REMEDIATION OF FIRST-YEAR MATHEMATICS STUDENTS' ALGEBRA DIFFICULTIES

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ABSTRACT

The pass rate of first-year university mathematics students at the University of KwaZulu-Natal (Pietermaritzburg Campus) has been low for many years. One cause may be weak algebra skills. At the time of this study, revision of high school algebra was not part of the major first year mathematics course. This study set out to investigate if it would be worthwhile to spend tutorial time on basic algebra when there is already an overcrowded calculus syllabus, or if students refresh their algebra skills sufficiently as they study first year mathematics. Since it was expected that remediation of algebra skills would be found to be worthwhile, two other questions were also investigated: Which remediation strategy is best? Which errors are the hardest to remediate?

Five tutorial groups for Math 130 were randomly assigned one of four remediation strategies, or no remediation. Three variations of using cognitive conflict to change students' misconceptions were used, as well as the strategy of practice. Pre- and post-tests in the form of multiple choice questionnaires with spaces for free responses were analysed. Comparisons between the remediated and non-remediated groups were made based on pre- and post-test results and Math 130 results. The most persistent errors were determined using an 8-category error classification developed for this purpose.

The best improvement from pre- to post-test was 12.1% for the group remediated with cognitive conflict over 5 weeks with explanations from the tutor. Drill and practice gave the next-best improvement of 8.1%, followed by self-guided cognitive conflict over 5 weeks (7.8% improvement). A once-off intervention using cognitive conflict gave a 5.9% improvement. The group with no remediation improved by 2.3%. The results showed that the use of tutor-intensive interventions more than doubled the improvement between pre-and post-tests but even after remediation, the highest group average was 80%, an unsatisfactory level for basic skills. The three most persistent errors were those involving technical or careless errors, errors from over-generalising and errors from applying a distorted algorithm, definition or theorem.

PREFACE

This thesis was supervised by Prof. Trevor Anderson, Prof. Iben Christiansen and Dr. Paddy Ewer at the University of KwaZulu-Natal, Pietermaritzburg Campus.

I, Anita Lee Campbell declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.
2. This thesis has not been submitted for any degree or examination at any other university.
3. This thesis does not contain other persons' data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.
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1. RATIONALE

This chapter describes the motivation factors and background for this study of basic algebra skills, and introduces the research questions. A description of the course and my connection to it are followed by an outline of the possible causes of the low pass rate under the headings, 'Is it us?', 'Is it them?' and 'Is it circumstances?' Finally the research questions are introduced and explained.

1.1. Background and context

The high failure rate common to tertiary institutions in South Africa and around the world (Hillel, 2001) is most likely due to a large number of factors, for example under-preparedness, ineffective teaching strategies, lack of motivation, poor work ethic, and/or difficult home circumstances. To compound the problem, first-generation tertiary students may hold beliefs that work against academic achievement, such as the belief that achievement is a threat to relationships with their family or community, and that life is governed by fate (C. L. Campbell, 2005). The choice of investigating students' prerequisite algebra skills in the first semester mathematics course (Math 130) at the University of KwaZulu-Natal (UKZN), Pietermaritzburg Campus was made because it was an area in which I, as someone not directly involved in the teaching of the course, felt I could make a measurable impact. If strength in algebra could be shown to be a significant factor for passing Math 130, then it would be worthwhile creating space for algebra revision in tutorials. If not, we would know that our efforts to improve the pass rate should be directed elsewhere.

About the course

Math 130: Introduction to Calculus is a first-semester course that runs over 13 weeks from February to May. It covers two sections: Discrete Mathematics and Calculus. The Discrete Mathematics is taught in the first 4 weeks from departmental course notes while a textbook, *Calculus: Concepts and Contexts* by James Stewart (2001), is used for the Calculus section.

Typically, the course has 2 to 4 different lecturers taking consecutive sections of the course. The lecturing style is mostly traditional, with the students sitting in a tiered lecture theatre taking notes while the lecturer presents material on the blackboard or overhead projector. Students do not ask many questions in class and the lengthy syllabus adds pressure on the lecturers to move rapidly through the material they present.

