A CLASSIFICATION OF SECOND ORDER EQUATIONS VIA NONLOCAL TRANSFORMATIONS

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Abstract

The study of second order ordinary differential equations is vital given their proliferation in mechanics. The group theoretic approach devised by Lie is one of the most successful techniques available for solving these equations. However, many second order equations cannot be reduced to quadratures due to the lack of a sufficient number of point symmetries. We observe that increasing the order will result in a third order differential equation which, when reduced via an alternate symmetry, may result in a solvable second order equation. Thus the original second order equation can be solved.

In this dissertation we will attempt to classify second order differential equations that can be solved in this manner. We also provide the nonlocal transformations between the original second order equations and the new solvable second order equations.

Our starting point is third order differential equations. Here we concentrate on those invariant under two- and three-dimensional Lie algebras.

Declaration

I declare that the contents of this dissertation are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

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Dedication

TO SONIA GINSBERG

Acknowledgments

First and foremost, I thank my supervisor, Dr K S Govinder, whose dedication, enthusiasm and knowledge has not only played an influential role in the direction of my life, but has opened my eyes to a world in which I have found my passion. His support and understanding has constantly set me back on track whenever the immensity of my task overwhelmed me. He has provided me with the courage and confidence to believe I can achieve my high goals, while his friendship and genuine understanding has continously inspired me to give of my best. He has definitely been a guiding light in my academic career.

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On a more personal note, I thank my father and mother whose love and unfailing support has enabled me to constantly strive to achieve a higher level of success; Erica, with his invincible spirit constantly challenging my competitive nature, and Max whose brilliant brain, untrained in mathematics, constantly puts my knowledge to the test. Thanks also to Samantha, Karen and Gary who in their own unselfish ways manage to *destress* me, and Ilan whose patience and understanding has been invaluable.

I have dedicated this dissertation to my grandmother, Sonia Ginsberg, whose total admiration and pride gives me the strength to believe in myself.

Contents

Chapter 1

Introduction

"HOW CAN IT BE THAT MATHEMATICS, A PRODUCT OF HUMAN THOUGHT INDEPENDENT OF EXPERIENCE, IS SO ADMIRABLY ADAPTED TO THE OBJECTS OF REALITY ?" -Albert Einstein [18, p.464]

1.1 Historical Overview

The phenomenally accurate results mathematical knowledge can derive with respect to almost everything in and beyond nature has occurred right through history. Not only were the earliest researchers denied the right to further their wisdom had they not a thorough understanding of mathematics, but in many old civilizations a mathematician's deduction was regarded with the same respect as the word of the Church [19]. As man began to modernise, one would naturally expect mathematical knowledge to reach even further dimensions.

Towards the turn of the seventeenth century, modern calculus developed encouraged by an era of outstanding scientists whose passion and dedication towards the abstract model of the physical world led them to the study of differential equations. Although during the next two centuries countless techniques were designed in an attempt to provide a wide variety of methods for solving these equations, it was not until the mid-1800's that these seemingly disconnected methods of integration, each corresponding to a particular class of equations, were unified [11].

Finally, slightly more than a century ago, the brilliant Norwegian mathematician, Sophus Marius Lie (1842–1899), made the profound and far reaching discovery that these special methods were, in fact, all special cases of a general integration procedure based on the invariance of the differential equation under a continuous group of transformations [30]. The many existing integration techniques could now be combined and replaced by a method in which solvability was to be determined by the available symmetries of the equation (which generated the relevant transformations). Inspired by the success of his earlier work, Lie devoted the remainder of his mathematical career to the development and application of his monumental theory of continuous groups (later to be known as Lie groups) [30].

The main idea behind solving an ordinary differential equation is to reduce the order of the original equation until it can be expressed in quadrature form. Unfortunately, Lie's success failed in the case of first order ordinary differential equations in that no systematic way could be found for finding the explicit symmetries necessary for their reduction [4]. (The occurrence of an infinite number of symmetries for these equations is of course due to him [20, p.114].) Thus, these equations could often only be reduced to algebraic expressions through a knowledge of the Lie group which was to be inherited via reduction of a higher order equation [3].

The concept of symmetry can easily be regarded as one of the most valuable intellectual discoveries of our time. Although it first showed itself in the form of art, it soon shed light on geometric theory which had become very prominent during this era. Strongly influenced by the famous geometers Darboux and Jordan, one is not surprised that symmetry was to be the basis of Lie's research [33, 12] .

Lie's theory was based on the infinitesimal properties of groups, and the classification of equations via these groups using symmetries (by this stage already known to be a central property of a differential equation). One was simply expected to determine the symmetry group of a system of differential equations, and the work of Lie would provide the foundations for solvability. Lie also investigated the close connection between continuous groups and specific algebra systems (later to be known as Lie algebras). His numerous books, papers and articles published during and after his lifetime portray the completeness and thoroughness of his research. Given his lifetime of dedication and brilliance, one is not surprised by the powerful results which emerged.

By the 1930's, the world's interest in Lie had substantially decreased following the decline of interest among the new young scholars of this time who considered the essence of his theory to be *old-fashioned* [33, p.107]. Following the invention of computers during the 1940's, the problem of solving differential equations took on a whole new light. Although the terms Lie groups and Lie algebras retained their respect within the scientific world, the solvability of differential equations was now to be determined by the methods of the newly growing group of computer scientists. Finally the 1970's saw first the physicists, and soon after, the mathematicians in the West, returning to the world of Lie, drawn back by the strong relationships he had encountered with symmetries. (Ovsiannikov had already rekindled interest in Lie's work in Novosibirsk in the 1940's.)

Much of the work related to Lie's group theory was not, however, researched and published by him. He relied greatly on enthusiastic students to develop his ideas and solve the problems he constructed in abundance. (The oft forgotten co-authorship of his books [20, 21, 22, 23] bear testament to this.) As one of the last great mathematicians of the nineteenth century, it was many years before his works were published, partly due to the vastness in content he had researched during his lifetime, as well as the depletion of collected funds following the rise of inflation in Germany during this time [33].

In surveying Lie's contribution to Science, one must not omit the strong and loyal friendship he maintained throughout his life with Felix Klein whose influence, both in and out of Lie's career, had a huge and important effect on all of his works. The strong bond they maintained and similar interest in their respective fields of studies benefitted both men, encouraging each of them to achieve outstanding results yet concurrently neglecting the competitive atmosphere one naturally sees to be inevitable.

Today, Lie's theory of extended groups plays an integral part in both the investigations as well as utilization of ordinary differential equations. Not only does Lie's work constitute an easily obtainable platform, a basis for the solvability of differential equations, but this relatively small piece of research has the potential to perhaps one day govern all the branches in the scientific world and it's surroundings.

1.2 Uses of symmetries

The symmetry group of a system of differential equations plays a major role in many of the applications associated with differential equations. Firstly, the symmetries of ordinary differential equations are used in the reduction process, enabling one to reduce higher order equations to quadratures. A symmetry cannot be used more than once within the reduction process and thus solvability of a differential equation is very much dependent on the original number of symmetries the equation to be solved possesses.

Symmetries can also be used in the transformation of equations into canonical form as well as to determine first integrals of differential equations. They provide a means of classifying different symmetry classes of solutions and families of differential equations.

Many equations appear to *loose* one or more symmetries when the order of the equation is decreased (increased). However, it has been shown that these symmetries are not *lost*, but instead become nonlocal. Equally, previously undiscovered nonlocal symmetries appear as point symmetries following a reduction (increase) in order of an equation [12].

Symmetries take on many forms. Here we are mainly concerned with point symmetries (where the transformations depend on the variables of the equation) and nonlocal symmetries (where the transformations depend on the variables of the equation and integrals thereof). Nonlocal symmetries can become point under the reduction (increase) of order of the equation. The resulting symmetries are called hidden symmetries. A Type I hidden symmetry can occur when the order of the equation increases, whereas decreasing the order of the differential equation may give rise to a Type II hidden symmetry [2]. (It is this approach which we exploit.)

Furthering this idea Abraham-Shrauner *et al* [1] found that equations possessing no Lie point symmetries could be reduced to quadrature form when an increase in the order of the equation resulted in a Type I hidden symmetry.

1.3 Outline

After introducing the salient terminology we provide an example which illustrates the importance of hidden (and hence nonlocal) symmetries. Thereafter we proceed to reduce the order of third order ordinary differential equations invariant under two- and three-dimensional Lie algebras to second order ordinary differential equations. By considering the Lie algebras of these new equations we provide a new classification of second order ordinary differential equations which can be solved via non-local transformations.

Chapter 2

Lie Theory of Differential Equations

We begin by introducing some of the concepts and terminologies we utilise in our analysis. Thereafter, we motivate the importance of hidden (and hence non-local) symmetries by way of an example.

2.1 Definitions

GROUP: A **group** G is a set of elements with a law of composition ϕ between elements satisfying the following properties[5]:

- CLOSURE PROPERTY: For any element a and b of G, $\phi(a, b)$ is an element of G.
- ASSOCIATIVE PROPERTY: For any elements a,b and c of G,

$$\phi(a, \phi(b, c)) = \phi(\phi(a, b), c).$$

- IDENTITY ELEMENT: There exists a unique identity element I of G such that, for any element a of G,

$$\phi(a, I) = \phi(I, a) = a.$$

- INVERSE ELEMENT: For any element a of G there exists a unique inverse element a^{-1} in G such that

$$\phi(a, a^{-1}) = \phi(a^{-1}, a) = I.$$

ABELIAN GROUP: A group G is **Abelian** if $\phi(a,b) = \phi(b,a)$ holds for all elements a and b in G.

GROUP OF TRANSFORMATIONS: The set of transformations

$$\bar{x} = X(x;\varepsilon),$$

defined for each x in the space $D \subset R$, depending on the parameter ε lying in the set $S \subset R$, with $\phi(\varepsilon,\delta)$ defining a composition of parameters ε and δ in S, forms a **group of transformations** on D if:

- For each parameter ε in S the transformations are one-to-one onto D.
- S with the law of composition ϕ forms a group.
- $\bar{x} = x$ when $\varepsilon = I$, *ie.*

$$X(x;I) = x.$$

- If $\bar{x} = X(x;\varepsilon)$, $\bar{\bar{x}} = X(\bar{x};\delta)$, then

$$\bar{\bar{x}} = X(x;\phi(\varepsilon,\delta)).$$

LIE GROUP OF TRANSFORMATIONS: A one-parameter **Lie group of transformations** is a group of transformations which, in addition to the above, satisfies the following:

- ε is a continuous parameter, *ie.* S is an interval in R. (Without loss of generality $\varepsilon = 0$ corresponds to the identity element I).
- X is infinitely differentiable with respect to x in D and an analytic function of ε in S.
- $\phi(\varepsilon,\delta)$ is an analytic function of ε and δ, $\varepsilon \in S$, $\delta \in S$.

SUBGROUP: A **subgroup** of G is a group formed by a subset of elements of G with the same law of composition.

EXAMPLES OF LIE GROUPS:

LINEAR GROUPS: The complex general linear group $GL(n, C)$ and the real general linear group $GL(n, R)$ consist of all nonsingular complex and real $n \times n$ matrices respectively [29]. (The latter may be considered as a subgroup of the former.) The complex special linear group $SL(n, C)$ is the subgroup of $GL(n, C)$ consisting of matrices with determinant one. The real special linear group $SL(n, R)$ is the intersection of these two subgroups, *ie.*

$$SL(n, R) = SL(n, C) \cap GL(n, R).$$

ROTATION GROUP: The **rotation group** $SO(n, R)$ is the special or proper real orthogonal group given by the intersection of the group of orthogonal matrices $O(n, R)$ and the complex special linear group, *ie.*

$$SO(n, R) = O(n, R) \cap SL(n, C).$$

LIE ALGEBRA: A **Lie algebra** L is a vector space together with a product $[x, y]$ that:

- is **Bilinear** (*ie.* linear in x and y separately),
- is **Anticommutative** (antisymmetric):

$$[x, y] = -[y, x],$$

  and,

- satisfies the **Jacobi Identity**

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for all vectors x, y, z in L.

ABELIAN LIE ALGEBRA: A Lie algebra L is called **Abelian** (equivalently commutative) if $\forall x, y \in L$, $[x, y] = 0$ [29].

SOLVABLE ALGEBRA: A Lie algebra L is called **solvable** if the derived series

$$\begin{aligned} L &\supseteq L' = [L, L] \\ &\supseteq L'' = [L', L'] \\ &\supseteq \ldots \\ &\supseteq L^k = [L^{k-1}, L^{k-1}] \end{aligned}$$

terminates with a null ideal *ie.* $L^k = 0$, $k > 0$ [17, 31]. Note: Any Abelian algebra is solvable and indeed any Lie algebra of dimension ≤ 3 is solvable except when $\dim L = 3 = \dim L'$.

A few comments about Lie algebras are now in order. The Jacobi identity plays the same role for Lie algebras that the associative law plays for associative algebras. While we can define a Lie algebra over any field, in practice it is usually considered over real and complex fields. We define the product associated with the Lie algebra as that of commutation, *ie.*

$$[X, Y] = XY - YX.$$

If a differential equation admits the operators X and Y, it also admits their commutator $[X, Y]$. Lie's main result [33] is the proof that it is always possible to assign to a continuous group (Lie group) a corresponding Lie algebra and vice versa. Thus for the real special linear group $SL(n, R)$ the corresponding Lie algebra is $sl(n, R)$ and for $SO(n, R)$, $so(n, R)$ [13, 14]

2.2 The Algorithm

An nth order ordinary differential equation

$$N(x, y, y', ..., y^n) = 0 \tag{2.1}$$

admits the one-parameter Lie group of transformations

$$\bar{x} = x + \epsilon\xi \tag{2.2}$$

$$\bar{y} = y + \epsilon\eta \tag{2.3}$$

with infinitesimal generator

$$G = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} \tag{2.4}$$

if

$$G^{[n]}N_{|_{N=0}} = 0, \tag{2.5}$$

where $G^{[n]}$ is the nth extension of G needed to transform the derivatives in (2.1) and is given by [25]

$$G^{[n]} = G + \sum_{i=1}^{n}\left[\eta^{(i)} - \sum_{j=0}^{i-1}\binom{i}{j}y^{(j+1)}\xi^{(i-j)}\right]\frac{\partial}{\partial y^{(i)}}. \tag{2.6}$$

Note that the superscripts in (2.6) denote total differentiation with respect to the independent variable. We say that (2.1) possesses the symmetry (group generator) (2.4) if (2.5) holds.

In the case of point symmetries we require that the coefficient functions ξ and η depend on the independent and dependent variables only (in this case x and y). The operation of (2.6) on (2.1) produces a single linear partial differential in ξ and η. As these functions are independent of derivatives of y (which appear in the partial differential equation) we separate by equating coefficients of powers of derivatives of y to zero. This leads to an overdetermined system of linear partial differential equations, the solution of which gives ξ and η.

Consider the equation

$$yy^{iv} + \frac{5}{2}y'y''' - \frac{1}{y^3} = 0 \tag{2.7}$$

which arises in cosmology [24]. Utilizing MULIE [15] (a computer package for the determination of symmetries of differential equations) the symmetries of this equation are easily found *viz.*

$$G_1 = \frac{\partial}{\partial x} \tag{2.8}$$

$$G_2 = x\frac{\partial}{\partial x} + \frac{4}{5}y\frac{\partial}{\partial y}. \tag{2.9}$$

Thus we are investigating a fourth order ordinary differential equation invariant under only two point symmetries. Ordinarily, the number of symmetries present (which is all we are concerned with if the Lie algebra is solvable) suggests that the equation cannot be reduced to quadratures. However, hidden symmetries arise during the reduction process, enabling one to fully reduce this fourth order equation with only two symmetries to quadratures.

The Lie Bracket relationship of (2.8) and (2.9) is

$$[G_1, G_2] = G_1 \tag{2.10}$$

Thus reduction via G_2 will result in the loss of G_1 as a point symmetry of the reduced equation [30, p.148].

We consider the reduction via G_1. Utilizing the associated Lagrange's system [28]

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}y'}{0} \tag{2.11}$$

we can easily see that both y and y' are characteristics, *ie.*

$$u = y \qquad\qquad v = y' \tag{2.12}$$

respectively. It follows that

$$y'' = v\frac{\mathrm{d}v}{\mathrm{d}u} \tag{2.13}$$

$$y''' = v^2\left(\frac{\mathrm{d}^2v}{\mathrm{d}u^2}\right) + v\left(\frac{\mathrm{d}v}{\mathrm{d}u}\right)^2 \tag{2.14}$$

$$y^{iv} = \frac{1}{u^4} - \frac{5v^2}{2u}\left[v\left(\frac{\mathrm{d}^2v}{\mathrm{d}u^2}\right) + \left(\frac{\mathrm{d}v}{\mathrm{d}u}\right)^2\right]. \tag{2.15}$$

Thus we have reduced (2.7) to the third order equation

$$v''' = -\left[\frac{5}{2u} + \frac{4v'}{v}\right]v'' - \frac{(v')^3}{v^2} - \frac{5(v')^2}{2uv} + \frac{1}{u^4v^3}. \tag{2.16}$$

As expected, it is possible to rewrite G_2 in terms of the new variables, *viz.*

$$X_2 = 4u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}. \tag{2.17}$$

Hence, X_2 is a symmetry of the newly formed third order equation. (We actually consider $G_2^{[1]}$, the first extension of G_2. Since x is not a variable relevant to the reduced equation, we ignore the $\frac{\partial}{\partial x}$ term.)

We use MULIE to determine whether any other symmetries of this third order equation exist. We discover that, in addition to (2.17),

$$X_1 = u^2\frac{\partial}{\partial u} + \frac{1}{2}uv\frac{\partial}{\partial v} \tag{2.18}$$

is also a symmetry. This is a Type II hidden symmetry [2].

