{M}_{\mathfrak{n}}(R)e$.* □

The next theorem is a partial converse to Theorem 2.

THEOREM 3. [AF74, Corollary 22.6, p265] *If R is a ring with identity and \mathfrak{n} is a finite nonzero cardinal then R and $\mathbb{M}_{\mathfrak{n}}(R)$ are Morita equivalent.* □

§7. ARTIN-WEDDERBURN THEOREM, GOLDIE'S THEOREMS AND RELATED RESULTS.

THEOREM 1. (Artin-Wedderburn) [Hun74, Theorem 3.3, p435 & Theorem 3.7, p439] *The following assertions are equivalent for a ring R :*

- (i) R is nonzero, semiprimitive and right artinian;
- (ii) there exist nonzero finite cardinals \mathfrak{k} and $\mathfrak{n}_1, \mathfrak{n}_2, \dots, \mathfrak{n}_{\mathfrak{k}}$ and division rings $D_1, D_2, \dots, D_{\mathfrak{k}}$ such that $R \cong \mathbb{M}_{\mathfrak{n}_1}(D_1) \times \mathbb{M}_{\mathfrak{n}_2}(D_2) \times \dots \times \mathbb{M}_{\mathfrak{n}_{\mathfrak{k}}}(D_{\mathfrak{k}})$;
- (iii) R has identity and every unital right R -module is semisimple;
- (iv) R has identity and R_R is isomorphic to a direct sum of finitely many simple right R -modules.

Moreover, assertion (ii) is equivalent to the left analogues of each of assertions (i), (iii) and (iv). □

Because of the left-right symmetry of the Artin-Wedderburn Theorem, it is customary to omit the prefix “right” in “semiprimitive right artinian” and to speak simply of a “semiprimitive artinian” ring. We warn the reader that the requirement that R have identity cannot be dispensed with in assertions (iii) and (iv) of Theorem 1; it is not difficult to construct rings R (necessarily without identity) with the property that $J(R) \neq 0$ yet R_R is semisimple artinian. In other words, for arbitrary rings, the conditions “semiprimitive artinian” and “semisimple artinian” are not equivalent; the former being stronger than the latter.

Recall that a subring R of a ring T is said to be a *right order* (resp. a *left order*) in T if, given any $t \in T$, there exists a unit u of T , contained in R , such that $tu \in R$ (resp. $ut \in R$).

LEMMA 2. [Row88, Lemma 3.1.10, p353] *The following assertions are equivalent for a subring R of a ring T :*

- (i) R is a right order in T ;
- (ii) given any finite nonempty subset X of T , there exists a unit u of T , contained in R , such that $Xu := \{xu : x \in X\} \subseteq R$. □

LEMMA 3. *If R is a right order in a ring T then $M_n(R)$ is a right order in $M_n(T)$ for all finite nonzero cardinals n .*

Proof. Suppose $0 < n < \aleph_0$ and choose $A \in M_n(T)$. Since R is a right order in T , there exists, by Lemma 2, a unit u of T , contained in R , such that $A_{\alpha\beta}u \in R$ for all $\alpha, \beta \in n$. Let $B \in M_n(T)$ denote the scalar matrix with main diagonal entry u . Certainly, B is a unit of $M_n(T)$ contained in $M_n(R)$. Moreover, it is not difficult to see that $AB \in M_n(R)$. Thus $M_n(R)$ is a right order in $M_n(T)$. □

Recall that a right R -module M is said to be *uniform* if every nonzero submodule of M is an essential submodule of M . More generally, if m is an arbitrary cardinal, then M is said to have *Goldie dimension* m if m is the supremum of all cardinals n such that M has an independent family of n nonzero submodules; we abbreviate this by writing $\dim M = m$. Note that M is nonzero and uniform if and only if $\dim M = 1$, and that $\dim 0 = 0$. It is easily shown that the Goldie dimension of a vector space coincides with its dimension in the classical sense. In [DF88, Theorem 6, p300], Dauns and Fuchs prove that if $\dim M = m$ is not an inaccessible cardinal then $\dim M$ is “attained” in M in the sense that M actually contains an independent family of m nonzero submodules. This fact extends the well known result that if a right R -module M contains no infinite independent family of nonzero submodules then $\dim M = n$ for some $n < \aleph_0$. The *right Goldie dimension* of a ring R refers to $\dim R_R$. Recall that a ring is said to satisfy the *ACC on right annihilators* if the poset $\{(0 :_r X) : X \subseteq R\}; \subseteq$ satisfies the ACC. Recall furthermore that a ring R is said to be a *right Goldie ring* if $\dim R_R < \aleph_0$ and R satisfies the ACC on right annihilators.

THEOREM 4. (Goldie's Second Theorem) [Her69, Theorem 4.7, p75 & Theorem 4.5, p70]

The following assertions are equivalent for a ring R :

- (i) R is isomorphic to a right order in a semiprimitive artinian ring;
- (ii) R is a semiprime right Goldie ring. □

THEOREM 5. (Goldie's First Theorem) [Her69, Theorems 4.4 & 4.5, p70] Let \mathfrak{n} be a finite nonzero cardinal. Then the following assertions are equivalent for a ring R :

- (i) R is isomorphic to a right order in $M_{\mathfrak{n}}(D)$ for some division ring D ;
- (ii) R is isomorphic to a right order in a simple artinian ring and $\dim R_R = \mathfrak{n}$;
- (iii) R is a prime right Goldie ring with $\dim R_R = \mathfrak{n}$. □

COROLLARY 6. The following assertions are equivalent for a domain R :

- (i) R is isomorphic to a right order in a division ring;
- (ii) $\dim R_R = 1$, i.e., R_R is uniform;
- (iii) $\dim R_R$ is finite. □

A domain which satisfies the equivalent conditions of Corollary 6 is called a *right Ore domain*.

THEOREM 7. (Faith-Utumi Theorem) [Lam66, Proposition, p114] Let \mathfrak{n} be a finite nonzero cardinal and D a division ring. Then the following assertions are equivalent for a ring R :

- (i) R is a right order in $M_{\mathfrak{n}}(D)$;
- (ii) there exists a right order T in D such that $M_{\mathfrak{n}}(T) \subseteq R \subseteq M_{\mathfrak{n}}(D)$ (as subrings). □

§8. RINGS OF FRACTIONS.

For the reader's convenience, we recall the basic definitions associated with rings of fractions. Suppose S is a multiplicatively closed set of regular elements of a ring R (a nonempty subset S of a ring R is said to be *multiplicatively closed* if $st \in S$ whenever $s, t \in S$). Let T be a ring with identity and $\varphi: R \rightarrow T$ a ring monomorphism. The pair (φ, T) is said to be a *right* (resp. *left*) *ring of fractions of R with respect to S* if the following two conditions are met:

RF1. $\varphi(s)$ is a unit of T for every $s \in S$;

RF2. Every $t \in T$ is expressible in the form $t = \varphi(r)\varphi(s)^{-1}$ (resp. $t = \varphi(s)^{-1}\varphi(r)$) for some $r \in R$ and some $s \in S$.

[It is customary in this situation to identify each $r \in R$ with its image in T and to write $t = rs^{-1}$ (resp. $t = s^{-1}r$).]

If (φ, T) is a right ring of fractions of R with respect to S and the monomorphism φ is

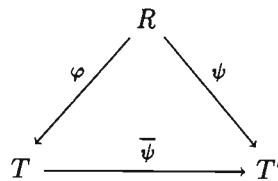
understood, it is also customary to refer to the ring T as a right ring of fractions of R with respect to S .

A *right denominator set* (resp. *left denominator set*) for a ring R is any multiplicatively closed set S of regular elements of R having the *common right multiple property* (resp. *common left multiple property*): for each $r \in R$ and each $s \in S$, there exist $r_1 \in R$ and $s_1 \in S$ such that $rs_1 = sr_1$ (resp. $s_1r = r_1s$). Use of the term “denominator” in the above definition is justified by the following theorem.

THEOREM 1. [Row88, Proposition 3.1.3 & Theorem 3.1.4, p349] *The following assertions are equivalent for a multiplicatively closed set S of regular elements of a ring R :*

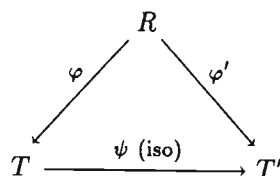
- (i) S is a right denominator set for R ;
- (ii) there exists a right ring of fractions of R with respect to S . □

THEOREM 2. (Universal Property) [Row88, Theorem 3.1.6, p352] *Let S be a right denominator set for a ring R and (φ, T) a right ring of fractions of R with respect to S . Suppose T' is a ring and $\psi: R \rightarrow T'$ a ring homomorphism such that $\psi(s)$ is a unit of T' for all $s \in S$. Then there exists a unique ring homomorphism $\bar{\psi}: T \rightarrow T'$ which makes the following diagram commute. Moreover, $\bar{\psi}$ is given by $\bar{\psi}(rs^{-1}) = \psi(r)\psi(s)^{-1}$ for all $rs^{-1} \in T$.*



□

It is, of course, a consequence of Theorem 2 that if S is a right denominator set for a ring R then a right ring of fractions of R with respect to S is unique in the sense that if (φ, T) and (φ', T') are any two right rings of fractions of R with respect to S , then there exists an isomorphism $\psi: T \rightarrow T'$ which makes the following diagram commute:



For each ring R , and each right denominator set S for R , we choose a fixed right ring of fractions of R with respect to S and denote it by (φ_S, RS^{-1}) . Note that if R is identified with its image in RS^{-1} , we obtain $RS^{-1} = \{rs^{-1} : r \in R, s \in S\}$.

There is an important connection between rings of fractions and right orders. It is obvious that the ring R is a right order in RS^{-1} whenever S is a right denominator set for R . It is perhaps less obvious, but is nevertheless easily established, that if R is a right order in a ring T then T is a right ring of fractions of R with respect to the right denominator set S consisting of all units of T contained in R .

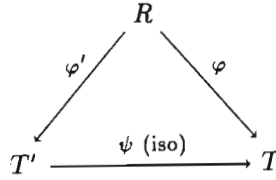
We shall now describe some important special cases. If the set S of all regular elements of a ring R is a right denominator set for R , we call RS^{-1} the *classical right ring of quotients* of R . Of course, not every ring possesses a classical right ring of quotients. It is easily checked that if R is a domain then $S = R \setminus \{0\}$ is a right denominator set for R if and only if R_R is uniform. It follows from Corollary 7.6, therefore, that a domain R has a classical right ring of quotients if and only if it is a right Ore domain. If R is a commutative ring then clearly every multiplicatively closed set of regular elements of R is a right and left denominator set for R . In this situation, we omit the prefixes “left” and “right” and speak simply of a “ring of fractions of R with respect to S ”. A particularly important special case arises if $S = R \setminus P$, where P is a prime ideal of a commutative domain R . In this case, it is customary to write R_P instead of RS^{-1} and to refer to R_P as the *localization of R at P* . (The term “localization” is appropriate here, since R_P is always a local ring with unique maximal proper ideal $P_P := \{ps^{-1} : p \in P, s \in S = R \setminus P\}$.) Note that if $P = 0$ then R_P is just the *field of quotients of R* .

We remind the reader of the definition of the so-called “maximal” ring of quotients. Suppose R and T are rings with $\varphi: R \rightarrow T$ a ring monomorphism. We shall call the pair (φ, T) a *right rational extension* of R if R_R is a dense submodule of T_R (here, R is identified with its image in T). (Recall that if $M \in \text{Mod-}R$ then a submodule N of M is said to be a *dense* submodule of M if, given $x, y \in M$ with $x \neq 0$, there exists $r \in R$ such that $xr \neq 0$ and $yr \in N$.) If, in this case, the monomorphism φ is understood, we call the ring T a *right rational extension* of R . It is easy to see, for example, that if R is a right order in T then T is a right rational extension of R . A right rational extension (φ, T) of a ring R is called a *maximal right ring of quotients* of R if, given any right rational extension (φ', T') of R , there exists a unique ring homomorphism $\psi: T' \rightarrow T$ which makes the following diagram commute.

$$\begin{array}{ccc}
 & R & \\
 \varphi' \swarrow & & \searrow \varphi \\
 T' & \xrightarrow{\psi} & T
 \end{array}$$

Since T'_R is an essential extension of R_R , it is not difficult to see that the homomorphism ψ in the above diagram must in fact be a monomorphism. A maximal right ring of quotients of R is thus, loosely speaking, a largest right rational extension of R . It follows from the definition that a maximal right ring of quotients of R , if it exists, is unique in the predictable sense: if (φ, T) and (φ', T') are

any two maximal right rings of quotients of R then there exists an isomorphism $\psi: T' \rightarrow T$ which makes the following diagram commute:



THEOREM 3. [Lam66, Proposition 8, p99] *Every ring with identity possesses a maximal right ring of quotients with identity.* □

Observe that if R is a ring with identity and (φ, T) a maximal right ring of quotients of R then T_R , being an essential extension of R_R , must embed in $E(R_R)$. Thus each right ring of quotients of R may be identified with a submodule of $E(R_R)$; we choose a fixed maximal right ring of quotients of R and denote it by $(\varphi_{\max}, Q_{\max}(R_R))$.

§9. VON NEUMANN REGULAR RINGS AND RELATED CONCEPTS.

Recall that a ring R is said to be *von Neumann regular* (or just *regular*) if for each $a \in R$, there exists $b \in R$ such that $aba = a$.

THEOREM 1. [Ker87, Theorems 13.8 & 13.9, p60] *The following assertions are equivalent for a ring R :*

- (i) R is regular;
- (ii) every principal right ideal of R is generated by an idempotent;
- (iii) every finitely generated right ideal of R is generated by an idempotent.

Moreover, assertion (i) is equivalent to each of assertions (ii) and (iii) with “right” replaced by “left”. □

THEOREM 2. *If R is a regular ring then:*

- (i) every right ideal and every left ideal of R is idempotent;
- (ii) the lattice $\langle \text{Id } R; \subseteq \rangle$ is distributive. □

Some explanation is in order here. The first assertion of Theorem 2 is well known and its easy proof does not require R to have identity. The second assertion is also well known, we believe, but a convenient reference in the setting of rings without identity eluded us. The following argument justifies it. An ideal I of a ring R is called *semiprime* if the ring R/I is a semiprime ring (see Proposition 5.2). The set $\mathfrak{S}(R)$ of all semiprime ideals of an arbitrary ring R is easily seen to be a closure system

on the complete lattice of subsets of R (ordered by set inclusion) and is, as such, a complete lattice in its own right, ordered by \subseteq . (It need not be an algebraic lattice in general.) Tominaga [Tom54, Theorem 2, p140] proved that $(\mathfrak{S}(R); \subseteq)$ is always a distributive lattice, and his proof makes no use of an identity element for R . If R is regular, however, it follows easily from Theorem 2(i) and Proposition 5.2 that $\mathfrak{S}(R) = \text{Id } R$. Consequently, for regular rings, (ii) is true.

We define a *left full linear ring* to be the ring $\text{End}_D V$ of all linear transformations (written on the right of their arguments) of any left vector space V over any division ring D . If the vector space V is finite dimensional, we call $\text{End}_D V$ a *finite dimensional (left) full linear ring*. A classical result of linear algebra asserts that if $\dim_D V = \mathfrak{m}$ then $\text{End}_D V$ (endomorphisms on the right) $\cong M_{\mathfrak{m}}(D)$. If \mathfrak{n} is a cardinal and $\varphi \in \text{End}_D V$, we say that φ has *rank* \mathfrak{n} , and write $\text{rank } \varphi = \mathfrak{n}$, if $\dim_D (\text{Im } \varphi) = \mathfrak{n}$.

PROPOSITION 3. [Jac56, Theorem 1, p93] *Let V be a left vector space of dimension $\mathfrak{m} \geq \aleph_0$ over a division ring D . Then I is a nonzero proper ideal of $\text{End}_D V$ (endomorphisms on the right) if and only if $I = \{\varphi \in \text{End}_D V : \text{rank } \varphi < \mathfrak{n}\}$ for some infinite cardinal $\mathfrak{n} \leq \mathfrak{m}$. In particular, the lattice of ideals of $\text{End}_D V$ is well ordered, i.e., order isomorphic to an ordinal.* □

Recall that a ring R is said to be *right self-injective* if the module R_R is injective.

THEOREM 4. [Goo79, Theorem 9.12, p100] *The following assertions are equivalent for a ring R with identity:*

- (i) R is isomorphic to a left full linear ring;
- (ii) R is prime, regular, left self-injective and $\text{soc } R \neq 0$. □

(It is not necessary to specify $\text{soc}(R_R)$ or $\text{soc}({}_R R)$ in (ii) above, in view of the well known fact that the left and right socles of a semiprime ring with identity coincide: see [Lam66, Proposition 4, p63].)

Recall that if $M \in \text{Mod-}R$ then the *singular submodule* of M , denoted $Z(M)$, is defined by

$$Z(M) = \{x \in M : (0:x) \text{ is an essential right ideal of } R\};$$

M is called *singular* if $Z(M) = M$ and *nonsingular* if $Z(M) = 0$. A ring R is said to be *right singular* (resp. *right nonsingular*) if the module R_R is singular (resp. nonsingular). (Of course, a ring with identity cannot be singular.)

PROPOSITION 5. [Goo79, Corollary 1.24, p12] *If R is a right nonsingular ring with identity then $Q_{\max}(R_R)$ is regular and right self-injective.* □

THEOREM 6. (Sandomierski's Theorem) [Goo76, Theorem 3.17, p83] *Let R be a ring with identity. Then the following assertions are equivalent:*

- (i) $\dim R_R$ is finite and R is right nonsingular;
- (ii) $Q_{\max}(R_R)$ is a semiprimitive artinian ring. □

§10. SEMIGROUPS, SEMIGROUP RINGS AND POWER SERIES RINGS.

We remark at the outset that in the context of semigroup and power series rings, we shall always adopt the multiplicative notation for (even commutative) semigroups. In accordance with this convention, we shall abbreviate products by juxtaposition. The identity element of a monoid H will always be denoted 1_H . We start by recalling the definitions of several classes of semigroups typically associated with semigroup rings.

Let $\langle H; \cdot \rangle$ be a semigroup. We say that H is *cancellative* if for any $x, y, z \in H$, we have $x = z$ whenever $xy = zy$ or $yx = yz$. We shall call H *linearly ordered* (resp. *strictly ordered*) if H admits a linear order \leq such that whenever $x, y, z \in H$ with $x < y$, we have $xz \leq yz$ and $zx \leq zy$ (resp. $xz < yz$ and $zx < zy$). We denote both structures by $\langle H; \cdot; \leq \rangle$. Of course, a strictly ordered semigroup is linearly ordered, while a linearly ordered cancellative semigroup is strictly ordered.

If X is a set, we shall denote by $\langle X \rangle$ the *free monoid on X* and by $\langle X, X^{-1} \rangle$ the *free group on X* . If $X = \{x\}$ is a singleton, we write $\langle x \rangle$ instead of $\langle \{x\} \rangle$ and $\langle x, x^{-1} \rangle$ instead of $\langle \{x\}, \{x\}^{-1} \rangle$. For the reader's convenience, we provide a rough sketch showing how the free group and free monoid on a set are constructed. (A more precise account may be found in [Rot88, Chapter 12].) Let X be a set. We form the set \hat{X} consisting of all formal "syllables" of the form x^n , where $x \in X$ and $n \in \mathbb{Z}$. Consider the set $W(\hat{X})$ of all "words" in \hat{X} (that is, finite sequences of elements of \hat{X}). Included is the empty word, which we denote by 1. A word $w \in W(\hat{X})$ is said to be in *reduced form* if $w = 1$ or $w = x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$, where k is a positive integer, the n_i are all nonzero integers and $x_i \neq x_{i+1}$ for all $i \in \{1, \dots, k-1\}$. Then the *free group on X* , $\langle X, X^{-1} \rangle$, has as its elements all words in reduced form. The monoid operation on $\langle X, X^{-1} \rangle$ is, loosely speaking, concatenation of words: if $w_1, w_2 \in \langle X, X^{-1} \rangle$, the product $w_1 \cdot w_2$ is obtained by juxtaposing w_1 and w_2 and then replacing each occurrence of $x^m x^n$ in $w_1 \cdot w_2$ by x^{m+n} and deleting each resultant occurrence of x^0 in $w_1 \cdot w_2$. If the set X is regarded as a subset of $\langle X, X^{-1} \rangle$ (identifying each $x \in X$ with the word x^1), the elements of X are called, variously, "indeterminates", "symbols" and "letters". The *free monoid on X* , $\langle X \rangle$, is the submonoid of $\langle X, X^{-1} \rangle$ consisting of 1 and all reduced words of the form $x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$, where k, n_i are positive integers for all $i \in \{1, \dots, k\}$. It is well known that $\langle X, X^{-1} \rangle$ may be ordered in such a way that $\langle X, X^{-1} \rangle$, and hence also $\langle X \rangle$, has the structure of a strictly ordered semigroup.

Recall that if H is a semigroup and R a ring then the *semigroup ring in H over R* , denoted RH , is defined to be the set of all formal "sums" of the form

$$\sum_{h \in H} r_h h \quad (r_h \in R \text{ for all } h \in H)$$

which have finite support. (If $x = \sum_{h \in H} r_h h$ is a formal sum, we define the *support* of x , abbreviated $\text{supp } x$, to be the support of $\{r_h\}_{h \in H} \in R^H$.) Addition and multiplication in RH are defined as follows: if $x = \sum_{h \in H} r_h h$ and $y = \sum_{h \in H} s_h h$ are elements of RH then

$$\begin{aligned} x + y &= \sum_{h \in H} (r_h + s_h)h, \text{ and} \\ xy &= \sum_{h \in H} t_h h, \end{aligned}$$

where $t_h = \sum \{r_{h'} s_{h''} : h', h'' \in H \text{ and } h'h'' = h\}$ for all $h \in H$. If $r \in R$ and $g \in H$, it is standard to use rg to denote the element $x = \sum_{h \in H} r_h h$ of RH for which $\text{supp } x \subseteq \{g\}$ and $r_g = r$. Clearly, every $x \in RH$ is expressible as a finite sum of elements of the form rg ($r \in R$, $g \in H$). If the ring R has identity then the map $h \mapsto 1_R h$ defines a semigroup monomorphism from H into $\langle RH; \cdot \rangle$. This allows us to regard H as a subsemigroup of $\langle RH; \cdot \rangle$. If $x = \sum_{h \in H} r_h h \in RH$, we call each r_h the *coefficient of h* in x . If H is a monoid with identity element 1_H and $x \in RH$ then we call the coefficient of 1_H in x the *constant term* of x . Also, if H is a linearly ordered semigroup and $0 \neq x \in RH$, we call the coefficient of the largest element in $\text{supp } x$ the *leading coefficient* of x . If H is a monoid (resp. a group) then RH is called the *monoid* (resp. *group*) *ring in H over R* ; in this case the map $r \mapsto r1_H$ defines a ring monomorphism from R into RH . This allows us to regard R as a subring of RH .

PROPOSITION 1. [Row88, Proposition 1.2.17, p46] *If D is a domain and H is a strictly ordered semigroup then DH is a domain.* □

We point out to the reader that if D is a domain and $\langle X \rangle$ denotes the free monoid on an arbitrary set X , then it is always possible to define a “degree function” $\partial: D\langle X \rangle \setminus \{0\} \rightarrow \mathbb{Z}$ which extends the classical degree function associated with polynomial rings: for each $f \in D\langle X \rangle \setminus \{0\}$, define ∂f to be the maximum number of occurrences of indeterminates from X in any single summand of f , this maximum being taken over all summands of f . (For example, $xy^2x^3z + 2xz^2y \in \mathbb{R}\langle x, y, z \rangle$ has degree 7.) This degree function satisfies the usual properties, viz., for all $f, g \in D\langle X \rangle \setminus \{0\}$,

$$\begin{aligned} \partial(f + g) &\leq \max\{\partial f, \partial g\} \text{ if } f + g \neq 0; \text{ and} \\ \partial(fg) &= \partial f + \partial g.^2 \end{aligned}$$

We also remark that if D is a domain and X a set such that $|X| \geq 2$, then $D\langle X \rangle$ is a domain which is neither left nor right Ore. Indeed, if x and y are any two distinct indeterminates from X then $xD \cap yD = 0 = Dx \cap Dy$. If F is a field and X an arbitrary set then $F\langle X \rangle$ is called the *free (associative) F -algebra on X* .

The reader will have noticed that if $X = \{x\}$ is a singleton then $R\langle X \rangle = R\langle x \rangle$ is just the *polynomial ring in x over R* , which we shall write as $R[x]$. More generally, if H is the free

²It is often useful to extend the domain of ∂ by assigning a degree of $-\infty$ to the zero element of $D\langle X \rangle$.

commutative monoid on a finite set $X = \{x_1, x_2, \dots, x_n\}$ of indeterminates (the *free commutative monoid on a set X* can be obtained by modifying, in the obvious fashion, the construction of the free (noncommutative) monoid $\langle X \rangle$, described earlier) then RH coincides with the *polynomial ring in (commuting) indeterminates x_1, x_2, \dots, x_n over R* ; we denote this ring by $R[x_1, x_2, \dots, x_n]$. It follows from Proposition 1 that $D[x_1, x_2, \dots, x_n]$ is a commutative domain whenever D is a commutative domain. In particular, if F is a field then the commutative domain $F[x_1, x_2, \dots, x_n]$ has a field of quotients which will be denoted by $F(x_1, x_2, \dots, x_n)$; it is customary to refer to $F(x_1, x_2, \dots, x_n)$ as the *field of rational functions in (commuting) indeterminates x_1, x_2, \dots, x_n over F* .

Let R be a ring and H a strictly ordered semigroup. Recall that the (*formal*) *power series ring in H over R* , denoted $R[[H]]$, is defined to be the set of all formal “series” of the form

$$x = \sum_{h \in H} r_h h \quad (r_h \in R \text{ for all } h \in H),$$

where $\text{supp } x$ is either \emptyset or a well ordered subset of $\langle H; \leq \rangle$. Addition and multiplication are defined as in RH ; the fact that the nonzero elements of $R[[H]]$ have well ordered support ensures that multiplication is well defined. Note that if $X = \{x\}$ is a singleton then $R[\langle X \rangle] = R[\langle x \rangle]$ is just the *power series ring in x over R* , which we shall write as $R[[x]]$.

PROPOSITION 2. [Row88, Corollary 1.2.23, p49] *If D is a domain and H a strictly ordered semigroup then $D[[H]]$ is a domain.* □

PROPOSITION 3. [Row88, Proposition 1.2.24, p49] *Suppose D is a domain with identity and H is a strictly ordered monoid. Then the following assertions are equivalent for a nonzero element $x = \sum_{h \in H} r_h h$ in $D[[H]]$:*

- (i) x is a unit of $D[[H]]$;
- (ii) if g is the smallest element in $\text{supp } x$ then r_g is a unit of D and g an invertible element of H . □

We shall now recall the definition of a “skew semigroup ring”, a “skew polynomial ring” and a “skew power series ring”. For convenience, we introduce some general notation first. If A is an algebraic structure (i.e., a nonempty set on which a set of operations and/or a set of relations have been specified; for the most part, A will be a ring or semigroup), we shall denote by $\text{Mon } A$ the monoid (under composition) of all monomorphisms from A to itself. Such maps must preserve all specified operations *and relations* on A . A semigroup Θ is said to *act* on A if Θ is isomorphic to a subsemigroup of $\text{Mon } A$.

Let R be a ring and Θ a semigroup which acts on R . The *right skew semigroup ring in Θ over R* , denoted $R[\Theta]$, is defined to be the set of all formal sums of the form $\sum_{\alpha \in \Theta} r_\alpha \alpha$ ($r_\alpha \in R$) with finite support. Addition is defined as in the semigroup ring $R\Theta$ and multiplication is defined

distributively, using the rule

$$\alpha \cdot r = (\alpha(r))\alpha \quad (\alpha \in \Theta, r \in R).$$

If Θ is a monoid (resp. group) which acts on R then $R[\Theta]$ is called a *right skew monoid ring* (resp. *right skew group ring*) in Θ over R . The *left skew semigroup ring* in Θ over R is defined, simply, as the opposite ring of $R[\Theta]$.

If R is a ring and $\sigma \in \text{Mon } R$, we define $R[x, \sigma]$ to be the ring consisting of all polynomials in x over R with addition defined as in $R[x]$ but with multiplication defined distributively, using the rule

$$x \cdot r = (\sigma(r))x \quad (r \in R).$$

We call $R[x, \sigma]$ a *right skew polynomial ring* in x over R . The reader will observe that if σ has infinite order and Θ is the (infinite) submonoid of $\text{Mon } R$ generated by σ (i.e., $\Theta = \{\sigma^k : k \geq 0\}$), then $R[x, \sigma]$ is just the right skew semigroup ring $R[\Theta]$.

PROPOSITION 4. *Let F be a field and $\sigma : F \rightarrow F$ a field monomorphism. Then:*

- (i) [Row88, Proposition 1.6.21, p94] $F[x, \sigma]$ is a principal left ideal domain;
- (ii) [Row88, Proposition 1.6.22, p95] if σ is not onto then $F[x, \sigma]$ is a left but not right Ore domain. □

If R is a ring and $\sigma \in \text{Mon } R$, we define $R[[x, \sigma]]$ to be the ring consisting of all formal power series in x over R with addition defined as in $R[[x]]$ and with multiplication defined distributively as in $R[x, \sigma]$. We call $R[[x, \sigma]]$ a *right skew power series ring* in x over R .

PROPOSITION 5. [Coh85, Theorem 4.14, p176] *Let F be a field and $\sigma : F \rightarrow F$ a field monomorphism. Define $D = F[[x, \sigma]]$. Then:*

- (i) every nonzero element r in D can be written in the form $r = ux^k$ for some unit u of D and nonnegative integer k . Hence every nonzero left ideal of D is two-sided and of the form Dx^k for some nonnegative integer k . Thus D is a local domain with unique maximal proper ideal Dx ;
- (ii) if σ is not onto then D is a left but not right Ore domain. □

It follows from the above result that the lattice of left ideals of the ring $D = F[[x, \sigma]]$ is linearly ordered by set inclusion; a ring with this property is called a *left chain ring*. Chain rings are the main topic of Chapter I. It is also clear from the description of ideals in D that Dx is the only nonzero prime ideal of D .

Chapter I

Right Chain Domains and Algebraic Chains

This chapter is concerned, in the main, with the question: how much freedom do we have to prescribe the ideal or right ideal lattice structure of a right chain ring? Our first aim is to prove the following representation theorem: *every algebraic chain is isomorphic to the ideal (i.e., congruence) lattice of a ring*. More precisely, we shall prove in Theorem 5.1 that the following conditions on an arbitrary chain L are equivalent:

- (i) L is algebraic (as a lattice);
- (ii) there is a right chain domain T , with identity, such that L is isomorphic to the chain of all proper (two-sided) ideals of T and all ideals of T are idempotent;
- (iii) L is isomorphic to the congruence lattice of a ring (not necessarily with identity).

(Here, a *right chain ring* means a ring whose lattice of right ideals is linearly ordered by set inclusion.)

Secondly, we shall prove in Theorem 5.3 that when constructing (as in Theorem 5.1) a right chain domain T with identity having any preassigned algebraic chain L of proper ideals (all idempotent), we may also require the chains of right ideals strictly between covering pairs of proper ideals of T to contain dually cofinal copies of further preassigned chains; these latter chains must be unbounded above, but may otherwise be quite arbitrary.

Both results seemed surprising to us; one might have expected the rich language and the relatively strong axioms of rings (in particular, right chain domains with identity) to militate against so general a pair of representation theorems. Some account of the motivation for these results and their position and significance in this dissertation is called for.

The investigation culminating in the above results grew out of a need to construct non-commutative chain rings, and the substance of this chapter is largely a development of techniques for the construction of such rings. The results that emerge are, we believe, of interest in their own right and inasmuch as they produce domains, they contribute to our understanding of prime rings, a major theme of this thesis. Our greatest need for these techniques, however, will arise in Chapter II. There, a natural class of "right \mathfrak{m} -closed rings" (\mathfrak{m} being a given uncountable cardinal) will be introduced and studied. This class contains all right artinian rings and it was necessary to show that it does not consist entirely of right artinian rings. It seemed to us, in trying to establish this, that the most economical and accessible counter-example (i.e., example of a right non-artinian right \mathfrak{m} -closed ring) would, if it exists, be a right chain ring R having just one nontrivial (two-sided) ideal, with the further property that the dual of the chain of nonzero right ideals of R contain a cofinal order isomorphic copy of the cardinal \mathfrak{m} (considered as a well ordered set in the usual way).

No constructions of such rings seemed to be to hand in the literature, although the theory of noncommutative chain rings has developed considerably in the last decade. (The interested reader should consult, e.g., the recent works of C. Bessenrodt, H.H. Brungs, N.I. Dubrovin, M. Fest, M. Schröder, G. Törner and A.R. Wadsworth.) There are several motivations for the study of right chain rings: according to Brungs ([Bru80],[Bru]) they “can be considered as noncommutative valuation rings and they occur (in geometry) as coordinate rings of Hjelmslev planes [Kli54], as localizations of rings with a distributive lattice of right ideals [Bru76], and in the construction of division rings that are not crossed products [Ami72],[JW86]”. Also, much of the activity in this field of study has been driven by a difficult open problem: *must a prime right chain ring be a domain?*

Our own investigations, apart from yielding a ring as prescribed by our needs in Chapter II, led to a proof of the much more general representation theorem stated at the beginning of this introduction, which is of independent ring theoretic interest. This consideration aside, the result also turns out to be of some interest from the point of view of universal algebra. For these two reasons, it seemed more efficient to present the result at the beginning of the thesis (most of the balance of which has a more consistently ring theoretic flavour), rather than take a lengthy detour at a later stage.

This chapter commences, therefore, with a summary of the very few universal algebraic prerequisites on which the presentation and the proofs of the main results depend. This account is given in §1 below, while §2 to §5 constitute a linear development of construction techniques culminating in the main theorems. In §5, we show also that as far as the representation of chains by rings is concerned, our results are, in all natural senses apparent to us, the best possible.

The universal algebraic motivations for Theorem 5.1 go somewhat deeper than the content of §1, however, and deserve some mention. Rather than delay or interrupt the proof of the main theorem, we defer a detailed discussion of its broader algebraic significance until the end of this chapter (§6). Nothing in the sequel to this chapter will depend on the universal algebraic discourse that occurs in §6.

In what follows, we assume the reader has had no previous exposure to universal algebra. Most of the main results of this chapter have been published in the paper [VR93].

§1. ALGEBRAIC PRELIMINARIES.

Recall that a *congruence relation* (or just a *congruence*) on a ring $\langle R; +, \cdot, -, 0 \rangle$ is an equivalence relation θ on the set R which is compatible with the binary operations $+$ and \cdot , with the unary operation ‘ $-$ ’ and with the nullary operation (or constant) 0 . (In fact, the last condition is redundant since it means no more than that $(0, 0) \in \theta$, which is true of any equivalence relation on R .) The set of all congruence relations on a ring R is a complete lattice when ordered by set inclusion. We shall denote this lattice by $\text{Con } R$.

One reason why ring theorists tend to ignore congruences on a ring R is that their theory is eclipsed by that of the ideals of R . Indeed, if $\theta \in \text{Con } R$ then the equivalence class of θ containing 0 , which we denote by $0/\theta$, is an ideal of R which determines θ completely and every ideal of R arises in this way as the 0 -class of a congruence on R . More strongly, the association $\theta \mapsto 0/\theta$ ($\theta \in \text{Con } R$) defines a lattice isomorphism from $\text{Con } R$ onto $\text{Id } R$, which means that within ring theory, no harm can come of working entirely with ideals and neglecting to consider congruences altogether. In exactly the same way, the congruences of groups are eclipsed by the normal subgroups, and all of this is very well known. An advantage of the notion of congruence, however, is that it may be generalized effortlessly to universal algebras of any type (and the generalization serves a useful purpose, because it allows us to form factor algebras). We remind the reader that a *similarity type* (briefly, a *type*) is a set F , whose elements are called *operation symbols*, together with a function ar from F to ω , the set of nonnegative integers. (The term *language* is a common synonym for “type” in the literature.) The function ar is called the *arity function*; if $f \in F$ then $ar(f)$ may be thought of as the number of elements that f (or, more precisely, a suitable interpretation of f as an operation on some set) should operate on. In particular, in the case of rings, $ar(+)=2=ar(\cdot)$; $ar(-)=1$ and $ar(0)=0$. Briefly, a universal algebraist would describe a ring (without identity) as an algebra of type $\langle 2, 2, 1, 0 \rangle$, satisfying suitable axioms. More generally, a (*universal*) *algebra of type* $\langle F; ar \rangle$ is a pair $\langle A; F^A \rangle$, where A is a nonempty set (called the *universe* of the algebra) and $F^A = \{f^A : f \in F\}$ is a family of finitary operations on A , indexed by F , such that for each operation symbol $f \in F$ with $ar(f) = n$, say, the operation f^A is n -ary, i.e., it is a function from A^n to A . We often abbreviate f^A as f in the absence of any possible confusion.

A *congruence* on such an algebra A is an equivalence relation θ on the set A , which is compatible with all of the operations in F^A (in other words, for each $f \in F$ with $ar(f) = n$, say, and for any $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in A$, we have $(f^A(a_1, a_2, \dots, a_n), f^A(b_1, b_2, \dots, b_n)) \in \theta$ whenever $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n) \in \theta$). In this case, the set A/θ of all equivalence classes a/θ of elements $a \in A$ under θ may be given, in a thoroughly natural way, the structure of an algebra of the same type as A , which we call a *factor algebra* of A . Indeed, for each $f \in F$ with $ar(f) = n$, say, and for any $a_1, a_2, \dots, a_n \in A$, we may define, unambiguously,

$$f^{A/\theta}(a_1/\theta, a_2/\theta, \dots, a_n/\theta) = [f^A(a_1, a_2, \dots, a_n)]/\theta.$$

The congruences on A form a complete lattice when ordered by set inclusion. We denote this lattice by $\text{Con } A$ and we call it the *congruence lattice* of A .

There is no similarly general analogue of the notion of an ideal for arbitrary algebras. At the very least, it seems that one has to insist that the type of the algebra under consideration possess a constant symbol. If this is the case then ideals are definable, but the most suitable definition is complicated and the ring-like correspondence between ideals and congruences breaks down in many instances, e.g., the ideal and congruence lattices of a lattice with a least element are not, in general, isomorphic. (The reader interested in a general algebraic theory of ideals should consult the recent

works of A. Ursini and, in particular, [GU84].)

A right module M over a ring R may be considered as an algebra $\langle M; +, -, 0, \bar{\cdot} (r \in R) \rangle$ where for each $r \in R$, $\bar{\cdot}$ is the unary operation on M defined by $\bar{\cdot}(m) = mr$ ($m \in M$). The equivalence classes $0/\theta$, $\theta \in \text{Con } M$, are precisely the submodules of M (different submodules corresponding to different θ) and the association $\theta \mapsto 0/\theta$ preserves and reflects order. Thus the lattice of submodules of M , ordered by set inclusion, is isomorphic to $\text{Con } M$. In particular, the right ideal lattice $\text{Id } R_R$ of R is isomorphic to $\text{Con } R_R$.

We start with the following classical result, which we shall need. We remind the reader that an element x of a complete lattice $\langle L; \leq \rangle$ is called *compact* if, whenever $X \subseteq L$ and $x \leq \bigvee X$, there exists a finite subset X' of X such that $x \leq \bigvee X'$. Recall also that a complete lattice $\langle L; \leq \rangle$ is called *algebraic* if every element of L is the join of a set of compact elements of L .

THEOREM 1. (Birkhoff and Frink) [Grä79, Theorem 2, p52] *The congruence lattice $\text{Con } A$ of a (universal) algebra A (of any similarity type) is an algebraic lattice.* □

In view of the isomorphism between the ideal (resp. submodule) and congruence lattices of any ring (resp. module), we obtain as a special case of Birkhoff and Frink's Theorem the following well known facts about rings (with or without identity) already mentioned in Chapter 0.

COROLLARY 2. *For any ring R and any right R -module M , the ideal lattice $\text{Id } R$ of R and the submodule lattice of M are algebraic lattices. In particular, the right ideal lattice $\text{Id } R_R$ of a ring R is algebraic.* □

In particular, therefore, the main results of this chapter cannot be improved by weakening the requirement that the chain L be algebraic.

In effect, Theorem 5.1 will say that the theory of algebraic chains is encoded in the (second order) theory of rings, so that rings are a broad enough context for the study of algebraic chains. Put differently, *there is as much freedom as there possibly could be, not only in the ideal structure of rings with linearly ordered ideal lattices, but even in the ideal structure of the considerably narrower class of right chain rings without zero divisors.* This result is perhaps most surprising when viewed in the context of the universal algebraic results to be presented in §6, one of which says that it is *not* possible to realise all (or even just all modular) algebraic lattices as the congruence lattices of algebras of any *fixed* type. Also, as we have already remarked, one might quite reasonably have expected rings (and especially right chain domains) to be too "highly structured" for their ideal lattices to include the linearly ordered congruence lattices of *all* algebras.

Before proceeding with the construction of chain rings, we need to introduce several notions concerning semigroups and monoids. An *ideal* of a semigroup $\langle H; \cdot \rangle$ is a nonempty subset I of H

such that $ab, ba \in I$ whenever $a \in I$ and $b \in H$. The lattice of ideals, ordered by set inclusion, of a semigroup H will be denoted by $\mathfrak{M}H$. We denote by IJ the product $\{ab: a \in I, b \in J\}$ of two semigroup ideals $I, J \in \mathfrak{M}H$, while I^2 shall abbreviate II .

Most of the lattices L under consideration have, as elements, *nonempty* subsets of a particular structure, and are ordered by set inclusion (for example, the ideal lattice $\mathfrak{M}H$ of a semigroup H). At times, we shall want to augment such a lattice L (where $\emptyset \notin L$) by adding \emptyset as a new least element. The resulting lattice will be denoted by L_\emptyset . It is an obvious fact that L_\emptyset will be algebraic whenever L is algebraic. It is also easily checked that $(\mathfrak{M}H)_\emptyset$ is algebraic for any semigroup H .

We remind the reader that if A is an algebraic structure (i.e., a nonempty set on which a set of operations and/or a set of relations have been specified) then $\text{Mon } A$ denotes the monoid (under composition) of all monomorphisms from A to itself (see remarks following Proposition 0.10.3). Recall also that a semigroup Θ is said to *act* on the structure A if Θ is isomorphic to a subsemigroup of $\text{Mon } A$. If $\Theta \subseteq \text{Mon } A$ and $B \subseteq A$, we say that B is Θ -*invariant* provided that $\alpha[B] \subseteq B$ for every $\alpha \in \Theta$; we call B *absolutely Θ -invariant* if, in addition, for every $a \in A$ and every $\alpha \in \Theta$, we have $a \in B$ whenever $\alpha(a) \in B$. For the most part, we shall be concerned with a group Θ (acting on A). In this case, Θ consists of automorphisms of A ; also, since inverses are available in Θ , any Θ -invariant subset B of A is absolutely Θ -invariant, and this means that $\alpha[B] = B$ for all $\alpha \in \Theta$. It is easy to see that the absolutely Θ -invariant ideals of a semigroup H form a sublattice of $\mathfrak{M}H$ (for any Θ); if Θ is a group, the Θ -invariant ideals of a ring R form a sublattice of $\text{Id } R$. We denote these sublattices by $\mathfrak{M}_\Theta H$ and $\text{Id}_\Theta R$, respectively.

Recall that $\langle H; \cdot; \leq \rangle$ is called a *linearly ordered* (resp. *strictly ordered*) *semigroup* if $\langle H; \cdot \rangle$ is a semigroup, $\langle H; \leq \rangle$ is a linearly ordered set, and whenever $x, y, z \in H$ with $x < y$, we have $xz \leq yz$ and $zx \leq zy$ (resp. $xz < yz$ and $zx < zy$). If $\langle G; \cdot, ^{-1}, 1_G; \leq \rangle$ is a linearly ordered group then the *positive cone* of G is defined to be the linearly ordered monoid $\langle H; \cdot, 1_H; \leq \rangle$ where $H = \{g \in G: g \geq 1_G\}$. For our purposes, these will be the most useful linearly ordered monoids. If H is the positive cone of a linearly ordered group $\langle G; \cdot, ^{-1}, 1_G; \leq \rangle$, then the monoid ideals of $\langle H; \cdot, 1_H \rangle$ clearly coincide with the order filters of $\langle H; \leq \rangle$ (i.e., $\mathfrak{M}H = \mathfrak{F}H$) and are just the nonempty dually hereditary subsets of H (i.e., those nonempty subsets K of H for which $a \in K$ whenever $a \in H, b \in K$ and $a \geq b$), hence $\mathfrak{M}H$ is linearly ordered.

§2. CONSTRUCTING COMMUTATIVE CHAIN DOMAINS.

We recall here a technique for constructing commutative chain domains with identity. The technique is well known but since some of its less well known features will be important in what follows, we give a comprehensive account of it.

Let F be a field and H the positive cone of a linearly ordered abelian group. Let $R_0 =$

$F[[H]]$ be the (formal) power series ring in H over F . (Recall that $F[[H]]$ is defined to be the set of all formal “series” of the form

$$x = \sum_{h \in H} a_h h \quad (a_h \in F \text{ for all } h \in H),$$

where $\text{supp } x$ is either \emptyset or a well ordered subset of $\langle H; \leq \rangle$. Addition and multiplication in $F[[H]]$ are defined as follows: if $x = \sum_{h \in H} a_h h$ and $y = \sum_{h \in H} b_h h$ are elements of $F[[H]]$ then

$$\begin{aligned} x + y &= \sum_{h \in H} (a_h + b_h)h, \text{ and} \\ xy &= \sum_{h \in H} c_h h, \end{aligned}$$

where $c_h = \sum \{a_{h'}, b_{h''} : h', h'' \in H \text{ and } h'h'' = h\}$ for all $h \in H$: see §10 of Chapter 0.) For each monoid ideal $I \in \mathfrak{M}H$, we define IR_0 to be the set of all finite sums of products (in R_0) of the form $(1_F h)r$, where $h \in I$ and $r \in R_0$. (Henceforth we shall identify $1_F h$ and h in R_0 .) Clearly $IR_0 \in \text{Id } R_0$. We also adopt the convention that $\emptyset R$ is the zero ideal of R , for any ring R . We remarked earlier that since H is the positive cone of a linearly ordered group the monoid ideals of H coincide with the order filters of H . In particular then if $x \in H$ then the principal monoid ideal of H generated by x (i.e., the smallest monoid ideal of H containing x) is just $[x] := \{y \in H : y \geq x\}$. The map $x \mapsto [x]$ ($x \in H$) is clearly an order reversing bijection from H to the chain of principal monoid ideals of H , and the set of all principal monoid ideals is clearly cofinal in the dual $\langle \mathfrak{M}H; \supseteq \rangle$ of the chain $\mathfrak{M}H$. Thus $\mathfrak{M}H$ contains a dually cofinal copy of H .

Notice also that if an abelian group Θ acts on H then Θ also acts on the ring $\langle R_0; +, \cdot, 0, 1_{R_0} \rangle$: for each $\alpha \in \Theta$, it is easily checked that the map defined by $\sum a_h h \mapsto \sum b_h h$ (where $b_h = a_{\alpha^{-1}(h)}$ for all $h \in H$), which we shall also call α , is a ring automorphism of R_0 .

PROPOSITION 1. *Let $R_0 = F[[H]]$, where F is a field and $\langle H; \cdot, 1_H; \leq \rangle$ is the positive cone of a linearly ordered abelian group. Let Θ be an abelian group acting on the strictly ordered monoid H . Then:*

- (i) *the map $I \mapsto IR_0$ ($I \in (\mathfrak{M}H)_\emptyset$) is a lattice isomorphism from the chain of monoid ideals of H , augmented by \emptyset , onto the ideal lattice $\text{Id } R_0$ of the ring R_0 ;*
- (ii) *R_0 is a commutative chain domain (with identity);*
- (iii) *the restriction to $(\mathfrak{M}_\Theta H)_\emptyset$ of the map $I \mapsto IR_0$ is a lattice isomorphism from the chain of (absolutely) Θ -invariant monoid ideals of H , augmented by \emptyset , onto the chain $\text{Id}_\Theta R_0$ of (absolutely) Θ -invariant ideals of R_0 .*

Proof. Note first that whenever $h' < h''$ in H , we have $h'' = h'\ell$ for some $\ell \in H$. This is because H is the positive cone of a linearly ordered group; in particular, H is a cancellative monoid, for which reason the above ℓ is unique. It follows that if $0 \neq r \in R_0$ and x is the smallest element in $\text{supp } r$, then r can be written as $r = xu$ where $u \in R_0$ and $1_H \in \text{supp } u$. By Proposition 0.10.3 such an element u must be a unit of R_0 . Consequently, every nonzero principal ideal of R_0 has the form xR_0 for some $x \in H$. More generally, each nonzero ideal J of R_0 has the form IR_0 for some ideal

I of H (the set of all $x \in H$ with $xR_0 \subseteq J$ generates such an I). Clearly the map $I \mapsto IR_0$ preserves order, so R_0 is a chain ring. Now if $x, y \in H$ and $z \in R_0$ are such that $x = yz$ (in R_0) then we must have $z \in H$, whereupon $xH \subseteq yH$. This establishes the injectivity of the map $x \mapsto xR_0$ ($x \in H$). Since ideals of R_0 are unions of principal ideals, the injectivity of the map $I \mapsto IR_0$ ($I \in (\mathfrak{M}H)_\emptyset$) also follows. In particular, for each $I \in (\mathfrak{M}H)_\emptyset$, we have $I = IR_0 \cap H$. We have proved (i) and (ii).

Now if $I \in \mathfrak{M}_\Theta H$, $\alpha \in \Theta$ and $r = xu \in R_0$ (where $x \in H$ and u is a unit of R_0) then $\alpha(u)$ is also a unit of R_0 (α being an automorphism), so $\alpha(r) \in IR_0$ if and only if $\alpha(x) \in I$, i.e., $x \in I$ (I being absolutely Θ -invariant), i.e., $r \in IR_0$. Conversely for $IR_0 \in \text{Id}_\Theta R_0$, where $I \in \mathfrak{M}H$, $x \in H$ and $\alpha \in \Theta$, we have $\alpha(x) \in I$ if and only if $\alpha(x) \in IR_0$, i.e., $x \in IR_0$, i.e., $x \in I$. This proves (iii), completing the proof of the proposition. \square

We have observed that, in the notation of the above proposition, the map $x \mapsto [x]$ ($x \in H$) is a lattice isomorphism from the chain $\langle H; \leq \rangle$ onto the chain of principal monoid ideals of H , ordered by *reversed* set inclusion \supseteq . Let $\text{Id}^P R_0$ denote the set of principal ideals of the ring R_0 . We have seen that $[x]R_0 = xR_0$ for all $x \in H$. It follows that the function $x \mapsto xR_0$ ($x \in H$) is a lattice embedding of the chain $\langle H; \leq \rangle$ into the chain $\langle (\text{Id}^P R_0) \setminus \{0\}; \supseteq \rangle$ of nonzero principal ideals, also ordered by *reversed* set inclusion, of the ring R_0 , and as noted in the proof of the above proposition, this function maps *onto* $(\text{Id}^P R_0) \setminus \{0\}$. Clearly, we also have $xR_0 yR_0 = xyR_0$ for all $x, y \in H$. Consequently, the nonzero principal ideals of R_0 form a strictly ordered monoid (with respect to ideal multiplication and *reversed* set inclusion and having R_0 as identity element) which is isomorphic to the strictly ordered monoid H ; formally, $\langle H; \cdot, 1_H; \leq \rangle \cong \langle (\text{Id}^P R_0) \setminus \{0\}; \cdot, R_0; \supseteq \rangle$. We shall need to use this fact in subsequent chapters, so we state it explicitly in the following (well known) corollary.

COROLLARY 2. *Given any linearly ordered abelian group G , there is a commutative chain domain R_0 , with identity, such that the strictly ordered monoid (with respect to ideal multiplication and reversed set inclusion) of nonzero principal ideals of R_0 is isomorphic to the positive cone of G .* \square

The linearly ordered abelian group G is uniquely determined by its positive cone and hence by the commutative chain domain R_0 (although G does not determine R_0 uniquely since, for example, the field F in Proposition 1 may be chosen arbitrarily). It is standard practice to refer to G as the *value group* of R_0 . Corollary 2 therefore says that commutative chain domains with arbitrary (linearly ordered and abelian) value groups may always be constructed.

Recall that a ring R is said to be a *right duo* ring if every right ideal of R is a (two-sided) ideal of R . The fact that the ring R_0 (in Proposition 1 and Corollary 2) is commutative illustrates a general theorem of Brungs and Törner about local rings, viz.:

THEOREM 3. [BT81, Corollary, p296] *The nonzero principal right ideals of a local ring R with identity form a semigroup with respect to ideal multiplication if and only if R is right duo.* \square

For this reason, most of the simpler constructions of right chain rings with preassigned chains of principal right ideals produce right duo rings. Right duo chain rings are insufficient for our purposes in this chapter. Indeed, it is very well known that a ring R with identity is right noetherian if it is right artinian (Theorem 0.5.6). Since the ordinal $\omega \oplus 1 := \{0, 1, 2, \dots, \omega\}$ (ordered naturally) is an algebraic chain satisfying the descending but not the ascending chain condition, it cannot be isomorphic to the lattice of proper ideals of any right or left duo ring with identity.

§3. CONSTRUCTING NONCOMMUTATIVE RIGHT CHAIN DOMAINS.

Clearly we shall need more powerful techniques for constructing noncommutative right chain rings. The development of such techniques has seen quite a lot of activity in the last decade. The technique to be used here, which starts with a commutative chain domain produced as in Proposition 2.1, owes much to [Bru86] for inspiration.

We start by recalling the definition of a right skew monoid ring. Given a ring R and a monoid Θ acting on R , the *right skew monoid ring in Θ over R* , denoted $R[\Theta]$, is defined to be the set of all formal sums of the form $\sum_{\alpha \in \Theta} a_\alpha \alpha$ ($a_\alpha \in R$) with finite support. Addition is natural and multiplication is defined distributively, using the rule

$$\alpha \cdot a = (\alpha(a))\alpha \quad (\alpha \in \Theta, a \in R).$$

(See §10 of Chapter 0.)

Our construction techniques shall also make use of rings of fractions. We have chosen therefore to recall, very briefly, some of the relevant definitions and results. (We refer the reader to §8 of Chapter 0 for a more detailed and precise account.) Let S be a multiplicatively closed set of regular elements of a ring R . A ring T with identity is called a *left ring of fractions of R with respect to S* if R is a subring of T with the property that every element of S is a unit of T and every $t \in T$ is expressible in the form $t = s^{-1}r$ for some $r \in R$ and $s \in S$. A *left denominator set* for a ring R is any (nonempty) multiplicatively closed set S of regular elements of R having the *common left multiple property*: for each $r \in R$ and each $s \in S$, there exist $r_1 \in R$ and $s_1 \in S$ such that $s_1 r = r_1 s$. In this case, R has a unique (up to isomorphism) left ring of fractions with respect to S (see Theorems 0.8.1 and 0.8.2). For each ring R , and each left denominator set S for R , we choose a fixed left ring of fractions of R with respect to S and denote it by $S^{-1}R$. We also point out that a ring R is a *left Ore domain* if and only if the set of its nonzero elements is a left denominator set for R (see Corollary 0.7.6 and the remarks following Theorem 0.8.2).

PROPOSITION 1. *Let R_0 be a left Ore domain and Θ a commutative strictly ordered monoid acting on R_0 . Then the right skew monoid ring $R = R_0[\Theta]$ is a left Ore domain.*

Proof. Consider nonzero elements $r = \sum_{\alpha \in \Theta} a_\alpha \alpha$ and $s = \sum_{\alpha \in \Theta} a'_\alpha \alpha$ of R and note that $rs = \sum_{\beta\gamma = \alpha} a_\beta(\beta(a'_\gamma))\beta\gamma$. If δ and ϵ are the smallest elements of the respective supports of r and s then the coefficient of $\delta\epsilon$ in rs is just $a_\delta(\delta(a'_\epsilon))$, as a consequence of the fact that Θ is strictly ordered. Now $\delta(a'_\epsilon)$ is nonzero since δ is a monomorphism, so $a_\delta(\delta(a'_\epsilon))$ is also nonzero since R_0 is a domain. This shows that R is a domain.

Observe that if Φ is a submonoid of Θ then Φ is strictly ordered and acts on R_0 , and that $R_0[\Phi]$, being a subring of $R_0[\Theta]$, is a domain. Since any pair of elements of R is contained in $R_0[\Phi]$ for a suitable finitely generated submonoid Φ of Θ , it suffices to prove that $R_0[\Phi] \setminus \{0\}$ satisfies the common left multiple property for every finitely generated submonoid Φ of Θ .

The proof is by induction on the number of generators of Φ . The result is true by assumption when there are no generators, i.e. when Φ is the trivial (one element) monoid, since $R_0[\Phi] \cong R_0$ in this case. Suppose the result holds for all submonoids of Θ on fewer than m generators, where m is a nonnegative integer.

Assume that Φ is a submonoid of Θ generated by an m -element subset $X \cup \{\alpha\}$ of Θ (with $\alpha \notin X$) and that Φ cannot be generated by fewer than m elements. Let Γ be the submonoid of Θ generated by X and $\langle \alpha \rangle$ the principal submonoid of Φ generated by α . Observe that $\langle \alpha \rangle$ acts on the ring $R_1 = R_0[\Gamma]$. Indeed, α may be extended to a ring monomorphism from R_1 to itself (also to be called α) defined by $\sum_{\gamma \in \Gamma} a_\gamma \gamma \mapsto \sum_{\gamma \in \Gamma} (\alpha(a_\gamma))\gamma$ ($a_\gamma \in R_0$) and the same applies to composite powers of α . Moreover, it is easily checked that with respect to this action and the obvious mapping, we have $R_1[\alpha] \cong R_0[\Phi]$; we identify these two rings and we set $R_2 = R_1[\alpha] = R_0[\Phi]$. Thus R_2 is a domain, every element of which may be expressed uniquely as a polynomial in α with coefficients from R_1 , and by the inductive hypothesis, R_1 is a left Ore domain.

Suppose that R_2 is not left Ore, i.e., that the set

$$\mathcal{A} = \{(a, b) : 0 \neq a, b \in R_2 \text{ and } R_2 a \cap R_2 b = 0\}$$

is nonempty. Choose an a in the domain of the relation \mathcal{A} , having minimum degree (as a polynomial in α over R_1), then choose a b having minimum degree such that $(a, b) \in \mathcal{A}$. We may assume without loss of generality that $a = \sum_{i=0}^n a_i \alpha^i$ and $b = \sum_{i=0}^{n+k} b_i \alpha^i$ for some integer $k \geq 0$ and some $a_i, b_i \in R_1$ with $a_n, b_{n+k} \neq 0$. Set $a'_i = \alpha^k(a_i) \in R_1$ for $i = 0, \dots, n$, and note that $a'_n \neq 0$, since $a_n \neq 0$ and α is a monomorphism. We have

$$\alpha^k a = \alpha^k \left(\sum_{i=0}^n a_i \alpha^i \right) = \sum_{i=0}^n \alpha^k(a_i) \alpha^k \alpha^i = \sum_{i=0}^n a'_i \alpha^{k+i}.$$

By the inductive hypothesis, there exist $p, q \in R_1$ (both nonzero, since R_1 is a domain) such that $pa'_n = qb_{n+k}$. Let $c = p(\alpha^k a) - qb$ and note that $c \neq 0$, otherwise we would have $0 \neq qb = (p\alpha^k a) \in R_2 a \cap R_2 b$, contradicting the fact that $(a, b) \in \mathcal{A}$. Clearly, c is a polynomial in α over R_1 of

degree less than $n+k$. We shall complete the proof by showing that $(a, c) \in \mathcal{A}$, contradicting the minimality of the degree of b . Suppose $d \in R_2a \cap R_2c$. Then

$$d = r'a = rc = r(p(\alpha^k a) - qb)$$

for some $r, r' \in R_2$, i.e., $(r' - r(p\alpha^k))a = rqb$. Since $(a, b) \in \mathcal{A}$, we must have $r(qb) = 0$, and $qb \neq 0$, so $r = 0$. We have shown that $R_2a \cap R_2c = 0$, hence $(a, c) \in \mathcal{A}$. \square

PROPOSITION 2. *Let R_0 be a commutative chain domain with identity, whose largest proper ideal is P . Let Θ be a commutative strictly ordered monoid acting on $(R_0; +, \cdot, 0, 1_{R_0})$ and let $R = R_0[\Theta]$ be the associated right skew monoid ring. Let $N = P[\Theta] := \{\sum_{\alpha \in \Theta} a_\alpha \alpha \in R : a_\alpha \in P\}$ and let $S = R \setminus N$. Suppose further that $\alpha[P] \subseteq P$ for all $\alpha \in \Theta$. Then:*

- (i) S is a left denominator set for R ;
- (ii) $S^{-1}R$ is a right chain domain whose unique maximal proper right ideal is $S^{-1}N := \{s^{-1}r : s \in S, r \in N\}$.

Proof. (i) By the previous proposition, R is a left Ore domain. The fact that P is an ideal of R_0 and that $\alpha[P] \subseteq P$ for all $\alpha \in \Theta$ clearly implies that N is an ideal of R .

We prove that S is a multiplicatively closed subset of R . Let $x = \sum_{i=1}^n a_i \theta_i$, $y = \sum_{i=1}^{n'} a'_i \theta'_i \in S$, where $a_i, a'_i \in R_0$ and $\theta_i, \theta'_i \in \Theta$, with $\theta_1 < \theta_2 < \dots < \theta_n$ and $\theta'_1 < \theta'_2 < \dots < \theta'_n$. Since $x \in S$, some a_i ($1 \leq i \leq n$) is not in P . This means, since R_0 is local, that a_i is a unit of R_0 (by Theorem 0.5.4). Choose $j \in \{1, \dots, n\}$ to be the least positive integer for which a_j is a unit of R_0 . Similarly there is a positive integer $k \in \{1, \dots, n'\}$ which is minimal such that a'_k is a unit of R_0 . It follows that $\theta_j(a'_k)$, and hence $a_j \theta_j(a'_k)$, is a unit of R_0 .

Consider the coefficient of $\theta_j \theta'_k$ in the product xy . This coefficient may clearly be expressed in the form $a_j \theta_j(a'_k) + p_1 + p_2 + \dots + p_t$, where each p_i is an element of the form $a_l \theta_l(a'_m)$ with either (1) $1 \leq l < j$ and $k < m \leq n'$, or (2) $j < l \leq n$ and $1 \leq m < k$. It follows from the minimality of j and k that $p_i \in P$ for $i = 1, \dots, t$. Consequently, the coefficient of $\theta_j \theta'_k$ is a unit of R_0 , and so $xy \in S$, as required. Notice also that $1_R \in S$.

Now let $r \in R$, $s \in S$. Since R is a left Ore domain and $s \neq 0$, there exist $p, q \in R$ such that $ps = qr$ and $q \neq 0$. Since R_0 is a (commutative) chain ring, we may write $p = a\bar{p}$ and $q = b\bar{q}$ where $a, b \in R_0$, $\bar{p}, \bar{q} \in S$ and, necessarily, $b \neq 0$. Indeed, we may choose a from among the R_0 -coefficients in the expansion of p in such a way that the right ideal of R_0 generated by a contains the right ideals generated by all of the other coefficients; the quotient \bar{p} obtained by extracting from p the common factor a has the identity element of R_0 as one of its coefficients, which forces $\bar{p} \in S$. A similar argument applies to q . We now have $a\bar{p}s = b\bar{q}r$. Because $\bar{p}s \in S$, some R_0 -coefficient of $\bar{p}s$ is a unit of R_0 . Equating coefficients in the equation $a\bar{p}s = b\bar{q}r$, we find that $a = bc$ for some $c \in R_0$. Therefore $bc\bar{p}s = b\bar{q}r$. Since R is a domain and $b \neq 0$, we must have

$(c\bar{p})s = \bar{q}r$, with $\bar{q} \in S$. This proves that S is a left denominator set for R .

(ii) Let $0 \neq x = s_1^{-1}r_1$ and $0 \neq y = s_2^{-1}r_2$, where $r_1, r_2 \in R$, $s_1, s_2 \in S$ and, necessarily, $r_1, r_2, s_1, s_2 \neq 0$. Then $s_3r_1 = r_3s_2$ for some nonzero $r_3 \in R$ and $s_3 \in S$. Consequently,

$$xy = s_1^{-1}s_3^{-1}s_3r_1s_2^{-1}r_2 = s_1^{-1}s_3^{-1}r_3s_2s_2^{-1}r_2 = s_1^{-1}s_3^{-1}r_3r_2 \neq 0,$$

because R is a domain. Certainly, therefore, $S^{-1}R$ is a domain.

By (i), there exist $u_1 \in S$ and $u_2 \in R$ such that $u_1s_1 = u_2s_2 = s$, say (which forces $u_2 \in S$, since N is an ideal of R). Setting $q_1 = u_1r_1$ and $q_2 = u_2r_2$, we obtain $x = s^{-1}q_1$ and $y = s^{-1}q_2$. We may also write $q_1 = a_1t_1$ and $q_2 = a_2t_2$ for suitable $a_1, a_2 \in R_0$ and $t_1, t_2 \in S$, since R_0 is a (commutative) chain ring. (Use a factorization argument similar to that employed in the proof of (i) above.) We therefore have $x = s^{-1}a_1t_1$ and $y = s^{-1}a_2t_2$. Again, since R_0 is a (commutative) chain ring, we lose no generality in assuming that $a_1 = a_2b$ for some $b \in R_0$. Now

$$\begin{aligned} x(S^{-1}R) &= s^{-1}a_1t_1(S^{-1}R) = s^{-1}a_1(S^{-1}R) = s^{-1}a_2b(S^{-1}R) \\ &\subseteq s^{-1}a_2(S^{-1}R) = s^{-1}a_2t_2(S^{-1}R) = y(S^{-1}R). \end{aligned}$$

This shows that $S^{-1}R$ is a right chain ring with identity.

Let $Q = S^{-1}N$. Of course, $0 = 1_R^{-1}0 \in Q$. If $s_1^{-1}n_1, s_2^{-1}n_2 \in Q$ and $s^{-1}r \in S^{-1}R$ with $s_1, s_2, s \in S$, $n_1, n_2 \in N$ and $r \in R$, then for some $t_1 \in S$ and some $t_2 \in R$, we have $t_1s_1 = t_2s_2$ (which entails $t_2 \in S$). It follows easily that $s_2^{-1} = s_1^{-1}t_1^{-1}t_2$. Then

$$s_1^{-1}n_1 - s_2^{-1}n_2 = s_1^{-1}t_1^{-1}t_1n_1 - s_1^{-1}t_1^{-1}t_2n_2 = (t_1s_1)^{-1}(t_1n_1 - t_2n_2) \in Q,$$

because $t_1s_1 \in S$ and $t_1n_1 - t_2n_2 \in N$ (since N is an ideal of R). Also, there exist $s_3 \in S$ and $n' \in R$ such that $s_3n_1 = n's$. Since N is an ideal of R and S is multiplicatively closed, this forces $n' \in N$. Thus,

$$s_1^{-1}n_1s^{-1}r = s_1^{-1}s_3^{-1}s_3n_1s^{-1}r = s_1^{-1}s_3^{-1}n'ss^{-1}r = (s_3s_1)^{-1}n'r \in Q,$$

which establishes that Q is a right ideal of $S^{-1}R$.

If $s \in S$, $t \in R$ and $s^{-1}t \notin Q$ then $t \notin N$, hence $t \in S$ and so $s^{-1}t$ is a unit of $S^{-1}R$. This shows that Q is the unique maximal proper right ideal (i.e., the Jacobson radical) of $S^{-1}R$, and is, in particular, an ideal of $S^{-1}R$ (see Theorem 0.5.4). \square

Inasmuch as we identify R_0 with the subring $\{r1_\Theta : r \in R_0\}$ of R (1_Θ denotes the identity element of the monoid Θ), which is a subring of $T = S^{-1}R$, each ideal I of R_0 generates a right ideal IT of T consisting of finite sums of products ab , with $a \in I$, $b \in T$. In fact, since T and R_0 are right chain rings, all such sums may be reduced to one product ab , $a \in I$, $b \in T$, using the factorization argument illustrated in the proof of the previous proposition. The association $I \mapsto IT$ ($I \in \text{Id } R_0$) clearly preserves order. It also turns out to be injective; to see this it suffices to check that whenever $a, b \in R_0$, $b \neq 0$, $aR_0 \subseteq bR_0$ and $aT = bT$, we have $aR_0 = bR_0$. Indeed, if $b \in aT$ and $a \in bR_0$, we may write $b = as^{-1}r$ and $a = bc$ for some $s \in S$, $r \in R$ and $c \in R_0$. Then $b = bcs^{-1}r$

and T is a domain, so $cs^{-1}r = 1_T$, hence c is a unit of T . Since units of T all have the form $s_1^{-1}t_1$ ($s_1, t_1 \in S$), we may write $s_1c = t_1$. But the expansion of t_1 in R has at least one coefficient $a_i \notin P$, i.e., a_i is a unit of R_0 . It follows easily that c is also a unit of R_0 , hence $aR_0 = bcR_0 = bR_0$, as required.

From the injectivity and isotonicity of the map $I \mapsto IT$ ($I \in \text{Id } R_0$) we may conclude that $I = IT \cap R_0$ for each $I \in \text{Id } R_0$. In order that $I \mapsto IT$ map onto the right ideal lattice $\text{Id } T_T$ of T , we appear to need stronger conditions on Θ . We shall now require Θ to be an abelian group. This has the effect that the elements of Θ , being automorphisms of R_0 , preserve and reflect units; thus the condition $\alpha[P] \subseteq P$ of Proposition 2 holds automatically for every $\alpha \in \Theta$.

PROPOSITION 3. *Let R_0, Θ, R and S be as in Proposition 2 but assume that Θ is a linearly ordered abelian group and let $T = S^{-1}R$. The map $I \mapsto IT$ ($I \in \text{Id } R_0$) is a lattice isomorphism from the (linearly ordered) ideal lattice of R_0 onto the right ideal lattice $\text{Id } T_T$ of T and the restriction of this map to $\text{Id}_{\Theta} R_0$ is a lattice isomorphism from the chain of (absolutely) Θ -invariant ideals of R_0 onto the ideal lattice $\text{Id } T$ of T .*

Proof. For the first assertion, it remains only to establish surjectivity, and for this it suffices, since R_0 and T are both right chain rings, to show that every principal right ideal of T has the form IT for a suitable principal ideal I of R_0 .

For each $r = \sum_{i=1}^n a_i \alpha_i \in R$ with $a_i \in R_0$ and $\alpha_i \in \Theta$, $i = 1, \dots, n$, define a map r^* on the chain L of principal ideals of R_0 by $r^*(K) = \sum_{i=1}^n a_i \alpha_i [K]$ ($K \in L$). The fact that Θ is a group, and in particular that each $\alpha_i: R_0 \rightarrow R_0$ is surjective, ensures that $\alpha_i[K]$ is an ideal of R_0 for each $K \in L$, and that $r^*: L \rightarrow L$. Clearly, r^* is order preserving. If $t = a\alpha$, $a \in R_0$, $\alpha \in \Theta$ and $bR_0 \subseteq aR_0$, say $b = ac$, $c \in R_0$, then $t^*(\alpha^{-1}(c)R_0) = bR_0$ so the image of t^* is just the order ideal (i.e., hereditary subset) of L generated by aR_0 (in other words, $\text{Im } t^* = \{K \in L: K \subseteq aR_0\}$). In particular, if a is a unit of R_0 then $t^*: L \rightarrow L$ is onto. Also, for $0 \neq a \in R_0$ and $\alpha \in \Theta$ with $t = a\alpha$, the map t^* is injective, since α is injective and R_0 is a chain domain.

Now suppose $s = \sum_{i=1}^n a_i \alpha_i \in S$ and $I \in L$. Then some a_j is a unit of R_0 , hence $I = a_j \alpha_j [K]$ for some $K \in L$. The fact that each $(a_i \alpha_i)^*$ is injective permits us to choose K to be minimal such that $a_i \alpha_i [K] = I$ for some i . We claim that $I = \bigcup_{i=1}^n a_i \alpha_i [K]$ ($= s^*(K)$). For if not, then $I \not\subseteq a_j \alpha_j [K]$ for some j , hence I belongs to the range of $(a_j \alpha_j)^*$, contradicting the minimality of K . Consequently, $s^*: L \rightarrow L$ is surjective.

Inasmuch as R_0 is a chain ring, every $r \in R$ may be written as $r = bs$ for some $b \in R_0$ and $s \in S$. Consequently, every principal right ideal of T has the form $s^{-1}bT$ for some $s \in S$ and $b \in R_0$. Let $0 \neq b \in R_0$ and $s = \sum_{i=1}^n a_i \alpha_i \in S$. Choose $a \in R_0$ with $s^*(aR_0) = bR_0$, i.e., $bR_0 = \bigcup_{i=1}^n (a_i \alpha_i(a))R_0 = (a_j \alpha_j(a))R_0$ for some j . Now $sa = \sum_{i=1}^n (a_i \alpha_i(a))\alpha_i = \sum_{i=1}^n (bc_i)\alpha_i$ for suitable $c_i \in R_0$, and since R_0 is a domain, we may choose c_j to be a unit of R_0 . Thus $sa = bt$, where

$t = \sum_{i=1}^n c_i \alpha_i \in S$. Now $s^{-1}bT = (s^{-1}bt)T = aT = (aR_0)T$. We have proved that the map $I \mapsto IT$ establishes a lattice isomorphism from $\text{Id } R_0$ onto $\text{Id } T_T$.

Now consider an ideal IT of T , where $I \in \text{Id } R_0$ and let $\alpha \in \Theta$. If $a \in I$ then $(\alpha(a))\alpha = \alpha \cdot a \in IT$, so $\alpha(a) = (\alpha(a))\alpha \cdot 1_R \alpha^{-1} \in IT \cap R_0 = I$. Thus I is (absolutely) Θ -invariant.

Conversely, let I be an (absolutely) Θ -invariant ideal of R_0 . We have to show that the right ideal IT of T is an ideal. Let $x = s^{-1}r \in T$ with $s = \sum_{i=1}^n b_i \beta_i \in S$, $r = \sum_{j=1}^m a_j \alpha_j \in R$, $b_i, a_j \in R_0$, $\beta_i, \alpha_j \in \Theta$. If $a \in I$ then $\alpha_j(a) \in I$ for each i , so $ra = \sum_{j=1}^m (a_j \alpha_j(a)) \alpha_j \in IT$. Thus $rIT \subseteq IT$, whence $xIT \subseteq s^{-1}IT$. Now the right ideal $s^{-1}IT$ has the form KT for some ideal K of R_0 , so $IT = sKT$. We show that $K \subseteq I$. If $d \in K$ then $sd = cy$ for some $c \in I$ and some $y \in T$. Also, since R_0 is a chain ring there is an index $\ell \in \{1, \dots, n\}$ such that $b_i \beta_i(d) \in (b_\ell \beta_\ell(d))R_0$, for $i = 1, \dots, n$. Thus $sd = \sum_{i=1}^n (b_i \beta_i(d)) \beta_i = b_\ell \beta_\ell(d)t$ for some $t \in S$. Since $s \in S$, some b_j is a unit of R_0 . If $d \notin I$ then $b_j \beta_j(d) \notin I$ (since I is absolutely Θ -invariant), hence $b_\ell \beta_\ell(d) \notin I$. But $b_\ell \beta_\ell(d) = sdt^{-1} \in sKT \cap R_0 = IT \cap R_0 = I$, a contradiction. Thus $d \in I$ and therefore $K \subseteq I$. Now $xIT \subseteq s^{-1}IT = KT \subseteq IT$. We conclude that IT is an ideal of T , and this completes the proof. \square

COROLLARY 4. *Let H be the positive cone of a linearly ordered abelian group G and let Θ be a linearly ordered abelian group which acts on H . Then there exists a right chain domain T , with identity, and a lattice isomorphism from the chain $(\mathfrak{M}H)_\emptyset$ onto the right ideal lattice $\text{Id } T_T$ of T , which restricts to a lattice isomorphism from the chain $(\mathfrak{M}_\Theta H)_\emptyset$ onto the ideal lattice $\text{Id } T$ of T . If, in addition, every monoid ideal of H is idempotent, then every ideal of the ring T is idempotent.*

Proof. Combine Proposition 2.1 and Proposition 3. For the last assertion, observe that a typical ideal of T has the form IT for a suitable (absolutely) Θ -invariant monoid ideal I of H . Clearly $I^2 T \subseteq (IT)^2$, since T has identity. (In fact we have equality, since $TI \subseteq IT$.) \square

To show that an algebraic chain is isomorphic to the lattice of proper ideals of a right chain domain with identity, it therefore suffices to prove that it is isomorphic to the lattice of proper (absolutely) Θ -invariant monoid ideals, augmented by \emptyset , of the positive cone H of some linearly ordered abelian group G , for some linearly ordered abelian group Θ acting on H . We look now to the construction of such G and Θ .

§4. CONSTRUCTING STRICTLY ORDERED COMMUTATIVE MONOIDS.

Throughout this section $\langle C; \leq \rangle$ shall denote a given chain and Θ a given nontrivial abelian group which acts on C . (Thus each $\alpha \in \Theta$ is an order preserving permutation of C .) Recall that the chain of (order) filters of the chain C , ordered by set inclusion, is denoted by $\mathfrak{F}C$. The subchain of

$\mathfrak{F}C$ consisting of the (absolutely) Θ -invariant filters of C will be denoted by $\mathfrak{F}_\Theta C$.

An interval A of C will be called a Θ -interval of C if (1) $\alpha[A] \subseteq A$ for every $\alpha \in \Theta$ (consequently, A is (absolutely) Θ -invariant) and (2) for each $a, b \in A$, there exists $\alpha \in \Theta$ such that $b < \alpha(a)$. Since Θ is a group, (2) is clearly equivalent to (2)' for each $a, b \in A$, there exists $\alpha \in \Theta$ such that $\alpha(a) < b$.

For each $c \in C$, let $\langle B(c); +, -, 0_c; \leq \rangle$ be a nontrivial additively written¹ linearly ordered abelian group with identity element 0_c . Observe that the direct sum $G = \bigoplus_{c \in C} B(c)$ of these groups is an abelian group which may also be linearly ordered. Indeed, the *antilexicographic* order on G (also denoted by \leq) may be defined as follows: if $g = \{g_c\}_{c \in C}$ and $h = \{h_c\}_{c \in C}$ are elements of G , then $g \leq h$ if and only if either $g = h$ or $g_c < h_c$ for the greatest $c \in C$ such that $g_c \neq h_c$. This makes G a linearly ordered abelian group and the positive cone of G is the set

$$\{0\} \cup \left\{ g = \{g_c\}_{c \in C} \in G : \text{if } c \in C \text{ is maximal such that } g_c \neq 0_c \text{ then } g_c > 0_c \right\},$$

considered as a strictly ordered monoid.

As a special case, we may take $\langle B(c); +, -, 0_c; \leq \rangle = \langle \mathbb{Z}; +, -, 0; \leq \rangle$ for all $c \in C$. Again let \leq denote the antilexicographic order on $G = \mathbb{Z}^{(C)}$. Let us establish the notation $M(C)$ for the positive cone of G , i.e.,

$$M(C) := \{0\} \cup \left\{ g = \{g_c\}_{c \in C} \in G : \text{if } c \in C \text{ is maximal such that } g_c \neq 0 \text{ then } g_c \geq 1 \right\},$$

considered as a strictly ordered monoid. Notice that each $\alpha \in \Theta$ induces an (order preserving) automorphism of $M(C)$, namely $\{g_c\}_{c \in C} \mapsto \{g_{\alpha^{-1}(c)}\}_{c \in C}$ ($\{g_c\}_{c \in C} \in M(C)$). We may therefore consider Θ to act on $M(C)$ also.

Now suppose that C is in fact the disjoint union of a family $\{A_d : d \in D\}$, where each of the subchains A_d of C is a Θ -interval of C , the index set D is also a chain, and $c_1 < c_2$ in C whenever $c_1 \in A_d$, $c_2 \in A_e$ and $d < e$ in D . With respect to these notational conventions, we have:

LEMMA 1. *The map $F \mapsto \bigcup_{d \in F} A_d$ ($F \in \mathfrak{F}D$) is a one-to-one lattice homomorphism from the chain $\mathfrak{F}D$ of order filters of the chain D into the chain $\mathfrak{F}C$ of order filters of the chain C , whose range is the chain $\mathfrak{F}_\Theta C$ of (absolutely) Θ -invariant order filters of the chain C .*

Proof. It is a virtually immediate consequence of definitions that the map $F \mapsto \bigcup_{d \in F} A_d$ ($F \in \mathfrak{F}D$) is an order preserving injection into $\mathfrak{F}C$ whose range is contained in $\mathfrak{F}_\Theta C$; it remains to prove that the map has range equal to $\mathfrak{F}_\Theta C$. For this it suffices to show that whenever we have $Q \in \mathfrak{F}_\Theta C$ and $d \in D$ with $A_d \cap Q \neq \emptyset$, we also have $A_d \subseteq Q$. Accordingly, suppose that $a \in A_d \cap Q$. We need only show that if $b \in A_d$ and $b < a$ then $b \in Q$. By (2)' above, we have $\alpha(a) < b$ for some $\alpha \in \Theta$, and by (1), $\alpha(a) \in Q$, hence $b \in Q$, as required. \square

¹Although not consistent with our general convention of using the multiplicative notation for semigroups, our use of the additive notation here is appropriate since, for the most part, $B(c)$ will be chosen to be the additive group of integers.

In general, we shall use the notation

$$g = \begin{pmatrix} z_1 & z_2 & \cdots & z_n \\ c_1 & c_2 & \cdots & c_n \end{pmatrix}$$

to indicate that $g = \{g_c\}_{c \in C}$ is the element of $M(C)$ such that $g_{c_i} = z_i \neq 0$ for $i = 1, \dots, n$, where $c_1 < c_2 < \dots < c_n$ in C and $g_c = 0$ for all other $c \in C$. (Necessarily $z_n \geq 1$.) In this case, observe that

$$\alpha(g) = \begin{pmatrix} z_1 & z_2 & \cdots & z_n \\ \alpha(c_1) & \alpha(c_2) & \cdots & \alpha(c_n) \end{pmatrix}$$

for any $\alpha \in \Theta$. For any nonempty subset X of C let M_X denote the subset of $M(C)$ defined by

$$M_X = \left\{ g = \begin{pmatrix} z_1 & z_2 & \cdots & z_n \\ c_1 & c_2 & \cdots & c_n \end{pmatrix} \in M(C) : c_n \in X \right\}.$$

LEMMA 2. *The map $F \mapsto M_F$ ($F \in \mathfrak{F}C$) is a one-to-one lattice homomorphism from the chain $\mathfrak{F}C$ of order filters of C into the chain $\mathfrak{M}(M(C)) \setminus \{M(C)\} = \mathfrak{F}(M(C)) \setminus \{M(C)\}$ of proper monoid ideals (i.e., order filters) of $M(C)$. Moreover, if C has the property that for each $c \in C$ there exists $\alpha \in \Theta$ such that $\alpha(c) < c$, then the aforementioned map restricts to a lattice isomorphism from the chain $\mathfrak{F}_\Theta C$ of (absolutely) Θ -invariant order filters of the chain C onto the chain $\mathfrak{M}_\Theta(M(C)) \setminus \{M(C)\}$ of proper (absolutely) Θ -invariant ideals of $M(C)$, and every Θ -invariant monoid ideal of $M(C)$ is idempotent.*

Proof. It follows easily from the definition that $M_F \in \mathfrak{M}(M(C)) \setminus \{M(C)\}$ whenever $F \in \mathfrak{F}C$. If $F_1, F_2 \in \mathfrak{F}C$ with $F_1 \subsetneq F_2$ then clearly $M_{F_1} \subseteq M_{F_2}$ while if $c \in F_2 \setminus F_1$ then $\binom{1}{c} \in M_{F_2} \setminus M_{F_1}$. This shows that the given map is a one-to-one lattice homomorphism. Moreover, if $\alpha \in \Theta$ and

$$g = \begin{pmatrix} z_1 & \cdots & z_n \\ c_1 & \cdots & c_n \end{pmatrix} \in M_F \quad (F \in \mathfrak{F}_\Theta C),$$

then

$$\alpha(g) = \begin{pmatrix} z_1 & z_2 & \cdots & z_n \\ \alpha(c_1) & \alpha(c_2) & \cdots & \alpha(c_n) \end{pmatrix} \in M_F$$

because $\alpha(c_n) \in F$. Thus $M_F \in \mathfrak{M}_\Theta(M(C)) \setminus \{M(C)\}$ whenever $F \in \mathfrak{F}_\Theta C$. We need to show that $\mathfrak{F}_\Theta C$ is mapped onto $\mathfrak{M}_\Theta(M(C)) \setminus \{M(C)\}$. Let $M(C) \neq I \in \mathfrak{F}_\Theta(M(C))$ and set

$$F = \left\{ c \in C : \left(\exists \begin{pmatrix} z_1 & \cdots & z_n \\ c_1 & \cdots & c_n \end{pmatrix} \in I \right) (c_n = c) \right\}.$$

It is easy to see that $F \in \mathfrak{F}_\Theta C$ and that $I \subseteq M_F$. To establish the reverse containment, choose

$$g = \begin{pmatrix} z_1 & \cdots & z_n \\ c_1 & \cdots & c_n \end{pmatrix} \in M_F.$$

Then $c_n \in F$, so there must exist some

$$g' = \begin{pmatrix} z'_1 & \cdots & z'_m \\ c'_1 & \cdots & c'_m \end{pmatrix} \in I$$

with $c'_m = c_n$. By hypothesis, $\alpha(c'_m) < c'_m = c_n$ for some $\alpha \in \Theta$, so

$$\alpha(g') = \begin{pmatrix} z'_1 & z'_2 & \cdots & z'_m \\ \alpha(c'_1) & \alpha(c'_2) & \cdots & \alpha(c'_m) \end{pmatrix} < g.$$

Since I is Θ -invariant we must have $\alpha(g') \in I$, so $g \in I$. Thus $I \supseteq M_F$, whence $I = M_F$.

It remains to show that each $I \in \mathfrak{F}_\Theta(M(C))$ is idempotent as a monoid ideal, i.e., that $I = I + I$, i.e., that $I \subseteq I + I$. Since $0 = 0 + 0$ is the only element of $M(C) \setminus \bigcup \{M_F : F \in \mathfrak{F}_\Theta C\}$, we may assume that I is proper. Consider

$$g = \begin{pmatrix} z_1 & \cdots & z_n \\ c_1 & \cdots & c_n \end{pmatrix} \in I$$

and choose $\alpha \in \Theta$ with $\alpha(c_n) < c_n$. Clearly $\alpha(g) \in I$ and $\alpha(g) + \alpha(g) < g$, so $g \in I + I$. Thus $I \subseteq I + I$, as required. \square

§5. CHAINS OF IDEALS AND RIGHT IDEALS IN RINGS: THE MAIN RESULTS.

We are equipped, at last, to prove the first of the two main results of this chapter.

THEOREM 1. *The following conditions on a chain L are equivalent:*

- (i) L is an algebraic lattice;
- (ii) there is a right chain domain T , with identity, such that L is isomorphic to the lattice of all proper ideals of T , ordered by set inclusion, and all ideals of T are idempotent;
- (iii) L is isomorphic to the congruence lattice of a ring (not necessarily with identity).

Proof. (i) \Rightarrow (ii): If $|L| = 1$ we take T to be any division ring. Now assume that $|L| > 1$. Let 0 denote the least element of L and consider the sublattice L^c of L consisting of all compact elements of L . Certainly, $0 \in L^c$. By Proposition 0.1.7 (iii), the map $x \mapsto \{y \in L^c : y \leq x\}$ ($x \in L$) is a lattice isomorphism from L onto the chain $\mathfrak{I}L^c$ of order ideals of L^c . Let $\langle D; \leq \rangle$ be the order dual of the subchain of L whose universe is $L^c \setminus \{0\}$. We thus have isomorphisms

$$(1) \quad L \cong \mathfrak{I}L^c \cong (\mathfrak{I}(L^c \setminus \{0\}))_\emptyset \cong (\mathfrak{I}D)_\emptyset.$$

Our task now is to produce a chain C and a linearly ordered group Θ acting on C , which satisfy the conditions of Lemma 4.1. Take $\langle \Theta; \cdot, ^{-1}, 1_\Theta; \leq \rangle$ to be any nontrivial linearly ordered abelian group (e.g., the integers $\langle \mathbb{Z}; +, -, 0; \leq \rangle$). Consider the chain $\langle C; \leq \rangle$ where $C = \Theta \times D$ and \leq denotes

the antilexicographic order on C (i.e., for any $\alpha_1, \alpha_2 \in \Theta$ and $d_1, d_2 \in D$, we have $(\alpha_1, d_1) < (\alpha_2, d_2)$ if and only if either (1) $d_1 < d_2$ or (2) $d_1 = d_2$ and $\alpha_1 < \alpha_2$). Now Θ acts on the chain C , the action being defined by $\alpha_2((\alpha_1, d_1)) = (\alpha_2 \cdot \alpha_1, d_1)$, $\alpha_1, \alpha_2 \in \Theta$, $d_1, d_2 \in D$. For each $d \in D$, the subset $A_d = \Theta \times \{d\}$ of C is clearly a Θ -interval of C , and C is the disjoint union of all such intervals. Thus C and Θ satisfy the conditions of Lemma 4.1. Observe that for each $c \in C$ there exists $\alpha \in \Theta$ such that $\alpha(c) < c$, hence C and Θ also satisfy the conditions of Lemma 4.2. The conjunction of these lemmas yields the isomorphisms

$$(2) \quad (\mathfrak{F}D)_\emptyset \cong (\mathfrak{F}_\Theta C)_\emptyset \cong (\mathfrak{M}_\Theta(M(C)))_\emptyset \setminus \{M(C)\}.$$

Taking H to be $M(C)$ in Corollary 3.4 we may conclude that there exists a right chain domain T , with identity, all of whose ideals are idempotent, and which is such that

$$(3) \quad (\mathfrak{M}_\Theta(M(C)))_\emptyset \setminus \{M(C)\} \cong \text{Id } T \setminus \{T\}.$$

The result now follows from (1), (2) and (3).