The class of approximately 120-150 students has four lectures together per week and the class is split into six groups of 20 – 25 for weekly three-hour tutorials, run by lecturers or post-graduate students in Mathematics or Applied Mathematics. The lecturer presenting a section selects tutorial questions from the notes or textbook. Students are told to attempt these questions at home and use tutorial times to ask their tutor for guidance on those that they could not complete. Every second or third week short quizzes are given to the students during tutorials to encourage them to keep up to date.

Math 130, together with the follow-on second semester course, Math 140: Calculus and Linear Algebra, serves two main purposes: preparing mathematics majors for further courses, and servicing other disciplines like Physics and Computer Science. Although teaching basic skills is not an aim of the course, it is expected by staff in other disciplines that students passing first year mathematics will have a near-perfect grasp of basic skills. Indeed, Math 130 and Math 140 lecturers expect this too.

I usually tutor one group of 20 – 25 students but in the year that this study was done, I was not involved in lecturing or tutoring the course.

There are five possible ways for students to be permitted to register for Math 130. Students matriculating prior to 2008 needed either an A symbol on standard grade mathematics or a D symbol on higher grade mathematics. In 2008 a National Senior Certificate (NSC, also referred to as 'matric') was introduced, with number scores from 0 – 7 replacing the symbol system. A level 5 pass in mathematics in the NSC was the entrance requirement from 2009. Another entry point for Math 130 was through the successful completion of a year-long foundation programme (the Science Foundation Programme) with at least 55% for

Mathematics. Finally, students not meeting the university entry requirements based on matric marks might qualify for the Access Programme, in which Math 130 might be one of the two first-year course they take with additional lectures to support their progress in their first year.

The course is assessed by a class mark and a 3-hour exam, weighted 1:2 in the final mark. The class mark is a weighted average of two 1½ -hour class tests written outside of class time and 20-minute quizzes written during tutorials. The pass mark is 50% and students with final marks from 40 – 49% have the option to write a supplementary exam 4 weeks later.

The pass rate before supplementary exams is typically 50% – 60% and after supplementary exams, it improves to 60% – 70%. As this course is a gateway course for students wishing to major in Mathematics, Applied Mathematics, Statistics, Computer Science and Physics, it is concerning that so many students who want to study mathematical disciplines cannot do so because they fail Math 130.

The concern over the low pass rate was a strong motivation factor for this project. Given the complexity of factors that can affect student performance (Mason & Johnston-Wilder, 2004) addressing algebra skills only is unlikely to be sufficient to achieve the student performance we hope for. What this investigation aims to do is show the impact of improving algebra skills on Math 130 performance so we can gauge how significant this factor is and how much effort should be directed towards improving algebra skills or in other directions.

Why do so many students fail Math 130?

Is it Us?

As described earlier, the predominant teaching style in Math 130 is lecturing. Diana Laurillard (2002) described the main shortcoming of lectures as relying on students having similar prior knowledge and capabilities, which is increasingly uncommon as access to universities becomes more open. The students in Math 130 come from many different schools, from well-resourced private schools to formerly black-only schools still battling to overcome a history of lack of resources, including qualified mathematics teachers.

Kieran, Forman & Sfard (2001 p. 1) speak of the 'traditional mathematics classroom, featuring one blackboard, one outspoken teacher and twenty to forty silent students' as seeming to belong to history. But this is generally not the case in South Africa. The mathematics course in this study is in some ways traditional: lectures are in tiered lecture theatres and most Math 130 lecturers have a presentational rather than interactive teaching style. Tutorials offer the chance for students to interact with their tutor and their class-mates but there are few demands made on students who remain disengaged in tutorials and do not ask for help.

Is a poor high school education a major factor in first-year students' failure in mathematics? Lynn Bowie (1998) noted that there appeared to be a lack of research on the basic mathematics skills of South African university students. A reason for this may be that university students are expected to enter university having mastered basic skills during high school and more research is focused on how students learn the main content (typically Calculus) in the first-year curriculum.

There is research to support the claim that doing something to try to improve the pass rate can make a measurable positive impact on student learning. Jill Larkin (2000) gives examples of studies in undergraduate mathematics and science courses where changes to the courses resulted in improved learning or decreased learning times by factors of 30% to 90%, e.g. Anderson, Corbett, Koedinger & Pelletier (1995).