The Lie Bracket relationship of (2.17) and (2.18) is

$$[X_1, X_2] = X_1. \tag{2.19}$$

As reduction via X_2 results in the loss of X_1 as a point symmetry of the reduced equation, we attempt reduction via X_1, in which case

$$p = \frac{v^2}{u} \qquad q = v^3\left(v' - \frac{v}{2u}\right). \tag{2.20}$$

Therefore

$$v'' = \frac{2uvv' - v^2}{u^2v^3}\left(\frac{\mathrm{d}q}{\mathrm{d}p}\right) - \frac{3(v')^2}{v} + \frac{2v'}{u} - \frac{v}{2u^2} \tag{2.21}$$

$$\begin{aligned} v''' &= \frac{(2uvv' - v^2)^2}{u^4v^3}\left(\frac{\mathrm{d}^2q}{\mathrm{d}p^2}\right) + \frac{\left(3u^2v^2(v')^2 + u^2v^3v'' - 2uv^3v' + \frac{1}{2}v^4\right)\left(2u(v')^2 + 2uvv''\right)}{u^2v^3\left(2uvv' - v^2\right)} \\ &\quad - \frac{6(v')^3}{v^2} - \frac{9v'v''}{v} \end{aligned} \tag{2.22}$$

and so (2.16) becomes

$$q'' = \frac{p^2}{4q^2} - \frac{5q}{2p^2} + \frac{7q'}{2p} - \frac{q'^2}{q}. \tag{2.23}$$

X_2 now becomes

$$U_2 = 6p\frac{\partial}{\partial p} + 8q\frac{\partial}{\partial q}. \tag{2.24}$$

Utilizing MULIE, we once again find a hidden symmetry, *viz.*

$$U_1 = \frac{q}{p^{\frac{1}{2}}}\frac{\partial}{\partial q} + p^{\frac{1}{2}}\frac{\partial}{\partial p}. \tag{2.25}$$

Conveniently, we notice two symmetries for the reduction of a second order ordinary differential equation, (2.23). However, once again the order in which we reduce is vital. The Lie Bracket relationship of (2.24) and (2.25) is

$$[U_1, U_2] = U_1. \tag{2.26}$$

Reducing via U_1 results in a first order ordinary differential equation with (at least) one point symmetry. Here,

$$s = \frac{q}{p} \qquad t = \frac{(pq' - q)^2}{p^3}, \tag{2.27}$$

which reduces (2.23) to

$$t' = \frac{1}{2s^2} - \frac{2t}{s}. \tag{2.28}$$

This is solved [16] (or using the integrating factor associated with the newly transformed U_2 [6]) to yield

$$t = \frac{\left[\sqrt{\frac{c}{s}} + 1\right]}{2s}. \tag{2.29}$$

Reversing the transformations (2.27), (2.20) and (2.12) and integrating the resulting first order ordinary differential equations will yield the solution to the original equation.

2.3 Summary

We have shown that an equation with fewer than the required number of point symmetries can still be reduced to quadratures. This was effected due to the discovery of hidden symmetries ((2.18) and (2.25)) of the reduced equations. These hidden symmetries are actually "useful" nonlocal symmetries [11] of the original equation, *viz.*

$$G_3 = \frac{3}{2}\left(\int y\mathrm{d}x\right)\frac{\partial}{\partial x} + 3y^2\frac{\partial}{\partial y} \tag{2.30}$$

$$G_4 = \frac{3}{4}\left[\int y\left(\int y^{\frac{-3}{2}}\mathrm{d}x\right)\partial x\right]\frac{\partial}{\partial x} + \frac{y^2}{2}\left(\int y^{\frac{-3}{2}}\mathrm{d}x\right)\frac{\partial}{\partial y}. \tag{2.31}$$

Clearly, if we knew that (2.7) possessed $G_1 - G_4$ originally, we would have realised that it could be reduced to quadratures. However, it is not a simple matter to find nonlocal symmetries of any ordinary differential equation although some progress was made in [10]. As a result, indirect means have been used to classify equations via nonlocal (hidden) symmetries [3, 11]. We consider one indirect method in the next chapter.

Chapter 3

Reduction of Third Order Equations

3.1 Introduction

In addition to the obvious importance of non-local transformations in reducing the order of ordinary differential equations, we have highlighted an additional use in §2.2. We now use these transformations to classify equations. Our classification scheme links second order ordinary differential equations with fewer than two symmetries to those with at least two symmetries. Thus the original equations which are thought to be unsolvable via the Lie approach are shown to be linked to solvable equations due to the presence of nonlocal symmetries.

The systematic application of the approach we follow was inadvertently initiated by González-Gáscon and González-López [9]. They analysed a system of second order ordinary differential equations not possessing any point symmetries and showed that it could still be integrated. Their system was later shown to possess point symmetries [1]. Six years later, Vawda [32] introduced a single second order ordinary differential equation which could be integrated while possessing no point symmetries. Abraham-Shrauner *et al* [1] showed that this equation possessed non-local symmetries. They achieved this by increasing the order of the equation to a third order ordinary differential equation with two point symmetries. This third order equation was reduced to a new second order linear ordinary differential equation with eight point symmetries. After reversing the two transformations, these point symmetries became non-local symmetries of the original second order ordinary differential equation. The utility of this approach is evident by the classification of second order ordinary differential equations with no point symmetries subsequently undertaken by Govinder and Leach [11]. We continue in that vein.

We start our analysis by considering third order ordinary differential equations invariant under two- and three-dimensional Lie algebras of point symmetries. We reduce each third order differential equation via each of its symmetries and consider the symmetries of the resulting second order differential equations. We then provide the nonlocal transformations linking these reduced second order differential equations. This work goes some way to filling the gap between [3] and [11].

All equations, Lie algebras and symmetries are taken from [26, 27]. The function Γ is an arbitrary function of its arguments. (When we refer to symmetries we mean point symmetries unless otherwise indicated. When we refer to the *loss* of symmetries we mean in the point sense.)

3.2 Two-Dimensional Lie Algebras

We begin by considering the two-dimensional Lie algebras. There are two two-dimensional Lie algebras (with two representations each) which leave third order equations invariant, *viz.*

$$2A_1^I: \quad y''' = \Gamma\,[y', y''] \tag{3.1}$$

$$2A_1^{II}: \quad y''' = \Gamma\,[x, y''] \tag{3.2}$$

$$A_2^I: \quad y''' = (y'')^2\Gamma\,[y', xy''] \tag{3.3}$$

$$A_2^{II}: \quad y''' = y''\Gamma\,[x, y''/y'] \tag{3.4}$$

We consider each in turn. We do not expect to reduce these third order equations to quadratures. Here we are only interested in the nonlocal transformations between the resulting second order ordinary differential equations.

3.2.1 $2A_1^I$

The first equation we investigate is

$$y''' = \Gamma\left[y', y''\right] \tag{3.5}$$

invariant under the two symmetries,

$$G_1 = \frac{\partial}{\partial x} \tag{3.6}$$
$$G_2 = \frac{\partial}{\partial y}. \tag{3.7}$$

Reducing via G_1,

$$u = y \qquad\qquad v = y' \tag{3.8}$$

from which we obtain

$$y'' = vv' \tag{3.9}$$
$$y''' = v^2v'' + vv'^2. \tag{3.10}$$

Substituting into (3.5) yields

$$v'' = -\frac{v'^2}{v} + \frac{1}{v^2}\Gamma\left[v, vv'\right]. \tag{3.11}$$

The remaining symmetry, G_2, can be rewritten in terms of the new coordinates. and is a symmetry of the newly formed second order ordinary differential equation, *ie.*

$$X_2 = \frac{\partial}{\partial u}. \tag{3.12}$$

Equation (3.11) does not possess any further symmetries and so can only be reduced to a first order differential equation.

Now consider the reduction of (3.5) via G_2. The invariants of the symmetry (3.7) are

$$u = x \qquad\qquad v = y' \tag{3.13}$$

and we use them as the new variables. Consequently

$$y'' = v' \tag{3.14}$$
$$y''' = v'' \tag{3.15}$$

and (3.5) reduces to

$$v'' = \Gamma\left[v, v'\right]. \tag{3.16}$$

G_1 takes on the form (3.12) and so becomes the only symmetry of (3.16).

For completeness, we provide the nonlocal transformation from (3.11) to (3.16):

$$u_1 = \int v_2 du_2 \tag{3.17}$$

$$v_1 = v_2. \tag{3.18}$$

(Here and in what follows, the variables u_i and v_i refer to the reduction variables obtained from the symmetry G_i.)

3.2.2 $2A_1^{II}$

Consider the equation

$$y''' = \Gamma\left[x, y''\right] \tag{3.19}$$

with symmetries

$$G_1 = \frac{\partial}{\partial y} \tag{3.20}$$

$$G_2 = x\frac{\partial}{\partial y}. \tag{3.21}$$

Reducing via G_1,

$$u = x \qquad v = y' \tag{3.22}$$

from which we obtain

$$y'' = v' \tag{3.23}$$

$$y''' = v''. \tag{3.24}$$

Substituting into (3.19) yields

$$v'' = \Gamma\left[u, v'\right]. \tag{3.25}$$

Symmetry G_2 now reduces to

$$X_2 = \frac{\partial}{\partial v}. \tag{3.26}$$

Since (3.25) is invariant under only (3.26), (3.19) cannot be reduced to quadratures using symmetries.

Reducing (3.19) via G_2 yields

$$u = x \qquad v = y' - \frac{y}{x} \tag{3.27}$$

and thus

$$y'' = v' + \frac{v}{u} \tag{3.28}$$

$$y''' = v'' + \frac{v'}{u} - \frac{v}{u^2}. \tag{3.29}$$

Hence, equation (3.19) reduces to

$$v'' = \Gamma\left[u, v' + \frac{v}{u}\right] - \frac{v'}{u} + \frac{v}{u^2}. \tag{3.30}$$

Now G_1 reduces to

$$X_1 = \frac{1}{u}\frac{\partial}{\partial v} \tag{3.31}$$

which is the only symmetry of (3.30)

Finally we investigate the nonlocal transformation from (3.25) to (3.30):

$$u_2 = u_1 \tag{3.32}$$

$$v_2 = v_1 - \frac{\int v_1 du_1}{u_1} \tag{3.33}$$

3.2.3 A_2^I

Consider the equation

$$y''' = (y'')^2 \Gamma\left[y', xy''\right] \tag{3.34}$$

which admits the symmetries

$$G_1 = \frac{\partial}{\partial y} \tag{3.35}$$

$$G_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{3.36}$$

Reducing via G_1,

$$u = x \qquad v = y' \tag{3.37}$$

from which we obtain

$$y'' = v' \tag{3.38}$$

$$y''' = v''. \tag{3.39}$$

Substituting into (3.34) yields

$$v'' = (v')^2\Gamma\left[v, uv'\right]. \tag{3.40}$$

G_2 reduces to the form

$$X_2 = u\frac{\partial}{\partial u}. \tag{3.41}$$

Equation (3.40) does not possess any further symmetries and so can only be reduced to a first order differential equation.

Now consider the reduction of (3.34) via G_2:

$$u = \frac{y}{x} \qquad v = y'. \tag{3.42}$$

Therefore

$$y'' = \frac{(xy' - y)\,v'}{x^2} \tag{3.43}$$

$$y''' = \frac{(xy' - y)^2\,v''}{x^4} - \frac{2y''}{x} + \frac{x(y'')^2}{(xy' - y)}. \tag{3.44}$$

Thus (3.34) reduces to

$$v'' = \left(\Gamma\left[v, v'\,(v - u)\right] - \frac{1}{v - u}\right)(v')^2 + \frac{2v'}{v - u}. \tag{3.45}$$

Unfortunately, (3.45) does not possess any symmetries (as expected). Thus the second order equation cannot be further reduced. This result is confirmed when one examines the Lie Bracket relationship between (3.35) and (3.36). In this particular case,

$$[G_1, G_2] = G_1. \tag{3.46}$$

The nonlocal transformation from (3.40) to (3.45) is

$$u_2 = \frac{\int v_1 du_1}{u_1} \tag{3.47}$$

$$v_2 = v_1. \tag{3.48}$$

3.2.4 A_2^{II}

It remains to consider

$$y''' = y''\Gamma\left[x, y''/y'\right] \tag{3.49}$$

with the corresponding symmetries

$$G_1 = \frac{\partial}{\partial y} \tag{3.50}$$

$$G_2 = y\frac{\partial}{\partial y}. \tag{3.51}$$

Reducing first with respect to G_1, we set

$$u = x \qquad v = y'. \tag{3.52}$$

Thus

$$y'' = v' \tag{3.53}$$

$$y''' = v'' \tag{3.54}$$

and therefore

$$v'' = v'\Gamma\left[u, \frac{v'}{v}\right]. \tag{3.55}$$

G_2 now takes on the form

$$X_2 = v\frac{\partial}{\partial v} \tag{3.56}$$

and can therefore be used in the reduction of (3.55). This is the only symmetry admitted by (3.55).

On the other hand, reducing via G_2, we set

$$u = x \qquad v = \frac{y'}{y}. \tag{3.57}$$

Hence

$$y'' = yv' + \frac{(y')^2}{y} \tag{3.58}$$

$$y''' = yv'' + \frac{y''y'}{y} + \frac{2y''y'}{y} - \frac{2(y')^3}{y^2} \tag{3.59}$$

from which it follows that

$$v'' = -3vv' - v^3 + \left(v' - v^2\right)\Gamma\left[u, \frac{v'}{v} + v\right]. \qquad (3.60)$$

In this case the remaining symmetry will be *lost.*

Equations (3.55) and (3.60) are linked via

$$\begin{aligned} u_2 &= u_1 \\ v_2 &= \frac{v_1}{\int v_1 du_1}. \end{aligned} \qquad (3.61)$$

The results obtained in this section show that the third order equation invariant only under the two dimensional Lie algebras (3.1)–(3.4) cannot be simply reduced to second order equations with more than one symmetry. A systematic treatment of the second order equations will be pursued elsewhere.

3.3 Three-dimensional Lie algebras

There are eleven three-dimensional Lie algebras associated with third order ordinary differential equations. These Lie algebras (with only the non-zero commutation relations given) are

$3A_1$			
$A_1 \oplus A_2$	$[G_1, G_3] = G_1$		
$A_{3,1}$	$[G_2, G_3] = G_1$		
$A_{3,2}$	$[G_1, G_3] = G_1$	$[G_2, G_3] = G_1 + G_2$	
$A_{3,3}$	$[G_1, G_3] = G_1$	$[G_2, G_3] = G_2$	
$A_{3,4}$	$[G_1, G_3] = G_1$	$[G_2, G_3] = -G_2$	
$A^a_{3,5}(0 < \|a\| < 1)$	$[G_1, G_3] = G_1$	$[G_2, G_3] = aG_2$	
$A_{3,6}$	$[G_1, G_3] = -G_2$	$[G_2, G_3] = G_1$	
$A^b_{3,7}(b > 0)$	$[G_1, G_3] = bG_1 - G_2$	$[G_2, G_3] = G_1 + bG_2$	
$A_{3,8}$	$[G_1, G_2] = G_1$	$[G_2, G_3] = G_3$	$[G_3, G_1] = -2G_2$
$A_{3,9}$	$[G_1, G_2] = G_3$	$[G_2, G_3] = G_1$	$[G_3, G_1] = G_2$

We ignore $3A_1$ and one representation of $A_{3,3}$ (*ie.* $A_{3,3}^{II}$) as the equations invariant under these Lie algebras are linear. As a result those equations admit larger classes of Lie algebras.

3.3.1 $A_{3,1}$

The algebra $A_{3,1}$ realises the equation

$$y''' = \Gamma[y''], \tag{3.62}$$

which admits the following three symmetries

$$\begin{aligned} G_1 &= \frac{\partial}{\partial y} && (3.63)\\ G_2 &= \frac{\partial}{\partial x} && (3.64)\\ G_3 &= x\frac{\partial}{\partial y}. && (3.65) \end{aligned}$$

Now, using each of the three symmetries, we reduce the canonical equation to second order equations.

Reduction via G_1 yields

$$u = x \qquad\qquad v = y' \tag{3.66}$$

$$\begin{aligned} y'' &= v' && (3.67)\\ y''' &= v''. && (3.68) \end{aligned}$$

Equation (3.62) becomes

$$v'' = \Gamma\left[v'\right]. \tag{3.69}$$

The reduction variables obtained via G_1 result in the symmetries G_2 and G_3 becoming

$$\begin{aligned} X_2 &= \frac{\partial}{\partial u} && (3.70)\\ X_3 &= \frac{\partial}{\partial v}. && (3.71) \end{aligned}$$

Since both X_2 and X_3 can be written in terms of u and v with a zero commutation relation, one can positively conclude that either of these two symmetries can be used in the further reduction of the new second order ordinary differential equation (3.69). More importantly, this allows us to reduce (3.69) (and hence (3.62)) to quadratures.

Reduction via G_2 yields

$$u = y \qquad\qquad v = y' \tag{3.72}$$

$$y'' = vv' \tag{3.73}$$

$$y''' = v^2 v'' + v(v')^2. \tag{3.74}$$

Thus (3.62) reduces to

$$v'' = -\frac{(v')^2}{v} + \frac{1}{v^2}\Gamma\left[vv'\right]. \tag{3.75}$$

From G_1, we obtain

$$X_1 = \frac{\partial}{\partial u} \tag{3.76}$$

but G_3 cannot not be written in terms of the new co-ordinates as a point symmetry.