(ii) \Rightarrow (iii): If T is as described in (ii) then T has a unique maximal proper right ideal J , and J is an ideal of T (Theorem 0.5.4). Clearly all proper ideals of T are ideals of the ring (without identity) J . Conversely, let K be an ideal of the ring J and let $\langle K \rangle$ be the ideal of T generated by K . Then $\langle K \rangle^3 \subseteq K$, by Andrunakievič's Lemma (Lemma 0.2.1), but $\langle K \rangle$ is idempotent by (ii), so $\langle K \rangle = \langle K \rangle^3 \subseteq K$. This means that $K = \langle K \rangle$, hence $K \in \text{Id } T$. Thus $(\text{Id } T) \setminus \{T\} = \text{Id } J \cong \text{Con } J$, so that $L \cong \text{Con } J$.

(iii) \Rightarrow (i) is just a special case of Corollary 1.2. □

In fact, Theorem 1 does not use the full force of the construction developed here. It turns out (again, we feel, surprisingly) that when constructing a right chain domain T with identity having any preassigned algebraic chain of proper ideals, *we may also require the chains of right ideals strictly between covering pairs of proper ideals of T to contain dually cofinal copies of further preassigned chains; these latter chains must be unbounded above, but may otherwise be quite arbitrary.* We shall have need of this freedom of construction in Chapter II. Theorem 3 is devoted to establishing that this is indeed available to us. We shall require a preliminary lemma.

LEMMA 2. *Let $\langle C; \leq \rangle$ be a chain. Suppose F_1, F_2 are order filters of C with $F_1 \subsetneq F_2$ and such that the interval $(F_1, F_2]$ of $\mathfrak{F}C$ has no least element. If X is a subset of $(F_1, F_2]$ with the property that X is cofinal in $(F_1, F_2]^{\text{du}}$ (that is to say, given any $F \in (F_1, F_2]$ there exists $F' \in X$ such that $F' \subseteq F$), then the image of X under the map from $\mathfrak{F}C$ to $\mathfrak{M}(M(C)) \setminus \{M(C)\}$ defined by $F \mapsto M_F$ ($F \in \mathfrak{F}C$) is cofinal in $(M_{F_1}, M_{F_2}]^{\text{du}}$.*

Proof. Let $I \in (M_{F_1}, M_{F_2}]$. Since $M_{F_1} \subsetneq I$ there must exist some $g \in I \setminus M_{F_1}$. Write

$$g = \begin{pmatrix} z_1 & \dots & z_n \\ c_1 & \dots & c_n \end{pmatrix}.$$

Since $g \notin M_{F_1}$, $c_n \notin F_1$. Consider the filter $[c_n]$ of C . Since $c_n \notin F_1$, $F_1 \not\subseteq [c_n]$. But

$$g = \begin{pmatrix} z_1 & \cdots & z_n \\ c_1 & \cdots & c_n \end{pmatrix} \in I \subseteq M_{F_2},$$

so $c_n \in F_2$. It follows that $[c_n] \subseteq F_2$, thus $F_1 \not\subseteq [c_n] \subseteq F_2$. Inasmuch as X is cofinal in $(F_1, F_2]^{\text{du}}$ and $(F_1, F_2]$ has no least element, we can choose $F' \in X$ such that $F' \not\subseteq [c_n]$. Take $g' \in M_{F'}$ and write

$$g' = \begin{pmatrix} z'_1 & \cdots & z'_m \\ c'_1 & \cdots & c'_m \end{pmatrix}.$$

Since $c'_m \in F' \not\subseteq [c_n]$ we cannot have $c'_m \leq c_n$, so $c'_m > c_n$. It follows that $g' > g \in I$. This shows that $M_{F_1} \not\subseteq M_{F'} \subseteq I$, thus $\{M_F : F \in X\}$ is cofinal in $(M_{F_1}, M_{F_2}]^{\text{du}}$, as required. \square

THEOREM 3. *Let L be an arbitrary algebraic chain and let Ω be the set of all covering pairs of L . Let $\{\langle C_\omega; \leq \rangle : \omega \in \Omega\}$ be a family of chains indexed by Ω , such that none of the C_ω has a greatest element. Then there exists a right chain domain T , with identity, such that all ideals of T are idempotent and the following further conditions are satisfied:*

- (i) *there is a lattice isomorphism f from L onto the chain $(\text{Id } T) \setminus \{T\}$ of all proper ideals of T (ordered by set inclusion);*
- (ii) *for each covering pair $\omega = (x, y)$ of L , the interval $(f(x), f(y))$ of $(\text{Id } T) \setminus \{T\}$ contains a dually cofinal copy of C_ω , that is to say, C_ω is embeddable into the dual $\langle (f(x), f(y)); \supseteq \rangle$ of the interval $(f(x), f(y))$ of $(\text{Id } T) \setminus \{T\}$ in such a way that the image of C_ω is cofinal in $\langle (f(x), f(y)); \supseteq \rangle$.*

Proof. With reference to assertion (ii) above, we point out that the open interval $(f(x), f(y))$ will contain a dually cofinal copy of the chain C_ω if it can be shown that the *half-open* interval $(f(x), f(y)]$ contains a dually cofinal copy of the chain C_ω augmented with the addition of a new least element. In establishing assertion (ii) therefore, no generality is lost if $(f(x), f(y)]$ is substituted for $(f(x), f(y))$.

If $|L| = 1$, we take T to be any division ring. Now assume that $|L| > 1$. Choose $\langle D; \leq \rangle$ as in the proof of Theorem 1 ((i) \Rightarrow (ii)). As noted in the proof of Theorem 1 ((i) \Rightarrow (ii)), we have $L \cong (\mathfrak{F}D)_\emptyset$. No generality is lost therefore if we assume that Ω is the set of all covering pairs of $(\mathfrak{F}D)_\emptyset$. Moreover, it is not difficult to show that if $F_1 = (d) := \{d' \in D : d' > d\}$ and $F_2 = [d] := \{d' \in D : d' \geq d\}$, then (F_1, F_2) is a covering pair of $(\mathfrak{F}D)_\emptyset$ and furthermore, every covering pair of $(\mathfrak{F}D)_\emptyset$ is of the form $((d), [d])$ for a suitable $d \in D$. This observation points to an obvious bijection from the set D onto Ω . We are clearly permitted, therefore, to index the family $\{C_\omega : \omega \in \Omega\}$ with elements of D instead of Ω . In the light of the above, our revised task is the following:

We seek a right chain domain T , with identity, such that all ideals of T are idempotent and the following further conditions are satisfied:

- (i)' *there is a lattice isomorphism f from $(\mathfrak{F}D)_\emptyset$ onto the chain $(\text{Id } T) \setminus \{T\}$;*
- (ii)' *for each $d \in D$ the interval $(f((d)), f([d]))$ of $(\text{Id } T) \setminus \{T\}$ contains a dually cofinal copy of C_d .*

We introduce a second simplifying assumption which results in a further revision of the task at hand. It follows from Corollary 3.4 that if C is a chain and Θ a linearly ordered abelian group which acts on C , then there exists a right chain domain T , with identity, all of whose ideals are idempotent, and which is such that the following diagram of chains and lattice homomorphisms commutes.

$$\begin{array}{ccc}
 \mathfrak{M}(M(C)) \setminus \{M(C)\} & \xrightarrow{\text{(iso)}} & (\text{Id } T_T) \setminus \{T\} \\
 \uparrow & & \uparrow \\
 \mathfrak{M}_\Theta(M(C)) \setminus \{M(C)\} & \xrightarrow{\text{(iso)}} & (\text{Id } T) \setminus \{T\}
 \end{array}$$

(The two vertical arrows represent inclusion maps.) In view of the above diagram it is easy to see that the task described in (i)' and (ii)' reduces to the following:

We seek a chain C and a linearly ordered abelian group Θ which acts on C such that the following two conditions are satisfied:

- (i)'' *there is a lattice isomorphism f from $(\mathfrak{F}D)_\emptyset$ onto the chain $\mathfrak{M}_\Theta(M(C)) \setminus \{M(C)\}$;*
- (ii)'' *for each $d \in D$ the interval $(f((d)), f([d]))$ of $\mathfrak{M}(M(C)) \setminus \{M(C)\}$ contains a dually cofinal copy of C_d .*

We now construct such a chain C and linearly ordered group Θ . Our choices for C and Θ will need to be more discriminating than those made in the proof of Theorem 1 ((i) \Rightarrow (ii)). Consider the integers as a linearly ordered group $\langle \mathbb{Z}; +, -, 0; \leq \rangle$, and take B_d to be $\mathbb{Z}^{(C_d)}$ for each $d \in D$. Each B_d , enriched with the antilexicographic order \leq_d , is to be considered structurally as a linearly ordered group. Let $\langle C; \leq_C \rangle$ be the "ordinal sum" of the chains $\langle B_d; \leq_d \rangle$ over the chain $\langle D; \leq \rangle$, i.e., $C := \bigcup_{d \in D} (B_d \times \{d\})$ and, whenever $d, e \in D$ with $b_1 \in B_d$ and $b_2 \in B_e$, we have $(b_1, d) <_C (b_2, e)$ if and only if either $d < e$ in D or $d = e$ and $b_1 <_d b_2$. We take Θ to be the (additively written) abelian group $\bigoplus_{d \in D} B_d$, enriched with its own antilexicographic order and considered structurally as a linearly ordered group. The group Θ acts on the chain C : if $\alpha = \{\alpha_d\}_{d \in D} \in \Theta$, the action of α is defined by $\alpha((b_d, d)) = (\alpha_d + b_d, d)$ for all $d \in D$ and $b_d \in B_d$. It is easy to check that each $A_d = B_d \times \{d\}$ is a Θ -interval of C and, of course, C is the disjoint union of the family $\{A_d : d \in D\}$. Moreover, it is obvious that for each $c \in C$ there exists $\alpha \in \Theta$ such that $\alpha(c) <_C c$. It follows that the chain C and linearly ordered group Θ thus defined, satisfy the conditions of Lemmas 4.1 and 4.2. Applying both these lemmas, we obtain the following commutative diagram of chains and lattice homomorphisms.

$$\begin{array}{ccc}
(\mathfrak{F}C)_\emptyset & \xrightarrow{(1-1)} & \mathfrak{M}(M(C)) \setminus \{M(C)\} \\
\uparrow & & \uparrow \\
(\mathfrak{F}D)_\emptyset & \xrightarrow{\text{(iso)}} & (\mathfrak{F}_\Theta C)_\emptyset \xrightarrow{\text{(iso)}} \mathfrak{M}_\Theta(M(C)) \setminus \{M(C)\}
\end{array}$$

(The two vertical arrows represent inclusion maps.) Our choice for f in property (i)'' is immediate: we take f to be the composition of the two isomorphisms on the bottom row of the above diagram. It remains to establish property (ii)''. Let $d \in D$ and consider the covering pair $((d), [d])$ of $(\mathfrak{F}D)_\emptyset$. We trace the images of (d) and $[d]$ under the action of the mapping described in Lemma 4.1:

$$\begin{aligned}
(d) &\mapsto \bigcup_{d' \in (d)} A_{d'} = \bigcup_{d' > d} A_{d'} = \bigcup_{d' > d} (B_{d'} \times \{d'\}) = F_1 \in (\mathfrak{F}_\Theta C)_\emptyset \\
\text{and } [d] &\mapsto \bigcup_{d' \in [d]} A_{d'} = \bigcup_{d' \geq d} A_{d'} = \bigcup_{d' \geq d} (B_{d'} \times \{d'\}) = F_2 \in (\mathfrak{F}_\Theta C)_\emptyset.
\end{aligned}$$

In view of Lemma 2, it suffices to show that the interval $(F_1, F_2]$ of $(\mathfrak{F}C)_\emptyset$ has no least element and contains a dually cofinal copy of the chain C_d . For each $d \in D$ and $c \in C_d$ define

$$F_{dc} = \left\{ c' \in C : c' \geq_C \left(\binom{1}{c}, d \right) \in \mathbb{Z}^{(C_d)} \times \{d\} = A_d \right\}.$$

It is clear that $F_{dc} \in (\mathfrak{F}C)_\emptyset$ and $F_1 \subsetneq F_{dc} \subseteq F_2$. Moreover, it is easily checked that for each $d \in D$, the set $\{F_{dc} : c \in C_d\}$, regarded as a chain under *reversed* set inclusion, is isomorphic to C_d . Take $F' \in (\mathfrak{F}C)_\emptyset$ such that $F_1 \subsetneq F' \subseteq F_2$. Let $c' \in F' \setminus F_1$ and write

$$c' = \left(\left(\begin{array}{ccc} z_1 & \cdots & z_n \\ c_1 & \cdots & c_n \end{array} \right), e \right) \text{ with } e \in D \text{ and } \left(\begin{array}{ccc} z_1 & \cdots & z_n \\ c_1 & \cdots & c_n \end{array} \right) \in \mathbb{Z}^{(C_e)} = B_e.$$

If $e > d$ then $c' \in \bigcup_{d' > d} (B_{d'} \times \{d'\}) = F_1$, a contradiction. Hence $e \leq d$. Since C_e has, by hypothesis, no greatest element, we can choose $c \in C_e$ such that $c > c_n$. Then $\left(\binom{1}{c}, e \right) >_C c'$. Consequently, $F_{ec} \subsetneq F'$. But $e \leq d$ so we must have $F_1 \subsetneq F_{ec}$. There are two conclusions that can be drawn from the above argument. First, the interval $(F_1, F_2]$ of $(\mathfrak{F}C)_\emptyset$ contains no least element and secondly, $(F_1, F_2]$ contains a dually cofinal copy of C_d . As noted earlier, this completes the proof in view of Lemma 2. \square

We cannot dispense with the requirement that the chains C_ω be unbounded above, except at the cost of the cofinality property in (ii) (without the prospect of which the assumption of upper unboundedness may, in any event, be made without any loss of generality). If in the statement of (ii), we replace the open intervals $(f(x), f(y))$ by the half-open intervals $[f(x), f(y))$, then the upper unboundedness requirement on the C_ω may be dropped without sacrificing the cofinality assertions, but then these assertions say no more than that the top elements of the C_ω map to the bottom elements of the corresponding intervals $[f(x), f(y))$; the resulting theorem therefore provides little insight, in practice, into the structure of $\text{Id } T_T$, beyond placing a lower bound on its cardinality. Of course, in proving (i) of Theorem 3, we have reproved Theorem 1 ((i) \Rightarrow (ii)). Our choice of the given

arrangement of this material (over that of a postponed corollary stating Theorem 1) was motivated by the belief that the arguments are more easily digestible in the above form.

We have already noted that the requirement in Theorem 1(i) and Theorem 3 that L be algebraic cannot be dropped, in view of Corollary 1.2. Several questions arise concerning the possibility of improving Theorem 1 in other natural ways. The remainder of this section is devoted to showing that in all natural senses apparent to us, our representation of algebraic chains by (ideals of) rings cannot be improved except, of course, by extending it (if possible) to more general algebraic lattices: see Question D below. There is a known limit, however, to the extent of such generalization: it is not possible to represent all modular algebraic lattices by the ideals of rings, in view of Theorem 6.2 below.

We pose some naturally arising questions explicitly.

QUESTION A. May we strengthen condition (iii) of Theorem 1 by requiring the ring to have identity?

Rather obviously, the answer is no. If R is any ring then the compact elements of $\text{Id } R$ are precisely the finitely generated ideals of R . It follows that if R is a ring with identity then $\text{Id } R$ contains a compact top element, namely R itself. On the other hand, it is easily checked that the ordinal $\omega \oplus 1$, ordered naturally, is an algebraic chain; its top element, ω , is not compact (because $\bigvee X < \omega$ whenever X is a finite subset of $\omega \oplus 1$ consisting of finite ordinals). Thus $\omega \oplus 1$ is not isomorphic to the ideal lattice of any ring with identity. It was for this reason that both of conditions (ii) and (iii) were included in Theorem 1.

QUESTION B. Since every ring has an (additive) abelian group as a reduct and every abelian group is the reduct of a ring (at worst, a ring with zero multiplication), it is natural to ask whether one may represent arbitrary algebraic chains by the congruence lattices of these simpler reducts of rings. In other words, is every algebraic chain isomorphic to the subgroup lattice of an abelian group?

Again, the answer is no, because of the following result.

PROPOSITION 4. *Let $\langle L; \leq \rangle$ be any infinite algebraic chain which has a compact top element, for example, the ordinal $\omega \oplus 2$, ordered naturally. Then L is not isomorphic to any interval in the subgroup lattice of any group. In particular, L is not isomorphic to (any interval in) the congruence lattice of any abelian group.*

Proof. Let c be the compact top element of L . Suppose that L is isomorphic to an interval in the subgroup lattice of a group G . Because c is compact, it must have an immediate predecessor $d \in L$, say. Any element of the subgroup of G corresponding to c which is not in the subgroup corresponding to d must generate the former subgroup, since (d, c) is a covering pair and L is a chain. This means that the subgroup corresponding to c is a cyclic group and is infinite, since L is.

In other words, c corresponds to a subgroup of G that is isomorphic to $\langle \mathbb{Z}; +, -, 0 \rangle$. It follows that the least element of L must correspond to a subgroup $m\mathbb{Z}$ of \mathbb{Z} which is contained by infinitely many other subgroups of \mathbb{Z} . This forces $m = 0$, so L must be isomorphic to the entire subgroup lattice of \mathbb{Z} , which is not a chain, a contradiction. \square

Question B touches tangentially on a recent representation result and an open problem, which we mention here as a matter of interest. Pálffy and Pudlák [PP80, Theorem 2, p23] have proved that the following two assertions are equivalent: (1) every finite lattice is isomorphic to the congruence lattice of a finite algebra; (2) every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group. It is *not* known, however, whether these assertions are true.

QUESTION C. We noted in the sequel to Theorem 2.3 that we cannot improve condition (ii) of Theorem 1 by requiring T to be commutative (nor even right or left duo). Is the same true of condition (iii) of Theorem 1? In other words, is every algebraic chain isomorphic to the ideal lattice of a commutative (or, failing that, a one-sided duo) ring, not necessarily with identity?

The argument used in the sequel to Theorem 2.3 cannot be re-used immediately to answer this question, since it relied on the fact that every right artinian ring with identity is right noetherian, whereas the analogous entailment for rings without identity is false (as is witnessed, for example, by the abelian group \mathbb{Z}_{p^∞} , p a positive prime, considered as a ring with zero multiplication: see Chapter 0, §4). Nevertheless, the answer to Question C is also negative, as the next result will show.

PROPOSITION 5. *There is no right duo (in particular, no commutative) ring R whose ideal lattice is isomorphic to the ordinal $\omega \oplus 3$, ordered naturally.*

Proof. Suppose, on the contrary, that such a ring R exists. Since R is right artinian (the chain $\omega \oplus 3$ satisfies the DCC), $J(R)^n = 0$ for some $n \geq 2$ by Proposition 0.5.5. Observe that if G is a subgroup of $\langle R; +, -, 0 \rangle$ such that $J(R)^k R \subseteq G \subseteq J(R)^k$ for some positive integer k , then G is also a right ideal of R . But R is right duo so G must be an ideal of R . Consequently, the interval $[J(R)^k R, J(R)^k]$ in the lattice $\text{Id } R$ is identical to the same interval in the subgroup lattice of $\langle R; +, -, 0 \rangle$ for all positive integers k .

Let P be the unique maximal proper (right) ideal of R . Observe that if $J(R) \subseteq P \subsetneq R$, then $R/J(R)$ is a nonzero semiprimitive artinian ring whose lattice of right ideals is a chain. It is an obvious consequence of the Artin-Wedderburn Theorem (Theorem 0.7.1) that in this instance, $R/J(R)$ must be a division ring and therefore $J(R) = P$. In other words, $J(R) = R$ or $J(R) = P$. We now divide our argument into two cases.

Case 1. $J(R) = R$. As noted above, the interval $[J(R)^k R, J(R)^k] = [R^{k+1}, R^k]$ in $\text{Id } R$ is identical to the same interval in the subgroup lattice of $\langle R; +, -, 0 \rangle$ for all positive integers k . Since $\text{Id } R \cong \omega \oplus 3$, it is clear that $[R^2, R] \cong \omega \oplus 3$ or $[R^2, R]$ is a subchain of the three-element chain. The

former possibility is excluded by Proposition 4 since $\omega \oplus 3$ has a compact top element, so we must have that $\langle R; +, -, 0 \rangle / \langle P; +, -, 0 \rangle$ and $\langle P; +, -, 0 \rangle / \langle R^2; +, -, 0 \rangle$ are simple abelian groups (the possibility exists of course, that the latter factor group is zero). Therefore, there must exist positive prime integers p and q (not necessarily distinct) such that $\langle R; +, -, 0 \rangle / \langle P; +, -, 0 \rangle \cong \mathbb{Z}_p$ and $\langle P; +, -, 0 \rangle / \langle R^2; +, -, 0 \rangle \lesssim \mathbb{Z}_q$, so $pR \subseteq P$ and $qP \subseteq R^2$. It follows that $pR^k \subseteq R^{k-1}P$ and $q(R^{k-1}P) \subseteq R^{k+1}$ for all $k \geq 2$. This means that $\langle R^k; +, -, 0 \rangle / \langle R^{k-1}P; +, -, 0 \rangle$ is isomorphic to a direct sum of copies of the abelian group \mathbb{Z}_p and $\langle R^{k-1}P; +, -, 0 \rangle / \langle R^{k+1}; +, -, 0 \rangle$ is isomorphic to a direct sum of copies of the abelian group \mathbb{Z}_q for all $k \geq 2$. Since $[R^{k+1}, R^k]$ is always a chain we must have $\langle R^k; +, -, 0 \rangle / \langle R^{k-1}P; +, -, 0 \rangle \lesssim \mathbb{Z}_p$ and $\langle R^{k-1}P; +, -, 0 \rangle / \langle R^{k+1}; +, -, 0 \rangle \lesssim \mathbb{Z}_q$ whenever $k \geq 2$. It follows that each of the intervals $[R^{k+1}, R^k]$ is finite. Consequently, the interval $[R^n, R] = [0, R]$ (because $R^n = J(R)^n = 0$) is finite, a contradiction.

Case 2. $J(R) = P$. Again, we start by noting that the interval $[P^k R, P^k]$ in $\text{Id } R$ is identical to the same interval in the subgroup lattice of $\langle R; +, -, 0 \rangle$ for all positive integers k . Since the interval $[0, P]$ of $\text{Id } R$ is isomorphic to $\omega \oplus 2$, it is clear that $[PR, P] \cong \omega \oplus 2$ or $[PR, P]$ is a subchain of the two-element chain. The former possibility is again excluded by Proposition 4, so we must have that $\langle P; +, -, 0 \rangle / \langle PR; +, -, 0 \rangle$ is a simple abelian group. An argument identical to that used in Case 1 above, shows that the interval $[P^k R, P^k]$ is finite for all $k \geq 1$. Our task now is to show that each of the intervals $[P^{k+1}, P^k R]$ is finite. Inasmuch as $R/J(R)$ is a ring with identity, it follows that $R/J(R) = (R/J(R))^2 = R^2/J(R)$, so $R^2 + J(R) = R$. Since R is a right chain ring this can only be satisfied if $R^2 = R$. Take $k \geq 1$ and set $I_k = (P^k R)/P^{k+1}$. Regarding I_k as a right R -module, it is clear that I_k is unital (because $I_k R = (P^k R^2)/P^{k+1} = (P^k R)/P^{k+1} = I_k$) and $I_k J(R) = I_k P = 0$. We may therefore regard I_k as a unital right module over the division ring $R/J(R)$. It follows that I_k is a semisimple R -module. Since the lattice of submodules of I_k is a chain, this can only be the case if I_k is simple. It follows that the interval $[P^{k+1}, P^k R]$ is finite. Consider now the descending chain

$$R \supseteq P \supseteq PR \supseteq P^2 \supseteq P^2 R \supseteq P^3 \supseteq \dots \supseteq P^n \supseteq P^n R = 0$$

of ideals of R . Since each interval of the form $[P^k R, P^k]$ or $[P^{k+1}, P^k R]$, $k \geq 1$, is finite, it follows that the interval $[0, R]$ is finite, a contradiction. \square

We conclude this section by stating a difficult problem the answer to which is certainly not known – indeed, the question may even have been neglected deliberately – although our Theorem 1 and some other results in the literature offer evidence in favour of an affirmative answer. (Some such evidence will be discussed in the next section, where the problem will also be placed in a natural universal algebraic context.)

QUESTION D. Can Theorem 1 be generalized to a characterization of algebraic distributive lattices as the congruence (i.e., ideal) lattices of (necessarily arithmetical) rings? (A ring R is called

arithmetical if it has a distributive lattice of ideals; this turns out to be equivalent to requiring R to satisfy a certain formulation of the so-called Chinese Remainder Theorem: see [ZS58, Theorem 18, p280].)

§6. A UNIVERSAL ALGEBRAIC PERSPECTIVE ON RIGHT CHAIN DOMAINS.

In this final section, we enlarge our perspective on Theorem 5.1, since it is of some interest also from the point of view of universal algebra.

Recall that Birkhoff and Frink's Theorem (Theorem 1.1) told us that the congruence lattice $\text{Con } A$ of an arbitrary (universal) algebra A (of any similarity type) is always an algebraic lattice. This result has a celebrated converse, viz. the *Grätzer-Schmidt Theorem*, which we quote as the next result.

THEOREM 1. (Grätzer and Schmidt) [Grä79, Theorem 3, p112] *Every algebraic lattice is isomorphic to the congruence lattice of a (universal) algebra of some similarity type.* \square

Let us say that a lattice L is *represented* by an algebra A if L is isomorphic to the congruence lattice of A . Jónsson [Jón79] formulated the following question, which is a natural response to the Grätzer-Schmidt Theorem: how much information (if any) is given by a similarity type $\tau = \langle F; ar \rangle$ about the congruence lattices of the algebras of type τ ? An answer to this rather general question was provided promptly by the following result of Freese, Lampe and Taylor.

THEOREM 2. [FLT79, Theorem 1, p62] *For every similarity type τ , there exists a modular algebraic lattice L such that for every universal algebra A of type τ , the congruence lattice $\text{Con } A$ is not isomorphic to L .* \square

Thus all similarity types τ give some information about the (modular) congruence lattices of algebras of type τ , answering Jónsson's initial question. Further related questions arise naturally, however. For example, it is not known whether similarity types provide any information about their associated *distributive* congruence lattices.

There is a strong connection between the aforementioned question and a conjecture of Grätzer and Schmidt, first proposed in the 1960s, to the effect that every algebraic distributive lattice is representable by a lattice². The problem has a natural lattice theoretic curiosity value, but researchers working on the conjecture are also motivated by the knowledge that from an affirmative solution one

²In 1993, it was reported that this conjecture had been settled affirmatively by Tischendorf and Tůma, and the preprint [TT93] was circulated, which contains an argument to that effect. During 1994 a flaw was found in this argument and the authors of the preprint, as well as specialists at the Universities of Hawaii and Manitoba attempted to circumvent the flaw. We have recently received reports that, to date, no such efforts have produced a proof. We understand, therefore, that the Grätzer-Schmidt Conjecture remains an open problem.

could infer that for any similarity type τ that is no poorer than $(2, 2)$ (i.e., that has at least two binary operation symbols), the specification of τ would provide *no* information about the distributive congruence lattices of algebras of type τ (contrasting sharply with the situation for the modular congruence lattices). Indeed, one could say that for any algebraic distributive lattice L , there is an algebra A of type $(2, 2)$, viz. a lattice, with $\text{Con } A \cong L$. One could then enrich A by defining on it arbitrarily chosen *constant* operations of appropriate arity for all operation symbols of τ other than any two of the binary operation symbols (these two being reserved for interpretation as \vee and \wedge). Since the new operations are constant, their introduction leaves the congruences of A unchanged, so the congruence lattice of the enriched algebra is still just $\text{Con } A$, and is therefore isomorphic to L . Since L was an arbitrary algebraic distributive lattice, τ clearly says nothing about its associated distributive congruence lattices.

An alternative approach to the problem of quantifying the information about distributive congruence lattices given by a similarity type would have been to try to prove an affirmative answer to Question D of the previous section – is every algebraic distributive lattice representable by a (necessarily arithmetical) ring? – then to apply a line of reasoning similar to that of the previous paragraph. But for the question to be considered seriously, two obvious tests must be passed. Indeed, since finiteness of a lattice implies algebraicity and linear order implies distributivity, one would want to know that (1) every finite distributive lattice, and (2) every algebraic chain is representable by a ring.

THEOREM 3. [KR80, Corollary, p1289] *Every finite distributive lattice is isomorphic to the ideal (i.e. congruence) lattice of a (von Neumann regular) ring.* □

The following qualification of this result is also interesting.

PROPOSITION 4. [Pál87, Proposition 3.1, p156] *There exists a finite distributive lattice which is not isomorphic to the ideal (i.e. congruence) lattice of any finite ring.* □

Thus Kim and Roush [KR80] proved that rings pass test (1) and we have proved in Theorem 5.1 that they also pass test (2), which would seem to vindicate the belief that Question D is worthy of investigation. Part of our motivation was to pursue this line of thought in the hope of contributing to a future ring theoretic proof that almost all similarity types provide no information about their distributive congruence lattices. The relative neglect to date of this question may be attributable to the fact that rings, not being congruence distributive in general, would seem at first glance an unlikely candidate class for the successful representation of all *distributive* algebraic lattices. Secondly, we have already alluded to the possibility of a subconscious expectation that rings, unlike lattices, are too highly structured by their language and too constrained by their axioms to stand any chance of providing representations for all possible algebraic distributive lattices. Such a prejudice, if it exists, would have been all the more justifiable given the absence, until now, of a construction for

representing even the smaller class of all algebraic chains by rings. Our Theorems 5.1 and 5.3 tend, however, to challenge the possible preconception that rings are “too highly structured”.

By contrast, it should be mentioned that there are other quite simple classes of algebras (of fixed type) whose linearly ordered congruence lattices exhaust the class of all algebraic chains. Indeed, every algebraic chain is isomorphic to the congruence lattice of a lattice (a rather trivial direct proof is possible); it is also isomorphic, as shown by a quite simple proof in [AN81, Claim 1, p301], to the congruence lattice of a special kind of groupoid, viz. a linearly ordered “Hilbert algebra” in the sense of [Die66] (known also as a “positive implicative *BCK*-algebra” [Cor82] – such algebras arise from logic as the algebraic counterparts of the implicational fragments of intuitionistic propositional calculi).

Because of the latter representation, it has been known for some time that *any non-unary similarity type τ gives no information about the linearly ordered congruence lattices of algebras of type τ* . (A type $\tau = \langle F; ar \rangle$ is called *non-unary* if it contains at least one operation symbol that is non-unary and non-nullary, i.e., at least one $f \in F$ has $ar(f) > 1$.) The proof follows precisely the line of argument used in the penultimate paragraph preceding Theorem 3 above. In this sense, therefore, our Theorem 5.1 does not add to the existing knowledge about the linearly ordered congruence lattices of algebras of a fixed non-unary type.

There is another sense, however, in which our result provides new information of this sort. To see this, we need to consider compatible binary relations on algebras that are weaker than congruences. A good many of these have been studied in the literature over the past two decades but the notion of generalized congruence that would seem to be the most useful is that of a “tolerance relation”. A binary relation on the universe of an algebra A is called a *tolerance relation* (or briefly, a *tolerance*) on A if it is reflexive, symmetric and compatible with all specified operations on A . The lattice $\text{Tol } A$ of all tolerances (ordered by set inclusion) on any given algebra A (of any similarity type) is well known to be algebraic. Conversely, in the spirit of Grätzer and Schmidt, it has recently been proved by Chajda and Czédli [CC92, Theorem (a)] that *every algebraic lattice is isomorphic to the tolerance lattice $\text{Tol } A$ of some (universal) algebra A of some type*. Naturally, this raises a tolerance analogue of Jónsson’s question(s).

An algebra A is said to be *tolerance trivial* if all of its tolerances are transitive, and therefore congruences, i.e., $\text{Tol } A = \text{Con } A$. It is very well known that all *rings are tolerance trivial*, as a consequence of much more general results in universal algebra (see, e.g., [Wer73, Theorem] or [Cha83, Theorem 1, p36], and [BS81, Theorem 12.2 & Example (1), pp78-79]). Thus our *Theorem 5.1 is also a tolerance representation result*. By contrast, linearly ordered Hilbert algebras and lattices with linearly ordered congruences need not be tolerance trivial, according to [RRS91, Remark 2.13 b, p407 & Remark 2.18 a, p408] and [Cha88, Examples 1 & 2, pp219-220], so the known results about them say nothing about tolerance representation. More importantly, when considered as a fact about tolerances, Theorem 5.1 has the following corollary, which is a new result that cannot (apparently) be deduced from the consideration of Hilbert algebras or lattices. Again, the proof follows exactly the line of

argument that was used in the penultimate paragraph preceding Theorem 3.

PROPOSITION 5. *Let τ be a similarity type that is no poorer than $\langle 2, 2 \rangle$. Then the specification of τ provides no information about the linearly ordered tolerance lattices of the algebras of type τ .* \square

We may use the type $\langle 2, 2 \rangle$ in this proposition, rather than the precise value $\langle 2, 2, 1, 0 \rangle$ of the type of rings without identity. Indeed, rings may be reconsidered as algebras $\langle R; -, \cdot \rangle$ of type $\langle 2, 2 \rangle$ without affecting their tolerance lattices since, in any ring, we have the identities $0 = x - x$, $-x = 0 - x$ and $x + y = x - (-y)$.

Chapter II

Torsion Preradicals

“Torsion theoretic” notions have been present for some time in the theory of rings with identity. They are approachable in any one of three equivalent ways. We shall describe these approaches precisely (and in greater generality) in the first two sections this chapter, but, roughly speaking, they are as follows. First, one may consider a “hereditary pretorsion class” (or more strongly, a “hereditary torsion class”), which means a class of modules over a given ring R which are “torsion” in a sense generalizing torsion abelian groups. Secondly, one may consider “topologizing filters” (or more strongly, “Gabriel filters”) on R : these are filters (in the lattice sense) of right ideals K of R with the property that the module R/K is, in a suitable sense, “torsion”. Lastly, one may study “torsion preradicals” (or more strongly, “torsion radicals”). These are functors on the ring’s module category that associate with a module its “torsion submodule”. These notions are interderivable; each may be taken as the starting point for a torsion theoretic investigation. They describe the same mathematical reality.

The hereditary pretorsion class of a torsion preradical is closed, by definition, under finite direct products; if closed under arbitrary direct products, the torsion preradical is called Jansian. The principal notion to be investigated in this chapter is that of an \mathfrak{m} -Jansian torsion preradical; this is a torsion preradical whose hereditary pretorsion class is closed under direct products of fewer than \mathfrak{m} modules, where \mathfrak{m} is a given regular cardinal. In Theorem 3.10 we shall prove that if \mathfrak{m} is any regular cardinal then a right topologizing filter \mathcal{F} on a ring R is closed under intersections of fewer than \mathfrak{m} right ideals if and only if its corresponding torsion preradical $\text{torsp } \mathcal{F}$ is \mathfrak{m} -Jansian. A topologizing filter \mathcal{F} satisfying the equivalent conditions of this theorem will also be called \mathfrak{m} -Jansian.

There are several reasons why these notions are of interest. In the first place, it is well known that the requirement that all right torsion preradicals on a ring with identity be Jansian characterizes the ring as right artinian. It is natural to ask what artinian-like properties survive if we weaken Jansian to \mathfrak{m} -Jansian (where $\mathfrak{m} \geq \aleph_1$). Secondly, for prime rings and regular cardinals $\mathfrak{m} \geq \aleph_0$, we shall be able to show that the behaviour of the \mathfrak{m} -Jansian torsion preradicals determines a “degree of primeness” of the ring which extends some well studied primeness conditions in the literature, such as “(bounded) strongly prime”. Moreover, the primeness conditions at issue have led to the investigation of special radicals in the category of rings, in which context it is unavoidable that we consider rings without identity¹. This being so, it is natural to ask whether the torsion theoretic results of interest to us (such as the aforementioned characterization of right artinian rings) remain true for rings without

¹This investigation will be carried out in Chapter IV. We wish to stress that radicals in the category of rings should not be confused with the radicals (and torsion radicals) on module categories defined in §1 of this chapter. They are quite different.

identity. Part of the purpose of this chapter is to show that for the most part, they do.

In §3 we examine Jansian and \mathfrak{m} -Jansian torsion preradicals and topologizing filters. In particular, we establish an important connection between the “lattice ordered semigroup” of ideals of an arbitrary ring R and the set of Jansian torsion preradicals on $\text{Mod-}R$. §4 is devoted to the study of right “ \mathfrak{m} -closed” rings, that is, rings all of whose right topologizing filters are \mathfrak{m} -Jansian. It turns out that for $\mathfrak{m} \geq \aleph_1$, an \mathfrak{m} -closed ring R enjoys many artinian-like properties, e.g., it satisfies the descending chain condition on (two-sided) ideals and is semilocal in the sense that $R/J(R)$ is a semisimple artinian ring, where $J(R)$ denotes the Jacobson radical of R . (Since $J(R/J(R)) = 0$, this means that $R/J(R)$ is either zero or a finite direct product of finite dimensional matrix rings over division rings.) In §5 we extend the classical notion of Morita equivalence to rings without identity. The main result of §5 asserts that for any regular cardinal \mathfrak{m} , the property “right \mathfrak{m} -closed” is invariant with respect to our extended notion of Morita equivalence. Finally, in §6, we focus on prime rings and demonstrate that in such rings the notion of an \mathfrak{m} -Jansian torsion preradical can be used to describe and determine a “degree of primeness” of the ring.

A version of the results of §4 (restricted to rings with identity) has been published in [VR94] and a similarly restricted version of parts of §3 and §6 has been published in [RV92].

§1. TORSION PRERADICALS ON A MODULE SUBCATEGORY.

Classical torsion theory is usually developed in the context of a category of unital modules over a ring with identity. This approach, however, is not quite general enough for our purposes. We need to develop a torsion theory in certain special subcategories of the category of all modules over an arbitrary associative ring, with or without identity. We point out that our approach is certainly not the most general available. Indeed, much pioneering work in torsion theory was done in the setting of an abelian category (see [Gab62] and [Ste71]). Most of the results of this first section of the present chapter are standard, at least at some level of generality. We have therefore chosen to include proofs only in cases where we believe a result to be less well known.

Recall that a subcategory \mathfrak{D} of \mathfrak{C} is called a *full subcategory* of \mathfrak{C} if $\text{Hom}_{\mathfrak{C}}(A, B) = \text{Hom}_{\mathfrak{D}}(A, B)$ for all objects A, B of \mathfrak{D} . Let R be an arbitrary ring. We shall call a full subcategory \mathfrak{C} of $\text{Mod-}R$ a *module subcategory* of $\text{Mod-}R$ if \mathfrak{C} is closed under homomorphic images, submodules, direct products (and therefore direct sums) and essential extensions. (It is easily shown that a module subcategory is necessarily closed under module extensions, that is to say, $M \in \mathfrak{C}$ whenever $N \leq M$ and $N, M/N \in \mathfrak{C}$.) For the most part, our choice of \mathfrak{C} will be $\text{Mod-}R$ itself.

Let \mathfrak{C} be a module subcategory of $\text{Mod-}R$. A functor $\tau: \mathfrak{C} \rightarrow \mathfrak{C}$ will be called a *preradical* on \mathfrak{C} if the following conditions are satisfied:

- P1. $\tau(M) \leq M$ for all $M \in \mathfrak{C}$;

P2. if $M, N \in \mathcal{C}$ and $f \in \text{Hom}_R(M, N)$ then $f[\tau(M)] \leq \tau(N)$ and $\tau(f) = f|_{\tau(M)}$ (the restriction of f to $\tau(M)$).

A preradical on \mathcal{C} is always a proper class so, strictly speaking, there is no such thing as the class of all preradicals on \mathcal{C} . Nevertheless, if, for preradicals τ and σ on \mathcal{C} , we write $\tau \leq \sigma$ to denote that $\tau(M) \subseteq \sigma(M)$ for all $M \in \mathcal{C}$, then \leq has the properties of a partial order. Loosely speaking, one might say that the “collection” of all preradicals on \mathcal{C} is partially ordered by the “relation” \leq . In fact, it is easily checked that this “partially ordered collection” has all of the properties of a complete modular lattice. Since, in the general case, we shall only ever need to say things about the “elements” of this “collection” (as opposed to making assertions about about the “collection” as an aggregate in its own right), we permit ourselves this and other order-theoretic abuses of nomenclature. Observe that the preradical 0 defined by $0(M) = 0$ for all $M \in \mathcal{C}$ is the smallest preradical on \mathcal{C} (with respect to \leq) and the identity functor $1_{\mathcal{C}}$ is the largest. Let τ be a preradical on \mathcal{C} and $M \in \mathcal{C}$. We say that M is τ -torsion (resp. τ -torsion-free) if $\tau(M) = M$ (resp. $\tau(M) = 0$). We denote the class of all τ -torsion (resp. τ -torsion-free) modules in \mathcal{C} by T_{τ} (resp. F_{τ}). Thus,

$$T_{\tau} := \{M \in \mathcal{C} : \tau(M) = M\}; \text{ and}$$

$$F_{\tau} := \{M \in \mathcal{C} : \tau(M) = 0\}.$$

PROPOSITION 1. [Ste75, Proposition 1.2, p137] *Let R be a ring and \mathcal{C} a module subcategory of $\text{Mod-}R$. If τ is a preradical on \mathcal{C} then:*

- (i) T_{τ} is closed under homomorphic images and direct sums;
- (ii) F_{τ} is closed under submodules and direct products. □

Using P2 and the fact that left multiplication by an element of the ring R is an endomorphism of the module R_R , we also obtain:

PROPOSITION 2. *Let R be a ring and \mathcal{C} a module subcategory of $\text{Mod-}R$. If τ is a preradical on \mathcal{C} and $R_R \in \mathcal{C}$ then $\tau(R_R)$ is an ideal of R .* □

Let \mathcal{C} be a module subcategory of $\text{Mod-}R$. We call a preradical τ on \mathcal{C} a *torsion preradical* if $\tau(N) = N \cap \tau(M)$ whenever $M, N \in \mathcal{C}$ with $N \leq M$. The “collection” of all torsion preradicals on \mathcal{C} will be denoted by $\text{torsp-}\mathcal{C}$. Trivially, the functors 0 and $1_{\mathcal{C}}$ belong to $\text{torsp-}\mathcal{C}$. If $\mathcal{C} = \text{Mod-}R$, we write $\text{torsp-}R$ for $\text{torsp-}\mathcal{C}$. Notice that if $\tau \in \text{torsp-}\mathcal{C}$ then for every $M \in \mathcal{C}$, we have $\tau(M) \in T_{\tau}$; in fact, $\tau(M)$ is the largest submodule of M that is an element of T_{τ} .

In Theorem 2.4 below we shall establish that with every torsion preradical τ on $\text{Mod-}R$, one may associate a set \mathcal{F} of right ideals of the Dorroh Extension R^* of R . (We refer the reader to Chapter 0, §4 for a definition of the Dorroh Extension.) Moreover, this set \mathcal{F} of right ideals determines τ uniquely. Since the class of all subsets of right ideals of a fixed ring is palpably a set, we

may treat $\text{torsp-}R$ as a set. In the sequel to Theorem 11 we shall show how every torsion preradical on \mathcal{C} may be identified with a torsion preradical on $\text{Mod-}R$. This allows us to treat $\text{torsp-}\mathcal{C}$ as a set as well. We reassure the reader that no loss of rigour results if, henceforth, $\text{torsp-}\mathcal{C}$ is treated as a set and \leq as a relation on $\text{torsp-}\mathcal{C}$.

PROPOSITION 3. [Ste75, Proposition 1.7, p138] *Let R be a ring and \mathcal{C} a module subcategory of $\text{Mod-}R$. If τ is a torsion preradical on \mathcal{C} then:*

- (i) T_τ is closed under homomorphic images, direct sums and submodules;
- (ii) F_τ is closed under submodules, direct products and essential extensions. □

A nonempty class \mathcal{A} of right R -modules is called a *hereditary pretorsion class* if \mathcal{A} is closed under homomorphic images, direct sums and submodules. Proposition 3 therefore asserts that if \mathcal{C} is a module subcategory of $\text{Mod-}R$ and $\tau \in \text{torsp-}\mathcal{C}$ then T_τ is a hereditary pretorsion class contained in \mathcal{C} . We shall refer to T_τ as the *hereditary pretorsion class of τ* . In fact, every hereditary pretorsion class \mathcal{A} contained in \mathcal{C} arises in this way. Indeed, it can be shown that the torsion preradical τ on \mathcal{C} defined by

$$\tau(M) = \sum \{N \leq M : N \in \mathcal{A}\} \text{ for all } M \in \mathcal{C},$$

is such that $T_\tau = \mathcal{A}$. Moreover, if $\sigma \in \text{torsp-}\mathcal{C}$ then it is possible to recover σ from T_σ , in the sense that

$$\sigma(M) = \sum \{N \leq M : N \in T_\sigma\} \text{ for all } M \in \mathcal{C}.$$

The next theorem follows immediately.

THEOREM 4. [Ste75, Corollary 1.8, p138] *Let R be a ring and \mathcal{C} a module subcategory of $\text{Mod-}R$. Then the map $\tau \mapsto T_\tau$ ($\tau \in \text{torsp-}\mathcal{C}$) defines a bijection from the set of all torsion preradicals on \mathcal{C} onto the set of all hereditary pretorsion classes of right R -modules contained in \mathcal{C} . □*

If \mathcal{C} is a module subcategory of $\text{Mod-}R$ then $\text{torsp-}\mathcal{C}$ has the structure of a complete lattice. If $\{\tau_i : i \in \Gamma\}$ is a subfamily of $\text{torsp-}\mathcal{C}$ then $\bigwedge_{i \in \Gamma} \tau_i \in \text{torsp-}\mathcal{C}$ is defined by $(\bigwedge_{i \in \Gamma} \tau_i)(M) = \bigcap_{i \in \Gamma} \tau_i(M)$ for all $M \in \mathcal{C}$. Since $\text{torsp-}\mathcal{C}$ is a complete lattice, given any nonempty subclass \mathcal{A} of \mathcal{C} , there is a (unique) smallest torsion preradical τ on \mathcal{C} such that $T_\tau \supseteq \mathcal{A}$. We call this τ the *torsion preradical on \mathcal{C} generated by \mathcal{A}* . Thus $\tau = \bigwedge \{\lambda \in \text{torsp-}\mathcal{C} : T_\lambda \supseteq \mathcal{A}\}$. If τ is an arbitrary torsion preradical on \mathcal{C} it is not difficult to show that τ is generated by a class \mathcal{A} of cyclic right R -modules contained in \mathcal{C} . More particularly, since every cyclic right R -module is isomorphic to an element of the set $\{R^*/I : I \leq R_R^*\}$, we may assume, without loss of generality, that \mathcal{A} is a set. Thus every $\tau \in \text{torsp-}\mathcal{C}$ is generated by a set of cyclic right R -modules. The notion of a torsion preradical generated by a class of modules has a dual. It is easily checked that the functor $\sigma : \mathcal{C} \rightarrow \mathcal{C}$ defined by

$$\sigma(M) = \bigcap \{ \text{Ker } f : f \in \text{Hom}_R(M, E(A)) \text{ for some } A \in \mathcal{A} \} \text{ for all } M \in \mathcal{C},$$

is the largest torsion preradical on \mathcal{C} such that $F_\sigma \supseteq \mathcal{A}$. Consequently, $\sigma = \bigvee \{ \lambda \in \text{torsp-}\mathcal{C} : F_\lambda \supseteq \mathcal{A} \}$. This σ is called the *torsion preradical on \mathcal{C} cogenerated by \mathcal{A}* . It also follows from the completeness of $\text{torsp-}\mathcal{C}$ that if τ is a preradical on \mathcal{C} then there is a smallest torsion preradical on \mathcal{C} , denoted by $\hat{\tau}$, such that $\tau \leq \hat{\tau}$. Thus $\hat{\tau} = \bigwedge \{ \lambda \in \text{torsp-}\mathcal{C} : \tau \leq \lambda \}$. The torsion preradical $\hat{\tau}$ has a more explicit description, however.

PROPOSITION 5. *Let R be a ring and \mathcal{C} a module subcategory of $\text{Mod-}R$. For each $M \in \mathcal{C}$, let $\iota_M: M \rightarrow E(M)$ denote the embedding of M in its injective hull, $E(M)$. If τ is a preradical on \mathcal{C} then*

$$\hat{\tau}(M) = \iota_M^{-1}[\iota_M[M] \cap \tau(E(M))] \text{ for all } M \in \mathcal{C}.$$

Proof. For each $M \in \mathcal{C}$, define $\lambda(M) = \iota_M^{-1}[\iota_M[M] \cap \tau(E(M))] \leq M$. Let $f \in \text{Hom}_R(N, M)$ with $N, M \in \mathcal{C}$. We claim that $f[\lambda(N)] \subseteq \lambda(M)$. Inasmuch as $E(M)$ is injective, there must exist a $g \in \text{Hom}_R(E(N), E(M))$ such that the following diagram commutes.

$$\begin{array}{ccc} E(N) & \xrightarrow{g} & E(M) \\ \iota_N \uparrow & & \uparrow \iota_M \\ N & \xrightarrow{f} & M \end{array}$$

Now $\iota_M[f[\lambda(N)]] = (\iota_M \circ f)[\lambda(N)] = (g \circ \iota_N)[\lambda(N)] = g[\iota_N[N] \cap \tau(E(N))] \subseteq g[\tau(E(N))] \subseteq \tau(E(M))$. Since $f[\lambda(N)] \leq M$, we must have that $f[\lambda(N)] \subseteq \iota_M^{-1}[\iota_M[M] \cap \tau(E(M))] = \lambda(M)$. We may therefore regard λ as a preradical on \mathcal{C} . Now suppose, in the context of the above diagram, that $N \leq M$ and that $f: N \rightarrow M$ is the inclusion map. Since ι_N is a monomorphism whose image is essential in $E(N)$ and $\iota_M \circ f$ is a monomorphism, we must have that g is a monomorphism. Now

$$\begin{aligned} \iota_M[N \cap \lambda(M)] &= (g \circ \iota_N)[N] \cap [\iota_M[M] \cap \tau(E(M))] \\ &= (g \circ \iota_N)[N] \cap \tau(E(M)) \text{ (since } (g \circ \iota_N)[N] \subseteq \iota_M[M] \text{)}. \end{aligned}$$

Since $g: E(N) \rightarrow E(M)$ is a monomorphism, it follows that $g[E(N)]$ is an injective submodule of $E(M)$, and hence a direct summand of $E(M)$. Write $E(M) = g[E(N)] \oplus K$ with $K \leq E(M)$. Then

$$\begin{aligned} \iota_M[N \cap \lambda(M)] &= (g \circ \iota_N)[N] \cap \tau(E(M)) = (g \circ \iota_N)[N] \cap \tau(g[E(N)] \oplus K) \\ &= (g \circ \iota_N)[N] \cap [\tau(g[E(N)]) \oplus \tau(K)] \\ &= (g \circ \iota_N)[N] \cap \tau(g[E(N)]) \text{ (since } K \cap (g \circ \iota_N)[N] \subseteq K \cap g[E(N)] = 0 \text{)} \\ &= (g \circ \iota_N)[N] \cap g[\tau(E(N))] \\ &= g[\iota_N[N] \cap \tau(E(N))] \\ &= (g \circ \iota_N)[\lambda(N)] \end{aligned}$$

$$= \iota_M[\lambda(N)].$$

Thus $N \cap \lambda(M) = \lambda(N)$. This shows that λ is a torsion preradical on \mathcal{C} . It is clear that if σ is any torsion preradical on \mathcal{C} such that $\tau \leq \sigma$, then $\sigma(M) \supseteq \iota_M^{-1}[\iota_M[M] \cap \tau(E(M))] = \lambda(M)$ for all $M \in \mathcal{C}$. Thus $\hat{\tau} = \lambda$. \square

We remark that if $M \in \text{Mod-}R$ and $\iota_M: M \rightarrow E(M)$ is an embedding of M into its injective hull $E(M)$, it is customary to identify M with its image in $E(M)$. Under this identification, Proposition 5 asserts that if τ is a preradical on \mathcal{C} then $\hat{\tau}(M) = M \cap \tau(E(M))$ for all $M \in \mathcal{C}$.

If \mathcal{C} is a module subcategory of $\text{Mod-}R$ then Proposition 5 can be used to give an explicit description of the join operation in $\text{torsp-}\mathcal{C}$. Indeed, if $\{\tau_i: i \in \Gamma\}$ is a family of torsion preradicals on \mathcal{C} then it is easily checked that the preradical τ on \mathcal{C} defined by

$$\tau(M) = \sum_{i \in \Gamma} \tau_i(M) \quad \text{for all } M \in \mathcal{C},$$

is the smallest preradical on \mathcal{C} for which $\tau_i \leq \tau$ for all $i \in \Gamma$. Clearly, then, $\hat{\tau}$ must be the join of $\{\tau_i: i \in \Gamma\}$ in $\text{torsp-}\mathcal{C}$. Thus,

$$(\bigvee_{i \in \Gamma} \tau_i)(M) = M \cap \sum_{i \in \Gamma} \tau_i(E(M)) \quad \text{for all } M \in \mathcal{C}.$$

Recall that an element x of a complete lattice $\langle L; \leq \rangle$ is called *compact* if, whenever $X \subseteq L$ and $x \leq \bigvee X$, there exists a finite subset X' of X such that $x \leq \bigvee X'$. Recall also that a complete lattice $\langle L; \leq \rangle$ is called *algebraic* if every element of L is the join of a set of compact elements of L . It can be shown that if \mathcal{C} is any module subcategory of $\text{Mod-}R$ then the compact elements of $\text{torsp-}\mathcal{C}$ are precisely those torsion preradicals on \mathcal{C} which are generated by a single, cyclic right R -module in \mathcal{C} . In the sequel to Theorem 4, however, we noted that every $\tau \in \text{torsp-}\mathcal{C}$ is generated by a set of cyclic right R -modules in \mathcal{C} . It follows, as a consequence, that every torsion preradical on \mathcal{C} is the join of a set of compact elements of $\text{torsp-}\mathcal{C}$. Thus $\text{torsp-}\mathcal{C}$ is an algebraic lattice.

Rather surprisingly, the following useful result does not appear in standard texts on hereditary pretorsion theories such as [Gol87] – not even at the concrete level where $\mathcal{C} = \text{Mod-}R$ (or the category of unital right R -modules) for a ring R with identity. In fact, the manner in which order theoretic properties of $\text{torsp-}R$ are reported in [Gol87] suggests that the following is not a known result.

PROPOSITION 6. *Let R be a ring and \mathcal{C} a module subcategory of $\text{Mod-}R$. Then $\text{torsp-}\mathcal{C}$ is a modular algebraic lattice (with respect to the relation \leq).*

Proof. The fact that $\text{torsp-}\mathcal{C}$ is algebraic is noted above. It remains to show that $\text{torsp-}\mathcal{C}$ satisfies the modular law. Suppose $\tau, \sigma, \lambda \in \text{torsp-}\mathcal{C}$ with $\tau \leq \sigma$. If $M \in \mathcal{C}$ then

$$[(\lambda \wedge \sigma) \vee \tau](M) = M \cap [(\lambda \wedge \sigma)(E(M)) + \tau(E(M))] = M \cap [(\lambda(E(M)) \cap \sigma(E(M))) + \tau(E(M))].$$

Since $\tau(E(M)) \subseteq \sigma(E(M))$, it follows from the modular law (applied to the lattice of submodules of

$E(M)$) that $[\lambda(E(M)) \cap \sigma(E(M))] + \tau(E(M)) = [\lambda(E(M)) + \tau(E(M))] \cap \sigma(E(M))$. Therefore,

$$\begin{aligned} [(\lambda \wedge \sigma) \vee \tau](M) &= M \cap [[\lambda(E(M)) + \tau(E(M))] \cap \sigma(E(M))] \\ &= M \cap [(\lambda \vee \tau)(E(M)) \cap \sigma(E(M))] \\ &= M \cap [(\lambda \vee \tau) \wedge \sigma](E(M)) \\ &= [(\lambda \vee \tau) \wedge \sigma](M). \end{aligned}$$

Thus $(\lambda \wedge \sigma) \vee \tau = (\lambda \vee \tau) \wedge \sigma$. □

If \mathcal{C} is a module subcategory of $\text{Mod-}R$ then it is possible to define a binary operation \cdot (to be thought of as a kind of multiplication) on $\text{torsp-}\mathcal{C}$ as follows. Take τ, σ in $\text{torsp-}\mathcal{C}$ and define $\tau \cdot \sigma \in \text{torsp-}\mathcal{C}$ by

$$(\tau \cdot \sigma)(M)/\sigma(M) = \tau(M/\sigma(M)) \quad \text{for all } M \in \mathcal{C}.$$

It can be checked, easily, that $\text{torsp-}\mathcal{C}$ is a monoid with respect to the operation \cdot , whose identity element is the zero functor $0 \in \text{torsp-}\mathcal{C}$. Note that $\tau \vee \sigma \leq \tau \cdot \sigma$.

Following Golan [Gol87, p43], we call an algebra $\langle H; \vee, \wedge, \cdot \rangle$ (of type $\langle 2, 2, 2 \rangle$) a *lattice ordered semigroup* if $\langle H; \vee, \wedge \rangle$ is a lattice and $\langle H; \cdot \rangle$ is a semigroup such that for all $x, y, z \in H$,

Q1. $x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$; and

Q2. $(x \vee y) \cdot z = (x \cdot z) \vee (y \cdot z)$.

The ring theorist's prototype of a lattice ordered semigroup is the lattice of ideals, $\text{Id } R$, of an arbitrary ring R , endowed with the semigroup operation \cdot of ideal multiplication. More precisely, if R is any ring then $\langle \text{Id } R; +, \cap, \cdot \rangle$ is a lattice ordered semigroup. (Of course, in this instance, the join operation corresponds with ideal addition and the meet operation with intersection.)

In the next theorem, we show that if \mathcal{C} is a module subcategory of $\text{Mod-}R$ then $[\text{torsp-}\mathcal{C}]^{\text{du}}$ (this is the dual of the lattice $\text{torsp-}\mathcal{C}$) has the structure of a lattice ordered semigroup. Recall that since we do not distinguish structures from their universes notationally, $[\text{torsp-}\mathcal{C}]^{\text{du}}$ must be understood to be ordered by the relation \leq^{-1} , i.e., for torsion preradicals τ, σ on \mathcal{C} , we have $\tau \leq^{-1} \sigma$ in $[\text{torsp-}\mathcal{C}]^{\text{du}}$ if and only if $\sigma \leq \tau$ in $\text{torsp-}\mathcal{C}$. More precisely,

$$[\text{torsp-}\mathcal{C}]^{\text{du}} = \langle [\text{torsp-}\mathcal{C}]; \vee, \wedge, \cdot \rangle^{\text{du}} := \langle [\text{torsp-}\mathcal{C}]; \wedge, \vee, \cdot \rangle,$$

and to avoid confusion, we shall always use ' \vee ' and ' \wedge ' to denote the join and meet operations in $\text{torsp-}\mathcal{C}$ and not in its dual. Note that $[\text{torsp-}\mathcal{C}]^{\text{du}}$ is still a complete modular lattice, since these order properties are preserved in taking duals. $[\text{torsp-}\mathcal{C}]^{\text{du}}$ is not algebraic, however; it may be called *dually algebraic*.

THEOREM 7. *Let R be a ring and \mathcal{C} a module subcategory of $\text{Mod-}R$. Then $[\text{torsp-}\mathcal{C}]^{\text{du}} := \langle [\text{torsp-}\mathcal{C}]; \wedge, \vee, \cdot \rangle$ is a lattice ordered semigroup.*

Proof. We have to show that

$$\mathbf{Q1}'. \quad \tau \cdot (\sigma \wedge \lambda) = (\tau \cdot \sigma) \wedge (\tau \cdot \lambda); \text{ and}$$

$$\mathbf{Q2}'. \quad (\tau \wedge \sigma) \cdot \lambda = (\tau \cdot \lambda) \wedge (\sigma \cdot \lambda)$$

for all $\tau, \sigma, \lambda \in \text{torsp-}\mathcal{C}$.

We first show that if $\mu, \eta \in \text{torsp-}\mathcal{C}$, with $\mu \geq \eta$, then $\tau \cdot \mu \geq \tau \cdot \eta$ and $\mu \cdot \tau \geq \eta \cdot \tau$. The second inequality follows trivially from the definition of \cdot . To establish the first inequality, take $M \in \mathcal{C}$ and let $\pi: M/\eta(M) \rightarrow M/\mu(M)$ denote the canonical epimorphism. Now

$$\pi[\tau(M/\eta(M))] = \pi[(\tau \cdot \eta)(M)/\eta(M)] = (\tau \cdot \eta)(M)/\mu(M) \subseteq \tau(M/\mu(M)) = (\tau \cdot \mu)(M)/\mu(M),$$

so $(\tau \cdot \eta)(M) \subseteq (\tau \cdot \mu)(M)$. Thus $\tau \cdot \eta \leq \tau \cdot \mu$. Since $\sigma \wedge \lambda \leq \sigma, \lambda$, it follows from the above that $\tau \cdot (\sigma \wedge \lambda) \leq \tau \cdot \sigma, \tau \cdot \lambda$, so $\tau \cdot (\sigma \wedge \lambda) \leq (\tau \cdot \sigma) \wedge (\tau \cdot \lambda)$. It remains to establish the reverse inequality.

Suppose $\mu \in \text{torsp-}\mathcal{C}$, with $\mu \leq \tau \cdot \sigma, \tau \cdot \lambda$. We must show that $\mu \leq \tau \cdot (\sigma \wedge \lambda)$. Let $M \in \mathcal{C}$.

Then

$$N = [\mu(M) + \sigma(M)]/\sigma(M) \subseteq (\tau \cdot \sigma)(M)/\sigma(M) \in T_\tau \text{ and}$$

$$N' = [\mu(M) + \lambda(M)]/\lambda(M) \subseteq (\tau \cdot \lambda)(M)/\lambda(M) \in T_\tau.$$

Furthermore,

$$[\mu(M) + (\sigma \wedge \lambda)(M)]/(\sigma \wedge \lambda)(M) = [\mu(M) + (\sigma(M) \cap \lambda(M))]/(\sigma(M) \cap \lambda(M))$$

embeds canonically in $N \oplus N' \in T_\tau$, so

$$\mu(M) + (\sigma \wedge \lambda)(M) \subseteq (\tau \cdot (\sigma \wedge \lambda))(M),$$

i.e., $\mu(M) \subseteq (\tau \cdot (\sigma \wedge \lambda))(M)$. Thus $\mu \leq \tau \cdot (\sigma \wedge \lambda)$. The verification of $\mathbf{Q2}'$ is entirely similar. \square

The next result follows easily from the definition of the operation \cdot .

PROPOSITION 8. *Let R be a ring and \mathcal{C} a module subcategory of $\text{Mod-}R$. Then the following assertions are equivalent for any torsion preradical σ on \mathcal{C} :*

(i) σ is idempotent, i.e., $\sigma^2 := \sigma \cdot \sigma = \sigma$;

(ii) $\sigma(M/\sigma(M)) = 0$ for all $M \in \mathcal{C}$. \square

A preradical σ on \mathcal{C} satisfying condition (ii) of Proposition 8 is called a *radical on \mathcal{C}* ; a torsion preradical σ on \mathcal{C} which satisfies the equivalent conditions of Proposition 8 above is called a *torsion radical on \mathcal{C}* . We denote the set of all torsion radicals on \mathcal{C} by $\text{tors-}\mathcal{C}$. Trivially, the functors 0 and $1_{\mathcal{C}}$ belong to $\text{tors-}\mathcal{C}$. If $\mathcal{C} = \text{Mod-}R$ then we abbreviate $\text{tors-}\mathcal{C}$ as $\text{tors-}R$.

Considered as an ordered subset of $\text{torsp-}\mathcal{C}$, $\text{tors-}\mathcal{C}$ is closed under arbitrary meets; $\text{tors-}\mathcal{C}$ is therefore a complete lattice. It follows that if $\sigma \in \text{torsp-}\mathcal{C}$ then there exists a smallest element of $\text{tors-}\mathcal{C}$ which is greater than or equal to σ ; we shall denote this element by $\bar{\sigma}$. Furthermore, if \mathcal{A} is any nonempty subclass of \mathcal{C} , it is possible to define, in the same way as was done for torsion

preradicals, the notions of the *torsion radical on \mathcal{C} generated by \mathcal{A}* and the *torsion radical on \mathcal{C} cogenerated by \mathcal{A}* . Although $\text{tors-}\mathcal{C}$ is closed under meets, the join, in $\text{torsp-}\mathcal{C}$, of even two torsion radicals need not be a radical, so $\text{tors-}\mathcal{C}$ is a meet-complete subsemilattice, but not a sublattice of $\text{torsp-}\mathcal{C}$. It may also be shown that, unlike $\text{torsp-}\mathcal{C}$, the lattice $\text{tors-}\mathcal{C}$ is always distributive. It is also well known that, unlike $\text{torsp-}\mathcal{C}$, the lattice $\text{tors-}\mathcal{C}$ is not, in general, algebraic. (In particular, even when \mathcal{C} is the category of all unital modules over a ring with identity, $\text{tors-}\mathcal{C}$ need not be algebraic.) There is, however, a lattice theoretic property weaker than “algebraic and distributive” which is possessed by $\text{tors-}\mathcal{C}$, viz., the lattice $\text{tors-}\mathcal{C}$ is Brouwerian, that is to say, $\text{tors-}\mathcal{C}$ satisfies the join-infinite distributive identity (see Chapter 0, §1). In general, the lattice $\text{tors-}\mathcal{C}$ is not continuous, i.e., it need not also satisfy the meet-infinite distributive identity (even if \mathcal{C} is the category of unital modules over a ring with identity). We also point out that $\text{tors-}\mathcal{C}$ is not, in general, a subsemigroup of $\text{torsp-}\mathcal{C}$, with respect to \cdot .

If \mathcal{C} is a module subcategory of $\text{Mod-}R$ then the “multiplication” operation \cdot can be extended to a calculus of transfinite powers of torsion preradicals. If $\sigma \in \text{torsp-}\mathcal{C}$ and α is a positive ordinal, we define:

$$\begin{aligned}\sigma^1 &= \sigma; \\ \sigma^{\alpha \oplus 1} &= \sigma^\alpha \cdot \sigma; \text{ and} \\ \sigma^\alpha &= \bigvee_{0 < \beta < \alpha} \sigma^\beta, \text{ if } \alpha \text{ is a limit ordinal.}\end{aligned}$$

Notice that if $\sigma \in \text{torsp-}\mathcal{C}$ then $\{\sigma^\alpha : \alpha > 0\}$ is an ascending chain of torsion preradicals on \mathcal{C} . Since the class of all positive ordinals is a proper class and $\text{torsp-}R$ is a set, there must be a least ordinal α for which $\sigma^\alpha = \sigma^{\alpha \oplus 1}$. It is clear that σ^α (for this least α) is idempotent and thus a torsion radical on \mathcal{C} . Moreover, it is not difficult to see that σ^α must be the smallest torsion radical on \mathcal{C} that is greater than or equal to σ . The next proposition is an obvious consequence of these remarks.

PROPOSITION 9. *Let R be a ring and \mathcal{C} a module subcategory of $\text{Mod-}R$. If σ is a torsion preradical on \mathcal{C} then $\bar{\sigma} = \sigma^\alpha$ for some ordinal α . □*

It is possible to describe $\bar{\sigma}$ more explicitly: it follows easily from Proposition 9 that for any torsion preradical σ on \mathcal{C} , we have

$$\bar{\sigma}(M) = \bigcap \{N \leq M : \sigma(M/N) = 0\} \text{ for all } M \in \mathcal{C}.$$

EXAMPLE 1. If R is a ring then the notion of the socle of a right R -module gives rise to a torsion preradical on $\text{Mod-}R$, which we shall denote by soc . For if $M, N \in \text{Mod-}R$ and $\varphi \in \text{Hom}_R(M, N)$ then $\varphi[\text{soc } M] \subseteq \text{soc } N$. Moreover, if $L \leq M$ then $\text{soc } L = L \cap \text{soc } M$. The torsion preradical soc is not a radical. The ascending chain of torsion preradicals $\{\text{soc}^\alpha : \alpha > 0\}$ is called the *extended socle series of $\text{Mod-}R$* and if $M \in \text{Mod-}R$, we call $\{\text{soc}^\alpha(M) : \alpha > 0\}$ the *extended socle series of M* . We use the word “extended” because the expression “socle series for M ” is usually

reserved for the subchain indexed by the finite ordinals, i.e., the *socle series* of M is $\{\text{soc}^\alpha(M) : 0 < \alpha < \aleph_0\}$. □

PROPOSITION 10. [Ste75, Propositions 3.1 & 3.2, p141] *Let R be a ring and \mathcal{C} a module subcategory of $\text{Mod-}R$. If τ is a torsion radical on \mathcal{C} then:*

- (i) T_τ is closed under homomorphic images, direct sums, submodules and module extensions;
- (ii) F_τ is closed under submodules, direct products, essential extensions and module extensions. □

A nonempty class \mathcal{A} of right R -modules is called a *hereditary torsion class* if \mathcal{A} is closed under homomorphic images, direct sums, submodules and module extensions. (Notice that every module subcategory of $\text{Mod-}R$ is a hereditary torsion class.) Proposition 10(i) asserts that if $\tau \in \text{tors-}\mathcal{C}$ then T_τ is a hereditary torsion class contained in \mathcal{C} . We know from Theorem 4 that the map $\tau \mapsto T_\tau$ defines a bijection from the set of all torsion preradicals on \mathcal{C} onto the set of all hereditary pretorsion classes of right R -modules contained in \mathcal{C} . It can be shown that this bijection restricts to a bijection from the set of all torsion radicals on \mathcal{C} onto the set of all hereditary torsion classes of right R -modules contained in \mathcal{C} . We therefore have:

THEOREM 11. [Ste75, Proposition 3.1, p141] *Let R be a ring and \mathcal{C} a module subcategory of $\text{Mod-}R$. Then the map $\tau \mapsto T_\tau$ ($\tau \in \text{tors-}R$) defines a bijection from the set of all torsion radicals on \mathcal{C} onto the set of all hereditary torsion classes of right R -modules contained in \mathcal{C} . □*

It is easy to show that the class $\text{Mod-}R(\text{zero})$ of all zero multiplication right R -modules (i.e., modules M_R such that $MR = 0$) is a hereditary pretorsion class in $\text{Mod-}R$. By Theorem 4, there exists a torsion preradical on $\text{Mod-}R$, denoted by τ_{zero} , whose hereditary pretorsion class is $\text{Mod-}R(\text{zero})$. Thus $\tau_{\text{zero}}(M) := \{x \in M : xR = 0\}$ for all $M \in \text{Mod-}R$. Notice that if the ring R is idempotent (i.e., $R^2 = R$) then τ_{zero} is idempotent because, for each $M \in \text{Mod-}R$,

$$\begin{aligned} x + \tau_{\text{zero}}(M) \in \tau_{\text{zero}}(M/\tau_{\text{zero}}(M)) &\Rightarrow (x + \tau_{\text{zero}}(M))R = xR + \tau_{\text{zero}}(M) = 0 \\ &\Rightarrow xR \subseteq \tau_{\text{zero}}(M) \\ &\Rightarrow (xR)R = xR = 0 \\ &\Rightarrow x \in \tau_{\text{zero}}(M). \end{aligned}$$

Consequently, $\tau_{\text{zero}}(M/\tau_{\text{zero}}(M)) = \tau_{\text{zero}}^2(M)/\tau_{\text{zero}}(M) = 0$.

The idempotence of both R and τ_{zero} is not coincidental. Indeed, in Theorem 3.5 we shall establish a connection between the multiplicative semigroup of ideals of R and the multiplicative monoid $\text{torsp-}R$.

If \mathcal{C} is a module subcategory of $\text{Mod-}R$ then there is an important connection between the

torsion preradicals on \mathcal{C} and those on $\text{Mod-}R$. If $\tau \in \text{torsp-}\mathcal{C}$ then, by Proposition 3, T_τ is a hereditary pretorsion class of right R -modules contained in \mathcal{C} . It follows from Theorem 4 that there exists a unique torsion preradical τ^* on $\text{Mod-}R$ such that $T_{\tau^*} = T_\tau$. Thus τ and τ^* have the same associated hereditary pretorsion class of modules. In other words, τ is just the “restriction” of τ^* to \mathcal{C} . The map from $\text{torsp-}\mathcal{C}$ to $\text{torsp-}R$ defined by $\tau \mapsto \tau^*$ ($\tau \in \text{torsp-}\mathcal{C}$), is thus structure preserving in every sense of the word; at the very least, it respects the lattice and semigroup structure of $\text{torsp-}\mathcal{C}$. This allows us to identify $\text{torsp-}\mathcal{C}$ with its image in $\text{torsp-}R$. Specifically, if $\tau_{\mathcal{C}}$ denotes the torsion radical on $\text{Mod-}R$ whose hereditary torsion class is \mathcal{C} , then $\text{torsp-}\mathcal{C}$ may be identified with the interval $[0, \tau_{\mathcal{C}}]$ of $\text{torsp-}R$. We highlight this situation by writing

$$\text{torsp-}\mathcal{C} = [0, \tau_{\mathcal{C}}] \subseteq \text{torsp-}R.$$

Suppose now that \mathcal{C} and \mathcal{D} are module subcategories of $\text{Mod-}R$. Denote by $\tau_{\mathcal{C}}$ and $\tau_{\mathcal{D}}$ the torsion radicals on $\text{Mod-}R$ whose hereditary torsion classes are \mathcal{C} and \mathcal{D} , respectively. We shall now consider the case where $\mathcal{C} \cap \mathcal{D} = \{0\}$ (i.e., $\tau_{\mathcal{C}} \wedge \tau_{\mathcal{D}} = 0$) and $M = \tau_{\mathcal{C}}(M) \oplus \tau_{\mathcal{D}}(M)$ for all $M \in \text{Mod-}R$. (This implies, but is not equivalent to, the assertion $\tau_{\mathcal{C}} \vee \tau_{\mathcal{D}} = 1_{\text{Mod-}R}$.)

PROPOSITION 12. *Let R be a ring and let \mathcal{C} and \mathcal{D} be module subcategories of $\text{Mod-}R$ such that $\mathcal{C} \cap \mathcal{D} = \{0\}$ and $M = \tau_{\mathcal{C}}(M) \oplus \tau_{\mathcal{D}}(M)$ for all $M \in \text{Mod-}R$. If $\tau, \sigma, \lambda \in \text{torsp-}R$ with $\tau \leq \tau_{\mathcal{C}}$ and $\sigma \leq \tau_{\mathcal{D}}$ then:*

- (i) $(\tau \vee \sigma)(M) = \tau(M) \oplus \sigma(M)$ for all $M \in \text{Mod-}R$;
- (ii) $[0, \tau_{\mathcal{C}}]$ and $[0, \tau_{\mathcal{D}}]$ are submonoids of $\text{torsp-}R$;
- (iii) $\tau \cdot \sigma = \sigma \cdot \tau = \tau \vee \sigma$;
- (iv) $(\tau \vee \sigma) \cdot \lambda = (\tau \cdot \lambda) \vee (\sigma \cdot \lambda)$;
- (v) $\lambda \cdot (\tau \vee \sigma) = (\lambda \cdot \tau) \vee (\lambda \cdot \sigma)$.

Proof. (i) It is obvious that, $\tau(M) \oplus \sigma(M) \subseteq (\tau \vee \sigma)(M)$ for all $M \in \text{Mod-}R$, so it remains to establish the reverse inequality. Let $M \in \text{Mod-}R$. By Proposition 5 (see also remarks following Proposition 5),

$$(\tau \vee \sigma)(M) = M \cap [\tau(E(M)) \oplus \sigma(E(M))].$$

Let $x \in (\tau \vee \sigma)(M)$ and write $x = y_1 + y_2$ with $y_1 \in \tau(E(M))$ and $y_2 \in \sigma(E(M))$. By hypothesis, $M = \tau_{\mathcal{C}}(M) \oplus \tau_{\mathcal{D}}(M)$, so $x = z_1 + z_2$ for some $z_1 \in \tau_{\mathcal{C}}(M)$ and $z_2 \in \tau_{\mathcal{D}}(M)$. Then

$$x = y_1 + y_2 = z_1 + z_2 \Rightarrow y_1 - z_1 = z_2 - y_2 \in \tau_{\mathcal{C}}(E(M)) \cap \tau_{\mathcal{D}}(E(M)) = 0.$$

It follows that $y_1 \in M \cap \tau(E(M)) = \tau(M)$ and $y_2 \in M \cap \sigma(E(M)) = \sigma(M)$. Thus $x \in \tau(M) \oplus \sigma(M)$.

(ii) By considerations of symmetry, it clearly suffices to show that $[0, \tau_{\mathcal{C}}]$ is a submonoid of $\text{torsp-}R$. Let $\eta, \mu \in [0, \tau_{\mathcal{C}}]$. Then $\eta \cdot \mu \leq \eta \cdot \tau_{\mathcal{C}} \leq \tau_{\mathcal{C}} \cdot \tau_{\mathcal{C}}$ (because $\eta, \mu \leq \tau_{\mathcal{C}}$). Since $\tau_{\mathcal{C}}$ is a torsion radical, $\tau_{\mathcal{C}} \cdot \tau_{\mathcal{C}} = \tau_{\mathcal{C}}$. Hence, $\eta \cdot \mu \leq \tau_{\mathcal{C}}$, i.e., $\eta \cdot \mu \in [0, \tau_{\mathcal{C}}]$.

(iii) Let $M \in \text{Mod-}R$. Then

$$\begin{aligned}
((\tau \cdot \sigma)(M))/\sigma(M) &= \tau(M/\sigma(M)) = \tau([\tau_{\mathcal{C}}(M) \oplus \tau_{\mathfrak{Q}}(M)]/\sigma(M)) \\
&= \tau([\tau_{\mathcal{C}}(M) \oplus \sigma(M)]/\sigma(M) \oplus \tau_{\mathfrak{Q}}(M)/\sigma(M)) \\
&= \tau([\tau_{\mathcal{C}}(M) \oplus \sigma(M)]/\sigma(M)) \oplus \tau(\tau_{\mathfrak{Q}}(M)/\sigma(M)).
\end{aligned}$$

Since $\tau_{\mathfrak{Q}}(M)/\sigma(M)$ is $\tau_{\mathfrak{Q}}$ -torsion and $\tau \wedge \tau_{\mathfrak{Q}} = 0$, we must have $\tau(\tau_{\mathfrak{Q}}(M)/\sigma(M)) = 0$. Thus $((\tau \cdot \sigma)(M))/\sigma(M) = \tau([\tau_{\mathcal{C}}(M) \oplus \sigma(M)]/\sigma(M))$. Let $\varphi: [\tau_{\mathcal{C}}(M) \oplus \sigma(M)]/\sigma(M) \rightarrow \tau_{\mathcal{C}}(M)$ denote the canonical isomorphism. Clearly we must have $\varphi[\tau([\tau_{\mathcal{C}}(M) \oplus \sigma(M)]/\sigma(M))] = \tau(\tau_{\mathcal{C}}(M)) = \tau(M)$. Therefore,

$$\tau([\tau_{\mathcal{C}}(M) \oplus \sigma(M)]/\sigma(M)) = \varphi^{-1}[\tau(M)] = [\tau(M) \oplus \sigma(M)]/\sigma(M),$$

and so $(\tau \cdot \sigma)(M) = \tau(M) \oplus \sigma(M) = (\tau \vee \sigma)(M)$. Thus $\tau \cdot \sigma = \tau \vee \sigma$. By symmetry, we obviously also have $\sigma \cdot \tau = \tau \vee \sigma$.

(iv) Since $\tau \vee \sigma \geq \tau$, we must have $(\tau \vee \sigma) \cdot \lambda \geq \tau \cdot \lambda$. Similarly, $(\tau \vee \sigma) \cdot \lambda \geq \sigma \cdot \lambda$, so $(\tau \vee \sigma) \cdot \lambda \geq (\tau \cdot \lambda) \vee (\sigma \cdot \lambda)$. It remains to establish the reverse inequality. Let $M \in \text{Mod-}R$. Then

$$\begin{aligned}
[(\tau \vee \sigma) \cdot \lambda](M)/\lambda(M) &= (\tau \vee \sigma)(M/\lambda(M)) \\
&= \tau(M/\lambda(M)) \oplus \sigma(M/\lambda(M)) \quad (\text{by (i)}) \\
&= [(\tau \cdot \lambda)(M)/\lambda(M)] \oplus [(\sigma \cdot \lambda)(M)/\lambda(M)] \\
&= [(\tau \cdot \lambda)(M) + (\sigma \cdot \lambda)(M)]/\lambda(M).
\end{aligned}$$

It follows that, $[(\tau \vee \sigma) \cdot \lambda](M) = (\tau \cdot \lambda)(M) + (\sigma \cdot \lambda)(M) \subseteq [(\tau \cdot \lambda) \vee (\sigma \cdot \lambda)](M)$. Hence, $(\tau \vee \sigma) \cdot \lambda \leq (\tau \cdot \lambda) \vee (\sigma \cdot \lambda)$.

(v) An argument similar to that used in (iv) above shows that $\lambda \cdot (\tau \vee \sigma) \geq (\lambda \cdot \tau) \vee (\lambda \cdot \sigma)$. We are therefore left to prove the reverse inequality. Let $M \in \text{Mod-}R$. Then

$$\begin{aligned}
[\lambda \cdot (\tau \vee \sigma)](M)/(\tau \vee \sigma)(M) &= \lambda(M/(\tau \vee \sigma)(M)) \\
&= \lambda(M/[\tau(M) \oplus \sigma(M)]) \quad (\text{by (i)})
\end{aligned}$$

Let $\varphi: M/[\tau(M) \oplus \sigma(M)] \rightarrow [\tau_{\mathcal{C}}(M)/\tau(M)] \oplus [\tau_{\mathfrak{Q}}(M)/\sigma(M)]$ be the canonical isomorphism defined by $\varphi(x + [\tau(M) \oplus \sigma(M)]) = (x_1 + \tau(M), x_2 + \sigma(M))$ for $x = x_1 + x_2 \in M$ with $x_1 \in \tau_{\mathcal{C}}(M)$, $x_2 \in \tau_{\mathfrak{Q}}(M)$. Now

$$\begin{aligned}
\varphi[\lambda(M/[\tau(M) \oplus \sigma(M)])] &= \lambda([\tau_{\mathcal{C}}(M)/\tau(M)] \oplus [\tau_{\mathfrak{Q}}(M)/\sigma(M)]) \\
&= \lambda(\tau_{\mathcal{C}}(M)/\tau(M)) \oplus \lambda(\tau_{\mathfrak{Q}}(M)/\sigma(M)) \\
&= \lambda(\tau_{\mathcal{C}}(M)/\tau(\tau_{\mathcal{C}}(M))) \oplus \lambda(\tau_{\mathfrak{Q}}(M)/\sigma(\tau_{\mathfrak{Q}}(M)))
\end{aligned}$$

(because $\tau(\tau_{\mathcal{C}}(M)) = \tau_{\mathcal{C}}(M) \cap \tau(M) = \tau(M)$ and $\sigma(\tau_{\mathfrak{Q}}(M)) = \tau_{\mathfrak{Q}}(M) \cap \sigma(M) = \sigma(M)$)

$$\begin{aligned}
&= (\lambda \cdot \tau)(\tau_{\mathcal{C}}(M))/\tau(\tau_{\mathcal{C}}(M)) \oplus (\lambda \cdot \sigma)(\tau_{\mathfrak{Q}}(M))/\sigma(\tau_{\mathfrak{Q}}(M)) \\
&= [(\lambda \cdot \tau)(M) \cap \tau_{\mathcal{C}}(M)]/\tau(M) \oplus [(\lambda \cdot \sigma)(M) \cap \tau_{\mathfrak{Q}}(M)]/\sigma(M) \\
&= [(\lambda \cdot \tau) \wedge \tau_{\mathcal{C}}](M)/\tau(M) \oplus [(\lambda \cdot \sigma) \wedge \tau_{\mathfrak{Q}}](M)/\sigma(M).
\end{aligned}$$

Hence, $\lambda(M/[\tau(M) \oplus \sigma(M)]) = \varphi^{-1}([(\lambda \cdot \tau) \wedge \tau_{\mathcal{C}}](M)/\tau(M) \oplus [(\lambda \cdot \sigma) \wedge \tau_{\mathfrak{Q}}](M)/\sigma(M))$
 $= ([(\lambda \cdot \tau) \wedge \tau_{\mathcal{C}}](M) \oplus [(\lambda \cdot \sigma) \wedge \tau_{\mathfrak{Q}}](M))/[\tau(M) \oplus \sigma(M)]$.

Therefore

$$[\lambda \cdot (\tau \vee \sigma)](M) \subseteq [(\lambda \cdot \tau) \wedge \tau_{\mathfrak{C}}](M) \oplus [(\lambda \cdot \sigma) \wedge \tau_{\mathfrak{D}}](M) \subseteq ([(\lambda \cdot \tau) \wedge \tau_{\mathfrak{C}}] \vee [(\lambda \cdot \sigma) \wedge \tau_{\mathfrak{D}}])(M).$$

It follows that $\lambda \cdot (\tau \vee \sigma) \leq (\lambda \cdot \tau) \vee (\lambda \cdot \sigma)$. □

PROPOSITION 13. *Let R be a ring and let \mathfrak{C} and \mathfrak{D} be module subcategories of $\text{Mod-}R$ such that $\mathfrak{C} \cap \mathfrak{D} = \{0\}$ and $M = \tau_{\mathfrak{C}}(M) \oplus \tau_{\mathfrak{D}}(M)$ for all $M \in \text{Mod-}R$. Then the function $\theta: [0, \tau_{\mathfrak{D}}] \rightarrow \text{torsp-}R$, defined by $\theta(\sigma) = \sigma \vee \tau_{\mathfrak{C}}$ ($\sigma \in [0, \tau_{\mathfrak{D}}]$) is one-to-one with image $[\tau_{\mathfrak{C}}, 1_{\text{Mod-}R}] \subseteq \text{torsp-}R$. Furthermore, θ is a complete lattice and monoid homomorphism which preserves transfinite powers, that is to say, $\theta(\sigma^\alpha) = \theta(\sigma)^\alpha$ for all ordinals $\alpha > 0$.*

Proof. Since $\text{torsp-}R$ is a complete modular lattice (Proposition 6), the fact that θ is one-to-one with image $[\tau_{\mathfrak{C}}, 1_{\text{Mod-}R}]$ and is a complete lattice homomorphism follows immediately from Proposition 0.1.3. Let $\tau, \sigma \in [0, \tau_{\mathfrak{D}}]$. Then

$$\begin{aligned} \theta(\tau) \cdot \theta(\sigma) &= (\tau \vee \tau_{\mathfrak{C}}) \cdot (\sigma \vee \tau_{\mathfrak{C}}) \\ &= [\tau \cdot (\sigma \vee \tau_{\mathfrak{C}})] \vee [\tau_{\mathfrak{C}} \cdot (\sigma \vee \tau_{\mathfrak{C}})] \quad (\text{by Proposition 12 (iv)}) \\ &= [(\tau \cdot \sigma) \vee (\tau \cdot \tau_{\mathfrak{C}})] \vee [(\tau_{\mathfrak{C}} \cdot \sigma) \vee (\tau_{\mathfrak{C}} \cdot \tau_{\mathfrak{C}})] \quad (\text{by Proposition 12 (v)}). \end{aligned}$$

By Proposition 12 (iii), $\tau \cdot \tau_{\mathfrak{C}} = \tau \vee \tau_{\mathfrak{C}}$ and $\tau_{\mathfrak{C}} \cdot \sigma = \tau_{\mathfrak{C}} \vee \sigma$. Moreover, $\tau_{\mathfrak{C}} \cdot \tau_{\mathfrak{C}} = \tau_{\mathfrak{C}}$. It follows that

$$\begin{aligned} \theta(\tau) \cdot \theta(\sigma) &= (\tau \cdot \sigma) \vee \tau_{\mathfrak{C}} \vee \sigma \\ &= (\tau \cdot \sigma) \vee \tau_{\mathfrak{C}} \quad (\text{since } \tau \cdot \sigma \geq \sigma) \\ &= \theta(\tau \cdot \sigma). \end{aligned}$$

This shows that θ is a monoid homomorphism.