Many universities have established resource centres, also known as maths labs or learning centres (Trillan, 1980). They have been found to be effective at addressing the diversity of students' mathematics backgrounds and allow students to work at an individual pace. Problems with resource centres include encouraging students who need help to attend, and the high cost of tutors. (See also the next section.)

Self-paced instruction involving mastery learning (needing to achieve 80% or more) and peer tutoring was found by Alice Trillan (1980) to result in material being more thoroughly learned than in conventional classes, although procrastination and poor reading ability were problems. The Science Foundation Mathematics course at UKZN-Pietermaritzburg gives students basic skills tests which must be re-taken until an average of 80% in two consecutive tests is achieved. However these students perform as poorly as new students in basic algebra tests in first year mathematics.

Is it them?

If students are unmotivated, it is likely that any attempt at improving their algebra skills will fail, unless it addresses their motivation to learn first. Glenda Anthony (2000) made a detailed investigation of the factors identified by students and lecturers as influencing students' success in a first-year mathematics course in New Zealand. Student and lecturer responses were classified into four broad categories: lectures, course, students and other external factors. Students placed more value on lectures and course design than lecturers did, while lecturers rated student factors as more significant in determining failure than students did. Interestingly, both students and lecturers rated self-motivation as the most important factor influencing success. Students ranked 'adequate background knowledge' as 36^{th} out of 40 factors which influence success, while lecturers ranked this as 10^{th} most important. 'Inadequate background knowledge' was ranked by students as 24^{th} out of 37 factors influencing students' failure, while lecturers ranked this factor in 11^{th} position. This suggests that any remediation strategy focusing on background knowledge would be improved by addressing many other factors, including how to help students become self-motivated.

A similar survey of factors influencing first-year students' success was done at the University of KwaZulu-Natal, Pietermaritzburg (Barnsley, 2003). Commenting on the reasons for their success in first-year science modules, the four most common responses chosen[1] by students were, being able to understand the work, handing in assignments on time, understanding the language used in class and keeping up to date with the work. The top reasons that students gave for their failure in modules were that exam questions were too difficult, the exams were too close together and too long and that they were unable to keep up with the pace of the module/s. From these comments, successful students seem more ready to take responsibility for their progress whereas unsuccessful students first blame external circumstances for their failure.

Fraser and Killen (2005) surveyed students and lecturers at two South African universities and identified factors that were felt to have the greatest influence on students' success. Their conclusions focused on three issues: (a) changing teaching practices (e.g. specifying outcomes and expected standards more clearly), (b) giving clear study guidelines for students (including what lecturers expect from them) and (c) changing administrative practices to ones that could better support students (e.g. allowing assessment other than examinations). Their study suggested that lecturers may blame students for poor pass rates although they failed to meet the needs of their students through ignorance about students' beliefs on learning, e.g. finding out why some students believe class attendance is not very important and addressing the reasons for the perception.

IS IT OTHER CIRCUMSTANCES?

South African high school mathematics

Since 1994, the South African high school mathematics syllabus has been in a state of transition from a segregated and unequally resourced system under apartheid towards one that provides for the needs of all South Africans (Case & Deaton, 1999). The major change has been the move towards an outcomes-based syllabus (Jansen, 1998) with greater emphasis on

[1] A list of 38 reasons for success or failure were obtained by open-ended questions in a pilot study. Students chose their reasons from this list.

understanding mathematical concepts and reasoning and consequently less emphasis on traditional formal algebraic manipulation. The same shift has happened in American schools (National Council of Teachers of Mathematics, 1989), where, for example, students now spend less time factorising polynomial expressions but instead are expected to understand the concept of finding roots, for which they may use factorising, graphical or numerical methods (Gordon, 2000 p. 74). Yet the transition has not leveled out the inequalities of the past and thus still impacts on the preparedness of many students.

Another problem with mathematics teaching in South African high schools is the severe lack of qualified teachers. The Education Labour Relations Council Report (2005) estimated a shortage of 32 000 to 34 000 teachers in 2008, with rural and poor urban school likely to experience the worst shortages. In 1996, severance packages were offered to historically advantaged teachers to speed up the transformation process. Many of the teachers who took up the offer were highly experienced and qualified in Mathematics and this policy was later acknowledged as a failure (South African Human Rights Commission, 2006).