Finally, considering G_3, we obtain

$$u = x \qquad\qquad v = y' - \frac{y}{x} \tag{3.77}$$

$$y'' = v' + \frac{v}{u} \tag{3.78}$$

$$y''' = v'' + \frac{v'}{u} - \frac{v}{u^2}. \tag{3.79}$$

The reduced equation is

$$v'' = \frac{v}{u^2} - \frac{v'}{u} + \Gamma\left[v' - \frac{v}{u}\right]. \tag{3.80}$$

G_1 takes on the new form

$$X_1 = \frac{1}{u}\frac{\partial}{\partial v}. \tag{3.81}$$

Once again, reduction results in the *loss* of a symmetry, this time G_2.

Thus, one can now see the extent to which solvability of a differential equation is dependent upon the order of the symmetries used in reduction. From the analysis above, it is now evident that the equation can only be reduced to quadratures if the third order equation is reduced initially by G_1. However, the two second order differential equations (3.75) and (3.80) are linked to (3.69). The nonlocal transformations between any two of these three equations are

$$\begin{aligned} u_2 &= \int v_1 \mathrm{d}u_1 \\ v_2 &= v_1, \end{aligned} \tag{3.82}$$

$$\begin{aligned} u_3 &= u_1 \\ v_3 &= v_1 - \frac{\int v_1 \mathrm{d}u_1}{u_1} \end{aligned} \tag{3.83}$$

and

$$\begin{aligned} u_3 &= \int \frac{1}{v_2} \mathrm{d}u_2 \\ v_3 &= v_2 - \frac{u_2}{\int \frac{1}{v_2} \mathrm{d}u_2}. \end{aligned} \tag{3.84}$$

Therefore, if one is given a second order equation of the form (3.75) or (3.80), one can increase its order by one, (thereby gaining a symmetry), and then reduce the third order differential equation via G_1. Alternatively, the transformations above can be used directly.

3.3.2 $A_{3,2}^I$

Consider the equation

$$y''' = (y'')^2 \Gamma \left[y'' \exp y' \right] \tag{3.85}$$

which admits the three symmetries

$$G_1 = \frac{\partial}{\partial y} \tag{3.86}$$

$$G_2 = \frac{\partial}{\partial x} \tag{3.87}$$

$$G_3 = x \frac{\partial}{\partial x} + (x + y) \frac{\partial}{\partial y}. \tag{3.88}$$

Reducing via G_1, we obtain

$$u = x \qquad\qquad v = y' \tag{3.89}$$

$$y'' = v' \tag{3.90}$$

$$y''' = v'' \tag{3.91}$$

and so (3.85) becomes

$$v'' = (v')^2 \Gamma \left[v' \exp v \right]. \tag{3.92}$$

Symmetries G_2 and G_3 transform to

$$X_2 = \frac{\partial}{\partial u} \tag{3.93}$$

$$X_3 = u\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \tag{3.94}$$

respectively. Thus, no symmetries have been *lost.*

Reducing via G_2, we have

$$u = y \qquad\qquad v = y' \tag{3.95}$$

$$y'' = vv' \tag{3.96}$$

$$y''' = v^2 v'' + v(v')^2. \tag{3.97}$$

Substituting into the original third order equation, we have

$$v'' = -\frac{(v')^2}{v} + (v')^2 \Gamma \left[vv' \exp v \right]. \tag{3.98}$$

G_1 and G_3 are now transformed to

$$X_1 = \frac{\partial}{\partial u} \tag{3.99}$$

$$X_3 = u\frac{\partial}{\partial u} + \frac{\partial}{\partial v}. \tag{3.100}$$

Once again reduction does not result in the *loss* of any of the symmetries.

Finally, reducing via G_3, we set

$$u = \frac{y}{x} - \log x \qquad\qquad v = y' - \log x. \tag{3.101}$$

Here

$$y'' = \left[\frac{y'}{x} - \frac{y}{x^2} - \frac{1}{x} \right] v' + \frac{1}{x} \tag{3.102}$$

$$y''' = \frac{(xy' - y - x)^2 v''}{x^4} + \frac{\left(y'' - \frac{1}{x}\right)(xy'' - 1)}{(xy' - y - x)} + \frac{1}{x^2} - \frac{2y''}{x}. \tag{3.103}$$

This implies that

$$v'' = \frac{(v')^2 + 2v'}{v-u-1} + \frac{1}{(v-u-1)^2} + \left[\frac{[(v-u-1)v'+1]^2}{(v-u-1)^2}\right]\Gamma\left[\exp v\,(v-u-1)\right]. \qquad (3.104)$$

Since G_1 as well as G_2 fail to be written in terms of the new coordinates, u and v, the second order differential equation (3.104) will *loose* both remaining symmetries.

The three second order equations are linked as follows:

$$\begin{aligned} u_2 &= \int v_1 \mathrm{d}u_1 \\ v_2 &= v_1, \end{aligned} \qquad (3.105)$$

$$\begin{aligned} u_3 &= \frac{\int v_1 \mathrm{d}u_1}{u_1} - \log u_1 \\ v_3 &= v_1 - \log u_1 \end{aligned} \qquad (3.106)$$

and

$$\begin{aligned} u_3 &= \frac{u_2}{\int \frac{1}{v_2}\mathrm{d}u_2} - \log\left[\int \frac{1}{v_2}\mathrm{d}u_2\right] \\ v_3 &= v_2 - \log\left[\int \frac{1}{v_2}\mathrm{d}u_2\right]. \end{aligned} \qquad (3.107)$$

Thus, given equation (3.104) which cannot be reduced to quadratures, one need only utilise the transformations given in (3.106) and/or (3.107) enabling this apparently unsolvable second order equation to now take on the form of (3.92) or (3.98). In both cases the new second order equations contain two symmetries and can be reduced to quadratures.

3.3.3 $A^{II}_{3,2}$

Here, we consider the equation

$$y''' = y''\Gamma\left[\exp x/y''\right] \qquad (3.108)$$

with the following symmetries

$$G_1 = -\frac{\partial}{\partial y} \tag{3.109}$$

$$G_2 = x\frac{\partial}{\partial y} \tag{3.110}$$

$$G_3 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{3.111}$$

Reducing via G_1, we utilise the substitution

$$u = x \qquad v = y' \tag{3.112}$$

Thus

$$y'' = v' \tag{3.113}$$

$$y''' = v''. \tag{3.114}$$

Substituting for y'' and y''' will reduce (3.108) to the form

$$v'' = v'\Gamma\left[\frac{\exp u}{v'}\right]. \tag{3.115}$$

G_2 and G_3 are now transformed to

$$X_2 = \frac{\partial}{\partial v} \tag{3.116}$$

$$X_3 = \frac{\partial}{\partial u} + v\frac{\partial}{\partial v}. \tag{3.117}$$

Hence, neither of the remaining symmetries will be *lost* via this reduction route.

Reducing via G_2 yields

$$u = x \qquad v = y' - \frac{y}{x}. \tag{3.118}$$

Here

$$y'' = v' - \frac{v}{u} \tag{3.119}$$

$$y''' = v'' + \frac{v'}{u} - \frac{v}{u^2} \tag{3.120}$$

which reduces the original equation to

$$v'' = -\frac{v'}{u} + \frac{v}{u^2} + \left(v' + \frac{v}{u}\right)\Gamma\left[\frac{\exp u}{v' + (v/u)}\right]. \tag{3.121}$$

Since G_3 cannot be expressed in terms of these co-ordinates, we consider only G_1, which now takes on the form

$$X_1 = \frac{1}{u}\frac{\partial}{\partial v}. \tag{3.122}$$

In the case of reduction via G_3,

$$u = \log y - x \qquad v = \frac{y'}{y}. \tag{3.123}$$

Thus

$$\begin{aligned} y'' &= (y'-y)\,v' - \frac{(y')^2}{y} & (3.124)\\ y''' &= \frac{(y'-y)^2}{y}v'' + \frac{((y'')^2 - y'y'')}{(y'-y)} + \frac{2y'y''}{y} - \frac{(y')^2\,(yy''-(y')^2)}{y^2\,(y'-y)} & (3.125)\end{aligned}$$

which leads to

$$v'' = -\frac{(v')^2}{(v-1)} - \frac{5v}{(v-1)}\left[\frac{v^2}{(v-1)} - v'\right] + \frac{[(v-1)\,v' - v^2]}{(v-1)^2}\Gamma\left[\frac{\exp(-u)}{(v-1)\,v' - v^2}\right]. \tag{3.126}$$

Neither G_1 nor G_2 can be expressed in terms of these co-coordinates, and so the resulting equation will be of second order with no point symmetries present, *ie*, an irreducible equation.

Finally, we consider the linking transformations:

$$\begin{aligned} u_2 &= u_1\\ v_2 &= v_1 - \frac{\int v_1 \mathrm{d}u_1}{u_1}, \end{aligned} \tag{3.127}$$

$$\begin{aligned} u_3 &= \log\left[\int v_1 \mathrm{d}u_1\right] - u_1\\ v_3 &= \frac{v_1}{\int v_1 \mathrm{d}u_1} \end{aligned} \tag{3.128}$$

and

$$\begin{aligned} u_3 &= \log\left[u_2 \int \frac{v_2}{u_2}\mathrm{d}u_2\right] - u_2\\ v_3 &= \frac{v_2 + \int (v_2/u_2)\mathrm{d}u_2}{u_2 \int (v_2/u_2)\mathrm{d}u_2}. \end{aligned} \tag{3.129}$$

Both (3.121) and (3.126) must take on the form of (3.115) to transform into a second order differential equation with two symmetries. The above transformations allow for this.

3.3.4 $A^I_{3,3}$

The equation associated with this Lie algebra is

$$y''' = (y'')^2\Gamma\left[y'\right] \tag{3.130}$$

and the symmetries associated with this equation are

$$G_1 = \frac{\partial}{\partial x} \tag{3.131}$$

$$G_2 = \frac{\partial}{\partial y} \tag{3.132}$$

$$G_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{3.133}$$

Now, reducing via G_1 yields

$$u = y \qquad\qquad v = y' \tag{3.134}$$

from which we obtain

$$y'' = vv' \tag{3.135}$$

$$y''' = v^2v'' + v(v')^2v'' \tag{3.136}$$

and it thus follows that

$$v'' = \frac{-(v')^2}{v} + (v')^2\Gamma\left[v\right]. \tag{3.137}$$

Here, both G_2 and G_3 can be reduced to symmetries of equation (3.137), *viz.*

$$X_2 = \frac{\partial}{\partial u} \tag{3.138}$$

$$X_3 = u\frac{\partial}{\partial u}. \tag{3.139}$$

Reducing (3.130) by G_2, *ie.* via

$$u = x \qquad\qquad v = y' \tag{3.140}$$

yields

$$y'' = v' \tag{3.141}$$

$$y''' = v'' \tag{3.142}$$

which gives the equation

$$v'' = (v')^2\Gamma[v]. \tag{3.143}$$

Once again, both G_1 and G_3 become point symmetries of the new second order differential equation as they transform as follows:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial u} && (3.144) \\ X_3 &= u\frac{\partial}{\partial u} && (3.145) \end{aligned}$$

Finally, we reduce the original equation by G_3. Here

$$u = \frac{y}{x} \qquad v = y'. \tag{3.146}$$

Now

$$\begin{aligned} y'' &= \frac{(xy'-y)}{x^2}v' && (3.147) \\ y''' &= \frac{(xy'-y)^2}{x^4}v'' + \frac{x(y'')^2}{(xy'-y)} - \frac{2y''}{x}. && (3.148) \end{aligned}$$

This results in

$$v'' = \frac{-(v')^2 + 2v'}{(v-u)} + (v')^2\Gamma[v]. \tag{3.149}$$

Since neither G_1 nor G_2 transform to symmetries in the new variables, the newly formed second order ordinary differential equation cannot be further reduced.

The transformations linking the second order equations are

$$\begin{aligned} u_1 &= \int v_2 \mathrm{d}u_2 \\ v_1 &= v_2, && (3.150) \end{aligned}$$

$$\begin{aligned} u_3 &= \frac{u_1}{\int \frac{1}{v_1}\mathrm{d}u_1} \\ v_3 &= v_1 && (3.151) \end{aligned}$$

and

$$\begin{aligned} u_3 &= \frac{\int v_2 \mathrm{d}u_2}{u_2} \\ v_3 &= v_2. && (3.152) \end{aligned}$$

Thus, if one is presented with an equation of the form (3.149), one need only perform the above transformations to convert it into a form that can be reduced to quadratures.

3.3.5 $A_1 \oplus A_2^I$

The equation invariant under this Lie algebra is

$$y''' = (y'')^{\frac{3}{2}}\Gamma\left[y''(y')^{-2}\right] \tag{3.153}$$

and the corresponding symmetries are as follows:

$$G_1 = \frac{\partial}{\partial x} \tag{3.154}$$

$$G_2 = \frac{\partial}{\partial y} \tag{3.155}$$

$$G_3 = x\frac{\partial}{\partial x}. \tag{3.156}$$

We first reduce (3.153) via G_1:

$$u = y \qquad v = y' \tag{3.157}$$

and so we find

$$y'' = vv' \tag{3.158}$$

$$y''' = v(v')^2 + v^2v'' \tag{3.159}$$

followed by

$$v'' = -\frac{(v')^2}{v} + \left(\frac{(v')^3}{v}\right)^{\frac{1}{2}}\Gamma\left[\frac{vv'}{u^2}\right]. \tag{3.160}$$

Both G_2 as well as G_3 can be transformed to point symmetries of (3.160), *viz.*

$$X_2 = \frac{\partial}{\partial u} \tag{3.161}$$

$$X_3 = v\frac{\partial}{\partial v} \tag{3.162}$$

respectively.

We now consider reduction via G_2:

$$u = x \qquad v = y'. \tag{3.163}$$

We have

$$y'' = v' \tag{3.164}$$
$$y''' = v'' \tag{3.165}$$

and

$$v'' = (v')^{\frac{3}{2}}\Gamma\left[\frac{v'}{v^2}\right] \tag{3.166}$$

follows.

Both remaining symmetries can be reduced to symmetries of the new equation, *ie.*

$$X_1 = \frac{\partial}{\partial u} \tag{3.167}$$
$$X_3 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}. \tag{3.168}$$

Continuing our investigations, we reduce via G_3. In this case,

$$u = y \qquad v = xy'. \tag{3.169}$$

Hence

$$y'' = \frac{y'}{x}(v' - 1) \tag{3.170}$$
$$y''' = \frac{(y')^2 v''}{x} - \frac{y''}{x} + \frac{(y'')^2}{y'}, \tag{3.171}$$

and

$$v'' = -\frac{(v')^2}{v} + \frac{3v'}{v} - \frac{2}{v} + \frac{(v'-1)^{\frac{3}{2}}}{v^{\frac{1}{2}}}\Gamma\left[\frac{v'-1}{v}\right]. \tag{3.172}$$

G_1 cannot be expressed in terms of the above coordinates. However G_2 will reduce to

$$X_2 = \frac{\partial}{\partial u}. \tag{3.173}$$

Now, let us investigate the links between the three transformations:

$$u_1 = \int v_2 \mathrm{d}u_2$$
$$v_1 = v_2, \tag{3.174}$$

$$
\begin{aligned}
u_3 &= u_1 \\
v_3 &= v_1 \int \frac{1}{v_1} \mathrm{d}u_1
\end{aligned}
\tag{3.175}
$$

and

$$
\begin{aligned}
u_3 &= \int v_2 \mathrm{d}u_2 \\
v_3 &= u_2 v_2.
\end{aligned}
\tag{3.176}
$$

Here, (3.172) can take on the form (3.160) or (3.166) via a nonlocal transformation and so can be solved.

3.3.6 $A_{3,4}^I$

Next we investigate the equation

$$y''' = (y'')^{\frac{4}{3}} \Gamma \left[y''(y')^{-\frac{3}{2}} \right] \tag{3.177}$$

with the three symmetries

$$G_1 = \frac{\partial}{\partial x} \tag{3.178}$$

$$G_2 = \frac{\partial}{\partial y} \tag{3.179}$$

$$G_3 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}. \tag{3.180}$$

Continuing as before, we begin with reduction via G_1:

$$u = y \qquad v = y'. \tag{3.181}$$

This yields

$$y'' = vv' \tag{3.182}$$

$$y''' = v^2 v'' + v(v')^2 \tag{3.183}$$

and therefore

$$v'' = -\frac{(v')^2}{v} + \left(\frac{(v')^2}{v}\right)^{\frac{2}{3}} \Gamma\left[\frac{v'}{v^{\frac{1}{2}}}\right]. \tag{3.184}$$

Here, the symmetries G_2 and G_3 become

$$X_2 = \frac{\partial}{\partial u} \tag{3.185}$$

$$X_3 = u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v}. \tag{3.186}$$

Thus, neither G_2 nor G_3 are *lost* during reduction.

Reducing via G_2 yields

$$u = x \qquad v = y'. \tag{3.187}$$

Thus

$$y'' = v' \tag{3.188}$$

$$y''' = v'' \tag{3.189}$$

are easily determined, and therefore

$$v'' = (v')^{\frac{4}{3}}\Gamma\left[\frac{v'}{v^{\frac{3}{2}}}\right]. \tag{3.190}$$

Again, we find both G_1 and G_3 to be symmetries of the reduced equation

$$X_1 = \frac{\partial}{\partial u} \tag{3.191}$$

$$X_3 = u\frac{\partial}{\partial u} - 2v\frac{\partial}{\partial v}. \tag{3.192}$$

Finally, reducing via G_3, we obtain

$$u = xy \qquad v = \frac{y'}{y^2}. \tag{3.193}$$

As a result,

$$y'' = y^2 (xy' + y)\, v' + \frac{2(y')^2}{y} \tag{3.194}$$

$$y''' = y^2 (xy' + y)^2 v'' + \frac{(yy'' - 2(y')^2)(xy'' + 2y')}{y(xy' + y)} + \frac{6y'y''}{y} - \frac{(y')^3}{y^2}. \tag{3.195}$$

Using (3.193), (3.194) and (3.195), (3.177) reduces to

$$v'' = -\frac{u(v')^2 + 2vv' - 6}{uv+1} + \frac{18v^3}{(uv+1)^2} + \frac{((uv+1)\,v' + 2v^2)^{\frac{4}{3}}}{(uv+1)^2}\Gamma\left[\left(\frac{u}{v^{\frac{1}{2}}} + \frac{1}{v^{\frac{3}{2}}}\right)v' + 2v^{\frac{1}{2}}\right]. \quad (3.196)$$

Reduction via G_3 results in the *loss* of both other symmetries.