It remains to show that θ preserves transfinite powers. We shall prove, using transfinite induction on α , that $\theta(\sigma^\alpha) = \theta(\sigma)^\alpha$ for all ordinals $\alpha > 0$. The result is trivial if $\alpha = 1$. Suppose that $\theta(\sigma^\alpha) = \theta(\sigma)^\alpha$ for some ordinal α . Then

$$\begin{aligned} \theta(\sigma^{\alpha \oplus 1}) &= \theta(\sigma^\alpha \cdot \sigma) = \theta(\sigma^\alpha) \cdot \theta(\sigma) \quad (\text{since } \theta \text{ is a monoid homomorphism}) \\ &= \theta(\sigma)^\alpha \cdot \theta(\sigma) \quad (\text{by the inductive hypothesis}) \\ &= \theta(\sigma)^{\alpha \oplus 1}. \end{aligned}$$

Now suppose that α is a limit ordinal and that $\theta(\sigma^\beta) = \theta(\sigma)^\beta$ for all ordinals $\beta < \alpha$. Then

$$\begin{aligned} \theta(\sigma^\alpha) &= \theta(\bigvee_{0 < \beta < \alpha} \sigma^\beta) = \bigvee_{0 < \beta < \alpha} \theta(\sigma^\beta) \quad (\text{since } \theta \text{ is a complete lattice homomorphism}) \\ &= \bigvee_{0 < \beta < \alpha} \theta(\sigma)^\beta \quad (\text{by the inductive hypothesis}) \\ &= \theta(\sigma)^\alpha. \end{aligned}$$

□

We shall now examine a fundamentally important special case. Suppose R is a ring with identity. In this case, it can be shown that the class $\text{Mod-}R(\text{zero})$ of all zero multiplication right R -modules and the class $\text{Mod-}R(\text{unital})$ of all unital right R -modules are module subcategories of $\text{Mod-}R$. (This is not always true for rings without identity.) For brevity, let us denote the set of all torsion preradicals (resp. torsion radicals) on $\text{Mod-}R(\text{zero})$ by $\text{torsp-}R(\text{zero})$ (resp. $\text{tors-}R(\text{zero})$)

and the set of all torsion preradicals (resp. torsion radicals) on $\text{Mod-}R(\text{unital})$ by $\text{torsp-}R(\text{unital})$ (resp. $\text{tors-}R(\text{unital})$). We have already used τ_{zero} to denote the torsion radical on $\text{Mod-}R$ whose hereditary torsion class is $\text{Mod-}R(\text{zero})$. Let τ_{unit} denote the torsion radical on $\text{Mod-}R$ whose hereditary torsion class is $\text{Mod-}R(\text{unital})$. It follows easily from Theorem 0.3.1 that $M = \tau_{\text{zero}}(M) \oplus \tau_{\text{unit}}(M)$ for all $M \in \text{Mod-}R$. The hypotheses of Propositions 12 and 13 are therefore satisfied, so we may conclude that the mapping $\sigma \mapsto \sigma \vee \tau_{\text{zero}}$ defines a complete lattice and monoid isomorphism from the interval $[0, \tau_{\text{unit}}]$ of $\text{torsp-}R$ onto $[\tau_{\text{zero}}, 1_{\text{Mod-}R}] \subseteq \text{torsp-}R$.

Now suppose that R is an arbitrary ring and consider its Dorroh Extension R^* . Replacing R by R^* in the above isomorphism, we obtain

$$\text{torsp-}R^*(\text{unital}) \cong [\tau_{\text{zero}}, 1_{\text{Mod-}R^*}] \subseteq \text{torsp-}R^*.$$

Since $\text{Mod-}R$ may be identified with $\text{Mod-}R^*(\text{unital})$, $\text{torsp-}R^*(\text{unital})$ may be replaced by $\text{torsp-}R$ in the above isomorphism to yield

$$\text{torsp-}R \cong [\tau_{\text{zero}}, 1_{\text{Mod-}R^*}] \subseteq \text{torsp-}R^*.$$

§2. TOPOLOGIZING FILTERS.

It is a well known fact that much of the theory of torsion preradicals can be recast in terms of “topologizing filters”. This different approach has some advantages. There are, for example, many problems relating to torsion preradicals which are made more transparent when expressed in terms of topologizing filters. Historically, topologizing filters have always been studied in the context of rings with identity. Since our approach is slightly more general than this, we have chosen again to include some proofs.

Let R be a ring (not necessarily with identity). A nonempty set \mathcal{F} of right ideals of R is said to be a *right topologizing² filter* on R if the following three conditions are met:

- T1.** if $I \in \mathcal{F}$ and $I \subseteq J \leq R_R$ then $J \in \mathcal{F}$;
- T2.** if $I, J \in \mathcal{F}$ then $I \cap J \in \mathcal{F}$;
- T3.** if $I \in \mathcal{F}$ then $(I; \tau, a) \in \mathcal{F}$ for all $a \in R$.

Notice that properties **T1** and **T2** just say that \mathcal{F} is a filter, in the usual lattice theoretic sense, on the lattice of right ideals of R . We denote by $\text{Fil-}R$ the set of all right topologizing filters on R . It is easily shown that $\text{Fil-}R$ is closed under arbitrary intersections, so $\text{Fil-}R$ is a complete lattice, when partially ordered by set inclusion \subseteq . It is also not difficult to show that the union of

²By way of motivation, we mention (but shall not need) the fact that filters of the right ideal lattice of R satisfying condition **T3** below are called *topologizing* because they are just the sets of right ideals of R that form neighbourhood bases at 0 for the so-called *linear topologies* on R . Here, a topology τ on R is called (*right*) *linear* if the binary operation $+$, the unary operation $-$ and, for each $r \in R$, the operation $a \mapsto ra$ ($a \in R$) are continuous in τ , and there exists a neighbourhood base at 0 for τ that consists of right ideals of R . For further details, see [Gol87].

any directed set of elements of $\text{Fil-}R$ is an element of $\text{Fil-}R$, so by Corollary 0.1.6, $\text{Fil-}R$ is an algebraic lattice with respect to \subseteq . If X is a nonempty set of right ideals of R then, because $\text{Fil-}R$ is a complete lattice, there is a smallest right topologizing filter \mathcal{F} on R such that $X \subseteq \mathcal{F}$. We call \mathcal{F} the *right topologizing filter on R generated by X* .

It is possible to define a binary operation \cdot on $\text{Fil-}R$ as follows. If $\mathcal{F}, \mathcal{G} \in \text{Fil-}R$, we define

$$\mathcal{F} \cdot \mathcal{G} = \{K \leq R_R : \text{there exists some } H \in \mathcal{F} \text{ such that } K \subseteq H \text{ and } (K :_r a) \in \mathcal{G} \text{ for all } a \in H\}.$$

It is clear from the definition that $\mathcal{F} \cdot \mathcal{G} \supseteq \mathcal{F}, \mathcal{G}$ so, $\mathcal{F} \cdot \mathcal{G} \supseteq \mathcal{F} \vee \mathcal{G}$ (the join of \mathcal{F} and \mathcal{G} in the lattice $(\text{Fil-}R; \subseteq)$). It is easily checked that $\text{Fil-}R$ is a semigroup with respect to the operation \cdot . We point out that, in general, there is no identity element for $\text{Fil-}R$ with respect to \cdot , so $\text{Fil-}R$ is not a monoid under this operation. If, however, the ring R possesses an identity element then the trivial filter $\{R\} \in \text{Fil-}R$ acts as an identity with respect to \cdot . A simple example illustrating the action of the operation \cdot is the following: for each ideal I of R , define $\eta(I) = \{A \leq R_R : A \supseteq I\}$. It is easily checked that $\eta(I) \in \text{Fil-}R$ and that $\eta(I) \cdot \eta(J) = \eta(IJ)$ for all ideals I, J of R . Observe that the filter $\eta(I)$ is idempotent (in the sense that $\eta(I)^2 := \eta(I) \cdot \eta(I) = \eta(I)$) if and only if the ideal I is idempotent, i.e., if and only if $I^2 = I$.

PROPOSITION 1. [Gol87, p55] *The following conditions are equivalent for a right topologizing filter \mathcal{F} on a ring R :*

- (i) \mathcal{F} is idempotent, i.e., $\mathcal{F}^2 = \mathcal{F}$;
- (ii) if $I \leq R_R$ and there exists some $J \in \mathcal{F}$ such that $(I :_r a) \in \mathcal{F}$ for all $a \in J$, then $I \in \mathcal{F}$. □

A right topologizing filter \mathcal{F} on R which satisfies the equivalent conditions of Proposition 1 above is called a *right Gabriel filter* on R . We denote the set of all right Gabriel filters on R by $\text{Gab-}R$. It can be shown that $\text{Gab-}R$ is closed under arbitrary intersections. Thus $\text{Gab-}R$ is a meet-complete subsemilattice of $\text{Fil-}R$ and is a complete lattice with respect to \subseteq . Consequently, if $\mathcal{F} \in \text{Fil-}R$ then there exists a smallest element of $\text{Gab-}R$ which is greater than or equal to \mathcal{F} ; we shall denote this element by $\overline{\mathcal{F}}$. Furthermore, if X is any nonempty set of right ideals of R , it is possible to define, in the same way as was done for topologizing filters, the notion of the *right Gabriel filter on R generated by X* for any nonempty set X of right ideals of R . We remark that $\text{Gab-}R$ need not, in general, be a sublattice of $\text{Fil-}R$, nor is $\text{Gab-}R$ always a subsemigroup of $\text{Fil-}R$ with respect to \cdot . As in the case of torsion preradicals, the operation \cdot on $\text{Fil-}R$ may be extended to allow for transfinite powers. If $\mathcal{F} \in \text{Fil-}R$ and α is a positive ordinal, we define:

$$\begin{aligned} \mathcal{F}^1 &= \mathcal{F}; \\ \mathcal{F}^{\alpha \oplus 1} &= \mathcal{F}^\alpha \cdot \mathcal{F}; \\ \mathcal{F}^\alpha &= \bigvee_{0 < \beta < \alpha} \mathcal{F}^\beta, \text{ if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

Notice that if $\mathcal{F} \in \text{Fil-}R$ then $\{\mathcal{F}^\alpha : \alpha > 0\}$ is an ascending chain of right topologizing filters on R . If

α is the smallest ordinal for which $\mathfrak{F}^\alpha = \mathfrak{F}^{\alpha \oplus 1}$, it can easily be shown that \mathfrak{F}^α is the smallest element of $\text{Gab-}R$ which contains \mathfrak{F} . The next proposition follows immediately.

PROPOSITION 2. *Let R be a ring and \mathfrak{F} a right topologizing filter on R . Then $\overline{\mathfrak{F}} = \mathfrak{F}^\alpha$ for some ordinal α . □*

Given a ring R and $\mathfrak{F} \in \text{Fil-}R$, we associate with \mathfrak{F} a torsion preradical $\text{torsp } \mathfrak{F}$ on $\text{Mod-}R$ defined by

$$(\text{torsp } \mathfrak{F})(M) = \{x \in M : (0:x) \in \mathfrak{F}\}, \quad M \in \text{Mod-}R.$$

To show that $\text{torsp } \mathfrak{F}$ is indeed a torsion preradical on $\text{Mod-}R$, take $a \in R$ and $x, y \in (\text{torsp } \mathfrak{F})(M)$ with $M \in \text{Mod-}R$. Then $(0:x-y) \supseteq (0:x) \cap (0:y) \in \mathfrak{F}$ since $(0:x), (0:y) \in \mathfrak{F}$. Consequently, $x-y \in (\text{torsp } \mathfrak{F})(M)$. Also, $(0:xa) \supseteq ((0:x):_r a) \in \mathfrak{F}$, so $(0:xa) \in \mathfrak{F}$, i.e., $xa \in (\text{torsp } \mathfrak{F})(M)$. This shows that $(\text{torsp } \mathfrak{F})(M) \leq M$. Now suppose that $N \in \text{Mod-}R$ and $\varphi \in \text{Hom}_R(M, N)$. If $x \in (\text{torsp } \mathfrak{F})(M)$ then $(0:\varphi(x)) \supseteq (0:x) \in \mathfrak{F}$, so $(0:\varphi(x)) \in \mathfrak{F}$. It follows that $\varphi(x) \in (\text{torsp } \mathfrak{F})(N)$. Consequently, $\varphi[(\text{torsp } \mathfrak{F})(M)] \subseteq (\text{torsp } \mathfrak{F})(N)$. It is obvious that if $L \leq M$ then $(\text{torsp } \mathfrak{F})(L) = L \cap (\text{torsp } \mathfrak{F})(M)$, so $\text{torsp } \mathfrak{F} \in \text{torsp-}R$, as claimed. Henceforth, we shall regard “torsp” as a map from $\text{Fil-}R$ to $\text{torsp-}R$. We remind the reader that R^* denotes the Dorroh Extension of a ring R .

LEMMA 3. *Let R be a ring and \mathfrak{F} a right topologizing filter on R . Then the following conditions are equivalent for a right ideal K of R :*

- (i) R_R^*/K is $(\text{torsp } \mathfrak{F})$ -torsion;
- (ii) $K \in \mathfrak{F}$.

Proof. Notice that if $x = 1_{R^*} + K \in R_R^*/K$ then $(0:x) = K$. This being so, (i) \Rightarrow (ii) follows immediately. Conversely, if $K \in \mathfrak{F}$ then $x \in (\text{torsp } \mathfrak{F})(R_R^*/K)$. Now R_R^*/K is the smallest submodule of R_R^*/K containing x , so we must have $(\text{torsp } \mathfrak{F})(R_R^*/K) = R_R^*/K$. Thus (ii) \Rightarrow (i) holds. □

The following important theorem establishes the connection between topologizing filters and torsion preradicals alluded to in the introduction to this section. Repeated use will be made of this result.

THEOREM 4. *If R is a ring then the map $\text{torsp}: \text{Fil-}R \rightarrow \text{torsp-}R$ (defined by $\mathfrak{F} \mapsto \text{torsp } \mathfrak{F}$) is one-to-one, with image $[\tau_{\text{zero}}, 1_{\text{Mod-}R}] \subseteq \text{torsp-}R$. Furthermore, torsp is a complete lattice and semigroup homomorphism which preserves transfinite powers, that is to say, $\text{torsp}(\mathfrak{F}^\alpha) = (\text{torsp } \mathfrak{F})^\alpha$ for all $\mathfrak{F} \in \text{Fil-}R$ and all ordinals $\alpha > 0$.*

Proof. It follows easily from the previous lemma that if $\mathfrak{F}, \mathfrak{G} \in \text{Fil-}R$ and $\text{torsp } \mathfrak{F} = \text{torsp } \mathfrak{G}$ then for any right ideal K of R , we have $K \in \mathfrak{F}$ if and only if $K \in \mathfrak{G}$, so $\mathfrak{F} = \mathfrak{G}$. This shows that the map torsp is one-to-one. We show now that the range of torsp is $[\tau_{\text{zero}}, 1_{\text{Mod-}R}]$. Choose $\sigma \in$

$[\tau_{\text{zero}}, 1_{\text{Mod-}R}]$ and define $\mathfrak{F} = \{K \leq R_R : R_R^*/K \in T_\sigma\}$. Since T_σ is closed under homomorphic images (Proposition 1.1), it is easy to see that if $K \in \mathfrak{F}$ and $K \subseteq L \leq R_R$, then $L \in \mathfrak{F}$. If $K, L \in \mathfrak{F}$ then $R_R^*/K, R_R^*/L \in T_\sigma$. Since T_σ is closed under submodules and direct sums (Proposition 1.3) and $R_R^*/(K \cap L) \lesssim (R_R^*/K) \oplus (R_R^*/L)$, we must have $K \cap L \in \mathfrak{F}$. Finally, let $a \in R$ and $K \in \mathfrak{F}$. Define $\varphi: R_R^* \rightarrow (R_R^*/K) \oplus \mathbb{Z}_{\text{zero}}$ by $\varphi(1_{R^*} \cdot r + 1_{R^*} \cdot m) = (1_{R^*} \cdot [ar + am] + K, m)$ for $r \in R, m \in \mathbb{Z}$. (Recall that $\mathbb{Z}_{\text{zero}} \in \text{Mod-}R$ is just the abelian group $\langle \mathbb{Z}; +, -, 0 \rangle$ endowed with the zero multiplication.) It is easily checked that φ is an R -module homomorphism. Now,

$$\begin{aligned} \text{Ker } \varphi &= \{1_{R^*} \cdot r + 1_{R^*} \cdot m : ar + am \in K \text{ and } m = 0\} \\ &= \{1_{R^*} \cdot r + 1_{R^*} \cdot m : ar \in K \text{ and } m = 0\} \\ &= \{r \in R : ar \in K\} = (K :_r a). \end{aligned}$$

Also, $R_R^*/(K :_r a) = R_R^*/\text{Ker } \varphi \cong \text{Im } \varphi \lesssim (R_R^*/K) \oplus \mathbb{Z}_{\text{zero}}$. Since $\sigma \geq \tau_{\text{zero}}$, we must have $(R_R^*/K) \oplus \mathbb{Z}_{\text{zero}} \in T_\sigma$, whence $R_R^*/(K :_r a) \in T_\sigma$, i.e., $(K :_r a) \in \mathfrak{F}$. This shows that $\mathfrak{F} \in \text{Fil-}R$. We claim that $\text{torsp } \mathfrak{F} = \sigma$. For suppose that $M \in \text{Mod-}R$ and $x \in \sigma(M)$. Define

$$(0 : x)^* = \{1_{R^*} \cdot r + 1_{R^*} \cdot m : xr + xm = 0\}.$$

Then

$$\begin{aligned} R_R^*/(0 : x) &= R_R^*/[(0 : x)^* \cap R] \\ &\lesssim xR^* \oplus (R_R^*/R_R) \\ &\cong xR^* \oplus \mathbb{Z}_{\text{zero}} \\ &\lesssim \sigma(M) \oplus \mathbb{Z}_{\text{zero}} \in T_\sigma \text{ (because } \sigma \geq \tau_{\text{zero}}). \end{aligned}$$

Thus $(0 : x) \in \mathfrak{F}$. Since $(\text{torsp } \mathfrak{F})(M) := \{x \in M : (0 : x) \in \mathfrak{F}\}$, it follows that $\sigma(M) \subseteq (\text{torsp } \mathfrak{F})(M)$. Consequently, $\sigma \leq \text{torsp } \mathfrak{F}$. Conversely, if $x \in (\text{torsp } \mathfrak{F})(M)$ then $(0 : x) \in \mathfrak{F}$ and so $R_R^*/(0 : x) \in T_\sigma$. Since T_σ is closed under homomorphic images and $(0 : x)^* \supseteq (0 : x)$, we must have $xR^* \cong R_R^*/(0 : x)^* \in T_\sigma$. Therefore, $x \in \sigma(M)$. Thus $\text{torsp } \mathfrak{F} \leq \sigma$ and so $\text{torsp } \mathfrak{F} = \sigma$, as claimed.

Since the range of torsp is a complete sublattice of $\text{torsp-}R$, to show that torsp is a complete lattice homomorphism, it suffices to show that if $\mathfrak{F}, \mathfrak{G} \in \text{Fil-}R$ then $\text{torsp } \mathfrak{F} \leq \text{torsp } \mathfrak{G}$ if and only if $\mathfrak{F} \subseteq \mathfrak{G}$. This assertion is easily verified; indeed, sufficiency is trivial while necessity follows from Lemma 3. We show now that torsp is a semigroup homomorphism. Let $\mathfrak{F}, \mathfrak{G} \in \text{Fil-}R$ and $M \in \text{Mod-}R$. Then

$$\begin{aligned} (\text{torsp } (\mathfrak{F} \cdot \mathfrak{G}))(M) &= \{x \in M : (0 : x) \in \mathfrak{F} \cdot \mathfrak{G}\} \\ &= \{x \in M : (\exists H \in \mathfrak{F}) [(H \supseteq (0 : x)) \ \& \ (\forall a \in H) (((0 : x) :_r a) \in \mathfrak{G})]\} \\ &= \{x \in M : (\exists H \in \mathfrak{F}) [(H \supseteq (0 : x)) \ \& \ (\forall a \in H) (xa \in (\text{torsp } \mathfrak{G})(M))]\} \\ &\quad \text{(since } ((0 : x) :_r a) = (0 : xa)\text{)} \\ &= \{x \in M : (\exists H \in \mathfrak{F}) [(H \supseteq (0 : x)) \ \& \ (xH \subseteq (\text{torsp } \mathfrak{G})(M))]\} \\ &= \{x \in M : (\exists H \in \mathfrak{F}) [(H \supseteq (0 : x)) \ \& \ ((0 : x + (\text{torsp } \mathfrak{G})(M)) \supseteq H)]\}. \end{aligned}$$

Therefore, if $x \in (\text{torsp } (\mathfrak{F} \cdot \mathfrak{G}))(M)$ then $(0 : x + (\text{torsp } \mathfrak{G})(M)) \supseteq H$ for some $H \in \mathfrak{F}$. It follows that $(0 : x + (\text{torsp } \mathfrak{G})(M)) \in \mathfrak{F}$, so

$$x + (\text{torsp } \mathfrak{G})(M) \in (\text{torsp } \mathfrak{F})(M / (\text{torsp } \mathfrak{G})(M)) = ((\text{torsp } \mathfrak{F}) \cdot (\text{torsp } \mathfrak{G}))(M) / (\text{torsp } \mathfrak{G})(M).$$

If, on the other hand, it is the case that $x + (\text{torsp } \mathfrak{G})(M) \in (\text{torsp } \mathfrak{F})(M / (\text{torsp } \mathfrak{G})(M))$ then by taking $H = (0 : x + (\text{torsp } \mathfrak{G})(M))$, we obtain $H \in \mathfrak{F}$ and $(0 : x + (\text{torsp } \mathfrak{G})(M)) = H \supseteq (0 : x)$, so by the above, $x \in (\text{torsp } (\mathfrak{F} \cdot \mathfrak{G}))(M)$. Thus $(\text{torsp } (\mathfrak{F} \cdot \mathfrak{G}))(M) = ((\text{torsp } \mathfrak{F}) \cdot (\text{torsp } \mathfrak{G}))(M)$, and so $\text{torsp } (\mathfrak{F} \cdot \mathfrak{G}) = (\text{torsp } \mathfrak{F}) \cdot (\text{torsp } \mathfrak{G})$, as required.

It remains to show that torsp preserves transfinite powers. We shall prove, using transfinite induction on α , that $\text{torsp } (\mathfrak{F}^\alpha) = (\text{torsp } \mathfrak{F})^\alpha$ for all $\mathfrak{F} \in \text{Fil-}R$ and all ordinals $\alpha > 0$. The result is trivial if $\alpha = 1$. Suppose the result holds for some ordinal $\alpha > 0$. Then

$$\begin{aligned} \text{torsp } (\mathfrak{F}^{\alpha \oplus 1}) &= \text{torsp } (\mathfrak{F}^\alpha \cdot \mathfrak{F}) \\ &= \text{torsp } (\mathfrak{F}^\alpha) \cdot \text{torsp } \mathfrak{F} \quad (\text{since } \text{torsp} \text{ is a semigroup homomorphism}) \\ &= (\text{torsp } \mathfrak{F})^\alpha \cdot \text{torsp } \mathfrak{F} \quad (\text{by the inductive hypothesis}) \\ &= (\text{torsp } \mathfrak{F})^{\alpha \oplus 1}. \end{aligned}$$

If α is a limit ordinal and $\text{torsp } (\mathfrak{F}^\beta) = (\text{torsp } \mathfrak{F})^\beta$ for all positive ordinals $\beta < \alpha$ then

$$\begin{aligned} \text{torsp } (\mathfrak{F}^\alpha) &= \text{torsp } (\bigvee_{0 < \beta < \alpha} \mathfrak{F}^\beta) \\ &= \bigvee_{0 < \beta < \alpha} \text{torsp } (\mathfrak{F}^\beta) \quad (\text{since } \text{torsp} \text{ is a complete lattice homomorphism}) \\ &= \bigvee_{0 < \beta < \alpha} (\text{torsp } \mathfrak{F})^\beta \quad (\text{by the inductive hypothesis}) \\ &= (\text{torsp } \mathfrak{F})^\alpha. \end{aligned} \quad \square$$

The following result is an immediate consequence of Theorem 4 and Theorem 1.7.

COROLLARY 5. *If R is a ring then $[\text{Fil-}R]^{\text{du}} := \langle [\text{Fil-}R]; \wedge, \vee, \cdot \rangle$ is a lattice ordered semigroup.* \square

In the sequel to Proposition 1.13, we remarked that if R is a ring with identity then $\text{torsp-}R$ (unital) is isomorphic (as a complete lattice and as a monoid) to the interval $[\tau_{\text{zero}}, 1_{\text{Mod-}R}]$ of $\text{torsp-}R$. We may therefore conclude from Theorem 4 that

$$\text{Fil-}R \cong \text{torsp-}R(\text{unital}).$$

We also remarked that if R is an arbitrary ring then

$$\text{torsp-}R \cong [\tau_{\text{zero}}, 1_{\text{Mod-}R^*}] \subseteq \text{torsp-}R^*.$$

Since $\text{Fil-}R^* \cong [\tau_{\text{zero}}, 1_{\text{Mod-}R^*}]$ (by Theorem 4), it follows that

$$\text{Fil-}R^* \cong \text{torsp-}R.$$

The above isomorphism has an important corollary (already anticipated in the previous section): since $\text{Fil-}R^*$ is clearly a set, we may regard $\text{torsp-}R$ (and hence also $\text{tors-}R$) as a set. Furthermore, if \mathfrak{C} is any module subcategory of $\text{Mod-}R$ we may also regard $\text{torsp-}\mathfrak{C}$ and $\text{tors-}\mathfrak{C}$ as sets since both structures embed in $\text{torsp-}R$.

EXAMPLE 1. Let R be a ring and let \mathcal{F} denote the set of all essential right ideals of R . It is well known and not hard to prove that \mathcal{F} is a right topologizing filter on R . Notice that if $M \in \text{Mod-}R$ then $(\text{torsp } \mathcal{F})(M) := \{x \in M : (0 : x) \text{ is essential in } R_R\}$ is precisely the singular submodule $Z(M)$ of M (see remarks preceding Proposition 0.9.5). Consequently, we write

$$Z = \text{torsp } \mathcal{F}.$$

It so happens that the ascending chain $\{Z^\alpha : \alpha > 0\}$ terminates at an early stage; in fact it can be shown that $Z^2 = \bar{Z}$: see [Ste75, Proposition 6.2, p148]. The torsion radical \bar{Z} is called the (*right*) *Goldie torsion radical* and is sometimes also denoted by G .

§3. JANSIAN AND \mathfrak{m} -JANSIAN PRERADICALS AND TOPOLOGIZING FILTERS.

The main purpose of this section is to introduce the notions of a Jansian torsion preradical and Jansian topologizing filter, and thereafter, the more refined notions of an \mathfrak{m} -Jansian torsion preradical and \mathfrak{m} -Jansian topologizing filter where \mathfrak{m} is an arbitrary regular cardinal. As stated in the introduction to this chapter, our major investment will be in the latter two types of notion. Indeed, many of the results on Jansian preradicals and topologizing filters presented are intended merely to motivate “ \mathfrak{m} -Jansian” analogues. Nevertheless, some results pertaining to the “Jansian” property are important in their own right; Proposition 5, for example, shows that the lattice of ideals of an arbitrary ring R embeds in the dual of the lattice of all (right) topologizing filters on R . It follows therefore that the set of Jansian torsion preradicals on $\text{Mod-}R$ carries at least as much information about the ring R as does its ideal lattice. We shall have cause to exploit this fact in §5.

A preradical τ on $\text{Mod-}R$ is said to be *Jansian* if its hereditary pretorsion class T_τ is closed under direct products.

PROPOSITION 1. *The following conditions are equivalent for a preradical τ on $\text{Mod-}R$:*

- (i) τ is Jansian;
- (ii) $\tau(\prod_{i \in \Gamma} M_i) = \prod_{i \in \Gamma} \tau(M_i)$ for every family $\{M_i : i \in \Gamma\}$ of right R -modules.

Proof. (i) \Rightarrow (ii): Let $\{M_i : i \in \Gamma\}$ be a family of right R -modules. Then $\prod_{i \in \Gamma} \tau(M_i)$ is a τ -torsion submodule of $\prod_{i \in \Gamma} M_i$, so $\prod_{i \in \Gamma} \tau(M_i) \subseteq \tau(\prod_{i \in \Gamma} M_i)$. But the inclusion $\tau(\prod_{j \in \Gamma'} N_j) \subseteq \prod_{j \in \Gamma'} \tau(N_j)$ holds for any preradical τ and any family of right R -modules $\{N_j : j \in \Gamma'\}$, so we must have $\tau(\prod_{i \in \Gamma} M_i) = \prod_{i \in \Gamma} \tau(M_i)$.

(ii) \Rightarrow (i): Let $\{M_i : i \in \Gamma\}$ be a subfamily of T_τ . Then $\prod_{i \in \Gamma} M_i = \prod_{i \in \Gamma} \tau(M_i) = \tau(\prod_{i \in \Gamma} M_i)$, so $\prod_{i \in \Gamma} M_i \in T_\tau$. Thus τ is Jansian. \square

Recall that if τ is a preradical on $\text{Mod-}R$ then $\hat{\tau}$ denotes the smallest torsion preradical on $\text{Mod-}R$ such that $\hat{\tau} \geq \tau$.

LEMMA 2. *Let τ be a preradical on $\text{Mod-}R$. If τ is Jansian then so is $\hat{\tau}$.*

Proof. By Proposition 1.5 $\hat{\tau}(M) = M \cap \tau(E(M))$ for all $M \in \text{Mod-}R$. Let $\{M_i : i \in \Gamma\}$ be a subfamily of $T_{\hat{\tau}}$. Since $M_i \in T_{\hat{\tau}}$ for all $i \in \Gamma$ we must have $M_i \subseteq \tau(E(M_i))$ for all $i \in \Gamma$. Hence $\prod_{i \in \Gamma} M_i \subseteq \prod_{i \in \Gamma} \tau(E(M_i)) = \tau(\prod_{i \in \Gamma} E(M_i)) \in T_{\hat{\tau}}$ (by Proposition 1), so $\prod_{i \in \Gamma} M_i \in T_{\hat{\tau}}$. Thus $\hat{\tau}$ is Jansian. \square

Henceforth we shall denote by $\text{Jans-}R$ the set of all Jansian torsion preradicals on $\text{Mod-}R$.

PROPOSITION 3. *$\text{Jans-}R$ is closed under arbitrary meets and finite joins.*

Proof. Let $\Delta \subseteq \text{Jans-}R$ and set $\tau = \Lambda\Delta$. Let $\{M_i : i \in \Gamma\}$ be a subfamily of T_{τ} . Note that for each $\sigma \in \Delta$, we have

$$\sigma(\prod_{i \in \Gamma} M_i) = \prod_{i \in \Gamma} \sigma(M_i) \supseteq \prod_{i \in \Gamma} \tau(M_i) = \prod_{i \in \Gamma} M_i.$$

Consequently, $\tau(\prod_{i \in \Gamma} M_i) = \bigcap_{\sigma \in \Delta} \sigma(\prod_{i \in \Gamma} M_i) = \prod_{i \in \Gamma} M_i$, and so $\prod_{i \in \Gamma} M_i \in T_{\tau}$. This shows that $\tau = \Lambda\Delta \in \text{Jans-}R$. Thus $\text{Jans-}R$ is closed under arbitrary meets. Now suppose that Δ is a finite subset of $\text{Jans-}R$. It is easily verified that the preradical ρ defined by $\rho(M) = \sum_{\sigma \in \Delta} \sigma(M)$ for all $M \in \text{Mod-}R$, is the smallest preradical on $\text{Mod-}R$ for which $\rho \geq \sigma$ for all $\sigma \in \Delta$. (The preradical ρ can be thought of as the join of Δ in the "lattice" of all preradicals.) Certainly then $\hat{\rho} = V\Delta$. To show that $V\Delta$ is Jansian, it suffices, in view of the previous lemma, to show that ρ is Jansian. Let $\{M_i : i \in \Gamma\}$ be a subfamily of T_{ρ} . We have

$$\begin{aligned} \rho(\prod_{i \in \Gamma} M_i) &= \sum_{\sigma \in \Delta} \sigma(\prod_{i \in \Gamma} M_i) \\ &= \sum_{\sigma \in \Delta} (\prod_{i \in \Gamma} \sigma(M_i)) \quad (\text{by Proposition 1}) \\ &= \prod_{i \in \Gamma} (\sum_{\sigma \in \Delta} \sigma(M_i)) \quad (\text{because } \Delta \text{ is finite}) \\ &= \prod_{i \in \Gamma} \rho(M_i) = \prod_{i \in \Gamma} M_i \quad (\text{because } M_i \in T_{\rho} \text{ for all } i \in \Gamma). \end{aligned}$$

Thus ρ is Jansian, as required. \square

It follows from the above that $\text{Jans-}R$ is a complete lattice as well as being a sublattice of $\text{torsp-}R$. In fact, $\text{Jans-}R$ is a meet-complete subsemilattice (but not necessarily a complete sublattice) of $\text{torsp-}R$. If \mathcal{A} is a nonempty subclass of $\text{Mod-}R$, we shall call $\Lambda\{\tau \in \text{Jans-}R : T_{\tau} \supseteq \mathcal{A}\}$ the *Jansian torsion preradical on $\text{Mod-}R$ generated by \mathcal{A}* .

If I is an arbitrary ideal of a ring R , we define $\eta(I) = \{A \leq R_R : A \supseteq I\}$. It is easily verified that $\eta(I)$ is a right topologizing filter on R which is closed under *arbitrary* (rather than merely finite) intersections. It turns out, in fact, that every right topologizing filter on R which is closed under arbitrary intersections arises in this way.

THEOREM 4. *The following conditions are equivalent for a right topologizing filter \mathfrak{F} on a ring R :*

- (i) \mathcal{F} is closed under arbitrary intersections;
- (ii) $\bigcap \mathcal{F} \in \mathcal{F}$;
- (iii) $\mathcal{F} = \{A \leq R_R : A \supseteq I\} := \eta(I)$ for some ideal I of R ;
- (iv) $\mathcal{F} = \{A \leq R_R : A \supseteq (0:M)\}$ for some $M \in \text{Mod-}R$;
- (v) $\text{torsp } \mathcal{F}$ is Jansian.

Moreover, if \mathcal{F} is as in (iv), then $\text{torsp } \mathcal{F}$ is the Jansian torsion preradical on $\text{Mod-}R$ generated by $M \oplus \mathbb{Z}_{\text{zero}}$.

Proof. (i) \Rightarrow (ii) is obvious and (ii) \Rightarrow (iii) is proved by setting $I = \bigcap \mathcal{F}$. The equivalence of (iii) and (iv) follows immediately from the fact that if I is an ideal of R then the annihilator of the right R -module $M = R_R^*/I$ is I . (Recall that R^* denotes the Dorroh Extension of R , which was defined in Chapter 0, §4.)

(iii) \Rightarrow (v): Set $\tau = \text{torsp } \mathcal{F} = \text{torsp } \eta(I)$. For each $M \in \text{Mod-}R$, we have

$$\tau(M) := \{x \in M : (0:x) \in \mathcal{F}\} = \{x \in M : xI = 0\},$$

so $T_\tau = \{M \in \text{Mod-}R : MI = 0\}$, which is clearly closed under arbitrary direct products. Thus $\tau \in \text{Jans-}R$.

(v) \Rightarrow (i): Let $\{I_i : i \in \Gamma\}$ be a subfamily of \mathcal{F} and set $\tau = \text{torsp } \mathcal{F}$. Then $R_R^*/I_i \in T_\tau$ for all $i \in \Gamma$, so by hypothesis, $\prod_{i \in \Gamma} R_R^*/I_i \in T_\tau$. Let $x = \{1_{R^*} + I_i\}_{i \in \Gamma} \in \prod_{i \in \Gamma} R_R^*/I_i$. Notice that

$$\bigcap_{i \in \Gamma} I_i = (0 : \{1_{R^*} + I_i\}_{i \in \Gamma}) = (0:x) \in \mathcal{F}.$$

Thus \mathcal{F} is closed under arbitrary intersections.

Finally, let $\tau = \text{torsp } \mathcal{F}$ with \mathcal{F} as in (iv) above. Trivially, $\mathbb{Z}_{\text{zero}} \in T_\tau$. Also, if $x \in M$ then $(0:x) \in \mathcal{F}$, so $x \in \tau(M)$. Thus $M \in T_\tau$. It follows that $M \oplus \mathbb{Z}_{\text{zero}} \in T_\tau$. Now suppose that $\sigma \in \text{Jans-}R$ and $M \oplus \mathbb{Z}_{\text{zero}} \in T_\sigma$. We shall demonstrate that $\sigma \geq \tau$. For each $x \in M$, let $(0:x)^*$ denote the right annihilator of x in R^* . Then

$$\begin{aligned} R_R^*/(0:M) &= R_R^*/\bigcap_{x \in M} (0:x) = R_R^*/([\bigcap_{x \in M} (0:x)^*] \cap R) \\ &\lesssim [\prod_{x \in M} R_R^*/(0:x)^*] \oplus (R_R^*/R_R) \cong [\prod_{x \in M} xR^*] \oplus \mathbb{Z}_{\text{zero}} \\ &\lesssim M^M \oplus \mathbb{Z}_{\text{zero}} \in T_\sigma \quad (\text{because } \sigma \text{ is Jansian}). \end{aligned}$$

Now let $N \in \text{Mod-}R$ and $y \in \tau(N)$. Since $(0:y)^* \supseteq (0:y) \supseteq (0:M)$ and $R_R^*/(0:M) \in T_\sigma$, it follows that $yR^* \cong R_R^*/(0:y)^* \in T_\sigma$, i.e., $y \in \sigma(N)$. Hence $\tau(N) \subseteq \sigma(N)$. This shows that $\tau \leq \sigma$, as required. \square

A right topologizing filter \mathcal{F} on a ring R will be called a *Jansian topologizing filter* if it satisfies the equivalent conditions of the above theorem. Inasmuch as every $\tau \in [\tau_{\text{zero}}, 1_{\text{Mod-}R}]$ is of the form $\text{torsp } \mathcal{F}$ for a suitable $\mathcal{F} \in \text{Fil-}R$ (Theorem 2.4), Theorem 4 shows that every Jansian $\tau \in [\tau_{\text{zero}}, 1_{\text{Mod-}R}]$ is of the form $\text{torsp } \eta(I)$ for a suitable ideal I of R . Since $\text{Jans-}R$ is a sublattice and a meet-complete subsemilattice of $\text{torsp-}R$, it follows from Theorem 4 and Theorem 2.4

that $\{\mathfrak{F} \in \text{Fil-}R : \mathfrak{F} \text{ is Jansian}\}$ is a sublattice and a meet-complete subsemilattice of $\text{Fil-}R$.

The reader will recall that in §1 and §2 we showed that $[\text{torsp-}R]^{\text{du}}$ and $[\text{Fil-}R]^{\text{du}}$ have the structure of lattice ordered semigroups which admit transfinite products (see Theorem 1.7 and Corollary 2.5). Likewise, it is possible to define transfinite products in $\text{Id } R$, the lattice ordered semigroup of ideals of an arbitrary ring R . If $I \in \text{Id } R$ and α is a positive ordinal, we define:

$$\begin{aligned} I^1 &= I; \\ I^{\alpha \oplus 1} &= I^\alpha \cdot I; \text{ and} \\ I^\alpha &= \bigcap_{0 < \beta < \alpha} I^\beta, \text{ if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

It turns out that the structures $[\text{Fil-}R]^{\text{du}}$ and $\text{Id } R$ are closely connected. The connection is made clear in Proposition 5 below. Henceforth, we shall regard η as a map from $\text{Id } R$ to $[\text{Fil-}R]^{\text{du}}$.

PROPOSITION 5. *If R is a ring then the map $\eta: \text{Id } R \rightarrow [\text{Fil-}R]^{\text{du}}$ is one-to-one and its image is the set of all Jansian elements of $[\text{Fil-}R]^{\text{du}}$. Furthermore, η is a lattice and join-complete semilattice homomorphism as well as a semigroup homomorphism.*

Proof. The fact that η is one-to-one with image $\{\mathfrak{F} \in [\text{Fil-}R]^{\text{du}} : \mathfrak{F} \text{ is Jansian}\}$ is an immediate consequence of Theorem 4. Clearly, for $I, J \in \text{Id } R$, we have $\eta(I) \subseteq \eta(J)$ if and only if $I \supseteq J$. Since $\{\mathfrak{F} \in [\text{Fil-}R]^{\text{du}} : \mathfrak{F} \text{ is Jansian}\}$ is a sublattice and a join-complete subsemilattice of $[\text{Fil-}R]^{\text{du}}$, it follows that η is a lattice and join-complete semilattice homomorphism.

It remains to show that η is a semigroup homomorphism. If $I, J \in \text{Id } R$ then

$$\begin{aligned} \eta(I) \cdot \eta(J) &= \{A \leq R_R : (\exists K \in \eta(I))((K \supseteq A) \ \& \ (\forall s \in K)((A :_r s) \in \eta(J)))\} \\ &= \{A \leq R_R : (\exists K \leq R_R)((K \supseteq I) \ \& \ (K \supseteq A \supseteq KJ))\}. \end{aligned}$$

If $A \in \eta(I) \cdot \eta(J)$ then certainly $A \supseteq IJ$, so $\eta(I) \cdot \eta(J) \subseteq \eta(IJ)$. Conversely, if $A \in \eta(IJ)$ then, choosing $K = I + A$, we have that $K \supseteq I$ and $K \supseteq A \supseteq KJ$, so $A \in \eta(I) \cdot \eta(J)$. Thus $\eta(I) \cdot \eta(J) = \eta(IJ)$, as required. \square

It follows from the above proposition and Theorem 2.4 that the composition of the maps

$$\text{Id } R \xrightarrow{\eta} [\text{Fil-}R]^{\text{du}} \xrightarrow{\text{torsp}} [\text{torsp-}R]^{\text{du}}$$

is a lattice and semigroup monomorphism $\text{torsp} \circ \eta$ from $\text{Id } R$ into $[\text{torsp-}R]^{\text{du}}$, whose image is $\{\tau \in \text{Jans-}R : \tau \geq \tau_{\text{zero}}\}^{\text{du}}$.

Inasmuch as $\text{Gab-}R$ denotes the set of all idempotent elements of $\text{Fil-}R$ (see Proposition 2.1) and $\eta: \text{Id } R \rightarrow [\text{Fil-}R]^{\text{du}}$ is a semigroup monomorphism, the next result is obvious.

PROPOSITION 6. *The following conditions are equivalent for an ideal I of a ring R :*

- (i) I is idempotent;
- (ii) $\eta(I) \in \text{Gab-}R$.

\square

PROPOSITION 7. *If I is an ideal of a ring R then $\eta(I^\alpha) \supseteq \eta(I)^\alpha$ for all ordinals $\alpha > 0$, with equality holding whenever α is finite.*

Proof. Since $\eta: \text{Id } R \rightarrow [\text{Fil-}R]^{\text{du}}$ is a semigroup monomorphism, a straightforward inductive argument shows that $\eta(I^\alpha) = \eta(I)^\alpha$ for all finite ordinals $\alpha > 0$. To establish the inequality $\eta(I^\alpha) \supseteq \eta(I)^\alpha$ for arbitrary ordinals α , we proceed by transfinite induction. If $\alpha = 1$, there is nothing to prove. Suppose that $\eta(I^\beta) \supseteq \eta(I)^\beta$ for all positive $\beta < \alpha$. If α is a successor ordinal, say $\alpha = \gamma \oplus 1$, then

$$\begin{aligned} \eta(I^\alpha) &= \eta(I^\gamma I) = \eta(I^\gamma) \cdot \eta(I) \quad (\text{since } \eta \text{ is a semigroup homomorphism}) \\ &\supseteq \eta(I)^\gamma \cdot \eta(I) \quad (\text{by the inductive hypothesis}) \\ &= \eta(I)^\alpha, \end{aligned}$$

as required. If α is a limit ordinal then certainly $I^\alpha \subseteq I^\beta$ whenever $0 < \beta < \alpha$. Since the mapping $\eta: \text{Id } R \rightarrow [\text{Fil-}R]^{\text{du}}$ is order preserving, it follows that $\eta(I^\alpha) \supseteq \eta(I^\beta) \supseteq \eta(I)^\beta$ whenever $0 < \beta < \alpha$. Consequently, $\eta(I^\alpha) \supseteq \bigvee_{0 < \beta < \alpha} \eta(I)^\beta = \eta(I)^\alpha$, as required. \square

The following example shows that the inequality in Proposition 7 may be strict in some cases.

EXAMPLE 1. Consider the ring $R = \mathbb{Z}$ and take $I = 2\mathbb{Z}$. Clearly $I^\omega = \bigcap_{0 < \alpha < \omega} 2^\alpha \mathbb{Z} = 0$ so $\eta(I^\omega) = \eta(0)$. However, if \mathfrak{F} denotes the topologizing filter of all nonzero ideals then $\mathfrak{F} \supseteq \eta(J)$ for all nonzero ideals J of \mathbb{Z} . In particular then, $\mathfrak{F} \supseteq \bigvee_{0 < \alpha < \omega} \eta(I^\alpha)$. Therefore,

$$\eta(I^\omega) = \eta(0) \not\supseteq \bigvee_{0 < \alpha < \omega} \eta(I^\alpha) = \bigvee_{0 < \alpha < \omega} \eta(I)^\alpha = \eta(I)^\omega. \quad \square$$

Let \mathfrak{m} be an infinite cardinal. A preradical τ on $\text{Mod-}R$ is said to be \mathfrak{m} -Jansian if $\prod_{i \in \Gamma} M_i \in T_\tau$ whenever $\{M_i : i \in \Gamma\}$ is a subfamily of T_τ with $|\Gamma| < \mathfrak{m}$, i.e., T_τ is closed under direct products of fewer than \mathfrak{m} modules. Of course every preradical on $\text{Mod-}R$ is \aleph_0 -Jansian. Suppose τ is \mathfrak{m} -Jansian with \mathfrak{m} a singular cardinal and let $\{M_i : i \in \Gamma\}$ be a subfamily of T_τ with $|\Gamma| = \mathfrak{m}$. Since \mathfrak{m} is singular, Γ has a partition $\{\Gamma_x : x \in X\}$ such that $|\Gamma_x| < \mathfrak{m}$ for all $x \in X$ and $|X| < \mathfrak{m}$. It follows from the definition that $\prod_{i \in \Gamma} M_i = \prod_{x \in X} (\prod_{i \in \Gamma_x} M_i) \in T_\tau$. Hence τ is \mathfrak{m}^+ -Jansian. Inasmuch as \mathfrak{m}^+ is regular, the previous definition loses no generality by insisting that \mathfrak{m} be regular.

A simple adaptation of the proof of Proposition 1 yields the following.

PROPOSITION 8. *Let \mathfrak{m} be a regular cardinal. Then the following conditions are equivalent for a preradical τ on $\text{Mod-}R$:*

- (i) τ is \mathfrak{m} -Jansian;
 - (ii) $\tau(\prod_{i \in \Gamma} M_i) = \prod_{i \in \Gamma} \tau(M_i)$ for every family $\{M_i : i \in \Gamma\}$ of right R -modules such that $|\Gamma| < \mathfrak{m}$.
- \square

If \mathfrak{m} is a regular cardinal we shall denote by $\mathfrak{m}\text{-Jans-}R$ the set of all \mathfrak{m} -Jansian elements of $\text{torsp-}R$. Clearly then, $\aleph_0\text{-Jans-}R = \text{torsp-}R$ and

$$\text{Jans-}R = \bigcap_{\mathfrak{m} \geq \aleph_0} \mathfrak{m}\text{-Jans-}R.$$

Again, an obvious adaptation of the proofs of Lemma 2 and Proposition 3 yield the following.

PROPOSITION 9. *Let \mathfrak{m} be a regular cardinal. Then $\mathfrak{m}\text{-Jans-}R$ is closed under arbitrary meets and finite joins. \square*

If \mathfrak{m} is a regular cardinal and \mathcal{A} a nonempty subclass of $\text{Mod-}R$ then Proposition 9 allows us to define the \mathfrak{m} -Jansian torsion preradical on $\text{Mod-}R$ generated by \mathcal{A} as $\Lambda\{\tau \in \mathfrak{m}\text{-Jans-}R: T_\tau \supseteq \mathcal{A}\}$.

THEOREM 10. *Let \mathfrak{m} be a regular cardinal. Then the following assertions are equivalent for a right topologizing filter \mathfrak{F} on a ring R :*

- (i) \mathfrak{F} is closed under intersections of fewer than \mathfrak{m} right ideals;
- (ii) there exists $M \in \text{Mod-}R$ such that $\mathfrak{F} = \{A \leq R_R: A \supseteq (0:X) \text{ for some subset } X \text{ of } M \text{ with } |X| < \mathfrak{m}\}$;
- (iii) $\text{torsp}\mathfrak{F}$ is \mathfrak{m} -Jansian.

Moreover, if \mathfrak{F} is as in (ii), then $\text{torsp}\mathfrak{F}$ is the \mathfrak{m} -Jansian torsion preradical on $\text{Mod-}R$ generated by $M \oplus \mathbb{Z}_{\text{zero}}$.

Proof. (i) \Rightarrow (iii): Suppose $\mathfrak{F} \in \text{Fil-}R$ satisfies (i) and set $\tau = \text{torsp}\mathfrak{F}$. Let $\{M_i: i \in \Gamma\}$ be a subfamily of T_τ with $|\Gamma| < \mathfrak{m}$. If $x = \{x_i\}_{i \in \Gamma} \in \prod_{i \in \Gamma} M_i$ then $(0:x) = \bigcap_{i \in \Gamma} (0:x_i) \in \mathfrak{F}$ so $x \in \tau(\prod_{i \in \Gamma} M_i)$. This shows that $\prod_{i \in \Gamma} M_i \in T_\tau$. Thus $\tau = \text{torsp}\mathfrak{F}$ is \mathfrak{m} -Jansian.

(iii) \Rightarrow (ii): Set $\tau = \text{torsp}\mathfrak{F}$ and $M = \bigoplus_{A \in \mathfrak{F}} R_R^*/A$. Define $\mathfrak{G} = \{A \leq R_R: A \supseteq (0:X) \text{ for some subset } X \text{ of } M \text{ with } |X| < \mathfrak{m}\}$. For each $A \in \mathfrak{F}$, we have $R_R^*/A \in T_\tau$. Consequently, $(0:x) \in \mathfrak{F}$ for all $x \in M$. Let $X = \{x_i: i \in \Gamma\}$ be a subfamily of M with $|\Gamma| < \mathfrak{m}$. Since $\tau \in \mathfrak{m}\text{-Jans-}R$, we have $x = \{x_i\}_{i \in \Gamma} \in M^\Gamma \in T_\tau$, so $(0:x) = (0:X) \in \mathfrak{F}$. This shows that $\mathfrak{G} \subseteq \mathfrak{F}$. Now choose $K \in \mathfrak{F}$ and $x = \{x_A\}_{A \in \mathfrak{F}} \in M$ with

$$x_A := \begin{cases} 0, & \text{if } A \notin K; \\ 1_{R^*} + A, & \text{if } A = K. \end{cases}$$

Then $K = (0:x) \in \mathfrak{G}$, so $\mathfrak{F} \subseteq \mathfrak{G}$. We conclude that $\mathfrak{F} = \mathfrak{G}$, as required.

(ii) \Rightarrow (i): Suppose $\{A_i: i \in \Gamma\}$ is a subfamily of \mathfrak{F} with $|\Gamma| < \mathfrak{m}$. Then there exist subsets X_i of M such that $A_i \supseteq (0:X_i)$ and $|X_i| < \mathfrak{m}$ for all $i \in \Gamma$. Therefore $\bigcap_{i \in \Gamma} A_i \supseteq (0:\bigcup_{i \in \Gamma} X_i)$ and $|\bigcup_{i \in \Gamma} X_i| < \mathfrak{m}$ (because \mathfrak{m} is regular). Thus $\bigcap_{i \in \Gamma} A_i \in \mathfrak{F}$, so (i) holds.

Finally, let $\tau = \text{torsp}\mathfrak{F}$ with \mathfrak{F} as in (ii) above. Trivially, $\mathbb{Z}_{\text{zero}} \in T_\tau$, also if $x \in M$ then

$(0:x) \in \mathfrak{F}$, so $x \in \tau(M)$. Thus $M \in T_\tau$. It follows that $M \oplus \mathbb{Z}_{\text{zero}} \in T_\tau$. Suppose that $\sigma \in \mathfrak{m}\text{-Jans-}R$ and $M \oplus \mathbb{Z}_{\text{zero}} \in T_\sigma$. We shall demonstrate that $\sigma \geq \tau$. Suppose first that $X \subseteq M$ with $|X| < \mathfrak{m}$. For each $x \in M$, let $(0:x)^*$ denote the right annihilator of x in R^* . Then

$$\begin{aligned} R_R^*/(0:X) &= R_R^*/\bigcap_{x \in X} (0:x) = R_R^*/([\bigcap_{x \in X} (0:x)^*] \cap R) \\ &\lesssim [\prod_{x \in X} R_R^*/(0:x)^*] \oplus (R_R^*/R_R) \cong [\prod_{x \in X} xR^*] \oplus \mathbb{Z}_{\text{zero}} \\ &\lesssim M^X \oplus \mathbb{Z}_{\text{zero}} \in T_\sigma \quad (\text{because } \sigma \text{ is } \mathfrak{m}\text{-Jansian}). \end{aligned}$$

Suppose now that $N \in \text{Mod-}R$ and $y \in \tau(N)$. Then $(0:y) \supseteq (0:X)$ for some subset X of M with $|X| < \mathfrak{m}$. Since $(0:y)^* \supseteq (0:y) \supseteq (0:X)$ and $R_R^*/(0:X) \in T_\sigma$, it follows that $yR^* \cong R_R^*/(0:y)^* \in T_\sigma$, i.e., $y \in \sigma(N)$. Thus $\tau(N) \subseteq \sigma(N)$. This shows that $\tau \leq \sigma$, as required. \square

A right topologizing filter \mathfrak{F} on a ring R which satisfies the equivalent conditions of Theorem 10 will be called an \mathfrak{m} -Jansian topologizing filter.

PROPOSITION 11. *Let \mathfrak{m} be a regular cardinal. Then the following assertions are equivalent for an arbitrary ring R :*

- (i) every $\sigma \in [\tau_{\text{zero}}, 1_{\text{Mod-}R}]$ is \mathfrak{m} -Jansian;
- (ii) every $\mathfrak{F} \in \text{Fil-}R$ is \mathfrak{m} -Jansian;
- (iii) for every chain \mathcal{A} of right ideals of R with $|\mathcal{A}| < \mathfrak{m}$, there exist $D \in \mathcal{A}$ and a finite $Y \subseteq R_R^*$ such that $(D:{}_r Y) \subseteq \bigcap \mathcal{A}$ (where $(D:{}_r Y) := \{r \in R: Yr \subseteq D\}$);
- (iv) for every nonzero ordinal $\beta < \mathfrak{m}$ and every descending chain $\{A_\alpha: \alpha \in \beta\}$ of right ideals of R , there exist $\gamma \in \beta$ and a finite $Y \subseteq R_R^*$ such that $(A_\gamma:{}_r Y) \subseteq \bigcap_{\alpha \in \beta} A_\alpha$ (where $(A_\gamma:{}_r Y) := \{r \in R: Yr \subseteq A_\gamma\}$);
- (v) for every $M \in \text{Mod-}R$ and every subset X of M with $|X| < \mathfrak{m}$, there exists a finite subset Y of M with $(0:Y) \subseteq (0:X)$.

Proof. The equivalence of (i) and (ii) follows immediately from Theorem 10 and Theorem 2.4.

(ii) \Rightarrow (iii): Given \mathcal{A} as in (iii), let $\mathfrak{F} = \{K \leq R_R: K \supseteq (A:{}_r Y) \text{ for some } A \in \mathcal{A} \text{ and some finite } Y \subseteq R_R^*\}$. Then $\mathfrak{F} \in \text{Fil-}R$ so \mathfrak{F} is \mathfrak{m} -Jansian, by (ii). Now $\mathcal{A} \subseteq \mathfrak{F}$, since $A = (A:{}_r \{1_{R^*}\})$ for every $A \in \mathcal{A}$, and $|\mathcal{A}| < \mathfrak{m}$ so $\bigcap \mathcal{A} \in \mathfrak{F}$, which is the desired result.

(iii) \Rightarrow (iv) is immediate.

(iv) \Rightarrow (v): Let $M \in \text{Mod-}R$ and consider $X \subseteq M$ with $|X| < \mathfrak{m}$. Let $X = \{x_\alpha: \alpha \in \beta\}$ be a well ordering of X where, necessarily, the ordinal β is less than \mathfrak{m} . Let $X_0 = \{0\} \subseteq M$ and for each positive $\alpha \leq \beta$, let $X_\alpha = \{x_\delta: \delta < \alpha\}$. Set $A_\alpha = (0:X_\alpha)$ for each $\alpha \leq \beta$. We claim that for every $\alpha \leq \beta$, there is a finite subset Z_α of M such that $A_\alpha \supseteq (0:Z_\alpha)$. The proof is by transfinite induction on α . The claim is true for $\alpha = 0$, since $A_0 = R$; if it is true for an ordinal $\alpha < \beta$, it is also true for the ordinal $\alpha \oplus 1$, since we may take $Z_{\alpha \oplus 1} = Z_\alpha \cup \{x_\alpha\}$. Now suppose that α is a limit

ordinal and that the claim is true for all ordinals $\delta < \alpha$. By (iv), there is an ordinal $\gamma < \alpha$ and a finite $U \subseteq R_R^*$ such that $(A_\gamma :_r U) \subseteq \bigcap_{\delta \in \alpha} A_\delta = A_\alpha$. Note that Z_γ is finite, by the inductive hypothesis. Let $U = \{1_{R^*} \cdot a_i + 1_{R^*} \cdot m_i : i = 1, 2, \dots, n\}$ with $a_i \in R$ and $m_i \in \mathbb{Z}$ for all $i \in \{1, 2, \dots, n\}$. Define $Z_\alpha = \{za_i : z \in Z_\gamma \text{ and } 1 \leq i \leq n\} \cup \{zm_i : z \in Z_\gamma \text{ and } 1 \leq i \leq n\}$. Then Z_α is a finite subset of M and the right annihilator in R of Z_α is contained in $(A_\gamma :_r U)$, and therefore in A_α , as required. Our claim is thus vindicated and in particular, we have $(0 : X) = A_\beta \supseteq (0 : Y)$ for some finite subset Y of M . This completes the proof of (v).

(v) \Rightarrow (ii): Let $\mathfrak{F} \in \text{Fil-}R$. Certainly, \mathfrak{F} is \aleph_0 -Jansian, so by Theorem 10, there must exist an $M \in \text{Mod-}R$ such that $\mathfrak{F} = \{A \leq R_R : A \supseteq (0 : X) \text{ for some finite subset } X \text{ of } M\}$. It follows from (v) and Theorem 10 that \mathfrak{F} is necessarily \mathfrak{m} -Jansian. \square

Proposition 11 remains true if the ring R has identity, the interval $[\tau_{\text{zero}}, 1_{\text{Mod-}R}]$ in condition (i) is replaced by $\text{torsp-}R(\text{unital})$, and we replace all occurrences of R^* by R and all occurrences of $\text{Mod-}R$ by $\text{Mod-}R(\text{unital})$.

A nonzero ring R satisfying the equivalent conditions in the above proposition is said to be *right \mathfrak{m} -closed*. (Of course, every nonzero ring is right \aleph_0 -closed.) Inasmuch as every nonempty family of right ideals of a right artinian ring has a minimal member, it is easy to see that condition (ii) of Proposition 11 holds for all regular cardinals \mathfrak{m} whenever R is right artinian. Thus every nonzero right artinian ring is right \mathfrak{m} -closed for all regular cardinals \mathfrak{m} . It is natural to wonder whether there are cardinals \mathfrak{m} for which the converse is true. In particular, must a right \aleph_1 -closed ring be right artinian? We shall address this question (and some other natural questions) in the next section.

§4. \mathfrak{m} -CLOSED RINGS.

The study of \mathfrak{m} -closed rings (for $\mathfrak{m} \geq \aleph_1$) and the comparison of their properties with those of artinian rings is motivated by the fact that the following three conditions on a ring R with identity are equivalent (see [BB78, Corollary 3.3, p25]):

- (i) every $\mathfrak{F} \in \text{Fil-}R$ is Jansian;
- (ii) every right R -module is finitely annihilated;
- (iii) R is right artinian.

Two questions arise naturally. First, does the above result hold for nonzero rings without identity? We remarked in the sequel to Proposition 3.11 that if R is a nonzero right artinian ring then every $\mathfrak{F} \in \text{Fil-}R$ is \mathfrak{m} -Jansian for all regular cardinals \mathfrak{m} . Thus (iii) \Rightarrow (i) is certainly valid for rings without identity, but does (i) \Rightarrow (ii) \Rightarrow (iii) hold in the more general setting of rings without identity? Secondly, are there regular cardinals \mathfrak{m} such that every right \mathfrak{m} -closed ring is right artinian? In particular, must a right \aleph_1 -closed ring be right artinian?

In Theorem 16 we answer the first question in the affirmative while in Theorem 20 we provide a negative answer to the second. More specifically, we show that for every regular cardinal \mathfrak{m} , there is a ring R which is right \mathfrak{m} -closed but not right \mathfrak{n} -closed for any regular cardinal $\mathfrak{n} > \mathfrak{m}$. In fact, we shall be able to choose R to be a right chain domain with identity, having just one nontrivial (two-sided) ideal, such that the chain of all nonzero right ideals of R contains a dually cofinal copy of the cardinal \mathfrak{m} (considered as a well ordered set). It is here that the construction techniques developed in Chapter I find application.

THEOREM 1. *Every right \aleph_1 -closed ring enjoys the ascending chain condition on right topologizing filters.*

Proof. Let R be a right \aleph_1 -closed ring and suppose that, contrary to the statement of the theorem, there is a strictly ascending chain $\mathfrak{F}_0 \subsetneq \mathfrak{F}_1 \subsetneq \mathfrak{F}_2 \subsetneq \dots$ in $\text{Fil-}R$. For each $\alpha \in \aleph_0$, choose $I_\alpha \in \mathfrak{F}_{\alpha+1} \setminus \mathfrak{F}_\alpha$. Consider $M = \bigoplus_{\alpha \in \aleph_0} R_R^*/I_\alpha$. For each $\beta \in \aleph_0$, define $x(\beta) = \{x(\beta)_\alpha\}_{\alpha \in \aleph_0} \in M$ by

$$x(\beta)_\alpha = \begin{cases} 1_{R^*} + I_\beta, & \text{if } \alpha = \beta; \\ 0, & \text{if } \alpha \neq \beta, \end{cases}$$

and set $X = \{x(\beta) : \beta \in \aleph_0\}$. By Proposition 3.11, there exists a finite subset Y of M such that

$$(0:Y) \subseteq (0:X) = \bigcap_{\beta \in \aleph_0} (0:x(\beta)) = \bigcap_{\beta \in \aleph_0} I_\beta.$$

Since Y is finite, we must have $Y \subseteq \bigoplus_{\alpha \in \Gamma} R_R^*/I_\alpha$ for some finite subset Γ of \aleph_0 . It follows that $\bigcap_{\beta \in \aleph_0} I_\beta \supseteq (0:Y) \in \mathfrak{F}_\alpha$ for some (sufficiently large) $\alpha \in \aleph_0$, a contradiction. \square

In any complete lattice, the ACC is equivalent to the condition that every element of the lattice be compact (in the sense of Chapter 0, §1). A right topologizing filter \mathfrak{F} on a ring R is compact if and only if there is a right ideal I of R such that \mathfrak{F} is the smallest topologizing filter on R with $I \in \mathfrak{F}$. Such an \mathfrak{F} has an explicit description as $\mathfrak{F} = \{A \leq R_R : A \supseteq (I :_r X)\}$ for some finite $X \subseteq R_R^*$. It follows from Theorem 3.10 that $\text{torsp } \mathfrak{F}$ is the torsion preradical on $\text{Mod-}R$ generated by R_R^*/I . (The component \mathbb{Z}_{zero} referred to in Theorem 3.10 is clearly superfluous in this case, since \mathbb{Z}_{zero} is an epimorphic image of R_R^*/I .) Also, a routine modification of [Gol87, Proposition 2.18, p22] shows that under these conditions, the hereditary pretorsion class of $\text{torsp } \mathfrak{F}$ consists precisely of the homomorphic images of submodules of direct sums of copies of R_R^*/I . As usual, when R has identity, we may refine all of the foregoing observations, simply by substituting R for R^* , $\text{Mod-}R(\text{unital})$ for $\text{Mod-}R$, and regarding $\text{torsp } \mathfrak{F}$ as an element of $\text{torsp-}R(\text{unital})$.

In Proposition 3.5, we showed that the map $\eta: \text{Id } R \rightarrow [\text{Fil-}R]^{\text{du}}$ is one-to-one and order preserving. The following corollary to Theorem 1 is therefore immediate.

COROLLARY 2. *Every right \aleph_1 -closed ring enjoys the descending chain condition on (two-sided) ideals.* \square

The next two corollaries are trivial consequences of the above result.

COROLLARY 3. *Every right duo (in particular, every commutative) right \aleph_1 -closed ring is right artinian.* \square

COROLLARY 4. *If I is an ideal of a right \aleph_1 -closed ring R and α is an arbitrary nonzero ordinal then $I^\alpha = I^\beta$ for some finite ordinal β .* \square

The next corollary also follows easily from Theorem 1.

COROLLARY 5. *Let R be a right \aleph_1 -closed ring. If $\mathfrak{F} \in \text{Fil-}R$ and α is an arbitrary nonzero ordinal then $\mathfrak{F}^\alpha = \mathfrak{F}^\beta$ for some finite ordinal β .* \square

Since $\eta: \text{Id } R \rightarrow [\text{Fil-}R]^{\text{du}}$ is a semigroup monomorphism, Corollary 5 is just an extension of Corollary 4. Recall that if $\tau \in \text{torsp-}R$ then there is a (unique) smallest torsion radical greater than or equal to τ which is expressible in the form τ^α for some ordinal α (Proposition 1.9). It follows from Corollary 5 that if R is a right \aleph_1 -closed ring and $\tau \in [\tau_{\text{zero}}, 1_{\text{Mod-}R}]$ then $\bar{\tau} = \tau^\beta$ for some finite ordinal β .

PROPOSITION 6. *Given a ring R , let $\mathfrak{G} = \{A \leq R_R : R_R/A \text{ is artinian}\}$. Then \mathfrak{G} is a right topologizing filter on R and the following conditions are equivalent for any right topologizing filter \mathfrak{F} on R such that $\mathfrak{F} \subseteq \mathfrak{G}$:*

- (i) \mathfrak{F} satisfies the descending chain condition (i.e., there is no infinite strictly descending chain of right ideals in \mathfrak{F});
- (ii) \mathfrak{F} is Jansian;
- (iii) \mathfrak{F} is \aleph_1 -Jansian.

Proof. The first assertion is well known and follows easily from the fact that the class of artinian right R -modules is closed under homomorphic images, submodules and finite direct products. (Indeed, if $A, B \in \mathfrak{G}$ and $A \subseteq C \leq R_R$ then R_R/C is a homomorphic image of R_R/A , while $R_R/(A \cap B) \lesssim (R_R/A) \oplus (R_R/B)$ and $R_R/(A :_r a) \lesssim R_R/A$ for all $r \in R$.)

It is easy to see that (i) \Rightarrow (ii) \Rightarrow (iii) hold for any $\mathfrak{F} \in \text{Fil-}R$.

(iii) \Rightarrow (i): Suppose, contrary to (i), that \mathfrak{F} contains a strictly descending infinite chain of right ideals, say $A_0 \supsetneq A_1 \supsetneq A_2 \supsetneq \dots$. If $I = \bigcap_{\alpha \in \aleph_0} A_\alpha$ then $I \in \mathfrak{F} \subseteq \mathfrak{G}$, hence R_R/I is artinian. But $A_0/I \supsetneq A_1/I \supsetneq A_2/I \supsetneq \dots$ is a strictly descending infinite chain of submodules of R_R/I , a

contradiction. □

Define $\mathfrak{F} = \{A \leq R_R : A \text{ contains a finite intersection of maximal proper right ideals of } R\}$. It is easily checked that \mathfrak{F} is a right topologizing filter on R . We remark that for a proper submodule N of a module M_R , the module M/N is finitely generated and semisimple if and only if N is a finite intersection of maximal proper submodules of M . Indeed, if N is a finite intersection of maximal proper submodules L_0, L_1, \dots, L_k of M then $M/N \simeq \bigoplus_{i=0}^k M/L_i$. Since $\bigoplus_{i=0}^k M/L_i$ is finitely generated and semisimple, M/N is finitely generated and semisimple. Conversely, if M/N is nonzero, finitely generated and semisimple then $M/N = \bigoplus_{i=0}^k S_i/N$ for suitable submodules S_i of M with $N \subsetneq S_i$ such that each S_i/N is a simple submodule of M/N . It may be checked routinely that $L_j = \sum_{i \neq j} S_i$ is a maximal proper submodule of M for $j = 0, 1, \dots, k$, and that $\bigcap_{j=0}^k L_j = N$. Consequently, we could have defined \mathfrak{F} as $\{A \leq R_R : R_R/A \text{ is finitely generated and semisimple}\}$.

PROPOSITION 7. *Let $\mathfrak{F} = \{A \leq R_R : A \text{ contains a finite intersection of maximal proper right ideals of } R\}$. Then the following assertions are equivalent:*

- (i) \mathfrak{F} is \aleph_1 -Jansian;
- (ii) $R_R / \bigcap \mathfrak{F}$ is a finitely generated semisimple module.

Proof. It follows from the above remarks that (ii) holds if and only if $\bigcap \mathfrak{F} \in \mathfrak{F}$, i.e., \mathfrak{F} is Jansian. By Proposition 6, \mathfrak{F} is Jansian if and only if (i) holds. □

We remind the reader that $J(R)$ denotes the Jacobson radical of a ring R .

COROLLARY 8. *If R is a right \aleph_1 -closed ring then $R/J(R)$ is a semisimple artinian ring,³ and is therefore either zero or isomorphic to a finite direct product of finite dimensional matrix rings over division rings.*

Proof. If \mathfrak{F} is chosen as in Proposition 7, it is obvious that $\bigcap \mathfrak{F} \subseteq J(R)$, so $R/J(R)$ is a semisimple (right) artinian ring. Also, of course, $J(R/J(R)) = 0$. If $R/J(R) \neq 0$ then by the Artin-Wedderburn Theorem (Theorem 0.7.1), we must have $R/J(R)$ isomorphic to a finite direct product of finite dimensional matrix rings $M_{\mathbf{n}_i}(D_i)$ over division rings D_i for some finite nonzero cardinals \mathbf{n}_i . □

The next three results will be required in Theorem 15.

LEMMA 9. *Let N be a submodule of a right R -module M and let \mathfrak{F}_N (resp. \mathfrak{F}_M) be the set of all essential submodules of N (resp. M). Let \mathfrak{m} be an arbitrary infinite cardinal. If \mathfrak{F}_M is closed under intersections of fewer than \mathfrak{m} submodules then so is \mathfrak{F}_N .*

Proof. Let L be a fixed orthogonal complement for N in M . It is known that $N \oplus L$ is

³A ring R with identity such that $R/J(R)$ is semisimple artinian is called a *semilocal* ring - see remarks following Theorem 0.5.4.

essential in M . If $A \in \mathfrak{F}_N$ then $A \oplus L$ is essential in $N \oplus L$ which is, in turn, essential in M , so $A \oplus L \in \mathfrak{F}_M$. The association $A \mapsto A \oplus L$ ($A \in \mathfrak{F}_N$) therefore defines a map from \mathfrak{F}_N to \mathfrak{F}_M . Let $\{A_i : i \in \Gamma\}$ be a subfamily of \mathfrak{F}_N with $|\Gamma| < \mathfrak{m}$. Then $\{A_i \oplus L : i \in \Gamma\}$ is a subfamily of \mathfrak{F}_M and so, by hypothesis, $\bigcap_{i \in \Gamma} (A_i \oplus L) = (\bigcap_{i \in \Gamma} A_i) \oplus L \in \mathfrak{F}_M$. Clearly, we must have $\bigcap_{i \in \Gamma} A_i \in \mathfrak{F}_N$, so \mathfrak{F}_N is closed under intersections of fewer than \mathfrak{m} submodules. \square

A subgroup H of an (additively written) group G will be called an *essential subgroup* of G if $H \cap A \neq 0$ for all nonzero subgroups A of G .

LEMMA 10. *Let G be a nonzero (additively written) abelian group and let \mathfrak{F} denote the set of all essential subgroups of G . Then the following assertions are equivalent:*

- (i) \mathfrak{F} is closed under countable intersections;
- (ii) \mathfrak{F} is closed under arbitrary intersections;
- (iii) $\text{soc } G_{\mathbb{Z}}$ is essential in $G_{\mathbb{Z}}$;
- (iv) G is a torsion abelian group;
- (v) $E(G_{\mathbb{Z}}) \cong \bigoplus_{i \in \Gamma} E(\mathbb{Z}_{p_i}) \cong \bigoplus_{i \in \Gamma} \mathbb{Z}_{p_i}^{\infty}$ for suitable (not necessarily distinct) positive prime integers p_i , indexed by a suitable (not necessarily finite) set Γ .

Proof. (ii) \Rightarrow (i) is trivial. The equivalence of (ii) and (iii) follows from Proposition 0.3.2. It follows from Theorem 0.4.3 that every injective \mathbb{Z} -module (i.e., divisible abelian group) is isomorphic to $\mathbb{Q}_{\mathbb{Z}}^{(\Gamma)} \oplus (\bigoplus_{i \in \Gamma} \mathbb{Z}_{p_i}^{\infty})$ for some set Γ and some family $\{p_i : i \in \Gamma\}$ of (not necessarily distinct) positive prime integers. In view of the fact that G is torsion if and only if $E(G_{\mathbb{Z}})$ is torsion (as an abelian group), we may infer the equivalence of (iv) and (v).

(i) \Rightarrow (iv): Suppose G is not torsion. Then the group $\langle \mathbb{Z}; +, -, 0 \rangle$ embeds in G . Since \mathfrak{F} is closed under countable intersections, it follows from the previous lemma that the set of all nonzero subgroups of $\langle \mathbb{Z}; +, -, 0 \rangle$ is closed under countable intersections, a contradiction.

(iv) \Rightarrow (iii): Suppose G is torsion and let $0 \neq x \in G$. Suppose x has order $n \in \mathbb{N}$. Write $n = mp$ with $m, p \in \mathbb{N}$ and p prime. Clearly $y = xm$ has prime order so $x\mathbb{Z} \cap \text{soc } G_{\mathbb{Z}} \neq 0$. This shows that $\text{soc } G_{\mathbb{Z}}$ is essential in $G_{\mathbb{Z}}$. \square

If I is an ideal of a ring R then it is a routine matter to check that the map from $\text{Fil-}R$ to $\text{Fil-}(R/I)$ defined by

$$\mathfrak{F} \mapsto \{A/I : A \in \mathfrak{F}\} \quad (\mathfrak{F} \in \text{Fil-}R)$$

is onto and preserves the \mathfrak{m} -Jansian property. This establishes the first assertion of the next result.

PROPOSITION 11. *Let \mathfrak{m} be a regular cardinal. Then:*

- (i) all nonzero homomorphic images of right \mathfrak{m} -closed rings are right \mathfrak{m} -closed;
- (ii) every right \mathfrak{m} -closed ring is isomorphic to a finite direct product of

indecomposable right \mathfrak{m} -closed rings.

Proof. (i) has been explained above and (ii) follows easily from (i) and Corollary 2. \square

The converse of (ii) is true for rings with identity and is easy to prove, e.g., by using the criteria for \mathfrak{m} -closure given in Proposition 3.11. In other words, a finite direct product of right \mathfrak{m} -closed rings with identity is right \mathfrak{m} -closed. Consequently, a ring with identity is right \mathfrak{m} -closed if and only if it is isomorphic to a finite direct product of indecomposable right \mathfrak{m} -closed rings (with identity).

In Corollaries 2 and 8, we exhibited a number of “artinian-like” properties of \mathfrak{m} -closed rings (for $\mathfrak{m} \geq \aleph_1$). The implied similarity between artinian and \mathfrak{m} -closed rings will be reinforced by Theorem 15, which shows that several large classes of right \mathfrak{m} -closed rings ($\mathfrak{m} \geq \aleph_1$) are necessarily right artinian. We first need to introduce the notion of “Gabriel dimension”.

Gabriel dimension has as an ancestor the notion of classical Krull dimension, defined originally for finite ordinals and commutative rings only (a definition appears in Chapter III, §4), has been generalized to arbitrary ordinals and to modules over an arbitrary ring. Gabriel dimension, which bears the name of its inventor P. Gabriel [Gab62], is one of several such generalizations in the literature.

Let R be an arbitrary ring. We define a chain $\{\sigma_\alpha\}_\alpha$ (indexed by the ordinals) of torsion radicals on $\text{Mod-}R$ as follows:

(i) $\sigma_0 = 0$.

(ii) Suppose that σ_α has been defined. We call a right R -module M α -simple if $M \in F_{\sigma_\alpha}$, yet $M/N \in T_{\sigma_\alpha}$ for all nonzero submodules N of M . Let \mathcal{F}_α denote the class of all α -simple right R -modules and let $\tau(\mathcal{F}_\alpha)$ denote the torsion radical on $\text{Mod-}R$ generated by \mathcal{F}_α . We define $\sigma_{\alpha \oplus 1} = \sigma_\alpha \vee \tau(\mathcal{F}_\alpha)$ (the join is calculated in $\text{tors-}R$).

(iii) If α is a limit ordinal and σ_β has been defined for all ordinals $\beta < \alpha$, then we define $\sigma_\alpha = \bigvee_{\beta < \alpha} \sigma_\beta$ (the join is again, calculated in $\text{tors-}R$).

We call the chain $\{\sigma_\alpha\}_\alpha$ the *Gabriel filtration* on $\text{Mod-}R$. If $M \in \text{Mod-}R$, we say that M has *Gabriel dimension* if the set of ordinals β for which $M \in T_{\sigma_\beta}$ is nonempty; if, in addition, α is the least element of this set of ordinals, we say that M has *Gabriel dimension* α , abbreviated $G\text{-dim } M = \alpha$. Thus $G\text{-dim } M \leq \alpha$ if and only if $M \in T_{\sigma_\alpha}$. It is possible that there is no ordinal α for which $M \in T_{\sigma_\alpha}$. In this case we say that M has *no Gabriel dimension*. The ring R is said to have *right Gabriel dimension* α if $G\text{-dim } R_R = \alpha$. If this is the case for some ordinal α , then R is said to have *right Gabriel dimension*.

Observe that the 0-simple right R -modules are precisely the nonzero simple modules. It follows that σ_1 is the torsion radical on $\text{Mod-}R$ generated by the class of all nonzero simple right R -modules. In other words, σ_1 is the smallest torsion radical on $\text{Mod-}R$ for which $\sigma_1(M) \supseteq \text{soc } M$ for all $M \in \text{Mod-}R$. Inasmuch as $\overline{\text{soc}}$ denotes the smallest torsion radical on $\text{Mod-}R$ for which

$\overline{\text{soc}} \geq \text{soc} \in \text{torsp-}R$, the following lemma is immediate.

LEMMA 12. *If $\{\sigma_\alpha\}_\alpha$ denotes the Gabriel filtration on $\text{Mod-}R$, then $\sigma_1 = \overline{\text{soc}}$.* □

PROPOSITION 13. *The following assertions are equivalent for any right R -module M :*

- (i) *every nonzero homomorphic image of M has a nonzero socle;*
- (ii) $G\text{-dim } M \leq 1$;
- (iii) $\text{soc}^\alpha M = M$ for some ordinal α .

Proof. The equivalence of (ii) and (iii) follows because $G\text{-dim } M \leq 1$ if and only if $M \in T_{\sigma_1}$ and $\sigma_1 = \overline{\text{soc}} = \text{soc}^\alpha$ for some ordinal α (by Proposition 1.9).

(i) \Rightarrow (iii): By Proposition 1.9, we may write $\overline{\text{soc}}$ as soc^α for some ordinal α . Now $\text{soc}(M/\overline{\text{soc}}M) \subseteq \overline{\text{soc}}(M/\overline{\text{soc}}M) = 0$, so by (i), $M/\overline{\text{soc}}M = 0$, i.e., $\text{soc}^\alpha M = M$.

(iii) \Rightarrow (i): Suppose, contrary to (i), that $\text{soc}(M/N) = 0$ for some proper submodule N of M . A routine transfinite induction argument shows that $\text{soc}^\beta(M/N) = 0$ for all ordinals $\beta > 0$. In particular, $\text{soc}^\alpha(M/N) = 0$. It follows that $(\text{soc}^\alpha M)/N \subseteq \text{soc}^\alpha(M/N) = 0$, so $M = \text{soc}^\alpha M = N$, a contradiction. □

A right R -module M which satisfies the equivalent conditions in Proposition 13 above is called *semiartinian*.

We introduce another well known generalization of the classical Krull dimension, which is related to Gabriel dimension.

Let R be an arbitrary ring and $M \in \text{Mod-}R$. The (generalized) *Krull dimension* of M , abbreviated $K\text{-dim } M$, is defined as follows:

- (i) If $M = 0$, then $K\text{-dim } M = -1$.
- (ii) If α is an ordinal and $K\text{-dim } M \not\leq \alpha$, then $K\text{-dim } M = \alpha$ provided that every infinite descending chain $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$ satisfies $K\text{-dim}(M_\beta/M_{\beta \oplus 1}) < \alpha$ for some $\beta \geq 0$.

It is possible that there is no ordinal α such that $K\text{-dim } M = \alpha$, in which case we say that M has *no Krull dimension*. We say that M has *Krull dimension* if it has Krull dimension α for some ordinal α or for $\alpha = -1$. The ring R is said to have *right Krull dimension* α if $K\text{-dim } R_R = \alpha$, and R is said to have *right Krull dimension* if this is true for some ordinal α or for $\alpha = -1$.

We refer the reader to Gordon and Robson's papers [GR73],[GR74] and to [NV87] for a detailed exposition on Gabriel and Krull dimension.

It is clear from the definition that the right R -modules having Krull dimension 0 are precisely the nonzero artinian modules. Note also that $K\text{-dim } \mathbb{Z}_{\mathbb{Z}} = 1$, since every proper factor module of $\mathbb{Z}_{\mathbb{Z}}$ is finite. It is known, more generally, that if R is an arbitrary ring then every noetherian right

R -module has (not necessarily finite) Krull dimension (see [GR73, Proposition 1.3, p7] or [NV87, Corollary 3.1.8, p124]).

The following result, which is due to Gordon and Robson, shows that a module with Krull dimension must have Gabriel dimension.

PROPOSITION 14. [GR74, Theorem 2.4, p464] *If M is a right R -module with Krull dimension then M has Gabriel dimension and*

$$\text{K-dim } M \leq \text{G-dim } M \leq (\text{K-dim } M) \oplus 1. \quad \square$$

Following [Fai76, p155], we call a ring R a *right B-ring* if every nonzero unital right R -module contains a maximal proper submodule.

THEOREM 15. *The following conditions are equivalent for a right \aleph_1 -closed ring R :*

- (i) R_R has Gabriel dimension;
- (ii) there exists a right R -module M with Gabriel dimension such that $(0 : M) \subseteq (0 :_r R)$;
- (iii) $J(R)$ is nilpotent;
- (iv) R is a right B-ring;
- (v) R is right artinian.

Proof. (v) \Rightarrow (i) and (i) \Rightarrow (ii) are obvious.

(ii) \Rightarrow (iii): It follows from Corollary 4 that $J(R)^\gamma$ is idempotent for some finite ordinal γ . We shall demonstrate that $M \cdot J(R)^\gamma = 0$ for all right R -modules M with Gabriel dimension.

We use transfinite induction on $\text{G-dim } M$. Let $\{\sigma_\alpha\}_\alpha$ be the Gabriel filtration on $\text{Mod-}R$. Recall that if α is an ordinal then $\text{G-dim } M \leq \alpha$ if and only if $M \in T_{\sigma_\alpha}$. If $\text{G-dim } M = 0$ then $M = 0$ and so $M \cdot J(R)^\gamma = 0$. Now suppose that $N \cdot J(R)^\gamma = 0$ for all $N \in T_{\sigma_\beta}$ and all $\beta < \alpha$. If α is a limit ordinal then $T_{\sigma_\alpha} = \bigcup_{\beta < \alpha} T_{\sigma_\beta}$ and therefore $N \cdot J(R)^\gamma = 0$ for all $N \in T_{\sigma_\alpha}$. Suppose α is a successor ordinal, say $\alpha = \delta \oplus 1$, and let M be a δ -simple right R -module. Then $M/N \in T_{\sigma_\delta}$ for all nonzero submodules N of M . By the inductive hypothesis, $(M/N) \cdot J(R)^\gamma = 0$ for all nonzero submodules N of M , i.e.,

$$M \cdot J(R)^\gamma \subseteq \bigcap \{N \leq M_R : N \neq 0\} = L.$$

If $L \neq 0$ then L is a simple module, in which case, $L \cdot J(R)^\gamma \subseteq L \cdot J(R) = 0$. Therefore,

$$M \cdot J(R)^\gamma = M \cdot J(R)^\gamma \cdot J(R)^\gamma \subseteq L \cdot J(R)^\gamma = 0.$$

We have thus shown that $M \cdot J(R)^\gamma = 0$ for all $M \in T_{\sigma_\delta} \cup \{N \in \text{Mod-}R : N \text{ is } \delta\text{-simple}\} = \mathcal{C}_\delta$, or equivalently, $\mathcal{C}_\delta \subseteq \{M \in \text{Mod-}R : M \cdot J(R)^\gamma = 0\}$, which is the hereditary pretorsion class of $\text{torsp } \eta(J(R)^\gamma)$. Since σ_α is, by definition, the torsion radical on $\text{Mod-}R$ generated by \mathcal{C}_δ , and $\text{torsp } \eta(J(R)^\gamma)$ is itself a torsion radical on $\text{Mod-}R$ (Proposition 3.6), it follows that $T_{\sigma_\alpha} \subseteq$

$\{M \in \text{Mod-}R : M \cdot J(R)^\gamma = 0\}$. This completes the inductive argument.

By hypothesis, we may choose $M \in \text{Mod-}R$ such that $M \cdot J(R)^\gamma = 0$ and $(0 : M) \subseteq (0 : {}_r R)$. It follows that $R \cdot J(R)^\gamma = 0$ so $J(R)^\gamma \cdot J(R)^\gamma = J(R)^\gamma = 0$. Thus $J(R)$ is nilpotent.

(iii) \Rightarrow (iv): Suppose $J(R)^\gamma = 0$ for some finite ordinal $\gamma > 0$ and let M be a nonzero unital right R -module. Note that $M \cdot J(R) \subsetneq M$, otherwise

$$M = M \cdot J(R) = M \cdot J(R)^2 = \dots = M \cdot J(R)^\gamma = 0,$$

a contradiction. Consider the nonzero right R -module $N = M/(M \cdot J(R))$. Since $N \cdot J(R) = 0$, we may regard N as a unital right module over the ring $R/J(R)$. Inasmuch as $M \cdot J(R) \subsetneq M = MR$, we cannot have $R/J(R) = 0$. Since $R/J(R)$ is a nonzero semisimple artinian ring (Corollary 8), it follows that N is a semisimple right R -module. Consequently, N must contain a maximal proper submodule, and it follows that the same is true of M . We conclude that R is a right B -ring.

(iv) \Rightarrow (v): Consider the right topologizing filter $\mathfrak{G} = \{A \leq R_R : R_R/A \text{ is artinian}\}$ on R . Since \mathfrak{G} is Jansian (by Proposition 6), $I = \bigcap \mathfrak{G} \in \mathfrak{G}$. Suppose $I \neq 0$. If $IR = I$ then, since R is a right B -ring, I_R has a maximal proper submodule, say K . Now R_R/I is artinian since $I \in \mathfrak{G}$, while of course I_R/K is also artinian, so R_R/K is artinian (by Theorem 0.3.3), i.e., $K \in \mathfrak{G}$, contradicting the definition of I . Consequently, we must have that $I \supsetneq IR$.

Consider the nonzero ring $\bar{R} = R/IR$. By Proposition 11, \bar{R} is right \aleph_1 -closed while \bar{R} is obviously a right B -ring, since the \bar{R} -submodule structure of any right \bar{R} -module coincides with its R -module structure. Moreover, $\{A \leq \bar{R}_{\bar{R}} : \bar{R}_{\bar{R}}/A \text{ is artinian}\}$ is a Jansian right topologizing filter on \bar{R} with smallest element $\bar{I} = I/IR \neq 0$. We lose no generality, therefore, in identifying R and \bar{R} and identifying I and \bar{I} and assuming that $I \supsetneq IR = 0$.