To cope with the lack of qualified mathematics teachers, some schools appoint unqualified teachers to teach mathematics. Teachers who may not have studied Mathematics beyond the level that they have to teach are unlikely to have great confidence and more likely to perpetuate commonly held beliefs such as mathematics should be learnt by rote, there is only one correct way to get an answer, and some people can never understand mathematics (P. Johnson, 2004).

Differences between secondary and tertiary education

Most algebra studies have targeted high school algebra learning or pre-algebra learning (Kieran, 2006) but these studies cannot be discounted for dealing with secondary rather than tertiary students. Coady and Pegg (1991) investigated the algebra knowledge of first-year tertiary students and found that their misconceptions and errors were very similar to those of junior secondary students. This does not mean that remediation strategies would be equally successful with each group. Tertiary students may have years of reinforcement of the wrong ideas, making them resistant to strategies that work on students only beginning to learn algebra.

Tertiary students may know basic facts relevant to solving a problem but choose the less cognitively demanding task of recalling a rule rather than considering the basic facts or attempting to reason constructively. For example, a student may know that $a^3 = a.a.a$ but may think that $a^5.a^3 = a^{15}$ (Lithner, 2003). This kind of thinking is common in undergraduate students (Lithner, 2000). William Cox (1994) and Tomas Bergqvist *et al.* (2008) suggest that good high school grades can be obtained using strategic but superficial learning of routine topics.

The inability of tertiary students to recognize and act on cues needed to solve a mathematical task was shown by a study of University of Natal students by Anita Craig and Paul Winter (1990). They point out that students need to develop the ability to think about their engagement with a task and they recommend providing the learner with interruption rules or 'negative cues' to prevent the spontaneous application of erroneous rules to a task (*ibid.* p. 62). A list of learners' incorrect rules and teaching strategies to address them were given (*ibid.* p. 66).

A study by Edward Barbeau (1995) found that more rigour is needed at tertiary level compared with high school and that students may not be prepared for this, resulting in difficulties. If a mechanical approach to algebra has been a student's norm in high school, an alternative and more meaningful way of thinking about algebraic manipulations could be of great benefit. Students have to be willing to change their existing 'schema', hence the need to motivate them.

1.2. The research questions

Research question 1

Is it worthwhile to implement algebra-improvement strategies in first-year mathematics tutorials?

The answer to this question will be of value to first-year mathematics lecturers who are faced with tight budgets, a lengthy syllabus and students who make frustrating errors with basic algebra in tests and exams. The sub-questions to be examined by reviewing existing literature are:

- Is algebraic manipulation still important when computers can be used?
- How are mathematical concepts learnt?
- In what aspects of learning algebra does research suggest intervention would be most helpful?

Three more sub-questions that the results of this study hope to answer are:

- Does remediation improve algebra skills significantly more than having no remediation?
- Are good algebra skills necessary for success in Math 130?
- Do the changes in the types of answers to individual questions support the use of remediation?

Research question 2

Which remediation strategy works best?

Suggestions for fixing problems with basic algebra were sought in a literature search of remediation programmes at tertiary institutions and more specific remediation ideas to target particular errors in algebra. They were selected for use in this study based on what promise they have for remediation as well as how they could be implemented in Math 130 given the constraints of how the course would be run. The answers to this research question will be found by comparing the improvement in pre- and post-tests between the different types of remediation tested.

Research question 3

Which types of errors are the hardest to remediate?

A method for classifying errors was developed, drawing from error classifications and reports on common errors in algebra found in the literature. A statistical analysis of the frequency of occurrence of these errors in the pre- and post-tests would show if there are errors that respond well to remediation and others that are persistent regardless of the type of remediation.

1.3. Chapter-by-chapter Outline

The Literature Review that follows in Chapter 2 begins with descriptions of algebra and a short overview of research on the teaching and learning of algebra. The need for manipulative algebraic skills in tertiary mathematics is considered in the light of increasingly available computer power that can be used for algebraic simplifications. Error classifications and errors particular to algebra are reviewed and recommendations for addressing errors are presented.