In conclusion of this algebra, let us once again consider the transformations between the second order differential equations obtained from the reduction via the three symmetries:

$$\begin{aligned} u_1 &= \int v_2 \mathrm{d}u_2 \\ v_1 &= v_2, \end{aligned} \quad (3.197)$$

$$\begin{aligned} u_3 &= u_1 \int \frac{1}{v_1} \mathrm{d}u_1 \\ v_3 &= \frac{v_1}{u_1^2} \end{aligned} \quad (3.198)$$

and

$$\begin{aligned} u_3 &= u_2 \int v_2 \mathrm{d}u_2 \\ v_3 &= \frac{v_2}{\left(\int v_2 \mathrm{d}u_2\right)^2}. \end{aligned} \quad (3.199)$$

Transforming the second order differential equation (3.196) to the form of (3.184) or (3.190) will increase the number of symmetries it contains to two. Reduction to quadratures is now possible.

3.3.7 $A_{3,5}^{aI}$

In this group $a \neq 1, 2$ is a constant. The relevant equation is defined as follows

$$y''' = (y'')^{\frac{a-3}{a-2}}\Gamma\left[y''(y')^{\frac{2-a}{a-1}}\right] \qquad (0 < |a| < 1) \quad (3.200)$$

and can be reduced via its three symmetries

$$G_1 = \frac{\partial}{\partial x} \quad (3.201)$$

$$G_2 = \frac{\partial}{\partial y} \quad (3.202)$$

$$G_3 = x\frac{\partial}{\partial x} + ay\frac{\partial}{\partial y}. \quad (3.203)$$

Let us begin by reducing via G_1:

$$u = y \qquad v = y'. \tag{3.204}$$

Thus

$$y'' = vv' \tag{3.205}$$

$$y''' = v^2 v'' + v(v')^2, \tag{3.206}$$

and so

$$v'' = -\frac{v'}{v} + \frac{(vv')^{\frac{a-3}{a-2}}}{v^2}\Gamma\left[v' v^{\frac{1}{a-1}}\right]. \tag{3.207}$$

G_2 and G_3 remain symmetries of (3.207), transforming to

$$X_2 = \frac{\partial}{\partial u} \tag{3.208}$$

$$X_3 = au\frac{\partial}{\partial u} + (a-1)\,v\frac{\partial}{\partial v} \tag{3.209}$$

respectively.

Reducing via G_2 we now have

$$u = x \qquad v = y'. \tag{3.210}$$

From this transformation we obtain

$$y'' = v' \tag{3.211}$$

$$y''' = v'' \tag{3.212}$$

and therefore

$$v'' = (v')^{\frac{a-3}{a-2}}\Gamma\left[v' v^{\frac{2-a}{a-1}}\right]. \tag{3.213}$$

As before G_1 and G_3 can be transformed to become symmetries of (3.213), *viz.*

$$X_1 = \frac{\partial}{\partial u} \tag{3.214}$$

$$X_3 = u\frac{\partial}{\partial u} + (a-1)\,v\frac{\partial}{\partial v} \tag{3.215}$$

Let us finally consider the case in which reduction occurs via G_3. Here,

$$u = \frac{y}{x^a} \qquad v = \frac{y'}{x^{a-1}}. \tag{3.216}$$

Thus

$$\begin{aligned} y'' &= \frac{(xy' - ay)\, v'}{x^2} + \frac{(a-1)\, y'}{x} & (3.217) \\ y''' &= \frac{(x^a y' - a x^{a-1} y)\,(xy' - ay)\, v''}{x^{2(a+1)}} + \frac{(xy'' - (a-1)\, y')^2}{x\,(xy' - ay)} + \frac{(a-1)\, y'}{x^2} \\ &- \frac{(2x - (a-1)\, x)\, y''}{x^2}. & (3.218) \end{aligned}$$

Substituting into the third order equation yields

$$\begin{aligned} v'' &= -\frac{(a-2)\, v'}{v - au} - \frac{(a-2)\,(a-1)\, v}{(v-au)^2} \\ &+ \frac{[(v-au)\, v' + v\,(a-1)]^{\frac{a-3}{a-2}}}{(v-au)^2} \Gamma \left[(v-u)\, v' v^{\frac{2-a}{a-1}} + (a-1)\, v^{1-a}\right]. & (3.219) \end{aligned}$$

Neither G_1 nor G_2 can be rewritten in terms of variables of (3.219) and are therefore *lost* when the equation is reduced using G_3.

As before we consider the linking transformations:

$$\begin{aligned} u_1 &= \int v_2 \mathrm{d}u_2 \\ v_1 &= v_2, & (3.220) \end{aligned}$$

$$\begin{aligned} u_3 &= \frac{u_1}{\left(\int \frac{1}{v_1} \mathrm{d}u_1\right)^a} \\ v_3 &= \frac{v_1}{\left(\int \frac{1}{v_1} \mathrm{d}u_1\right)^{a-1}} & (3.221) \end{aligned}$$

and

$$\begin{aligned} u_3 &= \frac{\int v_2 \mathrm{d}u_2}{u_1^a} \\ v_3 &= \frac{v_2}{u_1^{a-1}}. & (3.222) \end{aligned}$$

We can use (3.221) and/or (3.222) to transform (3.219) to the form (3.207) or (3.213) in an attempt to further reduce the unsolvable second order equation.

3.3.8 $A_{3,5}^{\frac{1}{2}I}$

Here, we consider the equation

$$y'y''' = \Gamma\left[y''\right] \tag{3.223}$$

for which the following symmetries can be found:

$$G_1 = \frac{\partial}{\partial x} \tag{3.224}$$

$$G_2 = \frac{\partial}{\partial y} \tag{3.225}$$

$$G_3 = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}. \tag{3.226}$$

Following the now familiar route, we begin by reducing the equation (3.223) via the symmetry G_1:

$$u = y \qquad v = y'. \tag{3.227}$$

Thus

$$y'' = vv' \tag{3.228}$$

$$y''' = v^2v'' + v(v')^2 \tag{3.229}$$

which reduces the third order differential equation to

$$v'' = -\frac{(v')^2}{v} + \frac{1}{v^3}\Gamma\left[vv'\right]. \tag{3.230}$$

The remaining symmetries become

$$X_2 = \frac{\partial}{\partial u} \tag{3.231}$$

$$X_3 = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}. \tag{3.232}$$

Therefore (3.230) has two point symmetries and so can be further reduced to quadratures.

To reduce (3.223) via G_2 we set

$$u = x \qquad v = y' \tag{3.233}$$

from which

$$y'' = v' \tag{3.234}$$

$$y''' = v''. \tag{3.235}$$

When substituted into (3.223) this results in

$$v'' = \frac{1}{v}\Gamma\left[v'\right]. \tag{3.236}$$

The symmetries G_1 and G_3 transform to

$$X_1 = \frac{\partial}{\partial u} \tag{3.237}$$

$$X_3 = u\frac{\partial}{\partial u} - \frac{1}{2}v\frac{\partial}{\partial v} \tag{3.238}$$

and so we do not *loose* any symmetries.

In the last reduction we use the symmetry G_3 which implies

$$u = \frac{y}{x^2} \qquad v = \frac{y'}{x}. \tag{3.239}$$

It follows that

$$y'' = \frac{(xy' - 2y)\,v'}{x^2} + \frac{y'}{x} \tag{3.240}$$

$$y''' = \frac{(xy' - 2y)^2\,v''}{x^5} + \frac{(xy'' - y')^2}{x\,(xy' - 2y)} - \frac{y''}{x} + \frac{y'}{x^2} \tag{3.241}$$

and thus

$$v'' = -\left(v' - 1\right)v' + \frac{1}{v\,(v - 2u)}\Gamma\left[(v - 2u)\,v' + v\right]. \tag{3.242}$$

Neither of the two remaining symmetries, namely G_1 nor G_2, can be rewritten in terms of these new coordinates and therefore reduction via G_3 cannot be the optimal method for solving the differential equation (3.223).

With respect to the linking expressions, we have

$$\begin{aligned} u_1 &= \int v_2 \mathrm{d}u_2 \\ v_1 &= v_2, \end{aligned} \tag{3.243}$$

$$
\begin{aligned}
u_3 &= \frac{u_1}{\left(\int \frac{1}{v_1} du_1\right)^2} \\
v_3 &= \frac{v_1}{\int \frac{1}{v_1} du_1}
\end{aligned} \tag{3.244}
$$

and

$$
\begin{aligned}
u_3 &= \frac{\int v_2 du_2}{u_2^2} \\
v_3 &= \frac{v_2}{u_2}.
\end{aligned} \tag{3.245}
$$

In order to solve an equation of the form (3.242) one of the transformations above should be used. This second order differential equation should then take on the form (3.230) or (3.236).

3.3.9 $A_1 \oplus A_2^{II}$

The equation relevant to this algebra is

$$
y''' = (y'')^2 \Gamma\left[xy''\right] \tag{3.246}
$$

and the symmetries are as follows:

$$
G_1 = \frac{\partial}{\partial y} \tag{3.247}
$$

$$
G_2 = x\frac{\partial}{\partial y} \tag{3.248}
$$

$$
G_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{3.249}
$$

To begin reduce (3.246) via G_1:

$$
u = x \qquad v = y'. \tag{3.250}
$$

Therefore

$$
y'' = v' \tag{3.251}
$$

$$
y''' = v'' \tag{3.252}
$$

and

$$v'' = (v')^2\Gamma\left[uv'\right]. \tag{3.253}$$

G_2 as well as G_3 can be written as symmetries of (3.253) and now take on the form

$$X_2 = \frac{\partial}{\partial v} \tag{3.254}$$

$$X_3 = u\frac{\partial}{\partial u}. \tag{3.255}$$

Reducing via G_2 yields

$$u = x \qquad v = y' - \frac{y}{x}. \tag{3.256}$$

Thus,

$$y'' = v' + \frac{v}{u} \tag{3.257}$$

$$y''' = v'' + \frac{v'}{u} - \frac{v}{u^2} \tag{3.258}$$

and

$$v'' = -\frac{v'}{u} + \frac{v}{u^2} + \left(v' + \frac{v}{u}\right)^2 \Gamma\left[uv' + v\right]. \tag{3.259}$$

Both G_1 and G_3 can be reduced to symmetries of (3.259), *viz.*

$$X_1 = \frac{1}{u}\frac{\partial}{\partial v} \tag{3.260}$$

$$X_3 = u\frac{\partial}{\partial u} \tag{3.261}$$

Thirdly, we reduce via the final symmetry G_3. Here

$$u = \frac{y}{x} \qquad v = y' \tag{3.262}$$

and therefore

$$y'' = \frac{(xy' - y)\,v'}{x^2} \tag{3.263}$$

$$y''' = \frac{(xy' - y)^2\,v''}{x^4} + \frac{x(y'')^2}{xy' - y} - \frac{2y''}{x}. \tag{3.264}$$

Substituting into (3.246) we obtain

$$v'' = \frac{-(v')^2 + 2v'}{v - u} + (v')^2\Gamma\left[(v - u)\,v'\right]. \tag{3.265}$$

G_1 cannot be written in terms of u and v. Only G_2 can be reduced to a point symmetry of the reduced equation, *viz.*

$$X_2 = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}. \tag{3.266}$$

The transformations linking the three second order equations are

$$\begin{aligned} u_2 &= u_1 \\ v_2 &= v_1 - \frac{\int v_1 du_1}{u_1}, \end{aligned} \tag{3.267}$$

$$\begin{aligned} u_3 &= \int v_1 du_1 \\ v_3 &= v_1 \end{aligned} \tag{3.268}$$

and

$$\begin{aligned} u_3 &= \int \frac{v_2}{u_2} du_2 \\ v_3 &= v_2 + \int \frac{v_2}{u_2} du_2. \end{aligned} \tag{3.269}$$

From this we observe that the easiest way in which (3.265) can be solved is by transforming it to the form (3.253) or (3.259).

3.3.10 $A_{3,4}^{II}$

The equation invariant under this Lie algebra is

$$y''' = (y'')^{\frac{5}{3}} \Gamma\left[x^{\frac{3}{2}} y''\right] \tag{3.270}$$

and the associated symmetries are

$$G_1 = \frac{\partial}{\partial y} \tag{3.271}$$

$$G_2 = x\frac{\partial}{\partial y} \tag{3.272}$$

$$G_3 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{3.273}$$

Reducing first with respect to G_1, we find

$$u = x \qquad v = y' \tag{3.274}$$

and so

$$y'' = v' \tag{3.275}$$
$$y''' = v'', \tag{3.276}$$

from which it follows that

$$v'' = (v')^{\frac{5}{3}}\Gamma\left[u^{\frac{3}{2}}v'\right]. \tag{3.277}$$

Both G_2 and G_3 can be reduced to symmetries of (3.277), *viz.*

$$X_2 = \frac{\partial}{\partial v} \tag{3.278}$$
$$X_3 = 2u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}. \tag{3.279}$$

Now, let us consider reduction via G_2:

$$u = x \qquad v = y' - \frac{y}{x}. \tag{3.280}$$

Thus

$$y'' = v' + \frac{v}{u} \tag{3.281}$$
$$y''' = v'' + \frac{v}{u} - \frac{v}{u^2} \tag{3.282}$$

which reduces (3.270) to the second order differential equation

$$v'' = -\frac{v'}{u} + \frac{v}{u^2} + \left(v' + \frac{v}{u}\right)^{\frac{5}{3}}\Gamma\left[u^{\frac{3}{2}}v' + u^{\frac{1}{2}}v\right]. \tag{3.283}$$

Neither of the remaining symmetries are *lost*, *ie.* they transform to

$$X_1 = \frac{1}{u}\frac{\partial}{\partial v} \tag{3.284}$$
$$X_3 = 2u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} \tag{3.285}$$

Finally, reducing via G_3:

$$u = \frac{y^2}{x} \qquad v = yy'. \tag{3.286}$$

Hence

$$y'' = \frac{(2xy' - y)\, v'}{x^2} + \frac{(y')^2}{y} \tag{3.287}$$

$$y''' = \frac{(2xyy' - y^2)^2 v''}{x^4 y} - \frac{2\,(yy'' - (y')^2)}{xy} - \frac{2x\,((y')^4 - y^2(y'')^2)}{y\,(2xyy' - y^2)} + \frac{y'y''}{y} \tag{3.288}$$

and therefore

$$\begin{aligned} v'' = & -\frac{2(v')^2}{2v-u} - \frac{(2u^2 - 5uv + 6v^2)\, v'}{u\,(2v-u)^2} - \frac{v^3}{u^2\,(2v-u)^2} \\ & + \frac{\left[(2v-u)\, v' + \frac{v^2}{u}\right]^{\frac{5}{3}}}{u^{\frac{1}{3}}\,(2v-u)^2}\Gamma\left[\left(\frac{2v - u^{\frac{1}{2}}}{u^{\frac{1}{2}}}\right) v' + \frac{v^2}{u^{\frac{3}{2}}}\right]. \end{aligned} \tag{3.289}$$

G_1 and G_2 cannot be transformed into symmetries of (3.289).

Now, let us consider the linking expressions:

$$\begin{aligned} u_2 &= u_1 \\ v_2 &= v_1 - \frac{\int v_1 \mathrm{d}u_1}{u_1}, \end{aligned} \tag{3.290}$$

$$\begin{aligned} u_3 &= \frac{\left(\int v_1 \mathrm{d}u_1\right)^2}{u_1} \\ v_3 &= v_1 \int v_1 \mathrm{d}u_1 \end{aligned} \tag{3.291}$$

and

$$\begin{aligned} u_3 &= \frac{\left(u_2 \int v_2/u_2 \mathrm{d}u_2\right)^2}{u_2} \\ v_3 &= \left(v_2 + \int v_2/u_2 \mathrm{d}u_2\right)\left(u_2 \int \frac{v_2}{u_2} \mathrm{d}u_2\right). \end{aligned} \tag{3.292}$$

We convert an equation of the form of (3.289) to either (3.277) or (3.283) in an attempt to solve it.

3.3.11 $A_{3,5}^{II}$

Here, we have the equation

$$y''' = (y'')^{\frac{2-3a}{1-2a}}\Gamma\left[x^{\frac{2a-1}{a-1}} y''\right] \qquad (0 < |a| < 1) \tag{3.293}$$

with its three symmetries

$$G_1 = \frac{\partial}{\partial y} \tag{3.294}$$
$$G_2 = x\frac{\partial}{\partial y} \tag{3.295}$$
$$G_3 = (1-a)\,x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \tag{3.296}$$

where $a \neq 1, \frac{1}{2}$ is a constant.

We begin with reduction via G_1:

$$u = x \qquad\qquad v = y'. \tag{3.297}$$

Hence

$$y'' = v' \tag{3.298}$$
$$y''' = v'' \tag{3.299}$$

and (3.293) is reduced to

$$v'' = (v')^{\frac{2-3a}{1-2a}}\Gamma\left[u^{\frac{2a-1}{a-1}}v'\right]. \tag{3.300}$$

Here, the remaining symmetries transform to

$$X_2 = \frac{\partial}{\partial v} \tag{3.301}$$
$$X_3 = (1-a)\,u\frac{\partial}{\partial u} + av\frac{\partial}{\partial v} \tag{3.302}$$

and thus (3.300) contains two symmetries via which it can be reduced to quadratures.