Now consider $\mathfrak{F} = \{A \leq R_R : A \text{ contains an essential subgroup of the group } \langle I; +, -, 0 \rangle\}$. We shall show that $\mathfrak{F} \in \text{Fil-}R$. The only nontrivial step is verifying that $(A : {}_r a) \in \mathfrak{F}$ whenever $A \in \mathfrak{F}$ and $a \in R$. For each $a \in R$, consider the map $\phi_a : R_R \rightarrow R_R$ defined by $\phi_a(r) = ar$ for all $r \in R$. It is clear that $\phi_a \in \text{End}_R(R_R)$ and $\phi_a[I] \subseteq I$ (since I is an ideal of R). Furthermore, if $A \leq R_R$ and $A \supseteq G$ for some essential subgroup G of $\langle I; +, -, 0 \rangle$, then $(A : {}_r a) = \{r \in R : ar \in A\} = \phi_a^{-1}[A] \supseteq \phi_a^{-1}[G]$. Since G is essential in $\langle I; +, -, 0 \rangle$, it follows that $\phi_a^{-1}[G] \cap I$ is essential in $\langle I; +, -, 0 \rangle$. Thus $(A : {}_r a) \in \mathfrak{F}$, as required.

Observe, however, that the right ideals of R contained in I are precisely the subgroups of $\langle I; +, -, 0 \rangle$, because $IR = 0$. Consequently, setting $G = \langle I; +, -, 0 \rangle$, we have that every essential subgroup of G is an element of \mathfrak{F} . By hypothesis, though, \mathfrak{F} is \aleph_1 -Jansian, so the set of essential subgroups of G is closed under countable intersections. By Lemma 10, $E(G_{\mathbb{Z}}) \cong \bigoplus_{i \in \Gamma} \mathbb{Z}_{p_i}^\infty$ for suitable prime integers p_i , $i \in \Gamma$. For each finite subset Θ of Γ , set $G_\Theta = \bigoplus_{i \in \Gamma \setminus \Theta} \mathbb{Z}_{p_i}^\infty \leq E(G_{\mathbb{Z}})$. Since each $\mathbb{Z}_{p_i}^\infty$ satisfies the DCC on subgroups (see remarks preceding Theorem 0.4.3), it follows that $E(G_{\mathbb{Z}})/G_\Theta \cong \bigoplus_{i \in \Theta} \mathbb{Z}_{p_i}^\infty$ satisfies the DCC on subgroups. Now

$$G/(G \cap G_\Theta) \cong (G + G_\Theta)/G_\Theta \leq E(G_{\mathbb{Z}})/G_\Theta,$$

so $G/(G \cap G_\Theta)$ satisfies the DCC on subgroups. Again, since $IR = 0$, $G \cap G_\Theta$ is a right ideal of R and so $I_R/(G \cap G_\Theta)$ is an artinian module. Since R_R/I is artinian, we must have that $R_R/(G \cap G_\Theta)$ is artinian, which contradicts the minimality of I unless $G \cap G_\Theta = G$, i.e., $G \subseteq G_\Theta$. We therefore have $G \subseteq G_\Theta = \bigoplus_{i \in \Gamma \setminus \Theta} \mathbb{Z}_{p_i}^\infty$ for all finite subsets Θ of Γ . But this can only be the case if $G = 0$, contradicting our assumption that $I \not\supseteq IR = 0$. It follows that $0 = I \in \{A \leq R_R : R_R/A \text{ is artinian}\}$, i.e., R is right artinian. \square

It follows from the above theorem that *any ring with right Krull dimension (in particular, any right noetherian ring) which is right \aleph_1 -closed is right artinian.*

Condition (iii) of Theorem 15 points to the fact that \mathfrak{m} -closure is a highly asymmetric property: *if R is a ring which is right but not left artinian then R is right \mathfrak{m} -closed for all regular cardinals \mathfrak{m} , but since $J(R)$ is nilpotent (Proposition 0.5.5), R is left \mathfrak{m} -closed only for $\mathfrak{m} = \aleph_0$.* (We remind the reader that the upper triangular 2×2 matrix ring $\begin{pmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$ is an example of a right but not left artinian ring.)

The following result generalizes [BB78, Corollary 3.3, p25] to rings without identity. Its proof makes use of the previous theorem.

THEOREM 16. *The following conditions on a nonzero ring R are equivalent:*

- (i) every $\sigma \in [\tau_{\text{zero}}, 1_{\text{Mod-}R}]$ is Jansian;
- (ii) every $\mathfrak{F} \in \text{Fil-}R$ is Jansian;
- (iii) every right R -module is finitely annihilated;
- (iv) R is right artinian.

Proof. The equivalence of (i) and (ii) follows immediately from Theorems 2.4 and 3.4.

(iv) \Rightarrow (iii): Let $M \in \text{Mod-}R$. Since R is right artinian, the set $\{(0:Y) : Y \text{ is a finite subset of } M\}$ must have a minimal member, say $(0:X)$. The minimality of $(0:X)$ clearly implies that $(0:X) = (0:M)$. Thus M is finitely annihilated.

(iii) \Rightarrow (ii): Let $\mathfrak{F} \in \text{Fil-}R$. By Theorem 3.10 ((i) \Rightarrow (ii)), there exists $M \in \text{Mod-}R$ such that $\mathfrak{F} = \{A \leq R_R : A \supseteq (0:X) \text{ for some finite } X \subseteq M\}$. By hypothesis, $\bigcap \mathfrak{F} = (0:M) = (0:X)$ for some finite subset X of M , so $\bigcap \mathfrak{F} \in \mathfrak{F}$, i.e., \mathfrak{F} is Jansian.

(ii) \Rightarrow (iv): Let I be an arbitrary proper ideal of R . Let \mathfrak{F}_I denote the set of all essential submodules of R_R/I and define $\mathfrak{F} = \{A \leq R_R : A \supseteq I \text{ and } A/I \in \mathfrak{F}_I\}$. It is a routine matter to check that $\mathfrak{F} \in \text{Fil-}R$ and that $(\bigcap \mathfrak{F})/I = \bigcap \mathfrak{F}_I$. By hypothesis, $\bigcap \mathfrak{F} \in \mathfrak{F}$, whence $\bigcap \mathfrak{F}_I \in \mathfrak{F}_I$, i.e., $\bigcap \mathfrak{F}_I$ is an essential submodule of R_R/I . Recalling, however, that $\bigcap \mathfrak{F}_I = \text{soc}(R_R/I)$ (Proposition 0.3.2), we find that we have proved that $\text{soc}(R_R/I) \neq 0$ for all proper ideals I of R . Now consider the torsion preradical soc on $\text{Mod-}R$. By Proposition 1.9, $\overline{\text{soc}} = \text{soc}^\alpha \in \text{tors-}R$ for

some ordinal α . Since $\text{soc}(R_R/\overline{\text{soc}}(R_R)) = 0$, it follows from the above argument that $R_R = \overline{\text{soc}}(R_R) = \text{soc}^\alpha(R_R)$. By Proposition 13, $\text{G-dim } R_R = 1$. The fact that R is right artinian follows from Theorem 15. \square

If \mathfrak{m} is an arbitrary nonzero cardinal and R an arbitrary ring, we shall say that a right R -module M is \mathfrak{m} -annihilated if there exists a subset X of M such that $|X| < \mathfrak{m} + 1$ and $(0 : X) = (0 : M)$. Thus every module M_R is \mathfrak{m} -annihilated for some $\mathfrak{m} \leq |M|^+$ and an \aleph_0 -annihilated module is just a finitely annihilated module. The following lemma is an immediate consequence of Proposition 3.11(v).

LEMMA 17. *Let \mathfrak{m} be a regular cardinal. If a ring R is right \mathfrak{m} -closed then every \mathfrak{m} -annihilated right R -module is finitely annihilated.* \square

PROPOSITION 18. *Let \mathfrak{m} be a regular cardinal. Then the following assertions are equivalent for a ring R :*

- (i) R is right artinian;
- (ii) R is right \mathfrak{m} -closed and satisfies the right \mathfrak{m} -DCC.

Proof. One implication is trivial. Conversely, assume that (ii) holds and let $M \in \text{Mod-}R$. We claim that M is finitely annihilated which, since M was arbitrary, will establish the result (in view of Theorem 16((iii) \Leftrightarrow (iv))).

By the previous lemma, it suffices to show that M is \mathfrak{m} -annihilated. Select $x_0 \in M$ and set $X_0 = \{x_0\}$ and $I_0 = (0 : X_0)$. If $I_\alpha = (0 : X_\alpha)$ has been defined, set $I_{\alpha \oplus 1} = I_\alpha$ if $I_\alpha = (0 : M)$; otherwise choose $x_{\alpha \oplus 1} \in M$ such that $(0 : X_\alpha \cup \{x_{\alpha \oplus 1}\}) \subsetneq (0 : X_\alpha)$ and set $X_{\alpha \oplus 1} = X_\alpha \cup \{x_{\alpha \oplus 1}\}$ and $I_{\alpha \oplus 1} = (0 : X_{\alpha \oplus 1})$. If α is a limit ordinal, set $X_\alpha = \bigcup_{\gamma < \alpha} X_\gamma$ and $I_\alpha = (0 : X_\alpha)$. The \mathfrak{m} -DCC implies the existence of a $\gamma < \mathfrak{m}$ such that $I_\gamma = I_\alpha$ for all $\alpha > \gamma$. Thus $(0 : M) = I_\gamma = (0 : X_\gamma)$ and $|X_\gamma| < \mathfrak{m}$, by the regularity of \mathfrak{m} . \square

For the results of this section to be considered significant, one must be able to cite examples of right \mathfrak{m} -closed rings which are not right artinian, at least for $\mathfrak{m} = \aleph_1$. The following (known) result suggests right chain rings as a convenient source of examples, because it says that their right topologizing filters are very conspicuous. We include the proof for the sake of completeness.

PROPOSITION 19. [DV88, Lemma 6, p24] *Let R be a nonzero right chain ring. If I is a proper ideal of R then $\overline{\eta}(I) := \{A \leq R_R : A \supsetneq I\} \in \text{Fil-}R$. Moreover, every $\mathcal{F} \in \text{Fil-}R$ is of the form $\mathcal{F} = \eta(I) = \{A \leq R_R : A \supseteq I\}$ or $\mathcal{F} = \overline{\eta}(I)$ for some ideal I of R .*

Proof. Let I be a proper ideal of R . Obviously, if $A \in \overline{\eta}(I)$ and $A \subseteq B \leq R_R$ then $B \in \overline{\eta}(I)$. Since R is a right chain ring, $A \cap B = B$ or $A \cap B = A$ and in either case, trivially,

$A \cap B \in \bar{\eta}(I)$. Suppose $A \in \bar{\eta}(I)$ and $s \in R$. If $sR \subseteq A$ then $(A :_r s) = R \supseteq I$ so $(A :_r s) \in \bar{\eta}(I)$. If $sR \not\subseteq A$ then $sR \supseteq A$. Take $t \in A \setminus I$ and write $t = sb$ with $b \in R$. Clearly, $b \in (A :_r s)$, yet $b \notin I$ (since $t \notin I$). Therefore $(A :_r s) \supseteq I$, i.e., $(A :_r s) \in \bar{\eta}(I)$. This shows that $\bar{\eta}(I) \in \text{Fil-}R$.

If $\mathcal{F} \in \text{Fil-}R$ then $I = \bigcap \mathcal{F}$ is an ideal of R and $\mathcal{F} \subseteq \eta(I)$. Suppose $\mathcal{F} \subsetneq \eta(I)$, i.e., $I \notin \mathcal{F}$. Given any $A \in \bar{\eta}(I)$, we must have $A \supseteq I = \bigcap \mathcal{F}$, so there must exist some $K \in \mathcal{F}$ such that $A \not\subseteq K$. But then $A \supseteq K$, by hypothesis, so $A \in \mathcal{F}$. It follows that $\bar{\eta}(I) \subseteq \mathcal{F} \subsetneq \eta(I)$. This forces $\mathcal{F} = \bar{\eta}(I)$. \square

THEOREM 20. *Let \mathfrak{m} be an arbitrary regular cardinal. Then there exists a ring with identity which is right \mathfrak{m} -closed but not right \mathfrak{n} -closed for any regular cardinal $\mathfrak{n} > \mathfrak{m}$.*

Proof. Let R be a right chain domain with identity which contains a unique nontrivial ideal P , such that the chain of nonzero right ideals of R contains a dually cofinal copy of the well ordered set \mathfrak{m} , ordered in the natural way, as an ordinal. (Note that it follows from Lemma 0.1.2 and the regularity of \mathfrak{m} that the chain of nonzero right ideals of R , ordered by *reversed* set inclusion, has cofinality \mathfrak{m} .) A ring R with these properties exists, by Theorem I.5.3: take L to be the two-element chain $\{a, b\}$ with $a < b$ and take $C_{(a,b)}$ to be \mathfrak{m} . It follows that $\eta(P) = \{P, R\}$ and $\bar{\eta}(0) = \{A \leq R_R : A \neq 0\}$ are the only nontrivial right topologizing filters on R . Trivially, $\eta(P)$ is Jansian, hence all elements of $\text{Fil-}R$ other than $\bar{\eta}(0)$ are Jansian. We claim that the filter $\bar{\eta}(0)$ is \mathfrak{m} -Jansian. Indeed, if $\mathcal{G} \subseteq \bar{\eta}(0)$ with $|\mathcal{G}| < \mathfrak{m}$ then since \mathfrak{m} is the cofinality of $\langle \bar{\eta}(0); \supseteq \rangle$, the set \mathcal{G} must be bounded above in $\langle \bar{\eta}(0); \supseteq \rangle$ (i.e., bounded below in $\langle \bar{\eta}(0); \subseteq \rangle$), which means that $\bigcap \mathcal{G} \in \bar{\eta}(0)$. This shows that $\bar{\eta}(0)$ is \mathfrak{m} -Jansian.

Let $\mathcal{A} = \langle \{A_\alpha : \alpha \in \mathfrak{m}\}; \supseteq \rangle$ be a subchain of the chain $\langle \bar{\eta}(0); \supseteq \rangle$, such that \mathcal{A} is order isomorphic to $\langle \mathfrak{m}; \subseteq \rangle$ and $\{A_\alpha : \alpha \in \mathfrak{m}\}$ is cofinal in $\langle \bar{\eta}(0); \supseteq \rangle$. By the assumption about cofinality, we have $\bigcap_{\alpha \in \mathfrak{m}} A_\alpha = 0 \notin \bar{\eta}(0)$. Therefore, for any cardinal $\mathfrak{n} > \mathfrak{m}$ (even a singular cardinal \mathfrak{n}), the filter $\bar{\eta}(0)$ is not closed under intersections of fewer than \mathfrak{n} right ideals. In particular, for every regular cardinal $\mathfrak{n} > \mathfrak{m}$, $\bar{\eta}(0)$ is not \mathfrak{n} -Jansian. This shows that R is right \mathfrak{m} -closed and is not right \mathfrak{n} -closed for any regular $\mathfrak{n} > \mathfrak{m}$. \square

§5. MORITA EQUIVALENCE.

The reader will recall that if R and S are rings with identity then R and S are said to be (*right*) *Morita equivalent* if the categories of unital right R -modules and unital right S -modules are equivalent in the usual category theoretic sense. (A detailed exposition on Morita equivalence for rings with identity may be found in [AF74].)

We shall need to have a more general definition of Morita equivalence for rings without identity. This raises the question: if the rings R and S lack identity and are to be called Morita equivalent in some general sense, which subcategories of their module categories should be required by

our definition to be equivalent categories? The obvious candidates are the (respective) subcategories consisting of all unital modules (recall that a module M_R is called unital if and only if $MR = M$). A difficulty with this choice is that the subcategory of all unital modules over the ring R is not, in general, closed under subobjects and direct products. One way of circumventing this problem is to restrict further the subcategory of modules in question and, if necessary, to impose certain “possession of identity”-like conditions on the rings R and S . There are several approaches to Morita equivalence for rings without identity which follow this strategy, each of which gives rise to a slightly different notion of Morita equivalence. (See for example, [Ful74], [Gar91], [Gar], [GSa89], [GSi91], [XST] and [Abr83].) A shortcoming of some of these approaches is the lack of left-right symmetry. Indeed, unlike the classical notion of Morita equivalence, which is left-right symmetric, some of the generalized notions are asymmetric. Again, this can be remedied, but only at the cost of imposing further conditions on the rings.

We have therefore chosen to adopt our own definition of (generalized) Morita equivalence, which appears to be as general as any of the definitions in the above references; it is certainly more general than some of them. This has the effect that where we prove a ring theoretic property to be Morita invariant, we obtain a theorem that is as strong as would seem to be possible. The account we give is intended to be as self-contained as possible.

We shall call rings R and S (*right*) *Morita \ast -equivalent* if the categories $\text{Mod-}R$ and $\text{Mod-}S$ are equivalent categories, that is to say, there exist additive covariant functors $F: \text{Mod-}R \rightarrow \text{Mod-}S$ and $G: \text{Mod-}S \rightarrow \text{Mod-}R$ such that $GF \cong 1_{\text{Mod-}R}$ and $FG \cong 1_{\text{Mod-}S}$. Recall that such F and G are referred to as *inverse (additive) category equivalences*. Recall also that R^\ast denotes the Dorroh Extension of R : see Chapter 0, §4. Identifying $\text{Mod-}R$ with $\text{Mod-}R^\ast$ (unital), we observe that rings R and S are (*right*) Morita \ast -equivalent if and only if R^\ast and S^\ast are (*right*) Morita equivalent in the classical sense. (This explains our choice of the expression “Morita \ast -equivalent”.) Since the classical notion of Morita equivalence is left-right symmetric, this observation serves to show that Morita \ast -equivalence is also a left-right symmetric notion. For this reason we may omit the prefix “right” and speak of Morita \ast -equivalent rings. Furthermore, it can be shown that if the rings R and S possess identities, then R and S will be Morita \ast -equivalent if and only if they are Morita equivalent in the classical sense. Thus our definition of equivalence extends the classical one.

Suppose R and S are Morita \ast -equivalent rings whose equivalence is established by inverse additive category equivalences $F: \text{Mod-}R \rightarrow \text{Mod-}S$ and $G: \text{Mod-}S \rightarrow \text{Mod-}R$. Inasmuch as these category equivalences preserve monomorphisms, epimorphisms and direct sums [AF74, Proposition 21.2, p252 & Proposition 21.5, p255]⁴, the mapping from the set of all hereditary pretorsion classes of $\text{Mod-}R$ to the set of all hereditary pretorsion classes of $\text{Mod-}S$, defined by $\mathcal{C} \mapsto F[\mathcal{C}]$, is a

⁴Although Anderson and Fuller assume their rings to have identity, it is easy to see that Propositions 21.2, 21.4 and 21.5 of [AF74] are valid for rings without identity also. This is because a category equivalence F from $\text{Mod-}R$ to $\text{Mod-}S$ may be regarded as a functor from $\text{Mod-}R^\ast$ (unital) to $\text{Mod-}S^\ast$ (unital).

bijection. Furthermore, it is clear that the inverse of this mapping is defined by $\mathfrak{D} \mapsto G[\mathfrak{D}]$, \mathfrak{D} a hereditary pretorsion class of $\text{Mod-}S$. Notice also that if \mathfrak{C} is a hereditary pretorsion class of $\text{Mod-}R$ then F restricts to an additive category equivalence from the full subcategory of $\text{Mod-}R$ on \mathfrak{C} to the full subcategory of $\text{Mod-}S$ on $F[\mathfrak{C}]$.

An additive category equivalence $F: \text{Mod-}R \rightarrow \text{Mod-}S$ induces a map $\widehat{F}: \text{torsp-}R \rightarrow \text{torsp-}S$ defined as follows. Let $\sigma \in \text{torsp-}R$. Then T_σ is a hereditary pretorsion class on $\text{Mod-}R$, by Proposition 1.3. As noted above, $F[T_\sigma]$ is a hereditary pretorsion class on $\text{Mod-}S$ which, by Theorem 1.4, corresponds with the hereditary pretorsion class of a unique torsion preradical on $\text{Mod-}S$, which we shall denote by $\widehat{F}(\sigma)$. Thus $F[T_\sigma] = T_{\widehat{F}(\sigma)}$ for all $\sigma \in \text{torsp-}R$. We also point out that since F preserves monomorphisms [AF74, Proposition 21.2, p252], if $\tau \in \text{torsp-}R$, $M \in \text{Mod-}R$ and $\iota: \tau(M) \rightarrow M$ denotes the natural embedding, then $F(\tau(M))$ can be identified, via the monomorphism $F(\iota)$, with a submodule of $F(M)$. This interpretation is implicit in the following lemma.

LEMMA 1. *Let R and S be Morita $*$ -equivalent rings and let $F: \text{Mod-}R \rightarrow \text{Mod-}S$ and $G: \text{Mod-}S \rightarrow \text{Mod-}R$ be inverse additive category equivalences. If $\tau \in \text{torsp-}R$ then $F(\tau(M)) = \widehat{F}(\tau)(F(M))$ for all $M \in \text{Mod-}R$.*

Proof. Inasmuch as $\widehat{F}(\tau)(F(M))$ is the unique maximal submodule of $F(M)$ contained in $T_{\widehat{F}(\tau)}$, it suffices to show that, given any $N \in T_{\widehat{F}(\tau)}$ and any monomorphism $\alpha: N \rightarrow F(M)$, there exists a monomorphism $\beta: N \rightarrow F(\tau(M))$ which makes the following diagram commute.

$$\begin{array}{ccc}
 & F(M) & \\
 \alpha \nearrow & & \nwarrow F(\iota) \\
 N & \xrightarrow{\beta} & F(\tau(M))
 \end{array}$$

Since $FG \cong 1_{\text{Mod-}S}$, there exists an isomorphism $\zeta_N: FG(N) \rightarrow N$. By [AF74, Proposition 21.2, p252], the restriction of F to $\text{Hom}_R(G(N), M)$ defines a bijection from $\text{Hom}_R(G(N), M)$ to $\text{Hom}_S(FG(N), F(M))$ with the property that for every $\rho \in \text{Hom}_R(G(N), M)$, $F(\rho)$ is a monomorphism if and only if ρ is a monomorphism. Consequently, there must exist a monomorphism $\rho \in \text{Hom}_R(G(N), M)$ such that $\alpha \zeta_N = F(\rho)$. Now consider the following diagram in $\text{Mod-}R$.

$$\begin{array}{ccc}
 & M & \\
 \rho \nearrow & & \nwarrow \iota \\
 G(N) & \xrightarrow{\gamma} & \tau(M)
 \end{array}$$

Since $FG(N) \cong N \in T_{\widehat{F}(\tau)} = F[T_\tau]$, it follows that $G(N) \in T_\tau$ and so there must exist a mono-

morphism $\gamma: G(N) \rightarrow \tau(M)$ which makes the above diagram commute. We therefore obtain the following commutative diagram in $\text{Mod-}S$.

$$\begin{array}{ccc}
 N & \xrightarrow{\alpha} & F(M) \\
 \uparrow \zeta_N \text{ (iso)} & \nearrow F(\rho) & \uparrow F(\iota) \\
 FG(N) & \xrightarrow{F(\gamma)} & F(\tau(M))
 \end{array}$$

Clearly then, $\beta = F(\gamma)\zeta_N^{-1}: N \rightarrow F(\tau(M))$ is the required monomorphism. Thus $F(\tau(M)) = \widehat{F}(\tau)(F(M))$. \square

THEOREM 2. *Let R and S be Morita $*$ -equivalent rings and let $F: \text{Mod-}R \rightarrow \text{Mod-}S$ and $G: \text{Mod-}S \rightarrow \text{Mod-}R$ be inverse additive category equivalences. Then the maps $\widehat{F}: \text{torsp-}R \rightarrow \text{torsp-}S$ and $\widehat{G}: \text{torsp-}S \rightarrow \text{torsp-}R$ are mutually inverse lattice and monoid isomorphisms.*

Proof. It was pointed out in the remarks preceding Lemma 1 that \widehat{F} and \widehat{G} are mutually inverse bijections. It is also obvious that \widehat{F} and \widehat{G} are order preserving. It remains to show that \widehat{F} and \widehat{G} are semigroup homomorphisms. Let $\sigma, \tau \in \text{torsp-}R$ and suppose that $N \in T_{\widehat{F}(\sigma \cdot \tau)} = F[T_{\sigma \cdot \tau}]$. Then $N = F(M)$ for some $M \in T_{\sigma \cdot \tau}$. Consider the exact sequence

$$0 \longrightarrow \tau(M) \xrightarrow{\iota} M \xrightarrow{\pi} M/\tau(M) \longrightarrow 0$$

in $\text{Mod-}R$. This induces an exact sequence

$$0 \longrightarrow F(\tau(M)) \xrightarrow{F(\iota)} F(M) \xrightarrow{F(\pi)} F(M/\tau(M)) \longrightarrow 0$$

in $\text{Mod-}S$ (see [AF74, Proposition 21.4, p254]). Since $M \in T_{\sigma \cdot \tau}$, we have $M/\tau(M) \in T_{\sigma}$, and so

$$F(M)/F(\tau(M)) \cong F(M/\tau(M)) \in F[T_{\sigma}] = T_{\widehat{F}(\sigma)}.$$

By the previous lemma, $F(\tau(M)) = \widehat{F}(\tau)(F(M))$. Hence $F(M)/\widehat{F}(\tau)(F(M)) \in T_{\widehat{F}(\sigma)}$, from which it follows that $N = F(M) \in T_{\widehat{F}(\sigma) \cdot \widehat{F}(\tau)}$. Thus $T_{\widehat{F}(\sigma \cdot \tau)} \subseteq T_{\widehat{F}(\sigma) \cdot \widehat{F}(\tau)}$, i.e., $\widehat{F}(\sigma \cdot \tau) \leq \widehat{F}(\sigma) \cdot \widehat{F}(\tau)$. By symmetry, we have that

$$\sigma \cdot \tau = \widehat{G}\widehat{F}(\sigma \cdot \tau) \leq \widehat{G}(\widehat{F}(\sigma) \cdot \widehat{F}(\tau)) \leq \widehat{G}\widehat{F}(\sigma) \cdot \widehat{G}\widehat{F}(\tau) = \sigma \cdot \tau,$$

consequently, $\sigma \cdot \tau = \widehat{G}(\widehat{F}(\sigma) \cdot \widehat{F}(\tau))$. But then

$$\widehat{F}(\sigma \cdot \tau) = \widehat{F}\widehat{G}(\widehat{F}(\sigma) \cdot \widehat{F}(\tau)) = \widehat{F}(\sigma) \cdot \widehat{F}(\tau).$$

Thus $\widehat{F}(\sigma \cdot \tau) = \widehat{F}(\sigma) \cdot \widehat{F}(\tau)$, as required. A similar argument shows that \widehat{G} is also a semigroup homomorphism. \square

In the next result we demonstrate that the map $\widehat{F}: \text{torsp-}R \rightarrow \text{torsp-}S$ preserves, in addition,

the \mathfrak{m} -Jansian property.

PROPOSITION 3. *Let R and S be Morita $*$ -equivalent rings with $F: \text{Mod-}R \rightarrow \text{Mod-}S$ an additive category equivalence. Let \mathfrak{m} be a regular cardinal. Then $\sigma \in \text{torsp-}R$ is \mathfrak{m} -Jansian if and only if $\widehat{F}(\sigma) \in \text{torsp-}S$ is \mathfrak{m} -Jansian.*

Proof. By symmetry, it suffices to show that $\widehat{F}(\sigma) \in \text{torsp-}S$ is \mathfrak{m} -Jansian whenever $\sigma \in \text{torsp-}R$ is \mathfrak{m} -Jansian. Suppose that $\sigma \in \text{torsp-}R$ is \mathfrak{m} -Jansian. Let $\{N_i: i \in \Gamma\}$ be a subfamily of $T_{\widehat{F}(\sigma)} = F[T_\sigma]$ with $|\Gamma| < \mathfrak{m}$. Write each N_i as $F(M_i)$, $M_i \in T_\sigma$. By hypothesis, σ is \mathfrak{m} -Jansian, so $\prod_{i \in \Gamma} M_i \in T_\sigma$. Since F preserves direct products [AF74, Proposition 21.5, p255], it follows that $\prod_{i \in \Gamma} N_i = \prod_{i \in \Gamma} F(M_i) \cong F(\prod_{i \in \Gamma} M_i) \in F[T_\sigma] = T_{\widehat{F}(\sigma)}$. This shows that $\widehat{F}(\sigma)$ is \mathfrak{m} -Jansian, as required. \square

Let R be an arbitrary ring and consider its Dorroh Extension R^* . Working in $\text{Mod-}R^*$, we deduce from Proposition 3.5 that the map $\text{torsp} \circ \eta: \text{Id } R^* \rightarrow \{\tau \in \text{Jans-}R^*: \tau \geq \tau_{\text{zero}}\}^{\text{du}}$ is an isomorphism of lattice ordered monoids. Inasmuch as $[\tau_{\text{zero}}, 1_{\text{Mod-}R^*}] \subseteq \text{torsp-}R^*$ can be identified with $\text{torsp-}R$ (see remarks at end of §1), we may identify $\{\tau \in \text{Jans-}R^*: \tau \geq \tau_{\text{zero}}\}^{\text{du}}$ with $[\text{Jans-}R]^{\text{du}}$. Under this identification, $\text{torsp} \circ \eta$ constitutes an isomorphism of lattice ordered monoids from $\text{Id } R^*$ to $[\text{Jans-}R]^{\text{du}}$. (The action of $\text{torsp} \circ \eta$ can be described explicitly: if I is an ideal of R^* and $M \in \text{Mod-}R$ then $((\text{torsp} \circ \eta)(I))(M) = \{x \in M_{R^*}: xI = 0\}$.) Suppose now that R and S are Morita $*$ -equivalent rings and that $F: \text{Mod-}R \rightarrow \text{Mod-}S$ and $G: \text{Mod-}S \rightarrow \text{Mod-}R$ are inverse additive category equivalences. It follows from Theorem 2 and Proposition 3 that $\widehat{F}: \text{torsp-}R \rightarrow \text{torsp-}S$ restricts to an isomorphism of lattice ordered monoids from $\text{Jans-}R$ to $\text{Jans-}S$. The next result is self-evident if one considers the following diagram.

$$\begin{array}{ccc}
 \text{Id } R^* & \xrightarrow[\text{(iso)}]{\text{torsp} \circ \eta} & [\text{Jans-}R]^{\text{du}} \\
 \text{(iso)} \downarrow & & \downarrow \widehat{F} \text{(iso)} \\
 \text{Id } S^* & \xrightarrow[\text{(iso)}]{\text{torsp} \circ \eta} & [\text{Jans-}S]^{\text{du}}
 \end{array}$$

PROPOSITION 4. *If R and S are Morita $*$ -equivalent rings then $\text{Id } R^*$ and $\text{Id } S^*$ are isomorphic as lattice ordered monoids.* \square

Our main objective is to prove Theorem 10, which asserts that if \mathfrak{m} is an arbitrary regular cardinal then the ring theoretic property of being right \mathfrak{m} -closed is *Morita $*$ -invariant*, by which we mean that the class of all right \mathfrak{m} -closed rings is closed under Morita $*$ -equivalence. Proposition 4 above is a key result in our approach to this goal. In using Proposition 4, however, we shall encounter

the problem of how to distinguish R and S from other ideals in their respective Dorroh Extensions, in purely lattice and semigroup theoretic terms. This difficulty will be overcome only in Lemma 9 and requires several preparatory lemmas.

We call two ideals A and B of a ring *incomparable* if $A \not\subseteq B$ and $B \not\subseteq A$.

LEMMA 5. *If a ring R satisfies the descending chain condition on ideals then any set of pairwise incomparable prime ideals of R is finite.*

Proof. Suppose, on the contrary, that R is a ring with the DCC on ideals which contains an infinite sequence of pairwise incomparable prime ideals P_1, P_2, P_3, \dots . Consider the descending chain of ideals

$$P_1 \supseteq P_1 \cap P_2 \supseteq P_1 \cap P_2 \cap P_3 \supseteq \dots$$

Clearly the containment $P_1 \supseteq P_1 \cap P_2$ must be strict, since $P_1 \not\subseteq P_2$. If $P_1 \cap P_2 = P_1 \cap P_2 \cap P_3$ then $P_3 \supseteq P_1 \cap P_2 \supseteq P_1 P_2$, which implies $P_3 \supseteq P_1$ or $P_3 \supseteq P_2$ since P_3 is prime. This again contradicts our assumption that the P_i are pairwise incomparable, so we must have $P_1 \cap P_2 \not\supseteq P_1 \cap P_2 \cap P_3$. A routine inductive argument can be used to show that $P_1 \cap P_2 \cap \dots \cap P_n \not\supseteq P_1 \cap P_2 \cap \dots \cap P_{n+1}$ for all positive integers n , violating the DCC on ideals. \square

The requirement of pairwise incomparability may not be dropped from the above lemma. For example, it follows from Proposition 0.9.3, that if V is any left vector space of dimension \aleph_ω over a division ring D , then the ideal lattice of the left full linear ring $R = \text{End}_D V$, is an infinite chain isomorphic to the ordinal $\omega \oplus 1$. Therefore R enjoys the DCC on ideals. Since R is also (von Neumann) regular (Theorem 0.9.4), every right ideal of R is idempotent by Theorem 0.9.2, from which it follows easily that every proper ideal of R is prime.

We remind the reader that whereas every maximal proper ideal in a ring with identity is a prime ideal, a ring without identity may possess non-prime maximal proper ideals.

LEMMA 6. *If M is a maximal proper ideal of a ring R but M is not a prime ideal of R then $R^2 \subseteq M$.*

Proof. Since M is not a prime ideal of R , there exist ideals A and B of R such that $AB \subseteq M$ but $A \not\subseteq M$ and $B \not\subseteq M$. It follows that $(A + M)(B + M) \subseteq M$. By hypothesis, since $A, B \not\subseteq M$, we must have $A + M = B + M = R$. Thus $R^2 \subseteq M$. \square

LEMMA 7. *Let G be an abelian group satisfying the descending chain condition on subgroups. Then G has only finitely many maximal proper subgroups.*

Proof. Let \mathcal{M} denote the family of all finite intersections of maximal proper subgroups of G . Inasmuch as G satisfies the DCC on subgroups, \mathcal{M} has a minimal member, say $H =$

$M_1 \cap M_2 \cap \dots \cap M_n$, with each M_i a maximal proper subgroup of G . Every maximal proper subgroup of G must contain H (by minimality of H). Consider the factor group G/H . Notice that G/H embeds canonically in a finite direct sum of simple abelian groups, namely $\bigoplus_{i=1}^n G/M_i$. Consequently, G/H is finite, so G/H has only finitely many maximal proper subgroups. Since every maximal proper subgroup of G contains H , it follows that G has only finitely many maximal proper subgroups. \square

Lemma 7 is really just a special case (when $R^2 = 0$) of the following result.

PROPOSITION 8. *A right \aleph_1 -closed ring contains only finitely many maximal proper ideals.*

Proof. Suppose, on the contrary, that R is a right \aleph_1 -closed ring containing infinitely many maximal proper ideals. Since R satisfies the DCC on ideals (Corollary 4.2), it follows from Lemma 5 that R has only finitely many maximal proper ideals which are prime. Consequently, the set of non-prime maximal proper ideals of R , say $\{M_i : i \in \Gamma\}$, is infinite. Because R satisfies the DCC on ideals, the set of all finite intersections of non-prime maximal proper ideals has a smallest member, say $A = \bigcap_{i \in \Gamma'} M_i$, with Γ' a finite subset of Γ . By Lemma 6, $R^2 \subseteq A$. Now consider the ring $\bar{R} = R/A$. By Proposition 4.11, \bar{R} is right \aleph_1 -closed. Moreover, $\{M_i/A : i \in \Gamma\}$ is an infinite set of maximal proper ideals of \bar{R} . Notice also that $\bar{R}^2 = 0$ because $A \supseteq R^2$. It follows that the subgroups of $\langle \bar{R}; +, -, 0 \rangle$ are precisely the ideals of \bar{R} . Consequently, the abelian group $\langle \bar{R}; +, -, 0 \rangle$ satisfies the DCC on subgroups, and so, by Lemma 7, has only finitely many maximal proper subgroups. But this implies that \bar{R} has only finitely many maximal proper ideals, a contradiction. \square

LEMMA 9. *Let R and S be rings and suppose that $\phi : \text{Id } R^* \rightarrow \text{Id } S^*$ is a lattice isomorphism. If S contains only finitely many maximal proper ideals then $\phi(R) = S$ and ϕ restricts to an isomorphism from $\text{Id } R$ onto $\text{Id } S$.*

Proof. Let \mathfrak{P} denote the set of positive prime integers. For each $p \in \mathfrak{P}$, set $I_p = \{1_{R^*} \cdot r + 1_{R^*} \cdot m : r \in R, m \in p\mathbb{Z}\} \subseteq R^*$. Observe that I_p is a maximal proper ideal of R^* , from which it follows that $\phi(I_p)$ is a maximal proper ideal of S^* . Let $\mathfrak{P}' = \{p \in \mathfrak{P} : \phi(I_p) \supseteq S\}$. We claim that \mathfrak{P}' is infinite. For suppose \mathfrak{P}' is finite. Then $|\mathfrak{P} \setminus \mathfrak{P}'| = \aleph_0$. Take $p, q \in \mathfrak{P} \setminus \mathfrak{P}'$ with $p \neq q$, and suppose that $\phi(I_p) \cap S = \phi(I_q) \cap S$. Then $S_S / (\phi(I_p) \cap S) \neq 0$ (because $\phi(I_p) \not\supseteq S$) and

$$S_S / (\phi(I_p) \cap S) \cong (S_S + \phi(I_p)) / \phi(I_p) = S_S^* / \phi(I_p),$$

because of the maximality of $\phi(I_p)$. Similarly,

$$S_S^* / \phi(I_q) \cong S_S / (\phi(I_q) \cap S) = S_S / (\phi(I_p) \cap S) \cong S_S^* / \phi(I_p).$$

Since $S_S^* / \phi(I_q)$ and $S_S^* / \phi(I_p)$ are isomorphic right S -modules, we must have that

$$\phi(I_q) = (0 :_r S_S^* / \phi(I_q)) = (0 :_r S_S^* / \phi(I_p)) = \phi(I_p),$$

so $I_q = I_p$, a contradiction. We conclude that $|\{\phi(I_p) \cap S : p \in \mathcal{P} \setminus \mathcal{P}'\}| = |\mathcal{P} \setminus \mathcal{P}'| = \aleph_0$. Since each $\phi(I_p) \cap S$ is a maximal proper ideal of S , this contradicts our hypothesis. Therefore \mathcal{P}' is infinite, as claimed. Inasmuch as any infinite family of prime ideals of the ring \mathbb{Z} has trivial intersection, it follows that $R = \bigcap_{p \in \mathcal{P}'} I_p$, so $\phi(R) = \phi(\bigcap_{p \in \mathcal{P}'} I_p) = \bigcap_{p \in \mathcal{P}'} \phi(I_p) \supseteq S$. Since the intervals $[R, R^*]$ and $[S, S^*]$ in $\text{Id } R^*$ and $\text{Id } S^*$ (respectively) are both isomorphic to $\text{Id } \mathbb{Z}$, we clearly cannot have $\phi(R) \supsetneq S$, so $\phi(R) = S$. It follows that ϕ restricts to a lattice isomorphism from $[0, R]$ onto $[0, S]$. \square

THEOREM 10. *Let R and S be Morita $*$ -equivalent rings and let \mathfrak{m} be a regular cardinal. If S is right \mathfrak{m} -closed then so is R . In other words, the property of being right \mathfrak{m} -closed is Morita $*$ -invariant.*

Proof. Let $F: \text{Mod-}R \rightarrow \text{Mod-}S$ be an additive category equivalence. We know from Theorem 2 that $\widehat{F}: \text{torsp-}R \rightarrow \text{torsp-}S$ is a lattice and monoid isomorphism. Consider the following diagram.

$$\begin{array}{ccc} \text{Id } R^* & \xrightarrow{\text{torsp} \circ \eta} & [\text{torsp-}R]^{\text{du}} \\ \text{(iso)} \downarrow & & \downarrow \text{(iso)} \widehat{F} \\ \text{Id } S^* & \xrightarrow{\text{torsp} \circ \eta} & [\text{torsp-}S]^{\text{du}} \end{array}$$

We also know that the isomorphism $\widehat{F}: \text{torsp-}R \rightarrow \text{torsp-}S$ induces an isomorphism from $\text{Id } R^*$ to $\text{Id } S^*$ which makes the above diagram commute (see the remarks preceding Proposition 4). By the previous lemma, this isomorphism sends $R \in \text{Id } R^*$ to $S \in \text{Id } S^*$. It follows that

$$\widehat{F}(\tau_{\text{zero}}) = \widehat{F}((\text{torsp} \circ \eta)(R)) = (\text{torsp} \circ \eta)(S) = \tau_{\text{zero}} \in \text{torsp-}S.$$

Thus \widehat{F} restricts to a map from $[\tau_{\text{zero}}, 1_{\text{Mod-}R}]$ onto $[\tau_{\text{zero}}, 1_{\text{Mod-}S}]$. Since S is right \mathfrak{m} -closed, we may conclude, using Proposition 3, that every $\sigma \in [\tau_{\text{zero}}, 1_{\text{Mod-}R}]$ is \mathfrak{m} -Jansian. Thus R is right \mathfrak{m} -closed. \square

We remarked earlier that if R and S are rings with identity then R and S will be Morita $*$ -equivalent if and only if they are equivalent in the classical Morita sense. The next result is therefore an immediate consequence of Theorem 10.

COROLLARY 11. *Let R and S be rings with identity which are Morita equivalent and let \mathfrak{m} be a regular cardinal. If S is right \mathfrak{m} -closed then so is R .* \square

It is also possible to prove Corollary 11 using the following more direct argument. Any given additive category equivalence from $\text{Mod-}R(\text{unital})$ to $\text{Mod-}S(\text{unital})$ can be extended, in a natural way, to an additive category equivalence $F: \text{Mod-}R \rightarrow \text{Mod-}S$. Since $F[\text{Mod-}R(\text{unital})] = \text{Mod-}S(\text{unital})$, it is not difficult to show that $F[\text{Mod-}R(\text{zero})] = \text{Mod-}S(\text{zero})$. Consequently, the induced isomorphism $\widehat{F}: \text{torsp-}R \rightarrow \text{torsp-}S$ sends $\tau_{\text{zero}} \in \text{torsp-}R$ to $\tau_{\text{zero}} \in \text{torsp-}S$, so \widehat{F} restricts to an isomorphism from $[\tau_{\text{zero}}, 1_{\text{Mod-}R}]$ onto $[\tau_{\text{zero}}, 1_{\text{Mod-}S}]$. For any regular cardinal \mathfrak{m} , therefore, it follows that R is right \mathfrak{m} -closed if and only if S is right \mathfrak{m} -closed.

§6. \mathfrak{m} -JANSKIAN PRERADICALS ON PRIME RINGS.

The principal notion to be introduced and studied in this section is that of a ring which is “right prime of bound \mathfrak{m} ”, for a fixed cardinal \mathfrak{m} . In fact, most of the remainder of this dissertation is devoted to an investigation of rings with this property. This notion of primeness relative to cardinal bounds induces a partition of the class of all prime rings into subclasses corresponding to the positive cardinals. The resulting classification of prime rings (first proposed in [Raf87]) extends the classification of strongly prime rings found in [GH75], [GHL74] and [HL75].

The notion of a ring which is “right prime of bound \mathfrak{m} ” owes its definition to Theorem 3, which is the main result of this section. In it we establish a number of torsion theoretic characterizations of rings which are right prime of bound \mathfrak{m} while in Theorem 6 we characterize the same rings in categorical terms.

Let $M \in \text{Mod-}R$. Recall that if \mathfrak{m} is an arbitrary cardinal then $\prod_{\Gamma}^{(\mathfrak{m})} M$ denotes the submodule of M^{Γ} consisting of all $x \in M^{\Gamma}$ for which $|\text{supp } x| < \max\{\mathfrak{m}, \aleph_0\}$. Recall also that M is said to be *faithful* if $(0:M) = 0$. If \mathfrak{m} is an arbitrary nonzero cardinal we shall call M *\mathfrak{m} -faithful* if there is a subset X of M with $|X| < \mathfrak{m} + 1$, such that $(0:X) = 0$; we call an ideal I of R *right \mathfrak{m} -faithful* if the module I_R is \mathfrak{m} -faithful. Clearly, if M is faithful then M is \mathfrak{m} -faithful for some $\mathfrak{m} \leq |M|^+$, and M is \mathfrak{m} -faithful if and only if M is faithful and \mathfrak{m} -annihilated. An \aleph_0 -faithful module is often called *cofaithful* in the literature.

PROPOSITION 1. *Let \mathfrak{m} be an arbitrary nonzero cardinal. Then the following conditions are equivalent for a right R -module M :*

- (i) M is \mathfrak{m} -faithful;
- (ii) $\prod_{\mathfrak{m}}^{(\mathfrak{m})} M$ contains an element with zero right annihilator;
- (iii) R_R^* embeds into $(\prod_{\mathfrak{m}}^{(\mathfrak{m})} M) \oplus \mathbf{Z}_{\text{zero}}$.

Moreover, if R possesses an element e such that $(0;_r e) = 0$ (e.g., if R has an identity), then the above conditions are equivalent also to:

- (iv) R_R embeds into $\prod_{\mathfrak{m}}^{(\mathfrak{m})} M$.

Proof. (i) \Rightarrow (ii): Let $X \subseteq M$ with $|X| < \mathfrak{m} + 1$ and $(0:X) = 0$. Write $X = \{x_i : i \in \Gamma\}$

with $|\Gamma| < \mathfrak{m} + 1$. Define $x \in M^\Gamma$ by $x = \{x_i\}_{i \in \Gamma}$. Then $(0 : x) = \bigcap_{i \in \Gamma} (0 : x_i) = (0 : X) = 0$. Clearly, since $|\Gamma| < \mathfrak{m} + 1$, M^Γ embeds into $\prod_{\mathfrak{m}}^{(\mathfrak{m})} M$, so (ii) is satisfied.

(ii) \Rightarrow (iii): Suppose $x \in \prod_{\mathfrak{m}}^{(\mathfrak{m})} M$ has zero annihilator in R . Define a map

$$\varphi : R_R^* \rightarrow (\prod_{\mathfrak{m}}^{(\mathfrak{m})} M) \oplus \mathbb{Z}_{\text{zero}}$$

by $\varphi(1_{R^*} \cdot r + 1_{R^*} \cdot m) = (xr + xm, m)$ for $r \in R$, $m \in \mathbb{Z}$. It is straightforward to check that φ is an R -module homomorphism. Moreover,

$$\begin{aligned} \varphi(1_{R^*} \cdot r + 1_{R^*} \cdot m) = 0 &\Rightarrow (xr + xm, m) = 0 \\ &\Rightarrow m = 0 \text{ and } xr = 0 \\ &\Rightarrow m = 0 \text{ and } x = 0 \text{ (because } (0 : x) = \emptyset) \\ &\Rightarrow 1_{R^*} \cdot r + 1_{R^*} \cdot m = 0. \end{aligned}$$

Thus φ is a monomorphism, proving (iii).

Note that the restriction of the map φ to R_R is also an R -monomorphism from R_R into $\prod_{\mathfrak{m}}^{(\mathfrak{m})} M$ (defined by $\varphi(r) = xr$ for all $r \in R$), so we have also proved that (ii) \Rightarrow (iv) in general. Conversely, it is obvious that if R possesses an element e such that $(0 : e) = 0$, then (iv) \Rightarrow (ii).

(iii) \Rightarrow (i): Let $\varphi : R_R^* \rightarrow (\prod_{\mathfrak{m}}^{(\mathfrak{m})} M) \oplus \mathbb{Z}_{\text{zero}}$ be an R -module monomorphism. Set $\varphi(1_{R^*}) = (x, m)$ with $x = \{x_\alpha\}_{\alpha \in \mathfrak{m}} \in \prod_{\mathfrak{m}}^{(\mathfrak{m})} M$ and $m \in \mathbb{Z}_{\text{zero}}$. Put $X = \{x_\alpha : \alpha \in \text{supp } x\}$. Clearly, $|X| < \mathfrak{m} + 1$. Note that

$$s \in (0 : X) \Rightarrow xs = 0 \Rightarrow (x, m)s = \varphi(1_{R^*})s = \varphi(s) = 0 \Rightarrow s = 0.$$

Thus $(0 : X) = 0$, so (i) is satisfied. □

If \mathfrak{m} is a regular cardinal, it turns out that the \mathfrak{m} -Jansian torsion preradicals can be used to characterize \mathfrak{m} -faithful modules.

PROPOSITION 2. *Let \mathfrak{m} be a regular cardinal. Then the following conditions are equivalent for a right R -module M :*

- (i) M is \mathfrak{m} -faithful;
- (ii) if $\mathfrak{F} = \{A \leq R_R : A \supseteq (0 : X) \text{ for some subset } X \text{ of } M \text{ with } |X| < \mathfrak{m}\}$, then $\text{torsp } \mathfrak{F} = 1_{\text{Mod-}R}$;
- (iii) the \mathfrak{m} -Jansian torsion preradical on $\text{Mod-}R$ generated by $M \oplus \mathbb{Z}_{\text{zero}}$ is $1_{\text{Mod-}R}$;
- (iv) $\sigma(M) \subsetneq M$ for all $\sigma \in \{\tau \in \mathfrak{m}\text{-Jans-}R : \tau_{\text{zero}} \leq \tau < 1_{\text{Mod-}R}\}$.

Proof. (i) \Rightarrow (ii) is trivial while (ii) \Rightarrow (iii) is an immediate consequence of Theorem 3.10.

(iii) \Rightarrow (iv): Let $\sigma \in \{\tau \in \mathfrak{m}\text{-Jans-}R : \tau_{\text{zero}} \leq \tau < 1_{\text{Mod-}R}\}$ and suppose, contrary to (iv), that $\sigma(M) = M$. We know from Theorem 3.10 that if \mathfrak{F} is chosen as in (ii) above, then $\text{torsp } \mathfrak{F}$ is the \mathfrak{m} -Jansian torsion preradical on $\text{Mod-}R$ generated by $M \oplus \mathbb{Z}_{\text{zero}}$. Since $M, \mathbb{Z}_{\text{zero}} \in T_\sigma$, it follows that $\text{torsp } \mathfrak{F} \leq \sigma$, so $\sigma = 1_{\text{Mod-}R}$, a contradiction.

(iv) \Rightarrow (i): If \mathcal{F} is chosen as in (ii) above and $\sigma = \text{torsp } \mathcal{F}$ then certainly, $M \oplus \mathbb{Z}_{\text{zero}} \in T_\sigma$. By hypothesis, we must therefore have $\sigma = 1_{\text{Mod-}R}$, so $0 \in \mathcal{F}$, i.e., $(0: X) = 0$ for some subset X of M with $|X| < \mathfrak{m}$. Thus (i) holds. \square

THEOREM 3. *Let \mathfrak{m} be an arbitrary nonzero cardinal. Then the following assertions are equivalent for a ring R :*

- (i) every nonzero right ideal of R is \mathfrak{m} -faithful;
- (ii) for every nonzero $a \in R$, there exists a subset X of R such that $|X| < \mathfrak{m} + 1$ and $(0: {}_r aX) = 0$.

If $\mathfrak{m} \geq \aleph_0$, then the above two conditions are each equivalent to:

- (iii) every nonzero ideal of R is right \mathfrak{m} -faithful.

Furthermore, if \mathfrak{m} is regular then the above three conditions are all equivalent to each of the following assertions:

- (iv) $(\text{torsp } \mathcal{F})(R_R) = 0$ for all proper \mathfrak{m} -Jansian right topologizing filters \mathcal{F} on R ;
- (v) $\sigma(R_R) = 0$ for all proper \mathfrak{m} -Jansian torsion preradicals $\sigma \geq \tau_{\text{zero}}$ on $\text{Mod-}R$;
- (vi) $\sigma(E(R_R)) = 0$ or $\sigma(E) = E$ for all injective right R -modules E , whenever σ is an \mathfrak{m} -Jansian preradical on $\text{Mod-}R$ such that $\sigma \geq \tau_{\text{zero}}$;
- (vii) $\sigma(E(R_R)) = 0$ for all proper \mathfrak{m} -Jansian torsion preradicals $\sigma \geq \tau_{\text{zero}}$ on $\text{Mod-}R$.

Proof. (i) \Rightarrow (ii): Let $0 \neq a \in R$ and consider the right ideal $aR^* = \{ar + am : r \in R, m \in \mathbb{Z}\}$ of R . Notice that we cannot have $aR = 0$, otherwise $(aR^*)R = 0$, so aR^* is not faithful, a contradiction. Since aR is a nonzero right ideal of R , there must exist a subset X of R such that $|X| < \mathfrak{m} + 1$ and $(0: {}_r aX) = 0$. Thus (ii) holds.

(ii) \Rightarrow (i): Let A be a nonzero right ideal of R . Certainly, $A \supseteq aR \neq 0$ for some nonzero $a \in A$. It follows from (ii) that aR is \mathfrak{m} -faithful. Since $A \supseteq aR$, we conclude that A is \mathfrak{m} -faithful.

Now suppose that $\mathfrak{m} \geq \aleph_0$. (i) \Rightarrow (iii) is trivial.

(iii) \Rightarrow (ii): Let $0 \neq a \in R$ and consider the ideal

$$R^*aR^* = \left\{ \sum_{i=1}^n r_i a s_i + (ra + as + am) : r_i, s_i, r, s \in R, m \in \mathbb{Z} \right\}$$

of R . Since R^*aR^* is faithful, we cannot have $aR = 0$. Let $I = aR + Ra \neq 0$ and consider the ideal RaR of R . If $RaR \neq 0$, then by (iii), there is a subset X of RaR , with $|X| < \mathfrak{m}$, such that $(0: {}_r X) = 0$. Write

$$X = \left\{ \sum_{j=1}^{n(i)} r_{ij} a s_{ij} : i \in \Gamma \right\} \text{ where } |\Gamma| < \mathfrak{m}.$$

If we set $Y = \{s_{ij} : j = 1, 2, \dots, n(i); i \in \Gamma\}$ then it is clear that $(0: {}_r aY) \subseteq (0: {}_r X) = 0$ and $|Y| \leq |\Gamma| \cdot \sup \{n(i) : i \in \Gamma\} < \mathfrak{m}$.

Now suppose that $RaR = 0$. Since $I \neq 0$, it follows from (iii) that there exists a subset Z of

I such that $(0:_{\mathfrak{r}}Z) = 0$ and $|Z| < \mathfrak{m}$. Write $Z = \{ar_i + s_i a : i \in \Delta\}$ with $|\Delta| < \mathfrak{m}$. Set $W = \{r_i : i \in \Delta\}$. If $t \in (0:_{\mathfrak{r}}aW)$ then $(ar_i + s_i a)t = 0$ for all $i \in \Delta$, i.e., $t \in (0:_{\mathfrak{r}}Z) = 0$. Thus $(0:_{\mathfrak{r}}aW) = 0$ and $|W| \leq |\Delta| < \mathfrak{m}$.

Now suppose that \mathfrak{m} is regular. The equivalence of (iv) and (v) is an easy consequence of Theorems 2.4 and 3.10.

Inasmuch as R_R is an essential submodule of $E(R_R)$, it follows that for any $\sigma \in \text{torsp-}R$, we have $\sigma(R_R) = 0$ if and only if $\sigma(E(R_R)) = 0$. The equivalence of (v) and (vii) is an immediate consequence of this observation.

(i) \Rightarrow (v): Let $\sigma \in \{\tau \in \mathfrak{m}\text{-Jans-}R : \tau_{\text{zero}} \leq \tau < 1_{\text{Mod-}R}\}$ and consider $\sigma(R_R) \leq R_R$. Suppose $\sigma(R_R) \neq 0$. Since, by hypothesis, $\sigma(R_R)$ is \mathfrak{m} -faithful, it follows from Proposition 2 ((i) \Leftrightarrow (iv)) that $\sigma(R_R) = \sigma(\sigma(R_R)) \subsetneq \sigma(R_R)$, an impossibility. Thus $\sigma(R_R) = 0$, as required.

(v) \Rightarrow (vi): Let σ be an \mathfrak{m} -Jansian preradical on $\text{Mod-}R$ with $\sigma \geq \tau_{\text{zero}}$ and let $\hat{\sigma}$ denote the smallest torsion preradical on $\text{Mod-}R$ such that $\hat{\sigma} \geq \sigma$. An argument similar to that used in Lemma 3.2 shows that $\hat{\sigma}$ is \mathfrak{m} -Jansian. Clearly, $\hat{\sigma} \geq \tau_{\text{zero}}$. If $\hat{\sigma} = 1_{\text{Mod-}R}$ and M is an arbitrary injective right R -module, then

$$\begin{aligned} M &= \hat{\sigma}(M) = M \cap \sigma(E(M)) \quad (\text{by Proposition 1.5}) \\ &= M \cap \sigma(M) \quad (\text{because } E(M) = M) \\ &= \sigma(M). \end{aligned}$$

Otherwise $\hat{\sigma}(R_R) = 0$, by (v), and since R_R is essential in $E(R_R)$, this means that $\sigma(E(R_R)) = 0$.

(vi) \Rightarrow (vii): If $\sigma \in \{\tau \in \mathfrak{m}\text{-Jans-}R : \tau_{\text{zero}} \leq \tau < 1_{\text{Mod-}R}\}$ then, since T_{σ} is closed under submodules and $R_R^* \notin T_{\sigma}$, we cannot have $\sigma(E(R_R^*)) = E(R_R^*)$. Therefore by (vi), we must have $\sigma(E(R_R)) = 0$, as required.

(v) \Rightarrow (i): Let $0 \neq K \leq R_R$ and let σ denote the \mathfrak{m} -Jansian torsion preradical on $\text{Mod-}R$ generated by $K \oplus \mathbb{Z}_{\text{zero}}$. By hypothesis, since $\sigma \geq \tau_{\text{zero}}$ and $\sigma(R_R) \supseteq K \neq 0$, we must have that $\sigma = 1_{\text{Mod-}R}$. By Proposition 2 ((i) \Leftrightarrow (iii)), K is \mathfrak{m} -faithful. \square

The reader will observe that a ring R satisfying condition (ii) of the previous theorem is necessarily prime. (Recall that a ring R is prime if and only if for any nonzero elements $a, b \in R$, we have $aRb \neq 0$.) In particular, if \mathfrak{m} is chosen to be \aleph_0 , then a ring R satisfying condition (ii) is, by definition, *right strongly prime* in the sense of Handelman and Lawrence: see [HL75]. (In [HL75], rings are assumed to have identity, but the term “strongly prime” has been used by several other authors to refer to arbitrary rings satisfying condition (ii), with $\mathfrak{m} = \aleph_0$.) Also, for $\mathfrak{m} = \aleph_0$, a ring R satisfies condition (v) of the previous theorem if and only if it is *right absolutely torsion-free* in the sense of Rubin [Rub73]. The equivalence of (ii) and (v) therefore yields, as a special case, the result of Handelman and Lawrence [HL75, Proposition V.4, p221] and Viola-Prioli [Vio75, Theorem 2.1 ((1) \Leftrightarrow (4)), p276]: *a ring (with identity) is right strongly prime if and only if it is*

right absolutely torsion-free. Furthermore, the equivalence of (ii), (vi) and (vii) for rings with identity and for $\mathfrak{m} = \aleph_0$ is part of a result of Katayama [Kat83, Theorem 3.2, p57].

Observe that every prime ring R satisfies condition (ii) of Theorem 3 for a sufficiently large cardinal \mathfrak{m} (at worst, $\mathfrak{m} = |R|^+$ will do, since we may take $X = R$ for every nonzero $a \in R$). Thus condition (ii) suggests a classification of arbitrary prime rings; we may assign to any such ring R a “degree of primeness”, viz. the least cardinal \mathfrak{m} such that for every $a \in R$, a subset X of R can be found satisfying (ii).

Accordingly, and following the terminology of Handelman and Lawrence [HL75], we call a subset X of a ring R a *right insulator* for a nonzero $a \in R$ if $(0 :_r aX) = 0$ and say that R is *right prime of bound \mathfrak{m}* if \mathfrak{m} is the least cardinal such that every nonzero element of R has a right insulator of cardinality less than $\mathfrak{m} + 1$. (Here we do not require \mathfrak{m} to be infinite.) We denote by $P_r(\mathfrak{m})$ the class of all rings that are prime of bound \mathfrak{m} . If $R \in P_r(\mathfrak{m})$, we call \mathfrak{m} the *right bound of primeness* of R . We also define $\bar{P}_r(\mathfrak{m})$ to be the class of all rings which are right prime of bound at most \mathfrak{m} , i.e., $\bar{P}_r(\mathfrak{m}) := \bigcup_{0 < \mathfrak{k} \leq \mathfrak{m}} P_r(\mathfrak{k})$. The left analogues of these classes are denoted by $P_l(\mathfrak{m})$ and $\bar{P}_l(\mathfrak{m})$, respectively, and expressions such as *left insulator* and *left bound of primeness* (of a prime ring) are defined in the obvious manner.

We have already observed that a prime ring R is a right insulator for each of its nonzero elements, hence $R \in P_r(\mathfrak{m})$ for some $\mathfrak{m} \leq |R|^+$. Examples will be provided in the next chapter to show that $P_r(\mathfrak{m})$ is nonempty for all $\mathfrak{m} > 0$. Thus the classes $P_r(\mathfrak{m})$, $\mathfrak{m} > 0$, partition the class of all prime rings. Note that $\bar{P}_r(\aleph_0)$ consists just of the *right strongly prime rings* of Handelman and Lawrence. In [HL75], the rings in $\bigcup_{0 < \mathfrak{n} < \aleph_0} P_r(\mathfrak{n})$ are called *right bounded strongly prime* – in particular, the rings in $P_r(\mathfrak{n})$, where $0 < \mathfrak{n} < \aleph_0$, are called *right bounded strongly prime of bound \mathfrak{n}* – while the elements of $P_r(\aleph_0)$ are called *right unbounded strongly prime rings*. Note that condition (ii) of Theorem 3 is equivalent to the assertion that $R \in \bar{P}_r(\mathfrak{m})$. This means that conditions (iv) to (vii) of Theorem 3 provide torsion theoretic characterizations of those rings which are prime of bound at most \mathfrak{m} , whenever \mathfrak{m} is a regular cardinal.

COROLLARY 4. *The following conditions on a ring R are equivalent:*

- (i) R is a prime ring;
- (ii) there is a regular cardinal \mathfrak{m} such that $\sigma(R_R) = 0$ for all proper \mathfrak{m} -Jansian torsion preradicals $\sigma \geq \tau_{\text{zero}}$ on $\text{Mod-}R$;
- (iii) $\sigma(R_R) = 0$ for all proper Jansian torsion preradicals $\sigma \geq \tau_{\text{zero}}$ on $\text{Mod-}R$.

Proof. (i) \Rightarrow (ii): Let $\mathfrak{m} = \max\{\aleph_0, |R|^+\}$. Then $R \in \bar{P}_r(\mathfrak{m})$ and the result follows from Theorem 3.

(ii) \Rightarrow (iii) is clear.

(iii) \Rightarrow (i): Let I, J be ideals of R with $IJ = 0$ and $J \neq 0$. Since $\eta(J)$ is Jansian and

$0 \notin \eta(J)$, it follows that $\sigma = \text{torsp } \eta(J)$ is Jansian and proper (Theorem 3.4), hence $\sigma(R_R) = \{a \in R : aJ = 0\} = 0$, by (iii). But $I \subseteq \sigma(R_R)$ so $I = 0$. It follows that R is a prime ring. \square

Our next objective is to prove Theorem 6, which provides a categorical characterization of the class $\bar{P}_r(\mathfrak{m})$ for every regular cardinal \mathfrak{m} .

Let \mathfrak{m} be an infinite cardinal and let $M, N \in \text{Mod-}R$. We say that N is \mathfrak{m} -cogenerated (resp. \mathfrak{m} -generated) by M if N is embeddable in (resp. N is an epimorphic image of) $\prod_{\Gamma}^{(\mathfrak{m})} M$ for some set Γ . We say that N is \mathfrak{m} -subgenerated by M if N is embeddable in some right R -module which is \mathfrak{m} -generated by M .

PROPOSITION 5. *Let \mathfrak{m} be an infinite cardinal. Then the following conditions are equivalent for a right R -module M :*

- (i) M is \mathfrak{m} -faithful;
- (ii) R_R^* is \mathfrak{m} -cogenerated by $M \oplus \mathbb{Z}_{\text{zero}}$;
- (iii) $E(R_R^*)$ is \mathfrak{m} -generated by $M \oplus \mathbb{Z}_{\text{zero}}$;
- (iv) R_R^* is \mathfrak{m} -subgenerated by $M \oplus \mathbb{Z}_{\text{zero}}$.

Proof. (i) \Rightarrow (ii) follows immediately from Proposition 1 while (iii) \Rightarrow (iv) is trivial.

(ii) \Rightarrow (iii): Suppose $R_R^* \lesssim \prod_{\Gamma}^{(\mathfrak{m})} (M \oplus \mathbb{Z}_{\text{zero}})$. Choose an epimorphism $\pi: (R_R^*)^{(\Gamma')} \rightarrow E(R_R^*)$. We assume, without loss of generality, that $|\Gamma'| \geq |\Gamma|, \aleph_0$. The embedding of R_R^* in $\prod_{\Gamma}^{(\mathfrak{m})} (M \oplus \mathbb{Z}_{\text{zero}})$ induces an embedding

$$(R_R^*)^{(\Gamma')} \lesssim \left(\prod_{\Gamma}^{(\mathfrak{m})} (M \oplus \mathbb{Z}_{\text{zero}}) \right)^{(\Gamma')} \cong \prod_{\Gamma'}^{(\mathfrak{m})} (M \oplus \mathbb{Z}_{\text{zero}}).$$

Since $E(R_R^*)$ is injective, π may be extended to an epimorphism from $\prod_{\Gamma'}^{(\mathfrak{m})} (M \oplus \mathbb{Z}_{\text{zero}})$ onto $E(R_R^*)$.

(iv) \Rightarrow (i): Suppose $\pi: \prod_{\Gamma}^{(\mathfrak{m})} (M \oplus \mathbb{Z}_{\text{zero}}) \rightarrow N$ is an epimorphism and $\iota: R_R^* \rightarrow N$ an embedding. Let $y = \{(x_i, m_i)\}_{i \in \Gamma}$ be a preimage under π of $\iota(1_{R^*})$. Put $x = \{x_i\}_{i \in \Gamma} \in \prod_{\Gamma}^{(\mathfrak{m})} M$. Then $X = \{x_i : i \in \text{supp } x\} \subseteq M$, $|X| < \mathfrak{m}$ and

$$\begin{aligned} s \in (0 : X) &\Rightarrow xs = 0 \Rightarrow ys = 0 \\ &\Rightarrow \pi(ys) = \pi(y)s = \iota(1_{R^*})s = \iota(s) = 0 \\ &\Rightarrow s = 0. \end{aligned}$$

Thus $(0 : X) = 0$. This shows that M is \mathfrak{m} -faithful. \square

An analogue of Proposition 5 is stated below for the case where the ring R has identity; the proof is entirely similar.

PROPOSITION 5a. *Let \mathfrak{m} be an infinite cardinal and R a ring with identity. Then the following conditions are equivalent for any unital right R -module M :*

- (i) M is \mathfrak{m} -faithful;

- (ii) R_R is \mathfrak{m} -cogenerated by M ;
- (iii) $E(R_R)$ is \mathfrak{m} -generated by M ;
- (iv) R_R is \mathfrak{m} -subgenerated by M .

□

THEOREM 6. *Let \mathfrak{m} be a regular cardinal. Then the following conditions are equivalent for any ring R :*

- (i) $R \in \bar{P}_r(\mathfrak{m})$;
- (ii) for every submodule P of $(R_R^*)^{(\Gamma)}$ and every nonzero submodule Q of $R_R^{(\Gamma)}$, P is \mathfrak{m} -cogenerated by $Q \oplus \mathbb{Z}_{\text{zero}}$;
- (iii) R_R^* is \mathfrak{m} -cogenerated by $A \oplus \mathbb{Z}_{\text{zero}}$ for all nonzero right ideals A of R ;
- (iv) $E(R_R^*)$ is \mathfrak{m} -generated by $A \oplus \mathbb{Z}_{\text{zero}}$ for all nonzero submodules A of $E(R_R)$;
- (v) $E(R_R^*)$ is \mathfrak{m} -generated by $A \oplus \mathbb{Z}_{\text{zero}}$ for all nonzero right ideals A of R ;
- (vi) R_R^* is \mathfrak{m} -subgenerated by $A \oplus \mathbb{Z}_{\text{zero}}$ for all nonzero right ideals A of R .

Proof. (i) \Rightarrow (ii): Suppose $0 \neq Q \leq R_R^{(\Gamma)}$ and let σ be the \mathfrak{m} -Jansian torsion preradical on $\text{Mod-}R$ generated by $Q \oplus \mathbb{Z}_{\text{zero}}$. Since $\sigma(Q) = Q \neq 0$, we cannot have $\sigma(R_R) = 0$, so by Theorem 3, $\sigma = 1_{\text{Mod-}R}$. It follows from Proposition 2((iii) \Rightarrow (i)) that Q is \mathfrak{m} -faithful. Therefore R_R^* is \mathfrak{m} -cogenerated by $Q \oplus \mathbb{Z}_{\text{zero}}$ (Proposition 5).

Suppose $R_R^* \lesssim \prod_{\Delta}^{(\mathfrak{m})}(Q \oplus \mathbb{Z}_{\text{zero}})$ and $P \leq (R_R^*)^{(\Gamma)}$. Then P embeds in $\left(\prod_{\Delta}^{(\mathfrak{m})}(Q \oplus \mathbb{Z}_{\text{zero}})\right)^{(\Gamma)}$ which is isomorphic to $\prod_{\Delta'}^{(\mathfrak{m})}(Q \oplus \mathbb{Z}_{\text{zero}})$ for a suitable index set Δ' . Thus P is \mathfrak{m} -cogenerated by $Q \oplus \mathbb{Z}_{\text{zero}}$, as required.

(ii) \Rightarrow (iii) is obvious (just take $|\Gamma| = 1$.)

(iii) \Rightarrow (iv): Suppose $0 \neq A \leq E(R_R)$. Then $A \cap R$ is a nonzero right ideal of R . By (iii) and Proposition 5, $E(R_R^*)$ is \mathfrak{m} -generated by $(A \cap R) \oplus \mathbb{Z}_{\text{zero}}$. Let

$$\pi : \prod_{\Gamma}^{(\mathfrak{m})}((A \cap R) \oplus \mathbb{Z}_{\text{zero}}) \rightarrow E(R_R^*)$$

be an epimorphism. Since $E(R_R^*)$ is injective, π extends to an epimorphism from $\prod_{\Gamma}^{(\mathfrak{m})}(A \oplus \mathbb{Z}_{\text{zero}})$ onto $E(R_R^*)$. Thus $E(R_R^*)$ is \mathfrak{m} -generated by $A \oplus \mathbb{Z}_{\text{zero}}$.

(iv) \Rightarrow (v) \Rightarrow (vi) are obvious.

(vi) \Rightarrow (i): By (vi) and Proposition 5, every nonzero right ideal of R is \mathfrak{m} -faithful. Hence $R \in \bar{P}_r(\mathfrak{m})$, by Theorem 3. □

The following analogue of Theorem 6 for rings with identity may be proved similarly, using Proposition 5a.

THEOREM 6a. *Let \mathfrak{m} be a regular cardinal. Then the following conditions are equivalent for any ring R with identity:*

- (i) $R \in \bar{P}_r(\mathfrak{m})$;
- (ii) for every projective unital right R -module P and every nonzero submodule Q of P , P is \mathfrak{m} -cogenerated by Q ;
- (iii) R_R is \mathfrak{m} -cogenerated by each of its nonzero right ideals;
- (iv) $E(R_R)$ is \mathfrak{m} -generated by each of its nonzero submodules;
- (v) $E(R_R)$ is \mathfrak{m} -generated by each nonzero right ideal of R ;
- (vi) R_R is \mathfrak{m} -subgenerated by each of its nonzero right ideals. □

We shall study the rings just characterized from a non-torsion theoretic point of view in the chapters which follow.

Chapter III

Prime Rings

One of the most important notions defined in Chapter II is that of a ring which is right prime of bound \mathfrak{m} , for a given nonzero cardinal \mathfrak{m} . The present chapter is devoted, primarily, to an investigation of this notion.

An important initial task is to show that, given an arbitrary nonzero cardinal \mathfrak{m} , there do exist rings which are right prime of bound \mathfrak{m} . Matrix rings turn out to play a useful role here. Indeed, in §1 we show that if D is a division ring and \mathfrak{m} an arbitrary nonzero cardinal, then the ring of all row-finite $\mathfrak{m} \times \mathfrak{m}$ matrices over D is (1) right prime of bound \mathfrak{m} , if \mathfrak{m} is finite; and (2) right prime of bound \mathfrak{m}^+ , if \mathfrak{m} is infinite. This gives one a way of characterizing, in purely ring theoretic terms, the rings of linear transformations of vector spaces (over division rings) of any fixed (finite or infinite) dimension: see Proposition 1.14. We show, furthermore, that if \mathfrak{m} is a limit cardinal then a suitable subring of the ring of all row-finite $\mathfrak{m} \times \mathfrak{m}$ matrices over D is right prime of bound \mathfrak{m} .

Special attention is also paid to the left bounds of primeness of matrix rings and many cases of left-right asymmetry are explored. We are almost able to say (Theorem 1.13) for exactly which pairs of cardinals $\mathfrak{m}, \mathfrak{n}$, there exist rings right prime of bound \mathfrak{m} and left prime of bound \mathfrak{n} ; the only obstacle to a complete solution of this problem is a longstanding (still open) question of Goodearl, Handelman and Lawrence: must a right unbounded strongly prime ring be left strongly prime?

A result deserving special mention is Theorem 1.2, which states that, for any finite cardinal $\mathfrak{m} > 1$, a ring which is right prime of bound \mathfrak{m} is necessarily a prime right Goldie ring, and is therefore isomorphic to a right order in a simple artinian ring (with identity), i.e., a right order in a finite dimensional matrix ring over a division ring. Although not new, this theorem is undoubtedly one of the cornerstones of this chapter.

Another of our objectives in this chapter is to examine the preservation of the property “right prime of bound at most \mathfrak{m} ” under various standard ring theoretic constructions. §1 deals with matrix rings; an interesting product of this investigation is Theorem 1.16 which asserts that if \mathfrak{m} is an infinite cardinal, then the property of being an element of $\bar{P}_r(\mathfrak{m})$ is Morita invariant within the class of all rings with identity. §2 looks, more generally, to closure properties of the class of all rings that are prime of a certain bound, with respect to the formation of subrings and overrings. As usual, overrings (e.g., essential extensions) behave well and subrings (even right orders) badly; the latter deficiency may be overcome, however, by considering a strong subring notion – that of a “right \mathfrak{m} -order”. §3 is more specific; in it, the bounds of primeness of certain semigroup (including monoid, group and polynomial) rings, and “monomial algebras”, are investigated, yielding several construction techniques. Many of the results of this section will find application in the radical theoretic study to be carried out

in Chapter IV.

§4 of this chapter is devoted mainly to the study of “uniformly strongly prime rings”. These are rings containing a finite subset which functions “uniformly” as a (right) insulator for all nonzero ring elements. The notion of a uniformly strongly prime ring and, more specifically, that of a ring which is “uniformly strongly prime of bound \mathbf{m} ”, for a given finite nonzero cardinal \mathbf{m} , was first defined by Handelman and Lawrence [HL75, p211]. Despite the prominence of the paper [HL75], these “uniform” notions have, rather surprisingly, been somewhat neglected in the literature. In §4, we attempt to give this topic a fuller and more deserved treatment. One of the main results (Theorem 4.4) asserts that a ring which is uniformly strongly prime of bound greater than 1 is necessarily prime right or left Goldie, thereby reducing the study of uniformly strongly prime rings to that of those such rings whose uniform bound is 1, and to prime Goldie rings. (The reader should compare this result with Theorem 1.2.) Finite dimensional matrix rings over division rings (which, incidentally, are always uniformly strongly prime) therefore play an important role here. Determining their uniform bounds of primeness can, however, be a surprisingly difficult task; this aspect of their study contrasts sharply with that of their right or left bounds of primeness, which are always equal to the dimensions of the matrices. Unexpected (if not also counter-intuitive) results like these may account to some extent for the relative neglect, to date, of the study of uniformly strongly prime rings. The results also show that for every finite nonzero cardinal \mathbf{n} , there exists a ring which is uniformly strongly prime of bound exactly \mathbf{n} (Corollary 4.15); this result may be generalized to infinite successor cardinals, but not to limit cardinals. Our study of uniformly strongly prime rings will also be continued in the radical theoretic context of Chapter IV.

Most of the results of this chapter have been published in the papers [RV92] and [VdB93] or will be published in [RV].

§1. MATRIX RINGS.

The ring of row-finite matrices over a division ring is our prototype of a prime ring. We shall show that by varying the dimensions of the matrices, it is possible to produce, for almost every cardinal \mathbf{m} , a ring which is prime of bound \mathbf{m} . First we consider the situation for finite cardinals.

Let R be a ring and \mathbf{m} an arbitrary nonzero cardinal. We remind the reader that if $A \in \mathbb{M}_{\mathbf{m}}(R)$, we use $A_{(\alpha)}$ (resp. $A^{(\alpha)}$) to denote the α -th row (resp. α -th column) of A , and $A_{\alpha\beta}$ to denote the (α, β) -th entry of A , where $\alpha, \beta \in \mathbf{m}$.

PROPOSITION 1. [HL75, Proposition I.2, p211] *Let \mathbf{n} be a finite nonzero cardinal. If D is a division ring then $\mathbb{M}_{\mathbf{n}}(D) \in P_r(\mathbf{n})$, i.e., $\mathbb{M}_{\mathbf{n}}(D)$ is (right) prime of bound \mathbf{n} .*

Proof. Let $0 \neq A \in \mathbb{M}_{\mathbf{n}}(D)$ and suppose $x = A_{\alpha\beta} \neq 0$ with $\alpha, \beta \in \mathbf{n}$. For each $\gamma, \delta \in \mathbf{n}$, let

$E_{\gamma,\delta}(r)$ denote the matrix in $\mathbb{M}_{\mathbf{n}}(D)$ with $r \in D$ in position (γ, δ) and zeros elsewhere. We claim that $X = \{E_{\beta,\gamma}(1_D) : \gamma < \mathbf{n}\}$ is a right insulator for A in $\mathbb{M}_{\mathbf{n}}(D)$. Indeed, suppose that $AXB = 0$ with $B \in \mathbb{M}_{\mathbf{n}}(D)$. Let $\gamma, \delta \in \mathbf{n}$. Then

$$0 = [AE_{\beta,\gamma}(1_D)B]_{\alpha\delta} = [AE_{\beta,\gamma}(1_D)]_{(\alpha)} \cdot B^{(\delta)} = xB_{\gamma\delta},$$

so $B_{\gamma\delta} = 0$. Since γ, δ were arbitrary, $B = 0$, vindicating our claim about X . Since $|X| = \mathbf{n}$, this shows that $\mathbb{M}_{\mathbf{n}}(D) \in \bar{P}_r(\mathbf{n})$.

We show now that the matrix unit $E_{0,0}(1_D)$ of $\mathbb{M}_{\mathbf{n}}(D)$ has no right insulator of cardinality less than \mathbf{n} in $\mathbb{M}_{\mathbf{n}}(D)$. Let X be an arbitrary nonempty subset of $\mathbb{M}_{\mathbf{n}}(D)$ with $|X| < \mathbf{n}$. Note that if $A \in \mathbb{M}_{\mathbf{n}}(D)$ then

$$[E_{0,0}(1_D)A]_{(\alpha)} = \begin{cases} 0, & \text{if } \alpha \neq 0; \\ A_{(0)}, & \text{if } \alpha = 0. \end{cases}$$

Choose $B \in \mathbb{M}_{\mathbf{n}}(D)$ such that $\{A_{(0)} : A \in X\} \subseteq \{B_{(\alpha)} : 0 \leq \alpha < \mathbf{n}\}$ and $\text{rank } B < \mathbf{n}$. Clearly, such a matrix B exists, since $|X| < \mathbf{n}$. Now

$$\begin{aligned} (0 :_r B) &= \{C \in \mathbb{M}_{\mathbf{n}}(D) : (\forall \alpha < \mathbf{n})(B_{(\alpha)}C = 0)\} \\ &\subseteq \{C \in \mathbb{M}_{\mathbf{n}}(D) : (\forall A \in X)(A_{(0)}C = 0)\} \\ &= \{C \in \mathbb{M}_{\mathbf{n}}(D) : (\forall A \in X)(E_{0,0}(1_D)AC = 0)\} \\ &= \bigcap_{A \in X} (0 :_r E_{0,0}(1_D)A) \end{aligned}$$

Thus $(0 :_r E_{0,0}(1_D)X) \supseteq (0 :_r B)$. Since $\text{rank } B < \mathbf{n}$, we must have $(0 :_r B) \neq 0$. Consequently, $(0 :_r E_{0,0}(1_D)X) \neq 0$. This shows that $\mathbb{M}_{\mathbf{n}}(D) \notin P_r(\mathbf{k})$ for any $\mathbf{k} < \mathbf{n}$. \square

It turns out that if $1 < \mathbf{n} < \aleph_0$ then every member of $P_r(\mathbf{n})$ embeds, in a rather special way, in $\mathbb{M}_{\mathbf{n}}(D)$ for some division ring D . The following result was first proved by Goodearl, Handelman and Lawrence for rings with identity [GHL74, Theorem 4.7, p27] (a more accessible reference is [GH75, Theorem 2.3, p803]) and was extended to rings without identity in [Raf87, Corollary 9, p263]. Its proof makes use of a number of prerequisite results, however, and will therefore be postponed until Chapter IV, §2.

Recall that a ring R is said to be a *right order* in a ring S (with identity) if R is a subring of S with the property that for every $s \in S$ there is a unit u of S , contained in R , such that $su \in R$.

THEOREM 2. *Let \mathbf{n} be a finite cardinal such that $\mathbf{n} > 1$. Then the following assertions are equivalent for a ring R :*

- (i) $R \in P_r(\mathbf{n})$;
- (ii) R is isomorphic to a right order in $\mathbb{M}_{\mathbf{n}}(D)$ for some division ring D . \square

The requirement that \mathbf{n} be greater than 1 cannot be dropped from Theorem 2, for if D is a

domain which is not right Ore then D is not isomorphic to a right order in a division ring but, of course, $D \in P_r(1)$. Indeed, there will be much evidence in the sequel to suggest that the members of $P_r(1)$ are more akin to those of the classes $P_r(\mathfrak{m})$, $\mathfrak{m} \geq \aleph_0$, than to the rings in any of the $P_r(\mathfrak{n})$ for finite $\mathfrak{n} > 1$. Inasmuch as a ring is prime right Goldie if and only if it is isomorphic to a right order in $\mathbb{M}_{\mathfrak{n}}(D)$ for some finite nonzero \mathfrak{n} and some division ring D (by Goldie's First Theorem – Theorem 0.7.5), it follows from Theorem 2 that a ring R is prime right Goldie if and only if R is a right Ore domain or $R \in P_r(\mathfrak{n})$ for some finite $\mathfrak{n} > 1$.

We turn now to arbitrary (possibly infinite) cardinals.

LEMMA 3. *Let \mathfrak{m} be an arbitrary nonzero cardinal and R a ring. If A is a nonzero element of $\mathbb{M}_{\mathfrak{m}}(R)$ and X is a right insulator for $A_{\alpha\beta}$ in R (for some $\alpha, \beta \in \mathfrak{m}$), then $\{E_{\beta, \gamma}(x) : x \in X, \gamma \in \mathfrak{m}\}$ is a right insulator for A in $\mathbb{M}_{\mathfrak{m}}(R)$.*

Proof. The proof is similar to that of Proposition 1. Set $t = A_{\alpha\beta}$, $Y = \{E_{\beta, \gamma}(x) : x \in X, \gamma \in \mathfrak{m}\}$ and suppose that $AYB = 0$ for some $B \in \mathbb{M}_{\mathfrak{m}}(R)$. Let $\gamma, \delta \in \mathfrak{m}$. Then

$$0 = [AE_{\beta, \gamma}(x)B]_{\alpha\delta} = [AE_{\beta, \gamma}(x)]_{(\alpha)} \cdot B^{(\delta)} = txB_{\gamma\delta},$$

for all $x \in X$. Since X is a right insulator for t in R we must have $B_{\gamma\delta} = 0$. Since γ, δ were arbitrary, this implies $B = 0$. Thus Y is a right insulator for A in $\mathbb{M}_{\mathfrak{m}}(R)$. \square

PROPOSITION 4. *Let $\mathfrak{m}, \mathfrak{n}$ be arbitrary nonzero cardinals and suppose that $R \in P_r(\mathfrak{n})$.*

Then:

$$\mathbb{M}_{\mathfrak{m}}(R) \in \begin{cases} P_r(\mathfrak{mn}), & \text{if } \mathfrak{m} < \aleph_0 \text{ and } 1 < \mathfrak{n} < \aleph_0; \\ P_r(\mathfrak{m}), & \text{if } \mathfrak{m} < \aleph_0, \mathfrak{n} = 1 \text{ and } R \text{ is a right Ore domain}; \\ P_r(1), & \text{if } \mathfrak{m} < \aleph_0, \mathfrak{n} = 1 \text{ and } R \text{ is not a right Ore domain}; \\ P_r(\mathfrak{m}^+), & \text{if } \mathfrak{m} \geq \aleph_0, \mathfrak{m} \geq \mathfrak{n}; \\ P_r(\mathfrak{n}), & \text{if } \mathfrak{n} > \mathfrak{m}, \mathfrak{n} \geq \aleph_0. \end{cases}$$

Proof. If $1 < \mathfrak{n} < \aleph_0$ then by Theorem 2, R is isomorphic to a right order in $\mathbb{M}_{\mathfrak{n}}(D)$ for some division ring D . By Lemma 0.7.3, $\mathbb{M}_{\mathfrak{m}}(R)$ is isomorphic to a right order in $\mathbb{M}_{\mathfrak{m}}(\mathbb{M}_{\mathfrak{n}}(D)) \cong \mathbb{M}_{\mathfrak{mn}}(D)$ whenever $\mathfrak{m} < \aleph_0$. By Theorem 2, $\mathbb{M}_{\mathfrak{m}}(R) \in P_r(\mathfrak{mn})$.

If $\mathfrak{m} < \aleph_0$ and R is a right Ore domain then $\mathbb{M}_{\mathfrak{m}}(R)$ is isomorphic to a right order in $\mathbb{M}_{\mathfrak{m}}(D)$ for some division ring D . By Theorem 2, $\mathbb{M}_{\mathfrak{m}}(R) \in P_r(\mathfrak{m})$.

If $\mathfrak{m} < \aleph_0$, $\mathfrak{n} = 1$ and R is not a right Ore domain, it follows from the previous lemma that every nonzero $A \in \mathbb{M}_{\mathfrak{m}}(R)$ has a right insulator of cardinality \mathfrak{m} . Consequently, $\mathbb{M}_{\mathfrak{m}}(R) \in \bar{P}_r(\mathfrak{m})$. Since R is not a right Ore domain, we cannot have $\mathbb{M}_{\mathfrak{m}}(R) \in P_r(\mathfrak{k})$ for any finite cardinal $\mathfrak{k} > 1$, by Theorem 2. It follows that $\mathbb{M}_{\mathfrak{m}}(R) \in P_r(1)$.

Let $0 \neq A \in \mathbb{M}_{\mathbf{m}}(R)$ and suppose that $A_{\alpha\beta} \neq 0$, with $\alpha, \beta \in \mathbf{m}$. Choose a right insulator X for $A_{\alpha\beta}$ in R with $|X| < \mathbf{n} + 1$ and set $Y = \{E_{\beta,\gamma}(x) : x \in X, \gamma \in \mathbf{m}\}$. We know from the previous lemma that Y is a right insulator for A in $\mathbb{M}_{\mathbf{m}}(R)$.

If $\mathbf{m} \geq \aleph_0$ and $\mathbf{m} \geq \mathbf{n}$, then $|Y| \leq \mathbf{m}|X| \leq \mathbf{m}(\mathbf{n} + 1) \leq \mathbf{m}$, and so $\mathbb{M}_{\mathbf{m}}(R) \in \bar{P}_r(\mathbf{m}^+)$. But in this case, if $0 \neq r \in R$, then $E_{0,0}(r)$ has no right insulator of cardinality less than \mathbf{m} in $\mathbb{M}_{\mathbf{m}}(R)$. For if $X \subseteq \mathbb{M}_{\mathbf{m}}(R)$ with $|X| = \mathbf{k} < \mathbf{m}$ then $E_{0,0}(r)X$ consists of at most \mathbf{k} ($\mathbf{m} \times \mathbf{m}$) matrices each of whose only nonzero entries are a finite number of first row entries. Together, the nonzero entries of the matrices in $E_{0,0}(r)X$ are fewer than \mathbf{m} in number, so it is easy to construct a nonzero matrix $B \in \mathbb{M}_{\mathbf{m}}(R)$ with $E_{0,0}(r)XB = 0$. It follows that $\mathbb{M}_{\mathbf{m}}(R) \notin \bar{P}_r(\mathbf{m})$, hence $\mathbb{M}_{\mathbf{m}}(R) \in P_r(\mathbf{m}^+)$.

Finally, if $\mathbf{n} \geq \aleph_0$ and $\mathbf{n} > \mathbf{m}$, then $|Y| \leq \mathbf{m}|X| < \mathbf{n}$, and so $\mathbb{M}_{\mathbf{m}}(R) \in \bar{P}_r(\mathbf{n})$. Certainly, if $\mathbf{n} = \aleph_0$ we cannot have $\mathbb{M}_{\mathbf{m}}(R) \in P_r(\mathbf{k})$ for any finite cardinal \mathbf{k} , so $\mathbb{M}_{\mathbf{m}}(R) \in P_r(\mathbf{n})$. Suppose $\mathbf{n} > \aleph_0$ and let \mathbf{k} be an infinite cardinal with $\mathbf{k} < \mathbf{n}$. Since $R \in P_r(\mathbf{n})$, there exists $a \in R$ such that a has no right insulator in R of cardinality less than \mathbf{k} . Then $E_{0,0}(a)$ has no right insulator in $\mathbb{M}_{\mathbf{m}}(R)$ of cardinality less than \mathbf{k} . For if $X \subseteq \mathbb{M}_{\mathbf{m}}(R)$ with $|X| < \mathbf{k}$ then the elements of $E_{0,0}(a)X$ have together fewer than \mathbf{k} nonzero entries, whence for some nonzero scalar matrix B we have $E_{0,0}(a)XB = 0$. It follows that $\mathbb{M}_{\mathbf{m}}(R) \notin \bar{P}_r(\mathbf{k})$, and so $\mathbb{M}_{\mathbf{m}}(R) \in P_r(\mathbf{n})$. \square

Notice, in the above result, that the bound of primeness of $\mathbb{M}_{\mathbf{m}}(R)$ is determined uniquely by the cardinals \mathbf{m} and \mathbf{n} , except in one instance, when \mathbf{m} is finite and $\mathbf{n} = 1$; in this instance the value of the relevant bound depends on whether R is a right Ore domain or not.

The conclusions of Proposition 4 are worth highlighting in the case where R is chosen to be a division ring D :

$$\mathbb{M}_{\mathbf{m}}(D) \in \begin{cases} P_r(\mathbf{m}), & \text{if } \mathbf{m} < \aleph_0; \\ P_r(\mathbf{m}^+), & \text{if } \mathbf{m} \geq \aleph_0. \end{cases}$$

It follows that the class $P_r(\mathbf{m})$ is nonempty for all successor cardinals \mathbf{m} . In the following two examples we complete the picture by showing that the class $P_r(\mathbf{m})$ is nonempty for all limit cardinals \mathbf{m} . The first example is due to Goodearl, Handelman and Lawrence (e.g., [GH75, Example(b), p815]).

EXAMPLE 1. We construct a ring R such that $R \in P_r(\aleph_0)$. Let D be a division ring and consider the ring $\mathbb{M}_{\aleph_0}(D)$. For each $\alpha \in \aleph_0$, let $\Theta_\alpha : \mathbb{M}_{2^\alpha}(D) \rightarrow \mathbb{M}_{\aleph_0}(D)$ be the map defined by

$$\Theta_\alpha(A) = \begin{bmatrix} A & 0 \\ 0 & A \\ & & \ddots \end{bmatrix} \in \mathbb{M}_{\aleph_0}(D) \quad (A \in \mathbb{M}_{2^\alpha}(D)).$$

Clearly, Θ_α is a ring monomorphism for all $\alpha \in \aleph_0$. Define

$$R = \bigcup_{\alpha \in \aleph_0} \Theta_\alpha[\mathbb{M}_{2^\alpha}(D)].^1$$

For notational convenience we shall identify each $\mathbb{M}_{2^\alpha}(D)$ with its image in R . Notice that under this convention, $\mathbb{M}_{2^\alpha}(D)$ is a subring of $\mathbb{M}_{2^\beta}(D)$ whenever $\alpha \leq \beta < \aleph_0$. Let $0 \neq A \in R$ and suppose that $A \in \mathbb{M}_{2^\alpha}(D)$ with $\alpha \in \aleph_0$. Inasmuch as $\mathbb{M}_{2^\alpha}(D) \in P_r(2^\alpha)$, by Proposition 1, we may choose a right insulator X for A in $\mathbb{M}_{2^\alpha}(D)$ with $|X| \leq 2^\alpha$. We claim that the image of X in R is a right insulator for A in R . For suppose $AXB = 0$ for some $B \in R$, say $B \in \mathbb{M}_{2^\beta}(D)$ with $\beta < \aleph_0$. Clearly, if $\beta \leq \alpha$ then $B \in \mathbb{M}_{2^\alpha}(D)$ and so $B = 0$. Suppose $\beta > \alpha$ and set $\mu = 2^{\beta-\alpha}$. Write B in block matrix form:

$$B = \begin{bmatrix} C(0,0) & C(0,1) & \dots & C(0,\mu-1) \\ C(1,0) & C(1,1) & \dots & C(1,\mu-1) \\ \vdots & \vdots & & \vdots \\ C(\mu-1,0) & C(\mu-1,1) & \dots & C(\mu-1,\mu-1) \end{bmatrix}$$

with each $C(\gamma, \delta) \in \mathbb{M}_{2^\alpha}(D)$. It follows from the equation $AXB = 0$ that $AMC(\gamma, \delta) = 0$ for all $M \in X$ and all ordinals $\gamma, \delta < \mu$. Since X is a right insulator for A in $\mathbb{M}_{2^\alpha}(D)$, we must have that $C(\gamma, \delta) = 0$ for all $\gamma, \delta < \mu$, i.e., $B = 0$. Thus X is a right insulator for A in R . This shows that $R \in \bar{P}_r(\aleph_0)$. On the other hand, it is easily checked that for each $\alpha \in \aleph_0$, the element $E_{0,0}(1_D) \in \mathbb{M}_{2^\alpha}(D)$ has no right insulator in R of cardinality less than 2^α . Consequently, $R \in P_r(\aleph_0)$. \square

EXAMPLE 2. Let \mathfrak{m} be an uncountable limit cardinal. We construct a ring R such that $R \in P_r(\mathfrak{m})$. This example is similar in spirit to the previous one but requires a slightly subtler application of ordinal arithmetic. Let D be a division ring. Recall that if \mathfrak{k} is a nonzero cardinal and $\mathfrak{k} < \mathfrak{m}$ then every ordinal $\gamma < \mathfrak{m}$ may be written uniquely in the form $\gamma = (\mathfrak{k} \odot \mu) \oplus \alpha$ for some ordinal $\alpha < \mathfrak{k}$ and some ordinal μ where, necessarily, $\mu \leq \gamma$ (Lemma 0.1.1). (Here, \oplus and \odot denote ordinal addition and multiplication.) Let us say that two ordinals $\gamma, \delta < \mathfrak{m}$ are \mathfrak{k} -related if there exist ordinals μ, α, β with $\alpha, \beta < \mathfrak{k}$ and $\gamma = (\mathfrak{k} \odot \mu) \oplus \alpha$, $\delta = (\mathfrak{k} \odot \mu) \oplus \beta$; otherwise we say that γ, δ are \mathfrak{k} -unrelated. Consider the ring $\mathbb{M}_{\mathfrak{m}}(D)$. For each infinite cardinal $\mathfrak{k} < \mathfrak{m}$, define a map $\Theta_{\mathfrak{k}}: \mathbb{M}_{\mathfrak{k}}(D) \rightarrow \mathbb{M}_{\mathfrak{m}}(D)$ as follows. If $A \in \mathbb{M}_{\mathfrak{k}}(D)$ and $\gamma, \delta < \mathfrak{m}$, let

$$(\Theta_{\mathfrak{k}}(A))_{\gamma\delta} = \begin{cases} A_{\alpha\beta}, & \text{if } \gamma \text{ and } \delta \text{ are } \mathfrak{k}\text{-related with } \gamma = (\mathfrak{k} \odot \mu) \oplus \alpha, \delta = (\mathfrak{k} \odot \mu) \oplus \beta \text{ for} \\ & \text{suitable ordinals } \mu < \mathfrak{m} \text{ and } \alpha, \beta < \mathfrak{k}; \\ 0, & \text{if } \gamma \text{ and } \delta \text{ are } \mathfrak{k}\text{-unrelated.} \end{cases}$$

It is easily checked that $\Theta_{\mathfrak{k}}$ is a ring monomorphism. (Loosely speaking, $\Theta_{\mathfrak{k}}$ is just the “diagonal embedding” of $\mathbb{M}_{\mathfrak{k}}(D)$ into $\mathbb{M}_{\mathfrak{m}}(D)$.) We define

$$R = \bigcup_{\aleph_0 \leq \mathfrak{k} < \mathfrak{m}} \Theta_{\mathfrak{k}}[\mathbb{M}_{\mathfrak{k}}(D)].$$

¹There is a slicker and perhaps more elegant way of constructing the ring R using direct limits, which is the approach adopted by Goodearl and Handelman in [GH75, Example (b), p815].

For convenience, we shall identify each $\mathbb{M}_{\mathbf{k}}(D)$, $\mathbf{k} < \mathbf{m}$, with its image in R . It is clear that if two ordinals $\gamma, \delta < \mathbf{m}$ are \mathbf{k} -related then they are \mathbf{n} -related for all cardinals \mathbf{n} such that $\mathbf{k} \leq \mathbf{n} < \mathbf{m}$. It follows that $\mathbb{M}_{\mathbf{k}}(D)$ is a subring of $\mathbb{M}_{\mathbf{n}}(D)$ whenever $\mathbf{k} \leq \mathbf{n} \leq \mathbf{m}$. Let $0 \neq A \in R$ and suppose that $A \in \mathbb{M}_{\mathbf{k}}(D)$ with $\mathbf{k} < \mathbf{m}$. Since $\mathbb{M}_{\mathbf{k}}(D) \in P_r(\mathbf{k}^+)$ (Proposition 4), we can choose a right insulator X for A in $\mathbb{M}_{\mathbf{k}}(D)$ with $|X| \leq \mathbf{k}$. We claim that the image of X in R is a right insulator for A in R . Indeed, suppose $AXB = 0$ for some $B \in R$, say $B \in \mathbb{M}_{\mathbf{n}}(D)$ with $\aleph_0 \leq \mathbf{n} < \mathbf{m}$. Clearly, if $\mathbf{n} \leq \mathbf{k}$ then $B \in \mathbb{M}_{\mathbf{k}}(D)$, and so $B = 0$. Suppose, therefore, that $\mathbf{n} > \mathbf{k}$. As in the previous example, we express B in block matrix form, defining, for each $\gamma, \delta < \mathbf{n}$, a matrix² $C(\gamma, \delta) \in \mathbb{M}_{\mathbf{k}}(D)$ by

$$C(\gamma, \delta)_{\alpha\beta} = B_{(\mathbf{k} \odot \gamma) \oplus \alpha, (\mathbf{k} \odot \delta) \oplus \beta} \text{ for all } \alpha, \beta < \mathbf{k}.$$

It follows from the equation $AXB = 0$ that $AMC(\gamma, \delta) = 0$ for all $M \in X$ and all $\gamma, \delta < \mathbf{k}$. But X is a right insulator for A in $\mathbb{M}_{\mathbf{k}}(D)$, so $C(\gamma, \delta) = 0$ for all $\gamma, \delta < \mathbf{k}$, i.e., $B = 0$. Thus X is a right insulator for A in R . This shows that $R \in \bar{P}_r(\mathbf{m})$. On the other hand, for each \mathbf{k} such that $\aleph_0 \leq \mathbf{k} < \mathbf{m}$, the element $E_{0,0}(1_D) \in \mathbb{M}_{\mathbf{k}}(D)$ has no right insulator in R of cardinality less than \mathbf{k} (by essentially the same argument as was used in the proof of Proposition 4). Therefore $R \notin P_r(\mathbf{k})$ for any $\mathbf{k} < \mathbf{m}$ and so $R \in P_r(\mathbf{m})$. \square

The following conclusion can be drawn from the previous two examples and the remarks preceding them.

PROPOSITION 5. *The class $P_r(\mathbf{m})$ is nonempty for all nonzero cardinals \mathbf{m} .* \square

The reader will observe that every prime ring R is a member of $P_r(\mathbf{m})$ and of $P_l(\mathbf{n})$ for suitable cardinals $\mathbf{m}, \mathbf{n} \leq |R|^+$. This observation leads naturally to the following question: what restrictions, if any, have to be placed on the cardinals \mathbf{m} and \mathbf{n} in order that the class $P_r(\mathbf{m}) \cap P_l(\mathbf{n})$ be nonempty? We shall show presently that if this class is to be nonempty for a pair of finite cardinals \mathbf{m}, \mathbf{n} then the possible values of \mathbf{m} and \mathbf{n} are rather limited while, by contrast, if \mathbf{m} and \mathbf{n} are large (uncountable, to be precise) then the class $P_r(\mathbf{m}) \cap P_l(\mathbf{n})$ is *always* nonempty.

We deal first with the symmetric case: $\mathbf{m} = \mathbf{n}$. The reader will be aware that if \mathbf{m} is a finite nonzero cardinal and R is any ring then the ring $\mathbb{M}_{\mathbf{m}}(R)$ is left-right symmetric, in the sense that $[\mathbb{M}_{\mathbf{m}}(R)]^{\text{opp}} \cong \mathbb{M}_{\mathbf{m}}(R^{\text{opp}})$. It follows from Proposition 1 (and its left analogue), therefore, that $\mathbb{M}_{\mathbf{m}}(D) \in P_r(\mathbf{m}) \cap P_l(\mathbf{m})$ for any division ring D and any finite nonzero cardinal \mathbf{m} .

Also, for $\mathbf{m} = \aleph_0$, the ring R of Example 1 is clearly left-right symmetric (essentially because $[\mathbb{M}_{2^\alpha}(D)]^{\text{opp}} \cong \mathbb{M}_{2^\alpha}(D^{\text{opp}})$ for all finite α), so $R \in P_r(\aleph_0) \cap P_l(\aleph_0)$.

The reader will also be aware that for $\mathbf{m} \geq \aleph_0$, the rings $[\mathbb{M}_{\mathbf{m}}(R)]^{\text{opp}}$ and $\mathbb{M}_{\mathbf{m}}(R^{\text{opp}})$ are *not*, in general, isomorphic. Indeed, we shall show presently (see Proposition 11) that even for a

²The matrices $C(\gamma, \delta)$ can be thought of as the "component $\mathbf{k} \times \mathbf{k}$ blocks" in a block matrix representation of B .

division ring D , when $\mathfrak{m} \geq \aleph_0$, it is possible for the right and left bounds of primeness of $\mathbb{M}_{\mathfrak{m}}(D)$ to differ quite markedly. This lack of symmetry may be overcome, however, if the ring $\mathbb{M}_{\mathfrak{m}}^*(D)$ is substituted for $\mathbb{M}_{\mathfrak{m}}(D)$ in parts of the text. (Recall that $\mathbb{M}_{\mathfrak{m}}^*(R)$ denotes the ring of all row- and column-finite $\mathfrak{m} \times \mathfrak{m}$ matrices over R and that $\mathbb{M}_{\mathfrak{m}}^*(R)$ is left-right symmetric, in the sense that $[\mathbb{M}_{\mathfrak{m}}^*(R)]^{\text{opp}} \cong \mathbb{M}_{\mathfrak{m}}^*(R^{\text{opp}})$ for all rings R and nonzero cardinals \mathfrak{m} .) In particular, an analogue of Proposition 4 may be obtained by substituting $\mathbb{M}_{\mathfrak{m}}^*(R)$ for $\mathbb{M}_{\mathfrak{m}}(R)$; virtually no changes to the proof are necessary. This allows us to conclude that if D is a division ring and \mathfrak{m} an infinite cardinal then $\mathbb{M}_{\mathfrak{m}}^*(D) \in P_r(\mathfrak{m}^+) \cap P_l(\mathfrak{m}^+)$. (A version of Example 2, modified in this way, may be found in [RV92, Example 10.3, p147].)

We may therefore sharpen Proposition 5 as follows.

PROPOSITION 5a. *The class $P_r(\mathfrak{m}) \cap P_l(\mathfrak{m})$ is nonempty for all nonzero cardinals \mathfrak{m} . \square*

We shall now investigate the possible left bounds of primeness of rings in $P_r(\mathfrak{m})$ for $1 < \mathfrak{m} < \aleph_0$.

It is evident from remarks in the literature that the following result is well known. It will be required in Proposition 7. In the absence of a convenient reference, we have chosen to provide a proof.

LEMMA 6. *If a ring R is isomorphic to a right order in Q_1 and to a left order in Q_2 , where Q_1 and Q_2 are simple artinian rings (necessarily with identity), then $Q_1 \cong Q_2$.*

Proof. Let $\alpha: R \rightarrow Q_1$ and $\beta: R \rightarrow Q_2$ be ring monomorphisms such that $\alpha[R]$ is a right order in Q_1 and $\beta[R]$ a left order in Q_2 . For convenience, we shall identify R with its images in Q_1 and Q_2 . Let S denote the set of all regular elements of R . (Recall that $s \in R$ is said to be *regular* if $(0 :_r s) = (0 :_l s) = 0$.) Since Q_1 is isomorphic to a finite dimensional matrix ring over a division ring (by the Artin-Wedderburn Theorem: Theorem 0.7.1), every element in Q_1 with trivial right annihilator is necessarily a unit of Q_1 . Consequently, every element of S is a unit of Q_1 . It follows that Q_1 is a right ring of fractions of R with respect to S . (We refer the reader to Chapter 0, §8 for details on rings of fractions.) Since every element of S is invertible in Q_2 (by symmetry) and Q_1 is a right ring of fractions of R with respect to S , it follows from the Universal Property for rings of fractions (Theorem 0.8.2) that there exists a unique ring homomorphism $\varphi: Q_1 \rightarrow Q_2$ which makes the following diagram commute.

$$\begin{array}{ccc}
 & R & \\
 \alpha \swarrow & & \searrow \beta \\
 Q_1 & \xrightarrow{\varphi} & Q_2 \\
 & \xleftarrow{\psi} &
 \end{array}$$

It remains to show that φ is an isomorphism. By symmetry, it may be argued that Q_2 is a left ring of fractions of R with respect to S and that as a consequence, there exists a unique ring homomorphism $\psi: Q_2 \rightarrow Q_1$ which makes the above diagram commute. The uniqueness condition in the Universal Property guarantees that $\psi \circ \varphi = 1_{Q_1}$ and $\varphi \circ \psi = 1_{Q_2}$. Thus φ and ψ are mutually inverse ring isomorphisms. \square

PROPOSITION 7. *Let \mathfrak{m} be a finite cardinal with $\mathfrak{m} > 1$. If $R \in P_r(\mathfrak{m})$ then precisely one of the following assertions is true:*

- (i) $R \in P_l(\mathfrak{m})$;
- (ii) $R \in P_l(1)$.

Proof. By Theorem 2, R is isomorphic to a right order in $\mathbb{M}_{\mathfrak{m}}(D)$ for some division ring D . We identify R with its image in $\mathbb{M}_{\mathfrak{m}}(D)$. By the Faith-Utumi Theorem (Theorem 0.7.7), there exists a right order T in D such that $\mathbb{M}_{\mathfrak{m}}(T) \subseteq R \subseteq \mathbb{M}_{\mathfrak{m}}(D)$. Let $0 \neq t \in T$ and define $X = \{E_{\alpha, \beta}(t) : \alpha, \beta < \mathfrak{m}\}$. An argument similar to that used in Proposition 1 shows that X is a left (and right) insulator for every nonzero element of $\mathbb{M}_{\mathfrak{m}}(D)$ and hence a left (and right) insulator for every nonzero element of R .³ Since $|X| = \mathfrak{m}^2 < \aleph_0$, it follows that R is left bounded strongly prime. Now suppose $R \notin P_l(1)$, i.e., (ii) is not satisfied. Then by the left analogue of Theorem 2, R is isomorphic to a left order in $\mathbb{M}_{\mathfrak{n}}(D')$ for some finite \mathfrak{n} and some division ring D' . By the previous lemma, we must have that $\mathbb{M}_{\mathfrak{m}}(D) \cong \mathbb{M}_{\mathfrak{n}}(D')$. A comparison of the right (or left) Goldie dimensions of $\mathbb{M}_{\mathfrak{m}}(D)$ and $\mathbb{M}_{\mathfrak{n}}(D')$ yields $\mathfrak{m} = \mathfrak{n}$. By Theorem 2, we deduce that $R \in P_l(\mathfrak{m})$, i.e., (i) is satisfied. \square

EXAMPLE 3. For each finite $\mathfrak{m} > 1$, we construct a ring in the class $P_r(\mathfrak{m}) \cap P_l(1)$. Define D to be the left skew polynomial ring $F[x, \sigma]$, where F is a field and $\sigma: F \rightarrow F$ a monomorphism which is not onto. Then D is a right Ore domain which is not left Ore (by the left analogue of Proposition 0.10.4(ii)). It follows from Proposition 4 (and its left analogue) that $\mathbb{M}_{\mathfrak{m}}(D) \in P_r(\mathfrak{m}) \cap P_l(1)$ whenever $0 < \mathfrak{m} < \aleph_0$. \square

If \mathfrak{m} is chosen to be 1, then the restrictions on the left bound of primeness described in Proposition 7 fall away. (This is perhaps not surprising, because of the distinctive character of the classes $P_r(\mathfrak{m})$ for finite $\mathfrak{m} > 1$.) Indeed, we shall show presently (Examples 4, 5 and 6) that the class $P_r(1) \cap P_l(\mathfrak{n})$ is nonempty for all nonzero cardinals \mathfrak{n} . In order to produce further examples, however, we need to be able to calculate the left bound of primeness of a matrix ring $\mathbb{M}_{\mathfrak{n}}(R)$ where R is a prime ring. If \mathfrak{n} is finite, we have left-right symmetry. (To be precise, $[\mathbb{M}_{\mathfrak{n}}(R)]^{\text{opp}} \cong \mathbb{M}_{\mathfrak{n}}(R^{\text{opp}})$ whenever $0 < \mathfrak{n} < \aleph_0$.) It therefore remains only to investigate the case where \mathfrak{n} is infinite. In the sequel to Proposition 4, we remarked that the right bound of primeness of $\mathbb{M}_{\mathfrak{n}}(R)$ is, in all but one instance, determined uniquely by \mathfrak{n} and the right bound of primeness of R . By contrast, we shall

³A subset X with this property is called a *uniform insulator* for the ring R and a ring which possesses a finite uniform insulator is called *uniformly strongly prime*. We shall study such rings in §4.

see that the left bound of primeness of R has no direct bearing on the left bound of primeness of $\mathbb{M}_{\mathbf{n}}(R)$, for infinite \mathbf{n} . The latter bound turns out to depend on other (related but more complex) properties of R . We require some preliminary definitions.

Given a right R -module M , we shall call a set $X \subseteq M$ a *free subset* of M if for any family $\{r_x : x \in X\}$ of elements $r_x \in R$, all but finitely many of which are zero, we have

$$\sum_{x \in X} xr_x = 0 \Rightarrow (\forall x \in X)(r_x = 0).$$

It is easy to see that when R contains an element with zero right annihilator (e.g., when R is a ring with identity), a module M_R contains a free subset of cardinality \mathbf{m} if and only if the module $R_R^{(\mathbf{m})}$ embeds in M . Notice also that if $x \in M$ then $\{x\}$ is a free subset of M if and only if $(0 : x) = 0$.

Let R be a prime ring and $0 \neq a \in R$. If $\mathbf{n} > 0$, we define $k_l(a, \mathbf{n})$ (resp. $k_r(a, \mathbf{n})$) to be the smallest nonzero cardinal for which ${}_R(Ra)^{k_l(a, \mathbf{n})}$ (resp. $(aR)_R^{k_r(a, \mathbf{n})}$) contains a free subset of cardinality \mathbf{n} . The primeness of R guarantees that ${}_R(Ra)$ (resp. $(aR)_R$) is a faithful left (resp. right) R -module, so such a cardinal always exists. An argument similar to that used in the proof of Proposition II.6.1 shows that for an arbitrary nonzero cardinal \mathbf{n} , the module ${}_R(Ra)$ contains a subset X of cardinality \mathbf{n} for which $(0 : {}_l X) = 0$, if and only if ${}_R(Ra)^{\mathbf{n}}$ contains an element with zero left annihilator. It follows that $k_l(a, 1)$ must coincide with the minimum cardinality of a left insulator for a in R . There is therefore an intimate connection between $\{k_l(a, 1) : 0 \neq a \in R\}$ and the left bound of primeness of R . In Proposition 9, we shall establish an analogous connection between $\{k_l(a, \mathbf{n}) : 0 \neq a \in R\}$ and the left bound of primeness of $\mathbb{M}_{\mathbf{n}}(R)$.

LEMMA 8. *Let \mathbf{n} be a nonzero cardinal and R a prime ring. Suppose A is a nonzero element of $\mathbb{M}_{\mathbf{n}}(R)$ with $a = A_{\gamma\delta} \neq 0$, $\gamma, \delta < \mathbf{n}$. Then A has a left insulator in $\mathbb{M}_{\mathbf{n}}(R)$ of cardinality $k_l(a, \mathbf{n})$. Moreover, if $A_{\gamma\delta}$ is the only nonzero entry of A then A has no left insulator in $\mathbb{M}_{\mathbf{n}}(R)$ of cardinality less than $k_l(a, \mathbf{n})$.*

Proof. Let $\mathbf{k} = k_l(a, \mathbf{n})$. By hypothesis, ${}_R(Ra)^{\mathbf{k}}$ contains a free subset of cardinality \mathbf{n} , say $\{x_\alpha : \alpha < \mathbf{n}\}$. Write each x_α as $\bar{x}_\alpha a$ with $\bar{x}_\alpha \in R^{\mathbf{k}}$. Define an $\mathbf{n} \times \mathbf{k}$ matrix C over R by setting $C_{(\alpha)} = \bar{x}_\alpha$ for all $\alpha < \mathbf{n}$. For each $\beta < \mathbf{k}$, define $B(\beta) \in \mathbb{M}_{\mathbf{n}}(R)$ by $B(\beta)^{(\alpha)} = C^{(\beta)}$ if $\alpha = \gamma$ and $B(\beta)^{(\alpha)} = 0$ if $\alpha \neq \gamma$. We claim that $X = \{B(\beta) : \beta < \mathbf{k}\}$ is a left insulator for A in $\mathbb{M}_{\mathbf{n}}(R)$. Indeed, suppose that $M \in \mathbb{M}_{\mathbf{n}}(R)$ and $MXA = 0$. If $\alpha < \mathbf{n}$ then

$$\begin{aligned} (\forall \beta < \mathbf{k}) ([MB(\beta)A]_{\alpha\delta} = 0) &\Leftrightarrow (\forall \beta < \mathbf{k}) (M_{(\alpha)} \cdot [B(\beta)A]^{(\delta)} = 0) \\ &\Leftrightarrow (\forall \beta < \mathbf{k}) (M_{(\alpha)} \cdot B(\beta)^{(\gamma)} a = 0) \\ &\Leftrightarrow (\forall \beta < \mathbf{k}) (M_{(\alpha)} \cdot C^{(\beta)} a = 0) \\ &\Leftrightarrow M_{(\alpha)} C a = 0. \end{aligned}$$

Writing $M_{(\alpha)} = \{r_\lambda\}_{\lambda \in \mathbf{n}} \in R_R^{\mathbf{n}}$, we obtain

$$M_{(\alpha)} C a = \sum_{\lambda \in \mathbf{n}} r_\lambda \bar{x}_\lambda a = \sum_{\lambda \in \mathbf{n}} r_\lambda x_\lambda = 0$$

so $M_{(\alpha)} = \{0\}_{\lambda \in \mathbf{n}}$. Since α was arbitrary, this shows that $M = 0$, which establishes our claim.

Now suppose that $a = A_{\gamma\delta}$ is the only nonzero entry of A . We lose no generality in supposing that $\gamma = \delta = 0$. Let Y be any subset of $\mathbb{M}_{\mathbf{n}}(R)$ of cardinality $\mathbf{m} < \mathbf{k} = k_l(a, \mathbf{n})$ and let C be an $\mathbf{n} \times \mathbf{m}$ matrix over R whose set of columns is $\{B^{(0)} : B \in Y\}$. Since ${}_R(Ra)^{\mathbf{m}}$ contains no free subset of cardinality \mathbf{n} , there is a nonzero $z = \{r_\alpha\}_{\alpha \in \mathbf{n}}$ in $R_R^{\mathbf{n}}$ such that $\sum_{\alpha \in \mathbf{n}} r_\alpha C_{(\alpha)} a = 0$. Define a matrix $M \in \mathbb{M}_{\mathbf{n}}(R)$ by $M_{(\alpha)} = 0$ if $\alpha \neq 0$ and $M_{(\alpha)} = z$ if $\alpha = 0$. Then $MCa = 0$, so $MYA = 0$. Thus A has no left insulator of cardinality less than \mathbf{k} . \square

The following result is therefore immediate.

PROPOSITION 9. *Let \mathbf{n} be a nonzero cardinal and R a prime ring. Set $\mathbf{k} = \sup \{k_l(a, \mathbf{n}) : 0 \neq a \in R\}$. Then:*

- (i) *if $\mathbf{k} < \aleph_0$, or \mathbf{k} is a limit cardinal and $k_l(a, \mathbf{n}) < \mathbf{k}$ for all nonzero $a \in R$, then $\mathbb{M}_{\mathbf{n}}(R) \in P_l(\mathbf{k})$;*
- (ii) *if \mathbf{k} is an infinite successor cardinal, or \mathbf{k} is a limit cardinal and $k_l(a, \mathbf{n}) = \mathbf{k}$ for some nonzero $a \in R$, then $\mathbb{M}_{\mathbf{n}}(R) \in P_l(\mathbf{k}^+)$.* \square

PROPOSITION 10. *Let \mathbf{m}, \mathbf{n} be nonzero cardinals with \mathbf{n} infinite. Set $\mathbf{k} = \sup \{k_l(a, \mathbf{n}) : 0 \neq a \in R\}$ and suppose that $R \in P_l(\mathbf{m})$. Then:*

- (i) *if $\mathbf{m} < \aleph_0$ then $\mathbf{m} \leq \mathbf{k} \leq \mathbf{n}$;*
- (ii) *if \mathbf{m} is a limit cardinal then $\mathbf{m} \leq \mathbf{k} \leq \mathbf{nm}$;*
- (iii) *if \mathbf{m} is an infinite successor cardinal, say $\mathbf{m} = \mathbf{p}^+$, then $\mathbf{p} \leq \mathbf{k} \leq \mathbf{np}$.*

Proof. Let $0 \neq a \in R$. Since ${}_R(Ra)^{k_l(a, 1)}$ contains a singleton free subset, it follows that $({}_R(Ra)^{k_l(a, 1)})^{\mathbf{n}} \cong {}_R(Ra)^{\mathbf{n} \cdot k_l(a, 1)}$ contains a free subset of cardinality \mathbf{n} . We may conclude that $k_l(a, 1) \leq k_l(a, \mathbf{n}) \leq \mathbf{n} \cdot k_l(a, 1)$, whence

$$(1) \quad \sup \{k_l(a, 1) : 0 \neq a \in R\} \leq \mathbf{k} \leq \sup \{\mathbf{n} \cdot k_l(a, 1) : 0 \neq a \in R\} = \mathbf{n} \cdot \sup \{k_l(a, 1) : 0 \neq a \in R\}.$$

Inasmuch as $k_l(a, 1)$ coincides with the minimum cardinality of a left insulator for a nonzero $a \in R$, it is easy to see that:

- (i) *if $\mathbf{m} < \aleph_0$ then $\sup \{k_l(a, 1) : 0 \neq a \in R\} = \mathbf{m}$ and $\mathbf{n} \cdot \sup \{k_l(a, 1) : 0 \neq a \in R\} = \mathbf{n}$;*
- (ii) *if \mathbf{m} is a limit cardinal then $\sup \{k_l(a, 1) : 0 \neq a \in R\} = \mathbf{m}$;*
- (iii) *if $\mathbf{m} = \mathbf{p}^+$ is the successor of an infinite cardinal \mathbf{p} then $\sup \{k_l(a, 1) : 0 \neq a \in R\} = \mathbf{p}$.*

The result is therefore a consequence of the inequality (1). \square

Now suppose that D is a division ring and that \mathbf{n} is an infinite cardinal. In this case, $k_l(a, \mathbf{n}) = k_l(1_D, \mathbf{n})$ for all nonzero $a \in D$, so $\mathbf{k} = \sup \{k_l(a, \mathbf{n}) : 0 \neq a \in D\} = k_l(1_D, \mathbf{n})$. It is easy to see that \mathbf{k} coincides with the smallest cardinal (necessarily infinite) for which $\dim({}_D D^{\mathbf{k}}) \geq \mathbf{n}$. Also, for any infinite cardinal \mathbf{m} , we have $\dim({}_D D^{\mathbf{m}}) = |D|^{\mathbf{m}}$. (A proof of this equality is given in the

Appendix.) We may conclude that \mathbf{k} is the smallest infinite cardinal for which $|D|^{\mathbf{k}} \geq \mathbf{n}$. Notice also that since $\mathbf{k} = \sup \{k_l(a, \mathbf{n}) : 0 \neq a \in D\} = k_l(1_D, \mathbf{n})$, the conditions of Proposition 9(ii) hold, so that $\mathbb{M}_{\mathbf{n}}(D) \in P_l(\mathbf{k}^+)$. Clearly, we may draw the conclusions of the following proposition.

PROPOSITION 11. *Let \mathbf{n} be an infinite cardinal and D a division ring. If \mathbf{k} is the smallest infinite cardinal for which $|D|^{\mathbf{k}} \geq \mathbf{n}$, then $\mathbb{M}_{\mathbf{n}}(D) \in P_l(\mathbf{k}^+)$. In particular:*

- (i) *if $|D| \geq \mathbf{n}$ then $\mathbf{k} = \aleph_0$, so $\mathbb{M}_{\mathbf{n}}(D) \in P_l(\aleph_1)$;*
- (ii) *if $|D| < \mathbf{n}$ and \mathbf{n} is a strong limit cardinal then $\mathbf{k} = \mathbf{n}$, so $\mathbb{M}_{\mathbf{n}}(D) \in P_l(\mathbf{n}^+)$;*
- (iii) *assuming the Generalized Continuum Hypothesis, if D is a countable division ring and \mathbf{n} is a successor cardinal then $\mathbb{M}_{\mathbf{n}}(D) \in P_l(\mathbf{n})$. □*

Also of interest is the case where D is chosen to be a domain which is not left Ore.

PROPOSITION 12. *Let \mathbf{n} be an infinite cardinal. If D is a domain which contains an independent family of \mathbf{n} nonzero left ideals then $\mathbb{M}_{\mathbf{n}}(D) \in P_l(1)$.*

Proof. Let $\{I_{\alpha} : \alpha \in \mathbf{n}\}$ be an independent family of \mathbf{n} nonzero left ideals of D . If, for each α , we choose a nonzero $x_{\alpha} \in I_{\alpha}$, then $\{x_{\alpha} : \alpha \in \mathbf{n}\}$ is a free subset of ${}_D D$. Consequently, $k_l(1_D, \mathbf{n}) = 1$. More generally, if $0 \neq a \in D$ then $\{I_{\alpha} a : \alpha \in \mathbf{n}\}$ is an independent family of \mathbf{n} nonzero submodules of ${}_D(Da)$, from which it follows similarly that $k_l(a, \mathbf{n}) = 1$. Therefore $\mathbf{k} = \sup \{k_l(a, \mathbf{n}) : 0 \neq a \in D\} = 1$, and the result follows from Proposition 9(i). □

Left analogues of Example 3 and Examples 4, 5 and 6 below show that the class $P_r(1) \cap P_l(\mathbf{n})$ is nonempty for all nonzero cardinals \mathbf{n} .

EXAMPLE 4. Let \mathbf{n} be an infinite cardinal. We construct a ring belonging to the class $P_r(\mathbf{n}^+) \cap P_l(1)$. Let F be a field and define D to be the free (associative) F -algebra on \mathbf{n} noncommuting indeterminates $\{x_{\alpha} : \alpha \in \mathbf{n}\}$. Certainly, D is a domain, and it is easy to see that the family $\{Dx_{\alpha} : \alpha \in \mathbf{n}\}$ of nonzero left ideals of D is independent. By Propositions 12 and 4, $\mathbb{M}_{\mathbf{n}}(D) \in P_r(\mathbf{n}^+) \cap P_l(1)$. □

EXAMPLE 5. We construct a ring R such that $R \in P_r(\aleph_0) \cap P_l(1)$. Let D be a domain which is right but not left Ore (e.g., the ring D in Example 3). As we observed in Example 3, we must have $\mathbb{M}_{\mathbf{m}}(D) \in P_r(\mathbf{m}) \cap P_l(1)$ whenever $0 < \mathbf{m} < \aleph_0$. Following the ring construction of Example 1, we define a subring R of $\mathbb{M}_{\aleph_0}(D)$ by setting

$$R = \bigcup_{\alpha \in \aleph_0} \Theta_{\alpha}[\mathbb{M}_{2^{\alpha}}(D)]$$

where, for each $\alpha \in \aleph_0$, $\Theta_{\alpha} : \mathbb{M}_{2^{\alpha}}(D) \rightarrow \mathbb{M}_{\aleph_0}(D)$ is the diagonal embedding

$$\Theta_\alpha(A) = \begin{bmatrix} A & 0 \\ 0 & A \\ & & \ddots \end{bmatrix} \in \mathbb{M}_{\aleph_0}(D) \quad (A \in \mathbb{M}_{2^\alpha}(D)).$$

The argument used in Example 1 shows that if $0 \neq A \in \mathbb{M}_{2^\alpha}(D) \subseteq R$ and X is a right (resp. left) insulator for A in $\mathbb{M}_{2^\alpha}(D)$ then the image of X in R is a right (resp. left) insulator for A in R . Since $\mathbb{M}_{2^\alpha}(D) \in P_r(2^\alpha) \cap P_l(1)$ for all $\alpha \in \aleph_0$, it follows that $R \in \bar{P}_r(\aleph_0) \cap P_l(1)$. If $0 \neq d \in D$, it is easily shown that the element $E_{0,0}(d) \in \mathbb{M}_{2^\alpha}(D)$ has no right insulator in R of cardinality less than 2^α . Consequently, $R \in P_r(\aleph_0) \cap P_l(1)$. \square

EXAMPLE 6. Let \mathfrak{m} be an uncountable limit cardinal. We construct a ring R such that $R \in P_r(\mathfrak{m}) \cap P_l(1)$. Let D be a domain which contains an independent family of \mathfrak{m} nonzero left ideals (see Example 4 for such a D). It follows from Propositions 12 and 4 that $\mathbb{M}_{\mathfrak{k}}(D) \in P_r(\mathfrak{k}^+) \cap P_l(1)$ whenever $\aleph_0 \leq \mathfrak{k} < \mathfrak{m}$. Following Example 2, on this occasion, we define a subring R of $\mathbb{M}_{\mathfrak{m}}(D)$ by setting

$$R = \bigcup_{\aleph_0 \leq \mathfrak{k} < \mathfrak{m}} \Theta_{\mathfrak{k}}[\mathbb{M}_{\mathfrak{k}}(D)]$$

where, for each infinite cardinal $\mathfrak{k} < \mathfrak{m}$, $\Theta_{\mathfrak{k}}: \mathbb{M}_{\mathfrak{k}}(D) \rightarrow \mathbb{M}_{\mathfrak{m}}(D)$ is the “diagonal” embedding described in Example 2. Just as in Example 2, the embeddings $\Theta_{\mathfrak{k}}$ send right (resp. left) insulators to right (resp. left) insulators so the same arguments yield that $R \in P_r(\mathfrak{m}) \cap P_l(1)$. \square

We now investigate the intersections $P_r(\mathfrak{m}) \cap P_l(\mathfrak{n})$, where $\mathfrak{m}, \mathfrak{n}$ are both infinite. The case where $\mathfrak{m} = \aleph_0$ deserves special comment. Examples 1 and 5 show that the classes $P_r(\aleph_0) \cap P_l(\aleph_0)$ and $P_r(\aleph_0) \cap P_l(1)$ are nonempty. A longstanding question of Goodearl, Handelman and Lawrence [GHL74, p121] asks whether every right unbounded strongly prime ring (i.e., member of $P_r(\aleph_0)$) is left strongly prime (i.e., is a member of $\bar{P}_l(\aleph_0)$). Since $P_l(\mathfrak{n}) \subseteq \bar{P}_r(\mathfrak{n})$ for all finite $\mathfrak{n} > 1$ (by the left analogue of Proposition 7), this question may be rephrased as: *is every member of $P_r(\aleph_0)$ necessarily a member of $P_l(\aleph_0)$ or a member of $P_l(1)$?* An affirmative answer would place a further constraint on the values of $\mathfrak{m}, \mathfrak{n}$ for which $P_r(\mathfrak{m}) \cap P_l(\mathfrak{n}) \neq \emptyset$ in the case where $\mathfrak{m} = \aleph_0$.

EXAMPLE 7. Let \mathfrak{m} and \mathfrak{n} be infinite cardinals with $\mathfrak{m} > \mathfrak{n}$. We construct a ring S such that $S \in P_r(\mathfrak{m}) \cap P_l(\mathfrak{n}^+)$. It follows from Examples 4 and 6 that there exists a ring $R \in P_r(\mathfrak{m}) \cap P_l(1)$. In the discussion following Proposition 5, we remarked that Proposition 4 remains valid if row-finite matrices are replaced by row- and column-finite matrices. It follows from this modification of Proposition 4 that if S is the ring $\mathbb{M}_{\mathfrak{n}}^*(R)$ then $S \in P_r(\mathfrak{m})$, and from the left analogue of the same result that $S \in P_l(\mathfrak{n}^+)$. Thus $S \in P_r(\mathfrak{m}) \cap P_l(\mathfrak{n}^+)$. \square

EXAMPLE 8. Let \mathfrak{m} and \mathfrak{n} be cardinals such that $\mathfrak{m} > \mathfrak{n}$ and \mathfrak{n} is an uncountable limit cardinal. We construct a ring S such that $S \in P_r(\mathfrak{m}) \cap P_l(\mathfrak{n})$. It follows from the previous example

that there exists a ring R such that $M_{\mathbf{k}}^*(R) \in P_r(\mathbf{m}) \cap P_l(\mathbf{k}^+)$ whenever $\aleph_0 \leq \mathbf{k} < \mathbf{n}$. As in Example 2, each $M_{\mathbf{k}}^*(R)$ is embeddable, as a subring, in $M_{\mathbf{n}}^*(R)$ and we may consider the subring S of $M_{\mathbf{n}}^*(R)$ consisting of the union of the images of the $M_{\mathbf{k}}^*(R)$'s. Similar arguments yield that $S \in P_r(\mathbf{m}) \cap P_l(\mathbf{n})$. \square

We are now able to state the following refinement of Propositions 5 and 5a.

THEOREM 13. (i) *The classes $P_r(\mathbf{m}) \cap P_l(\mathbf{m})$, $P_r(1) \cap P_l(\mathbf{n})$ and $P_r(\mathbf{m}) \cap P_l(1)$ are nonempty for all nonzero cardinals \mathbf{m} and \mathbf{n} .*

(ii) *The class $P_r(\mathbf{m}) \cap P_l(\mathbf{n})$ is nonempty for all uncountable cardinals \mathbf{m} and \mathbf{n} .*

Thus, beyond the constraint $P_r(\mathbf{m}) \subseteq P_l(\mathbf{m}) \cup P_l(1)$ for finite $\mathbf{m} > 1$ (established in Proposition 7) and the aforementioned conjecture $P_r(\aleph_0) \stackrel{?}{\subseteq} P_l(\aleph_0) \cup P_l(1)$ of Goodearl, Handelman and Lawrence, there are no further constraints on the existence of rings that are right prime of bound \mathbf{m} and left prime of bound \mathbf{n} , for $\mathbf{m}, \mathbf{n} > 0$.

Proof. (i) follows from Proposition 5a, the constructions in Examples 3, 4, 5 and 6, and the availability of opposite rings. In view of (i) and the consideration of opposite rings, we may assume without loss of generality in (ii) that $\mathbf{m} > \mathbf{n}$; then Examples 7 and 8 complete the proof. \square

A rather simple application of the classification of prime rings under discussion is the following. It is well known that a ring R with identity is isomorphic to a left full linear ring if and only if R is prime, regular and left self-injective, with nonzero socle (Theorem 0.9.4). If we consider Proposition 4, it is immediately clear that we may sharpen this characterization as follows:

PROPOSITION 14. *The following conditions are equivalent for any ring S with identity and any finite (resp. infinite) nonzero cardinal \mathbf{m} :*

- (i) *S is a regular left self-injective ring with nonzero socle and $S \in P_r(\mathbf{m})$ (resp. $S \in P_r(\mathbf{m}^+)$);*
- (ii) *S is isomorphic to the ring of all linear transformations (written on the right) of an \mathbf{m} -dimensional left vector space over some division ring.* \square

Our next goal is to prove Theorem 16, which asserts that if \mathbf{m} is an infinite cardinal, then the property of being an element of $\bar{P}_r(\mathbf{m})$ is Morita invariant within the class of all rings with identity.

LEMMA 15. *Let \mathbf{m} be a nonzero cardinal. If e is an idempotent element of a ring R and $R \in \bar{P}_r(\mathbf{m})$, then the subring $eRe := \{ere : r \in R\} \in \bar{P}_r(\mathbf{m})$.*

Proof. Let $0 \neq ere \in eRe$ and let X be a right insulator for ere in R with $|X| < \mathbf{m} + 1$. It is easily checked that eXe is a right insulator for ere in eRe . Since $|eXe| \leq |X| < \mathbf{m} + 1$ this shows that $eRe \in \bar{P}_r(\mathbf{m})$. \square

THEOREM 16. *Let R and S be Morita equivalent rings with identity and let \mathfrak{m} be an infinite cardinal. If R is an element of $\bar{P}_r(\mathfrak{m})$ then so is S .*

Proof. By Theorem 0.6.2, $S \cong eM_{\mathfrak{n}}(R)e$ for some finite nonzero cardinal \mathfrak{n} and idempotent $e \in M_{\mathfrak{n}}(R)$. The fact that $S \in \bar{P}_r(\mathfrak{m})$ follows from Proposition 4 and Lemma 15. \square

If D is a division ring and \mathfrak{m} a finite nonzero cardinal then D and $M_{\mathfrak{m}}(D)$ are Morita equivalent (by Theorem 0.6.3) yet $D \in P_r(1)$ and $M_{\mathfrak{m}}(D) \in P_r(\mathfrak{m})$. This shows that the requirement that \mathfrak{m} be infinite cannot be dispensed with in Theorem 16. It is also natural to ask whether Theorem 16 can be extended by dropping the requirement that the rings R and S have identity, and replacing the phrase “Morita equivalent” with “Morita $*$ -equivalent”. (The notion of Morita $*$ -equivalent rings is defined in §5 of Chapter II.) The next lemma and example show that this is not possible unless some restriction is placed on the rings R and S .

We remind the reader that R^* denotes the Dorroh Extension of a ring R .

LEMMA 17. *If R is an arbitrary ring then $(R^*)^* \cong (R \times \mathbb{Z})^*$.*

Proof. For notational convenience we shall write the elements of $(R^*)^*$ and $(R \times \mathbb{Z})^*$ as ordered triples (r, m, n) with $r \in R$ and $m, n \in \mathbb{Z}$. In both rings addition is defined componentwise. In $(R^*)^*$ multiplication is defined by

$$(r_1, m_1, n_1)(r_2, m_2, n_2) = (r_1r_2 + r_1m_2 + r_2m_1 + r_1n_2 + r_2n_1, m_1m_2 + m_1n_2 + m_2n_1, n_1n_2)$$

while in $(R \times \mathbb{Z})^*$ multiplication is defined by

$$(r_1, m_1, n_1)(r_2, m_2, n_2) = (r_1r_2 + r_1n_2 + r_2n_1, m_1m_2 + m_1n_2 + m_2n_1, n_1n_2).$$

A routine check shows that the map $\theta: (R^*)^* \rightarrow (R \times \mathbb{Z})^*$ defined by $\theta(r, m, n) = (r, -m, n + m)$ for all $(r, m, n) \in (R^*)^*$, is a ring isomorphism. \square

Recall that rings R and S are said to be Morita $*$ -equivalent if the categories $\text{Mod-}R$ and $\text{Mod-}S$ are equivalent. As noted in the introduction to §5 of Chapter II, R and S are Morita $*$ -equivalent if and only if R^* and S^* are Morita equivalent in the classical sense. It is an immediate consequence of Lemma 17, therefore, that R^* and $R \times \mathbb{Z}$ are Morita $*$ -equivalent for all rings R . Whereas $R \times \mathbb{Z}$ is clearly never prime (unless $R = 0$), the following example shows that R may be chosen such that R^* is prime. (The reader will observe that if R is prime then $R \times \mathbb{Z}$, although not prime, is at least semiprime. This is not coincidental; indeed, it is not difficult to show (using Proposition 0.5.2(ii)) that, in contrast to the property of being prime, the property of being semiprime is Morita $*$ -equivalent.)

EXAMPLE 9. We construct a ring R such that R^* is prime. Let R be a prime ring such that $\text{cen } R = 0$ and $\text{char } R = 0$. (For example, let F be a field with $\text{char } F = 0$ and take R to be

the ideal of the free (associative) F -algebra $F\langle x, y \rangle$ on two noncommuting indeterminates x and y , generated by x and y .) Suppose I and J are ideals of R^* such that $IJ = 0$. Since $(R \cap I)(R \cap J) = 0$ and R is prime, we may assume without loss of generality that $R \cap I = 0$. Let $t = 1_{R^*} \cdot s + 1_{R^*} \cdot m \in I$. Since $tR \cup Rt \subseteq R \cap I = 0$, we must have that $sr + rm = rs + rm = 0$ for all $r \in R$. It follows that $sr = rs$ for all $r \in R$, i.e., $s \in \text{cen } R = 0$. Hence $rm = 0$ for all $r \in R$. But $\text{char } R = 0$, so $m = 0$. Thus $t = 0$. This shows that $I = 0$ and so R^* must be prime. \square

Notwithstanding the limitations imposed by the previous example, we demonstrate in the next theorem that the property of being an element of $\bar{P}_r(\mathfrak{m})$ ($\mathfrak{m} \geq \aleph_0$) is Morita $*$ -equivalent within a large class of rings without identity. Its proof requires preparation, however, and will therefore be postponed until §2 of Chapter IV.

THEOREM 18. *Let R and S be Morita $*$ -equivalent prime rings and let \mathfrak{m} be an infinite cardinal. If R is an element of $\bar{P}_r(\mathfrak{m})$ then so is S . In other words, the property of being an element of $\bar{P}_r(\mathfrak{m})$ is Morita $*$ -invariant within the class of all prime rings.* \square

§2. SUBRINGS AND OVERRINGS.

Although §1 serves the purpose of supplying us amply with examples of prime rings having various unequal right and left bounds of primeness, it also attempts to address the equally important task of establishing connections between the left and right bounds of primeness of a prime ring R and those of matrix rings over R . We have seen, for example, that in some instances the values of both of the respective bounds coincide. The perspective offered by §1 therefore prompts the following general questions. Which prime overrings of a prime ring R have the same bound of primeness as R ? Conversely, which prime subrings of a prime ring S inherit the bound of primeness of S ? In the present section of this chapter, we provide some answers to these questions.

PROPOSITION 1. *Let \mathfrak{m} be a nonzero cardinal. If R is a subring of a ring S and S_R is an essential extension of R_R and $R \in P_r(\mathfrak{m})$ then $S \in \bar{P}_r(\mathfrak{m})$.*

Proof. Let $0 \neq x \in S$. If $xR \neq 0$ then $0 \neq s = xr \in R$ for some $r \in R$. Let X be a right insulator for s in R with $|X| < \mathfrak{m} + 1$. Set $Y = rX$ and suppose that $xYt = 0$ with $0 \neq t \in S$. Then $\{ta + tm : a \in R, m \in \mathbb{Z}\}$ is a nonzero submodule of S_R , so $0 \neq ta + tm \in R$ for some $a \in R, m \in \mathbb{Z}$. Now $sX(ta + tm) = xY(ta + tm) = 0$, a contradiction. Thus Y is a right insulator for x in S and $|Y| \leq |X| < \mathfrak{m} + 1$.

If, on the other hand, $xR = 0$, then $\{xm : m \in \mathbb{Z}\}$ is a nonzero submodule of S_R , so $0 \neq s = xm \in R$ for some $m \in \mathbb{Z}$. An argument similar to the one above shows that if X is a right insulator for s in R with $|X| < \mathfrak{m} + 1$, then X is also a right insulator for x in S .

In either case, therefore, x has a right insulator in S of cardinality less than $\mathfrak{m} + 1$. □

It follows from Proposition 1 that if \mathfrak{m} is any nonzero cardinal and R is a ring with identity such that $R \in P_r(\mathfrak{m})$, then $Q_{\max}(R_R)$, the maximal right ring of quotients of R , is an element of $\bar{P}_r(\mathfrak{m})$. If R is chosen to be the right unbounded strongly prime ring of Example 1.1, it can be shown that $Q_{\max}(R_R) \in P_r(1)$ (see [GH75, Example(e), p831]). Consequently, Proposition 1 cannot be sharpened by replacing $\bar{P}_r(\mathfrak{m})$ with $P_r(\mathfrak{m})$. We also point out that the converse of Proposition 1 is not valid. The ring $S = M_2(D)$, D a division ring, is right (and left) prime of bound 2 and has a subring R consisting of all upper triangular 2×2 matrices over D , such that S_R is an essential extension of R_R but R is not even semiprime. A partial converse to Proposition 1 may be obtained, however, if the requirement “ S_R is an essential extension of R_R ” is strengthened to “ R is a right order in S ”. Viola-Prioli [Vio75, Lemma 2.11, p280] proved that in this case, if S is right strongly prime then so is R . Any attempt to generalize Viola-Prioli’s result to a larger class of prime rings is certain to be limited by the next result, which is due to O’Meara.

Recall that a right R -module M is said to be *uniform* if every nonzero submodule of M is essential in M . If M contains a nonzero uniform submodule then the *uniform dimension* of M is defined to be the cardinality of any maximal independent set of nonzero uniform submodules of M (see e.g., [Miy65, Theorem 1.10, p163]).

PROPOSITION 2. [O’Me73, Corollary 3.3, p181] *If R is a left full linear ring (i.e., $R \cong M_{\mathfrak{m}}(D)$ for some nonzero cardinal \mathfrak{m} and division ring D), then the following assertions are equivalent:*

- (i) *every right order in R is prime;*
- (ii) *R_R has countable uniform dimension.* □

If \mathfrak{m} is an uncountable cardinal and D a division ring then the ring $R = M_{\mathfrak{m}}(D)$ clearly contains an independent set \mathcal{F} of minimal nonzero right ideals such that $|\mathcal{F}| > \aleph_0$. It follows that R_R has uncountable uniform dimension. By Proposition 2, there exists a right order in R which is not prime.

It is clear from the above that only rather special subrings of a prime ring S can be expected to be prime and to inherit the bound of primeness of S . This motivates the following definition.

Let \mathfrak{m} be an infinite cardinal and R a subring of a ring S with identity. We say that R is a *right \mathfrak{m} -order* in S if for every subset X of S with $|X| < \mathfrak{m}$, there is a unit u of S , contained in R , such that $Xu := \{xu : x \in X\} \subseteq R$. Obviously, if \mathfrak{n} is another infinite cardinal with $\mathfrak{m} < \mathfrak{n}$, then any right \mathfrak{n} -order in S is a right \mathfrak{m} -order in S . Notice also that if $\mathfrak{m} = \aleph_0$ then a right \mathfrak{m} -order in S is just a right order in S in the usual sense, by Lemma 0.7.2. As an obvious consequence, we have that for any infinite cardinal \mathfrak{m} , if R is a right \mathfrak{m} -order in S then R_R is an essential submodule of S_R . (This fact will be needed repeatedly.)

THEOREM 3. *Let \mathbf{k} and \mathbf{m} be nonzero cardinals with $\mathbf{m} \geq \mathbf{k}, \aleph_0$. If R is a right \mathbf{m} -order in a ring S then $R \in P_r(\mathbf{k})$ if and only if $S \in P_r(\mathbf{k})$.*

Proof. Suppose $S \in P_r(\mathbf{k})$ and let $0 \neq r \in R$. Choose a right insulator X for r in S such that $|X| < \mathbf{k} + 1$. Since R is a right \mathbf{m} -order in S and $|X| < \mathbf{k} + 1 \leq \mathbf{m}$, there exists a unit u of S in R such that $Xu \subseteq R$. If $t \in R$ and $rXut = 0$ then $ut = 0$ so $t = 0$; hence the right annihilator of rXu in R is zero. Since $Xu \subseteq R$ and $|Xu| = |X| < \mathbf{k} + 1$, this shows that $R \in \bar{P}_r(\mathbf{k})$. But R_R is an essential submodule of S_R (because R is a right \mathbf{m} -order in S), and therefore we cannot have $R \in P_r(\mathbf{n})$ for some $\mathbf{n} < \mathbf{k}$, by Proposition 1. Thus $R \in P_r(\mathbf{k})$.

Conversely, suppose $R \in P_r(\mathbf{k})$. Again, by Proposition 1, we must have $S \in P_r(\mathbf{n})$ for some $\mathbf{n} \leq \mathbf{k}$. The above argument shows that $R \in P_r(\mathbf{n})$, hence $\mathbf{n} = \mathbf{k}$ and $S \in P_r(\mathbf{k})$. \square

Notice that if \mathbf{m} is chosen to be \aleph_0 in the above argument, we obtain Viola-Prioli's result: a right order in a right strongly prime ring is right strongly prime.

We shall now describe conditions on a ring under which the notions of order and \mathbf{m} -order coincide. Recall that a *right Gabriel filter* on a ring R is a right topologizing filter \mathcal{F} on R (in the sense of Chapter II, §2) with the additional property that if $I \leq R_R$ and there exists some $J \in \mathcal{F}$ such that $(I :_r a) \in \mathcal{F}$ for all $a \in J$, then $I \in \mathcal{F}$. Recall also that a *right denominator set* for a ring R is a (nonempty) multiplicatively closed set S of regular elements of R having the *common right multiple property*: for each $r \in R$ and $s \in S$, there exist $r_1 \in R$ and $s_1 \in S$ such that $rs_1 = sr_1$. Moreover, if R is a right order in a ring T and S is the set of all units of T contained in R , then S is a right denominator set for R and T a right ring of fractions of R with respect to S . (We refer the reader to §8 of Chapter 0 for further details on denominator sets and rings of fractions.)

The next result appears to be well known.

PROPOSITION 4. *If S is a right denominator set in a ring R then the set of right ideals $\{A \leq R_R : A \cap S \neq \emptyset\}$ is a right Gabriel filter on R .*

Proof. Let $\mathcal{F} = \{A \leq R_R : A \cap S \neq \emptyset\}$. It is obvious that if $A \in \mathcal{F}$ and $A \subseteq B \leq R_R$, then $B \in \mathcal{F}$. If $A, B \in \mathcal{F}$, we may choose $s \in A \cap S$ and $t \in B \cap S$. Since S is a right denominator set, there exist $r_1 \in R$ and $s_1 \in S$ such that $sr_1 = ts_1$. Clearly, $sr_1 = ts_1 \in A \cap B \cap S$, so $A \cap B \in \mathcal{F}$. If $a \in R$ then there exist $r_2 \in R$ and $s_2 \in S$ such that $sr_2 = as_2$. Inasmuch as $sr_2 \in A$ we must have $s_2 \in (A :_r a) \cap S$, whence $(A :_r a) \in \mathcal{F}$. This shows that $\mathcal{F} \in \text{Fil-}R$. Finally, suppose that $A \leq R_R$, $B \in \mathcal{F}$ and $(A :_r t) \in \mathcal{F}$ for all $t \in B$. Take $t' \in B \cap S$. By hypothesis, $(A :_r t') \cap S \neq \emptyset$, so $t's' \in A$ for some $s' \in S$. Therefore $t's' \in A \cap S$, so $A \in \mathcal{F}$. Thus \mathcal{F} is a Gabriel filter. \square

If S is a right denominator set in a ring R , we shall denote the Gabriel filter $\{A \leq R_R : A \cap S \neq \emptyset\}$ by \mathcal{F}_S .

PROPOSITION 5. *Let \mathfrak{m} be an infinite cardinal. Let R be a right order in a ring T (with identity) and let S be the right denominator set of all units of T contained in R . Then the following conditions are equivalent:*

- (i) R is a right \mathfrak{m} -order in T ;
- (ii) the right Gabriel filter \mathfrak{F}_S is \mathfrak{m} -Jansian (i.e., \mathfrak{F}_S is closed under intersections of fewer than \mathfrak{m} right ideals).

Proof. (i) \Rightarrow (ii): Let $\mathcal{A} \subseteq \mathfrak{F}_S$ with $|\mathcal{A}| < \mathfrak{m}$. Choose $s_A \in A \cap S$ for each $A \in \mathcal{A}$. By (i), there exists $u \in S$ with $s_A^{-1}u \in R$ for all $A \in \mathcal{A}$, i.e., $u = s_A r_A$ for suitable $r_A \in R$, for all $A \in \mathcal{A}$. Then $u \in \bigcap \mathcal{A}$, so $\bigcap \mathcal{A} \in \mathfrak{F}_S$. Thus \mathfrak{F}_S is \mathfrak{m} -Jansian.

(ii) \Rightarrow (i): Let $\{b_i s_i^{-1} : i \in \Gamma\}$ be a subfamily of T with $b_i \in R$, $s_i \in S$ for all $i \in \Gamma$ and $|\Gamma| < \mathfrak{m}$. Consider the right ideals $s_i R$ of R . Since $s_i^2 \in s_i R \cap S$ for all $i \in \Gamma$, it follows that $s_i R \in \mathfrak{F}_S$ for all $i \in \Gamma$. By (ii), \mathfrak{F}_S is \mathfrak{m} -Jansian, so $\bigcap_{i \in \Gamma} s_i R \in \mathfrak{F}_S$. Let $t \in (\bigcap_{i \in \Gamma} s_i R) \cap S$. Then $t = s_i r_i$ for suitable $r_i \in R$ and all $i \in \Gamma$. Therefore $(b_i s_i^{-1})t = (b_i s_i^{-1})s_i r_i = b_i r_i \in R$ for all $i \in \Gamma$. Thus R is a right \mathfrak{m} -order in T . \square

We know from Chapter II, §3 (see remarks preceding Proposition II.3.8) that when dealing with \mathfrak{m} -Jansian topologizing filters (and \mathfrak{m} -Jansian torsion preradicals), we may as well confine our attention to regular cardinals \mathfrak{m} . Indeed, it is easily shown that if \mathfrak{m} is an infinite cardinal then a right topologizing filter \mathfrak{F} is closed under intersections of fewer than \mathfrak{m} right ideals if and only if \mathfrak{F} is closed under intersections of fewer than \mathfrak{n} right ideals, where \mathfrak{n} is the smallest regular cardinal not less than \mathfrak{m} . (This \mathfrak{n} is \mathfrak{m} if \mathfrak{m} is regular, and is \mathfrak{m}^+ if \mathfrak{m} is singular.) In view of the above result, therefore, we lose no generality in assuming that the cardinal \mathfrak{m} is regular whenever R is a right \mathfrak{m} -order in a ring T .

The next example shows that for any regular cardinals $\mathfrak{m}, \mathfrak{n}$, one may construct a ring R and overrings T_1, T_2 of R such that R is a right \mathfrak{m} -order in T_1 and a right \mathfrak{n} -order in T_2 .

EXAMPLE 1. Let \mathfrak{m} and \mathfrak{n} be regular cardinals and let G_1 and G_2 be linearly ordered abelian groups whose underlying ordered sets have cofinalities \mathfrak{m} and \mathfrak{n} , respectively. (For example, if the integers are considered as a linearly ordered group $\langle \mathbb{Z}; +, -, 0; \leq \rangle$, we may take $G_1 = \mathbb{Z}^{(\mathfrak{m})}$ and $G_2 = \mathbb{Z}^{(\mathfrak{n})}$, ordered antilexicographically. See Chapter I, §4 for the definition of this order.) Consider the abelian group $H = G_2 \oplus G_1$, also ordered antilexicographically. By Corollary I.2.2, there exists a commutative chain domain R such that the strictly ordered monoid (with respect to ideal multiplication and reversed set inclusion) of nonzero principal ideals of R is isomorphic to the positive cone H^+ of H . We shall use \leq to denote the antilexicographic order on H , as well as its restrictions to subsets of H , such as H^+ . It is easily checked that if X is a cofinal subset of the order reduct of G_1 then $\{(0, x) : x \in X\}$ is a cofinal subset of $\langle H; \leq \rangle$. On the other hand, it is equally obvious that if Y is a cofinal subset of $\langle H; \leq \rangle$ then the projection of Y onto G_1 is cofinal in the order reduct of

G_1 . Since the order reduct of G_1 has cofinality \mathfrak{m} we may conclude that the chains $\langle H; \leq \rangle$ and $\langle H^+; \leq \rangle$ both have cofinality \mathfrak{m} .

For notational convenience, we shall write $[g]$ for the principal ideal of R corresponding to an element g of H^+ . Define $S_1 = R \setminus \{0\}$ and $S_2 = R \setminus P$, where

$$P = \bigcup \{[(g_2, g_1)] : g_2 \in G_2, g_1 \in G_1, g_1 > 0\},$$

which is a prime ideal of R . Note that S_1 and S_2 are both multiplicatively closed (hence denominator) subsets of R , that RS_1^{-1} is the field of quotients of R and that RS_2^{-1} is the localization of R at the prime ideal P . Clearly, \mathfrak{F}_{S_1} consists precisely of the nonzero (right) ideals of R . We claim that \mathfrak{F}_{S_1} is \mathfrak{m} -Jansian. Let $\mathcal{A} \subseteq \mathfrak{F}_{S_1}$ with $|\mathcal{A}| < \mathfrak{m}$. We need to show that $\bigcap \mathcal{A} \in \mathfrak{F}_{S_1}$, i.e., that $\bigcap \mathcal{A} \neq 0$. No generality is lost if we assume that each ideal in \mathcal{A} is principal, since every nonzero ideal contains a nonzero principal ideal, so we may assume that $\mathcal{A} = \{[g] : g \in \mathcal{A}'\}$ for a subset \mathcal{A}' of H^+ . Since $\langle H^+; \leq \rangle$ has cofinality \mathfrak{m} and $|\mathcal{A}| < \mathfrak{m}$, the set \mathcal{A}' must be bounded above in $\langle H^+; \leq \rangle$. It follows that \mathcal{A} is bounded below in $\langle \mathfrak{F}_{S_1}; \subseteq \rangle$, i.e., $\bigcap \mathcal{A} \in \mathfrak{F}_{S_1}$.

To show that \mathfrak{F}_{S_2} is \mathfrak{n} -Jansian, let $\mathcal{A} \subseteq \mathfrak{F}_{S_2}$ with $|\mathcal{A}| < \mathfrak{n}$. For each $A \in \mathcal{A}$, choose $s_A \in A \cap S_2$. Since $s_A R \in \mathfrak{F}_{S_2}$ for all $A \in \mathcal{A}$, we may again assume without loss of generality that each ideal in \mathcal{A} is principal, and hence that $\mathcal{A} = \{[g] : g \in \mathcal{A}'\}$ for a subset \mathcal{A}' of H^+ . Taking $H_2 = \{(g_2, 0) : g_2 \in G_2^+\}$, where G_2^+ is the positive cone of G_2 , the reader will observe that the chain $\langle H_2; \leq \rangle$ must have cofinality \mathfrak{n} , since the order reduct of G_2 has cofinality \mathfrak{n} . Since $|\mathcal{A}| < \mathfrak{n}$, the set \mathcal{A}' must be bounded above in $\langle H_2; \leq \rangle$, i.e., bounded below in $\langle \mathfrak{F}_{S_2}; \subseteq \rangle$. This means that $\bigcap \mathcal{A} \in \mathfrak{F}_{S_2}$.

By Proposition 5, the ring R is a right \mathfrak{m} -order in RS_1^{-1} but a right \mathfrak{n} -order in RS_2^{-1} . \square

EXAMPLE 2. We demonstrate that if \mathfrak{m} is a regular cardinal and \mathfrak{n} a nonzero cardinal then there is a ring T and a subring S of T such that S is a right \mathfrak{m} -order in T and $S, T \in P_r(\mathfrak{n})$.

Let D be a commutative domain which is an \mathfrak{m} -order in its field of quotients F (see the previous example for such a D). Let R be the subring of $\mathbf{M}_{\mathfrak{n}}^*(F)$ consisting of all matrices which contain fewer than \mathfrak{m} nonzero entries and set $S = \mathbf{M}_{\mathfrak{n}}^*(D)$. Define $T = R + S \subseteq \mathbf{M}_{\mathfrak{n}}^*(F)$. Observe that if $\mathfrak{n} < \mathfrak{m}$ then $R = \mathbf{M}_{\mathfrak{n}}^*(F)$, so $T = \mathbf{M}_{\mathfrak{n}}^*(F)$. We claim that, in general, T is a subring of $\mathbf{M}_{\mathfrak{n}}^*(F)$. The only nontrivial step is to verify closure under multiplication. Let $M = M' + M''$ and $N = N' + N''$ with $M', N' \in R$ and $M'', N'' \in S$. Then $MN = M'N' + M'N'' + M''N' + M''N''$. Certainly $M'N' \in R$ and $M''N'' \in S$. Since M' has fewer than \mathfrak{m} nonzero rows, it follows that $M'N''$ has fewer than \mathfrak{m} nonzero rows. But every row of $M'N''$ contains only finitely many nonzero entries, so $M'N''$ has fewer than \mathfrak{m} nonzero entries, i.e., $M'N'' \in R$. An analogous argument involving columns shows that $M''N' \in R$. Thus $MN \in T$.

We show now that S is a right \mathfrak{m} -order in T . Let $X \subseteq T$ with $|X| < \mathfrak{m}$. Write each $M \in X$ as $M = M' + M''$ with $M' \in R$ and $M'' \in S$. Consider $Y = \{M'_{\alpha\beta} : \alpha, \beta \in \mathfrak{n} \text{ and } M \in X\}$.

Note that $|Y| < \mathfrak{m}$. Since D is a right \mathfrak{m} -order in F , there is a nonzero $u \in D$ such that $Yu \subseteq D$. Let $I(u)$ denote the scalar matrix in S with u on the main diagonal and zeros elsewhere. Clearly, $I(u)$ is a unit of T , and it is easy to see that $MI(u) = M'I(u) + M''I(u) \in S$ for all $M \in X$. Thus S is a right \mathfrak{m} -order in T . Note that $S, T \in P_r(\mathfrak{n})$ if $\mathfrak{n} < \aleph_0$ and $S, T \in P_r(\mathfrak{n}^+)$ if $\mathfrak{n} \geq \aleph_0$ (essentially by Proposition 1.4).

To realise the case $S, T \in P_r(\aleph_0)$, we modify the ring construction of Example 1.1. Choose F as above and let $\Theta_\alpha: \mathbb{M}_{2\alpha}(F) \rightarrow \mathbb{M}_{\aleph_0}(F)$ denote the diagonal embedding

$$\Theta_\alpha(A) = \begin{bmatrix} A & 0 \\ 0 & A \\ & & \ddots \end{bmatrix} \in \mathbb{M}_{\aleph_0}(F) \quad (A \in \mathbb{M}_{2\alpha}(F)).$$

described in Example 1.1. Define $T = \bigcup_{\alpha \in \aleph_0} \Theta_\alpha[\mathbb{M}_{2\alpha}(F)]$ and $S = \bigcup_{\alpha \in \aleph_0} \Theta_\alpha[\mathbb{M}_{2\alpha}(D)] \subseteq T$. An argument similar to the one used above shows that S is a right \mathfrak{m} -order in T and the argument of Example 1.1 may be re-used to show that $S, T \in P_r(\aleph_0)$.

Finally, to realise the case $S, T \in P_r(\mathfrak{n})$, \mathfrak{n} an uncountable limit cardinal, we modify the construction of Example 1.2. We again choose F as above and let $\Theta_{\mathbf{k}}: \mathbb{M}_{\mathbf{k}}^*(F) \rightarrow \mathbb{M}_{\mathfrak{m}}^*(F)$ denote the “diagonal” embedding described in Example 1.2. Define R to be the subring of $\bigcup_{\aleph_0 \leq \mathbf{k} < \mathfrak{n}} \Theta_{\mathbf{k}}[\mathbb{M}_{\mathbf{k}}^*(F)]$ consisting of all matrices M such that if \mathbf{k} is the smallest infinite cardinal for which $M \in \Theta_{\mathbf{k}}[\mathbb{M}_{\mathbf{k}}^*(F)]$, and $M = \Theta_{\mathbf{k}}(\bar{M})$ with $\bar{M} \in \mathbb{M}_{\mathbf{k}}^*(F)$, then \bar{M} has fewer than \mathfrak{m} nonzero entries. Set

$$S = \bigcup_{\aleph_0 \leq \mathbf{k} < \mathfrak{n}} \Theta_{\mathbf{k}}[\mathbb{M}_{\mathbf{k}}^*(D)]$$

and define $T = R + S$. Again, previous arguments may be modified to show that S is a right \mathfrak{m} -order in T and $S, T \in P_r(\mathfrak{n})$. □

§3. SEMIGROUP RINGS.

In this section we consider closure properties of the classes $P_r(\mathfrak{m})$ under the formation of semigroup rings (in particular, monoid, group and polynomial rings). We preserve the convention established in Chapter 0, §10, of using multiplicative notation for the binary operation of a semigroup (and in particular, abbreviating products by juxtaposition), except in cases where this is palpably unnatural.

A semigroup H which is not a monoid may be extended to a monoid H^M by choosing a symbol $1_H \notin H$, defining $H^M = H \cup \{1_H\}$, and defining the monoid operation on H^M to be the extension of the semigroup operation on H such that $1_H h = h = h 1_H$ for all $h \in H^M$. With an arbitrary semigroup H , we associate a monoid H^1 with universe:

$$H^1 = \begin{cases} H, & \text{if } H \text{ has an identity element;} \\ H^M, & \text{otherwise.} \end{cases}$$

We shall, at times, refer to 1_H without explicit mention of H^1 , even in cases where H does not have an identity.

PROPOSITION 1. *Let \mathfrak{m} be a cardinal greater than 1. If H is a semigroup and $RH \in P_r(\mathfrak{m})$ then $R \in \bar{P}_r(\mathfrak{m})$.*

Proof. Suppose first that $\mathfrak{m} \geq \aleph_0$. Let $0 \neq r \in R$ and suppose $X = \{\sum_{i=1}^{n(\gamma)} a_{i\gamma} g_{i\gamma} : \gamma \in \Gamma\}$ is a right insulator for r in RH , where $a_{i\gamma} \in R$ and $g_{i\gamma} \in H$ for $i = 1, \dots, n(\gamma)$, $\gamma \in \Gamma$, and $|\Gamma| < \mathfrak{m}$. Set $Y = \{a_{i\gamma} : i = 1, \dots, n(\gamma), \gamma \in \Gamma\}$. Observe that Y is a right insulator for r in R , for if $rYs = 0$ with $s \in R$, then $rXs = 0$, so $s = 0$. Since $|Y| < \mathfrak{m}$, this shows that $R \in \bar{P}_r(\mathfrak{m})$.

Now suppose that $1 < \mathfrak{m} < \aleph_0$. By Theorem 1.2, RH is isomorphic to a right order in $M_{\mathfrak{m}}(D)$ for some division ring D . It follows that $\dim RH_{RH} = \mathfrak{m}$. Furthermore, if $\{K_i : i = 1, \dots, n\}$ is an independent family of nonzero right ideals of R , it is easily checked that $\{K_i H : i = 1, \dots, n\}$ is an independent family of nonzero right ideals of RH . Consequently, $\dim R_R \leq \mathfrak{m}$. By Goldie's First Theorem (Theorem 0.7.5), RH satisfies the ACC on right annihilators. We assert that this implies that R also satisfies the ACC on right annihilators. Indeed, suppose that $(0 :_r X_0) \subsetneq (0 :_r X_1) \subsetneq \dots$ is a strictly ascending chain of right annihilators of subsets X_α ($\alpha \in \aleph_0$) of R . Choose $h \in H$ and define $Y_\alpha = X_\alpha h := \{rh : r \in X_\alpha\} \subseteq RH$, for each $\alpha \in \aleph_0$. It can be shown easily that $(0 :_r Y_0) \subsetneq (0 :_r Y_1) \subsetneq \dots$ is a strictly ascending chain of right annihilators in RH , a contradiction. We may conclude, therefore, that R is a prime right Goldie ring with $\dim R_R \leq \mathfrak{m}$. It follows from Goldie's First Theorem (Theorem 0.7.5) and Theorem 1.2 that $R \in \bar{P}_r(\mathfrak{m})$. \square

The following example shows that Proposition 1 would fail if we ceased to insist that $\mathfrak{m} > 1$.

EXAMPLE 1. Let $1 < \mathfrak{n} < \aleph_0$. Set $R = M_{\mathfrak{n}}(F)$, F any field, and let H be the free monoid on two (noncommuting) symbols. Clearly $R \in P_r(\mathfrak{n})$, yet $RH = [M_{\mathfrak{n}}(F)]H \cong M_{\mathfrak{n}}(FH)$: the natural map $\varphi : [M_{\mathfrak{n}}(F)]H \rightarrow M_{\mathfrak{n}}(FH)$ defined by

$$\left(\varphi\left(\sum_{k=1}^m A_k h_k\right)\right)_{\alpha\beta} = \sum_{k=1}^m (A_k)_{\alpha\beta} h_k \quad (\alpha, \beta < \mathfrak{n}, A_k \in M_{\mathfrak{n}}(F), h_k \in H, k = 1, \dots, m)$$

may be shown to be a ring isomorphism. Since $D = FH$ is a domain which is not right Ore (see remarks following Proposition 0.10.1), it follows from Proposition 1.4 that $RH \cong M_{\mathfrak{n}}(D) \in P_r(1)$. \square

In Proposition 1, if H is chosen to be a group then a result of Connell [Con63, Theorem 8, p675] asserts that RH is prime if and only if R is prime and H contains no finite normal subgroups except $\{1_H\}$. If RH is right strongly prime then R is right strongly prime and H has no locally finite normal subgroups except $\{1_H\}$ [HL75, Proposition III.1(a), p215]. (Recall that a group G is said to be *locally finite* if every finitely generated subgroup of G is finite.) The converse of this

result is an open problem. Some partial converses are known, for example, Hannah [Han77, Proposition 7, p342] showed that if F is a field and H a solvable group with no locally finite normal subgroups except $\{1_H\}$ then FH is right strongly prime.

Suppose A_1, A_2, \dots, A_n are nonempty subsets of a semigroup H . We call an element $x \in A_1 A_2 \dots A_n := \{a_1 a_2 \dots a_n : a_k \in A_k, k = 1, \dots, n\}$ a *unique product element* (abbreviated *u.p. element*) with respect to A_1, A_2, \dots, A_n (or just a *u.p. element* if the A_i are understood) if x is uniquely expressible in the form $x = a_1 a_2 \dots a_n$ where $a_i \in A_i$ for $i = 1, \dots, n$.

In [Sil92], Silva investigates conditions on a semigroup H which guarantee that the semigroup ring RH is prime whenever R is prime. Specifically, [Sil92, Theorem 1.1, p191] asserts that if H satisfies “Condition C” (defined below) then RH is prime whenever R is prime.

CONDITION C: For each pair of finite nonempty subsets A, B of the semigroup H , there exists $t \in H$ such that $A\{t\}B$ contains a u.p. element.

Since our interest is in the degree of primeness of a ring we choose to introduce two more specific conditions.

CONDITION D: For each finite nonempty subset A of the semigroup H , there exists a nonempty subset T_A of H^1 with the property that for each finite nonempty subset B of H , there exists $t \in T_A$ such that $A\{t\}B$ contains a u.p. element.

CONDITION E: For each finite nonempty subset A of the semigroup H , there exists a nonempty subset T_A of H^1 and $a \in A$ with the property that for each finite nonempty subset B of H , there exists $t \in T_A$ such that $A\{t\}B$ contains a u.p. element of the form atb for some $b \in B$.

Note that Condition E is stronger than Condition D. Following Passman [Pas77, §13.1], we call a semigroup H a *unique product semigroup* if, for each pair of finite nonempty subsets A, B of H , AB contains a u.p. element. Observe that if T_A may be chosen to be $\{1_H\}$ for all finite $A \subseteq H$ then Condition D amounts to saying that H is a unique product semigroup. On the other hand, if we may choose to have $T_A = H^1$ for all finite $A \subseteq H$ then Conditions D and C are equivalent for H .

LEMMA 2. Let R be a prime ring, H a semigroup and $z = \sum_{i=1}^n r_i a_i \in RH$ with $0 \neq r_i \in R$, $a_i \in H$ for $i = 1, \dots, n$. Then:

- (i) if H satisfies Condition D, $A = \{a_1, \dots, a_n\}$ and X_i is a right insulator for r_i in R , for $i = 1, \dots, n$, then $(\bigcup_{i=1}^n X_i)T_A \subseteq RH$ is a right insulator for z in RH ;
- (ii) if H satisfies Condition E, $A = \{a_1, \dots, a_n\}$ and X_i is a right insulator for r_i in R , for $i = 1, \dots, n$, then $X_j T_A \subseteq RH$ is a right insulator for z in RH for some $j \in \{1, \dots, n\}$.

Proof. (i) Suppose, on the contrary, that $z(\bigcup_{i=1}^n X_i)T_A w = 0$ with $0 \neq w = \sum_{i=1}^m s_i b_i \in RH$, $0 \neq s_i \in R$, $b_i \in H$ for $i = 1, \dots, m$. Taking $B = \{b_1, \dots, b_m\}$ there exists, by hypothesis, $t \in T_A$ such that $A\{t\}B$ contains a unique product element $a_k t b_j$ for some $k \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$. It follows that $r_k(\bigcup_{i=1}^n X_i)s_j = 0$, a contradiction.

(ii) Suppose the element a of Condition E is a_j , where $j \in \{1, \dots, n\}$. Then an argument similar to the one used above shows that $X_j T_A$ is a right insulator for z in RH . \square

A semigroup H is said to be *strictly right ordered* if it admits a linear order \leq such that for any $x, y, z \in H$, we have

$$x < y \Rightarrow xz < yz.$$

Strictly left ordered semigroups are defined analogously, and a semigroup that is strictly right and strictly left ordered is strictly ordered in the sense of Chapter 0, §10.

If $\{G_i : i \in \Gamma\}$ is a family of monoids, we denote their *free product* by $\star_{i \in \Gamma} G_i$. We remind the reader that the free product of a family of monoids always exists and we provide a crude outline of its construction. (A more precise account may be found in [Pas77, Theorem 9.2.9, p372]⁴.) We assume the sets $G_i \setminus \{1_{G_i}\}$ are mutually disjoint. We form $X = \bigcup_{i \in \Gamma} (G_i \setminus \{1_{G_i}\})$. Consider the set $W(X)$ of all “words” in X (i.e., finite sequences of elements of X , including the empty word, which we denote by 1, where we assume without loss of generality that $1 \notin X$). A word $w \in W(X)$ is said to be in *reduced form* if $w = 1$ or $w = g_1 g_2 \dots g_n$, where $n \geq 1$, each $g_j \in X$ and adjacent “letters” g_j, g_{j+1} lie in distinct G_i . Then the *free product* of the G_i has as its elements all words in reduced form and the monoid operation on the free product is just concatenation of words. We write $G_1 \star G_2 \star \dots \star G_n$ for $\star_{i \in \Gamma} G_i$ if $\Gamma = \{1, \dots, n\}$. Note, however, that as a binary operation, \star is commutative. We point out that this construction is similar to that of $\langle X \rangle$, the free monoid on a set X (see Chapter 0, §10). In fact, if X is any nonempty set then $\langle X \rangle = \star_{x \in X} \langle x \rangle$.

Generalizing [Gro79, Theorems 2.1 & 2.2, pp242-243], [Vio75, Proposition 2.14, p281] and [HL75, Proposition III.3, p216], we obtain:

PROPOSITION 3. *Let \mathbf{m} be a nonzero cardinal. Then:*

- (i) *if \mathbf{m} is infinite, H is a unique product semigroup and $R \in \bar{P}_r(\mathbf{m})$ then $RH \in \bar{P}_r(\mathbf{m})$;*
- (ii) *if H is a cancellative strictly right ordered semigroup and $R \in \bar{P}_r(\mathbf{m})$ then $RH \in \bar{P}_r(\mathbf{m})$;*
- (iii) *if $H = G \star G'$ is the free product of two nontrivial monoids G and G' and $R \in \bar{P}_r(\mathbf{m})$ then $RH \in \bar{P}_r(2\mathbf{m})$.*

Proof. (i) follows immediately from Lemma 2(i), taking $T_A = \{1_H\}$.

⁴Although Passman constructs a free product for a family of groups, his construction may be used, mutatis mutandis, for a family of monoids.

(ii) We show that a semigroup satisfying the conditions of (ii) must satisfy Condition E. Let $A = \{x_1, \dots, x_n\} \subseteq H$ with $x_1 < x_2 < \dots < x_n$. Take $a = x_n$ and $T_A = \{1_H\}$. Suppose $B = \{z_1, \dots, z_m\} \subseteq H$ with $z_1 < z_2 < \dots < z_m$. It suffices to show that $y = \max\{x_i z_j : i = 1, \dots, n; j = 1, \dots, m\}$ is uniquely expressible in the form $y = x_n z_j = a z_j$ for some $j \in \{1, \dots, m\}$. Observe first that since H is strictly right ordered, y may be written in the form $y = x_n z_j$ for some $j \in \{1, \dots, m\}$. If $y = x_n z_j = x_k z_l$ for some $k \in \{1, \dots, n\}$ and $l \in \{1, \dots, m\}$ then $n = k$, otherwise $y = x_k z_l < x_n z_l$, a contradiction. Since H is cancellative and $x_n z_j = x_n z_l$, we must have $j = l$. Now the result follows from Lemma 2 (ii).

(iii) For convenience, we first introduce some notation and terminology. We say that an element $h \in H$ has *length* $2n+1$ if it has reduced form $h = g_1 g'_1 g_2 g'_2 \dots g_n g'_n g_{n+1}$ where $g_i \in G \setminus \{1_G\}$, $g'_i \in G' \setminus \{1_{G'}\}$ for each i . Since the first factor of h is contained in G , we say that h has *type* G_- and since the last factor of h is contained in G , we say that h also has *type* $_G$. Elements of even length and of type G'_- and of type $_G'$ are defined in a similar manner.

It suffices, in view of Lemma 2 (ii), to show that H satisfies Condition E where, in particular, we are able to choose $|T_A| = 2$ for all finite nonempty subsets A of H . Suppose then that A is a finite nonempty subset of H . Let a be an element of maximal length in A and choose $g \in G \setminus \{1_G\}$, $g' \in G' \setminus \{1_{G'}\}$.

Case 1. If a has type $_G$ (or $a = 1_H$) we may take $T_A = \{g', g'g\}$. Let B be a finite nonempty subset of H and b an element of maximal length in B . If b has type G_- (or $b = 1_H$) then $ag'b$ is a u.p. element in $Ag'B$. If b has type G'_- then $ag'gb$ is a u.p. element in $Ag'B$.

Case 2. If a has type $_G'$ we take $T_A = \{g, gg'\}$. Let B be a finite nonempty subset of H and b an element of maximal length in B . If b has type G'_- (or $b = 1_H$) then agb is a u.p. element in AgB . If b has type G_- then $agg'b$ is a u.p. element in $Agg'B$. \square

If \mathfrak{m} is infinite and $H = G \star G'$ is the free product of nontrivial monoids G and G' then Proposition 3 (iii) yields: $R \in \bar{P}_r(\mathfrak{m}) \Rightarrow RH \in \bar{P}_r(\mathfrak{m})$. This implication fails, in general, for finite \mathfrak{m} (see Example 2 below). If we assume, however, that $|G| \geq 3$ or $|G'| \geq 3$ then $R \in \bar{P}_r(\mathfrak{m}) \Rightarrow RH \in \bar{P}_r(\mathfrak{m})$ holds for all nonzero cardinals \mathfrak{m} . For suppose $|G| \geq 3$. Let R be an arbitrary nonzero ring and let $H = G \star G'$. Choose $g_1, g_2 \in G \setminus \{1_G\}$ with $g_1 \neq g_2$ and $g' \in G' \setminus \{1_{G'}\}$. For each positive integer n , set $x_n = g'(g_1 g')^n g_2 + (g_1 g')^n g_2 g'$ and $I_n = x_n R H$. We claim that $\{I_n : n \geq 1\}$ is an independent family of (nonzero) right ideals of RH . For suppose $\sum_{i=1}^n x_i s_i = 0$ with $\{s_1, \dots, s_n\} \subseteq RH$. If $x_i s_i \neq 0$ for some $i \in \{1, \dots, n\}$, let h be an element of maximal length in $\bigcup_{i=1}^n \text{supp}(x_i s_i)$. We may assume without loss of generality that $h = g'(g_1 g')^k g_2 h_k$ where $1 \leq k \leq n$ and h_k is an element of maximal length in $\text{supp } s_k$ of type G'_- . Inasmuch as $\sum_{i=1}^n x_i s_i = 0$, we must have $h = g'(g_1 g')^k g_2 h_k = g'(g_1 g')^j g_2 h_j$ for some $j \neq k$, $j \in \{1, \dots, n\}$, where h_j is an element of maximal length in $\text{supp } s_j$ of type G'_- . Since $j \neq k$ and $g_1 \neq g_2$, this is clearly not possible, so $x_i s_i = 0$ for $i = 1, \dots, n$. Thus $\{I_n : n \geq 1\}$ is independent, from which it follows that $\dim RH_{RH} \geq \aleph_0$. Suppose

now that $R \in \bar{P}_r(\mathfrak{m})$ with $0 < \mathfrak{m} < \aleph_0$. By Proposition 3(iii), RH is right bounded strongly prime and since $\dim RH_{RH} \geq \aleph_0$, RH cannot be a prime right Goldie ring, so $RH \in P_r(1)$, by Theorem 1.2.