Reducing via G_2 yields

$$u = x \qquad\qquad v = y' - \frac{y}{x}. \tag{3.303}$$

Thus

$$y'' = v' + \frac{v}{u} \tag{3.304}$$
$$y''' = v'' + \frac{v'}{u} - \frac{v}{u^2} \tag{3.305}$$

and

$$v'' = -\frac{v'}{u} + \frac{v}{u^2} + \left(v' - \frac{v}{u}\right)^{\frac{2-3a}{1-2a}}\Gamma\left[u^{\frac{2a-1}{a-1}}v' + vu^{\frac{a}{a-1}}\right]. \tag{3.306}$$

As before, neither of the remaining symmetries are *lost*:

$$X_1 = \frac{1}{u}\frac{\partial}{\partial v} \tag{3.307}$$

$$X_3 = (1-a)\,u\frac{\partial}{\partial u} + av\frac{\partial}{\partial v} \tag{3.308}$$

Finally, reducing via G_3

$$u = \frac{y^{1-a}}{x} \qquad v = \frac{y'}{y^a}. \tag{3.309}$$

Hence

$$y'' = \frac{(xy'(1-a)-y)\,v'}{x^2} + \frac{a(y')^2}{y} \tag{3.310}$$

$$y''' = \frac{\left(x(1-a)\frac{y'}{y^a} - y^{1-a}\right)(xy'(1-a)-y)\,v''}{x^4} + \frac{\left(x^2y'' - \frac{ax^2(y')^2}{y}\right)(xy''(1-a)+ay')}{x^2(xy'(1-a)-y)} - \frac{2y''}{x} + \frac{2a(y')^2}{xy} + \frac{2ay'y''}{y} - \frac{a(y')^3}{y^2} \tag{3.311}$$

and

$$v'' = -\frac{(1-a)(v')^2}{[(1-a)v-u]} - \frac{[(1-a)(3av^2+2u^2(u-v)) - au^2v]\,v'}{u^2[(1-a)v-u]^2} - \frac{av^3(2a-1)}{u^2[(1-a)v-u]^2} + \frac{[(1-a)uvv'+u^2v'+av^2]^{\frac{2-3a}{1-2a}}}{u^2[(1-a)v-u]^2}\Gamma\left[vv'(1-a)u^{\frac{a}{1-a}} - u^{\frac{1}{1-a}}v' + av^2u^{\frac{2a-1}{1-a}}\right]. \tag{3.312}$$

Both symmetries are *lost*.

We now examine the linking transformations:

$$u_2 = u_1$$
$$v_2 = v_1 - \frac{\int v_1 \mathrm{d}u_1}{u_1}, \tag{3.313}$$

$$u_3 = \frac{\left[\int v_1 \mathrm{d}u_1\right]^{1-a}}{u_1}$$
$$v_3 = \frac{v_1}{\left[\int v_1 \mathrm{d}u_1\right]^a} \tag{3.314}$$

and

$$u_3 = \frac{\left[u_2\int \frac{v_2}{u_2}\mathrm{d}u_2\right]^{1-a}}{u_2}$$
$$v_3 = \frac{v_2 + \int \frac{v_2}{u_2}\mathrm{d}u_2}{\left[u_2\int \frac{v_2}{u_2}\mathrm{d}u_2\right]^a}. \tag{3.315}$$

Equation (3.312) can be further reduced if it is transformed to either (3.300) or alternatively to (3.306).

3.3.12 $A_{3,5}^{\frac{1}{2}II}$

Consider

$$y''' = \frac{1}{x}\Gamma\left[y''\right] \tag{3.316}$$

with the following symmetries

$$G_1 = \frac{\partial}{\partial y} \tag{3.317}$$

$$G_2 = x\frac{\partial}{\partial y} \tag{3.318}$$

$$G_3 = \frac{1}{2}x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{3.319}$$

Reducing via G_1 yields

$$u = x \qquad\qquad v = y'. \tag{3.320}$$

Thus

$$y'' = v' \tag{3.321}$$

$$y''' = v''. \tag{3.322}$$

Substituting into (3.316), we obtain

$$v'' = \frac{1}{u}\Gamma\left[v'\right]. \tag{3.323}$$

Both G_2 as well as G_3 become symmetries of equation (3.323), *viz.*

$$X_2 = \frac{\partial}{\partial v} \tag{3.324}$$

$$X_3 = \frac{1}{2}u\frac{\partial}{\partial u} + \frac{1}{2}v\frac{\partial}{\partial v}. \tag{3.325}$$

Now we reduce via G_2 by performing the substitutions

$$u = x \qquad\qquad v = y' - \frac{y}{x} \tag{3.326}$$

$$y'' = v' + \frac{v}{u} \tag{3.327}$$

$$y''' = v'' + \frac{v'}{u} - \frac{v}{u^2}. \tag{3.328}$$

Therefore

$$v'' = -\frac{v'}{u} + \frac{v}{u^2} + \frac{1}{u}\Gamma\left[v' + \frac{v}{u}\right]. \tag{3.329}$$

Rewriting G_1 and G_2 in terms of the new coordinates, we obtain

$$X_1 = \frac{1}{u}\frac{\partial}{\partial v} \tag{3.330}$$

$$X_3 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}. \tag{3.331}$$

Methodically, we now reduce via G_3, where

$$u = \frac{y}{x^2} \qquad v = \frac{y'}{x}. \tag{3.332}$$

Now,

$$y'' = (v - 2u)\, v' + v \tag{3.333}$$

$$y''' = \frac{(xy' - 2y)^2\, v''}{x^5} + \frac{(xy'' - y')^2}{x\,(xy' - 2y)} + \frac{y'}{x^2} - \frac{y''}{x}. \tag{3.334}$$

Hence

$$v'' = \frac{(-(v')^2 + v')}{(v - 2u)} + \frac{1}{(v - 2u)^2}\Gamma\left[(v - 2u)\, v' + v\right]. \tag{3.335}$$

Neither of the remaining symmetries can be rewritten in terms of the new coordinates.

Let us now investigate the linking expressions:

$$\begin{aligned} u_2 &= u_1 \\ v_2 &= v_1 - \frac{\int v_1 \mathrm{d}u_1}{u_1}, \end{aligned} \tag{3.336}$$

$$\begin{aligned} u_3 &= \frac{\left[\int v_2 \mathrm{d}u_2\right]^{\frac{1}{2}}}{u_2} \\ v_3 &= \frac{v_2}{\left[\int v_2 \mathrm{d}u_2\right]^{\frac{1}{2}}} \end{aligned} \tag{3.337}$$

and

$$\begin{aligned} u_3 &= \frac{\left[u_2 \int (v_2/u_2) \mathrm{d}u_2\right]^{\frac{1}{2}}}{u_2} \\ v_3 &= \frac{v_2 + \int (v_2/u_2) \mathrm{d}u_2}{\left[u_2 \int (v_2/u_2) \mathrm{d}u_2\right]^{\frac{1}{2}}}. \end{aligned} \tag{3.338}$$

In this case, equations of the form (3.335) must be transformed to second order equations of the forms (3.323) or (3.329) in order to be reduced to quadratures.

3.3.13 $A^I_{3,6}$

The relevant equation for this algebra is

$$y''' = \frac{3y'(y'')^2}{1+(y')^2} + \left(1+(y')^2\right)^2 \Gamma\left[\frac{y''}{(1+(y')^2)^{\frac{3}{2}}}\right] \tag{3.339}$$

with the corresponding symmetries

$$G_1 = \frac{\partial}{\partial x} \tag{3.340}$$

$$G_2 = \frac{\partial}{\partial y} \tag{3.341}$$

$$G_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{3.342}$$

Reducing via G_1, we obtain

$$u = y \qquad v = y'. \tag{3.343}$$

Thus

$$y'' = vv' \tag{3.344}$$

$$y''' = v^2 v'' + v(v')^2. \tag{3.345}$$

Substituting into (3.339) yields

$$v'' = -\frac{(v')^2}{v} + \frac{3v^3(v')^2}{v^2(1+v^2)} + \frac{(1+v^2)^2}{v^2}\Gamma\left[\frac{vv'}{(1+v^2)^{\frac{3}{2}}}\right]. \tag{3.346}$$

G_1 and G_3 transform to

$$X_2 = \frac{\partial}{\partial u} \tag{3.347}$$

$$X_3 = \left(1+v^2\right)\frac{\partial}{\partial v}. \tag{3.348}$$

Thus, for the reduction of equation (3.346) one can use the symmetries above.

Now consider G_2. The reduction variables are

$$u = x \qquad v = y' \tag{3.349}$$

from which we find

$$\begin{aligned} y'' &= v' && (3.350)\\ y''' &= v'' && (3.351) \end{aligned}$$

and thus

$$v'' = \frac{3v(v')^2}{1+v^2} + \left(1+v^2\right)^2 \Gamma\left[\frac{v'}{(1+v^2)^{\frac{3}{2}}}\right] \tag{3.352}$$

is the reduced equation.

Both G_1 and G_3 can now be written in terms of the new variables, *ie.*

$$\begin{aligned} X_1 &= \frac{\partial}{\partial u} && (3.353)\\ X_3 &= \left(1+v^2\right)\frac{\partial}{\partial v}. && (3.354) \end{aligned}$$

To complete this group, we reduce (3.339) via G_3:

$$\begin{aligned} u &= x^2+y^2 && (3.355)\\ v &= \frac{yy'+x}{y-xy'}. && (3.356) \end{aligned}$$

Thus

$$\begin{aligned} y'' &= \frac{(-y+xy')\left(1-2xyv'+2\left(x^2+y^2\right)y'v'+\left(1+2xyv'\right)y'^2\right)}{x^2+y^2} && (3.357)\\ y''' &= -\frac{1}{x^2+y^2}\left[4\left(y-xy'\right)^2\left(x+yy'\right)^2\left(-v''\right.\right.\\ &\quad +\left(-\left(y-xy'\right)^2\left(1+y'^2\right)^2+2\left(y-xy'\right)\left(x+y-\left(x-y\right)y'\right)\left(x-y+\left(x+y\right)y'\right)y''\right.\\ &\quad \left.\left.\left.+\left(x^2+y^2\right)\left(2x^2-y^2+3xyy'\right)y''^2\right)/\left(4\left(y-xy'\right)^3\left(x+yy'\right)^3\right)\right)\right] && (3.358) \end{aligned}$$

and therefore

$$v'' = \frac{\left(2v^2-1\right)v'^2}{v\left(1+v^2\right)} - \frac{v'}{u} + \frac{\left(1+v^2\right)\left(-v+u\left(1+v^2\right)\Gamma\left[\frac{2uvv'-v^2-1}{\left(u(1+v^2)^3\right)^{1/2}}\right]\right)}{4u^2v^2} \tag{3.359}$$

is the reduced equation. Here, both of the remaining symmetries are lost.

We now move our attention to the linking expressions:

$$\begin{aligned} u_1 &= \int v_2 \mathrm{d}u_2 \\ v_1 &= v_2, \end{aligned} \tag{3.360}$$

$$\begin{aligned} u_3 &= \left(\int \frac{1}{v_1}\mathrm{d}u_1\right)^2 + u_1^2 \\ v_3 &= \frac{u_1 v_1 + \int (1/v_1)\mathrm{d}u_1}{u_1 - v_1 \int (1/v_1)\mathrm{d}u_1} \end{aligned} \tag{3.361}$$

and

$$\begin{aligned} u_3 &= u_2^2 + \left[\int v_2 \mathrm{d}u_2\right]^2 \\ v_3 &= \frac{v_1 \int v_1 \mathrm{d}u_1 + u_1}{\int v_1 \mathrm{d}u_1 - u_1 v_1}. \end{aligned} \tag{3.362}$$

To solve an equation of the form (3.359), we must transform it using the above expressions into either (3.346) or (3.352).

3.3.14 $A^{bI}_{3,7}$

We next turn our attention to the third order equation invariant under $A^{bI}_{3,7}$

$$y''' = \frac{3y'(y'')^2}{1+(y')^2} + \left(1+(y')^2\right)^2 \exp\left(2b\tan^{-1} y'\right) \Gamma\left[\frac{y''}{(1+(y')^2)^{\frac{3}{2}}\left(\exp\left(b\tan^{-1} y'\right)\right)}\right] \tag{3.363}$$

which admits the symmetries

$$G_1 = \frac{\partial}{\partial x} \tag{3.364}$$

$$G_2 = \frac{\partial}{\partial y} \tag{3.365}$$

$$G_3 = (bx+y)\frac{\partial}{\partial x} + (by-x)\frac{\partial}{\partial y} \tag{3.366}$$

where $b > 0$ is a constant.

Reducing via G_1:

$$u = y \qquad v = y' \tag{3.367}$$

and so,

$$\begin{aligned} y'' &= vv' && (3.368)\\ y''' &= v^2 v'' + v(v')^2. && (3.369)\end{aligned}$$

Hence

$$v'' = -\frac{(1-2v^2)(v')^2}{1+v^2} + \frac{(1+v^2)^2 \exp(2b\tan^{-1} v)}{v}\Gamma\left[\frac{vv'\exp(-b\tan^{-1} v)}{(1+v^2)^{\frac{3}{2}}}\right] \tag{3.370}$$

is the reduced equation.

G_3 cannot be rewritten in terms of the new coordinates; only G_2 is transformed into a symmetry of equation (3.370), *viz.*

$$X_2 = \frac{\partial}{\partial u}. \tag{3.371}$$

Now reducing via G_2 yields

$$u = x \qquad v = y'. \tag{3.372}$$

Thus we have

$$\begin{aligned} y'' &= v' && (3.373)\\ y''' &= v'' && (3.374)\end{aligned}$$

and the new reduced equation is

$$v'' = \frac{3v(v')^2}{1+v^2} + \left(1+v^2\right)^2 \exp\left(2b\tan^{-1} v\right)\Gamma\left[\frac{v'\exp(-b\tan^{-1} v)}{(1+v^2)^{\frac{3}{2}}}\right]. \tag{3.375}$$

In this case, neither of the remaining symmetries are *lost*:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial u} && (3.376)\\ X_3 &= bu\frac{\partial}{\partial u} - \left(1+v^2\right)\frac{\partial}{\partial v}. && (3.377)\end{aligned}$$

Finally, we consider reduction via G_3, with

$$\begin{aligned} u &= \frac{1}{2}\log\left[x^2+y^2\right] + b\tan^{-1}\frac{y}{x} && (3.378)\\ v &= \frac{1}{b}\log\left[x^2+y^2\right] + 2\tan^{-1} y'. && (3.379)\end{aligned}$$

It follows that

$$y'' = \frac{[(x+yy')+b(xy'-y)](1+(y')^2)v'}{2(x^2+y^2)} - \frac{(x+yy')(1+(y')^2)}{b(x^2+y^2)} \tag{3.380}$$

$$\begin{aligned} y''' = & \left[\frac{(x+yy')+b(xy'-y)}{x^2+y^2}\right]^2 \frac{(1+(y')^2)v''}{2} - \frac{(1+(y')^2+yy'')(1+(y')^2)}{b(x^2+y^2)} \\ & + \frac{2(x+yy')^2(1+(y')^2)}{b(x^2+y^2)^2} + \frac{2y'(y'')^2}{(1+(y')^2)} \\ & - \left[\frac{x+yy'}{b(x^2+y^2)} + \frac{y''}{1+(y')^2}\right]\left[\frac{2(x+yy')}{x^2+y^2} - \frac{(1+(y')^2+y''(y+bx))(1+(y')^2)}{((x+yy')+b(xy'-y))}\right]. \end{aligned} \tag{3.381}$$

Equation (3.363) is now reduced to the second order differential equation

$$\begin{aligned} v'' = & -\frac{\left(b-\tan\left[\frac{v}{2}-\frac{u}{b}\right]\right)(v')^2}{2\left(1+b\tan\left[\frac{v}{2}-\frac{u}{b}\right]\right)} - \frac{\left(\tan\left[\frac{v}{2}-\frac{u}{b}\right]-b\right)v''}{b\left(1+b\tan\left[\frac{v}{2}-\frac{u}{b}\right]\right)} \\ & - \frac{2\left[b\left(1-\tan\left[\frac{v}{2}-\frac{u}{b}\right]\right)-\tan^2\left[\frac{v}{2}-\frac{u}{b}\right]\right]}{b^2\left(1+b\tan\left[\frac{v}{2}-\frac{u}{b}\right]\right)^2} \\ & + \frac{2\exp(vb)\sec^2\left[\frac{v}{2}-\frac{u}{b}\right]}{\left(1+b\tan\left[\frac{v}{2}-\frac{u}{b}\right]\right)^2}\Gamma\left[\frac{\left(1+b\tan\left[\frac{v}{2}-\frac{u}{b}\right]\right)v'-\frac{2}{b}}{2\exp\left(\frac{vb}{2}\right)\sec\left[\frac{v}{2}-\frac{u}{b}\right]}\right]. \end{aligned} \tag{3.382}$$

Both of the remaining symmetries are lost.