EXAMPLE 2. We show that if \mathfrak{m} is a finite nonzero cardinal and $H = G \star G'$,⁵ where G and G' are both (multiplicatively written) groups of order 2, generated by elements g and g' , respectively, then $R = \mathbb{M}_{\mathfrak{m}}(\mathbb{Z}_2)H \in P_r(2\mathfrak{m})$ yet $\mathbb{M}_{\mathfrak{m}}(\mathbb{Z}_2) \in P_r(\mathfrak{m})$. This shows that the implication $R \in \bar{P}_r(\mathfrak{m}) \Rightarrow RH \in \bar{P}_r(2\mathfrak{m})$ of Proposition 3(iii) cannot be sharpened. Note first that $R \cong \mathbb{M}_{\mathfrak{m}}(\mathbb{Z}_2H)$. (A suitable isomorphism is described in Example 1.) We show that $\mathbb{Z}_2H \in P_r(2)$. Certainly, by Proposition 3(iii), $\mathbb{Z}_2H \in \bar{P}_r(2)$. Consider the element $1+g \in \mathbb{Z}_2H$. Suppose $1+g$ has a singleton right insulator, $\{y\}$, say. Write $y = \sum_{i=1}^n h_i$ with $h_i \in H$, $i = 1, \dots, n$. Inasmuch as $(1+g)y = 1+g$, we clearly lose no generality in assuming that each h_i (distinct from 1_H) is of type G'_- . Set $z = (\sum_{i=1}^n h_i^{-1})(1+g)$. Clearly $z \neq 0$. Now

$$\begin{aligned} (1+g)yz &= (1+g)\left(\sum_{i=1}^n h_i\right)\left(\sum_{i=1}^n h_i^{-1}\right)(1+g) \\ &= (1+g)\left(\sum_{\substack{1 \leq i, j \leq n \\ i \neq j}} h_i h_j^{-1} + \sum_{i=1}^n h_i h_i^{-1}\right)(1+g) \\ &= (1+g)\left(\sum_{1 \leq i < j \leq n} [h_i h_j^{-1} + h_j h_i^{-1}] + n\right)(1+g). \end{aligned}$$

Since $(1+g)t(1+g) = (1+g)t^{-1}(1+g)$ for all $t \in H$, setting $t_{ij} = h_i h_j^{-1}$, $1 \leq i < j \leq n$, we obtain

$$\begin{aligned} (1+g)yz &= (1+g)\left(\sum_{1 \leq i < j \leq n} [t_{ij} + t_{ij}^{-1}] + n\right)(1+g) \\ &= (1+g)n(1+g) = 0, \end{aligned}$$

a contradiction. Thus $\mathbb{Z}_2H \in P_r(2)$. It follows that \mathbb{Z}_2H is isomorphic to a right order in $\mathbb{M}_2(D)$ for some division ring D , by Theorem 1.2. By Lemma 0.7.3, R is isomorphic to a right order in $\mathbb{M}_{\mathfrak{m}}(\mathbb{M}_2(D)) \cong \mathbb{M}_{2\mathfrak{m}}(D)$, so $R \in P_r(2\mathfrak{m})$, by Theorem 1.2. \square

We do not know whether the requirement that \mathfrak{m} be infinite is necessary in Proposition 3(i). As a consequence of Proposition 3(ii), we have:

COROLLARY 4. *Let \mathfrak{m} be a nonzero cardinal. If $R \in \bar{P}_r(\mathfrak{m})$ then $R[x] \in \bar{P}_r(\mathfrak{m})$.* \square

We conclude this section with a brief investigation of “monomial algebras”. Monomial algebras are very similar to certain types of semigroup rings (specifically, free algebras) and it is for this reason that we have chosen to discuss them here. They are a rich source of “pathological” prime rings, as is illustrated by an example at the end of this section.

Let F be a field and X a nonempty set whose elements will be called *indeterminates*. Consider the free F -algebra $F\langle X \rangle$ on X . (Recall that $F\langle X \rangle$ is, by definition, the semigroup ring

⁵This is the so-called *infinite dihedral group*, D_{∞} .

whose coefficient ring is F and whose semigroup is the free monoid $\langle X \rangle$ on X .) A *monomial* in $F\langle X \rangle$ is an element of the form aw where $a \in F$ and $w \in \langle X \rangle$. Any F -algebra of the form $F\langle X \rangle/I$, where I is an ideal of $F\langle X \rangle$ generated by monomials, is called a *monomial F -algebra*. We introduce some notation which will be used throughout the remainder of this section. If $w \in \langle X \rangle$, we denote the image of w in $F\langle X \rangle/I$ by \bar{w} , and the image of $\langle X \rangle$ in $F\langle X \rangle/I$ by $\overline{\langle X \rangle}$. It is also customary to use the term “monomial” to describe the image in $F\langle X \rangle/I$ of any monomial of $F\langle X \rangle$.

The following result shows that much of the “freedom” in the free F -algebra $F\langle X \rangle$ is passed on to the monomial algebra $F\langle X \rangle/I$. Although the result is well known, we have chosen to provide a proof, in the absence of a suitable reference.

PROPOSITION 5. *Let $A_F = F\langle X \rangle/I$, where F is a field, X a nonempty set, and I an ideal of $F\langle X \rangle$ generated by monomials. Then every nonzero $r \in A$ is uniquely expressible in the form $r = \sum_{i=1}^n a_i \bar{w}_i$, where the w_i are distinct elements of $\langle X \rangle \setminus (\langle X \rangle \cap I)$ and $0 \neq a_i \in F$ for $i = 1, \dots, n$.*

Proof. Suppose that the ideal I is generated by a nonempty set V of monomials. The fact that every nonzero $r \in A$ is expressible in the given form is obvious. To establish uniqueness, it clearly suffices to show that if $\{w_1, \dots, w_n\}$ is any set of distinct elements of $\langle X \rangle$ and $\{a_1, \dots, a_n\}$ any set of nonzero elements of F then $\bar{w}_i = 0$ for all $i \in \{1, \dots, n\}$, whenever $\sum_{i=1}^n a_i \bar{w}_i = 0$. Suppose then $\sum_{i=1}^n a_i \bar{w}_i = 0$ with the w_i and a_i as above. This implies that $\sum_{i=1}^n a_i w_i \in I$ which means that $\sum_{i=1}^n a_i w_i$ can be expressed in the form $\sum_{i=1}^m b_i u_i v_i u'_i$, where $0 \neq b_i \in F$, $u_i, u'_i \in \langle X \rangle$, $v_i \in V$ for $i = 1, \dots, m$, with the u_i, v_i, u'_i distinct. Now

$$\text{supp}\left(\sum_{i=1}^n a_i w_i\right) = \{w_i : i = 1, \dots, n\} \subseteq \{u_i v_i u'_i : i = 1, \dots, m\} = \text{supp}\left(\sum_{i=1}^m b_i u_i v_i u'_i\right),$$

so $\{w_i : i = 1, \dots, n\} \subseteq I$, whence $\bar{w}_i = 0$ for all $i \in \{1, \dots, n\}$, as required. \square

We remarked in the sequel to Proposition 0.10.1 that if F is a field and X a nonempty set then the free F -algebra $F\langle X \rangle$ admits a naturally defined degree function $\partial: F\langle X \rangle \setminus \{0\} \rightarrow \mathbb{Z}$ which extends the classical degree function associated with polynomial rings. It follows from Proposition 5 that the degree function ∂ , when restricted to nonzero monomials, does “survive” in the monomial algebra $F\langle X \rangle/I$. To be more precise, it is possible to define a function $\bar{\partial}: \overline{\langle X \rangle} \setminus \{0\} \rightarrow \mathbb{Z}$ by $\bar{\partial} \bar{w} = \partial w$ for all nonzero $\bar{w} \in \overline{\langle X \rangle}$; the point being that $\bar{\partial}$ is well defined because $\bar{w} = \bar{u}$ ($\bar{u}, \bar{w} \in \overline{\langle X \rangle}$) implies $u = w$ or $\bar{u} = \bar{w} = 0$, by Proposition 5. Notice that for all $\bar{u}, \bar{w} \in \overline{\langle X \rangle} \setminus \{0\}$, we have $\bar{\partial}(\bar{u}\bar{w}) = \bar{\partial}\bar{u} + \bar{\partial}\bar{w}$ or $\bar{u}\bar{w} = 0$.

There is a further notion associated with the free algebra $F\langle X \rangle$ which “survives” in the monomial algebra $F\langle X \rangle/I$, namely that of the support of a nonzero element. If $r = \sum_{i=1}^n a_i \bar{w}_i$ (with the w_i and a_i as in Proposition 5) is a nonzero element of the monomial algebra $F\langle X \rangle/I$, we define the *support* of r , abbreviated $\text{supp } r$, to be $\{\bar{w}_1, \bar{w}_2, \dots, \bar{w}_n\} \subseteq \overline{\langle X \rangle}$. Since, by Proposition 5, r is *uniquely* expressible in the form $\sum_{i=1}^n a_i \bar{w}_i$ (with the w_i and a_i as in Proposition 5), $\text{supp } r$ is clearly

uniquely determined by r .

LEMMA 6. *Let $A_F = F\langle X \rangle / I$, where F is a field, X a nonempty set, and I an ideal of $F\langle X \rangle$ generated by monomials. If $0 \neq r, s \in A$ with $rs = 0$, then $\bar{w}\bar{u} = 0$ whenever \bar{w} and \bar{u} are elements of maximum degree in $\text{supp } r$ and $\text{supp } s$, respectively, that is to say $\bar{w}\bar{u} = 0$ whenever $\bar{\partial}\bar{w} \geq \bar{\partial}\bar{w}'$ for all $\bar{w}' \in \text{supp } r$ and $\bar{\partial}\bar{u} \geq \bar{\partial}\bar{u}'$ for all $\bar{u}' \in \text{supp } s$.*

Proof. If $\bar{w}\bar{u} \neq 0$ then we must have $\bar{w}\bar{u} = \bar{w}'\bar{u}'$ for some $\bar{w}' \in \text{supp } r$, $\bar{w}' \neq \bar{w}$, and $\bar{u}' \in \text{supp } s$, $\bar{u}' \neq \bar{u}$. By Proposition 5, $wu = w'u'$. Since $\partial(wu) = \partial w + \partial u = \partial w' + \partial u'$ with $\partial w \geq \partial w'$ and $\partial u \geq \partial u'$, we must have $\partial w = \partial w'$ and $\partial u = \partial u'$. This clearly implies $w = w'$ and $u = u'$, a contradiction. Thus $\bar{w}\bar{u} = 0$. \square

The following result is a routine extension of [GHL74, Proposition 2.3, p8] and shows that the bound of primeness of a monomial algebra is determined almost completely by the behaviour of its monomials.

LEMMA 7. *Let \mathfrak{m} be a nonzero cardinal. Let $A_F = F\langle X \rangle / I$, where F is a field, X a nonempty set, and I an ideal of $F\langle X \rangle$ generated by monomials. Suppose that each nonzero $\bar{w} \in \overline{\langle X \rangle}$ has a right insulator in $\overline{\langle X \rangle}$ of cardinality less than $\mathfrak{m} + 1$, that is to say, there exists a subset Y of $\overline{\langle X \rangle}$ such that $|Y| < \mathfrak{m} + 1$ and $\bar{w}Y\bar{u} = 0$ implies $\bar{u} = 0$, whenever $\bar{u} \in \overline{\langle X \rangle}$. Then $A \in \bar{P}_r(\mathfrak{m})$.*

Proof. Let $0 \neq r \in A$ and let \bar{w} be an element of maximum degree in $\text{supp } r$. By hypothesis, we may choose $Y \subseteq \overline{\langle X \rangle}$ such that $|Y| < \mathfrak{m} + 1$ and $\bar{w}Y\bar{u} \neq 0$ for all nonzero $\bar{u} \in \overline{\langle X \rangle}$. We claim that Y is a right insulator for r in A . Indeed, if $rYs = 0$ for some nonzero $s \in A$ then, choosing an element \bar{u} of maximum degree in $\text{supp } s$, we infer from the previous lemma that $\bar{w}Y\bar{u} = 0$, a contradiction. \square

EXAMPLE 3. Let \mathfrak{m} be a regular cardinal and F a field. We construct a monomial F -algebra A such that $A \in P_r(1) \cap P_l(\mathfrak{m}^+)$.

Set $X = \mathfrak{m}$, i.e., X is the set of all ordinals α such that $\alpha < \mathfrak{m}$. Let I be the ideal of $F\langle X \rangle$ generated by all monomials of the form $\alpha\beta\gamma$ with $\alpha > \beta > \gamma$. Define $A_F = F\langle X \rangle / I$. Take $0 \neq \bar{w} \in \overline{\langle X \rangle}$ and suppose that \bar{w} terminates in $\bar{\beta}$, say, with $\beta \in X$. It is clear that $\{\bar{\beta}0\}$ is a right insulator for \bar{w} in $\overline{\langle X \rangle}$. By Lemma 7, $A \in P_r(1)$. Suppose \bar{w} has initial symbol $\bar{\alpha}$, say, with $\alpha \in X$. We claim that $Y = \{\bar{\gamma}\bar{\alpha} : \gamma \in X\}$ is a left insulator for \bar{w} in $\overline{\langle X \rangle}$. Indeed, if $0 \neq \bar{u} \in \overline{\langle X \rangle}$ and \bar{u} terminates in $\bar{\gamma}$ say, $\gamma \in X$, then it is easy to see that $\bar{u}(\bar{\gamma}\bar{\alpha})\bar{w} \neq 0$. Since $|Y| = \mathfrak{m}$, we may conclude from the left analogue of Lemma 7 that $A \in \bar{P}_l(\mathfrak{m}^+)$. To show that $A \notin P_l(\mathfrak{n})$ for any $\mathfrak{n} \leq \mathfrak{m}$, let Y be a subset of A with $|Y| < \mathfrak{m}$. Denote by Y' the set of all ordinals β (i.e., elements of X) for which $\bar{\beta}$ appears in Y . Since $|Y'| < \mathfrak{m}$ and $\text{cof } \mathfrak{m} = \mathfrak{m}$, there must exist some $\alpha \in X$

such that $\alpha > \beta$ for all $\beta \in Y'$. Clearly, therefore, $\overline{\alpha \oplus 1} \bar{\alpha} \in (0 : {}_l Y)$. Thus $A \in P_l(\mathfrak{m}^+)$. \square

§4. UNIFORMLY STRONGLY PRIME RINGS.

In [HL75], Handelman and Lawrence introduced the notion of a “uniformly strongly prime” ring and, more specifically, that of a ring “uniformly strongly prime of bound \mathfrak{n} ”, for a given finite nonzero cardinal \mathfrak{n} . Loosely speaking, these notions arise by insisting on the existence of a fixed subset of a ring which functions as an insulator for *all* nonzero ring elements, and by calculating the minimum cardinality of any such “uniform” insulator. (This cardinal will be called the “uniform bound of primeness” of the ring; more precise definitions follow below.) Surprisingly, given the prominence of Handelman and Lawrence’s work, it is only the more radical theoretic aspects of uniformly strongly prime rings that have received attention in the literature. In this section we hope to give the subject a fuller and more deserved treatment.

In Theorem 4, we show that the study of uniformly strongly prime rings can be reduced to that of (1) those such rings whose uniform bound of primeness is 1; and (2) prime Goldie rings (i.e., rings that are isomorphic to right or left orders in finite dimensional matrix rings over division rings). It is known that the property of being uniformly strongly prime is preserved in taking right orders (and this does not increase the uniform bound); we show in Corollary 7 that the passage to two-sided orders preserves and reflects the uniform bound itself. Thus the calculation of the uniform bounds of many strongly prime rings reduces to a consideration of properties of matrix rings $M_{\mathfrak{n}}(D)$, where \mathfrak{n} is a finite nonzero cardinal and D is a division ring. Surprisingly, however, the value of this bound turns out to depend, in general, on the division ring D , rather than being solely determined by the dimensions of the matrices, i.e., by \mathfrak{n} . (This contrasts with the situation for the right and left bounds of primeness of two-sided orders in simple artinian rings.)

We conclude the section with some examples, in which the uniform bounds of several prime Goldie rings are calculated. In particular, we show that for every finite nonzero cardinal \mathfrak{n} , there exist rings which are uniformly strongly prime of bound precisely \mathfrak{n} .

We point out that our study of uniformly strongly prime rings does not end here. Indeed, the class of uniformly strongly prime rings gives rise to a number of special classes in the category of rings which will be studied in Chapter IV.

A nonempty subset X of a ring R is said to be a *uniform insulator* for R if $aXb \neq 0$ whenever $0 \neq a, b \in R$. If R possesses a *finite* uniform insulator, we call R a *uniformly strongly prime ring* (in which case, of course, R is both a right and a left bounded strongly prime ring). A prime ring always possesses a uniform insulator – at worst, the ring itself functions as such. If \mathfrak{m} is the smallest cardinal such that a prime ring R possesses a uniform insulator X with $|X| < \mathfrak{m} + 1$, we call R *uniformly prime* (or *uniformly strongly prime*, if \mathfrak{m} is finite) *of bound \mathfrak{m}* , and we refer to \mathfrak{m} as the *uniform bound of primeness* of R . We shall denote by $UP(\mathfrak{m})$ (resp. $\overline{UP}(\mathfrak{m})$)

the class of rings which are uniformly prime of bound \mathfrak{m} (resp. of bound at most \mathfrak{m}). Thus $\overline{UP}(\aleph_0)$ is just the class of all uniformly strongly prime rings. Note that only a successor cardinal \mathfrak{m} may be the uniform bound of primeness of a ring, i.e., $UP(\mathfrak{m}) = \emptyset$ for all limit cardinals \mathfrak{m} . For any limit cardinal \mathfrak{m} , we have $\overline{UP}(\mathfrak{m}) = \bigcup_{0 < \mathfrak{n} < \mathfrak{m}} \overline{UP}(\mathfrak{n}) = \bigcup_{0 < \mathfrak{n} < \mathfrak{m}} UP(\mathfrak{n})$.

Observe that all of the “uniform” notions defined in the previous paragraph are left-right symmetric, unlike the corresponding notions associated with right and left bounds of primeness. The nonempty classes $UP(\mathfrak{m})$ partition the class of all prime rings since, as we noted in the previous paragraph, every prime ring R is uniformly prime of bound at most $|R|^+$. In fact, while we have chosen to pitch the above definition at a level of generality consistent with the investigations of this thesis as a whole, it is only for finite \mathfrak{m} that the classes $UP(\mathfrak{m})$ and $\overline{UP}(\mathfrak{m})$ have been studied in the literature and there are, at present, no known interesting characterizations of the classes $UP(\mathfrak{m})$ and $\overline{UP}(\mathfrak{m})$ for infinite \mathfrak{m} . The results of this section deal almost exclusively with the case where \mathfrak{m} is finite.

The following assertions are clearly equivalent for a nonempty subset X of a ring R :

- (i) X is a uniform insulator for R ;
- (ii) X is a right (resp. left) insulator for every nonzero $a \in R$.

It follows that if $R \in UP(\mathfrak{m})$ ($\mathfrak{m} > 0$), then $R \in \overline{P}_r(\mathfrak{m}) \cap \overline{P}_l(\mathfrak{m})$.

Clearly, every domain is a member of $UP(1)$. More generally, if \mathfrak{m} is a nonzero cardinal, D a domain and $0 \neq t \in D$, then $\{E_{\alpha,\beta}(t) : \alpha, \beta < \mathfrak{m}\}$ is a right insulator for every nonzero element of $\mathbb{M}_{\mathfrak{m}}(D)$ and hence a uniform insulator for $\mathbb{M}_{\mathfrak{m}}(D)$. (Recall that $E_{\alpha,\beta}(t)$ denotes the matrix in $\mathbb{M}_{\mathfrak{m}}(D)$ with t in position (α, β) and zeros elsewhere. The argument of Proposition 1.1 may be used virtually unchanged to justify the previous claim.) Thus $\mathbb{M}_{\mathfrak{m}}(D) \in \overline{UP}(\mathfrak{m}^2)$ whenever $0 < \mathfrak{m} < \aleph_0$ and $\mathbb{M}_{\mathfrak{m}}(D) \in \overline{UP}(\mathfrak{m}^+)$ whenever $\mathfrak{m} \geq \aleph_0$. We also know that for infinite \mathfrak{m} , $\mathbb{M}_{\mathfrak{m}}(D) \in P_r(\mathfrak{m}^+)$ (Proposition 1.4) so we may conclude that $\mathbb{M}_{\mathfrak{m}}(D) \in UP(\mathfrak{m}^+)$ in this case; actually, all of the above may be generalized effortlessly to the case where $D \in UP(1)$ with uniform insulator $\{t\}$. Since much of what follows will be an analysis of the uniform bounds of primeness of matrix rings $\mathbb{M}_{\mathfrak{m}}(D)$, the fact that, when \mathfrak{m} is infinite, this bound coincides with the right bound of primeness of the ring for “most” D makes us feel that the uniform bound of primeness of an infinite dimensional matrix ring is of little interest. This is part of the rationale for the more general emphasis, in what follows, on *finite* uniform bounds of primeness.

Now consider the case where \mathfrak{n} is a finite nonzero cardinal and D is a division ring. We established earlier that $\mathbb{M}_{\mathfrak{n}}(D) \in P_r(\mathfrak{n}) \cap P_l(\mathfrak{n})$, so we must have $\mathbb{M}_{\mathfrak{n}}(D) \in UP(\mathfrak{m})$ for some (finite) cardinal \mathfrak{m} with $\mathfrak{n} \leq \mathfrak{m} \leq \mathfrak{n}^2$. This fact is not new (see [HL75, Proposition I.2, p211]). It turns out, however, that the lastmentioned inequality can be sharpened to $\mathfrak{n} \leq \mathfrak{m} \leq 2\mathfrak{n} - 1$. Thus $\mathbb{M}_{\mathfrak{n}}(D) \in UP(\mathfrak{m})$ for some \mathfrak{m} such that $\mathfrak{n} \leq \mathfrak{m} \leq 2\mathfrak{n} - 1$, whenever D is a division ring. A slightly stronger assertion will be proved in Theorem 3. We require two lemmas. The first generalizes a result of Olson [Ols87, Theorem 10, p98], which shows that the property of being uniformly strongly prime is

preserved in taking right orders.

LEMMA 1. *Let $\mathfrak{m}, \mathfrak{n}$ be cardinals with $0 < \mathfrak{n} < \mathfrak{m}$ and $\mathfrak{m} \geq \aleph_0$. Let R be a right \mathfrak{m} -order in a ring S . If $S \in UP(\mathfrak{n})$ then $R \in \overline{UP}(\mathfrak{n})$. In particular, a right order in a uniformly strongly prime ring is uniformly strongly prime of bound no greater than that of the overring.*

Proof. Let X be a uniform insulator for S with $|X| < \mathfrak{n} + 1$. Since $|X| < \mathfrak{m}$ and R is a right \mathfrak{m} -order in S , there is a unit u of S , contained in R , such that $Xu \subseteq R$. If $a, b \in R$ with $a \neq 0$ and $aXub = 0$ then $ub = 0$ (because X is a uniform insulator for S), and so $b = 0$. This shows that Xu is a uniform insulator for R , and since $|Xu| = |X| < \mathfrak{n} + 1$, we have $R \in \overline{UP}(\mathfrak{n})$. The second assertion follows from the first, setting $\mathfrak{m} = \aleph_0$ and using Lemma 0.7.2. \square

It follows from the above result and the remarks preceding it that if a ring R is isomorphic to a right order in $\mathbb{M}_{\mathfrak{n}}(D)$, where $0 < \mathfrak{n} < \aleph_0$ and D is a division ring, then $R \in UP(\mathfrak{m})$ for some \mathfrak{m} such that $\mathfrak{n} \leq \mathfrak{m} \leq \mathfrak{n}^2$. Thus every prime right (and, by symmetry, left) Goldie ring is uniformly strongly prime. (The reader will recall that this fact was established in the proof of Proposition 1.7, using the Faith-Utumi Theorem (Theorem 0.7.7).)

Let M be an $\mathfrak{n} \times \mathfrak{m}$ matrix over any ring R , where $0 < \mathfrak{n} \leq \mathfrak{m} < \aleph_0$. Any $\mathfrak{n} \times \mathfrak{k}$ matrix (where $0 < \mathfrak{k} \leq \mathfrak{m}$) obtained from M by deleting $\mathfrak{m} - \mathfrak{k}$ columns of M will be called an $\mathfrak{n} \times \mathfrak{k}$ minor of M .

LEMMA 2. *Let \mathfrak{n} be a finite nonzero cardinal and D a division ring. Let $\{A_{\gamma} : \gamma < \mathfrak{m}\}$ be a finite subset of $\mathbb{M}_{\mathfrak{n}}(D)$, with $\mathfrak{m} \geq \mathfrak{n}$. For each \mathfrak{n} -tuple $\underline{x} \in D^{\mathfrak{n}}$ (regarded as a column vector), let $A(\underline{x})$ denote the $\mathfrak{n} \times \mathfrak{m}$ matrix whose γ -th column is $A_{\gamma}\underline{x}$ for each $\gamma \in \{0, \dots, \mathfrak{m} - 1\}$. Then the following assertions are equivalent:*

- (i) $\{A_{\gamma} : \gamma < \mathfrak{m}\}$ is a uniform insulator for $\mathbb{M}_{\mathfrak{n}}(D)$;
- (ii) $\text{rank } A(\underline{x}) = \mathfrak{n}$ for all nonzero $\underline{x} \in D^{\mathfrak{n}}$;
- (iii) for each nonzero $\underline{x} \in D^{\mathfrak{n}}$, there exists an $\mathfrak{n} \times \mathfrak{n}$ minor of $A(\underline{x})$ whose rank is \mathfrak{n} .

Proof. The equivalence of (ii) and (iii) is a basic result of linear algebra. For the sake of completeness, however, we provide a proof. The implication (iii) \Rightarrow (ii) is obvious. To establish (ii) \Rightarrow (iii), let $\underline{0} \neq \underline{x} \in D^{\mathfrak{n}}$. We can clearly regard the columns of $A(\underline{x})$ as elements of $D^{\mathfrak{n}}$. Let \mathcal{J} be a maximal independent set of columns of $A(\underline{x})$ and let $\langle \mathcal{J} \rangle$ denote the subspace of $D^{\mathfrak{n}}$ generated by \mathcal{J} . By the maximality of \mathcal{J} among independent sets of columns, $\langle \mathcal{J} \rangle$ is the column space of $A(\underline{x})$. By hypothesis, $\mathfrak{n} = \text{rank } A(\underline{x}) = \dim(\text{column space of } A(\underline{x})) = \dim \langle \mathcal{J} \rangle = |\mathcal{J}|$. Clearly, if B is chosen to be the $\mathfrak{n} \times \mathfrak{n}$ minor of $A(\underline{x})$ whose set of columns is \mathcal{J} , then $\text{rank } B = \mathfrak{n}$. Thus (iii) holds.

(i) \Rightarrow (ii): Suppose that $\text{rank } A(\underline{x}) < \mathfrak{n}$ for some nonzero $\underline{x} \in D^{\mathfrak{n}}$. Then $\underline{y}A(\underline{x}) = \underline{0}$ for some nonzero $\underline{y} \in D^{\mathfrak{n}}$ (regarded as a row vector). Define $Y, X \in \mathbb{M}_{\mathfrak{n}}(D)$ by:

$$Y_{(\alpha)} = \begin{cases} \underline{y} & \text{for } \alpha = 0; \\ \underline{0} & \text{for } \alpha > 0, \end{cases} \quad \text{and} \quad X^{(\beta)} = \begin{cases} \underline{x} & \text{for } \beta = 0; \\ \underline{0} & \text{for } \beta > 0. \end{cases}$$

Clearly, $Y, X \neq 0$, yet $YA_\gamma X = 0$ for all $\gamma \in \{0, \dots, m-1\}$. This contradicts (i).

(ii) \Rightarrow (i): Let $Y, X \in M_n(D)$ with $Y \neq 0$. Then

$$\begin{aligned} (\forall \gamma \in m) (YA_\gamma X = 0) &\Rightarrow (\forall \alpha \in n) (\forall \beta \in n) (Y_{(\alpha)} A(X^{(\beta)}) = 0) \\ &\Rightarrow (\forall \beta \in n) (\text{rank } A(X^{(\beta)}) < n) \\ &\Rightarrow (\forall \beta \in n) (X^{(\beta)} = \underline{0}) \text{ (by (ii))} \\ &\Rightarrow X = 0. \end{aligned}$$

This shows that $\{A_\gamma : \gamma < m\}$ is a uniform insulator for $M_n(D)$. □

Note that if $\underline{x} = (x_1, x_2, \dots, x_n) \in D^n$ then $A(\underline{x})$ (as defined above) is an $n \times m$ matrix each of whose entries is a linear homogeneous form (over D) in x_1, x_2, \dots, x_n .⁶ Conversely, it is equally clear that any such matrix can be written as $A(\underline{x})$ for a suitable subset $\{A_\gamma : \gamma < m\}$ of $M_n(D)$. If D is a field then, by the previous lemma, $\{A_\gamma : \gamma < m\}$ is a uniform insulator for $M_n(D)$ if and only if for each nonzero $\underline{x} \in D^n$, there exists an $n \times n$ minor of $A(\underline{x})$ whose determinant is nonzero. Note also that in this instance, the determinant of every $n \times n$ minor of $A(\underline{x})$ is an n -th degree homogeneous form in x_1, x_2, \dots, x_n .

THEOREM 3. *Let n be a finite nonzero cardinal and D a division ring. If R is isomorphic to a right order in $M_n(D)$ then $R \in UP(m)$ for some m such that $n \leq m \leq 2n-1$.*

Proof. Note first that we cannot have $R \in UP(m)$ for any $m < n$, since $R \in P_r(n)$ (Theorem 1.2). It suffices, in view of Lemma 1, to prove that $M_n(D) \in \overline{UP}(2n-1)$.

$$\text{Consider } A(\underline{x}) = \begin{bmatrix} x_1 & x_2 & \cdot & \cdot & x_{n-1} & x_n & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & x_1 & x_2 & \cdot & \cdot & x_{n-1} & x_n & \cdot & \cdot & \cdot & 0 \\ & & \cdot & & & & & \cdot & & & \\ & & & \cdot & & & & & \cdot & & \\ 0 & \cdot & \cdot & \cdot & x_1 & x_2 & \cdot & \cdot & x_{n-1} & x_n & 0 \\ 0 & 0 & \cdot & \cdot & 0 & x_1 & x_2 & \cdot & \cdot & x_{n-1} & x_n \end{bmatrix}$$

where $\underline{x} = (x_1, x_2, \dots, x_n)$. Notice that $A(\underline{x})$ is an $n \times (2n-1)$ matrix each of whose entries is a linear homogeneous form in x_1, x_2, \dots, x_n . Suppose that $\text{rank } A(\underline{x}) < n$ for some nonzero $\underline{x} \in D^n$. Then every $n \times n$ minor of $A(\underline{x})$ has rank $< n$. Whenever $\alpha < n$, we let $A(\underline{x}, \alpha)$ denote the $n \times n$ minor of $A(\underline{x})$ for which $A(\underline{x}, \alpha)^{(\beta)} = A(\underline{x})^{(\beta \oplus \alpha)}$ whenever $\beta < n$. Since $A(\underline{x}, 0)$ is an $n \times n$ matrix with zeros below the main diagonal and since $\text{rank } A(\underline{x}, 0) < n$, we must have $x_1 = 0$. Equally, since

⁶ It would be more accurate in this context to interpret $\underline{x} = (x_1, x_2, \dots, x_n)$ as an n -tuple of indeterminates (hence an element of $D[x_1, x_2, \dots, x_n]^n$), rather than as an n -tuple of elements in the division ring D . Nevertheless, for the sake of simplicity, we tolerate this ambiguity and regard $\underline{x} = (x_1, x_2, \dots, x_n)$ both as an n -tuple in D^n and as an n -tuple in $D[x_1, x_2, \dots, x_n]^n$.

$A(\underline{x}, 1)$ is an $n \times n$ matrix with zeros below the main diagonal and since $\text{rank } A(\underline{x}, 1) < n$, we must have $x_2 = 0$. Continuing in this way, we obtain $x_1 = x_2 = \dots = x_n = 0$, i.e., $\underline{x} = \underline{0}$, a contradiction. Hence $\text{rank } A(\underline{x}) = n$ for all nonzero $\underline{x} \in D^n$. By the previous lemma, $\mathcal{M}_n(D)$ has a uniform insulator of cardinality $2n - 1$. \square

We shall show presently, with the aid of several examples, that the inequality $n \leq m \leq 2n - 1$ of Theorem 3 cannot be sharpened further. This is an interesting fact which shows that the uniform bound of primeness of a right order in $\mathcal{M}_n(D)$ (where n is a finite nonzero cardinal and D a division ring) is not determined solely by n . This contrasts with the situation for the right and left bounds of primeness of such orders (see Theorem 1.2).

Theorem 4 below may be seen as a converse of Theorem 3. In it, we prove that every ring which is uniformly strongly prime of bound greater than 1 is necessarily prime right or left Goldie. Theorem 4 may be viewed, therefore, as a “uniformly strongly prime” analogue of Theorem 1.2. (Recall that Theorem 1.2 asserts that every ring which is right prime of bound greater than 1 is necessarily prime right Goldie.) We also remark that Theorem 3, which attempts to locate the uniform bound of primeness of a prime right Goldie ring, takes on greater significance in the light of Theorem 4.

THEOREM 4. *If n is a finite cardinal such that $n > 1$, and $R \in UP(n)$, then R is a prime right or left Goldie ring.*

Proof. Suppose $R \in P_r(1) \cap P_l(1)$. Let X be a uniform insulator for R with $|X| = n$ and choose $x \in X$. Since $R \notin UP(1)$, there must exist nonzero elements $r, s \in R$ such that $rxs = 0$. Since $R \in P_r(1) \cap P_l(1)$, we may choose $\bar{r}, \bar{s} \in R$ such that $(0 :_l \bar{r}r) = (0 :_r s\bar{s}) = 0$. Then $\bar{r}rXs\bar{s} \setminus \{0\}$ is a uniform insulator for R since, for any $a, b \in R$,

$$a(\bar{r}rXs\bar{s})b = 0 \Rightarrow a\bar{r}r = 0 \text{ or } s\bar{s}b = 0 \Rightarrow a = 0 \text{ or } b = 0.$$

But $|\bar{r}rXs\bar{s} \setminus \{0\}| < |X| = n$, a contradiction. We must therefore have that $R \in P_r(n)$ or $R \in P_l(n)$ (by Proposition 1.7), whereupon the result follows from Theorem 1.2. \square

In [HL75, Corollary to Proposition II.3, p214], Handelman and Lawrence prove that if R is a (von Neumann) regular ring with identity then the conditions

- (i) R is simple; and
- (ii) R is (right) strongly prime

are equivalent. In the next proposition, we show that uniformly strongly prime regular rings with identity have a very “simple” characterization.

PROPOSITION 5. *The following assertions are equivalent for any regular ring R with identity:*

- (i) R is a simple artinian ring;
- (ii) R is uniformly strongly prime.

Proof. (i) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (i): Suppose first that $R \in UP(1)$ and let $\{x\}$ be a uniform insulator for R . Since R is regular, $Rx = Re$ for some idempotent $e \in R$ (Theorem 0.9.1). Write $e = tx$ with $t \in R$. Since $(0 :_r e) = (0 :_r tx) = 0$, we must have $e = tx = 1_R$. Similarly, x is right invertible so x is a unit of R . This implies that R is a domain, since for any $a, b \in R$, we have

$$ab = 0 \Rightarrow axx^{-1}b = 0 \Rightarrow a = 0 \text{ or } x^{-1}b = 0 \Rightarrow a = 0 \text{ or } b = 0.$$

R must therefore be a division ring.

Suppose now that R is uniformly strongly prime of bound greater than 1. By Theorem 4, R is isomorphic to a right or left order in a simple artinian ring S . We claim that $R \cong S$. To see this, take $s \in S$. Identifying R with its image in S , we get that $su \in R$ for some unit u of S contained in R . Since u is a regular element of R and R is a regular ring, u must be a unit of R . Therefore, $s = suu^{-1} \in R$, so $R = S$. Thus R is a simple artinian ring. \square

Recall that in Proposition 2.1 we proved that if R is a subring of S such that R_R is essential in S_R and $R \in P_r(\mathfrak{m})$, then $S \in \overline{P}_r(\mathfrak{m})$. The proof is similar in spirit to that of the next proposition, which is the analogous result for uniform primeness.

PROPOSITION 6. *Let R be a subring of S and suppose that R_R and ${}_R R$ are essential submodules of S_R and ${}_R S$, respectively. For any nonzero cardinal \mathfrak{n} , if $R \in UP(\mathfrak{n})$ then $S \in \overline{UP}(\mathfrak{n})$.*

Proof. Let X be a uniform insulator for R with $|X| < \mathfrak{n} + 1$ and suppose that $aXb = 0$, where $0 \neq a, b \in S$. Since R_R is essential in S_R , it follows that $\{br + bn : r \in R, n \in \mathbb{Z}\} \cap R \neq 0$, so $0 \neq br + bn \in R$ for some $r \in R, n \in \mathbb{Z}$. Since R is a prime ring, $(br + bn)R \neq 0$, and therefore $bR \neq 0$. Then $R \cap bR \neq 0$, so $0 \neq bs \in R$ for some $s \in R$. Using the fact that ${}_R R$ is essential in ${}_R S$, a similar argument establishes the existence of some $t \in R$ such that $0 \neq ta \in R$. Then $taXbs = 0$ and so $ta = 0$ or $bs = 0$, a contradiction. Thus X is a uniform insulator for S , from which it follows that $S \in \overline{UP}(\mathfrak{n})$. \square

The following example shows that the requirement that ${}_R R$ be essential in ${}_R S$ cannot be dropped from the above result.

EXAMPLE 1. Let R be a ring with identity such that $R \in UP(1)$ and $\dim R_R \geq \aleph_0$ (e.g., a domain which is not right Ore). Consider the maximal right ring of quotients $Q = Q_{\max}(R_R)$. Certainly, R_R is essential in Q_R . We shall show that Q is not uniformly strongly prime. Since R is

right strongly prime, the singular submodule $Z(R_R)$ of R_R is zero, by [HL75, Proposition II.2, p213]. (This result is, of course, just a special case of Theorem II.6.3 ((ii) \Rightarrow (v)), where we set $\mathfrak{m} = \aleph_0$ and $\sigma = Z$.) By Proposition 0.9.5, Q is a regular, right self-injective ring. Since $\dim R_R$ is not finite, it follows from Sandomierski's Theorem (Theorem 0.9.6) that Q is not simple artinian. By Proposition 5, Q cannot be uniformly strongly prime. \square

In the sequel to Proposition 2.1, we remarked that if $S = \mathbb{M}_2(D)$ (D a division ring) and R is the subring of S consisting of all upper triangular 2×2 matrices over D , then R_R is essential in S_R but R is not even semiprime. A symmetrical argument shows that ${}_R R$ is essential in ${}_R S$. Since S is uniformly strongly prime, but R is not, the converse of Proposition 6 is certainly not true. As Corollary 7 below shows, more can be said if the hypotheses of Proposition 6 are strengthened. The result follows immediately from Lemma 1 and Proposition 6.

COROLLARY 7. *Let $\mathfrak{m}, \mathfrak{n}$ be cardinals with $0 < \mathfrak{n} < \mathfrak{m}$ and $\mathfrak{m} \geq \aleph_0$. Let R be a right \mathfrak{m} -order in a ring S and suppose further that ${}_R R$ is an essential submodule of ${}_R S$ (e.g., suppose R is also a left order in S). Then $R \in UP(\mathfrak{n})$ if and only if $S \in UP(\mathfrak{n})$. \square*

We point out again that the restrictions on R and S in the above result are necessary. In particular, we show in the following example that the requirement that ${}_R R$ be essential in ${}_R S$ cannot be dispensed with.

EXAMPLE 2. We construct a ring S with a subring R such that R is a right order in S and $R \in UP(1)$ but S is not uniformly strongly prime.

Let F be a field. Let x, y, z be distinct symbols and let H be the free product of monoids $H_1 \star H_2 \star H_3$, where H_1 and H_2 are the free monoids on $\{x\}$ and $\{z\}$, respectively, and H_3 is the free group on $\{y\}$. If $w \in H$, we define the *degree* of w , denoted ∂w , to be the sum of the absolute values of the powers of x, y and z occurring in w . (For example, $\partial(1_H) = 0$; $\partial(y^{-2}z^3) = 5$; $\partial(yx^{-3}z) = 6$, etc.) Clearly, $\partial(ww') \leq \partial w + \partial w'$ for all $w, w' \in H$. Because of the presence of negative powers of y , however, this inequality is strict in some cases, so ∂ is not a "degree function" in the strong sense associated with free algebras over fields and monomial algebras. Consider the monoid ring FH . For each $w \in H \setminus \{1_H\}$, choose a positive integer $n(w)$ such that $n(w) > \partial w$. Set $H' = H \setminus (\{1_H\} \cup \{z^n : n > 0\})$ and let I be the ideal of FH generated by $\{z^{n(w)}wz^{n(w)} : w \in H'\}$. Define $S = FH/I$. (S is not unlike the monomial algebras discussed in the previous section.) If $w \in H$, we denote the image of w in S by \bar{w} and the image of a subset X of H in S by \bar{X} .

We show first that S is not uniformly strongly prime. We claim that \bar{z} is not nilpotent. Indeed, suppose that $\bar{z}^m = 0$ for some positive integer m . Then $z^m \in I$. Inasmuch as every element of I is expressible in the form $\sum_{i=1}^n a_i v_i (z^{n(w_i)} w_i z^{n(w_i)}) v'_i$, where $a_i \in F$, $w_i \in H'$ and $v_i, v'_i \in H$ for all $i \in \{1, \dots, n\}$, it follows that $z^m = v z^{n(w)} w z^{n(w)} v'$ for some $w \in H'$ and $v, v' \in H$, an

impossibility. Thus $\bar{z}^m \neq 0$ for all $m > 0$. Let X be a finite nonempty subset of FH and set $Y = \{w \in H : w \in \text{supp } r \text{ for some } r \in X\}$. Since Y is finite, it is clear that we can choose n sufficiently large so that $z^n w z^n \in I$ for all $w \in Y$. It follows that $\bar{z}^n \bar{X} \bar{z}^n = 0$. Thus \bar{X} is not a uniform insulator for S . This shows that S is not uniformly strongly prime.

Let K be the submonoid of H generated by $\{x, y\} \cup \{wy^{n(w)} : w \in H\}$. We define R to be the image of FK in S . As such, R is a subring of S . We claim that R is a right order in S . To see this, note first that \bar{y} is a unit of S contained in R . Moreover, it is not difficult to see that if $s \in S$ then, for sufficiently large n , we have $s\bar{y}^n \in R$. This validates our claim.

Finally, we show that $\{\bar{x}\}$ is a uniform insulator for R , which clearly implies that $R \in UP(1)$. Suppose, on the contrary, that $\bar{r}_1 \bar{x} \bar{r}_2 = 0$, i.e., $r_1 x r_2 \in I$, for some $r_1, r_2 \in FK \setminus (FK \cap I)$. Clearly no generality is lost by our assuming that no element of $\text{supp } r_1$ or $\text{supp } r_2$ is in I . Certainly, $r_1 x r_2 \neq 0$. Indeed, if w_1 and w_2 are elements of maximum degree in $\text{supp } r_1$ and $\text{supp } r_2$, respectively, then $\partial(w_1 x w_2) = \partial w_1 + 1 + \partial w_2 \geq \partial(u x u') = \partial u + 1 + \partial u'$ for all $u \in \text{supp } r_1$ and $u' \in \text{supp } r_2$. It follows that $w_1 x w_2 \in \text{supp}(r_1 x r_2)$, so $r_1 x r_2 \neq 0$. Let $u_1 x u_2 \in \text{supp}(r_1 x r_2)$ with $u_1 \in \text{supp } r_1$ and $u_2 \in \text{supp } r_2$. Since $r_1 x r_2 \in I$, $u_1 x u_2$ is expressible in the form

$$(1) \quad u_1 x u_2 = v_1 z^{n(w)} w z^{n(w)} v_2 \quad (\text{for some } w \in H' \text{ and } v_1, v_2 \in H).$$

If $\partial(u_1 x) = \partial u_1 + 1 \leq \partial v_1$ then, cancelling $u_1 x$ on the left in (1), we obtain $u_2 = v_1' z^{n(w)} w z^{n(w)} v_2 \in I$ for some $v_1' \in H$, a contradiction. A similar contradiction results from supposing that $\partial(x u_2) = \partial u_2 + 1 \leq \partial v_2$. Suppose that $\partial u_1 \geq \partial v_1$ and $\partial u_2 \geq \partial v_2$. Cancelling v_1 and v_2 on the left and right hand sides of (1), respectively, we obtain

$$(2) \quad u_1' x u_2' = z^{n(w)} w z^{n(w)} \quad (\text{for some } u_1', u_2' \in H).$$

Clearly we must have $\partial u_1', \partial u_2' \geq \partial(z^{n(w)}) = n(w)$, so, cancelling $z^{n(w)}$ on both sides of (2), we obtain $w = u_1'' x u_2''$ for some $u_1'', u_2'' \in H$. Then $u_1 = v_1 u_1' = v_1 z^{n(w)} u_1''$. It is not difficult to see that in order that $v_1 z^{n(w)} u_1''$ be an element of K , u_1'' must contain a power of y whose degree exceeds $\partial(z^{n(w)}) = n(w)$. But this is impossible, because $\partial u_1'' \leq \partial w < n(w)$. Thus $u_1 = v_1 z^{n(w)} u_1'' \notin K$, a contradiction. We may therefore conclude that $\{\bar{x}\}$ is a uniform insulator for R . \square

The following result shows that rings which are uniformly strongly prime of bound 1 are easily constructed.

PROPOSITION 8. *Let R be a uniformly strongly prime ring. If G and G' are monoids with $|G| \geq 3$ or $|G'| \geq 3$, then $R[G \star G'] \in UP(1)$.*

Proof. Set $H = G \star G'$. In the proof of Proposition 3.3(iii), we showed that the semigroup H satisfies Condition E (see §3) where, in particular, if A is an arbitrary finite nonempty subset of H then T_A may be chosen to be $\{g', gg'\}$ or $\{g, gg'\}$, where g is any fixed element of $G \setminus \{1_G\}$ and g' any fixed element of $G' \setminus \{1_{G'}\}$. It follows from Lemma 3.2(ii) that if $0 \neq z \in RH$ and X is a

uniform insulator for R then $X\{g', g'g\}$ or $X\{g, gg'\}$ is a right insulator for z in RH . Consequently, $X\{g', g, g'g, gg'\}$ is a uniform insulator for RH . In the sequel to Proposition 3.3, we showed that $\dim RH_{RH} \geq \aleph_0$ and so RH cannot be a prime right Goldie ring. A similar argument shows that RH is not a prime left Goldie ring either. By Theorem 4, $RH \in UP(1)$. \square

Results proved so far suggest that the study of finite uniform bounds of primeness in general is not significantly more extensive than the study of uniform bounds of primeness of finite dimensional matrix rings over division rings. (We refer the reader to Theorem 4 in particular.) Nevertheless, as will be apparent in the sequel, the calculation of such bounds is, in many instances, a nontrivial task.

Our next main result is Theorem 11, which describes the uniform bound of primeness of the ring $M_n(F)$ for a finite nonzero cardinal n and an algebraically closed field F . Its proof utilises several standard notions and results from commutative algebra which will be presented here, for the sake of completeness and for the reader's convenience. A more detailed account may be found in standard texts such as [Mat86] or [Kap70].

Let R be a commutative ring with identity. A prime ideal P of R is said to have *height* n , where n is a nonnegative integer, if there exists a chain $P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n = P$ of prime ideals of R and no longer such chain exists. In this case we write $\text{ht } P = n$. It is quite possible that a given prime ideal may not have finite height. Dually, we say that a prime ideal P of R has *coheight* n , where n is a nonnegative integer, if there exists a chain $P = P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_n$ of prime ideals of R and no longer such chain exists. In this case we write $\text{coht } P = n$. Note that a prime ideal P of R is a maximal proper ideal of R if and only if $\text{coht } P = 0$. Again, we point out that a given prime ideal need not have finite coheight. If every prime ideal of R has finite height and there exists a prime ideal with maximal (finite) height n , then R is said to have *classical Krull dimension* n . In this case we write $\text{cl-K-dim } R = n$.

The so-called *Principal Ideal Theorem* is probably the most important theorem in commutative ring theory pertaining to classical Krull dimension. In its most general form, the theorem states that if R is a commutative noetherian ring with identity and I an ideal of R generated by n elements (n a positive integer) then any prime ideal P of R which is minimal among prime ideals of R containing I has height at most n . It follows from the Principal Ideal Theorem that every nonzero prime ideal in a principal ideal domain with identity has height 1, and therefore that every principal ideal domain with identity has classical Krull dimension 1. In particular, if F is a field then the polynomial ring $F[x]$ has classical Krull dimension 1. Moreover, a straightforward inductive argument shows that $\text{cl-K-dim}(F[x_1, x_2, \dots, x_n]) = n$, for every positive integer n . More strongly, it can be shown that $\text{ht } P + \text{coht } P = n$ for all prime ideals P of $F[x_1, x_2, \dots, x_n]$ and all positive integers n [Mat86, Ex 5.1, p37].

Another important consequence of the Principal Ideal Theorem is the fact that every prime ideal in a commutative noetherian ring with identity has finite height. This implies that if R is a

commutative noetherian ring with identity then $\text{Spec } R$ (set of prime ideals of R) satisfies the DCC. Interestingly, the dual assertion is not true: it is possible for a prime ideal in a commutative noetherian ring with identity not to have finite coheight – this despite the fact that $\text{Spec } R$ obviously satisfies the ACC (see [AM69, Ex 11.4, p126]).

Our next immediate goal is to state a famous result in classical algebraic geometry, the so-called *Nullstellensatz* (Zeros Theorem) of Hilbert. Classical algebraic geometry is the study of simultaneous solutions of polynomial equations

$$f(x_1, x_2, \dots, x_n) = 0, \quad f \in S,$$

where K is a field and $S \subseteq K[x_1, x_2, \dots, x_n]$, n a positive integer. A *solution* of this system is an n -tuple $(a_1, a_2, \dots, a_n) \in F^n$, where F is an algebraically closed extension field of K and $f(a_1, a_2, \dots, a_n) = 0$ for all $f \in S$. The set of all such $(a_1, a_2, \dots, a_n) \in F^n$ is called the *affine K -variety in F^n defined by S* and is denoted by $V(S)$. Thus $V(S) := \{(a_1, a_2, \dots, a_n) \in F^n : f(a_1, a_2, \dots, a_n) = 0 \text{ for all } f \in S\}$. Note that if I is the ideal of $K[x_1, x_2, \dots, x_n]$ generated by S , then $V(I) = V(S)$.

The assignment $S \mapsto V(S)$ clearly defines an order reversing function from the power set of $K[x_1, x_2, \dots, x_n]$ to the power set of F^n . Conversely, it is possible to define an order reversing function from the power set of F^n to the power set of $K[x_1, x_2, \dots, x_n]$ by $Y \mapsto J(Y)$, where $Y \subseteq F^n$ and $J(Y) := \{f \in K[x_1, x_2, \dots, x_n] : f(a_1, a_2, \dots, a_n) = 0 \text{ for all } (a_1, a_2, \dots, a_n) \in Y\}$. Note that $J(Y)$ is an ideal of $K[x_1, x_2, \dots, x_n]$.

If I is an ideal of a ring R , we define $\text{rad } I = \bigcap \{P \in \text{Spec } R : P \supseteq I\}$ and we call I a *radical ideal* of R if $\text{rad } I = I$. This terminology and notation is used here because it is very standard in classical algebraic geometry. We note, however, that an ideal I of a ring R is a radical ideal of R if and only if the ring R/I is semiprime, i.e., $\beta(R/I) = R/I$ (see Chapter 0, §5). A radical ideal is therefore sometimes called a *semiprime ideal* and the notions defined above are closely related to the theory of radicals in the category of rings, which will receive detailed treatment in Chapter IV.

THEOREM 9. (Hilbert's "Nullstellensatz") [Hun74, Proposition 7.4, p412] *Let K be a field and F an algebraically closed extension field of K . Let I be a proper ideal of $K[x_1, x_2, \dots, x_n]$, where n is a positive integer. Let $V(I) = \{(a_1, a_2, \dots, a_n) \in F^n : g(a_1, a_2, \dots, a_n) = 0 \text{ for all } g \in I\}$. Then $\text{rad } I = J(V(I)) = \{f \in K[x_1, x_2, \dots, x_n] : f(a_1, a_2, \dots, a_n) = 0 \text{ for all } (a_1, a_2, \dots, a_n) \in V(I)\}$. \square*

It follows from Hilbert's Nullstellensatz that if I_1 and I_2 are radical ideals of $K[x_1, x_2, \dots, x_n]$ (where K is a field and n a positive integer) then $V(I_1) = V(I_2)$ implies $I_1 = \text{rad } I_1 = J(V(I_1)) = J(V(I_2)) = \text{rad } I_2 = I_2$. This shows that the restriction of the operator V to the set of all radical ideals of $K[x_1, x_2, \dots, x_n]$ is strictly order reversing. It also follows from the

Nullstellensatz that if I is a proper ideal of $K[x_1, x_2, \dots, x_n]$ then $V(I) \neq \emptyset$, because $J(\emptyset) = K[x_1, x_2, \dots, x_n] \supsetneq \text{rad } I = J(V(I))$.

The next result is attributed by the author of [Mat86] to Macaulay [Mac16, p54]. Eagon has proved a more general version pertaining to arbitrary commutative noetherian rings with identity. A proof of his result may be found in [Mat86, Theorem 13.10, p104].

PROPOSITION 10. *Let $R = F[x_1, x_2, \dots, x_n]$, where F is a field and n a positive integer. Let A be a $k \times m$ matrix over R , for some positive integers k, m with $k \leq m$, and let I be the ideal of R generated by the determinants of all $k \times k$ minors of A . If P is minimal among prime ideals of R containing I then $\text{ht } P \leq m - k + 1$. \square*

With reference to Proposition 10, we point out that since the ring R is commutative and noetherian, with identity, there will always exist a prime ideal P of R which is minimal over I .

It follows easily from the Faith-Utumi Theorem (Theorem 0.7.7) that if R is a right order in $\mathbb{M}_{\mathbf{n}}(F)$, where F is a field and \mathbf{n} a finite nonzero cardinal, then R is also a left order in $\mathbb{M}_{\mathbf{n}}(F)$. By symmetry, we may conclude that the right orders in $\mathbb{M}_{\mathbf{n}}(F)$ are exactly the left orders in $\mathbb{M}_{\mathbf{n}}(F)$. This being so, we choose to omit the prefix “right” or “left” in this context, and to speak simply of an “order” in $\mathbb{M}_{\mathbf{n}}(F)$.

We are finally in a position to prove our next result on uniform bounds of primeness, which sharpens our knowledge of the values of such bounds in matrix rings.

THEOREM 11. *If F is an algebraically closed field, \mathbf{n} a finite nonzero cardinal and R an order in $\mathbb{M}_{\mathbf{n}}(F)$, then $R \in UP(2\mathbf{n} - 1)$.*

Proof. Clearly, we may assume that $\mathbf{n} > 1$. In view of Theorem 3 and Corollary 7, it suffices to show that $\mathbb{M}_{\mathbf{n}}(F)$ has no uniform insulator of cardinality less than $2\mathbf{n} - 1$. We use Lemma 2. Let $\{A_{\gamma} : \gamma < \mathbf{m}\}$ be an arbitrary subset of $\mathbb{M}_{\mathbf{n}}(F)$ with $1 < \mathbf{n} \leq \mathbf{m} < 2\mathbf{n} - 1$. Define $A(\underline{x}) = A(x_1, x_2, \dots, x_{\mathbf{n}})$ as in Lemma 2. Let I be the ideal of $K = F[x_1, x_2, \dots, x_{\mathbf{n}}]$ generated by the determinants of all $\mathbf{n} \times \mathbf{n}$ minors of $A(\underline{x})$. By Proposition 10, there is a prime ideal P , minimal among prime ideals of K containing I , such that $\text{ht } P \leq \mathbf{m} - \mathbf{n} + 1 < (2\mathbf{n} - 1) - \mathbf{n} + 1 = \mathbf{n}$. Now $\text{ht } P + \text{coht } P = \mathbf{n}$, so $\text{coht } P \geq 1$, i.e., P is not a maximal proper ideal of K . We can therefore choose a prime ideal Q of K such that $Q \supsetneq P$. For each ideal A of K , consider $V(A) = \{(y_1, y_2, \dots, y_{\mathbf{n}}) \in F^{\mathbf{n}} : f(y_1, y_2, \dots, y_{\mathbf{n}}) = 0 \text{ for all } f \in A\}$. By Hilbert’s Nullstellensatz (Theorem 9), $V(I) \supseteq V(P) \supsetneq V(Q) \neq \emptyset$, so $|V(I)| > 1$. It follows that $V(I)$ must contain a nonzero element $\underline{x} \in F^{\mathbf{n}}$. Therefore, every $\mathbf{n} \times \mathbf{n}$ minor of $A(\underline{x})$ has zero determinant. By Lemma 2, $\{A_{\gamma} : \gamma < \mathbf{m}\}$ is not a uniform insulator for $\mathbb{M}_{\mathbf{n}}(F)$. \square

The next lemma generalizes [Ols87, Lemma 8, p97].

LEMMA 12. Let \mathfrak{m} be a nonzero cardinal and let e be a nonzero idempotent in a ring R . If $R \in UP(\mathfrak{m})$ then $eRe \in \overline{UP}(\mathfrak{m})$.

Proof. It is easily checked that if X is a uniform insulator for R with $|X| < \mathfrak{m} + 1$, then eXe is a uniform insulator for eRe , and $|eXe| \leq |X| < \mathfrak{m} + 1$. \square

We now investigate the uniform bound of primeness of $\mathbb{M}_{\mathfrak{n}}(\mathbb{R})$ for various positive integers \mathfrak{n} . Interestingly, the parity of \mathfrak{n} plays a role here.

PROPOSITION 13. Let \mathfrak{n} be an odd integer greater than 1. If R is isomorphic to an order in $\mathbb{M}_{\mathfrak{n}}(\mathbb{R})$ then $R \in UP(\mathfrak{m})$ for some integer \mathfrak{m} such that $\mathfrak{n} < \mathfrak{m} \leq 2\mathfrak{n} - 1$.

Proof. Again, in view of Theorem 3 and Corollary 7, it suffices to show that $\mathbb{M}_{\mathfrak{n}}(\mathbb{R}) \notin UP(\mathfrak{n})$. Let $\{A_{\gamma} : \gamma < \mathfrak{n}\}$ be an arbitrary subset of $\mathbb{M}_{\mathfrak{n}}(\mathbb{R})$. Define $A(\underline{x}) = A(x_1, x_2, \dots, x_{\mathfrak{n}})$ as in Lemma 2. The result follows from Lemma 2 if we can show that $\det A(\underline{x}) = 0$ for some nonzero $\underline{x} \in \mathbb{R}^{\mathfrak{n}}$. Replace x_1 with x and each of $x_2, x_3, \dots, x_{\mathfrak{n}}$ with y . Notice that $\det A(\underline{x}) = \det(A(x, y, y, \dots, y)) = f(x, y)$ is an \mathfrak{n} -th degree homogeneous form in x and y over \mathbb{R} . It suffices to show that $f(x, y) = 0$ for some nonzero $(x, y) \in \mathbb{R} \times \mathbb{R}$. If y divides $f(x, y)$ then, clearly, $f(x, 0) = 0$ for all $x \in \mathbb{R}$. Suppose that y does not divide $f(x, y)$. Then $f(x, y)$ must contain the term $x^{\mathfrak{n}}$ (with a suitable nonzero coefficient). It follows that $f(x, 1)$ is an \mathfrak{n} -th degree polynomial in x over \mathbb{R} and, since \mathfrak{n} is odd, this polynomial must have a zero. Thus $f(x, 1) = 0$ for some $x \in \mathbb{R}$. We may therefore conclude that in either case, $f(x, y) = 0$ for some nonzero $(x, y) \in \mathbb{R} \times \mathbb{R}$, as required. \square

EXAMPLE 3. In $\mathbb{M}_2(\mathbb{R})$, if $A(\underline{x}) = A(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix}$ then $\det A(\underline{x}) = x_1^2 + x_2^2 = 0$ if and only if $\underline{x} = (x_1, x_2) = (0, 0)$. By Lemma 2, therefore, we have $\mathbb{M}_2(\mathbb{R}) \in UP(2)$.

In $\mathbb{M}_4(\mathbb{R})$, if $A(\underline{x}) = A(x_1, x_2, x_3, x_4) = \begin{bmatrix} x_1 & x_2 & x_3 & -x_4 \\ -x_2 & x_1 & x_4 & x_3 \\ x_3 & x_4 & -x_1 & x_2 \\ -x_4 & x_3 & -x_2 & -x_1 \end{bmatrix}$ then

$$\det A(\underline{x}) = x_1^4 + x_2^4 + x_3^4 + x_4^4 + 2(x_1^2 x_2^2 + x_1^2 x_3^2 + x_1^2 x_4^2 + x_2^2 x_3^2 + x_2^2 x_4^2 + x_3^2 x_4^2) \\ = 0 \text{ if and only if } \underline{x} = (x_1, x_2, x_3, x_4) = \underline{0},$$

so $\mathbb{M}_4(\mathbb{R}) \in UP(4)$.

Inasmuch as $\mathbb{M}_3(\mathbb{R}) \cong e\mathbb{M}_4(\mathbb{R})e$ for a suitable idempotent e in $\mathbb{M}_4(\mathbb{R})$, it follows from the above and Lemma 12 and Proposition 13 that $\mathbb{M}_3(\mathbb{R}) \in UP(4)$. \square

We remark that determining the uniform bound of primeness of $\mathbb{M}_{\mathfrak{n}}(\mathbb{R})$ for large finite values of \mathfrak{n} appears to be a difficult task.

An important theorem in the theory of commutative noetherian rings is the so-called *Krull Intersection Theorem*, which asserts that if I is a proper ideal of a commutative noetherian domain R , with identity, then $\bigcap_{0 < \mathbf{n} < \aleph_0} I^{\mathbf{n}} = 0$. It follows easily from this theorem that if $x, p \in R$ with $x \neq 0$, then there is a largest (finite) cardinal \mathbf{n} such that $p^{\mathbf{n}}$ divides x . We shall make use of this corollary in the next theorem, which determines, inter alia, the uniform bounds of primeness of $\mathbb{M}_{\mathbf{n}}(\mathbb{Z}), \mathbb{M}_{\mathbf{n}}(\mathbb{Q})$ and $\mathbb{M}_{\mathbf{n}}(F[y])$, where F is any field.

Recall that an element p of a commutative ring R is said to be *prime* provided that for any $a, b \in R$, the element p divides the product ab only if p divides a or p divides b .

THEOREM 14. *Let D be a commutative noetherian domain, with identity, which contains a nonzero prime element p and let Q denote the field of quotients of D . If R is isomorphic to an order in $\mathbb{M}_{\mathbf{n}}(Q)$ for some finite nonzero cardinal \mathbf{n} then $R \in UP(\mathbf{n})$.*

Proof. Again, it suffices to show that $\mathbb{M}_{\mathbf{n}}(Q) \in \overline{UP}(\mathbf{n})$. Define

$$A(\underline{x}) = A(x_1, x_2, \dots, x_{\mathbf{n}}) = \begin{bmatrix} x_1 & x_2 & \cdot & \cdot & \cdot & x_{\mathbf{n}} \\ px_{\mathbf{n}} & x_1 & x_2 & \cdot & \cdot & x_{\mathbf{n}-1} \\ px_{\mathbf{n}-1} & px_{\mathbf{n}} & x_1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ px_2 & px_3 & \cdot & \cdot & \cdot & x_1 \end{bmatrix}.$$

Suppose $\det A(\underline{x}) = 0$ for some $\underline{x} = (x_1, x_2, \dots, x_{\mathbf{n}}) \in Q^{\mathbf{n}}$. By multiplying the equation $\det A(\underline{x}) = 0$ through by a suitable nonzero element of D if necessary, we may assume that $\{x_1, x_2, \dots, x_{\mathbf{n}}\} \subseteq D$. It is easy to see that the product $(= x_1^{\mathbf{n}})$ of the main diagonal entries of $A(\underline{x})$ is the only term in the expansion of $\det A(\underline{x})$ which contains no entries of $A(\underline{x})$ below the main diagonal. Since p divides every entry of $A(\underline{x})$ below the main diagonal and $\det A(\underline{x}) = 0$, we must have that p divides $x_1^{\mathbf{n}}$, so p divides x_1 (because p is prime). Write $x_1 = px'_1$ with $x'_1 \in D$. Now we modify the matrix $A(\underline{x})$ by:

- (i) replacing x_1 by px'_1 ;
- (ii) dividing the first column by p ;
- (iii) replacing the i -th column by the $(i+1)$ -th column for $1 \leq i \leq \mathbf{n}-1$, and the \mathbf{n} -th column by the first column. This yields the matrix

$$B = \begin{bmatrix} x_2 & x_3 & \cdot & \cdot & \cdot & x'_1 \\ px'_1 & x_2 & x_3 & \cdot & \cdot & x_n \\ px_n & px'_1 & x_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ px_3 & px_4 & \cdot & \cdot & \cdot & x_2 \end{bmatrix}.$$

Since $\det A(\underline{x}) = 0$, it is clear from the operations performed that B also has a zero determinant. Notice also that B has the same basic form as $A(\underline{x})$. To be more precise, $B = A(x_2, x_3, \dots, x_n, x'_1)$. Repeating the above argument, we obtain $x_2 = px'_2$ for some $x'_2 \in D$ and $\det C = 0$, where $C = A(x_3, \dots, x_n, x'_1, x'_2)$. Continuing in this way, we get that $p^{\mathbf{m}}$ divides each element of $\{x_1, x_2, \dots, x_n\}$ for all finite nonzero cardinals \mathbf{m} . It follows from the Krull Intersection Theorem that this is possible only if $x_1 = x_2 = \dots = x_n = 0$. By Lemma 2, we have $\mathbb{M}_{\mathbf{n}}(Q) \in \overline{UP}(\mathbf{n})$. \square

We noted at the beginning of this section that if D is a domain then $\mathbb{M}_{\mathbf{m}}(D) \in UP(\mathbf{m}^+)$ for all infinite cardinals \mathbf{m} . Thus the classes $UP(\mathbf{m})$ are nonempty for all infinite successor cardinals \mathbf{m} ; we also know that $UP(\mathbf{m})$ is empty for all limit cardinals \mathbf{m} . The following corollary, which is an immediate consequence of Theorem 14, completes the picture.

COROLLARY 15. *The class $UP(\mathbf{n})$ is nonempty for all finite nonzero cardinals \mathbf{n} .* \square

The study of uniformly strongly prime rings and uniform bounds of primeness will be taken up again in Chapter IV, §5, in the context of the theory of special radicals on the category of rings.

In conclusion, we point out a further corollary to Theorems 11 and 14. It follows from the former theorem that if D is isomorphic to an order in an algebraically closed field then $\mathbb{M}_{\mathbf{n}}(D) \in UP(2\mathbf{n} - 1)$ whenever $0 < \mathbf{n} < \aleph_0$; the latter implies that if D is a commutative noetherian domain, with identity, which contains a nonzero prime element, then $\mathbb{M}_{\mathbf{n}}(D) \in UP(\mathbf{n})$ whenever $0 < \mathbf{n} < \aleph_0$. Inasmuch as the above two situations are mutually exclusive, we may draw the following conclusion:

an algebraically closed field cannot contain an order with identity which is noetherian and which contains a nonzero prime element.

Since $\mathbb{M}_{\mathbf{n}}(\mathbb{R}) \notin UP(\mathbf{n})$ for all odd integers $\mathbf{n} > 1$, the same argument applies to \mathbb{R} , i.e.,

\mathbb{R} contains no order S with identity such that S is noetherian and S contains a nonzero prime element.

These observations, though not deep, are interesting because they are easily deducible from theorems on uniformly strongly prime rings but have, themselves, no direct connection with uniformly strongly prime rings.

Chapter IV

Radicals

Two of the best known radicals in ring theory are the so-called Jacobson radical J and the Prime (or Lower Baer) radical β . Although each of these radicals may be characterized in a number of different ways, both admit a description involving intersections of prime ideals. To be precise, if R is any ring, then

$$\beta(R) = \bigcap \{I \in \text{Id } R : R/I \text{ is a prime ring}\}, \text{ while}$$

$$J(R) = \bigcap \{I \in \text{Id } R : R/I \text{ is a (right) primitive ring}\}.$$

The relationship between β and the class of all prime rings is therefore analogous to that between J and the class of all (right) primitive rings. This correspondence between these radicals and classes of prime rings underpins a more general notion of “radical”: roughly speaking, if \mathcal{M} is any class of prime rings enjoying certain closure properties then it is possible to define a “radical” \mathfrak{R} by $\mathfrak{R}(R) = \bigcap \{I \in \text{Id } R : R/I \in \mathcal{M}\}$ for all rings R . The radical \mathfrak{R} defined in this manner is usually referred to in the literature as the “upper radical determined by \mathcal{M} ” and is denoted by $\mathfrak{U}\mathcal{M}$. This idea is important. Indeed, it is easily distinguishable as a recurring motif in the general theory of radicals on rings.

In §1, we introduce the important notions of a “special class” of prime rings and a “special radical”; the latter being simply the upper radical determined by a special class of prime rings. Special radicals will be the primary objects of our study. Although rich in content, they are general enough to include almost all of the well known radicals. In §1, for example, we list ten classical radicals, showing how each is realizable as the upper radical $\mathfrak{U}\mathcal{M}$ determined by a special class \mathcal{M} of prime rings.

Subsequent to Handelman and Lawrence’s invention of strongly prime rings, it was proved, independently by Desale and Varadarajan [DV80, Proposition 3.3, p16] and by Groenewald and Heyman [GH81, Theorem 1.1, p140], that the class of all right strongly prime rings is a special class. This discovery led to the definition of a “right strongly prime radical” (also called “right Groenewald-Heyman radical”) which has attracted some interest in the past decade (see [DV80], [GH81], [PSW84] and [PPS84]). In [Raf87, Theorem 3, p259], Raftery extended [DV80, Proposition 3.3, p16] and [GH81, Theorem 1.1, p140] by showing that if \mathfrak{m} is an arbitrary nonzero cardinal then $P_r(\mathfrak{m})$ and $\bar{P}_r(\mathfrak{m})$ are special classes. As yet, however, no detailed study of the upper radicals determined by these special classes has been made. The main body of this chapter, which comprises §2, §3 and §4, is devoted to such a study. Specifically, in §2, we investigate whether these radicals have properties such as being “right hereditary”, “right stable” or “right strong”, and we also present several alternative descriptions of the radicals $\mathfrak{U}\bar{P}_r(\mathfrak{m})$.

In §3, we investigate the relationship between $\mathfrak{U}\bar{P}_r(\mathfrak{m})(R)$ and $\mathfrak{U}\bar{P}_r(\mathfrak{m})(RH)$ for any ring

R and monoid H . Following and extending the work of Ferrero and Parmenter [FP89], we also introduce the notion of a “ $\bar{P}_r(\mathfrak{m})$ -Jacobson ring” and prove that the class of all such rings is closed under the formation of polynomial rings.

Questions concerning the relationship between the radicals $\mathfrak{U}\bar{P}_r(\mathfrak{m})$ (and $\mathfrak{U}P_r(\mathfrak{m})$) and some of the better known classical radicals are addressed in §4. Much of the content of this section is captured in the diagram of radicals displayed in Figure 3. In this section and others, we draw on results established in the previous chapter, particularly concerning matrix rings.

In §5, we broaden our focus slightly with a brief examination of some special radicals associated with uniform bounds of primeness. (These were defined and discussed in the previous chapter.) The main result of this section asserts that the class of all rings which are uniformly prime of bound at most \mathfrak{m} , is a special class for all nonzero cardinals \mathfrak{m} . This answers in the affirmative a question posed by Olson and Veldsman in [OV88].

Most of the results of this chapter have been or will be published in [RV] and [VdB93].

§1. SOME PRELIMINARIES ON RADICAL THEORY.

The purpose of this section is to provide a brief summary of some aspects of radical theory (for rings) and to present a selection of classical results which will be needed later. We shall be concerned mainly with so-called “special classes” of prime rings and their associated radicals. Most of the results and examples given in this section may be found in standard texts such as [Div65].

A class \mathfrak{R} of rings is called a *radical in the category of rings* (henceforth, briefly, a *radical*) if the following three conditions are met:

- R1. \mathfrak{R} is closed under homomorphic images;
- R2. every ring R possesses an ideal, denoted $\mathfrak{R}(R)$, such that $\mathfrak{R}(R) \in \mathfrak{R}$ and $\mathfrak{R}(R)$ contains every ideal of R belonging to \mathfrak{R} ;
- R3. $\mathfrak{R}(R/\mathfrak{R}(R)) = 0$ for every ring R .

If \mathfrak{R} is a radical, it is customary to call $\mathfrak{R}(R)$ the \mathfrak{R} -*radical* of the ring R . If $\mathfrak{R}(R) = R$, we call R an \mathfrak{R} -*radical ring* while if $\mathfrak{R}(R) = 0$ then R is said to be \mathfrak{R} -*semisimple*. Property R3 above therefore asserts that $R/\mathfrak{R}(R)$ is \mathfrak{R} -semisimple for every ring R .

We should point out that a radical \mathfrak{R} is always a proper class (rather than a set). Strictly speaking, therefore, it is not permissible to consider the collection of all radicals as a class; nevertheless, it is customary to speak of it as though it were a class. No real harm comes of this because we are only ever concerned with its internal properties. (More precisely, any assertion about it that we shall make can be rephrased equivalently as an assertion about classes whose elements are sets.) In particular, inclusion \subseteq may be considered to partially order radicals and with respect to this order, the

“collection” of all radicals has the properties of a complete lattice, because any intersection of radicals is again a radical. (This is not immediately obvious from the definition, but follows easily from other characterizations of radicals, e.g., [Div65, Theorem 1, p4].) We therefore speak of the “lattice of all radicals”.

THEOREM 1. [Div65, Theorem 2 & Lemma 3, pp5-6] *Let \mathcal{M} be a class of rings such that every nonzero ideal of a ring in \mathcal{M} has a nonzero homomorphic image in \mathcal{M} . Define*

$$\mathcal{U}\mathcal{M} = \{R : R \text{ is a ring which has no nonzero homomorphic image in } \mathcal{M}\}.$$

Then $\mathcal{U}\mathcal{M}$ is a radical. Moreover, $\mathcal{U}\mathcal{M}$ is the largest radical with respect to which every member of \mathcal{M} is semisimple. □

We call the radical $\mathcal{U}\mathcal{M}$ described in Theorem 1 above the *upper radical determined by \mathcal{M}* . Notice that if $\mathcal{M}_1, \mathcal{M}_2$ are classes of rings satisfying the hypotheses of Theorem 1 and $\mathcal{M}_1 \subseteq \mathcal{M}_2$ then $\mathcal{U}\mathcal{M}_1 \supseteq \mathcal{U}\mathcal{M}_2$.

The following radical classes are well known. Each is described as the upper radical $\mathcal{U}\mathcal{M}$ determined by a class \mathcal{M} of rings. Recall that the *heart* of a ring R is defined to be the intersection of all nonzero ideals of R and that a ring with nonzero heart is called *subdirectly irreducible*. Recall also that a ring R is said to be *locally nilpotent* if every finite subset of R generates a nilpotent subring of R .

<u>RADICAL $\mathcal{U}\mathcal{M}$</u>	<u>CLASS \mathcal{M} OF RINGS</u>
β = the <i>Prime</i> (or <i>Lower Baer</i>) radical	the class of all prime rings
L = the <i>Levitzki</i> radical	the class of all prime rings possessing no nonzero locally nilpotent ideals
N = the <i>Nil</i> (or <i>Upper Baer</i> or <i>Koethe</i>) radical	the class of all prime rings possessing no nonzero nil ideals
J = the <i>Jacobson</i> radical	the class of all right primitive rings (or the class of all left primitive rings)
J_B = the <i>Behrens</i> radical	the class of all subdirectly irreducible rings with hearts which are idempotent and contain nonzero idempotent elements
G = the <i>Brown-McCoy</i> radical	the class of all simple rings with identity
T	the class of all full $\mathfrak{n} \times \mathfrak{n}$ matrix rings over division rings ($0 < \mathfrak{n} < \aleph_0$)
F	the class of all fields
N_g = the <i>Generalized Nil</i> radical	the class of all nonzero rings without zero divisors (i.e., the class of all domains)
β_ϕ = the <i>Antisimple</i> radical	the class of all subdirectly irreducible rings with hearts which are idempotent

Proofs of each of the assertions made in the following theorem may be found in [Div65, Chapter 7].

THEOREM 2. (i) $\beta \subsetneq N \subsetneq J \subsetneq J_B \subsetneq G \subsetneq T \subsetneq F$.

(ii) $N \subsetneq N_g \subsetneq F$.

(iii) N_g is incomparable with all radicals \mathfrak{R} such that $J \subseteq \mathfrak{R} \subseteq T$.

(iv) $L \subsetneq \beta_\phi \subsetneq J_B$.

(v) β_ϕ is incomparable with J .

(vi) $\beta_\phi \subsetneq G$. □

We remark that it is not known whether N and β_ϕ are comparable. It is known, however, that if the famous “Koethe Conjecture” holds then $N \subsetneq \beta_\phi$. There are many equivalent formulations of *Koethe’s Conjecture*, four of which we list below (see [Row88, p209]).

(i) If R is any ring then every nil (right) ideal of R is contained in $N(R)$.

(ii) The sum of two nil (right) ideals of any ring R is necessarily nil.

(iii) $N(\mathbf{M}_n(R)) = \mathbf{M}_n(N(R))$ for all rings R and all finite nonzero cardinals n .

(iv) There exists no nontrivial simple nil ring.

In Figure 1 we capture the content of Theorem 2 by means of a diagram in the lattice of radicals. Solid lines indicate strict containment and are justified by assertions (i), (ii), (iv) and (vi) of Theorem 2, while the incomparability relations implied by the absence of connecting lines are justified by assertions (iii) and (v) of Theorem 2. The broken line joining β_ϕ and N reflects the uncertainty associated with Koethe’s Conjecture.

We shall call an ideal J of a ring R *large* if $J \cap I \neq 0$ for all nonzero ideals I of R .

A nonempty class \mathcal{M} of prime rings is called a *special class* if the following two conditions hold:

S1. every nonzero ideal of a ring in \mathcal{M} is itself in \mathcal{M} ;

S2. whenever J is a large ideal of a ring R and $J \in \mathcal{M}$, we have $R \in \mathcal{M}$.

We should point out that the above definition of a special class is different from (although equivalent to) the original one which appears in [And66, p108] and [Div65, p138]. The equivalence of the two definitions was established by Heyman and Roos [HR77, Theorem 5, p345].

If \mathcal{M} is a special class of prime rings then \mathcal{M} clearly satisfies the hypotheses of Theorem 1. In this case, we call the upper radical $\mathfrak{U}\mathcal{M}$ determined by \mathcal{M} a *special radical*. As it happens, each of the classes \mathcal{M} of prime rings appearing in the table of examples compiled above is special (see [Div65, pp140-155]). It follows that the upper radical determined by each of these classes is special. In particular, we have:

THEOREM 3. *The Prime (or Lower Baer) radical β is a special radical.* □

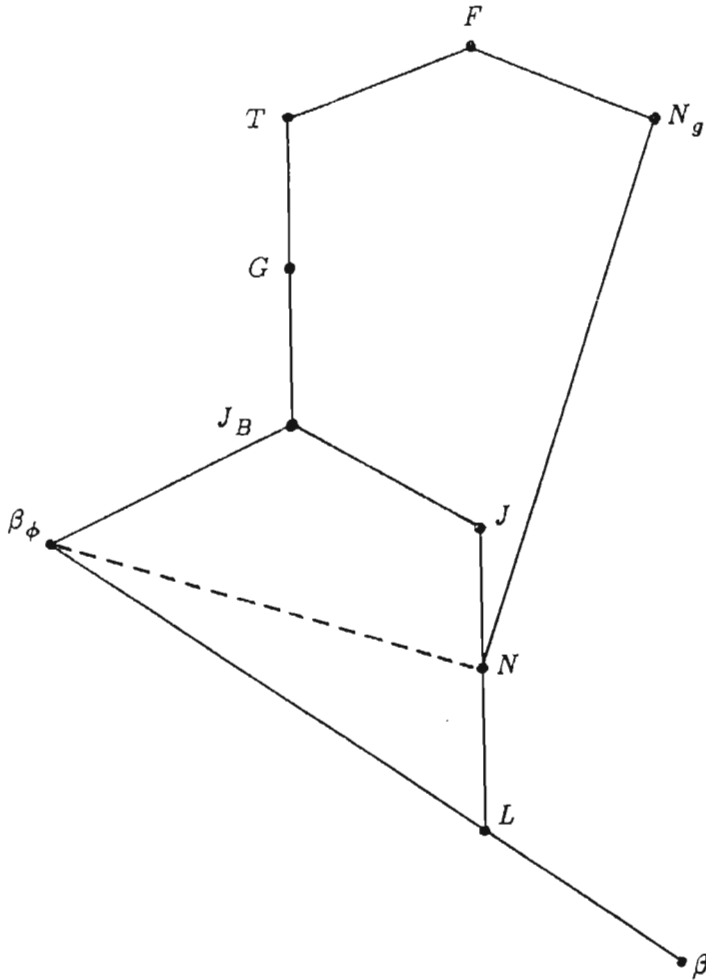


Figure 1

Let \mathfrak{R} be a radical and \mathcal{M} a class of rings. Then \mathfrak{R} is said to :

- (i) be *hereditary* (resp. *right hereditary*) if every ideal (resp. right ideal) of an \mathfrak{R} -radical ring is itself an \mathfrak{R} -radical ring ;
- (ii) have the *intersection property for \mathcal{M}* if $\mathfrak{R}(R) = \bigcap \{J \in \text{Id } R : R/J \in \mathcal{M}\}$ for all rings R .

THEOREM 4. [And66, Remark 14, p111] *If \mathcal{M} is a special class of prime rings then $\mathfrak{R}\mathcal{M}$ is hereditary and has the intersection property for \mathcal{M} .* □

Theorem 4 accounts, to a large extent, for the attention paid by radical theorists to special classes. The reader will observe that Theorem 4 implies the familiar characterizations of the Prime and Jacobson radicals given in the introduction to this chapter.

Many of the classical radicals such as the Prime, Jacobson and Nil radicals have useful elementwise characterizations. We recall here the well known characterization of the elements of the Prime radical $\beta(R)$ of a ring R .

A (countable) sequence a_0, a_1, a_2, \dots of elements of a ring R is called an *m-sequence* if there

exists a second sequence b_0, b_1, b_2, \dots of elements of R such that $a_{k+1} = a_k b_k a_k$ for all nonnegative integers k . We say that such an m -sequence *vanishes* if $a_k = 0$ for some nonnegative integer k . (Notice that in this case, $a_n = 0$ for all integers $n \geq k$.) It is not difficult to see that if an element a of a ring R , with identity, has the property that every m -sequence beginning with a vanishes, then a is necessarily nilpotent. Also, it is easily checked that every nilpotent element a of a commutative ring R satisfies the aforementioned property. (Consequently, in the literature on rings with identity, a ring element a is often called *strongly nilpotent* if every m -sequence commencing with a vanishes.) A proof of the following classical result may be found in [Sza81, Proposition 17.4, p98].

PROPOSITION 5. *If R is a ring then*

$$\beta(R) = \{a \in R : \text{every } m\text{-sequence commencing with } a \text{ vanishes}\}. \quad \square$$

It follows immediately from this elementwise characterization of β that β is a right and left hereditary radical.

PROPOSITION 6. [And66, Theorem 12, p122] *Let \mathfrak{n} be a finite nonzero cardinal. Suppose that \mathcal{M} is a special class of prime rings with the property that if R is a ring with identity, then $R \in \mathcal{M}$ if and only if $\mathbf{M}_{\mathfrak{n}}(R) \in \mathcal{M}$. Then*

$$\mathfrak{U}\mathcal{M}(\mathbf{M}_{\mathfrak{n}}(R)) = \mathbf{M}_{\mathfrak{n}}(\mathfrak{U}\mathcal{M}(R)) \text{ for all rings } R. \quad \square$$

We conclude this section with two corollaries to Proposition 6 which will be needed in the sequel.

We remarked earlier that the Jacobson radical J is the upper radical determined by the special class of all (right) primitive rings. Inasmuch as right primitivity is a Morita invariant property (that is to say, if R and S are Morita equivalent rings with identity, then R is right primitive if and only if S is right primitive: see [AF74, Corollary 21.9, p258]), it follows from Theorem 0.6.3 that if \mathfrak{n} is a finite nonzero cardinal and R a ring with identity, then R is right primitive if and only if $\mathbf{M}_{\mathfrak{n}}(R)$ is right primitive. The following corollary to Proposition 6 is immediate.

COROLLARY 7. *If \mathfrak{n} is a finite nonzero cardinal then*

$$J(\mathbf{M}_{\mathfrak{n}}(R)) = \mathbf{M}_{\mathfrak{n}}(J(R)) \text{ for all rings } R. \quad \square$$

If R is an arbitrary ring with identity and \mathfrak{n} a finite nonzero cardinal then every ideal of $\mathbf{M}_{\mathfrak{n}}(R)$ is of the form $\mathbf{M}_{\mathfrak{n}}(I)$ for some ideal I of R (Proposition 0.6.1). It is an easy consequence of this fact that if H is the heart of R then $\mathbf{M}_{\mathfrak{n}}(R)$ has heart $\mathbf{M}_{\mathfrak{n}}(H)$. It follows that the class of all subdirectly irreducible rings with hearts which are idempotent satisfies the hypotheses of Proposition 6. The following corollary is therefore immediate.

COROLLARY 8. *If \mathfrak{n} is a finite nonzero cardinal then*

$$\beta_\phi(\mathbb{M}_\mathfrak{n}(R)) = \mathbb{M}_\mathfrak{n}(\beta_\phi(R)) \text{ for all rings } R. \quad \square$$

§2. RADICALS ASSOCIATED WITH DEGREES OF PRIMENESS.

Our first and main objective is to prove that the class $P_r(\mathfrak{m})$ is special for all nonzero cardinals \mathfrak{m} (Theorem 4). This is a published result, but we include a proper treatment of it here because, apart from laying the basis for a study of certain radicals that is to occupy most of the balance of this chapter, it serves another important purpose also – it constitutes a vital step in the proofs of two important results, namely Theorems III.1.2 and III.1.18. (The reader will recall that both theorems were stated but not proved in Chapter III.) A proof of the former result is given after Lemma 7 while a proof of the latter result follows Lemma 9. Thereafter, we concentrate on a detailed radical theoretic study of the upper radicals $\mathfrak{U}P_r(\mathfrak{m})$ and $\mathfrak{U}\bar{P}_r(\mathfrak{m})$. We point out, however, that the task of locating each of these radicals in the lattice of radicals is undertaken only in §4.

Let R be a subring of a ring S and let \mathcal{A} be a nonempty class of rings. Recall that the inclusion $R \subseteq S$ is said to satisfy:

- (i) *GD* (“going down condition”) with respect to \mathcal{A} if $R/(R \cap A) \in \mathcal{A}$ whenever A is an ideal of S and $S/A \in \mathcal{A}$;
- (ii) *LO* (“lying over condition”) with respect to \mathcal{A} if for each ideal I of R such that $R/I \in \mathcal{A}$, there exists some ideal A of S such that $S/A \in \mathcal{A}$ and $I = R \cap A$.

LEMMA 1. [Raf87, Lemma 1, p259] *Let \mathfrak{m} be a nonzero cardinal. If I is a nonzero ideal of a ring R then $I \subseteq R$ satisfies GD with respect to $\bar{P}_r(\mathfrak{m}) \cup \{0\}$.*

Proof. Let A be an ideal of R such that $R/A \in \bar{P}_r(\mathfrak{m}) \cup \{0\}$. If $I/(I \cap A) = 0$, there is nothing to prove, so suppose $R/A \in \bar{P}_r(\mathfrak{m})$ and $I \not\supseteq I \cap A$. Take $b \in I \setminus (I \cap A)$. Since $R/A \in \bar{P}_r(\mathfrak{m})$ and $b \notin A$, there must exist a subset X of R with $|X| < \mathfrak{m} + 1$, such that if $c \in R \setminus A$ then $bxc \in R \setminus A$ for some $x \in X$. Let $d \in I \setminus (I \cap A)$ and choose $x \in X$ such that $bxd \in R \setminus A$. Insofar as $bxd \in I \setminus (I \cap A)$, by the same argument, there exists a subset Y of R with $|Y| < \mathfrak{m} + 1$ such that if $t \in I \setminus (I \cap A)$, then $bxdyt \in R \setminus A$ for some $y \in Y$. It follows that $bxdyt \in I \setminus (I \cap A)$. This shows that $Z = \{xdy + (I \cap A) : y \in Y\}$ is a right insulator for $b + (I \cap A)$ in the ring $I/(I \cap A)$. Since $|Z| \leq |Y| < \mathfrak{m} + 1$, we may conclude that $I/(I \cap A) \in \bar{P}_r(\mathfrak{m})$, as required. \square

The following result establishes the property **S1** in the definition of a special class, for the class $\bar{P}_r(\mathfrak{m})$, \mathfrak{m} an arbitrary nonzero cardinal.