Let us now investigate the linking expressions:

$$\begin{aligned} u_1 &= \int v_2 \mathrm{d}u_2 \\ v_1 &= v_2, \end{aligned} \tag{3.383}$$

$$\begin{aligned} u_3 &= \frac{1}{2}\log\left[\left(\int \frac{1}{v_1}\mathrm{d}u_1\right)^2 + u_1^2\right] + b\tan^{-1}\left(\frac{u_1}{\int \frac{1}{v_1}\mathrm{d}u_1}\right) \\ v_3 &= \frac{1}{b}\log\left[\left(\int \frac{1}{v_1}\mathrm{d}u_1\right)^2 + u_1^2\right] + 2\tan^{-1} v_1 \end{aligned} \tag{3.384}$$

and

$$\begin{aligned} u_3 &= \frac{1}{2}\log\left[u_2^2 + \left(\int v_2\mathrm{d}u_2\right)^2\right] + b\tan^{-1}\left[\frac{\int v_2\mathrm{d}u_2}{u_2}\right] \\ v_3 &= \frac{1}{b}\log\left[u_2^2 + \left(\int v_2\mathrm{d}u_2\right)^2\right] + 2\tan^{-1} v_2. \end{aligned} \tag{3.385}$$

Since the optimal process for reduction to quadratures is via symmetry G_2, both (3.370) and (3.382) should be transformed to the form of (3.375) to enable them to be solved.

3.3.15 $A_{3,6}^{II}$

The relevant equation is as follows

$$y''' = -\frac{3xy''}{1+x^2} + \frac{1}{(1+x^2)^{\frac{5}{2}}}\Gamma\left[y''\left(1+x^2\right)^{\frac{3}{2}}\right] \tag{3.386}$$

and the corresponding three symmetries are

$$G_1 = x\frac{\partial}{\partial y} \tag{3.387}$$

$$G_2 = \frac{\partial}{\partial y} \tag{3.388}$$

$$G_3 = \left(1+x^2\right)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \tag{3.389}$$

We first reduce via G_1 to obtain

$$u = x \qquad v = y' - \frac{y}{x}. \tag{3.390}$$

Proceeding as before we find

$$y'' = v' + \frac{v}{u} \tag{3.391}$$

$$y''' = v'' + \frac{v'}{u} - \frac{v}{u^2}. \tag{3.392}$$

Equation (3.386) is now reduced to the form

$$v'' = \frac{(v')^2}{v} + \frac{1}{v^2}\Gamma\left[vv'\right]. \tag{3.393}$$

Both G_2 and G_3 can be written as new symmetries to be used in the further reduction of (3.393), *ie.*

$$X_2 = \frac{1}{u}\frac{\partial}{\partial v} \tag{3.394}$$

$$X_3 = \left(1+u^2\right)\frac{\partial}{\partial u} + (1-uv)\frac{\partial}{\partial v} \tag{3.395}$$

We now attempt to reduce (3.386) via G_2 by setting

$$u = x \qquad v = y'. \tag{3.396}$$

Thus

$$\begin{aligned} y'' &= v' && (3.397)\\ y''' &= v'' && (3.398)\end{aligned}$$

and therefore

$$v'' = -\frac{3uv'}{1+u^2} + \frac{1}{(1+u^2)^{\frac{5}{2}}}\Gamma\left[v'\left(1+u^2\right)^{\frac{3}{2}}\right]. \tag{3.399}$$

Only G_1 can be rewritten in terms of the new coordinates (3.396) and thus used in the further reduction of the equation (3.399):

$$X_1 = \frac{\partial}{\partial v}. \tag{3.400}$$

Finally we reduce the third order equation via G_3. Here

$$\begin{aligned} u &= \frac{y^2}{1+x^2} && (3.401)\\ v &= \left(1+x^2\right)^{\frac{1}{2}} y' - \frac{xy}{(1+x^2)^{\frac{1}{2}}}. && (3.402)\end{aligned}$$

Thus it follows

$$\begin{aligned} y'' &= \frac{2y\left((1+x^2)\,y' - xy\right)v'}{(1+x^2)^{\frac{5}{2}}} + \frac{y}{(1+x^2)^2} && (3.403)\\ y''' &= \frac{(2y)^2\left[y'\,(1+x^2) - xy\right]^2}{(1+x^2)^{\frac{9}{2}}}v'' \\ &\quad + \left[\left(1+x^2\right)^{\frac{5}{2}} y'' - \left(1+x^2\right)^{\frac{3}{2}} y + x^2 y\left(1+x^2\right)^{\frac{1}{2}}\right] \\ &\quad \times \frac{y'\left[(1+x^2)\,y' - xy\right] + y\left[(1+x^2)\,y'' + xy' - y\right]}{y\,(1+x^2)^{\frac{5}{2}}\left[(1+x^2)\,y' - xy\right]}. && (3.404)\end{aligned}$$

Hence, the reduced equation is

$$v'' = -\frac{(v')^2}{v} - \frac{v'}{2u} - \frac{1}{4uv} + \frac{1}{4uv^2}\Gamma\left[u^{\frac{1}{2}}\left(2vv'+1\right)\right]. \tag{3.405}$$

Unfortunately, reducing via G_3, results in the *loss* of all remaining symmetries.

We now consider the linking transformations:

$$\begin{aligned} u_1 &= u_2 \\ v_1 &= v_2 - \frac{\int v_2 \mathrm{d}u_2}{u_2}, \end{aligned} \tag{3.406}$$

$$\begin{aligned} u_3 &= \frac{\left[u_1 \int \frac{v_1}{u_1} \mathrm{d}u_1\right]^2}{1+u_1^2} \\ v_3 &= \left(1+u_1^2\right)^{\frac{1}{2}} \left[v_1 + \int \frac{v_1}{u_1} \mathrm{d}u_1\right] - \frac{u_1^2 \left[\int \frac{v_1}{u_1} \mathrm{d}u_1\right]}{(1+u_1^2)^{\frac{1}{2}}} \end{aligned} \tag{3.407}$$

and

$$\begin{aligned} u_3 &= \frac{\left[\int v_2 \mathrm{d}u_2\right]^2}{1+u_2^2} \\ v_3 &= \left(1+u_2^2\right)^{\frac{1}{2}} v_2 - \frac{u_2 \int v_2 \mathrm{d}u_2}{(1+u_2^2)^{\frac{1}{2}}}. \end{aligned} \tag{3.408}$$

Thus, (3.399) as well as (3.405) must take on the form of (3.393) which is the only second order equation reduced from (3.386) to be invariant under two symmetries.

3.3.16 $A_{3,7}^{bII}$

Here we consider the equation

$$y''' = -\frac{3xy''}{1+x^2} + \frac{\exp\left(b\tan^{-1}x\right)}{(1+x^2)^{\frac{5}{2}}} \Gamma\left[y''\left(1+x^2\right)^{\frac{3}{2}} \exp\left(-b\tan^{-1}x\right)\right] \tag{3.409}$$

admitting the three symmetries

$$G_1 = x\frac{\partial}{\partial y} \tag{3.410}$$

$$G_2 = \frac{\partial}{\partial y} \tag{3.411}$$

$$G_3 = \left(1+x^2\right)\frac{\partial}{\partial x} + (xy+by)\frac{\partial}{\partial y}. \tag{3.412}$$

where $b > 0$ is once again a constant.

Reducing via G_1:

$$u = x \qquad v = y' - \frac{y}{x}. \tag{3.413}$$

Thus

$$y'' = v' + \frac{v}{u} \tag{3.414}$$

$$y''' = v'' + \frac{v'}{u} - \frac{v}{u^2} \tag{3.415}$$

and

$$v'' = -\frac{v'}{u} - \frac{v}{u^2} - \frac{3(uv'+v)}{1+u^2} + \frac{\exp(b\tan^{-1}u)}{(1+u^2)^{\frac{5}{2}}}\Gamma\left[\left(v' + \frac{v}{u}\right)\left(1+u^2\right)^{\frac{3}{2}}\exp\left(-b\tan^{-1}u\right)\right] \tag{3.416}$$

is the reduced equation.

Only G_2 can be further reduced to remain a symmetry of (3.416), *ie.*

$$X_2 = \frac{1}{u}\frac{\partial}{\partial v}. \tag{3.417}$$

Reducing (3.409) via G_2 yields

$$u = x \qquad\qquad v = y'. \tag{3.418}$$

Hence

$$y'' = v' \tag{3.419}$$

$$y''' = v'' \tag{3.420}$$

and therefore

$$v'' = -\frac{3uv'}{1+u^2} + \frac{\exp(b\tan^{-1}u)}{(1+u^2)^{\frac{5}{2}}}\Gamma\left[v'\left(1+u^2\right)^{\frac{3}{2}}\exp\left(-b\tan^{-1}u\right)\right]. \tag{3.421}$$

Both G_1 as well as G_3 are now rewritten as symmetries of (3.421), *viz.*

$$X_1 = \frac{\partial}{\partial v} \tag{3.422}$$

$$X_3 = \left(1+u^2\right)\frac{\partial}{\partial u} + (b-u)\,v\frac{\partial}{\partial v} \tag{3.423}$$

respectively.

Finally, reducing via G_3, we set

$$u = \log\left[\frac{y}{(1+x^2)^{\frac{1}{2}}}\right] - b\tan^{-1}x \tag{3.424}$$

$$v = \exp\left[-b\tan^{-1}x\right]\left[-xy\left(1+x^2\right)^{-\frac{1}{2}} + y'\left(1+x^2\right)^{\frac{1}{2}}\right]. \tag{3.425}$$

Thus we have

$$y'' = \left[\frac{(\exp(-u)\,v - b)\,v' + \exp(u) + bv}{(1+x^2)^{\frac{3}{2}}} +\right] \exp\left(b\tan^{-1}x\right) \tag{3.426}$$

$$\begin{aligned} y''' = & \left[\frac{y'(1+x^2) - y(x+b)}{y(1+x^2)}\right]^2 \frac{\exp(b\tan^{-1}x)}{(1+x^2)^{\frac{1}{2}}} v'' \\ & + \frac{y}{y'(1+x^2) - y(x+b)} \\ & \times \left[-(y+by') + \frac{xy(x+b)}{1+x^2} + y''\left(1+x^2\right)\right]\left[\frac{y''}{y} - \frac{(y')^2}{y^2} - \frac{1}{(1+x^2)} + \frac{2x(x+b)}{(1+x^2)^2}\right] \\ & + \frac{y' + y''(2b-x)}{1+x^2} - \frac{3xy + 2bxy' + 2yb + (x^2+b^2)\,y'}{(1+x^2)^2} \\ & + \frac{3x^2y(x+b) + bx^2y + xyb^2}{(1+x^2)^3}. \end{aligned} \tag{3.427}$$

Therefore

$$\begin{aligned} v'' = & -\frac{(v')^2}{\exp(u)\,(v\exp(-u) - b)} - \frac{(b - v\exp(-u))\,(v\exp(-u) - 2b)\,v'}{(v\exp(-u) - b)^2} \\ & - \frac{v\,(b^2+1)}{(v\exp(-u) - b)^2} \\ & + \frac{1}{(v\exp(-u) - b)^2}\Gamma\left[(\exp(-u)\,v - b)\,v' + \exp(u) + bv\right] \end{aligned} \tag{3.428}$$

is the reduced equation. Reduction via G_3 results in the *loss* of the other two symmetries.

Let us now consider the linking transformations:

$$\begin{aligned} u_1 &= u_2 \\ v_1 &= v_2 - \frac{\int v_2 \mathrm{d}u_2}{u_2}, \end{aligned} \tag{3.429}$$

$$\begin{aligned} u_3 &= \log\left[\frac{u_1 \int v_1/u_1 \mathrm{d}u_1}{(1+u_1^2)^{\frac{1}{2}}}\right] - b\tan^{-1}u_1 \\ v_3 &= \exp\left(-b\tan^{-1}u_1\right)\left[-\frac{u_1^2 \int \frac{u_1}{v_1}\mathrm{d}u_1}{(1+u_1^2)^{\frac{1}{2}}} + \left(1+u_1^2\right)^{\frac{1}{2}}\left(v_1 + \int \frac{v_1}{u_1}\mathrm{d}u_1\right)\right] \end{aligned} \tag{3.430}$$

and

$$\begin{aligned} u_3 &= \log\left[\frac{\int v_2 \mathrm{d}u_2}{(1+u_2^2)^{\frac{1}{2}}} - b\tan^{-1}u_2\right] \\ v_3 &= \exp\left(-b\tan^{-1}u_2\right)\left[\frac{-u_2\int v_2\mathrm{d}u_2}{(1+u_2^2)^{\frac{1}{2}}} + v_2\left(1+u_2^2\right)^{\frac{1}{2}}\right]. \end{aligned} \tag{3.431}$$

Here, we consider the optimal reduction of (3.409) to be via G_2. Thus equations of the form of (3.416) and (3.428) should be rewritten in the form of (3.421) via the nonlocal transformations above.

3.3.17 $A^{I}_{3,8}$

The relevant equation for this Lie algebra is

$$y'y''' = \frac{3}{2}(y'')^2 + (y')^2\Gamma\left[x\right] \tag{3.432}$$

and the associated symmetries are

$$G_1 = \frac{\partial}{\partial y} \tag{3.433}$$

$$G_2 = y\frac{\partial}{\partial y} \tag{3.434}$$

$$G_3 = y^2\frac{\partial}{\partial y}. \tag{3.435}$$

While this equation does admit the three symmetries (3.433)–(3.435), it also admits the three symmetries

$$G_4 = f\left(x\right)\frac{\partial}{\partial x} \tag{3.436}$$

where $f(x)$ is the solution to the third order ordinary differential equation

$$f''' + 2f'\Gamma[x] + f\Gamma[x] = 0, \tag{3.437}$$

a fact omitted in [26, 27]. Since (3.432) admits a six-dimensional Lie algebra it lies beyond the scope of this work.

3.3.18 $A^{II}_{3,8}$

For this algebra, we consider the equation

$$x^2y'y''' = 3x^2(y'')^2 + (y')^5\Gamma\left[\frac{xy'' + \frac{1}{2}y'}{(y')^3}\right] \tag{3.438}$$

and the corresponding symmetries

$$G_1 = \frac{\partial}{\partial y} \tag{3.439}$$

$$G_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{3.440}$$

$$G_3 = 2xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}. \tag{3.441}$$

Reducing (3.438) via G_1 yields

$$u = x \qquad v = y'. \tag{3.442}$$

Therefore

$$y'' = v' \tag{3.443}$$

$$y''' = v'' \tag{3.444}$$

and therefore we obtain

$$v'' = \frac{3(v')^2}{v} + \frac{v^4}{u^2}\Gamma\left[\frac{uv' + \frac{1}{2}v}{v^3}\right]. \tag{3.445}$$

Both G_2 and G_3 reduce to symmetries of (3.445), *viz.*

$$X_2 = u\frac{\partial}{\partial u} \tag{3.446}$$

$$X_3 = 2uv^2\frac{\partial}{\partial v}. \tag{3.447}$$

As a result, they can now be utilised in the reduction of (3.445).

We now reduce (3.438) via G_2:

$$u = \frac{y}{x} \qquad v = y'. \tag{3.448}$$

Thus

$$y'' = \frac{(xy' - y)\, v'}{x^2} \tag{3.449}$$

$$y''' = \frac{(xy' - y)^2\, v''}{x^4} + \frac{x(y'')^2}{xy' - y} - \frac{2y''}{x}. \tag{3.450}$$

This will reduce the original equation to the second order equation

$$v'' = \frac{-(v')^2 + 2v'}{(v-u)} + \frac{3(v')^2}{v} + \frac{v^4}{(v-u)^2}\Gamma\left[\frac{(v-u)\, v' + \frac{v}{2}}{v^3}\right]. \tag{3.451}$$

Reduction via G_2 results in the *loss* of the other two symmetries.

Finally, we consider reduction via G_3. With

$$u = \frac{y^2}{x} \qquad v = y - \frac{y^2}{2xy'}, \tag{3.452}$$

it follows that

$$\begin{aligned} y'' &= \left(v' + \frac{1}{2y'}\right)\frac{2(y')^2(2xy'-y)}{xy} - \frac{2(y')^3 x}{y^2} & (3.453) \\ y''' &= \frac{8(y')^4 v^2 v''}{xy^2} + \frac{(y')^3}{vy} + \frac{2x(y')^4}{y^3} + \left(\frac{y}{vy'} + \frac{2}{y'}\right)(y'')^2 - \frac{4(y')^3}{y^2} \\ &+ \left[\frac{2x(y')^2}{vy} + \frac{y}{2xv} - \frac{2x(y')^2}{y^2} - \frac{3y'}{y}\right] y''. & (3.454) \end{aligned}$$

Substituting (3.452), (3.453) and (3.454) into (3.438), we obtain

$$v'' = -\frac{(v')^2}{v} + \frac{2v'}{u} - \frac{v}{u^2} + \frac{u}{8v^2}\Gamma\left[\frac{4vv'}{u} - \frac{2v^2}{u^2}\right]. \tag{3.455}$$

On further investigation, we find that only G_2 remains after reduction, and takes on the form

$$X_2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}. \tag{3.456}$$

Once again the linking expressions are as follows:

$$\begin{aligned} u_2 &= \frac{\int v_1 \mathrm{d}u_1}{u_1} \\ v_2 &= v_1 \end{aligned} \tag{3.457}$$

and

$$\begin{aligned} u_3 &= \frac{\left[\int v_1 \mathrm{d}u_1\right]^2}{u_1} \\ v_3 &= \int v_1 \mathrm{d}u_1 - \frac{\left[\int v_1 \mathrm{d}u_1\right]^2}{2u_1 v_1}. \end{aligned} \tag{3.458}$$

Thus, equations (3.451) and (3.455) must be transformed to the form of equation (3.445) to be further reduced. We have not been able to find an explicit relationship between (u_2, v_2) and (u_3, v_3). Fortunately, this is not required to link the reducible equation to the irreducible ones

3.3.19 $A_{III}^{3,8}$

The relevant equation is

$$x^2\left(1+(y')^2\right)y''' = 3x^2(y'')^2y' + \left(1+(y')^2\right)^3\Gamma\left[\frac{xy''-y'-(y')^3}{(1+(y')^2)^{\frac{3}{2}}}\right] \tag{3.459}$$

and the corresponding symmetries are as follows:

$$G_1 = \frac{\partial}{\partial y} \tag{3.460}$$

$$G_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{3.461}$$

$$G_3 = 2xy\frac{\partial}{\partial x} + \left(y^2 - x^2\right)\frac{\partial}{\partial y}. \tag{3.462}$$

First consider reduction via G_1, which yields

$$u = x \qquad v = y'. \tag{3.463}$$

Thus

$$y'' = v' \tag{3.464}$$

$$y''' = v'' \tag{3.465}$$

and via substitution

$$v'' = \frac{3v(v')^2}{1+v^2} + \frac{(1+v^2)^2}{u^2}\Gamma\left[\frac{uv'-v-v^3}{(1+v^2)^{\frac{3}{2}}}\right]. \tag{3.466}$$

G_2 and G_3 now take on the forms

$$X_2 = u\frac{\partial}{\partial u} \tag{3.467}$$

$$X_3 = -2u\left(1+v^2\right)\frac{\partial}{\partial v} \tag{3.468}$$

respectively.