COROLLARY 2. [Raf87, Corollary 2, p259] *Let \mathfrak{m} be a nonzero cardinal. Then every nonzero ideal of a ring in $\bar{P}_r(\mathfrak{m})$ is itself in $\bar{P}_r(\mathfrak{m})$.*

Proof. Let $R \in \bar{P}_r(\mathfrak{m})$ and let I be a nonzero ideal of R . Since $I \subseteq R$ satisfies GD with respect to $\bar{P}_r(\mathfrak{m}) \cup \{0\}$ (by the previous lemma) and $R \in \bar{P}_r(\mathfrak{m})$, it follows that $I \in \bar{P}_r(\mathfrak{m}) \cup \{0\}$. By hypothesis, $I \neq 0$, so $I \in \bar{P}_r(\mathfrak{m})$. \square

The following lemma has been extracted from the proof of [Raf87, Theorem 3, p259].

LEMMA 3. *If I is a large ideal of a prime ring R then I_I and ${}_I I$ are essential submodules of R_I and ${}_I R$, respectively.*

Proof. Suppose $0 \neq K \leq R_I$ and let $\langle K \rangle$ denote the ideal of R generated by K . Then $\langle K \rangle I \neq 0$ (because R is prime), so $0 \neq KI \subseteq K \cap I$. Thus I_I is an essential submodule of R_I . Similarly, it can be shown that ${}_I I$ is essential in ${}_I R$. \square

THEOREM 4. [Raf87, Theorem 3, p259] *The class $P_r(\mathfrak{m})$ is a special class for all nonzero cardinals \mathfrak{m} .*

Proof. Let I be a large ideal of a ring R , and suppose that $I \in P_r(\mathfrak{m})$. Since the class of all prime rings is special (Theorem 1.3), we must have that R is prime. Let $0 \neq r \in R$. By the previous lemma, I_I is an essential submodule of R_I , so we must have $\{ra + rm : a \in I, m \in \mathbb{Z}\} \cap I_I \neq 0$. Choose $a \in I$ and $m \in \mathbb{Z}$ such that $0 \neq ra + rm \in I$. Notice that $ra, rm \in I$. Clearly, $ra \neq 0$ or $rm \neq 0$. Suppose first that $ra \neq 0$. Since $I \in P_r(\mathfrak{m})$, we can choose a right insulator Y for ra in I with $|Y| < \mathfrak{m} + 1$. Let $0 \neq t \in R$ and suppose $raYt = 0$. Again, since I_I is essential in R_I , we must have $0 \neq ta' + tn \in I$ for some $a' \in I$ and $n \in \mathbb{Z}$. It follows that $raY(ta' + tn) = 0$, which contradicts the fact that Y is a right insulator for ra in I . Thus aY must be a right insulator for r in R . Suppose now that $rm \neq 0$. Again, we choose a right insulator Y for rm in I with $|Y| < \mathfrak{m} + 1$. If $0 \neq t \in R$ and $rYt = 0$ then it follows, as above, that $0 \neq ta' + tn \in I$ for some $a' \in I$ and $n \in \mathbb{Z}$, and $rY(ta' + tn) = 0$. Certainly then, $rmY(ta' + tn) = 0$, which again contradicts the fact that Y is a right insulator for rm in I . Thus Y must also be a right insulator for r in R . We have shown that $R \in \bar{P}_r(\mathfrak{m})$. Consequently, $R \in P_r(\mathfrak{k})$ for some $\mathfrak{k} \leq \mathfrak{m}$. By Corollary 2, we must have that $I \in \bar{P}_r(\mathfrak{k})$. This forces $\mathfrak{k} = \mathfrak{m}$. Thus $R \in P_r(\mathfrak{m})$. This establishes property S2 of the definition of a special class.

Now suppose that I is a nonzero ideal of R and $R \in P_r(\mathfrak{m})$. By Corollary 2, $I \in P_r(\mathfrak{k})$ for some $\mathfrak{k} \leq \mathfrak{m}$. Since R is a prime ring, I must be large in R . Therefore, by the above argument, $R \in P_r(\mathfrak{k})$. It follows that $\mathfrak{k} = \mathfrak{m}$ and hence that $I \in P_r(\mathfrak{m})$. This establishes property S1 of a special class. \square

Inasmuch as any nonempty union of special classes is clearly special, it follows from Theorem 4 that if \mathfrak{m} is an arbitrary nonzero cardinal then the classes $\bar{P}_r(\mathfrak{m})$ and $\bigcup_{0 < \mathfrak{k} < \mathfrak{m}} P_r(\mathfrak{k})$ (in the case where \mathfrak{m} is a limit cardinal) are also special classes. In accordance with the notation of §1, we let

$\mathfrak{U}P_r(\mathfrak{m})$ (resp. $\mathfrak{U}\bar{P}_r(\mathfrak{m})$) denote the (special) upper radical determined by the class $P_r(\mathfrak{m})$ (resp. $\bar{P}_r(\mathfrak{m})$) and if \mathfrak{m} is a limit cardinal, we let $B_r(\mathfrak{m})$ denote the (special) upper radical determined by the class $\bigcup_{0 < \mathfrak{k} < \mathfrak{m}} P_r(\mathfrak{k})$. The radical $\mathfrak{U}\bar{P}_r(\aleph_0)$ is known in the literature as the *right strongly prime radical* or the *right Groenewald-Heyman radical* and has been studied in [GH81], [PPS84] and [PSW84]. In [DV80, Proposition 3.3, p16] and [GH81, Theorem 1.1, p140], it is proved that the right Groenewald-Heyman radical is special. Notice that this is a special case of the aforementioned immediate corollary of Theorem 4.

Equipped with Theorem 4 and two further preparatory lemmas, we shall be able to present a proof of Theorem III.1.2. We shall take, as a point of departure, Goodearl, Handelman and Lawrence's well known version of Theorem III.1.2 for rings with identity, which asserts that the following conditions are equivalent for a ring R with identity and a finite cardinal $\mathfrak{n} > 1$:

- (i) $R \in P_r(\mathfrak{n})$;
- (ii) R is isomorphic to a right order in $\mathbf{M}_{\mathfrak{n}}(D)$ for some division ring D .

A proof of this result may be found in [GHL74, Theorem 4.7, p27] (or, more accessibly, in [GH75, Theorem 2.3, p803]). We shall show, following [Raf87], how to extend the above result to rings without identity. The crucial step is to show that every prime ring R (possibly without identity) embeds in a prime ring S (with identity) which has the same right bound of primeness as R . Theorem 4 suggests that the ring S should be chosen in such a way that R embeds as a large ideal in S . The following result shows that such a ring S may be constructed by modifying the Dorroh Extension R^* of R (see Chapter 0, §4).

LEMMA 5. [Raf87, Lemma 6, p261] *Let R be a ring and let A be the left annihilator of R in R^* , i.e., $A = \{1_{R^*} \cdot r + 1_{R^*} \cdot m \in R^* : rs + sm = 0 \text{ for all } s \in R\}$. If $R \cap A = 0$ then R embeds as an ideal in the ring $S = R^*/A$ such that R_R is an essential submodule of S_R .*

Proof. Let $\varphi: R \rightarrow S$ denote the composition of the canonical ring embedding from R into R^* with the natural epimorphism from R^* onto R^*/A . Inasmuch as R is an ideal of R^* , $\varphi[R]$ is certainly an ideal of S . Moreover, φ is a monomorphism, since $\varphi(r) = 1_{R^*} \cdot r + A = 0$ implies $1_{R^*} \cdot r \in A$, i.e., $rR = 0$, whence $r = 0$ (because $R \cap A = 0$). Thus R embeds as an ideal in S . We identify R with its image $\varphi[R]$ in S . To show that R_R is essential in S_R , take $0 \neq (1_{R^*} \cdot r + 1_{R^*} \cdot m) + A \in S$. Since $1_{R^*} \cdot r + 1_{R^*} \cdot m \notin A$, there must exist some $s \in R$ for which $rs + sm \neq 0$. Therefore

$$[(1_{R^*} \cdot r + 1_{R^*} \cdot m) + A][1_{R^*} \cdot s + A] = 1_{R^*} \cdot (rs + sm) + A \in R_R.$$

This shows that R_R is essential in S_R . □

The following corollary is immediate, in view of Theorem 4.

COROLLARY 6. [Raf87, Theorem 7, p262] *Let \mathfrak{m} be a nonzero cardinal. If R is a prime ring and A the left annihilator of R in R^* then $R \in P_r(\mathfrak{m})$ if and only if $R^*/A \in P_r(\mathfrak{m})$. \square*

We require one further lemma.

LEMMA 7. [Gol60, Theorem 5.4, p216] *Let R be a right order in a nonzero semiprimitive artinian ring Q . If I is a nonzero ideal of R then I is a right order in the ideal IQ of Q . \square*

THEOREM III.1.2. [GH75, Theorem 2.3, p803], [Raf87, Corollary 9, p263] *Let \mathfrak{n} be a finite cardinal such that $\mathfrak{n} > 1$. Then the following assertions are equivalent:*

- (i) $R \in P_r(\mathfrak{n})$;
- (ii) R is isomorphic to a right order in $\mathbb{M}_{\mathfrak{n}}(D)$ for some division ring D .

Proof. (ii) \Rightarrow (i) is an immediate consequence of Proposition III.1.1 and Theorem III.2.3.

(i) \Rightarrow (ii): Let A be the left annihilator of R in R^* and put $S = R^*/A$. Inasmuch as S is a ring with identity and $S \in P_r(\mathfrak{n})$ (by Corollary 6), it follows from Goodearl, Handelman and Lawrence's result [GHL74, Theorem 4.7, p27] that S is isomorphic to a right order in $Q = \mathbb{M}_{\mathfrak{n}}(D)$ for some division ring D . If R is identified with its image in S and S with its image in Q then it follows from Lemma 7 that R is a right order in the ideal RQ of Q . But $Q = \mathbb{M}_{\mathfrak{n}}(D)$ is a simple ring so we must have $RQ = Q$. Thus R is a right order in $\mathbb{M}_{\mathfrak{n}}(D)$. \square

Our next task is to prove Theorem III.1.18. For the reader's convenience we start by providing a brief summary of some important concepts and results from Chapter II. If R is a ring then $\text{torsp-}R$ denotes the lattice of all torsion preradicals on $\text{Mod-}R$, and $\text{Fil-}R$ the lattice of all right topologizing filters on R . It is possible to define semigroup operations on $\text{torsp-}R$ and $\text{Fil-}R$ in such a way that $[\text{torsp-}R]^{\text{du}}$ and $[\text{Fil-}R]^{\text{du}}$ have the structure of lattice ordered semigroups (see Theorem II.1.7 and Corollary II.2.5). The map $\eta: \text{Id } R \rightarrow [\text{Fil-}R]^{\text{du}}$ defined by $\eta(I) = \{A \leq R_R : A \supseteq I\}$ ($I \in \text{Id } R$) is a lattice and semigroup monomorphism whose image consists precisely of all Jansian right topologizing filters on R (see Proposition II.3.5). Also, the map $\text{torsp}: \text{Fil-}R \rightarrow \text{torsp-}R$ defined by $(\text{torsp } \mathcal{F})(M) = \{x \in M : (0:x) \in \mathcal{F}\}$ for $M \in \text{Mod-}R$ and $\mathcal{F} \in \text{Fil-}R$, is a lattice and semigroup monomorphism whose image is the interval $[\tau_{\text{zero}}, 1_{\text{Mod-}R}]$ of $\text{torsp-}R$ (see remarks following Proposition II.2.2 and Theorem II.2.4). The composition of the maps

$$\text{Id } R \xrightarrow{\eta} [\text{Fil-}R]^{\text{du}} \xrightarrow{\text{torsp}} [\text{torsp-}R]^{\text{du}}$$

is a lattice and semigroup monomorphism $\text{torsp} \circ \eta$ from $\text{Id } R$ into $[\text{torsp-}R]^{\text{du}}$, whose image is $\{\tau \in \text{Jans-}R : \tau \geq \tau_{\text{zero}}\}^{\text{du}}$ ($\text{Jans-}R$ denotes the set of all Jansian torsion preradicals on $\text{Mod-}R$). As noted in the sequel to Proposition II.5.3, if R is replaced by R^* and $\{\tau \in \text{Jans-}R^* : \tau \geq \tau_{\text{zero}}\}^{\text{du}}$ identified with $[\text{Jans-}R]^{\text{du}}$, then $\text{torsp} \circ \eta$ constitutes an isomorphism of lattice ordered semigroups from $\text{Id } R^*$ to $[\text{Jans-}R]^{\text{du}}$. It is important to note the explicit manner in which the map $\text{torsp} \circ \eta$

acts: if I is an ideal of R^* and $M \in \text{Mod-}R$ then $((\text{torsp} \circ \eta)(I))(M) = \{x \in M_{R^*} : xI = 0\}$. Suppose now that R and S are Morita $*$ -equivalent rings and that $F: \text{Mod-}R \rightarrow \text{Mod-}S$ is an additive category equivalence. If $\sigma \in \text{torsp-}R$ and T_σ denotes the hereditary pretorsion class of σ then $F[T_\sigma]$ is a hereditary pretorsion class on $\text{Mod-}S$ which corresponds with the hereditary pretorsion class of a unique torsion preradical on $\text{Mod-}S$, which we denote by $\widehat{F}(\sigma)$. Hence $F[T_\sigma] = T_{\widehat{F}(\sigma)}$. Thus the association $\sigma \mapsto \widehat{F}(\sigma)$ defines a map from $\text{torsp-}R$ to $\text{torsp-}S$ (see remarks preceding Lemma II.5.1). The map \widehat{F} is structure preserving in several respects: it defines an isomorphism of lattice ordered semigroups from $[\text{torsp-}R]^{\text{du}}$ to $[\text{torsp-}S]^{\text{du}}$ (Theorem II.5.2) and also preserves the Jansian property, that is to say, $\sigma \in \text{torsp-}R$ is Jansian if and only if $\widehat{F}(\sigma) \in \text{torsp-}S$ is Jansian (Proposition II.5.3). It follows that \widehat{F} restricts to an isomorphism from $[\text{Jans-}R]^{\text{du}}$ to $[\text{Jans-}S]^{\text{du}}$. This isomorphism induces, in turn, an isomorphism $\theta: \text{Id } R^* \rightarrow \text{Id } S^*$. Thus we obtain the following diagram (see also sequel to Proposition II.5.3).

$$\begin{array}{ccc}
 \text{Id } R^* & \xrightarrow[\text{(iso)}]{\text{torsp} \circ \eta} & [\text{Jans-}R]^{\text{du}} \\
 \text{(iso)} \downarrow \theta & & \text{(iso)} \downarrow \widehat{F} \\
 \text{Id } S^* & \xrightarrow[\text{(iso)}]{\text{torsp} \circ \eta} & [\text{Jans-}S]^{\text{du}}
 \end{array}$$

Observe that if $I \in \text{Id } R^*$, $J = \theta(I) \in \text{Id } S^*$ and $\sigma = (\text{torsp} \circ \eta)(I)$, then $T_\sigma = \{M \in \text{Mod-}R : MI = 0\}$ and $T_{\widehat{F}(\sigma)} = \{M \in \text{Mod-}S : MJ = 0\}$. Inasmuch as $F[T_\sigma] = T_{\widehat{F}(\sigma)}$, we must have $F[\{M \in \text{Mod-}R : MI = 0\}] = \{M \in \text{Mod-}S : MJ = 0\}$. The functor F restricts, therefore, to an additive category equivalence from the full subcategory of $\text{Mod-}R$ on $\{M \in \text{Mod-}R : MI = 0\}$ to the full subcategory of $\text{Mod-}S$ on $\{M \in \text{Mod-}S : MJ = 0\}$. It is not difficult to see, however, that the full subcategory of $\text{Mod-}R$ on $\{M \in \text{Mod-}R : MI = 0\}$ is isomorphic to, and hence identifiable with, the category $\text{Mod-}R^*/I$ (unital). Similarly, $\{M \in \text{Mod-}S : MJ = 0\}$ is identifiable with $\text{Mod-}S^*/J$ (unital). We may conclude, therefore, that F induces an additive category equivalence from $\text{Mod-}R^*/I$ (unital) to $\text{Mod-}S^*/J$ (unital); but this means precisely that the rings R^*/I and S^*/J are Morita equivalent in the classical sense. We shall make use of this fact in proving Theorem III.1.18.

LEMMA 8. *Let R be a prime ring and K an ideal of R^* such that $K \not\subseteq R$. If $(0:{}_l K)$ denotes the left annihilator of K in R^* then either $(0:{}_l K) = 0$ or $(0:{}_l K) = R$.*

Proof. Let $1_{R^*} \cdot r + 1_{R^*} \cdot m \in K \setminus (K \cap R)$. Then $m \neq 0$. If $1_{R^*} \cdot s + 1_{R^*} \cdot n \in (0:{}_l K)$ then

$$(1_{R^*} \cdot s + 1_{R^*} \cdot n)(1_{R^*} \cdot r + 1_{R^*} \cdot m) = 1_{R^*} \cdot (sr + sm + rn) + 1_{R^*} \cdot (nm) = 0,$$

so $n = 0$, whence $1_{R^*} \cdot s + 1_{R^*} \cdot n = 1_{R^*} \cdot s \in R$. This shows that $(0:{}_l K) \subseteq R$. Since RK and

$(0:{}_l K)$ are ideals of R and $(0:{}_l K)(RK) = 0$, it follows from the primeness of R that $(0:{}_l K) = 0$ or $RK = 0$, i.e., $(0:{}_l K) \supseteq R$. \square

LEMMA 9. *Let R and S be rings and suppose $\theta: \text{Id } R^* \rightarrow \text{Id } S^*$ is a lattice and semigroup isomorphism. Let $(0:{}_l R)$ denote the left annihilator of R in R^* and $(0:{}_l S)$ the left annihilator of S in S^* . If R is prime then $(0:{}_l S) \in \{0, \theta(R), (0:{}_l \theta(R))\}$.*

Proof. Note first that if $I \in \text{Id } R$ and $(0:{}_l I)$ denotes the left annihilator of I in R^* , then $(0:{}_l I)$ is characterizable as the unique largest ideal of R^* for which $(0:{}_l I)I = 0$. Since θ is a lattice and semigroup isomorphism it follows, therefore, that $\theta((0:{}_l I)) = (0:{}_l \theta(I))$ (in S^*).

Suppose $(0:{}_l S) \neq 0$. If $\theta(R) = S$ then $(0:{}_l S) = (0:{}_l \theta(R))$. It suffices, therefore, to prove that $(0:{}_l S) = \theta(R)$ given $\theta(R) \neq S$. If $S \subsetneq \theta(R)$ then $\theta(R) = \{1_{S^*} \cdot s + 1_{S^*} \cdot (mn) : s \in S, n \in \mathbb{Z}\}$ for some $m \in \mathbb{Z}, m \neq 0$. But $|\{I \in \text{Id } R^* : I \supseteq R\}|$ is infinite and $|\{I \in \text{Id } S^* : I \supseteq \theta(R)\}|$ is finite which contradicts the fact that θ is a lattice isomorphism. Consequently, we must have $S \not\subseteq \theta(R)$ i.e., $\theta^{-1}(S) \not\subseteq R$. Since $(0:{}_l S) \neq 0$ it follows that $\theta^{-1}((0:{}_l S)) = (0:{}_l \theta^{-1}(S)) \neq 0$. By the previous lemma $\theta^{-1}((0:{}_l S)) = (0:{}_l \theta^{-1}(S)) = R$, whence $(0:{}_l S) = \theta(R)$, as required. \square

THEOREM III.1.18. *Let R and S be Morita $*$ -equivalent prime rings and let \mathfrak{m} be an infinite cardinal. If $R \in \bar{P}_r(\mathfrak{m})$ then $S \in \bar{P}_r(\mathfrak{m})$. In other words, the property of being an element of $\bar{P}_r(\mathfrak{m})$ is Morita $*$ -invariant within the class of all prime rings.*

Proof. In this proof we shall use $(0:{}_l T)$ to denote the left annihilator of any ring T in its Dorroh Extension. It was noted in the discussion preceding Lemma 8, that any additive category equivalence from $\text{Mod-}R$ to $\text{Mod-}S$ induces a lattice and semigroup isomorphism $\theta: \text{Id } R^* \rightarrow \text{Id } S^*$, and moreover, if I is any ideal of R^* then R^*/I and $S^*/\theta(I)$ are Morita equivalent in the classical sense. Consider the ideal $(0:{}_l S)$ of S^* . Since R is prime, $(0:{}_l S) \in \{0, \theta(R), (0:{}_l \theta(R))\} \subseteq \text{Id } S^*$, by Lemma 9. Suppose first that $(0:{}_l S) = (0:{}_l \theta(R)) = \theta((0:{}_l R))$. Then $R^*/(0:{}_l R)$ and $S^*/(0:{}_l S)$ are Morita equivalent. The fact that $S \in \bar{P}_r(\mathfrak{m})$ follows from Theorem III.1.16 and Corollary 6. Now suppose that $(0:{}_l S) \neq (0:{}_l \theta(R))$. If $(0:{}_l S) = 0$ then it follows from Corollary 6 that S^* is a prime ring. This, in turn, implies R^* is prime by Theorem III.1.16. Since R^* is prime we must have $(0:{}_l R) = 0$, so $(0:{}_l \theta(R)) = \theta((0:{}_l R)) = 0 = (0:{}_l S)$, a contradiction. The only remaining possibility is $(0:{}_l S) = \theta(R)$. In this case $R^*/R \cong \mathbb{Z}$ is Morita equivalent to $S^*/(0:{}_l S)$. Again, we may conclude from Theorem III.1.16 and Corollary 6 that $S \in \bar{P}_r(\mathfrak{m})$, as required. \square

LEMMA 10. [DH84, Proposition] *The following assertions are equivalent for a radical \mathfrak{R} :*

- (i) $\mathfrak{R}(R^*) = \mathfrak{R}(R)$ for all rings R ;
- (ii) $\mathfrak{R}(\mathbb{Z}) = 0$.

\square

PROPOSITION 11. *If R is any ring then $\mathfrak{U}\bar{P}_r(\mathfrak{m})(R) = \mathfrak{U}\bar{P}_r(\mathfrak{m})(R^*)$ for all nonzero cardinals \mathfrak{m} and $B_r(\mathfrak{m})(R) = B_r(\mathfrak{m})(R^*)$ for all limit cardinals \mathfrak{m} . If $\mathfrak{m} > 1$, then there exist rings R such that $\mathfrak{U}P_r(\mathfrak{m})(R) \subsetneq \mathfrak{U}P_r(\mathfrak{m})(R^*)$.*

Proof. Inasmuch as $\mathbb{Z} \in \bar{P}_r(\mathfrak{m})$ for all $\mathfrak{m} > 0$, it follows from the previous lemma that $\mathfrak{U}\bar{P}_r(\mathfrak{m})(R) = \mathfrak{U}\bar{P}_r(\mathfrak{m})(R^*)$ and $B_r(\mathfrak{m})(R) = B_r(\mathfrak{m})(R^*)$ (in the case where \mathfrak{m} is a limit cardinal) for all rings R .

Since the factor rings of \mathbb{Z} are either trivial, non-prime or fields, it follows that if $\mathfrak{m} > 1$ then \mathbb{Z} has no nonzero homomorphic image in $P_r(\mathfrak{m})$. Thus $\mathbb{Z} \in \mathfrak{U}P_r(\mathfrak{m})$ and so $\mathfrak{U}P_r(\mathfrak{m})(\mathbb{Z}) \neq 0$. By the previous lemma, there must exist rings R for which $\mathfrak{U}\bar{P}_r(\mathfrak{m})(R) \subsetneq \mathfrak{U}\bar{P}_r(\mathfrak{m})(R^*)$. \square

EXAMPLE 1. We show that $\mathfrak{U}\bar{P}_r(\mathfrak{m})$ is neither right nor left hereditary for any nonzero cardinal \mathfrak{m} . Let \mathfrak{m} be an arbitrary nonzero cardinal and \mathfrak{k} an infinite cardinal such that $\mathfrak{k} > \mathfrak{m}$. Let D be a division ring and put $R = \mathbb{M}_{\mathfrak{k}}(D)$. Recall that R is isomorphic to the left full linear ring $\text{End}_D V$ of all linear transformations (written on the right) of a \mathfrak{k} -dimensional left vector space over D . By Proposition 0.9.3, $\text{End}_D V$ has a unique minimal nonzero ideal consisting precisely of all $\varphi \in \text{End}_D V$ for which $\text{rank } \varphi < \aleph_0$. If A denotes the image of this ideal in R , it is not difficult to see that A consists precisely of all matrices with only finitely many nonzero columns. By Proposition III.1.4, $R \in P_r(\mathfrak{k}^+)$, so $R \notin \bar{P}_r(\mathfrak{m})$. Since $\mathfrak{U}\bar{P}_r(\mathfrak{m})$ has the intersection property for $\bar{P}_r(\mathfrak{m})$ (Theorem 1.4 and Theorem 4) and A is the unique minimal nonzero ideal of R , we must have that $A \subseteq \mathfrak{U}\bar{P}_r(\mathfrak{m})(R)$, so $A \in \mathfrak{U}\bar{P}_r(\mathfrak{m})$. Let I be the set of all matrices in R all of whose nonzero entries are in the first row. Clearly, I is a right ideal of R contained in A . If $K = \{M \in I : M_{00} = 0\}$ then K is an ideal of I and $I/K \cong D \in P_r(1) \subseteq \bar{P}_r(\mathfrak{m})$, so $I \notin \mathfrak{U}\bar{P}_r(\mathfrak{m})$. This shows that $\mathfrak{U}\bar{P}_r(\mathfrak{m})$ is not right hereditary.

If I is chosen instead to be the set of all matrices in R all of whose nonzero entries are in the first column, then an argument similar to the above shows that $\mathfrak{U}\bar{P}_r(\mathfrak{m})$ is not left hereditary. The above argument can also be used to show that $B_r(\mathfrak{m})$ is neither right nor left hereditary for any limit cardinal \mathfrak{m} . \square

A radical \mathfrak{R} is said to be:

- (i) *right stable* if, for every ring R and every right ideal K of R , we have $\mathfrak{R}(K) \subseteq \mathfrak{R}(R)$;
- (ii) *right strong* if, for every ring R , the ideal $\mathfrak{R}(R)$ contains all right ideals of R which are members of \mathfrak{R} .

Clearly every right stable radical is right strong. It is known that β is right and left strong (this follows, incidentally, from our Proposition 14) and that $\mathfrak{U}\bar{P}_r(\aleph_0)$ is right but not left strong [Puc86, Corollary 1 & Example 1, p2]. In Proposition 14, we shall sharpen and extend Puczyłowski's result [Puc86, Corollary 1, p2] by showing that $\mathfrak{U}\bar{P}_r(\mathfrak{m})$ is right strong for all nonzero cardinals \mathfrak{m} . By [PR92, Lemma 2.3 (i), p974], if a radical \mathfrak{R} is right stable and $\beta \subseteq \mathfrak{R}$ then \mathfrak{R} must contain the

Generalized Nil radical N_g . It is clear that $\beta \subseteq \mathfrak{U}\bar{P}_r(\mathfrak{m}) \subseteq N_g$ (for every $\mathfrak{m} > 0$) and we shall show in Proposition 4.6 that both of these inclusions are sharp. Consequently, the radical $\mathfrak{U}\bar{P}_r(\mathfrak{m})$ cannot be right stable for any nonzero cardinal \mathfrak{m} .

PROPOSITION 12. [DKS71, Theorem 9, p385] *Let \mathcal{M} be a class of rings with the property that every nonzero right ideal of a ring in \mathcal{M} has a nonzero homomorphic image in \mathcal{M} . Then $\mathfrak{U}\mathcal{M}$ is a right strong radical.* □

PROPOSITION 13. *Let \mathfrak{m} be a nonzero cardinal. If A is a nonzero right ideal of a ring $R \in \bar{P}_r(\mathfrak{m})$ then $A/J \in \bar{P}_r(\mathfrak{m})$ for some proper ideal J of the ring A .*

Proof. Define $J = \{a \in A : aA = 0\}$. Note that J is an ideal of A . We claim that $A/J \in \bar{P}_r(\mathfrak{m})$. Let $0 \neq a + J \in A/J$. Then $aa' \neq 0$ for some $a' \in A$. Since $R \in \bar{P}_r(\mathfrak{m})$, there exists a subset X of R such that $(0 :_r aa'X) = 0$ (in R) and $|X| < \mathfrak{m} + 1$. But $Y = a'X \subseteq A$ and $|Y| \leq |X| < \mathfrak{m} + 1$. Thus Y is a right insulator for a in A of cardinality less than $\mathfrak{m} + 1$. Suppose now that $(a + J)(Y/J)(t + J) = 0$ for some $t \in A$. Then $aYt \subseteq J$, i.e., $aytA = ay(tA) = 0$ for all $y \in Y$ and so $tA = 0$, i.e., $t \in J$. Thus $Y/J := \{y + J : y \in Y\}$ is a right insulator for $a + J$ in A/J and $|Y/J| < \mathfrak{m} + 1$. It follows that $A/J \in \bar{P}_r(\mathfrak{m})$. □

PROPOSITION 14. $\mathfrak{U}\bar{P}_r(\mathfrak{m})$ is right strong for all nonzero cardinals \mathfrak{m} and $B_r(\mathfrak{m})$ is right strong for all limit cardinals \mathfrak{m} .

Proof. It follows from Proposition 13 that if $R \in \bar{P}_r(\mathfrak{m})$ (resp. $R \in \bigcup_{0 < \mathfrak{k} < \mathfrak{m}} P_r(\mathfrak{k})$, \mathfrak{m} a limit cardinal) then every nonzero right ideal of R has a nonzero homomorphic image in $\bar{P}_r(\mathfrak{m})$ (resp. $\bigcup_{0 < \mathfrak{k} < \mathfrak{m}} P_r(\mathfrak{k})$). By Proposition 12, $\mathfrak{U}\bar{P}_r(\mathfrak{m})$ (resp. $B_r(\mathfrak{m})$, \mathfrak{m} a limit cardinal) is right strong. □

The following example, which was motivated by [Puc86, Example 1, p2], shows that $\mathfrak{U}\bar{P}_r(\mathfrak{m})$ is not left strong for any nonzero cardinal \mathfrak{m} .

EXAMPLE 2. We first construct a simple domain D , with identity, which contains an independent family of \mathfrak{k} nonzero right ideals, where \mathfrak{k} is an arbitrary infinite cardinal.

Let G be the free group on \mathfrak{k} generators, say x_α ($\alpha \in \mathfrak{k}$), and let F be a field on which G acts.¹ Let $D = F[G]$ be the right skew monoid ring in G over F . Recall that

¹Such a field F always exists. Indeed, if K is an arbitrary field and X an arbitrary nonempty set then every bijection $\alpha : X \rightarrow X$ extends naturally to a field automorphism $\bar{\alpha} : K(X) \rightarrow K(X)$. (Recall that $K(X)$ denotes the field of rational functions in X over K : see Chapter 0, §10) The association $\alpha \mapsto \bar{\alpha}$ defines an embedding of the permutation group S_X into the group of automorphisms of the field $K(X)$. Inasmuch as every group embeds in a permutation group (Cayley's Theorem), it follows that every group embeds in the group of automorphisms of a suitable field.

$$D = F[G] := \{ \sum_{g \in G} a_g g : a_g \in F \},$$

where sums are formal and are assumed to have finite support, addition is natural and multiplication is defined distributively, using the rule $g \cdot a = (g(a))g$ ($g \in G$, $a \in F$) (see Chapter 0, §10). Since G is a strictly ordered monoid, D is a domain (see the proof of Proposition I.3.1). If $g \in G$, we define the *degree* of g , denoted ∂g , to be the sum of the absolute values of the powers of x_α 's occurring in g . (For example, $\partial(1_G) = 0$ and for any $\alpha, \beta, \gamma \in \mathbf{k}$, $\partial(x_\alpha^{-2} x_\beta^3) = 5$; $\partial(x_\alpha x_\beta x_\alpha^{-3} x_\gamma) = 6$, etc. See also Example III.4.2.) We warn the reader that, although $\partial(gg') \leq \partial g + \partial g'$ for all $g, g' \in G$, this inequality is strict in many instances, so ∂ is not a "degree function" in the strong sense associated with free algebras over fields and monomial algebras. To show that D is a simple domain, suppose I is a nonzero ideal of D and let d be a nonzero element of I such that $|\text{supp } d|$ is minimal. By multiplying d on the right by a suitable element of G , we may assume, without any loss of generality, that $1_G \in \text{supp } d$. Suppose that $\text{supp } d \supsetneq \{1_G\}$ and let $g \in (\text{supp } d) \setminus \{1_G\}$. Since $g \neq 1_G$, we can choose $a \in F$ such that $g(a) \neq a$. It follows that $g \in \text{supp } d'$, where $d' = ad - da$, so $d' \neq 0$. Since $d' \in I$ and $|\text{supp } d'| < |\text{supp } d|$, the minimality of $|\text{supp } d|$ is contradicted. We may conclude that $\text{supp } d = \{1_G\}$, so d is a unit of D . Thus D is simple.

For each $\alpha \in \mathbf{k}$, set $y_\alpha = x_\alpha + x_\alpha^{-1}$. We claim that $\{y_\alpha D : \alpha \in \mathbf{k}\}$ is an independent family of (nonzero) right ideals of D . For suppose, on the contrary, that $\sum_{i=1}^n y_{\alpha_i} d_i = 0$, where $\{\alpha_1, \dots, \alpha_n\}$ is a set of distinct elements of \mathbf{k} and $\{d_1, \dots, d_n\}$ a set of nonzero elements of D . Let g be an element of maximal degree in $\bigcup_{i=1}^n \text{supp } d_i$. Suppose $g \in \text{supp } d_j$, where $1 \leq j \leq n$. It is not difficult to see that either $\partial(x_{\alpha_j} g) = \partial g + 1$, in which case $x_{\alpha_j} g \in \text{supp}(y_{\alpha_j} d_j)$, or $\partial(x_{\alpha_j}^{-1} g) = \partial g + 1$, in which case $x_{\alpha_j}^{-1} g \in \text{supp}(y_{\alpha_j} d_j)$. Note, however, that if $\partial(x_{\alpha_j} g) = \partial g + 1$ then

$$x_{\alpha_j} g \notin \left[\bigcup_{\substack{i=1 \\ i \neq j}}^n x_{\alpha_i} \text{supp } d_i \right] \cup \left[\bigcup_{\substack{i=1 \\ i \neq j}}^n x_{\alpha_i}^{-1} \text{supp } d_i \right] \supseteq \text{supp} \left(\sum_{\substack{i=1 \\ i \neq j}}^n y_{\alpha_i} d_i \right).$$

Similarly, if $\partial(x_{\alpha_j}^{-1} g) = \partial g + 1$ then

$$x_{\alpha_j}^{-1} g \notin \left[\bigcup_{\substack{i=1 \\ i \neq j}}^n x_{\alpha_i} \text{supp } d_i \right] \cup \left[\bigcup_{\substack{i=1 \\ i \neq j}}^n x_{\alpha_i}^{-1} \text{supp } d_i \right] \supseteq \text{supp} \left(\sum_{\substack{i=1 \\ i \neq j}}^n y_{\alpha_i} d_i \right).$$

Then $\text{supp}(y_{\alpha_j} d_j) \not\subseteq \text{supp} \left(\sum_{\substack{i=1 \\ i \neq j}}^n y_{\alpha_i} d_i \right)$, which contradicts the fact that

$$\sum_{i=1}^n y_{\alpha_i} d_i = y_{\alpha_j} d_j + \sum_{\substack{i=1 \\ i \neq j}}^n y_{\alpha_i} d_i = 0.$$

This shows that $\{y_\alpha D : \alpha \in \mathbf{k}\}$ is independent.

Now let R denote the ring of all $\mathbf{k} \times \mathbf{k}$ matrices over D containing only finitely many nonzero rows. We show first that $R \in P_r(1)$. Let $0 \neq r \in R$ and suppose that $r_{\alpha\beta} = x \neq 0$ for some $\alpha, \beta \in \mathbf{k}$. Define $s \in R$ by (1) $s_{\epsilon\delta} = 0$ if $\epsilon \neq \beta$; and (2) $s_{\epsilon\delta} = y_\delta = x_\delta + x_\delta^{-1}$ if $\epsilon = \beta$ ($\epsilon, \delta \in \mathbf{k}$). Then $\{s\}$ is a right insulator for r in R . For suppose $rst = 0$, $t \in R$. Then $\sum_{\delta \in \mathbf{k}} x y_\delta t_{\delta\epsilon} = x(\sum_{\delta \in \mathbf{k}} y_\delta t_{\delta\epsilon}) = 0$ for all $\epsilon \in \mathbf{k}$, so $\sum_{\delta \in \mathbf{k}} y_\delta t_{\delta\epsilon} = 0$ for all $\epsilon \in \mathbf{k}$. Since $\{y_\delta D : \delta \in \mathbf{k}\}$ is independent, we must have $t_{\delta\epsilon} = 0$ for all $\delta, \epsilon \in \mathbf{k}$, so $t = 0$. This shows that $R \in P_r(1)$. Certainly

then, $\mathfrak{U}\bar{P}_r(\mathfrak{m})(R) = 0$ for all cardinals $\mathfrak{m} > 0$.

Let L denote the set of all matrices in R which have only finitely many nonzero entries. Note that L is a left ideal of R . (Indeed, it is easy to see that if A is a $\mathfrak{k} \times \mathfrak{k}$ matrix over D with finitely many nonzero rows and B is a $\mathfrak{k} \times \mathfrak{k}$ matrix over D with finitely many nonzero columns, then AB has only finitely many nonzero entries.) We shall demonstrate that, as a ring, L is simple. Suppose that J is a nonzero ideal of L and choose a nonzero $r \in J$. Then $r_{\alpha\beta} = x \neq 0$ for some $\alpha, \beta \in \mathfrak{k}$. Since D is a simple ring with identity, we lose no generality in assuming that $axb = 1_D$ for some $a, b \in D$. If $\delta, \varepsilon \in \mathfrak{k}$ and $y \in D$, we shall denote by $E_{\delta, \varepsilon}(y)$ the matrix in L which has y in position (δ, ε) and zeros elsewhere. For each $\delta, \varepsilon \in \mathfrak{k}$, $E_{\delta, \alpha}(a)rE_{\beta, \varepsilon}(b) = E_{\delta, \varepsilon}(1_D)$. Therefore, $\{E_{\delta, \varepsilon}(1_D) : \delta, \varepsilon \in \mathfrak{k}\} \subseteq J$, so $J = L$. Thus L is simple. If $X \subseteq L$ with $|X| < \mathfrak{k}$ then, since each matrix in X has only finitely many nonzero entries, there must exist some $\alpha \in \mathfrak{k}$ such that $A^{(\alpha)}$ (the α -th column of A) = 0 for all $A \in X$. It follows that $XE_{\alpha, \alpha}(1_D) = 0$, so $L \notin \bar{P}_r(\mathfrak{k})$. Since L is simple, we may conclude that $L \in \mathfrak{U}\bar{P}_r(\mathfrak{m})$ for all $\mathfrak{m} \leq \mathfrak{k}$. Inasmuch as $\mathfrak{U}\bar{P}_r(\mathfrak{m})(R) = 0$ for all $\mathfrak{m} > 0$, this shows that $\mathfrak{U}\bar{P}_r(\mathfrak{m})$ is not left strong for all $\mathfrak{m} \leq \mathfrak{k}$. Since \mathfrak{k} was an arbitrary infinite cardinal, it follows that $\mathfrak{U}\bar{P}_r(\mathfrak{m})$ is not left strong for any nonzero cardinal \mathfrak{m} .

The above argument also shows that $B_r(\mathfrak{m})$ is not left strong for all limit cardinals \mathfrak{m} . \square

In [DV80, Theorem 3.15, p21] and [Vel86, Proposition 2.3, p284], an internal characterization of the elements of the right Groenewald-Heyman radical $\mathfrak{U}\bar{P}_r(\aleph_0)(R)$ of an arbitrary ring R is given. We generalize these to yield elementwise characterizations of $\mathfrak{U}\bar{P}_r(\mathfrak{m})(R)$ and $\mathfrak{U}P_r(\mathfrak{m})(R)$ for all nonzero cardinals \mathfrak{m} . If $\mathfrak{m} > 0$ then a subset X of R is called a *right $(f, \bar{\mathfrak{m}})$ -system in R* if, for each $x \in X$, there is a subset F of R with $|F| < \mathfrak{m} + 1$ such that

$$(\forall r \in R) (xFr \subseteq R \setminus X \Rightarrow r \in R \setminus X).$$

A right $(f, \bar{\mathfrak{m}})$ -system X in R is called a *right (f, \mathfrak{m}) -system in R* if X is not a right $(f, \bar{\mathfrak{n}})$ -system in R for any nonzero cardinal $\mathfrak{n} < \mathfrak{m}$.

LEMMA 15. *Let \mathfrak{m} be a nonzero cardinal. If A is an ideal of a ring R then $R/A \in \bar{P}_r(\mathfrak{m})$ (resp. $R/A \in P_r(\mathfrak{m})$) if and only if $R \setminus A$ is a right $(f, \bar{\mathfrak{m}})$ -system (resp. a right (f, \mathfrak{m}) -system) in R .*

Proof. Suppose that $R/A \in \bar{P}_r(\mathfrak{m})$ and let $x \in R \setminus A$. By hypothesis, there exists a subset F of R such that $|F| < \mathfrak{m} + 1$ and F/A is a right insulator for $x + A$ in R/A . It follows that if $r + A \in R/A$ and $(x + A)(F/A)(r + A) = 0$ then $r + A = 0$. In other words, $xFr \subseteq R \setminus (R \setminus A)$ implies $r \in R \setminus (R \setminus A)$. This shows that $R \setminus A$ is a right $(f, \bar{\mathfrak{m}})$ -system in R .

Conversely, suppose that $R \setminus A$ is a right $(f, \bar{\mathfrak{m}})$ -system in R and let $0 \neq x + A \in R/A$. By hypothesis, there exists a subset F of A with $|F| < \mathfrak{m} + 1$ such that $xFr \subseteq R \setminus (R \setminus A)$ implies $r \in R \setminus (R \setminus A)$, whenever $r \in R$. This is clearly equivalent to the assertion: $(x + A)(F/A)(r + A) = 0$

implies $r + A = 0$, whenever $r \in R$. In other words, F/A is a right insulator for $x + A$ in R/A . Since $|F/A| \leq |F| < \mathfrak{m} + 1$, we may conclude that $R/A \in \bar{P}_r(\mathfrak{m})$.

It follows from the above argument that for a nonzero cardinal $\mathfrak{n} < \mathfrak{m}$, we have $R/A \notin \bar{P}_r(\mathfrak{n})$ if and only if $R \setminus A$ is not a right $(f, \bar{\mathfrak{n}})$ -system in R . We may conclude therefore that $R/A \in P_r(\mathfrak{m})$ if and only if $R \setminus A$ is a right (f, \mathfrak{m}) -system in R . \square

If \mathfrak{m} is a nonzero cardinal then a subset Y of a ring R is called a *right $(g, \bar{\mathfrak{m}})$ -system* (resp. a *right (g, \mathfrak{m}) -system*) in R if there exists an ideal A of R such that $A \cap Y = \emptyset$ and $R \setminus A$ is a right $(f, \bar{\mathfrak{m}})$ -system (resp. a right (f, \mathfrak{m}) -system) in R .

PROPOSITION 16. *Let \mathfrak{m} be a nonzero cardinal. If R is a ring then:*

- (i) $\mathfrak{U}\bar{P}_r(\mathfrak{m})(R) = \{a \in R : \text{there is no right } (g, \bar{\mathfrak{m}})\text{-system } G \text{ in } R \text{ with } a \in G\}$;
- (ii) $\mathfrak{U}P_r(\mathfrak{m})(R) = \{a \in R : \text{there is no right } (g, \mathfrak{m})\text{-system } G \text{ in } R \text{ with } a \in G\}$.

Proof. (i) $\{a \in R : \text{there is no right } (g, \bar{\mathfrak{m}})\text{-system } G \text{ in } R \text{ with } a \in G\}$
 $= \{a \in R : \{a\} \text{ is not a right } (g, \bar{\mathfrak{m}})\text{-system in } R\}$
 $= \{a \in R : \text{if } A \in \text{Id } R \text{ and } R \setminus A \text{ is a right } (f, \bar{\mathfrak{m}})\text{-system in } R \text{ then } a \in A\}$
 $= \{a \in R : \text{if } A \in \text{Id } R \text{ and } R/A \in \bar{P}_r(\mathfrak{m}) \text{ then } a \in A\}$ (by Lemma 15)
 $= \bigcap \{A \in \text{Id } R : R/A \in \bar{P}_r(\mathfrak{m})\}$
 $= \mathfrak{U}\bar{P}_r(\mathfrak{m})(R)$ (since $\mathfrak{U}\bar{P}_r(\mathfrak{m})$ has the intersection property for $\bar{P}_r(\mathfrak{m})$).

(ii) is proved similarly. \square

Recall that if \mathfrak{m} is a nonzero cardinal then a right R -module M is said to be *\mathfrak{m} -faithful* if there exists a subset X of M such that $|X| < \mathfrak{m} + 1$ and $(0 :_r X) = 0$. An ideal I of R is said to be *right \mathfrak{m} -faithful* if the module I_R is \mathfrak{m} -faithful.

PROPOSITION 17. *Let \mathfrak{m} be an infinite cardinal and let I be a proper ideal of a ring R . If I is maximal among ideals of R which are not right \mathfrak{m} -faithful then $R/I \in \bar{P}_r(\mathfrak{m})$.*

Proof. Let J be a nonzero ideal of R such that $J \not\subseteq I$. It follows from the maximality of I that there is a subset X of J such that $|X| < \mathfrak{m}$ and $(0 :_r X) = 0$ (in R). We assert that X/I has zero right annihilator in R/I . Suppose, on the contrary, that $Xr \subseteq I$ for some $r \in R \setminus I$. If $\langle r \rangle$ denotes the ideal of R generated by r then $I + \langle r \rangle$ is a right \mathfrak{m} -faithful ideal of R so we can find $Y \subseteq I + \langle r \rangle$ with $|Y| < \mathfrak{m}$ and $(0 :_r Y) = 0$ (in R). We lose no generality in supposing that $Y = \{r\} \cup \{rb_i : i \in \Gamma\} \cup \{a_i : i \in \Gamma\}$, where $|\Gamma| < \mathfrak{m}$ and $a_i \in I$, $b_i \in R$ for all $i \in \Gamma$. It follows that $XY := \{xy : x \in X, y \in Y\} \subseteq I$. Since $|X|, |Y| < \mathfrak{m}$, we must have $|XY| < \mathfrak{m}$. Since I is not right \mathfrak{m} -faithful, we must have that $XYt = 0$ for some nonzero $t \in R$. Then $Yt \subseteq (0 :_r X) = 0$, so $t \in (0 :_r Y) = 0$, a contradiction. Thus every nonzero ideal of R/I is right \mathfrak{m} -faithful and so $R/I \in \bar{P}_r(\mathfrak{m})$ by Theorem II.6.3. \square

The following corollary is immediate.

COROLLARY 18. *Let \mathfrak{m} be an infinite cardinal. If a ring R has a proper ideal I which is maximal among ideals of R which are not right \mathfrak{m} -faithful then $\mathfrak{U}\bar{P}_r(\mathfrak{m})(R) \subseteq I$. In this case, therefore, $\mathfrak{U}\bar{P}_r(\mathfrak{m})(R)$ is not right \mathfrak{m} -faithful. \square*

In the case $\mathfrak{m} = \aleph_0$, if R is a ring such that R_R is \aleph_0 -faithful (e.g., if R has identity) then, by Zorn's Lemma, it is always possible to choose an ideal I as described in Proposition 17. The next example shows that this ceases to be true for $\mathfrak{m} > \aleph_0$, even if R is commutative with identity.

EXAMPLE 3. Let R be a commutative chain domain whose value group is the linearly ordered abelian group $\langle \mathbb{R}; +, -, 0; \leq \rangle$. (Such a ring R exists by Corollary I.2.2.) As in Example III.2.1, we shall write $[x]$ for the principal ideal of R corresponding to an element x of the positive cone of $\langle \mathbb{R}; +, -, 0; \leq \rangle$. Note that R has a unique maximal proper ideal $P = \bigcup \{[x] : x \in \mathbb{R}, x > 0\}$. Let $I = [1]$ and set $\bar{R} = R/I$, $\bar{P} = P/I$. If $a \in R$, let \bar{a} denote the image of a in \bar{R} . Note that \bar{P} has zero (right) annihilator in \bar{R} , since we can have $y + z \geq 1$ for all $z \in \{x \in \mathbb{R} : x > 0\}$ only if $y \geq 1$. Clearly, if x_0, x_1, x_2, \dots is a (countable) sequence in $\{x \in \mathbb{R} : x > 0\}$ such that $\lim_{k \rightarrow \infty} x_k = 0$ then $P = \bigcup_{k \geq 0} [x_k]$. Inasmuch as \bar{P} has zero annihilator in \bar{R} it follows that $\bar{P}_{\bar{R}}$ is \aleph_1 -faithful (but it is clearly not \aleph_0 -faithful). Also, every ideal of \bar{R} properly contained in \bar{P} is nilpotent and therefore not \mathfrak{m} -faithful for any cardinal \mathfrak{m} . Furthermore, since $\{x \in \mathbb{R} : x > 0\}$ has no positive lower bound in \mathbb{R} , there is no maximal non- \mathfrak{m} -faithful ideal in \bar{R} if $\mathfrak{m} \geq \aleph_1$. \square

If \mathfrak{m} is an infinite cardinal, we define:

- (i) $\mathcal{T}_{\mathfrak{m}} = \{R : R \text{ is a ring and } R_R \text{ is not } \mathfrak{m}\text{-faithful}\}$; and
- (ii) $H(\mathcal{T}_{\mathfrak{m}}) = \{R : R \text{ is a ring and every nonzero homomorphic image of } R \text{ is in } \mathcal{T}_{\mathfrak{m}}\}$.

Thus $H(\mathcal{T}_{\mathfrak{m}})$ is the largest homomorphically closed subclass of $\mathcal{T}_{\mathfrak{m}} \cup \{0\}$.

PROPOSITION 19. *Let \mathfrak{m} be an infinite cardinal. Then $H(\mathcal{T}_{\mathfrak{m}}) \subseteq \mathfrak{U}\bar{P}_r(\mathfrak{m})$.*

Proof. Let $R \in H(\mathcal{T}_{\mathfrak{m}})$. Suppose $R/I \in \bar{P}_r(\mathfrak{m})$ for some ideal I of R . Then $R/I \neq 0$ so $R/I \in \mathcal{T}_{\mathfrak{m}}$. But this contradicts Theorem II.6.3, so R has no (nonzero) homomorphic image in $\bar{P}_r(\mathfrak{m})$, i.e., $R \in \mathfrak{U}\bar{P}_r(\mathfrak{m})$. \square

Parmenter, Stewart and Wiegandt proved the following stronger version of Proposition 19 in the case where $\mathfrak{m} = \aleph_0$.

PROPOSITION 20. [PSW84, Theorem 1, p227] $H(\mathcal{T}_{\aleph_0}) = \mathfrak{U}\bar{P}_r(\aleph_0)$. \square

The following example shows, however, that for all uncountable cardinals \mathfrak{m} , we have

$$\mathcal{U}\bar{P}_r(\mathfrak{m}) \not\subseteq \mathcal{T}_\mathfrak{m}.$$

EXAMPLE 4. Let \bar{R} and \bar{P} be as in Example 3. Since $\bar{P}_{\bar{R}}$ is \aleph_1 -faithful, $\bar{P}_{\bar{P}}$ is \mathfrak{m} -faithful for all $\mathfrak{m} \geq \aleph_1$. Thus $\bar{P} \notin \mathcal{T}_\mathfrak{m}$ for all $\mathfrak{m} \geq \aleph_1$. Inasmuch as \bar{P} is commutative and nil, it follows from Proposition 1.5 that $\bar{P} \in \beta \subseteq \mathcal{U}\bar{P}_r(\mathfrak{m})$ for all $\mathfrak{m} > 0$. \square

PROPOSITION 21. Let \mathfrak{n} be a finite nonzero cardinal. If \mathfrak{m} is an infinite cardinal then $\mathcal{U}\bar{P}_r(\mathfrak{m})(\mathbb{M}_\mathfrak{n}(R)) = \mathbb{M}_\mathfrak{n}(\mathcal{U}\bar{P}_r(\mathfrak{m})(R))$ for all rings R and if \mathfrak{m} is a limit cardinal then $B_r(\mathfrak{m})(\mathbb{M}_\mathfrak{n}(R)) = \mathbb{M}_\mathfrak{n}(B_r(\mathfrak{m})(R))$ for all rings R .

Proof. If $\mathfrak{m} \geq \aleph_0$, then it follows from Proposition III.1.4 that $R \in P_r(\mathfrak{m})$ if and only if $\mathbb{M}_\mathfrak{n}(R) \in P_r(\mathfrak{m})$. Since $\mathcal{U}\bar{P}_r(\mathfrak{m})$ and $B_r(\mathfrak{m})$ (if \mathfrak{m} is a limit cardinal) are special radicals (Theorem 4), the results follow from Proposition 1.6. \square

The requirement that \mathfrak{m} be infinite cannot be dropped from the above result since, if $0 < \mathfrak{m}, \mathfrak{n} < \aleph_0$, $\mathfrak{n} > 1$ and $R = \mathbb{M}_\mathfrak{m}(F)$ with F a field, then

$$\mathcal{U}\bar{P}_r(\mathfrak{m})(\mathbb{M}_\mathfrak{n}(R)) = \mathbb{M}_\mathfrak{n}(R), \text{ whereas } \mathbb{M}_\mathfrak{n}(\mathcal{U}\bar{P}_r(\mathfrak{m})(R)) = 0.$$

§3. THE $\mathcal{U}\bar{P}_r(\mathfrak{m})$ -RADICALS OF MONOID RINGS.

Our main task in this section is to compare the $\mathcal{U}\bar{P}_r(\mathfrak{m})$ -radical of a ring R with the $\mathcal{U}\bar{P}_r(\mathfrak{m})$ -radical of a monoid ring RH over R . Many of the results of Chapter III, §3 find application here. Indeed, this section may be regarded as a natural sequel to Chapter III, §3. We also introduce the notion of an " \mathcal{M} -Jacobson ring", where \mathcal{M} is a nonempty class of rings and prove that the class of rings which are $\bar{P}_r(\mathfrak{m})$ -Jacobson ($\mathfrak{m} \geq \aleph_0$) is closed under the formation of polynomial rings (Theorem 12). This extends a result of Ferrero and Parmenter [FP89, Theorem 5, p281].

It turns out that the "going down" and "lying over" conditions defined at the beginning of the previous section play an important role in the present section. The following result is presumably well known.

PROPOSITION 1. If R is a subring of a ring S and \mathcal{M} is a special class of prime rings then the upper radical $\mathcal{U}\mathcal{M}$ determined by \mathcal{M} has the following properties:

- (i) $\mathcal{U}\mathcal{M}(R) \subseteq \mathcal{U}\mathcal{M}(S)$ if $R \subseteq S$ satisfies GD with respect to \mathcal{M} ;
- (ii) $\mathcal{U}\mathcal{M}(R) \supseteq R \cap \mathcal{U}\mathcal{M}(S)$ if $R \subseteq S$ satisfies LO with respect to \mathcal{M} .

Proof. This follows immediately from the fact that $\mathcal{U}\mathcal{M}$ has the intersection property for \mathcal{M} . \square

Following [Row88, p291], we call a subring R of a ring S a (right) ideal compatible subring

of S if the left ideal of S generated by I is an ideal of S whenever I is an ideal of R . For example, for any monoid H , the ring R is always an ideal compatible subring of RH ; in particular, R is an ideal compatible subring of the polynomial ring $R[x]$.

PROPOSITION 2. *Let \mathfrak{m} be an infinite cardinal. If R is an ideal compatible subring of a ring S then $R \subseteq S$ satisfies GD with respect to $\bar{P}_r(\mathfrak{m})$.*

Proof. Let A be an ideal of S such that $S/A \in \bar{P}_r(\mathfrak{m})$ and let $K/(R \cap A)$ be a nonzero ideal of $R/(R \cap A)$. Since R is ideal compatible in S , it follows that $(SK + K + A)/A$ is an ideal of S/A . If $SK + K + A = A$ then $K \subseteq R \cap A$, a contradiction, so $(SK + K + A)/A$ is nonzero. Since $S/A \in \bar{P}_r(\mathfrak{m})$, $(SK + K + A)/A$ is right \mathfrak{m} -faithful (Theorem II.6.3). Let $X \subseteq SK + K + A$ be such that $(0 :_r X/A) = 0$ (in S/A) and $|X| < \mathfrak{m}$. Suppose

$$X = \left\{ \left(\sum_{i=1}^{n(\gamma)} s_{i\gamma} b_{i\gamma} \right) + b_\gamma : \gamma \in \Gamma \right\} \text{ where } |\Gamma| < \mathfrak{m}, b_{i\gamma}, b_\gamma \in K \text{ and } s_{i\gamma} \in S \text{ for all } \gamma \in \Gamma, \\ i = 1, \dots, n(\gamma).$$

Let $Y = \{b_{i\gamma} : \gamma \in \Gamma, i = 1, \dots, n(\gamma)\} \cup \{b_\gamma : \gamma \in \Gamma\}$. Note that $|Y| < \mathfrak{m}$. We claim that $Y/(R \cap A)$ has zero right annihilator in $R/(R \cap A)$. Indeed, suppose $Yr \subseteq R \cap A$ for some $r \in R$. Then $Xr \subseteq S(R \cap A) + (R \cap A) \subseteq A$, so $r + A \in (0 :_r X/A) = 0$, i.e., $r \in R \cap A$. Thus $K/(R \cap A)$ is a right \mathfrak{m} -faithful ideal of $R/(R \cap A)$. By Theorem II.6.3, $R/(R \cap A) \in \bar{P}_r(\mathfrak{m})$. \square

The next corollary follows from the previous two results.

COROLLARY 3. *Let R be a ring and H a monoid. Then $\mathfrak{U}\bar{P}_r(\mathfrak{m})(R) \subseteq \mathfrak{U}\bar{P}_r(\mathfrak{m})(RH)$ for all infinite cardinals \mathfrak{m} . Moreover, if \mathfrak{m} is an uncountable limit cardinal then $B_r(\mathfrak{m})(R) \subseteq B_r(\mathfrak{m})(RH)$.* \square

The following example shows that the restrictions on \mathfrak{m} in the above corollary cannot be dispensed with.

EXAMPLE 1. Choose R, H and \mathfrak{n} as in Example III.3.1. If $0 < \mathfrak{m} < \mathfrak{n}$ then $R \in \mathfrak{U}\bar{P}_r(\mathfrak{m})$ and $\mathfrak{U}\bar{P}_r(\mathfrak{m})(RH) = 0$. Thus $\mathfrak{U}\bar{P}_r(\mathfrak{m})(R) \not\subseteq \mathfrak{U}\bar{P}_r(\mathfrak{m})(RH)$.

Suppose $\mathfrak{m} = \aleph_0$ and let F be a field. For each $\alpha \in \aleph_0$, let $\Theta_\alpha : \mathbf{M}_{2^\alpha}(F) \rightarrow \mathbf{M}_{\aleph_0}(F)$ denote the diagonal embedding

$$(1) \quad \Theta_\alpha(A) = \begin{bmatrix} A & 0 & & \\ 0 & A & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix} \in \mathbf{M}_{\aleph_0}(F) \quad (A \in \mathbf{M}_{2^\alpha}(F)).$$

Define $R = \bigcup_{\alpha \in \aleph_0} \Theta_\alpha[\mathbf{M}_{2^\alpha}(F)]$. As shown in Example III.1.1, $R \in P_r(\aleph_0)$. Moreover, R is simple since it is the union of an ascending chain of simple subrings of $\mathbf{M}_{\aleph_0}(F)$. Consequently, $R \in B_r(\aleph_0)$. If, in (1) above, F is replaced by FH we obtain, for each $\alpha \in \aleph_0$, a diagonal

embedding $\Theta'_\alpha: \mathbb{M}_{2\alpha}(FH) \rightarrow \mathbb{M}_{\aleph_0}(FH)$. Note, however, that $RH \cong \bigcup_{\alpha \in \aleph_0} \Theta'_\alpha[\mathbb{M}_{2\alpha}(FH)] \in P_r(1)$ since $\mathbb{M}_{2\alpha}(FH) \in P_r(1)$ for all $\alpha \in \aleph_0$ (see Example III.3.1), so $B_r(\aleph_0)(RH) = 0$. Thus $B_r(\aleph_0)(R) \not\subseteq B_r(\aleph_0)(RH)$. \square

Let H be an arbitrary monoid and let $\Phi: RH \rightarrow R$ denote the canonical ring epimorphism defined by

$$\Phi\left(\sum_{i=1}^n r_i g_i\right) = \sum_{i=1}^n r_i \quad \left(\sum_{i=1}^n r_i g_i \in RH\right).$$

Note that Φ restricts to the identity map on R . If I is an ideal of R and $K = IH + \text{Ker } \Phi$ then it is not difficult to show that $RH/K \cong R/I$ and $I = R \cap K$. Consequently, the inclusion $R \subseteq RH$ satisfies LO with respect to any class of rings that is closed under isomorphic images. By Proposition 1 (ii), $\mathfrak{U}\bar{P}_r(\mathfrak{m})(R) \supseteq R \cap \mathfrak{U}\bar{P}_r(\mathfrak{m})(RH)$ for all cardinals $\mathfrak{m} > 0$. A more direct line of reasoning yields more: if \mathfrak{R} is any radical then $R \cap \mathfrak{R}(RH) = \Phi(R \cap \mathfrak{R}(RH)) \subseteq \Phi(\mathfrak{R}(RH)) \subseteq \mathfrak{R}(R)$. In any case, the following result is immediate.

PROPOSITION 4. *Let R be a ring and H a monoid. Then:*

- (i) $\mathfrak{U}\bar{P}_r(\mathfrak{m})(R) \supseteq R \cap \mathfrak{U}\bar{P}_r(\mathfrak{m})(RH)$ for all nonzero cardinals \mathfrak{m} ; for all limit cardinals \mathfrak{m} , we have $B_r(\mathfrak{m})(R) \supseteq R \cap B_r(\mathfrak{m})(RH)$;
- (ii) $\mathfrak{U}\bar{P}_r(\mathfrak{m})(R) = R \cap \mathfrak{U}\bar{P}_r(\mathfrak{m})(RH)$ for all infinite cardinals \mathfrak{m} ; for uncountable limit cardinals \mathfrak{m} , we have $B_r(\mathfrak{m})(R) = R \cap B_r(\mathfrak{m})(RH)$. \square

In some instances, the equality $\mathfrak{U}\bar{P}_r(\mathfrak{m})(R) = R \cap \mathfrak{U}\bar{P}_r(\mathfrak{m})(RH)$ can be sharpened to $[\mathfrak{U}\bar{P}_r(\mathfrak{m})(R)]H = \mathfrak{U}\bar{P}_r(\mathfrak{m})(RH)$. Parmenter, Passman and Stewart [PPS84, Remark 5.3, p1110] observe, however, that even for $\mathfrak{m} = \aleph_0$, this equality need not hold unless some restriction is placed on the monoid H .

PROPOSITION 5. *Let \mathfrak{m} be a nonzero cardinal. Suppose H is a monoid with the property that for all rings R , $R \in \bar{P}_r(\mathfrak{m})$ implies $RH \in \bar{P}_r(\mathfrak{m})$ (resp., in the case where \mathfrak{m} is a limit cardinal, $R \in \bigcup_{0 < \mathfrak{k} < \mathfrak{m}} P_r(\mathfrak{k})$ implies $RH \in \bigcup_{0 < \mathfrak{k} < \mathfrak{m}} P_r(\mathfrak{k})$). Then:*

- (i) $[\mathfrak{U}\bar{P}_r(\mathfrak{m})(R)]H \supseteq \mathfrak{U}\bar{P}_r(\mathfrak{m})(RH)$ (resp., in the case where \mathfrak{m} is a limit cardinal, $[B_r(\mathfrak{m})(R)]H \supseteq B_r(\mathfrak{m})(RH)$);
- (ii) if \mathfrak{m} is infinite then $[\mathfrak{U}\bar{P}_r(\mathfrak{m})(R)]H = \mathfrak{U}\bar{P}_r(\mathfrak{m})(RH)$ (resp., in the case where \mathfrak{m} is an uncountable limit cardinal, $[B_r(\mathfrak{m})(R)]H = B_r(\mathfrak{m})(RH)$).

Proof. Suppose I is an ideal of R and $R/I \in \bar{P}_r(\mathfrak{m})$. Then $RH/IH \cong (R/I)H \in \bar{P}_r(\mathfrak{m})$ and $I = R \cap IH$. Therefore,

$$\begin{aligned} [\mathfrak{U}\bar{P}_r(\mathfrak{m})(R)]H &= [\bigcap \{K \in \text{Id } R : R/K \in \bar{P}_r(\mathfrak{m})\}]H \\ &= \bigcap \{KH : K \in \text{Id } R \text{ and } R/K \in \bar{P}_r(\mathfrak{m})\} \\ &\supseteq \mathfrak{U}\bar{P}_r(\mathfrak{m})(RH). \end{aligned}$$

If $\mathfrak{m} \geq \aleph_0$ then by Corollary 3, $\mathfrak{U}\bar{P}_r(\mathfrak{m})(RH) \supseteq [\mathfrak{U}\bar{P}_r(\mathfrak{m})(R)]H$. It follows that $[\mathfrak{U}\bar{P}_r(\mathfrak{m})(R)]H = \mathfrak{U}\bar{P}_r(\mathfrak{m})(RH)$. The assertions involving $B_r(\mathfrak{m})$ are proved similarly. \square

The next two results follow easily from Proposition III.3.3, Corollary III.3.4 and Proposition 5.

COROLLARY 6. *Let \mathfrak{m} be an infinite cardinal and R a ring. If H is a unique product monoid, a cancellative strictly right ordered monoid or the free product of two nontrivial monoids, then $[\mathfrak{U}\bar{P}_r(\mathfrak{m})(R)]H = \mathfrak{U}\bar{P}_r(\mathfrak{m})(RH)$ and if \mathfrak{m} is an uncountable limit cardinal, $[B_r(\mathfrak{m})(R)]H = B_r(\mathfrak{m})(RH)$.* \square

COROLLARY 7. *If \mathfrak{m} is an infinite cardinal and R a ring then $[\mathfrak{U}\bar{P}_r(\mathfrak{m})(R)][x] = \mathfrak{U}\bar{P}_r(\mathfrak{m})(R[x])$ and if \mathfrak{m} is an uncountable limit cardinal, $[B_r(\mathfrak{m})(R)][x] = B_r(\mathfrak{m})(R[x])$.* \square

Let \mathcal{M} be a nonempty class of prime rings. Following [FP89], we say that a ring R is \mathcal{M} -Jacobson if every prime ideal of R is an intersection of ideals A of R such that $R/A \in \mathcal{M}$. We point out at the outset that this notion does have a bearing on radical theory. Indeed, it is a consequence of the intersection property for special radicals that if \mathcal{M} is any special class of prime rings and R is an \mathcal{M} -Jacobson ring then $\mathfrak{U}\mathcal{M}(R) = \beta(R)$. In [FP89], Ferrero and Parmenter address the question of what conditions on a nonempty class \mathcal{M} of prime rings guarantee that the class of \mathcal{M} -Jacobson rings is closed under the formation of polynomial rings. Their main theorem is [FP89, Theorem 5, p281], which establishes this closure property in the case where \mathcal{M} satisfies a certain ‘‘Condition A’’ (to be defined presently). Ferrero and Parmenter then proceed to show that several well known classes of prime rings, and in particular the class $\bar{P}_r(\aleph_0)$ of all right strongly prime rings, satisfy Condition A (see [FP89, Propositions 6, 7 and 9 and Corollary 10, pp284-286]). We extend [FP89, Proposition 6(ii), p284] by proving that the class $\bar{P}_r(\mathfrak{m})$ satisfies Condition A for all infinite cardinals \mathfrak{m} (Proposition 10). A difficulty that we face, however, in applying Ferrero and Parmenter’s main result is that these authors assume their rings to have identity. This raises the question of whether [FP89, Theorem 5, p281] holds for rings without identity. We don’t attempt to answer this question. Instead, we circumvent it by proving directly that the class of $\bar{P}_r(\mathfrak{m})$ -Jacobson rings ($\mathfrak{m} \geq \aleph_0$) is closed under the formation of polynomial rings (Theorem 12). We point out, however, that in our proof of Theorem 12, we make use of the fact that $\bar{P}_r(\mathfrak{m})$ ($\mathfrak{m} \geq \aleph_0$) satisfies Condition A; we also utilize many ingredients of Ferrero and Parmenter’s proof of [FP89, Theorem 5, p281].

We say that an ideal K of a polynomial ring $R[x]$ is R -disjoint if $K \neq 0$ and $R \cap K = 0$. The next two results are attributed to G. Bergman by the authors of [FP89]. Variants of the second result appear in several papers, e.g., [Wat75, Lemma 3, p305] and [FK85, Lemma 3.3, p293] and a more general result for \mathbb{Z} -graded rings is to be found in [Pass89, Theorem 22.5, p224]. Proofs are included here for the reader’s convenience.

If $f \in R[x]$, ∂f shall denote the degree of f and we adopt the convention that the degree of

the zero polynomial is $-\infty$. If I is a nonzero ideal of $R[x]$, we shall use $\text{Min } I$ to abbreviate $\min \{\partial f : 0 \neq f \in I\}$.

LEMMA 8. *Let R be a ring and I a nonzero ideal of $S = R[x]$. Suppose that $f = a_0 + a_1x + \dots + a_mx^m$ is a nonzero element of minimal degree in I , that $a_m = \bar{a} \neq 0$ and that $m \geq 1$. Let $0 \neq g \in I$, say $g = b_0 + b_1x + \dots + b_nx^n$ with $n \geq m$ and $b_n \neq 0$. If $r_1, r_2, \dots, r_{n-m} \in R$ then there exists $h \in S$ such that $\partial h \leq n - m$ and*

$$htf = g(r_{n-m}\bar{a})(r_{n-m-1}\bar{a}) \dots (r_1\bar{a})t\bar{a} \text{ for all } t \in R.$$

Proof. The proof is by induction on $\partial g = n$. If $n = m$, take $h = b_m$ and consider $g' = htf - g\bar{a}$. Clearly, $\partial g' \leq m - 1$ and $g' \in I$, which contradicts the minimality of ∂f unless $g' = 0$, so $htf = g\bar{a}$ for all $t \in R$, as required.

Suppose now that the result is true whenever $\partial g < n$ and consider the case where $\partial g = n > m$. Consider the polynomial $g' = b_nx^{n-m}r_{n-m}f - gr_{n-m}\bar{a}$. Notice that $g' \in I$ and $\partial g' \leq n - 1$. By the inductive hypothesis, there exists $h' \in S$ such that $\partial h' \leq (n - 1) - m$ and

$$h'tf = g'(r_{n-m-1}\bar{a})(r_{n-m-2}\bar{a}) \dots (r_1\bar{a})t\bar{a} \text{ for all } t \in R, \text{ i.e.,}$$

$$(1) \quad h'tf = [b_nx^{n-m}r_{n-m}f(r_{n-m-1}\bar{a}) \dots (r_1\bar{a})]t\bar{a} - g(r_{n-m}\bar{a}) \dots (r_1\bar{a})t\bar{a} \text{ for all } t \in R.$$

Consider the polynomial $g'' = b_nr_{n-m}f(r_{n-m-1}\bar{a}) \dots (r_1\bar{a})$. Since $\partial g'' \leq \partial f = m$ (in fact $g'' = 0$ or $\partial g'' = m$), it follows that there exists $h'' \in S$ with $\partial h'' \leq 0$ such that $h''tf = g''\bar{a}$ for all $t \in R$, i.e., $h''tf = b_nr_{n-m}f(r_{n-m-1}\bar{a}) \dots (r_1\bar{a})t\bar{a}$ so $x^{n-m}h''tf = b_nx^{n-m}r_{n-m}f(r_{n-m-1}\bar{a}) \dots (r_1\bar{a})t\bar{a}$ for all $t \in R$. Substituting into equation (1), we obtain $h'tf = x^{n-m}h''tf - g(r_{n-m}\bar{a}) \dots (r_1\bar{a})t\bar{a}$ so $(x^{n-m}h'' - h')tf = g(r_{n-m}\bar{a}) \dots (r_1\bar{a})t\bar{a}$ for all $t \in R$. Setting $h = x^{n-m}h'' - h'$, note that $\partial h \leq n - m$ (because $\partial h' \leq n - m - 1$ and $\partial h'' \leq 0$) while $htf = g(r_{n-m}\bar{a}) \dots (r_1\bar{a})t\bar{a}$ for all $t \in R$, as required. \square

LEMMA 9. *Let R be a prime ring and P an R -disjoint ideal of $R[x]$. Then the following statements are equivalent:*

- (i) P is a prime ideal of $R[x]$;
- (ii) P is maximal in the set of all R -disjoint ideals of $R[x]$.

Proof. (ii) \Rightarrow (i): Suppose, on the contrary, that P is maximal R -disjoint but not prime. Then there exist ideals A, B of $R[x]$ such that $AB \subseteq P$ yet $A \not\subseteq P$ and $B \not\subseteq P$. Since $A + P$ and $B + P$ are not R -disjoint, we can choose nonzero elements $r \in (A + P) \cap R$ and $s \in (B + P) \cap R$. Then $0 \neq rRs$ (since R is prime) and $rRs \subseteq (A + P)(B + P) \subseteq P$, which contradicts the fact that P is R -disjoint.

(i) \Rightarrow (ii): Note first that $S = R[x]$ is prime, since R is prime (Corollary III.3.4). Let I be an ideal of S , containing P , which is maximal R -disjoint. Let $f = a_0 + a_1x + \dots + a_mx^m$ be a

nonzero polynomial of minimal degree m in I . Since I is R -disjoint, $m \geq 1$ and $\bar{a} = a_m \notin I$. Let g be a nonzero polynomial of minimal degree in P . Suppose that $\partial g = n > m$. Since $g, \bar{a} \neq 0$, it follows from the primeness of S that $gS\bar{a} \neq 0$, so $gR\bar{a} \neq 0$. Choose $r_{n-m} \in R$ such that $gr_{n-m}\bar{a} \neq 0$. Then $(gr_{n-m}\bar{a})S\bar{a} \neq 0$ so $(gr_{n-m}\bar{a})R\bar{a} \neq 0$, hence we can choose $r_{n-m-1} \in R$ such that $(gr_{n-m}\bar{a})(r_{n-m-1}\bar{a}) \neq 0$. Continuing in this way, we can produce $r_{n-m}, r_{n-m-1}, \dots, r_1 \in R$ such that $h' = (gr_{n-m}\bar{a}) \dots (r_1\bar{a}) \neq 0$. By the previous lemma, we may choose $h \in S$ such that $\partial h \leq n - m$ and $htf = (gr_{n-m}\bar{a}) \dots (r_1\bar{a})t\bar{a} \in P$ for all $t \in R$. Moreover, since $h', \bar{a} \neq 0$, we must have $h't\bar{a} \neq 0$ for some $t \in R$, so $h \neq 0$. We also have that $hRf \subseteq P$, whence $hSf \subseteq P$. Since P is prime and $f \notin P$, we must have $0 \neq h \in P$. But $\partial h \leq n - m < n$, which contradicts the minimality of ∂g . It follows that $\partial g = m$.

In view of the above argument, we lose no generality in assuming that $f = a_0 + a_1x + \dots + a_mx^m \in P$. Now let $0 \neq f' \in I$ with $\partial f' = l \geq m$, and let $r_{l-m}, r_{l-m-1}, \dots, r_1, r_0 \in R$. By the previous lemma, there exists $h \in S$ such that $hr_0f = f'(r_{l-m}\bar{a}) \dots (r_1\bar{a})(r_0\bar{a})$. Since $f \in P$, we have $f'(r_{l-m}\bar{a}) \dots (r_1\bar{a})(r_0\bar{a}) \in P$. Since the r_i were arbitrary, we may conclude that $f'(R\bar{a}) \dots (R\bar{a})(R\bar{a}) \subseteq P$ so $f'(S\bar{a}) \dots (S\bar{a})(S\bar{a}) \subseteq P$. But $\bar{a} \notin P$, so $f' \in P$. Thus $I \subseteq P$. \square

Following [FP89], we say that a nonempty class \mathcal{M} of prime rings satisfies *Condition A* if, whenever $R \in \mathcal{M}$, we have $R[x]/P \in \mathcal{M}$ for every R -disjoint prime ideal P of $R[x]$.

PROPOSITION 10. $\bar{P}_r(\mathfrak{m})$ satisfies Condition A for all infinite cardinals \mathfrak{m} .

Proof. Suppose $R \in \bar{P}_r(\mathfrak{m})$. Let P be an R -disjoint prime ideal of $R[x]$ and let K/P be a nonzero ideal of $R[x]/P$. By the previous lemma, P is maximal R -disjoint, so $R \cap K \neq 0$. By Theorem II.6.3, $R \cap K$ is right \mathfrak{m} -faithful, so $(0 :_r Y) = 0$ (in R) for some subset Y of $R \cap K$ with $|Y| < \mathfrak{m}$. Suppose $P \not\subseteq (P :_r Y)$ (in $R[x]$). Let g be a polynomial of minimal degree for which $Yg \subseteq P$ and $g \notin P$. If $\partial g < \text{Min } P$ then $yg = 0$ for all $y \in Y$. Furthermore, if b is any nonzero coefficient of g then we may deduce that $yb = 0$ for all $y \in Y$, which contradicts the fact that $(0 :_r Y) = 0$. We must therefore have $\partial g \geq \text{Min } P = n$, say.

Suppose $g = a_0 + a_1x + \dots + a_mx^m$ with $a_m \neq 0$ and that $f = b_0 + b_1x + \dots + b_nx^n$, $b_n \neq 0$, is a nonzero polynomial of minimal degree in P . Since P is prime and R -disjoint, we have that $gR[x]b_n \not\subseteq P$ and so $grb_n \notin P$ for some $r \in R$. Set $g' = grb_n - a_mr^fx^{m-n}$. Note that $g' \notin P$, $\partial g' < \partial g = m$ and $Yg' \subseteq P$, which contradicts the minimality of ∂g . It follows that Y/P has zero right annihilator in $R[x]/P$, so K/P is right \mathfrak{m} -faithful. By Theorem II.6.3, $R[x]/P \in \bar{P}_r(\mathfrak{m})$. \square

Let R be a ring and I a nonzero ideal of $R[x]$. It is easily checked that the subset of R consisting of 0 and all leading coefficients of all nonzero polynomials of minimal degree in I is an ideal of R . We denote this ideal by $\tau(I)$.

The following lemma is due to Ferrero and Parmenter [FP89]. Although they stated the

result for rings with identity, their proof is valid for arbitrary rings and is included for the reader's convenience.

LEMMA 11. [FP89, Lemma 3, p283] *Let R be a ring and P an R -disjoint prime ideal of $R[x]$. If Q is a nonzero prime ideal of R such that $\tau(P) \not\subseteq Q$, then $R \cap (P + Q[x]) = Q$.*

Proof. Suppose, on the contrary, that $R \cap (P + Q[x]) \neq Q$. Since it is always the case that $R \cap (P + Q[x]) \supseteq Q$, we must have $R \cap (P + Q[x]) \supsetneq Q$. We can therefore choose $r \in R \setminus Q$ such that $r = h_1 + h_2$ for some $h_1 \in P$ and $h_2 \in Q[x]$. Consider $h_1 = r - h_2 \in P$. Clearly the constant term of h_1 cannot be contained in Q , since this would imply $r \in Q$, a contradiction. Consequently, P must contain a polynomial of the form $g = b_0 + \dots + b_m x^m$ with $b_i \in Q$ for all $i \in \{1, \dots, m\}$ and $b_0 \notin Q$. Take g of minimal degree with respect to these conditions. Since $\tau(P) \not\subseteq Q$, there exists a polynomial $f = a_0 + \dots + a_n x^n$ of minimal degree in P whose leading coefficient $a_n \notin Q$. Since Q is prime and $b_0, a_n \notin Q$, we must have $b_0 R a_n \not\subseteq Q$, so $b_0 c a_n \notin Q$ for some $c \in R$. Put $h = g c a_n - x^{m-n} b_m c f \in P$. Note that the constant term, d say, of h is equal to $b_0 c a_n$, or possibly $b_0 c a_n - b_n c a_0$ if $m = n$. In either case, $d \notin Q$. Note also that all other coefficients of h are contained in Q , since $b_1, b_2, \dots, b_m \in Q$. Inasmuch as $\partial h < \partial g$, this contradicts the minimality of ∂g . \square

THEOREM 12. *Let \mathfrak{m} be an infinite cardinal. If R is a $\bar{P}_r(\mathfrak{m})$ -Jacobson ring then so is $R[x]$.*

Proof. Suppose that P is a prime ideal of $R[x]$. We need to show that P is an intersection of ideals A of $R[x]$ such that $R/A \in \bar{P}_r(\mathfrak{m})$. Consider the epimorphism from $R[x]$ onto $(R/R \cap P)[x]$ defined by $\sum_{i=0}^n a_i x^i \mapsto \sum_{i=0}^n \bar{a}_i x^i$, where $\bar{a}_i = a_i + (R \cap P)$ for $i = 0, \dots, n$. For each ideal K of $R[x]$, let \bar{K} denote the image of K in $(R/R \cap P)[x]$. Note that \bar{P} is a prime ideal of $(R/R \cap P)[x] = \bar{R}[x]$ and that $\bar{R} \cap \bar{P} = 0$. Since R is $\bar{P}_r(\mathfrak{m})$ -Jacobson, it follows that \bar{R} is $\bar{P}_r(\mathfrak{m})$ -Jacobson. It clearly suffices to show that \bar{P} is an intersection of ideals A of $\bar{R}[x]$ such that $\bar{R}[x]/A \in \bar{P}_r(\mathfrak{m})$. We therefore lose no generality in supposing that P is a prime ideal of $R[x]$ such that $R \cap P = 0$. Since R is an ideal compatible subring of $R[x]$, it follows from Proposition 2 that the inclusion $R \subseteq R[x]$ satisfies GD with respect to $\bar{P}_r(\mathfrak{k})$ for all cardinals $\mathfrak{k} \geq \aleph_0$. Therefore, $R \cap P = 0$ is a prime ideal of R , i.e., R is a prime ring. Put $\mathfrak{R} = \{Q \in \text{Id } R : R/Q \in \bar{P}_r(\mathfrak{m})\}$. Since R is, by assumption, $\bar{P}_r(\mathfrak{m})$ -Jacobson, we have $\bigcap \mathfrak{R} = 0$. This clearly implies that $\bigcap \{Q[x] : Q \in \mathfrak{R}\} = 0$. We know from Corollary III.3.4 that $R[x]/Q[x] \cong (R/Q)[x] \in \bar{P}_r(\mathfrak{m})$ whenever Q is an ideal of R such that $R/Q \in \bar{P}_r(\mathfrak{m})$. We may conclude that the zero ideal of $R[x]$ is expressible as an intersection of ideals A of $R[x]$ such that $R[x]/A \in \bar{P}_r(\mathfrak{m})$. This deals with the case $P = 0$.

Now suppose $P \neq 0$. Since P is prime and $R \cap P = 0$, it follows from Lemma 9 that P is maximal in the set of all R -disjoint ideals of $R[x]$. Put $\mathfrak{S} = \{Q \in \mathfrak{R} : R \cap (P + Q[x]) = Q\}$. Inasmuch as $(\bigcap \mathfrak{S}) \cap (\bigcap (\mathfrak{R} \setminus \mathfrak{S})) = \bigcap \mathfrak{R} = 0$, the primeness of R implies that either $\bigcap \mathfrak{S} = 0$ or $\bigcap (\mathfrak{R} \setminus \mathfrak{S}) = 0$. If $Q \in \mathfrak{R} \setminus \mathfrak{S}$ then $R \cap (P + Q[x]) \neq Q$ and therefore, by the previous lemma, we cannot

have $\tau(P) \not\subseteq Q$, so $\tau(P) \subseteq Q$. Consequently, $\bigcap (\mathfrak{R} \setminus \mathfrak{S}) \supseteq \tau(P) \neq 0$ and so $\bigcap \mathfrak{S} = 0$. For each $Q \in \mathfrak{S}$, consider the epimorphism from $R[x]$ onto $(R/Q)[x]$ defined by $\sum_{i=0}^n a_i x^i \mapsto \sum_{i=0}^n \bar{a}_i x^i$, where $\bar{a}_i = a_i + Q$ for $i = 0, \dots, n$. Again, we denote by \bar{K} the image of the ideal K of $R[x]$ in $(R/Q)[x]$. Inasmuch as $Q \in \mathfrak{R}$, it follows from Corollary III.3.4 that $\bar{R}[x] = (R/Q)[x] \in \bar{P}_r(\mathfrak{m})$. Observe that $\bar{R} \cap \bar{P} = \overline{R \cap (P + Q[x])} = \bar{Q} = 0$. Choose an ideal P_Q of $R[x]$ which is maximal such that $P_Q \supseteq Q$, $P_Q \supseteq P$ and $\bar{R} \cap \bar{P}_Q = 0$. We claim that $R[x]/P_Q \in \bar{P}_r(\mathfrak{m})$. If $\bar{P}_Q = 0$, that is to say $P_Q = Q[x]$, then $R[x]/P_Q = R[x]/Q[x] \cong (R/Q)[x] = \bar{R}[x] \in \bar{P}_r(\mathfrak{m})$. If $\bar{P}_Q \neq 0$ then \bar{P}_Q is a maximal \bar{R} -disjoint ideal of $\bar{R}[x]$. Since \bar{R} is prime, it follows from Lemma 9 that \bar{P}_Q is a prime ideal of $\bar{R}[x]$. Since $\bar{R} \in \bar{P}_r(\mathfrak{m})$ and $\bar{P}_r(\mathfrak{m})$ satisfies Condition A (Proposition 10), we must have that $R[x]/P_Q \cong \bar{R}[x]/\bar{P}_Q \in \bar{P}_r(\mathfrak{m})$. This establishes our claim. Notice that $\bigcap_{Q \in \mathfrak{S}} P_Q$ is an R -disjoint ideal of $R[x]$ since $R \cap (\bigcap_{Q \in \mathfrak{S}} P_Q) = \bigcap_{Q \in \mathfrak{S}} (R \cap P_Q) = \bigcap_{Q \in \mathfrak{S}} Q = 0$. However, since P is a maximal R -disjoint ideal of $R[x]$ and $P \subseteq \bigcap_{Q \in \mathfrak{S}} P_Q$, we must have $P = \bigcap_{Q \in \mathfrak{S}} P_Q$. This shows that P is an intersection of ideals A of $R[x]$ such that $R[x]/A \in \bar{P}_r(\mathfrak{m})$, as required. \square

The following example, which is motivated by [FP89, Remark 2, p286], shows that the requirement that \mathfrak{m} be infinite cannot be dropped from the above theorem.

EXAMPLE 2. Let \mathbb{H} denote the division ring of real quaternions. For the purposes of this example, we shall identify \mathbb{H} with the subring

$$\left\{ \left[\begin{array}{cc} u & v \\ -\bar{v} & \bar{u} \end{array} \right] : u, v \in \mathbb{C} \right\}$$

of $\mathbb{M}_2(\mathbb{C})$ (here \bar{u} denotes the complex conjugate of u). Consider the ideal $\langle 1 + x^2 \rangle$ of $\mathbb{H}[x]$ generated by $1 + x^2$. Clearly $\langle 1 + x^2 \rangle$ is prime and \mathbb{H} -disjoint. It can easily be verified that the map from $\mathbb{H}[x]/\langle 1 + x^2 \rangle$ to $\mathbb{M}_2(\mathbb{C})$ defined by

$$(h_0 + h_1 x) + \langle 1 + x^2 \rangle \mapsto h_0 + h_1 \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \in \mathbb{M}_2(\mathbb{C}) \quad (h_0, h_1 \in \mathbb{H})$$

is a ring isomorphism. Inasmuch as $\mathbb{H} \in P_r(1)$ and $\mathbb{M}_2(\mathbb{C}) \in P_r(2)$, this shows that $P_r(1)$ fails to satisfy Condition A. More generally, if \mathfrak{n} is a finite nonzero cardinal, I is the $\mathfrak{n} \times \mathfrak{n}$ identity matrix (over \mathbb{H}) and $\langle I + Ix^2 \rangle$ the ideal of $\mathbb{M}_{\mathfrak{n}}(\mathbb{H})[x]$ generated by $I + Ix^2$, then $\mathbb{M}_{\mathfrak{n}}(\mathbb{H})[x]/\langle I + Ix^2 \rangle$ is canonically isomorphic to $\mathbb{M}_{\mathfrak{n}}(\mathbb{H}[x]/\langle 1 + x^2 \rangle) \cong \mathbb{M}_{\mathfrak{n}}(\mathbb{M}_2(\mathbb{C})) \cong \mathbb{M}_{2\mathfrak{n}}(\mathbb{C}) \in P_r(2\mathfrak{n})$. The ideal $\langle I + Ix^2 \rangle$ is certainly prime and proper. Since $\mathbb{M}_{\mathfrak{n}}(\mathbb{H})$ is simple, we must have $\langle I + Ix^2 \rangle \cap \mathbb{M}_{\mathfrak{n}}(\mathbb{H}) = 0$. Thus $\langle I + Ix^2 \rangle$ is $\mathbb{M}_{\mathfrak{n}}(\mathbb{H})$ -disjoint. Inasmuch as $\mathbb{M}_{\mathfrak{n}}(\mathbb{H}) \in P_r(\mathfrak{n})$, this shows that the class $\bar{P}_r(\mathfrak{n})$ fails to satisfy Condition A.

§4. COMPARISONS WITH OTHER RADICALS.

In this section, we shall locate the radicals $\mathfrak{U}\bar{P}_r(\mathfrak{m})$, $\mathfrak{U}P_r(\mathfrak{m})$ and $B_r(\mathfrak{m})$ in the lattice of radicals by comparing them with the well known radicals depicted in Figure 1 of §1. The diagram of

radicals we intend constructing is made more illuminating by the inclusion of two further special radicals. These radicals are not new (see [Ols87] and [Vel87]) but are perhaps less well known and deserve an introduction.

THEOREM 1. [Ols87, Theorem 16, p100] *The class of all uniformly strongly prime rings is special.* □

Following Olson [Ols87], we denote by US the (special) upper radical determined by the class of all uniformly strongly prime rings.

A nonzero ring R is said to be *right superprime* (see [VdW] or [Vel87]) if every nonzero ideal of R contains an element x such that $(0 :_r x) = 0$.

PROPOSITION 2. [Vel87, Theorem 3.1, p183] *The class of all right superprime rings is special.* □

We shall use the notation of Veldsman [Vel87] and denote by σ_r (resp. σ_l) the (special) upper radical determined by the class of all right (resp. left) superprime rings.

Suppose \mathfrak{m} is an infinite cardinal and D a division ring. In Example 2.1, we noted that, as a consequence of Proposition 0.9.3, the ring $R = \mathbf{M}_{\mathfrak{m}}(D)$ contains a unique minimal nonzero ideal consisting precisely of all matrices with only finitely many nonzero columns. We shall, henceforth, denote this ideal by $I_{\aleph_0, \mathfrak{m}}(D)$. We shall make frequent use of the next lemma.

LEMMA 3. *Let \mathfrak{m} be a nonzero cardinal. Then:*

- (i) *if \mathfrak{m} is countable then there exists a simple ring R , with identity, such that $R \in P_r(\mathfrak{m})$;*
- (ii) *if \mathfrak{m} is uncountable then there exists a simple ring R which contains nonzero idempotent elements, such that $R \in P_r(\mathfrak{m})$.*

Proof. (i) If $\mathfrak{m} < \aleph_0$, we take $R = \mathbf{M}_{\mathfrak{m}}(D)$, with D any division ring. Suppose $\mathfrak{m} = \aleph_0$. Since the union of any ascending chain of simple subrings of a given ring is simple, it is easy to see that the right unbounded strongly prime ring R of Example III.1.1 is simple. Since R has identity, this establishes (i).

(ii) Let D be a division ring and consider $I = I_{\aleph_0, \mathfrak{m}}(D)$. By Proposition III.1.4, $\mathbf{M}_{\mathfrak{m}}(D) \in P_r(\mathfrak{m}^+)$. Since $P_r(\mathfrak{m}^+)$ is a special class (Theorem 2.4) and I is an ideal of $\mathbf{M}_{\mathfrak{m}}(D)$, we must have $I \in P_r(\mathfrak{m}^+)$. Suppose K is a nonzero ideal of I and let $\langle K \rangle$ denote the ideal of $\mathbf{M}_{\mathfrak{m}}(D)$ generated by K . Since $\mathbf{M}_{\mathfrak{m}}(D)$ is (von Neumann) regular, $\langle K \rangle$ is idempotent (Theorem 0.9.2(i)). By Andrunakievič's Lemma (Lemma 0.2.1), $\langle K \rangle = \langle K \rangle^3 \subseteq K$, so $I = \langle K \rangle = K$. Thus I is a simple ring. Clearly, I contains nonzero idempotent elements, so (ii) is satisfied whenever \mathfrak{m} is a successor

cardinal.

Suppose \mathfrak{m} is an uncountable limit cardinal. Consider the ring R of Example III.1.2. Recall that R is the union of an ascending chain of subrings of $\mathbb{M}_{\mathfrak{m}}(D)$ where D is a division ring. Specifically, $R = \bigcup_{\aleph_0 \leq \mathfrak{k} < \mathfrak{m}} \Theta_{\mathfrak{k}}[\mathbb{M}_{\mathfrak{k}}(D)]$ where, for each infinite cardinal $\mathfrak{k} < \mathfrak{m}$, $\Theta_{\mathfrak{k}}: \mathbb{M}_{\mathfrak{k}}(D) \rightarrow \mathbb{M}_{\mathfrak{m}}(D)$ is the “diagonal embedding” of $\mathbb{M}_{\mathfrak{k}}(D)$ into $\mathbb{M}_{\mathfrak{m}}(D)$. Consider the ideal $I = \bigcup_{\aleph_0 \leq \mathfrak{k} < \mathfrak{m}} \Theta_{\mathfrak{k}}[I_{\aleph_0, \mathfrak{k}}(D)]$ of R . Since $P_r(\mathfrak{m})$ is a special class and $R \in P_r(\mathfrak{m})$, we must have $I \in P_r(\mathfrak{m})$. Again, since the union of any ascending chain of simple subrings is simple, it follows that I is a simple ring. Moreover, I clearly contains nonzero idempotent elements. \square

PROPOSITION 4. *Every right bounded strongly prime ring is right superprime.*

Proof. Let $R \in P_r(\mathfrak{n})$ with $0 < \mathfrak{n} < \aleph_0$ and let I be a nonzero ideal of R . Suppose first that $\mathfrak{n} = 1$. If $0 \neq a \in I$ and $\{b\}$ is a right insulator for a in R then $(0:{}_r ab) = 0$ (in R) and $ab \in I$. Thus R is right superprime. Now suppose $\mathfrak{n} > 1$. By Theorem III.1.2, R is isomorphic to a right order in $\mathbb{M}_{\mathfrak{n}}(D)$ for some division ring D . If R is identified with its image in $\mathbb{M}_{\mathfrak{n}}(D)$, we may conclude from Lemma 2.7 that I is a right order in $I\mathbb{M}_{\mathfrak{n}}(D) = \mathbb{M}_{\mathfrak{n}}(D)$. Clearly then, I must contain a unit u of $\mathbb{M}_{\mathfrak{n}}(D)$, in which case, $(0:{}_r u) = 0$ (in R). \square

Because every simple ring with identity is right superprime (trivially), Lemma 3 (i) implies the existence of a right superprime ring which is not right bounded strongly prime. Therefore, the converse of Proposition 4 is not valid. Furthermore, if R is a right strongly prime ring which contains a nonzero nil ideal (an example of such a ring is given in [GHL74, Ex 2.5, p12]), then R cannot be right superprime.

In the following result we locate the radicals $\mathfrak{U}\bar{P}_r(\aleph_0), B_r(\aleph_0), US$ and σ_r . We should point out that Proposition 5 below represents a synthesis of several known, but recent results as well as several new results. In most cases, proofs have been included although in some instances, we refer the reader to [PSW84] for proof details.

PROPOSITION 5. (i) $N, \mathfrak{U}\bar{P}_r(\aleph_0) \subsetneq \sigma_r \subsetneq B_r(\aleph_0) \subsetneq US \subsetneq T, N_g$.

(ii) $\sigma_r \subsetneq G$.

(iii) $L \subsetneq \mathfrak{U}\bar{P}_r(\aleph_0)$.

(iv) $US \not\supseteq J \cap \beta_{\phi}$.

(v) $G \not\supseteq B_r(\aleph_0)$.

(vi) $J_B \not\supseteq \mathfrak{U}\bar{P}_r(\aleph_0)$.

(vii) $\mathfrak{U}\bar{P}_r(\aleph_0) \not\supseteq \beta_{\phi} \cap N$.

Proof. (i) Inasmuch as every right superprime ring is right strongly prime, every right bounded strongly prime ring is right superprime (Proposition 4), every uniformly strongly prime ring is

right bounded strongly prime and every domain is uniformly strongly prime, we must have that

$$\mathfrak{U}\bar{P}_r(\aleph_0) \subseteq \sigma_r \subseteq B_r(\aleph_0) \subseteq US \subseteq N_g.$$

Since $\mathbb{M}_n(D)$ is uniformly strongly prime whenever D is a division ring and $1 \leq n < \aleph_0$ (Theorem III.4.3), we must also have $US \subseteq T$. The inequality $N \subseteq \sigma_r$ follows because a right superprime ring cannot contain a nonzero nil ideal. It remains to show that the inequalities are strict.

Since T and N_g are known to be incomparable (Theorem 1.2 (iii)), we must have that $US \subsetneq T, N_g$. Similarly, to show that $N, \mathfrak{U}\bar{P}_r(\aleph_0) \subsetneq \sigma_r$, it suffices to show that N and $\mathfrak{U}\bar{P}_r(\aleph_0)$ are incomparable. Let R be a right strongly prime ring with a nonzero nil ideal I (see [GHL74, Ex 2.5, p12]). Since $\bar{P}_r(\aleph_0)$ is a special class (Theorem 2.4), it follows that $I \in \bar{P}_r(\aleph_0)$. Inasmuch as every homomorphic image of I is nil, we must have $I \in N$. Thus $\mathfrak{U}\bar{P}_r(\aleph_0) \not\subseteq N$. If, on the other hand, R is chosen to be a simple ring which contains nonzero idempotent elements but which is not strongly prime (such a ring exists by Lemma 3 (ii)), then $R \notin N$ yet $R \in \mathfrak{U}\bar{P}_r(\aleph_0)$. Thus $N \not\subseteq \mathfrak{U}\bar{P}_r(\aleph_0)$.

To show that $\sigma_r \subsetneq B_r(\aleph_0)$, choose a simple ring R with identity such that $R \in P_r(\aleph_0)$ (such a ring exists by Lemma 3 (i)). Clearly R is right superprime, yet $R \in B_r(\aleph_0)$. Thus $\sigma_r \subsetneq B_r(\aleph_0)$.

We now show that $B_r(\aleph_0) \subsetneq US$. Choose rings R and Q as in Example III.4.1. Since $R \in P_r(1)$ and R_R is essential in Q_R , it follows from Proposition III.2.1 that $Q \in P_r(1)$. It was shown in Example III.4.1 that Q is a (von Neumann) regular ring with identity which is not uniformly strongly prime. Since a right strongly prime regular ring with identity is necessarily simple (see the remarks preceding Proposition III.4.5), it follows that Q is simple. Clearly then, $Q \in US \setminus B_r(\aleph_0)$, so $B_r(\aleph_0) \subsetneq US$.

(ii) Since every simple ring with identity is right superprime, we must have $\sigma_r \subseteq G$. This inequality must be strict since, by Theorem 1.2 (iii), G and N_g are incomparable and $\sigma_r \subseteq G, N_g$.

(iii) was proved in [PSW84, Theorem 2, p229].

(iv) It clearly suffices to show that $N_g \not\subseteq J \cap \beta_\phi$. Let R be a commutative chain domain with value group $\langle \mathbb{Z}; +, -, 0; \leq \rangle$. For example, we could take $R = F[[x]]$, the power series ring in x over any field F . Let P be the unique maximal proper ideal of R . Since P is a domain, we must have $P \notin N_g$. Let I be a nonzero ideal of P and let $\langle I \rangle$ denote the ideal of R generated by I . Then $\langle I \rangle = P^n$ for some positive integer n . By Andrunakievič's Lemma (Lemma 0.2.1), $\langle I \rangle^3 \subseteq I$ so $(P/I)^{3n} = (P^n/I)^3 = \langle I \rangle^3/I = 0$, so P/I is nilpotent. Thus every nonzero homomorphic image of P has zero heart or nilpotent heart. Consequently, $P \in \beta_\phi$. Since P is the unique maximal proper ideal of R , certainly $P \in J$. Thus $P \in J \cap \beta_\phi$.

(v) By Lemma 3 (i), there exists a simple ring R , with identity, such that $R \in P_r(\aleph_0)$. Clearly $R \notin G$, yet $R \in B_r(\aleph_0)$. Thus $G \not\subseteq B_r(\aleph_0)$.

(vi) and (vii) are consequences of [PSW84, Theorem 2, p229]. □

Figure 2 is the diagram obtained by introducing $\mathfrak{U}\bar{P}_r(\aleph_0), B_r(\aleph_0), US$ and σ_r into our previous diagram of classical radicals (Figure 1). The strict inclusions indicated are justified by assertions (i) – (iii) of Proposition 5, together with Figure 1. The implied incomparabilities are justified by assertions (iv) – (vii) of Proposition 5, together with Figure 1.

Our next task is to locate the radicals $\mathfrak{U}\bar{P}_r(\mathfrak{m})$ and $B_r(\mathfrak{m})$ for $\mathfrak{m} > \aleph_0$.

PROPOSITION 6. *Let $\mathfrak{m}, \mathfrak{n}$ be nonzero cardinals with $\mathfrak{m} > \mathfrak{n}$. Then:*

- (i) $\beta \subsetneq \mathfrak{U}\bar{P}_r(\mathfrak{m}) \subsetneq \mathfrak{U}\bar{P}_r(\mathfrak{n}) \subsetneq N_g$;
- (ii) if \mathfrak{m} is a limit cardinal then $\mathfrak{U}\bar{P}_r(\mathfrak{m}) \subsetneq B_r(\mathfrak{m}) \subsetneq \mathfrak{U}\bar{P}_r(\mathfrak{n})$.

Proof. (i) The containments $\beta \subseteq \mathfrak{U}\bar{P}_r(\mathfrak{m}) \subseteq \mathfrak{U}\bar{P}_r(\mathfrak{n}) \subseteq N_g$ are obvious, so it remains to demonstrate that they are strict. To show that $\mathfrak{U}\bar{P}_r(\mathfrak{n}) \subsetneq N_g$, it clearly suffices to show that $\mathfrak{U}P_r(1) \subsetneq N_g$. Let R be any ring with identity which is not a domain but which is a member of $P_r(1)$. (For example, take $R = \mathbb{M}_{\mathfrak{k}}(D)$ where $0 < \mathfrak{k} < \aleph_0$ and D is a domain which is not right Ore. By Proposition III.1.4, $R \in P_r(1)$.) The argument used in Example III.4.1 and the proof of Proposition 5 (i) ($B_r(\aleph_0) \subsetneq US$) shows that $Q = Q_{\max}(R_R)$ is a simple ring with identity which is not uniformly strongly prime but which is a member of $P_r(1)$. It follows that $Q \in N_g \setminus \mathfrak{U}P_r(1)$, hence $\mathfrak{U}P_r(1) \subsetneq N_g$, as required.

By Lemma 3, there exists a simple ring R such that $R \in P_r(\mathfrak{m})$. Since $R \in \mathfrak{U}\bar{P}_r(\mathfrak{n})$, we must have $\mathfrak{U}\bar{P}_r(\mathfrak{m}) \subsetneq \mathfrak{U}\bar{P}_r(\mathfrak{n})$. Inasmuch as $\beta = \bigcap_{0 < \mathfrak{k} < \aleph_0} \mathfrak{U}\bar{P}_r(\mathfrak{k})$, the inequality $\beta \subseteq \mathfrak{U}\bar{P}_r(\mathfrak{m})$ is obviously strict.

(ii) Again, the containments $\mathfrak{U}\bar{P}_r(\mathfrak{m}) \subseteq B_r(\mathfrak{m}) \subseteq \mathfrak{U}\bar{P}_r(\mathfrak{n})$ are self evident, while Lemma 3 may be used to show that $\mathfrak{U}\bar{P}_r(\mathfrak{m}) \subsetneq B_r(\mathfrak{m})$. Since \mathfrak{m} is a limit cardinal, we can choose a cardinal \mathfrak{k} such that $\mathfrak{m} > \mathfrak{k} > \mathfrak{n}$. Then $B_r(\mathfrak{m}) \subseteq \mathfrak{U}\bar{P}_r(\mathfrak{k}) \subsetneq \mathfrak{U}\bar{P}_r(\mathfrak{n})$. □

PROPOSITION 7. (i) $T \not\subseteq \mathfrak{U}\bar{P}_r(\mathfrak{n})$ for all finite nonzero cardinals \mathfrak{n} .

- (ii) $\mathfrak{U}P_r(1) \not\subseteq US$.
- (iii) $\mathfrak{U}\bar{P}_r(\aleph_1) \not\subseteq L$.
- (iv) $J_B \not\subseteq \mathfrak{U}\bar{P}_r(\mathfrak{m})$ for all nonzero cardinals \mathfrak{m} .

Proof. (i) Let D be a division ring and choose \mathfrak{k} such that $\mathfrak{n} < \mathfrak{k} < \aleph_0$. Then $\mathbb{M}_{\mathfrak{k}}(D) \in \mathfrak{U}\bar{P}_r(\mathfrak{n})$, but obviously, $\mathbb{M}_{\mathfrak{k}}(D) \notin T$. Thus $T \not\subseteq \mathfrak{U}\bar{P}_r(\mathfrak{n})$.

(ii) In the proof of Proposition 5 (i) ($B_r(\aleph_0) \subsetneq US$), we noted that the ring Q of Example III.4.1 is a member both of $P_r(1)$ and of US . It follows that $\mathfrak{U}P_r(1) \not\subseteq US$.

A stronger version of (iii) will be proved in Proposition 9 (vi).

(iv) If $\mathfrak{m} < \aleph_0$ then the result follows from (i). Suppose $\mathfrak{m} \geq \aleph_0$. By Lemma 3, there exists a simple ring R which contains nonzero idempotent elements such that $R \in P_r(\mathfrak{m}^+)$. Certainly then,

R must be idempotent, so $R \notin J_B$. Thus $J_B \not\subseteq \mathfrak{U}\bar{P}_r(\mathfrak{m})$. □

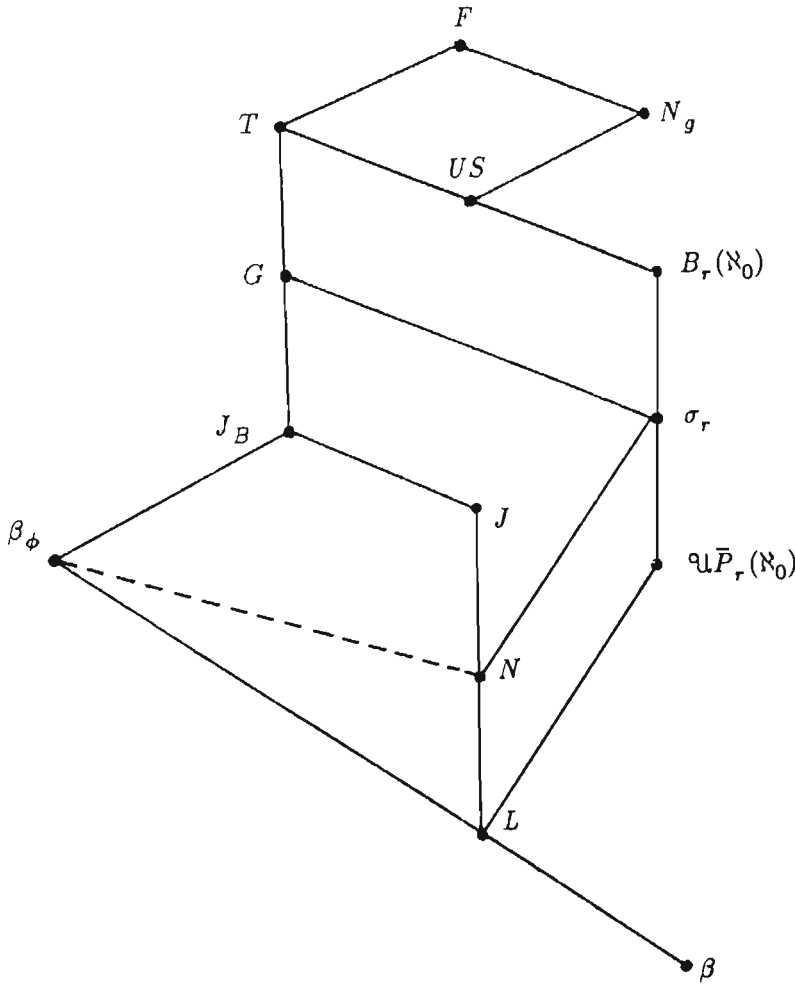


Figure 2

Propositions 6 and 7 allow us to locate the radicals $\mathfrak{U}\bar{P}_r(\mathfrak{m})$ and $B_r(\mathfrak{m})$ in the lattice of radicals. Before constructing a diagram, we locate the radicals $\mathfrak{U}P_r(\mathfrak{m})$.

LEMMA 8. *Let R be a ring, with identity, which satisfies the ACC on ideals. For each finite ordinal α , let $\Theta_\alpha: \mathbb{M}_{2^\alpha}(R) \rightarrow \mathbb{M}_{N_0}(R)$ denote the diagonal embedding defined by*

$$\Theta_\alpha(A) = \begin{bmatrix} A & 0 \\ 0 & A \\ & & \ddots \end{bmatrix} \in \mathbb{M}_{N_0}(R) \quad (A \in \mathbb{M}_{2^\alpha}(R)).$$

(Also see Example III.1.1.) Define $S = \bigcup_{\alpha \in N_0} \Theta_\alpha[\mathbb{M}_{2^\alpha}(R)]$. Then every ideal of S is of the form $\bigcup_{\alpha \in N_0} \Theta_\alpha[\mathbb{M}_{2^\alpha}(K)]$ for some ideal K of R .

Proof. Let I be an ideal of S . For each $\alpha \in N_0$, set $S_\alpha = \Theta_\alpha[\mathbb{M}_{2^\alpha}(R)]$ and $I_\alpha = S_\alpha \cap I$. Note that I_α is an ideal of S_α for each $\alpha \in N_0$ and that $I = S \cap I = (\bigcup_{\alpha \in N_0} S_\alpha) \cap I = \bigcup_{\alpha \in N_0} (S_\alpha \cap I) = \bigcup_{\alpha \in N_0} I_\alpha$. It follows from Proposition 0.6.1 that each I_α is equal to

$\Theta_\alpha[\mathbb{M}_{2^\alpha}(K_\alpha)]$ for some ideal K_α of R . Since $I_\alpha \subseteq I_\beta$ whenever $\alpha \leq \beta < \aleph_0$, we must have $K_\alpha \subseteq K_\beta$ whenever $\alpha \leq \beta < \aleph_0$. Since R satisfies the ACC on ideals, it follows that $\{K_\alpha : \alpha \in \aleph_0\}$ has a maximal member, K , say. Then $I = \bigcup_{\alpha \in \aleph_0} \Theta_\alpha[\mathbb{M}_{2^\alpha}(K)]$, as required. \square

EXAMPLE 1. Let \mathfrak{m} be a nonzero countable cardinal. We construct a ring S such that $S \in P_r(\mathfrak{m})$ and $(J \cap \beta_\phi)(S) \neq 0$.

Let R be a commutative chain domain (with identity) with value group $\langle \mathbb{Z}; +, -, 0; \leq \rangle$ and with unique maximal proper ideal P . As shown in the proof of Proposition 5(iv), $P \in J \cap \beta_\phi$. If $\mathfrak{m} < \aleph_0$, we take $S = \mathbb{M}_\mathfrak{m}(R)$. Certainly $S \in P_r(\mathfrak{m})$ and by Corollaries 1.7 and 1.8, $J(S) = \beta_\phi(S) = \mathbb{M}_\mathfrak{m}(P) \neq 0$.

Now suppose that $\mathfrak{m} = \aleph_0$. Take R as above and S as in the previous lemma. Since R is a commutative domain, the argument used in Example III.1.1 shows that $S \in P_r(\aleph_0)$. It remains to show that $(J \cap \beta_\phi)(S) \neq 0$. It follows from the previous lemma that every prime ideal of S is of the form $\bigcup_{\alpha \in \aleph_0} \Theta_\alpha[\mathbb{M}_{2^\alpha}(Q)]$, where Q is some prime ideal of R . Inasmuch as R is a principal ideal domain with identity, it is not difficult to see that P is the only nonzero prime ideal of R . Consequently, $I = \bigcup_{\alpha \in \aleph_0} \Theta_\alpha[\mathbb{M}_{2^\alpha}(P)]$ is the only nonzero prime ideal of S . Since R is not subdirectly irreducible, S cannot be subdirectly irreducible. We may conclude, therefore, that $\beta_\phi(S) = I$. Moreover, it is not difficult to see that $S \setminus I$ consists entirely of units of S , so $J(S) = I$. Thus $(J \cap \beta_\phi)(S) = I \neq 0$. \square

PROPOSITION 9. (i) $\mathfrak{U}\bar{P}_r(\mathfrak{m}) \subsetneq \mathfrak{U}P_r(\mathfrak{m})$ for all cardinals $\mathfrak{m} > 1$.

(ii) $US \subsetneq \mathfrak{U}P_r(\mathfrak{m})$ for all finite cardinals $\mathfrak{m} > 1$.

(iii) $F \not\subseteq \mathfrak{U}P_r(\mathfrak{m})$ for all cardinals $\mathfrak{m} > 1$.

(iv) $\mathfrak{U}P_r(\mathfrak{m}) \not\subseteq J \cap \beta_\phi$ for all nonzero cardinals \mathfrak{m} .

(v) $\mathfrak{U}P_r(\aleph_0) \not\subseteq N$.

(vi) $\mathfrak{U}P_r(\mathfrak{m}) \not\subseteq L$ if \mathfrak{m} is an uncountable cardinal.

(vii) $\mathfrak{U}P_r(\mathfrak{m}) \not\subseteq \mathfrak{U}P_r(\mathfrak{n})$ if $\mathfrak{m}, \mathfrak{n}$ are distinct nonzero cardinals.

(viii) $\mathfrak{U}P_r(\mathfrak{m}) \not\subseteq \mathfrak{U}\bar{P}_r(\mathfrak{n})$ if $\mathfrak{m}, \mathfrak{n}$ are nonzero cardinals such that $\mathfrak{m} > \mathfrak{n}$.

(ix) $\mathfrak{U}P_r(\mathfrak{m}) \not\subseteq B_r(\mathfrak{m})$ if \mathfrak{m} is a limit cardinal.

Proof. (i), (vii), (viii) and (ix) are immediate consequences of Lemma 3.

(ii) If $R \in P_r(\mathfrak{m})$, where $1 < \mathfrak{m} < \aleph_0$, then by Theorem III.1.2, R is isomorphic to a right order in $\mathbb{M}_\mathfrak{m}(D)$ for some division ring D , whence R is uniformly strongly prime (Theorem III.4.3). Thus $US \subseteq \mathfrak{U}P_r(\mathfrak{m})$ whenever $1 < \mathfrak{m} < \aleph_0$. If D is any division ring then obviously $D \in \mathfrak{U}P_r(\mathfrak{m}) \setminus US$ for all $\mathfrak{m} > 1$. Thus $US \subsetneq \mathfrak{U}P_r(\mathfrak{m})$ whenever $1 < \mathfrak{m} < \aleph_0$.

(iii) If D is any field then obviously $D \notin F$, yet $D \in \mathfrak{U}P_r(\mathfrak{m})$ for all $\mathfrak{m} > 1$. Thus $F \not\subseteq \mathfrak{U}P_r(\mathfrak{m})$ for all $\mathfrak{m} > 1$.

(vi) Choose a ring $R \in P_r(\mathfrak{m})$, $\mathfrak{m} \geq \aleph_1$. For each $\alpha \in \aleph_0$, let $\Theta_\alpha: \mathbb{M}_{2^\alpha}(R) \rightarrow \mathbb{M}_{\aleph_0}(R)$ denote the diagonal embedding described in Example III.1.1. (Also see Lemma 8.) Consider the subring $C = \bigcup_{\alpha \in \aleph_0} \Theta_\alpha[\mathbb{M}_{2^\alpha}(R)]$ of $\mathbb{M}_{\aleph_0}(R)$. As in Example III.1.1, we choose to identify each $\mathbb{M}_{2^\alpha}(R)$ with its image in C . For each $\alpha \in \aleph_0$, let $\Delta_{2^\alpha}(R)$ denote the subring of $\mathbb{M}_{2^\alpha}(R)$ consisting of all upper triangular matrices with zero main diagonal entries. We claim that the ring $S = \bigcup_{\alpha \in \aleph_0} \Theta_\alpha[\Delta_{2^\alpha}(R)]$ is locally nilpotent and a member of $P_r(\mathfrak{m})$. The former fact is obvious since S is the union of an ascending chain of nilpotent subrings. It follows that $S \in L$. To prove the latter claim, let $0 \neq s \in S$, say $s \in \Delta_{2^\alpha}(R)$, $\alpha \in \aleph_0$. Since $\mathbb{M}_{2^\alpha}(R) \in P_r(\mathfrak{m})$ (Proposition III.1.4), we can choose a right insulator X for s in $\mathbb{M}_{2^\alpha}(R)$ with $|X| < \mathfrak{m}$. For each finite ordinal $\beta \geq \alpha$, let X_β denote the image of X in $\mathbb{M}_{2^\beta}(R)$ (via the obvious diagonal embedding), set

$$\bar{X}_\beta = \left\{ \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} : A \in X_\beta \right\} \subseteq \Delta_{2^{\beta+1}}(R)$$

and put $\bar{X} = \bigcup_{\alpha \leq \beta < \aleph_0} \bar{X}_\beta \subseteq S$. We claim that \bar{X} is a right insulator for s in S . For suppose that $s\bar{X}t = 0$ for some $t \in S$. By choosing β sufficiently large, we may regard s and t as elements of $\Delta_{2^\beta}(R)$ with $\beta \geq \alpha$. Since $s\bar{X}t = 0$, we must have $s\bar{X}_\beta t = 0$, from which it follows that $sX_\beta t = 0$, so $t = 0$. Since $|\bar{X}| < \mathfrak{m}$, this shows that $S \in \bar{P}_r(\mathfrak{m})$. Inasmuch as $R \notin P_r(\mathfrak{k})$ for any $\mathfrak{k} < \mathfrak{m}$, it is not difficult to see that $S \in P_r(\mathfrak{m})$. Thus $\mathfrak{U}P_r(\mathfrak{m}) \not\subseteq L$ whenever $\mathfrak{m} \geq \aleph_1$.

(iv) If $\mathfrak{m} > \aleph_0$, the result follows from (vi) above. Suppose $\mathfrak{m} \leq \aleph_0$ and consider the ring S of Example 1. Since $(J \cap \beta_\phi)(S) \not\supseteq \mathfrak{U}P_r(\mathfrak{m})(S) = 0$ we must have that $\mathfrak{U}P_r(\mathfrak{m}) \not\subseteq J \cap \beta_\phi$ for all $\mathfrak{m} > 0$.

(v) Let R be a right strongly prime ring with a nonzero nil ideal I (see [GHL74, Ex 2.5, p12]). We noted in the sequel to Proposition 4 that R cannot be right superprime and is therefore not right bounded strongly prime (by Proposition 4). It follows that $R \in P_r(\aleph_0)$, whence $I \in P_r(\aleph_0)$. Since I is nil, we must have $I \in N$. Thus $\mathfrak{U}P_r(\aleph_0) \not\subseteq N$. \square

We are finally in a position to present a diagram depicting the positions of the radicals $\mathfrak{U}\bar{P}_r(\mathfrak{m})$, $\mathfrak{U}P_r(\mathfrak{m})$ and $B_r(\mathfrak{m})$. This is done in Figure 3. We point out again that, modulo the Koethe Conjecture, all indicated inclusions are strict and that the absence of indicated comparability implies known incomparability.

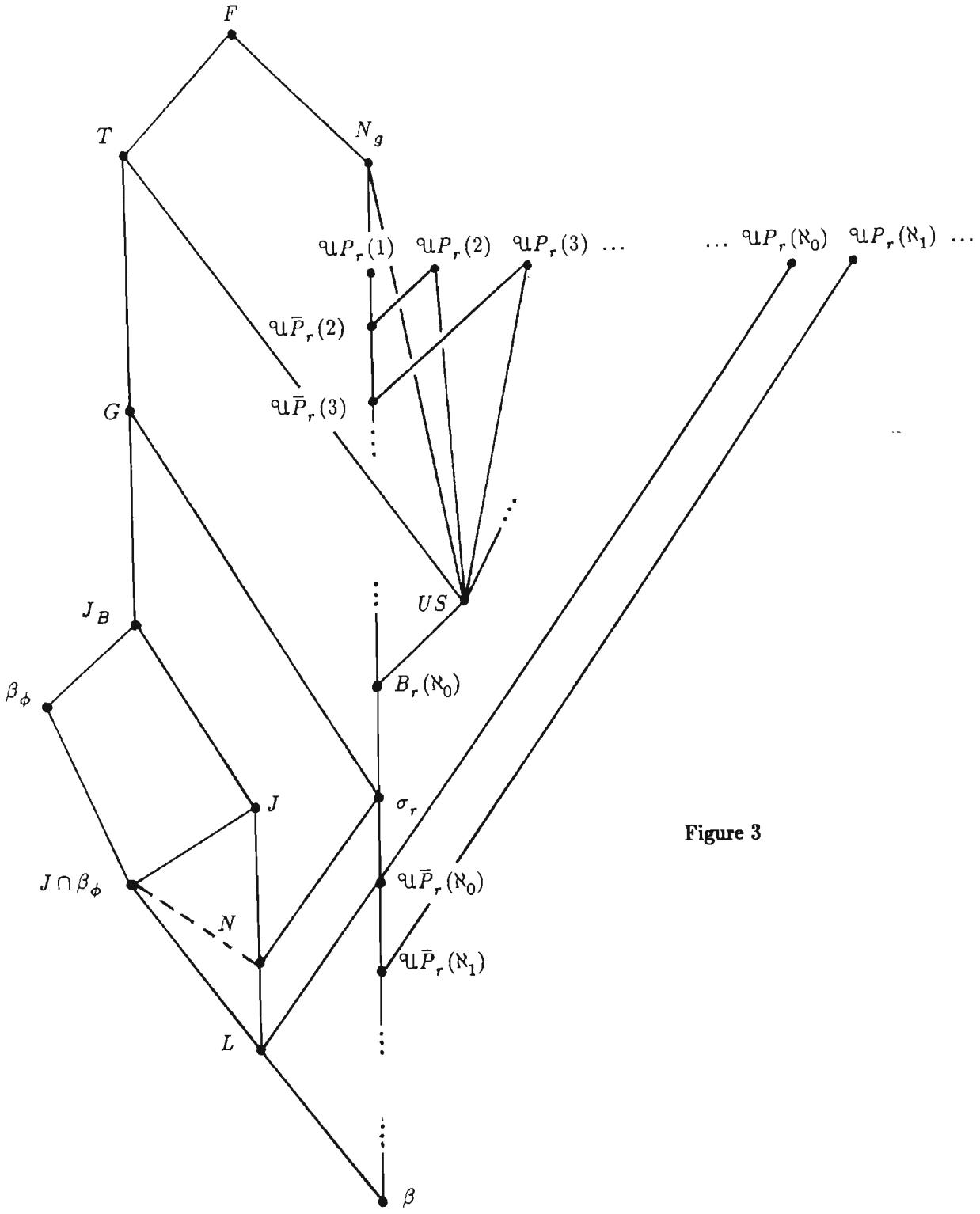


Figure 3

Lastly, we consider some questions about left-right symmetry. We start by refining Lemma 3.

LEMMA 10. (i) *If \mathfrak{m} is an arbitrary nonzero cardinal then there exists a simple ring R such that $R \in P_r(\mathfrak{m}) \cap P_l(\mathfrak{m})$.*

(ii) *If \mathfrak{m} is an uncountable cardinal then there exists a simple ring R such that*

$$R \in P_r(\mathfrak{m}) \cap P_l(1).$$

Proof. (i) The rings R described in the proof of Lemma 3(i) are clearly left-right symmetric. It remains to establish (i) in the case where $\mathfrak{m} > \aleph_0$. Let D be a division ring and let I denote the ideal of $\mathbb{M}_{\mathfrak{m}}^*(D)$ consisting of all matrices which contain only finitely many nonzero columns (i.e., $I = I_{\aleph_0, \mathfrak{m}}(D) \cap \mathbb{M}_{\mathfrak{m}}^*(D)$). If I is viewed, alternatively, as the ideal of $\mathbb{M}_{\mathfrak{m}}^*(D)$ consisting precisely of all matrices with only finitely many nonzero entries then it is not difficult to see that I is idempotent and is the unique minimal nonzero ideal of $\mathbb{M}_{\mathfrak{m}}^*(D)$. By the argument used in the proof of Lemma 3(ii), I is simple and $I \in P_r(\mathfrak{m}^+) \cap P_l(\mathfrak{m}^+)$. Assertion (i) is therefore true whenever \mathfrak{m} is a successor cardinal. An adaptation of the remaining part of the proof of Lemma 3(ii) (roughly speaking, we substitute row- and column-finite matrices for row-finite matrices) can be used to establish (i) in the case where \mathfrak{m} is a limit cardinal.

(ii) Let D be a simple domain, with identity, which contains an independent family of \mathfrak{m} nonzero left ideals (for example, construct D as in Example 2.2). By Propositions III.1.4 and III.1.12, $\mathbb{M}_{\mathfrak{m}}(D) \in P_r(\mathfrak{m}^+) \cap P_l(1)$. We claim that $I = I_{\aleph_0, \mathfrak{m}}(D)$ is idempotent and is the unique minimal nonzero ideal of $\mathbb{M}_{\mathfrak{m}}(D)$. The former claim is obvious. To verify the latter claim, suppose K is a nonzero ideal of $\mathbb{M}_{\mathfrak{m}}(D)$. Let $0 \neq A \in K$ and suppose $x = A_{\alpha\beta} \neq 0$, with $\alpha, \beta \in \mathfrak{m}$. For each $\gamma, \delta \in \mathfrak{m}$, let $E_{\gamma, \delta}(r)$ denote the matrix in $\mathbb{M}_{\mathfrak{m}}(D)$ with $r \in D$ in position (γ, δ) and zeros elsewhere. Since D is simple (with identity), we must have $\sum_{i=1}^n a_i x b_i = 1_D$ for suitable $a_i, b_i \in D$, for $i = 1, \dots, n$. We lose no generality in supposing that $n = 1$ and $a_1 x b_1 = 1_D$. Notice that for each $\gamma, \delta \in \mathfrak{m}$, we have $E_{\gamma, \alpha}(a_1) A E_{\beta, \delta}(b_1) = E_{\gamma, \delta}(1_D)$. Consequently, $\{E_{\gamma, \delta}(1_D) : \gamma, \delta \in \mathfrak{m}\} \subseteq K$. Now let $0 \neq B \in I$ and set $\Gamma = \{\gamma \in \mathfrak{m} : B^{(\gamma)} \neq 0\}$. Certainly $|\Gamma| < \aleph_0$, since $B \in I$. Observe that $B = \sum_{\gamma \in \Gamma} B E_{\gamma, \gamma}(1_D) \in K$, so $I \subseteq K$. This establishes our claim. Since $\mathbb{M}_{\mathfrak{m}}(D) \in P_r(\mathfrak{m}^+) \cap P_l(1)$ and I is an ideal of $\mathbb{M}_{\mathfrak{m}}(D)$, we must have $I \in P_r(\mathfrak{m}^+) \cap P_l(1)$. A routine application of Andrunakievič's Lemma (Lemma 0.2.1) (as in the proof of Lemma 3(ii)) shows that I is, moreover, simple.

Again, by adapting the remaining part of the proof of Lemma 3(ii), we can show that (ii) above also holds in the case where \mathfrak{m} is an uncountable limit cardinal. \square

PROPOSITION 11. *Let $\mathfrak{m}, \mathfrak{n}$ be arbitrary (not necessarily distinct) nonzero cardinals.*

Then:

- (i) $\mathfrak{U}\bar{P}_r(\mathfrak{m})$ and $\mathfrak{U}\bar{P}_l(\mathfrak{n})$ are incomparable;
- (ii) $\mathfrak{U}P_r(\mathfrak{m})$ and $\mathfrak{U}P_l(\mathfrak{n})$ are incomparable.

Proof. It suffices, by symmetry, to show that $\mathfrak{U}\bar{P}_r(\mathfrak{m}) \not\subseteq \mathfrak{U}\bar{P}_l(\mathfrak{n})$ and $\mathfrak{U}P_r(\mathfrak{m}) \not\subseteq \mathfrak{U}P_l(\mathfrak{n})$ for all $\mathfrak{m}, \mathfrak{n} > 0$.

(i) Choose a cardinal \mathfrak{k} such that $\mathfrak{k} > \max\{\mathfrak{m}, \aleph_0\}$. By Lemma 10(ii), there exists a simple ring R such that $R \in P_r(\mathfrak{k}) \cap P_l(1)$. Then $R \in \mathfrak{U}\bar{P}_r(\mathfrak{m})$ and $R \notin \mathfrak{U}\bar{P}_l(\mathfrak{n})$ for all $\mathfrak{n} > 0$. Thus $\mathfrak{U}\bar{P}_r(\mathfrak{m}) \not\subseteq \mathfrak{U}\bar{P}_l(\mathfrak{n})$ for all $\mathfrak{m}, \mathfrak{n} > 0$.

(ii) It follows from Lemma 10(i) that $\mathfrak{U}P_r(\mathfrak{m}) \not\subseteq \mathfrak{U}P_l(\mathfrak{n})$ whenever $\mathfrak{m} \neq \mathfrak{n}$, while the left analogue of Lemma 10(ii) yields $\mathfrak{U}P_r(\mathfrak{m}) \not\subseteq \mathfrak{U}P_l(\mathfrak{m})$ whenever $\mathfrak{m} > \aleph_0$. It remains to show that $\mathfrak{U}P_r(\mathfrak{m}) \not\subseteq \mathfrak{U}P_l(\mathfrak{m})$ whenever $1 < \mathfrak{m} \leq \aleph_0$. Define D to be the right skew power series ring $F[[x, \sigma]]$, where F is a field and $\sigma: F \rightarrow F$ a monomorphism which is not onto. Then D is a left Ore domain which is not right Ore (Proposition 0.10.5(ii)) and every nonzero ideal of D is of the form Dx^k for some nonnegative integer k (Proposition 0.10.5(i)). By Proposition III.1.4 (and its left analogue), $\mathbb{M}_{\mathfrak{m}}(D) \in P_r(1) \cap P_l(\mathfrak{m})$ whenever $1 < \mathfrak{m} < \aleph_0$. Since Dx is the only nonzero prime ideal of D and since every ideal of $\mathbb{M}_{\mathfrak{m}}(D)$ is of the form $\mathbb{M}_{\mathfrak{m}}(K)$ for some ideal K of D (Proposition 0.6.1), it is not difficult to see that $I = \mathbb{M}_{\mathfrak{m}}(Dx)$ is the only nonzero prime ideal of $\mathbb{M}_{\mathfrak{m}}(D)$. Consequently, $\mathfrak{U}P_r(\mathfrak{m})(\mathbb{M}_{\mathfrak{m}}(D)) \supseteq I \not\supseteq \mathfrak{U}P_l(\mathfrak{m})(\mathbb{M}_{\mathfrak{m}}(D)) = 0$. It follows that $\mathfrak{U}P_r(\mathfrak{m}) \not\subseteq \mathfrak{U}P_l(\mathfrak{m})$ whenever $1 < \mathfrak{m} < \aleph_0$.

Now suppose $\mathfrak{m} = \aleph_0$. Choose D as above. For each $\alpha \in \aleph_0$, let $\Theta_\alpha: \mathbb{M}_{2^\alpha}(D) \rightarrow \mathbb{M}_{\aleph_0}(D)$ denote the usual diagonal embedding. Define $R = \bigcup_{\alpha \in \aleph_0} \Theta_\alpha[\mathbb{M}_{2^\alpha}(D)]$. We may infer from the left analogue of Example III.1.5 that $R \in P_r(1) \cap P_l(\aleph_0)$. Since D possesses an identity element and satisfies the ACC on ideals, it follows from Lemma 8 that every prime ideal of R is of the form $\bigcup_{\alpha \in \aleph_0} \Theta_\alpha[\mathbb{M}_{2^\alpha}(Q)]$ for some prime ideal Q of D . Inasmuch as Dx is the only nonzero prime ideal of D , we may conclude that $I = \bigcup_{\alpha \in \aleph_0} \Theta_\alpha[\mathbb{M}_{2^\alpha}(Dx)]$ is the only nonzero prime ideal of R , so $\mathfrak{U}P_r(\aleph_0)(R) \supseteq I \not\supseteq \mathfrak{U}P_l(\aleph_0)(R) = 0$. Thus $\mathfrak{U}P_r(\aleph_0) \not\subseteq \mathfrak{U}P_l(\aleph_0)$. \square

The precise relationship between the radicals $\mathfrak{U}P_r(\aleph_0)$ and $\mathfrak{U}\bar{P}_l(\aleph_0)$ is unclear. We alluded earlier to an open question from [GHL74, p121] which asks whether $P_r(\aleph_0) \subseteq \bar{P}_l(\aleph_0)$. If this is the case then of course $\mathfrak{U}P_r(\aleph_0) \supseteq \mathfrak{U}\bar{P}_l(\aleph_0)$.

§5. RADICALS ASSOCIATED WITH UNIFORM BOUNDS OF PRIMENESS.

In §4, attention was given to Olson's uniformly strongly prime radical US , this being the upper radical determined by the class of all uniformly strongly prime rings. In [OV88, Theorem 17, p450], Olson and Veldsman proved that the class of rings which are uniformly strongly prime of bound 1 is special and posed the question: is the class of rings which are uniformly strongly prime of bound at most \mathfrak{m} special for all finite nonzero cardinals \mathfrak{m} ? (See remarks following [OV88, Theorem 19, p451].) More generally, is the class $\overline{UP}(\mathfrak{m})$ special for all nonzero \mathfrak{m} ? (We remind the reader that if \mathfrak{m} is a nonzero cardinal then $UP(\mathfrak{m})$ (resp. $\overline{UP}(\mathfrak{m})$) denotes the class of all rings which are uniformly prime of bound \mathfrak{m} (resp. of bound at most \mathfrak{m}), and that we usually replace all occurrences of the word "prime" by "strongly prime" when \mathfrak{m} is finite.)

Our main theorem of the present section answers this question in the affirmative, thereby identifying denumerably many special classes within the class of all uniformly strongly prime rings. We shall locate the corresponding special radicals in the lattice of radicals, finding that for every finite nonzero cardinal \mathfrak{k} , the upper radical determined by $\overline{UP}(\mathfrak{k})$ lies strictly between a pair of radicals of

the form $\mathcal{U}\bar{P}_r(\mathbf{m})$ and $\mathcal{U}P_r(\mathbf{n})$ for suitable finite nonzero cardinals \mathbf{m} and \mathbf{n} . This relationship breaks down for infinite \mathbf{k} and it is not yet clear that the radicals associated with $\overline{UP}(\mathbf{k})$, $\mathbf{k} > \aleph_0$, are of much interest. We leave any further study of properties of these radicals to future researchers.

THEOREM 1. $\overline{UP}(\mathbf{m})$ is a special class for all nonzero cardinals \mathbf{m} .

Proof. Suppose $R \in \overline{UP}(\mathbf{m})$ and let I be a nonzero ideal of R . Let X be a uniform insulator for R with $|X| < \mathbf{m} + 1$. If $\mathbf{m} \leq \aleph_0$ then by Proposition 4.4, there exists $a \in I$ such that $(0 :_r a) = 0$ (in R). We claim that $Xa \subseteq I$ is a uniform insulator for I . Suppose $s(Xa)t = 0$ for some $s, t \in I$ with $s \neq 0$. Since X is a uniform insulator for R , we must have $at = 0$. Therefore, $t \in (0 :_r a) = 0$, validating our claim. Since $|Xa| \leq |X| \leq \mathbf{m}$, we may conclude that $I \in \overline{UP}(\mathbf{m})$. If $\mathbf{m} > \aleph_0$ then I is right \mathbf{m} -faithful by Theorem II.6.3 (iii), so we may choose a subset Y of I with $|Y| < \mathbf{m}$ such that $(0 :_r Y) = 0$ (in R). Consider $XY := \{xy : x \in X, y \in Y\}$. Notice that $XY \subseteq I$ and $|XY| \leq \max\{\aleph_0, |X|, |Y|\} < \mathbf{m}$. If $sXYt = 0$ for some $s, t \in I$ with $s \neq 0$, then $sXyt = 0$ for all $y \in Y$ so, since X is a uniform insulator for R , we must have $yt = 0$ for all $y \in Y$, i.e., $t \in (0 :_r Y) = 0$. Consequently, $I \in \overline{UP}(\mathbf{m})$. This establishes property **S1** in the definition of a special class.

To show that $\overline{UP}(\mathbf{m})$ satisfies property **S2**, suppose I is a large ideal of a ring R and $I \in \overline{UP}(\mathbf{m})$. Since the class of all prime rings is special (Theorem 1.3), we must have that R is prime. By Lemma 2.3, I_I and ${}_I I$ are essential submodules of R_I and ${}_I R$, respectively. It follows from this and Proposition III.4.6 that $R \in \overline{UP}(\mathbf{m})$, as required. \square

We shall adopt Olson and Veldsman's notation [OV88] and denote by $\tau_{\mathbf{m}}$ the upper radical determined by the special class $\overline{UP}(\mathbf{m})$ for each nonzero cardinal \mathbf{m} . It follows from the definitions that $US = \tau_{\aleph_0}$ and that $\tau_{\mathbf{m}} = \bigcap_{0 < \mathbf{k} < \mathbf{m}} \tau_{\mathbf{k}}$ for all limit cardinals \mathbf{m} . In the next result we compare the $\tau_{\mathbf{n}}$'s (for finite \mathbf{n}) with some of the radicals studied earlier in this chapter.

- PROPOSITION 2.** (i) $N_g \supseteq \tau_1 \supseteq \tau_2 \supseteq \dots \supseteq US$.
(ii) $\tau_{\mathbf{n}+1} \not\supseteq \mathcal{U}\bar{P}_r(\mathbf{n})$ for all finite nonzero cardinals \mathbf{n} .
(iii) $\mathcal{U}P_r(\mathbf{n}) \supseteq \tau_{2\mathbf{n}-1}$ for all finite cardinals $\mathbf{n} > 1$.
(iv) $\mathcal{U}P_r(\mathbf{n}) \not\supseteq \tau_{2\mathbf{n}-2}$ for all finite cardinals $\mathbf{n} > 1$.
(v) $\tau_{\mathbf{n}} \supseteq \mathcal{U}\bar{P}_r(\mathbf{n})$ for all finite nonzero cardinals \mathbf{n} .

Proof. (i) The only nontrivial step is to show that the containments are strict. To show that $N_g \supseteq \tau_1$, let $1 < \mathbf{n} < \aleph_0$ and let D be a simple domain, with identity, which is neither right nor left Ore (see Example 2.2). By Proposition 0.6.1, $\mathbf{M}_{\mathbf{n}}(D)$ must also be simple. Since $\mathbf{M}_{\mathbf{n}}(D)$ is clearly uniformly strongly prime but neither right nor left Goldie, it follows from Theorem III.4.4 that $\mathbf{M}_{\mathbf{n}}(D) \in UP(1)$. Clearly, $\mathbf{M}_{\mathbf{n}}(D) \in N_g \setminus \tau_1$, so $N_g \supseteq \tau_1$.

It is clear from Theorem III.4.14 that for each finite $\mathbf{n} > 1$, there exists a simple (matrix) ring

R belonging to $UP(\mathbf{n})$. Certainly then, $R \in \tau_{\mathbf{n}-1} \setminus \tau_{\mathbf{n}}$. Thus $\tau_{\mathbf{n}-1} \not\supseteq \tau_{\mathbf{n}}$. Inasmuch as $US = \tau_{\aleph_0} = \bigcap_{0 < \mathbf{k} < \aleph_0} \tau_{\mathbf{k}}$ we must have $\tau_{\mathbf{n}} \not\supseteq US$ whenever $0 < \mathbf{n} < \aleph_0$.

(ii) It follows from Theorem III.4.14 that $\mathbb{M}_{\mathbf{n}+1}(\mathbb{Q}) \in P_r(\mathbf{n}+1) \cap UP(\mathbf{n}+1)$. Certainly then, $\mathbb{M}_{\mathbf{n}+1}(\mathbb{Q}) \notin \tau_{\mathbf{n}+1}$, but $\mathbb{M}_{\mathbf{n}+1}(\mathbb{Q})$ is simple, so we must have $\mathbb{M}_{\mathbf{n}+1}(\mathbb{Q}) \in \mathcal{U}\bar{P}_r(\mathbf{n})$. Thus $\tau_{\mathbf{n}+1} \not\supseteq \mathcal{U}\bar{P}_r(\mathbf{n})$ for all finite $\mathbf{n} > 0$.

(iii) The containment $\mathcal{U}P_r(\mathbf{n}) \supseteq \tau_{2\mathbf{n}-1}$ holds because of Theorems III.1.2 and III.4.3. This containment must be strict since $\mathcal{U}P_r(\mathbf{n}) = \tau_{2\mathbf{n}-1}$ would imply $\tau_{\mathbf{n}+1} \supseteq \tau_{2\mathbf{n}-1} = \mathcal{U}P_r(\mathbf{n}) \not\supseteq \mathcal{U}\bar{P}_r(\mathbf{n})$, contradicting (ii) above.

(iv) Let F be an algebraically closed field. By Theorem III.4.11, $\mathbb{M}_{\mathbf{n}}(F) \in UP(2\mathbf{n}-1)$. Certainly $\mathbb{M}_{\mathbf{n}}(F) \notin \mathcal{U}P_r(\mathbf{n})$, yet $\mathbb{M}_{\mathbf{n}}(F) \in \tau_{2\mathbf{n}-2}$, because $\mathbb{M}_{\mathbf{n}}(F)$ is simple. Thus $\mathcal{U}P_r(\mathbf{n}) \not\supseteq \tau_{2\mathbf{n}-2}$ for all finite $\mathbf{n} > 1$.

(v) Clearly only the strictness of the inclusion is at issue here. Suppose, contrary to (v), that $\tau_{\mathbf{n}} = \mathcal{U}\bar{P}_r(\mathbf{n})$ for some finite $\mathbf{n} > 0$. If $\mathbf{n} > 1$ then $\mathcal{U}P_r(\mathbf{n}) \not\supseteq \mathcal{U}\bar{P}_r(\mathbf{n}) = \tau_{\mathbf{n}} \supseteq \tau_{2\mathbf{n}-2}$ which contradicts (iv). If $\mathbf{n} = 1$ then $\mathcal{U}P_r(1) = \tau_1 \not\supseteq US$ which contradicts Proposition 4.7(ii). \square

Proposition 2 allows us to incorporate the radicals $\tau_{\mathbf{m}}$ ($\mathbf{m} < \aleph_0$) into the previous Hasse diagram (Figure 3). We avoid presenting an unwieldy diagram and exhibit, in Figure 4, the location of the $\tau_{\mathbf{m}}$'s relative to a portion of the lattice of radicals. It is important to note that no information regarding relative positions is lost. Indeed, a diagram displaying all instances of comparability between radicals can be obtained by superimposing the diagrams in Figures 3 and 4.

Finally, we remark that since the class $\overline{UP}(\mathbf{m})$ is special for *all* nonzero cardinals \mathbf{m} , it follows easily that the class $UP(\mathbf{m})$ is also special for every successor cardinal \mathbf{m} . Initial investigations suggest that the (special) upper radicals determined by these classes have pathological properties somewhat similar to those of the radicals $\mathcal{U}P_r(\mathbf{m})$, and are incomparable with most standard radicals. We have therefore chosen not to investigate them further here.

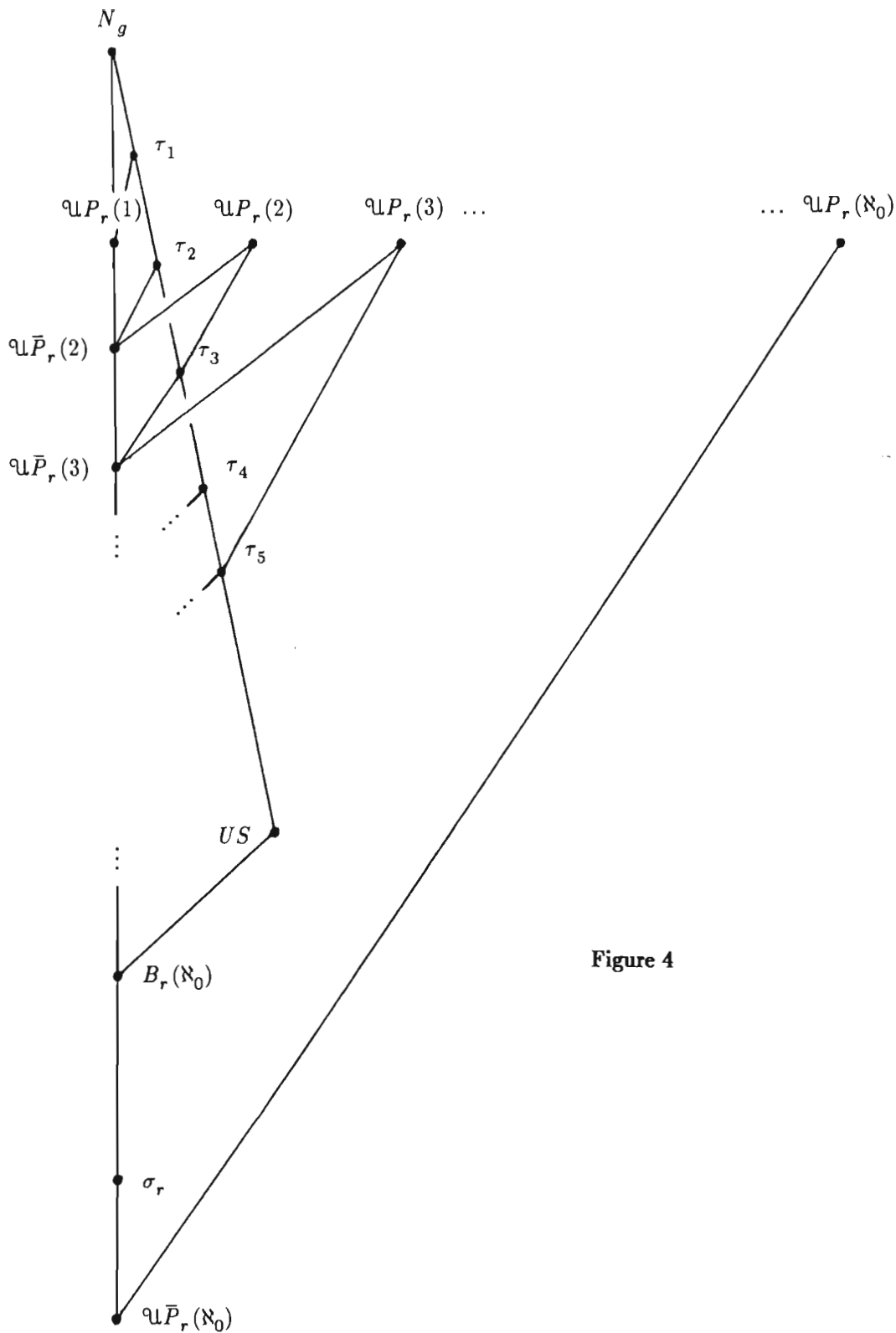


Figure 4

Appendix

In Chapter III, §1, Proposition 11, we make use of the fact that for any division ring D , and any infinite cardinal \mathfrak{m} , the dimension of the left vector space ${}_D D^{\mathfrak{m}}$ ($:= \prod_{\mathfrak{m}} D$) over D is $|D|^{\mathfrak{m}}$. This fact should be well known, but we could find no proof of it in the literature. Since it is essentially a result of set theory and linear algebra, rather than ring theory, we have chosen to offer a proof in an appendix, rather than in the main body of the text. The result depends on the following lemma. We remind the reader that a subset X of a right R -module M is called *free* if, for any family $\{r_x : x \in X\}$ of elements $r_x \in R$, all but finitely many of which are zero, we have

$$\sum_{x \in X} x r_x = 0 \Rightarrow (\forall x \in X)(r_x = 0).$$

If $(0 : x) = 0$ for all $x \in X$, then it follows immediately from the definition that X is a free subset of M if and only if $\{xR : x \in X\}$ is an independent family of submodules of M . We shall make use of this fact in the lemma below.

LEMMA. *Let D be an infinite left Ore domain and \mathfrak{m} an infinite cardinal. Then the left D -module ${}_D D^{\mathfrak{m}}$ has an independent family of $|D|$ nonzero submodules. Consequently, the Goldie dimension of ${}_D D^{\mathfrak{m}}$ is at least $|D|$.*

Proof. Since ${}_D D^{\aleph_0}$ is isomorphic to a submodule of ${}_D D^{\mathfrak{m}}$, it clearly suffices to prove the result when $\mathfrak{m} = \aleph_0$. Let $|D| = \mathfrak{k} \geq \aleph_0$.

We claim that there is a \mathfrak{k} -sequence $\mathfrak{U} = \langle d_\gamma : \gamma \in \mathfrak{k} \rangle$, where $d_\gamma = (d_{\gamma_0}, d_{\gamma_1}, d_{\gamma_2}, \dots) \in {}_D D^{\aleph_0}$ for each $\gamma \in \mathfrak{k}$, such that for every finite nonzero cardinal \mathfrak{n} , every \mathfrak{n} -element subset of $\{(d_{\gamma_0}, d_{\gamma_1}, d_{\gamma_2}, \dots, d_{\gamma_{\mathfrak{n}-1}}) : \gamma \in \mathfrak{k}\}$ is free in ${}_D D^{\mathfrak{m}}$. This would imply that $\text{range } \mathfrak{U}$ is a free subset of ${}_D D^{\aleph_0}$, and hence that ${}_D D^{\aleph_0}$ contains an independent family of \mathfrak{k} nonzero submodules, so establishing the truth of the claim will prove the lemma.

We construct \mathfrak{U} recursively. Indeed, it suffices to show that for every finite nonzero cardinal \mathfrak{n} , there exists a \mathfrak{k} -sequence $\mathfrak{U}_{\mathfrak{n}} = \langle (d_{\gamma_0}, d_{\gamma_1}, d_{\gamma_2}, \dots, d_{\gamma_{\mathfrak{n}-1}}) : \gamma \in \mathfrak{k} \rangle$ of elements of $D^{\mathfrak{n}}$ such that for each nonzero cardinal $\mathfrak{p} \leq \mathfrak{n}$, every \mathfrak{p} -element subset of $\text{range } \mathfrak{U}_{\mathfrak{p}} = \{(d_{\gamma_0}, d_{\gamma_1}, \dots, d_{\gamma_{\mathfrak{p}-1}}) : \gamma \in \mathfrak{k}\}$ is free in ${}_D D^{\mathfrak{p}}$. For $\mathfrak{n} = 1$, we just take $\mathfrak{U}_1 = \langle d_{\gamma_0} : \gamma \in \mathfrak{k} \rangle$ to be any one-to-one \mathfrak{k} -sequence of elements of D . (Such a \mathfrak{k} -sequence certainly exists, since $|D| = \mathfrak{k}$ and D may be well ordered, by the axiom of choice.) Suppose that a \mathfrak{k} -sequence $\mathfrak{U}_{\mathfrak{q}} = \langle (d_{\gamma_0}, d_{\gamma_1}, \dots, d_{\gamma_{\mathfrak{q}-1}}) : \gamma \in \mathfrak{k} \rangle$ of elements of ${}_D D^{\mathfrak{q}}$ has been constructed (where \mathfrak{q} is a finite nonzero cardinal) with the property that for each nonzero cardinal $\mathfrak{p} \leq \mathfrak{q}$, every \mathfrak{p} -element subset of $\text{range } \mathfrak{U}_{\mathfrak{p}} = \{(d_{\gamma_0}, d_{\gamma_1}, \dots, d_{\gamma_{\mathfrak{p}-1}}) : \gamma \in \mathfrak{k}\}$ is free in ${}_D D^{\mathfrak{p}}$. We have to define a $d_{\gamma_{\mathfrak{q}}} \in D$, for each $\gamma \in \mathfrak{k}$, which we do by transfinite recursion on γ . Take $d_{0_{\mathfrak{q}}}$ to be any element of D . Suppose that for some ordinal $\alpha < \mathfrak{k}$, we have already defined $d_{\gamma_{\mathfrak{q}}} \in D$ for all ordinals $\gamma < \alpha$, in such a way that for every nonzero cardinal $\mathfrak{p} \leq \mathfrak{q} + 1$, every

\mathfrak{p} -element subset of $\{(d_{\gamma_0}, d_{\gamma_1}, d_{\gamma_2}, \dots, d_{\gamma_{\mathfrak{p}-1}}) : \gamma \in \alpha\}$ is free in ${}_D D^{\mathfrak{p}}$.

We have to choose $d_{\alpha \mathfrak{q}} \in D$ in such a way that for any strictly ascending chain of ordinals $1 \leq \lambda_0 < \lambda_1 < \dots < \lambda_{\mathfrak{p}-1} < \alpha$, with $\mathfrak{p} \leq \mathfrak{q}$, the set

$$\{(d_{\lambda_i 0}, d_{\lambda_i 1}, \dots, d_{\lambda_i \mathfrak{q}}) : i = 0, 1, \dots, \mathfrak{p} - 1\} \cup \{(d_{\alpha 0}, d_{\alpha 1}, \dots, d_{\alpha \mathfrak{q}})\}$$

is free in ${}_D D^{\mathfrak{q}+1}$. For such a chain, define

$$\mathfrak{B}(\lambda_0, \lambda_1, \dots, \lambda_{\mathfrak{p}-1}) = \left\{ x \in D : \{(d_{\lambda_i 0}, d_{\lambda_i 1}, \dots, d_{\lambda_i \mathfrak{q}}) : i = 0, 1, \dots, \mathfrak{p} - 1\} \cup \{(d_{\alpha 0}, d_{\alpha 1}, \dots, d_{\alpha \mathfrak{q}-1}, x)\} \text{ is not a free subset of } {}_D D^{\mathfrak{q}+1} \right\}.$$

We claim that $|\mathfrak{B}(\lambda_0, \lambda_1, \dots, \lambda_{\mathfrak{p}-1})| \leq 1$. Suppose, on the contrary, that $x, y \in \mathfrak{B}(\lambda_0, \lambda_1, \dots, \lambda_{\mathfrak{p}-1})$ with $x \neq y$. Then there must exist nonzero elements $(r_0, r_1, \dots, r_{\mathfrak{p}}), (s_0, s_1, \dots, s_{\mathfrak{p}}) \in D^{\mathfrak{p}+1}$ such that

$$r_0(d_{\lambda_0 0}, \dots, d_{\lambda_0 \mathfrak{q}}) + \dots + r_{\mathfrak{p}-1}(d_{\lambda_{\mathfrak{p}-1} 0}, \dots, d_{\lambda_{\mathfrak{p}-1} \mathfrak{q}}) + r_{\mathfrak{p}}(d_{\alpha 0}, \dots, d_{\alpha \mathfrak{q}-1}, x) = \underline{0},$$

and

$$s_0(d_{\lambda_0 0}, \dots, d_{\lambda_0 \mathfrak{q}}) + \dots + s_{\mathfrak{p}-1}(d_{\lambda_{\mathfrak{p}-1} 0}, \dots, d_{\lambda_{\mathfrak{p}-1} \mathfrak{q}}) + s_{\mathfrak{p}}(d_{\alpha 0}, \dots, d_{\alpha \mathfrak{q}-1}, y) = \underline{0}.$$

Since $\{(d_{\lambda_i 0}, d_{\lambda_i 1}, \dots, d_{\lambda_i \mathfrak{q}}) : i = 0, 1, \dots, \mathfrak{p} - 1\}$ is a free subset of ${}_D D^{\mathfrak{q}+1}$, we must have $r_{\mathfrak{p}}, s_{\mathfrak{p}} \neq 0$. Since D is a left Ore domain, we may choose $a, b \in D$ such that $ar_{\mathfrak{p}} = bs_{\mathfrak{p}} \neq 0$. Then

$$(ar_0 - bs_0)(d_{\lambda_0 0}, \dots, d_{\lambda_0 \mathfrak{q}}) + \dots + (ar_{\mathfrak{p}-1} - bs_{\mathfrak{p}-1})(d_{\lambda_{\mathfrak{p}-1} 0}, \dots, d_{\lambda_{\mathfrak{p}-1} \mathfrak{q}}) + (0, 0, \dots, 0, ar_{\mathfrak{p}}x - bs_{\mathfrak{p}}y) = \underline{0}.$$

Again, since $\{(d_{\lambda_i 0}, d_{\lambda_i 1}, \dots, d_{\lambda_i \mathfrak{q}}) : i = 0, 1, \dots, \mathfrak{p} - 1\}$ is free, we must have $ar_i - bs_i = 0$ for $i = 1, \dots, \mathfrak{p} - 1$. Hence, $(0, 0, \dots, 0, ar_{\mathfrak{p}}x - bs_{\mathfrak{p}}y) = \underline{0}$, so $ar_{\mathfrak{p}}x - bs_{\mathfrak{p}}y = 0$. Since D is a domain, it follows that $x = y$, a contradiction. This verifies our claim that $|\mathfrak{B}(\lambda_0, \lambda_1, \dots, \lambda_{\mathfrak{p}-1})| \leq 1$.

Now the number of possible choices of $\lambda_0, \lambda_1, \dots, \lambda_{\mathfrak{p}-1}$ as described above is at most $|\alpha|^{\mathfrak{q}}$, which is either finite or equal to $|\alpha|$, and is therefore strictly less than \mathfrak{k} . It follows (using the axiom of choice) that we may choose $d_{\alpha \mathfrak{q}} \in D$ so that for all possible choices of $\lambda_0, \lambda_1, \dots, \lambda_{\mathfrak{p}-1}$, we have $d_{\alpha \mathfrak{q}} \notin \mathfrak{B}(\lambda_0, \lambda_1, \dots, \lambda_{\mathfrak{p}-1})$. Such a $d_{\alpha \mathfrak{q}}$ has the property that for every nonzero cardinal $\mathfrak{p} \leq \mathfrak{q} + 1$, every \mathfrak{p} -element subset of $\{(d_{\gamma_0}, d_{\gamma_1}, \dots, d_{\gamma_{\mathfrak{q}}}) : \gamma \leq \alpha\}$ is free in ${}_D D^{\mathfrak{q}+1}$. By transfinite induction, it follows that we may find, for every $\gamma < \mathfrak{k}$, an element $d_{\gamma \mathfrak{q}} \in D$ such that for every nonzero cardinal $\mathfrak{p} \leq \mathfrak{q} + 1$, every \mathfrak{p} -element subset of $\{(d_{\gamma_0}, d_{\gamma_1}, \dots, d_{\gamma_{\mathfrak{q}}}) : \gamma \in \mathfrak{k}\}$ is free in ${}_D D^{\mathfrak{q}+1}$.

The existence and properties of the \mathfrak{k} -sequence \mathfrak{U} described at the beginning of this proof follow by (ordinary) induction. \square

COROLLARY. *If D is a division ring then, for any infinite cardinal \mathfrak{m} , the dimension of the left vector space ${}_D D^{\mathfrak{m}}$ over D is $|D|^{\mathfrak{m}}$.*

Proof. Let \mathfrak{n} be the dimension of ${}_D D^{\mathfrak{m}}$ over D . Certainly, \mathfrak{n} is infinite, and we have ${}_D D^{\mathfrak{m}} \cong {}_D D^{(\mathfrak{n})} (:= \bigoplus_{\mathfrak{n}} D)$. We claim that

$$(1) \quad |D|^{\mathfrak{n}} = \mathfrak{n}|D|.$$

The left hand side of this equation is the cardinality of ${}_D D^{\mathfrak{n}}$, while $|{}_D D^{(\mathfrak{n})}| \geq \mathfrak{n}$. If D is finite then, since the number of finite subsets of an \mathfrak{n} -element set is \mathfrak{n} , it follows that $|{}_D D^{(\mathfrak{n})}| \leq \mathfrak{n}\aleph_0 = \mathfrak{n}$, so $|{}_D D^{(\mathfrak{n})}| = \mathfrak{n} = \mathfrak{n}|D|$, proving (1) in this case. If D is infinite, then again, since the number of finite subsets of an \mathfrak{n} -element set is \mathfrak{n} , it follows that $|{}_D D^{(\mathfrak{n})}| = \mathfrak{n}|D|$, establishing (1). Now, we claim that $\mathfrak{n} \geq |D|$. This is trivially true if D is finite, while for infinite D , it follows from the Lemma. Thus the right hand side of (1) is \mathfrak{n} (in both cases), so $\mathfrak{n} = |D|^{\mathfrak{n}}$, as required. \square

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		subring,
		successor cardinal,
		sum of cardinals,
		—, — ordinals,
		superprime ring,
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		—, —, —, —, —, — monomial algebra,
		—, —, —, —, —, — semigroup ring,
		symbol (see indeterminate)
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		τ -torsion-free module,
		Θ -interval of a chain,
		Θ -invariant subset,
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List of symbols and abbreviations

$X \subseteq Y$ (X is a subset of Y)	8	$\exp S$ (power set)	1
$Y \setminus X$ (complement of X in Y)	8	$\langle P_1; \leq \rangle \cong \langle P_2; \leq \rangle$ (isomorphic posets)	1
$ X $ (cardinality of X)	8	$\langle P_1; \leq \rangle \lesssim \langle P_2; \leq \rangle$ (poset embedding)	1
$\varphi[X']$ (image of X')	8	$\mathfrak{F}P$ (filters of $\langle P; \leq \rangle$)	1
$\varphi^{-1}[Y']$ (preimage of Y')	8	$\mathfrak{I}P$ (ideals of $\langle P; \leq \rangle$)	1
$\varphi _{X'}$ (restriction of φ to X')	8	L^c (compact elements of L)	1
1_X (identity map on X)	8	\mathbb{Z} (integers)	1
\mathbf{m}^+ (cardinal successor of \mathbf{m})	8	\mathbb{N} (natural numbers)	1
\mathbf{m}^- (cardinal predecessor of \mathbf{m})	8	\mathbb{Z}_n (integers modulo n)	1
$\mathbf{m} + \mathbf{n}$ (cardinal sum)	8	\mathbb{Q} (rational numbers)	1
$\mathbf{m} \cdot \mathbf{n}$ or $\mathbf{m}\mathbf{n}$ (cardinal product)	8	\mathbb{R} (real numbers)	1
$\mathbf{m}^{\mathbf{n}}$ (cardinal exponentiation)	8	\mathbb{C} (complex numbers)	1
$\sum_{i \in \Gamma} \mathbf{m}_i$ (sum of cardinals \mathbf{m}_i)	8	1_A (identity element of ring or monoid A)	15, 3
ω (smallest infinite ordinal)	9	R^{opp} (opposite ring)	1
$\alpha \oplus \beta$ (ordinal sum)	9	$\text{cen } R$ (centre of R)	1
$\alpha \odot \beta$ (ordinal product)	9	$\text{char } R$ (characteristic of R)	1
$\prod_{i \in \Gamma} X_i$ (cartesian product of X_i)	9	$A_1 A_2 \dots A_n$ (product of subgroups A_i of a ring)	1
$\{x_i\}_{i \in \Gamma}$ (element of $\prod_{i \in \Gamma} X_i$)	9	A^n (product of A 's, A subgroup of a ring)	1
$\prod_{\Gamma} X$ or X^{Γ} (cartesian power)	9	$\text{Id } R$ (two-sided ideals of R)	1
$[x, y]$ (bounded interval)	9	$\text{Id } R_R$ (right ideals of R)	1
(x) (open half-unbounded interval)	9	$\text{Id } {}_R R$ (left ideals of R)	1
$[x)$ (closed half-unbounded interval)	9	$\langle X \rangle$ (ideal of a ring generated by X or a free monoid on X)	16, 3
\leq^{-1} (dual order)	9	$\langle x \rangle$ (free monoid on $\{x\}$)	3
$\langle P; \leq \rangle^{\text{du}}$ (dual poset)	9	$\text{Hom}_{\mathcal{C}}(A, B)$ (category morphisms)	3
0 (bottom element of a poset, zero ring, zero ideal, zero element, zero module, zero submodule, zero functor)	10, 15, 65	$1_{\mathcal{C}}$ (identity functor)	3
1 (top element of a poset)	10	MA (product of module and subgroup of ring)	3
$\text{cof } P$ (cofinality of P)	10	$\text{Mod-}R$ (all right R -modules)	3
\mathbf{m} -ACC (\mathbf{m} -ascending chain condition)	10	$\text{Mod-}R(\text{zero})$ (right zero multiplication R -modules)	3
\mathbf{m} -DCC (\mathbf{m} -descending chain condition)	10	$\text{Mod-}R(\text{unital})$ (right unital R -modules)	3
ACC (ascending chain condition)	10	$R\text{-Mod}$ (all left R -modules)	3
DCC (descending chain condition)	10	$R\text{-Mod}(\text{zero})$ (left zero multiplication R -modules)	3
$\inf X, \wedge_P X, \wedge X, x \wedge y$ (infimum or meet)	10	$R\text{-Mod}(\text{unital})$ (left unital R -modules)	3
$\sup X, \vee_P X, \vee X, x \vee y$ (supremum or join)	10	\mathbb{Z}_{zero} (\mathbb{Z} with zero multiplication)	3

M_R (right R -module)	17	$F_1 \cong F_2$ (natural isomorphism between functors)	
${}_R M$ (left R -module)	17	$\dim M$ (Goldie dimension of M)	
$N \leq M$ (submodule)	17	(φ_S, RS^{-1}) (right ring of fractions)	
$N \lesssim M$ (module embedding)	17	RS^{-1} (right ring of fractions)	
$\text{Hom}_R(M, N)$ (R -module homomorphisms)	17	$S^{-1}R$ (left ring of fractions)	
$\text{End}_R M$ (R -module endomorphisms)	17	R_P (localization of R at P)	
$\text{Im } \varphi$ (image of φ)	17	$(\varphi_{\max}, Q_{\max}(R_R))$ (maximal right ring of quotients)	
$\text{Ker } \varphi$ (kernel of φ)	17	$Q_{\max}(R_R)$ (maximal right ring of quotients)	
$\varphi: M \cong N$ (isomorphism φ)	17	$\mathfrak{S}(R)$ (semiprime ideals of R)	
$\text{supp } x$ (support of x)	17, 33, 141	$\text{rank } \varphi$ (rank of a linear transformation φ)	
$\bigoplus_{i \in \Gamma} M_i$ (direct sum of M_i)	17	$Z(M)$ (singular submodule of M)	
$\bigoplus_{\Gamma} M$ or $M^{(\Gamma)}$ (direct sum of M 's)	17	$\langle X, X^{-1} \rangle$ (free group on X)	
$\prod_{i \in \Gamma}^{(\mathbf{m})} M_i$ (elements in $\prod_{i \in \Gamma} M_i$ with $ \text{support} < \max\{\mathbf{m}, \aleph_0\}$)	17	$\langle x, x^{-1} \rangle$ (free group on $\{x\}$)	
$\prod_{\Gamma}^{(\mathbf{m})} M$ (elements in M^{Γ} with $ \text{support} < \max\{\mathbf{m}, \aleph_0\}$)	17	RH (semigroup or monoid or group ring)	32, 3
$(X:Y)$	18	rg (element of RH)	
$(X:{}_r Y)$	18	$F\langle X \rangle$ (free associative F -algebra)	
$(X:{}_l Y)$	18	∂ (degree function)	33, 149, 1
$(0:y)$ (annihilator)	18	$R[x]$ (polynomial ring in x)	
$(0:{}_r y)$ (right annihilator)	18	$R[x_1, x_2, \dots, x_n]$ (polynomial ring in x_1, x_2, \dots, x_n)	
$(0:{}_l y)$ (left annihilator)	18	$F(x_1, x_2, \dots, x_n)$ (field of rational functions in x_1, x_2, \dots, x_n)	
$\text{soc } M$ (socle of M)	18	$R[[x]]$ (power series ring in x)	
R^* (Dorroh Extension)	19	$R[[H]]$ (power series ring in H)	
1_{R^*} (identity element of R^*)	19	$\text{Mon } A$ (monomorphisms on A)	
$(X:Y)^*$	20	$R[\Theta]$ (right skew semigroup ring in Θ)	
R_0^*	20	$R[x, \sigma]$ (right skew polynomial ring in x)	
$(\iota_M, E(M))$ (injective hull)	21	$R[[x, \sigma]]$ (right skew power series ring in x)	
$E(M)$ (injective hull)	21	$\text{Con } A$ (congruence relations on A)	37, 38,
\mathbb{Z}_{p^∞}	22	$0/\theta$ (0 -class of a congruence relation)	
$\text{Spec } R$ (spectrum of R)	22	a/θ (equivalence class of θ containing a)	
β (Prime (or Lower Baer) radical)	22	A/θ (factor algebra of A)	
J (Jacobson radical)	23	ar (arity function)	
$\mathbb{M}_{\mathbf{m}}(R)$ (row-finite matrix ring)	24	$\langle F; ar \rangle$ (type of a universal algebra)	
$\mathbb{M}_{\mathbf{m}}^*(R)$ (row- and column-finite matrix ring)	24	$\mathfrak{M}H$ (ideals of a semigroup H)	
$A_{\alpha\beta}$ ((α, β) -th entry of a matrix A)	24	L_{\emptyset} (lattice L augmented by \emptyset)	
$A^{(\alpha)}$ (α -th column of a matrix A)	24	$\mathfrak{M}_{\Theta}H$ (absolutely Θ -invariant ideals of a monoid H)	
$A_{(\alpha)}$ (α -th row of a matrix A)	24		

$\text{Id}_\Theta R$	(absolutely Θ -invariant ideals of a ring R)	40	$\mathcal{F} \cdot \mathcal{G}$	(product of topologizing filters)	
$\emptyset R$		41	$\eta(I)$	(smallest topologizing filter containing I)	
$\text{Id}^P R$	(principal ideals of R)	42	$\text{Gab-}R$	(Gabriel filters on R)	
$\mathfrak{F}_\Theta C$	(absolutely Θ -invariant filters of a chain C)	49	$\overline{\mathfrak{F}}$	(smallest Gabriel filter $\supseteq \mathfrak{F}$)	
$M(C)$	(positive cone associated with chain C)	49	\mathfrak{F}^α	(transfinite power of a topologizing filter)	
$\begin{pmatrix} z_1 & z_2 & \dots & z_n \\ c_1 & c_2 & \dots & c_n \end{pmatrix}$	(element of $M(C)$)	50	torsp	(map from $\text{Fil-}R$ to $\text{torsp-}R$)	
M_X	(subset of $M(C)$ associated with subset X of C)	50	$\text{torsp } \mathfrak{F}$	(image of \mathfrak{F} under torsp)	
$\text{Tol } A$	(tolerance relations on A)	61	$(0 : x)^*$	(annihilator of x in R^*)	
T_τ	(τ -torsion modules)	65	\overline{Z}	(Goldie torsion radical)	
F_τ	(τ -torsion-free modules)	65	$\text{Jans-}R$	(Jansian torsion preradicals on $\text{Mod-}R$)	
$\text{torsp-}\mathcal{C}$	(torsion preradicals on \mathcal{C})	65	I^α	(transfinite power of an ideal)	
$\text{torsp-}R$	(torsion preradicals on $\text{Mod-}R$)	65	η	(map from $\text{Id } R$ to $[\text{Fil-}R]^{\text{du}}$)	
$\widehat{\tau}$	(smallest torsion preradical $\geq \tau$)	67	$\mathfrak{m-} \text{Jans-}R$	(\mathfrak{m} -Jansian torsion preradicals on $\text{Mod-}R$)	
$\tau \cdot \sigma$	(product of torsion preradicals)	69	$\{\sigma_\alpha\}_\alpha$	(Gabriel filtration)	
$\text{tors-}\mathcal{C}$	(torsion radicals on \mathcal{C})	70	$\text{G-dim } M$	(Gabriel dimension of M)	
$\text{tors-}R$	(torsion radicals on $\text{Mod-}R$)	70	$\text{K-dim } M$	(Krull dimension of M)	
$\overline{\sigma}$	(smallest torsion radical $\geq \sigma$)	70	\widehat{F}	(functor induced map from $\text{torsp-}R$ to $\text{torsp-}S$)	
σ^α	(transfinite power of a torsion preradical)	71	$P_\tau(\mathfrak{m})$	(rings right prime of bound \mathfrak{m})	
soc	(socle as a torsion preradical)	71	$\overline{P}_\tau(\mathfrak{m})$	(rings right prime of bound at most \mathfrak{m})	
$\{\text{soc}^\alpha : \alpha > 0\}$	(extended socle series of $\text{Mod-}R$)	71	$P_l(\mathfrak{m})$	(rings left prime of bound \mathfrak{m})	
$\{\text{soc}^\alpha(M) : \alpha > 0\}$	(extended socle series of M)	71	$\overline{P}_l(\mathfrak{m})$	(rings left prime of bound at most \mathfrak{m})	
τ_{zero}	(torsion preradical associated with $\text{Mod-}R$ (zero))	72	$E_{\gamma,\delta}(r)$	(matrix with r in position (γ, δ) and zeros elsewhere)	
τ^*	(extension to torsion preradical on $\text{Mod-}R$)	73	\mathfrak{F}_S	(Gabriel filter associated with denominator set S)	
$\tau_{\mathcal{C}}$	(torsion radical associated with module subcategory \mathcal{C})	73	H^+	(positive cone of H)	
$\text{torsp-}R(\text{zero})$	(torsion preradicals on $\text{Mod-}R(\text{zero})$)	75	H^M	(monoid extension of semigroup H)	
$\text{tors-}R(\text{zero})$	(torsion radicals on $\text{Mod-}R(\text{zero})$)	75	H^1	(monoid extension of semigroup H)	
$\text{torsp-}R(\text{unital})$	(torsion preradicals on $\text{Mod-}R(\text{unital})$)	76	T_A	(subset of H^1 in Conditions D and E)	
$\text{tors-}R(\text{unital})$	(torsion radicals on $\text{Mod-}R(\text{unital})$)	76	$\star_{i \in \Gamma} G_i$	(free product of monoids G_i)	
τ_{unit}	(torsion radical associated with $\text{Mod-}R(\text{unital})$)	76	$G_1 \star G_2 \star \dots \star G_n$	(free product of monoids G_i)	
$\text{Fil-}R$	(topologizing filters on R)	76	D_∞	(infinite dihedral group)	
			$\overline{\partial}$	(degree function on monomials)	
			$UP(\mathfrak{m})$	(rings uniformly prime of bound \mathfrak{m})	
			$\overline{UP}(\mathfrak{m})$	(rings uniformly prime of bound at most \mathfrak{m})	
			$A(\underline{x})$	($\mathfrak{n} \times \mathfrak{m}$ matrix associated with \mathfrak{m} -element subset of $\mathbf{M}_{\mathfrak{n}}(D)$)	

$\text{ht } P$	(height of prime ideal P)	151
$\text{coht } P$	(coheight of prime ideal P)	151
$\text{cl-K-dim } R$	(classical Krull dimension of R)	151
$V(S)$	(affine variety defined by S)	152
$J(Y)$	(subset of $K[x_1, x_2, \dots, x_n]$ associated with subset Y of F^n)	152
$\text{rad } I$	(intersection of all prime ideals containing I)	152
$\det A$	(determinant of A)	154
$\mathfrak{U}\mathcal{A}$	(upper radical determined by \mathcal{A})	159
L	(Levitzki radical)	159
N	(Nil (or Upper Baer or Koethe) radical)	159
J_B	(Behrens radical)	159
G	(Brown-McCoy radical)	159
T	(radical associated with the class of matrix rings over division rings)	159
F	(radical associated with the class of fields)	159
N_g	(Generalized Nil radical)	159
β_ϕ	(Antisimple radical)	159
GD	(going down condition)	163
LO	(lying over condition)	163
$\mathfrak{U}P_r(\mathfrak{m})$	(upper radical determined by $P_r(\mathfrak{m})$)	165
$\mathfrak{U}\bar{P}_r(\mathfrak{m})$	(upper radical determined by $\bar{P}_r(\mathfrak{m})$)	165
$B_r(\mathfrak{m})$	(upper radical determined by $\bigcup_{0 < \mathfrak{k} < \mathfrak{m}} P_r(\mathfrak{k})$)	165
$\mathfrak{U}\bar{P}_r(\aleph_0)$	(right strongly prime (or Groenewald-Heyman) radical)	165
$\mathfrak{T}_\mathfrak{m}$	(rings R for which R_R is not \mathfrak{m} -faithful)	174
$H(\mathfrak{T}_\mathfrak{m})$	(largest homomorphically closed subclass of $\mathfrak{T}_\mathfrak{m} \cup \{0\}$)	174
Min I	(“degree” associated with an ideal I of a polynomial ring)	179
$\tau(I)$	(ideal of R associated with ideal I of $R[x]$)	180
\mathbb{H}	(division ring of real quaternions)	182
US	(uniformly strongly prime radical)	183
σ_r	(right superprime radical)	183
σ_l	(left superprime radical)	183
$I_{\aleph_0, \mathfrak{m}}(D)$	(minimal nonzero ideal of $\mathbb{M}_\mathfrak{m}(D)$)	183
$\tau_\mathfrak{m}$	(upper radical determined by $\overline{UP}(\mathfrak{m})$)	193