Reducing via symmetry G_2 we have

$$u = \frac{y}{x} \qquad v = y'. \tag{3.469}$$

Hence

$$
\begin{aligned}
y'' &= \frac{(xy'-y)\,v'}{x^2} && (3.470)\\
y''' &= \frac{(xy'-y)^2\,v''}{x^4} + \frac{x(y'')^2}{xy'-y} - \frac{2y''}{x} && (3.471)
\end{aligned}
$$

and it therefore follows that

$$
v'' = -\frac{(v')^2 - 2v'}{v-u} + \frac{3v(v')^2}{(1+v^2)(v-u)} + \frac{(1+v^2)^2}{(v-u)^2}\Gamma\left[\frac{(v-u)\,v' - v - v^3}{(1+v^2)^{\frac{3}{2}}}\right]. \quad (3.472)
$$

Further reduction is not possible due to the fact that neither of the remaining symmetries can be rewritten in terms of the new variables.

Finally, reducing via G_3, we have

$$
\begin{aligned}
u &= \frac{y^2}{x} + x && (3.473)\\
v &= \frac{2xyy' + x^2 - y^2}{2xy - y'(x^2-y^2)}. && (3.474)
\end{aligned}
$$

Thus

$$
\begin{aligned}
y'' &= \frac{(x^2-y^2+2xyy')\,(-2xy+(x^2-y^2)\,y')^2}{x^2\,(x^2+y^2)^2}v' - \frac{2x^2\,(x^2+y^2)\,(y-xy')\,(1+y'^2)}{x^2\,(x^2+y^2)^2} && (3.475)\\
y''' &= -\frac{1}{x^4\,(x^2+y^2)^2}\left(\left(x^2-y^2+2xyy'\right)^2\left(-2xy+\left(x^2-y^2\right)^2y'\right)^2(-v''\right.\\
&\quad +\left(x^3\left(4\,(y-xy')\left(1+y'^2\right)\left(4x^3y^3+\left(x^6-9x^4y^2+3x^2y^4+y^6\right)y'\right.\right.\right.\\
&\quad +3x\,(x-y)\,y\,(x+y)\left(3x^2+y^2\right)y'^2 - x^2\left(x^4-6x^2-3y^4\right)y'^3\Big)\\
&\quad -2\left(x^2+y^2\right)\left(2xy\left(x^4-6x^2y^2+y^4\right) - \left(3x^6-29x^4y^2+17x^2y^4+y^6\right)y'\right.\\
&\quad -2xy\left(11x^4-14x^2y^2-y^4\right)y'^2 + 2x^2\left(x^4-8x^2y^2-y^4\right)y'^3\Big)\,y''\\
&\quad -2x\left(x^2+y^2\right)^2\left(x^4-4x^2y^2+y^4+3xy\left(x^2-y^2\right)y'y''^2\right)\Big)\\
&\quad \Big/\left(\left(x^2-y^2+2xyy'\right)^3\left(-2xy+\left(x^2-y^2\right)y'\right)^3\right)\Big)\Big)\Big) && (3.476)
\end{aligned}
$$

from which it follows

$$
v'' = \frac{(2v^2-1)\,v'^2}{v\,(1+v^2)} - \frac{2v'}{u} + \frac{(1+v^2)^2}{uv^2}\Gamma\left[\frac{1+v^2-uvv'}{\left(u\,(1+v^2)^3\right)^{1/2}}\right]. \quad (3.477)
$$

Reduction via G_3 will result in the loss of the other two symmetries.

For completeness, let us consider the linking transformations:

$$\begin{aligned} u_2 &= \frac{\int v_1 du_1}{u_1} \\ v_2 &= v_1 \end{aligned} \tag{3.478}$$

and

$$\begin{aligned} u_3 &= \frac{[\int v_1 du_1]^2}{u_1} + u_1 \\ v_3 &= \frac{[(\int v_1 du_1)^2 - u_1^2]v_1 + 2u_1 \int v_1 du_1}{[(\int v_1 du_1)^2 - u_1^2] - v_1}. \end{aligned} \tag{3.479}$$

Thus, once again reduction to quadratures is only possible if the second order differential equations, (3.472) and (3.477), are transformed to an equation of the form (3.466) using the above transformations. The note at the end of §3.3.18 applies here as well.

3.3.20 $A^I_{3,9}$

The final equation we investigate is

$$y'y''' = \frac{3}{2}(y'')^2 - \frac{1}{2}(y')^4 + (y')^2\Gamma[x] \tag{3.480}$$

which has the three symmetries,

$$G_1 = i(\sin y)\frac{\partial}{\partial y} \tag{3.481}$$

$$G_2 = \frac{\partial}{\partial y} \tag{3.482}$$

$$G_3 = -i(\cos y)\frac{\partial}{\partial y}. \tag{3.483}$$

While this equation does admit the three symmetries (3.481)–(3.483), it also admits the three symmetries

$$G_4 = f(x)\frac{\partial}{\partial x} \tag{3.484}$$

where $f(x)$ is the solution to the third order ordinary differential equation

$$f''' + 2f'\Gamma[x] + f\Gamma[x] = 0, \tag{3.485}$$

a fact omitted in [26, 27]. Since (3.480) admits a six-dimensional Lie algebra it lies beyond the scope of this work.

Chapter 4

Conclusion

"Mathematics is a model of exact reasoning, an absorbing challenge to the mind, an esthetic experience for creators and some students, a nightmarish experience to other students, and an outlet for the egotistic display of mental power. But historically, intellectually, and practically, mathematics is primarily man's finest creation for the investigation of nature." [18, p.vii]

In this dissertation we have attempted to illustrate the connection between second order ordinary differential equations which are derived from a common third order equation. In this way unsolvable second order differential equations can be converted to a form from where reduction to quadratures is possible. The nonlocal transformations between the resulting second order equations with two and fewer than two symmetries will inevitably save both in time and effort as the third order differential equation no longer need even be considered. These results have been submitted for publication [7].

While we have only considered reduction variables arising from point symmetries we note that exponential nonlocal symmetries can also be used to effect reduction [8].

Below follows a list of the second order equations with fewer than two symmetries that can be solved via nonlocal transformations. Each set is followed by the nonlocal transformation linking them to an equation(s) that has two point symmetries. We remind the reader that all these results are up to an arbitrary point transformation.

$A_{3,1}$

$$v_2'' = -\frac{(v_2')^2}{v_2} + \frac{1}{v_2^2}\Gamma\left[v_2 v_2'\right] \tag{4.1}$$

$$v_3'' = \frac{v_3}{u_3^2} - \frac{v_3'}{u_3} + \Gamma\left[v_3' - \frac{v_3}{u_3}\right] \tag{4.2}$$

$$u_2 = \int v_1 \mathrm{d}u_1 \tag{4.3}$$

$$v_2 = v_1 \tag{4.4}$$

$$u_3 = u_1 \tag{4.5}$$

$$v_3 = v_1 - \frac{\int v_1 \mathrm{d}u_1}{u_1} \tag{4.6}$$

$$v_1'' = \Gamma\left[v_1'\right] \tag{4.7}$$

$A_{3,2}^I$

$$\begin{aligned} v_3'' &= \frac{(v_3')^2 + 2v_3'}{v_3 - u_3 - 1} + \frac{1}{(v_3 - u_3 - 1)^2} \\ &\quad + \left[\frac{\left[(v_3 - u_3 - 1)\, v_3' + 1\right]^2}{(v_3 - u_3 - 1)^2}\right] \Gamma\left[\exp v_3 \,(v_3 - u_3 - 1)\right] \end{aligned} \tag{4.8}$$

$$u_3 = \frac{\int v_1 \mathrm{d}u_1}{u_1} - \log u_1 \tag{4.9}$$

$$v_3 = v_1 - \log u_1 \tag{4.10}$$

$$u_3 = \frac{u_2}{\int \frac{1}{v_2} \mathrm{d}u_2} - \log\left[\int \frac{1}{v_2} \mathrm{d}u_2\right] \tag{4.11}$$

$$v_3 = v_2 - \log\left[\int \frac{1}{v_2} \mathrm{d}u_2\right] \tag{4.12}$$

$$v_1'' = (v_1')^2 \Gamma\left[v_1' \exp v_1\right] \tag{4.13}$$

$$v_2'' = -\frac{(v_2')^2}{v_2} + (v_2')^2 \Gamma\left[v_2 v_2' \exp v_2\right] \tag{4.14}$$

$A^{II}_{3,2}$

$$v_2'' = -\frac{v_2'}{u_2} + \frac{v_2}{u_2^2} + \left(v_2' + \frac{v_2}{u_2}\right)\Gamma\left[\frac{\exp u_2}{v_2' + (v_2/u_2)}\right] \tag{4.15}$$

$$v_3'' = -\frac{(v_3')^2}{(v_3-1)} - 5\frac{v_3}{(v_3-1)}\left[\frac{v_3^2}{(v_3-1)} - v_3'\right] + \frac{[(v_3-1)\,v_3' - v_3^2]}{(v_3-1)^2}\Gamma\left[\frac{\exp(-u_3)}{(v_3-1)\,v_3' - (v_3)^2}\right] \tag{4.16}$$

$$u_2 = u_1 \tag{4.17}$$

$$v_2 = v_1 - \frac{\int v_1 \mathrm{d}u_1}{u_1} \tag{4.18}$$

$$u_3 = \log\left[\int v_1 \mathrm{d}u_1\right] - u_1 \tag{4.19}$$

$$v_3 = \frac{v_1}{\int v_1 \mathrm{d}u_1} \tag{4.20}$$

$$v_1'' = v_1'\Gamma\left[\frac{\exp u_1}{v_1'}\right] \tag{4.21}$$

$A^{I}_{3,3}$

$$v_3'' = \frac{-(v_3')^2 + 2v_3'}{(v_3 - u_3)} + (v_3')^2\Gamma[v_3] \tag{4.22}$$

$$u_3 = \frac{u_1}{\int \frac{1}{v_1}\mathrm{d}u_1} \tag{4.23}$$

$$v_3 = v_1 \tag{4.24}$$

$$u_3 = \frac{\int v_2 \mathrm{d}u_2}{u_2} \tag{4.25}$$

$$v_3 = v_2 \tag{4.26}$$

$$v_1'' = \frac{-(v_1')^2}{v_1} + (v_1')^2\Gamma[v_1] \tag{4.27}$$

$$v_2'' = (v_2')^2\Gamma[v_2] \tag{4.28}$$

$A_1 \oplus A_2^I$

$$v_3'' = -\frac{(v_3')^2}{v_3} + \frac{3v_3'}{v_3} - \frac{2}{v_3} + \frac{(v_3'-1)^{\frac{3}{2}}}{(v_3)^{\frac{1}{2}}}\Gamma\left[\frac{v_3'-1}{v_3}\right] \tag{4.29}$$

$$u_3 = u_1 \tag{4.30}$$
$$v_3 = v_1\int \frac{1}{v_1}\mathrm{d}u_1 \tag{4.31}$$
$$u_3 = \int v_2 \mathrm{d}u_2 \tag{4.32}$$
$$v_3 = u_2 v_2 \tag{4.33}$$

$$v_1'' = -\frac{(v_1')^2}{v_1} + \left(\frac{(v_1')^3}{v_1}\right)^{\frac{1}{2}}\Gamma\left[\frac{v_1 v_1'}{(u_1)^2}\right] \tag{4.34}$$
$$v_2'' = (v_2')^{\frac{3}{2}}\Gamma\left[\frac{v_2'}{(v_2)^2}\right] \tag{4.35}$$

$A_{3,4}^I$

$$v_3'' = -\frac{u_3(v_3')^2 + 2v_3v_3' - 6}{u_3v_3+1} + \frac{18(v_3)^3}{(u_3v_3+1)^2} + \frac{((u_3v_3+1)\,v_3' + 2(v_3)^2)^{\frac{4}{3}}}{(u_3v_3+1)^2} \times \Gamma\left[\left(\frac{u_3}{(v_3)^{\frac{1}{2}}} + \frac{1}{(v_3)^{\frac{3}{2}}}\right)v_3' + 2(v_3)^{\frac{1}{2}}\right] \tag{4.36}$$

$$u_3 = u_1\int \frac{1}{v_1}\mathrm{d}u_1 \tag{4.37}$$
$$v_3 = \frac{v_1}{u_1^2} \tag{4.38}$$
$$u_3 = u_2\int v_2 \mathrm{d}u_2 \tag{4.39}$$
$$v_3 = \frac{v_2}{(\int v_2 \mathrm{d}u_2)^2} \tag{4.40}$$

$$v_1'' = -\frac{(v_1')^2}{v_1} + \left(\frac{(v_1')^2}{v_1}\right)^{\frac{2}{3}}\Gamma\left[\frac{v_1'}{(v_1)^{\frac{1}{2}}}\right] \tag{4.41}$$
$$v_2'' = (v_2')^{\frac{4}{3}}\Gamma\left[\frac{v_2'}{(v_2)^{\frac{3}{2}}}\right] \tag{4.42}$$

$A_{3,5}^{aI}$

$$v_3'' = -\frac{(a-2)\,v_3'}{v_3 - au_3} - \frac{(a-2)\,(a-1)\,v_3}{(v_3 - au_3)^2} + \frac{\left[(v_3 - au_3)\,v_3' + v_3\,(a-1)\right]^{\frac{a-3}{a-2}}}{(v_3 - au_3)^2}\Gamma\left[(v_3 - u_3)\,v_3'(v_3)^{\frac{2-a}{a-1}} + (a-1)\,(v_3)^{1-a}\right] \quad (4.43)$$

$$u_3 = \frac{u_1}{\left(\int \frac{1}{v_1}\mathrm{d}u_1\right)^a} \quad (4.44)$$

$$v_3 = \frac{v_1}{\left(\int \frac{1}{v_1}\mathrm{d}u_1\right)^{a-1}} \quad (4.45)$$

$$u_3 = \frac{\int v_2 \mathrm{d}u_2}{u_1^a} \quad (4.46)$$

$$v_3 = \frac{v_2}{u_1^{a-1}} \quad (4.47)$$

$$v_1'' = -\frac{v_1'}{v_1} + \frac{(v_1 v_1')^{\frac{a-3}{a-2}}}{v_1^2}\Gamma\left[v_1'(v_1)^{\frac{1}{a-1}}\right] \quad (4.48)$$

$$v_2'' = (v_2')^{\frac{a-3}{a-2}}\Gamma\left[v_2'(v_2)^{\frac{2-a}{a-1}}\right] \quad (4.49)$$

$A_{3,5}^{\frac{1}{2}I}$

$$v_3'' = -\,(v_3' - 1)\,v_3' + \frac{1}{v_3\,(v_3 - 2u_3)}\Gamma\left[(v_3 - 2u_3)\,v_3' + v_3\right] \quad (4.50)$$

$$u_3 = \frac{u_1}{\left(\int \frac{1}{v_1}\mathrm{d}u_1\right)^2} \quad (4.51)$$

$$v_3 = \frac{v_1}{\int \frac{1}{v_1}\mathrm{d}u_1} \quad (4.52)$$

$$u_3 = \frac{\int v_2 \mathrm{d}u_2}{u_2^2} \quad (4.53)$$

$$v_3 = \frac{v_2}{u_2} \quad (4.54)$$

$$v_1'' = -\frac{(v_1')^2}{v_1} + \frac{1}{(v_1)^3}\Gamma\left[v_1 v_1'\right] \tag{4.55}$$

$$v_2'' = \frac{1}{v_2}\Gamma\left[v_2'\right] \tag{4.56}$$

$A_1 \oplus A_2^{II}$

$$v_3'' = \frac{-(v_3')^2 + 2v_3'}{v_3 - u_3} + (v_3')^2\Gamma\left[(v_3 - u_3)\, v_3'\right] \tag{4.57}$$

$$u_3 = \int v_1 \mathrm{d}u_1 \tag{4.58}$$

$$v_3 = v_1 \tag{4.59}$$

$$u_3 = \int \frac{v_2}{u_2}\mathrm{d}u_2 \tag{4.60}$$

$$v_3 = v_2 + \int \frac{v_2}{u_2}\mathrm{d}u_2 \tag{4.61}$$

$$v_1'' = (v_1')^2\Gamma\left[u_1 v_1'\right] \tag{4.62}$$

$$v_2'' = -\frac{v_2'}{u_2} + \frac{v_2}{u_2^2} + \left(v_2' + \frac{v_2}{u_2}\right)^2\Gamma\left[u_2 v_2' + v_2\right] \tag{4.63}$$

$A_{3,4}^{II}$

$$\begin{aligned} v_3'' = & -\frac{2(v_3')^2}{2v_3 - u_3} - \frac{\left(2(u_3)^2 - 5u_3v_3 + 6(v_3)^2\right) v_3'}{u_3\left(2v_3 - u_3\right)^2} - \frac{(v_3)^3}{(u_3)^2\left(2v_3 - u_3\right)^2} \\ & + \frac{\left[(2v_3 - u_3)\, v_3' + ((v_3)^2/u_3)\right]^{\frac{5}{3}}}{u_3^{\frac{1}{3}}\left(2v_3 - u_3\right)^2}\Gamma\left[\left(\frac{2v_3 - (u_3)^{\frac{1}{2}}}{(u_3)^{\frac{1}{2}}}\right) v_3' + \frac{(v_3)^2}{(u_3)^{\frac{3}{2}}}\right] \end{aligned} \tag{4.64}$$

$$u_3 = \frac{\left(\int v_1 \mathrm{d}u_1\right)^2}{u_1} \tag{4.65}$$

$$v_3 = v_1 \int v_1 \mathrm{d}u_1 \tag{4.66}$$

$$u_3 = \frac{\left(u_2 \int \frac{v_2}{u_2}\mathrm{d}u_2\right)^2}{u_2} \tag{4.67}$$

$$v_3 = \left(v_2 + \int \frac{v_2}{u_2}\mathrm{d}u_2\right)\left(u_2 \int \frac{v_2}{u_2}\mathrm{d}u_2\right) \tag{4.68}$$

$$v_1'' = (v_1')^{\frac{5}{3}}\Gamma\left[(u_1)^{\frac{3}{2}}v_1'\right] \tag{4.69}$$

$$v_2'' = -\frac{v_2'}{u_2}+\frac{v_2}{u_2^2}+\left(v_2'+\frac{v_2}{u_2}\right)^{\frac{5}{3}}\Gamma\left[(u_2)^{\frac{3}{2}}v_2'+(u_2)^{\frac{1}{2}}v_2\right] \tag{4.70}$$

$A_{3,5}^{II}$

$$\begin{aligned} v_3'' = & -\frac{(1-a)(v_3')^2}{[(1-a)v_3-u_3]}-\frac{[(1-a)(3a(v_3)^2+2(u_3)^2(u_3-v_3))-a(u_3)^2v_3]v_3'}{(u_3)^2[(1-a)v_3-u_3]^2} \\ & -\frac{av_3^3(2a-1)}{u_3^2[(1-a)v_3-u_3]^2}+\frac{[(1-a)u_3v_3v_3'+(u_3)^2v_3'+a(v_3)^2]^{\frac{2-3a}{1-2a}}}{(u_3)^2[(1-a)v_3-u_3]^2}\times \\ & \Gamma\left[v_3v_3'(1-a)u_3^{\frac{a}{1-a}}-u_3^{\frac{1}{1-a}}v_3'+a(v_3)^2u_3^{\frac{2a-1}{1-a}}\right] \end{aligned} \tag{4.71}$$

$$u_3 = \frac{\left[\int v_1 \mathrm{d}u_1\right]^{1-a}}{u_1} \tag{4.72}$$

$$v_3 = \frac{v_1}{\left[\int v_1 \mathrm{d}u_1\right]^{a}} \tag{4.73}$$

$$u_3 = \frac{\left[u_2\int (v_2/u_2)\mathrm{d}u_2\right]^{1-a}}{u_2} \tag{4.74}$$

$$v_3 = \frac{v_2+\int (v_2/u_2)\mathrm{d}u_2}{\left[u_2\int (v_2/u_2)\mathrm{d}u_2\right]^{a}} \tag{4.75}$$

$$v_1'' = v_1'^{\frac{2-3a}{1-2a}}\Gamma\left[u_1^{\frac{2a-1}{a-1}}v_1'\right] \tag{4.76}$$

$$v_2'' = -\frac{v_2'}{u_2}+\frac{v_2}{u_2^2}+\left(v_2'-\frac{v_2}{u_2}\right)^{\frac{2-3a}{1-2a}}\Gamma\left[u_2^{\frac{2a-1}{a-1}}v_2'+v_2u_2^{\frac{a}{a-1}}\right] \tag{4.77}$$

$A_{3,5}^{\frac{1}{2}II}$

$$v_3'' = \frac{(-(v_3')^2+v_3')}{(v_3-2u_3)}+\frac{1}{(v_3-2u_3)^2}\Gamma\left[(v_3-2u_3)v_3'+v_3\right] \tag{4.78}$$

$$u_3 = \frac{\left[\int v_2 \mathrm{d}u_2\right]^{\frac{1}{2}}}{u_2} \tag{4.79}$$

$$v_3 = \frac{v_2}{\left[\int v_2 \mathrm{d}u_2\right]^{\frac{1}{2}}} \tag{4.80}$$

$$u_3 = \frac{\left[u_2 \int (v_2/u_2)\mathrm{d}u_2\right]^{\frac{1}{2}}}{u_2} \tag{4.81}$$

$$v_3 = \frac{v_2 + \int (v_2/u_2)\mathrm{d}u_2}{\left[u_2 \int (v_2/u_2)\mathrm{d}u_2\right]^{\frac{1}{2}}} \tag{4.82}$$

$$v_1'' = \frac{1}{u_1}\Gamma\left[v_1'\right] \tag{4.83}$$

$$v_2'' = -\frac{v_2'}{u_2} + \frac{v_2}{u_2^2} + \frac{1}{u_2}\Gamma\left[v_2' + \frac{v_2}{u_2}\right] \tag{4.84}$$

$A^I_{3,6}$

$$v_3'' = \frac{\left(2v_3^2 - 1\right){v_3'}^2}{v_3\left(1+v_3^2\right)} - \frac{v_3'}{u_3} + \frac{\left(1+v_3^2\right)\left(-v_3 + u_3\left(1+v_3^2\right)\Gamma\left[\frac{2u_3v_3v_3' - v_3^2 - 1}{\left(u_3\left(1+v_3^2\right)^3\right)^{1/2}}\right]\right)}{4u_3^2v_3^2} \tag{4.85}$$

$$u_3 = \left(\int \frac{1}{v_1}\mathrm{d}u_1\right)^2 + u_1^2 \tag{4.86}$$

$$v_3 = \frac{u_1v_1 + \int \frac{1}{v_1}\mathrm{d}u_1}{u_1 - v_1\int \frac{1}{v_1}\mathrm{d}u_1} \tag{4.87}$$

$$u_3 = u_2^2 + \left[\int v_2 \mathrm{d}u_2\right]^2 \tag{4.88}$$

$$v_3 = \frac{v_1 \int v_1 \mathrm{d}u_1 + u_1}{\int v_1 \mathrm{d}u_1 - u_1 v_1} \tag{4.89}$$

$$v_1'' = -\frac{(v_1')^2}{v_1} + \frac{3(v_1)^3(v_1')^2}{(v_1)^2\left(1+(v_1)^2\right)} + \frac{\left(1+(v_1)^2\right)^2}{(v_1)^2}\Gamma\left[\frac{v_1v_1'}{\left(1+(v_1)^2\right)^{\frac{3}{2}}}\right] \tag{4.90}$$

$$v_2'' = \frac{3v_2(v_2')^2}{1+(v_2)^2} + \left(1+(v_2)^2\right)^2\Gamma\left[\frac{v_2'}{\left(1+(v_2)^2\right)^{\frac{3}{2}}}\right] \tag{4.91}$$

$A^{bI}_{3,7}$

$$v_1'' = -\frac{\left(1-2v_1^2\right)(v_1')^2}{1+(v_1)^2} + \frac{\left(1+(v_1)^2\right)^2\exp\left(2b\tan^{-1}v_1\right)}{v_1}\Gamma\left[\frac{v_1v_1'\exp\left(-b\tan^{-1}v_1\right)}{\left(1+(v_1)^2\right)^{\frac{3}{2}}}\right] \tag{4.92}$$

$$
\begin{aligned}
v_3'' = & -\frac{\left(b-\tan\left[\frac{v_3}{2}-\frac{u_3}{b}\right]\right)(v_3')^2}{2\left(1+b\tan\left[\frac{v_3}{2}-\frac{u_3}{b}\right]\right)} - \frac{\left(\tan\left[\frac{v_3}{2}-\frac{u_3}{b}\right]-b\right)v_3'}{b\left(1+b\tan\left[\frac{v_3}{2}-\frac{u_3}{b}\right]\right)} \\
& -\frac{2\left[b\left(1-\tan\left[\frac{v_3}{2}-\frac{u_3}{b}\right]\right)-\tan^2\left[\frac{v_3}{2}-\frac{u_3}{b}\right]\right]}{b^2\left(1+b\tan\left[\frac{v_3}{2}-\frac{u_3}{b}\right]\right)^2} \\
& +\frac{2\exp(v_3 b)\sec^2\left[\frac{v_3}{2}-\frac{u_3}{b}\right]}{\left(1+b\tan\left[\frac{v_3}{2}-\frac{u_3}{b}\right]\right)^2}\Gamma\left[\frac{\left(1+b\tan\left[\frac{v_3}{2}-\frac{u_3}{b}\right]\right)v_3'-\frac{2}{b}}{2\exp\left(\frac{v_3 b}{2}\right)\sec\left[\frac{v_3}{2}-\frac{u_3}{b}\right]}\right]
\end{aligned} \tag{4.93}
$$

$$
u_1 = \int v_2 \mathrm{d}u_2 \tag{4.94}
$$

$$
v_1 = v_2 \tag{4.95}
$$

$$
u_3 = \frac{1}{2}\log\left[u_2^2+\left(\int v_2\mathrm{d}u_2\right)^2\right]+b\tan^{-1}\left[\frac{\int v_2\mathrm{d}u_2}{u_2}\right] \tag{4.96}
$$

$$
v_3 = \frac{1}{b}\log\left[u_2^2+\left(\int v_2\mathrm{d}u_2\right)^2\right]+2\tan^{-1}v_2 \tag{4.97}
$$

$$
v_2'' = \frac{3v_2(v_2')^2}{1+(v_2)^2}+\left(1+(v_2)^2\right)^2\exp\left(2b\tan^{-1}v_2\right)\Gamma\left[\frac{v_2'\exp\left(-b\tan^{-1}v_2\right)}{(1+(v_2)^2)^{\frac{3}{2}}}\right] \tag{4.98}
$$

$A_{3,6}^{II}$

$$
v_2'' = -\frac{3u_2v_2'}{1+(u_2)^2}+\frac{1}{(1+(u_2)^2)^{\frac{5}{2}}}\Gamma\left[v_2'\left(1+(u_2)^2\right)^{\frac{3}{2}}\right] \tag{4.99}
$$

$$
v_3'' = -\frac{(v_3')^2}{v_3}-\frac{v_3'}{2u_3}-\frac{1}{4u_3v_3}+\frac{1}{4u_3(v_3)^2}\Gamma\left[(u_3)^{\frac{1}{2}}(2v_3v_3'+1)\right] \tag{4.100}
$$

$$
u_1 = u_2 \tag{4.101}
$$

$$
v_1 = v_2-\frac{\int v_2\mathrm{d}u_2}{u_2} \tag{4.102}
$$

$$
u_3 = \frac{\left[u_1\int(v_1/u_1)\mathrm{d}u_1\right]^2}{1+(u_1)^2} \tag{4.103}
$$

$$
v_3 = \left(1+(u_1)^2\right)^{\frac{1}{2}}\left[v_1+\int\frac{v_1}{u_1}\mathrm{d}u_1\right]-\frac{(u_1)^2\left[\int(v_1/u_1)\mathrm{d}u_1\right]}{(1+(u_1)^2)^{\frac{1}{2}}} \tag{4.104}
$$

$$
v_1'' = \frac{(v_1')^2}{v_1}+\frac{1}{(v_1)^2}\Gamma\left[v_1v_1'\right] \tag{4.105}
$$

$A_{3,7}^{bII}$

$$v_1'' = -\frac{v_1'}{u_1} - \frac{v_1}{(u_1)^2} - \frac{3(u_1v_1'+v_1)}{1+(u_1)^2} + \frac{\exp(b\tan^{-1}u_1)}{(1+(u_1)^2)^{\frac{5}{2}}} \times$$
$$\Gamma\left[\left(v_1' + \frac{v_1}{u_1}\right)\left(1+(u_1)^2\right)^{\frac{3}{2}}\exp\left(-b\tan^{-1}u_1\right)\right] \quad (4.106)$$
$$v_3'' = -\frac{(v_3')^2}{\exp(u_3)(v_3\exp(-u_3)-b)} - \frac{(b-v_3\exp(-u_3))(v_3\exp(-u_3)-2b)v_3'}{(v_3\exp(-u_3)-b)^2}$$
$$-\frac{v_3(b^2+1)}{(v_3\exp(-u_3)-b)^2}$$
$$+\frac{1}{(v_3\exp(-u_3)-b)^2}\Gamma[(\exp(-u_3)v_3-b)v_3'+\exp(u_3)+bv_3] \quad (4.107)$$

$$u_1 = u_2 \quad (4.108)$$
$$v_1 = v_2 - \frac{\int v_2 \mathrm{d}u_2}{u_2} \quad (4.109)$$
$$u_3 = \log\left[\frac{\int v_2\mathrm{d}u_2}{(1+u_2^2)^{\frac{1}{2}}} - b\tan^{-1}u_2\right] \quad (4.110)$$
$$v_3 = \exp\left(-b\tan^{-1}u_2\right)\left[\frac{-u_2\int v_2\mathrm{d}u_2}{(1+(u_2)^2)^{\frac{1}{2}}} + v_2\left(1+(u_2)^2\right)^{\frac{1}{2}}\right] \quad (4.111)$$

$$v_2'' = -\frac{3u_2v_2'}{1+(u_2)^2} \quad (4.112)$$
$$+\frac{\exp(b\tan^{-1}u_2)}{(1+(u_2)^2)^{\frac{5}{2}}}\Gamma\left[v_2'\left(1+(u_2)^2\right)^{\frac{3}{2}}\exp\left(-b\tan^{-1}u_2\right)\right] \quad (4.113)$$

$A^{II}_{3,8}$

$$v_2'' = \frac{-(v_2')^2 + 2v_2'}{(v_2 - u_2)} + \frac{3(v_2')^2}{v_2} + \frac{v_2^4}{(v_2 - u_2)^2}\Gamma\left[\frac{(v_2 - u_2)\,v_2' + \frac{v_2}{2}}{(v_2)^3}\right] \tag{4.114}$$

$$v_3'' = -\frac{(v_3')^2}{v_3} + \frac{2v_3'}{u_3} - \frac{v_3}{(u_3)^2} + \frac{u_3}{8(v_3)^2}\Gamma\left[\frac{4v_3v_3'}{u_3} - \frac{2(v_3)^2}{(u_3)^2}\right] \tag{4.115}$$

$$u_2 = \frac{\int v_1 \mathrm{d}u_1}{u_1} \tag{4.116}$$

$$v_2 = v_1 \tag{4.117}$$

$$u_3 = \frac{\left[\int v_1 \mathrm{d}u_1\right]^2}{u_1} \tag{4.118}$$

$$v_3 = \int v_1 \mathrm{d}u_1 - \frac{\left[\int v_1 \mathrm{d}u_1\right]^2}{2u_1v_1} \tag{4.119}$$

$$v_1'' = \frac{3(v_1')^2}{v_1} + \frac{(v_1)^4}{(u_1)^2}\Gamma\left[\frac{u_1v_1' + \frac{1}{2}v_1}{(v_1)^3}\right] \tag{4.120}$$

$A^{3,8}_{III}$

$$\begin{aligned} v_2'' &= -\frac{(v_2')^2 - 2v_2'}{v_2 - u_2} + \frac{3v_2(v_2')^2}{(1+(v_2)^2)\,(v_2 - u_2)} \\ &\quad + \frac{\left(1+(v_2)^2\right)^2}{(v_2 - u_2)^2}\Gamma\left[\frac{(v_2 - u_2)\,v_2' - v_2 - (v_2)^3}{(1+(v_2)^2)^{\frac{3}{2}}}\right] \end{aligned} \tag{4.121}$$

$$v_3'' = \frac{(2v_3^2 - 1)\,v_3'^2}{v_3\,(1+v_3^2)} - \frac{2v_3'}{u_3} + \frac{\left(1+v_3^2\right)^2}{u_3v_3^2}\Gamma\left[\frac{1 + v_3^2 - u_3v_3v_3'}{\left(u_3\,(1+v_3^2)^3\right)^{1/2}}\right] \tag{4.122}$$

$$u_2 = \frac{\int v_1 \mathrm{d}u_1}{u_1} \tag{4.123}$$

$$v_2 = v_1 \tag{4.124}$$

$$u_3 = \frac{\left[\int v_1 \mathrm{d}u_1\right]^2}{u_1} + u_1 \tag{4.125}$$

$$v_3 = \frac{[(\int v_1 \mathrm{d}u_1)^2 - u_1^2]v_1 + 2u_1\int v_1 \mathrm{d}u_1}{[(\int v_1 \mathrm{d}u_1)^2 - u_1^2] - v_1} \tag{4.126}$$

$$v_1'' = \frac{3v_1(v_1')^2}{1+(v_1)^2} + \frac{(1+(v_1)^2)^2}{(u_1)^2}\Gamma\left[\frac{u_1v_1' - v_1 - (v_1)^3}{(1+(v_1)^2)^{\frac{3}{2}}}\right] \tag{4.127}$$

We submit this list of equations as a contribution to the class of second order ordinary differential equations that can be reduced to quadratures. It remains to consider other third order equations invariant under larger (> 3) dimensional Lie groups. Mahomed [26, 27] lists the appropriate Lie algebras together with the relevant symmetries for these equations. However, he does not list the equations invariant under those algebras. That is the next step in this work - a project that is ongoing. The second order equations obtained in the two-dimensional case also warrants further investigation.

We conclude by noting that mathematics is indeed a vital and necessary step in the exploration of the physical world. The preciseness of the results it produces plays a major role in time reduction, an uncontrollable variable, often considered man's greatest enemy.

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