Chapter 3, Theoretical Framework, is arranged in two sections, each on topics relevant to the first and second research questions. Three learning theories are considered for question one, 'Is remediation worthwhile?': Reification, the Process-Object view of algebra, and the Pseudo-structural approach. For research question two, 'Which remediation strategy is best?' the learning theories of behaviourism, constructivism, and participationism are considered.

Chapter 4, Method, describes how teaching interventions to address each of the research questions were devised and implemented, and considers issues of validity. An error classification scheme, based on the findings of the literature found in chapter 2, is devised to help answer research question 3: 'Which types of errors are the hardest to remediate?'

Chapter 5, Results and Discussion, examines the findings relating to each of the remediation questions.

Chapter 6, Conclusions, summarizes answers to the remediation questions, considers the limitations of the study and gives recommendations for further research.

2. LITERATURE REVIEW

In chapter one, the reasons for this investigation of basic algebra skills were presented. This chapter reviews literature that can contribute towards the answers to the three research questions that have been introduced.

The first main section considers descriptions of algebra and how research on teaching and learning algebra has been affected by the way it is perceived. Secondly, the question 'Are manipulative algebra skills necessary for students to cope with tertiary mathematics?' is considered and the 'reformed mathematics' movement is discussed.

The third section reports on error classifications and other particular errors in algebra. This is followed in the fourth section with general and more specific recommendations for remediating algebra, with a view to designing remediation strategies to address the errors reported on earlier.

The theoretical framework developed in the next chapter will describe the theoretical standpoints that will guide the design, implementation and analysis of remediation strategies to address the research questions. In Chapter 4, Method, an error classification scheme will be developed based on the errors discussed in this chapter.

2.1. Algebra

Descriptions and classifications of algebra

Dictionaries provide concise definitions of algebra, such as this one from the Britannica World Language Edition of the Oxford Dictionary:

> **Algebra: "The part of mathematics which investigates the relations and properties of numbers by means of general symbols"**

Mathematicians, however, have not reached consensus on a short definition of the vast and growing field of algebra. A working group at the 2001 ICMI[2] conference came up with six descriptions of algebra (Stacey & Chick, 2004 p.335):

a) A way of expressing generality and pattern.

b) A study of symbol manipulation and equation solving.

c) A study of functions and their transformations.

d) A way to solve problems beyond the reach of arithmetic methods.

e) A way to interpret the world through modeling real situations.

f) A formal system, possibly dealing with set theory, logical operations, and operations on entities other than real numbers.

Although all aspects of algebra are important for a student studying mathematics, I chose to focus this study on the view described by point (b) above: algebra as the study of symbol manipulation and equation solving. I felt that this aspect of algebra related most closely with what most concerned mathematics lecturers at UKZN regarding the skills that first year mathematics students were expected to be competent in. In later reflections on this assumption, I came to feel symbol manipulation skills might be improved by expanding students' understanding of algebra to some of the other views listed above. For example, strengthening students' understanding of aspect (c) above, a study of functions and their transformations, e.g. considering the graphs of the functions $f(x) = e^x$ and $g(x) = e^{-x} = 1/e^x$ could help students to have a deeper understanding of manipulating exponents.

Six other descriptions of algebra were found in the literature. Sigrid Wagner and Carolyn Kieran (1989) and Barbara von Ameron (2003) describe perspectives of viewing algebra; Mark Saul (2001) describes stages of understanding algebra; Eric Love (1986), Alan Bell (1995) and

[2]ICMI is the International Commission on Mathematics Instruction

Carolyn Kieran (1996; 2007) describe algebra according to activities. These descriptions are each described below.

The view of algebra as symbol manipulation and solving equations can be seen in the description of algebra as a set of rules, which is one of the three major perspectives of algebra identified by Wagner and Kieran (1989 p. 221) as prevalent in algebra research up to 1989. They see algebra as (a) generalised arithmetic, (b) a set of rules and (c) a representation system. Barbara von Ameron (2003) and Stacey, Chick and Kendal (2004) give a similar but extended perspective. They distinguish four basic perspectives of algebra: (a) as generalised arithmetic, (b) as a tool to solve certain problems, (c) as the study of relationships, and (d) as the study of structures. Compared to Wagner and Kieran's description, this one seems to de-emphasize the view commonly held by less able learners of algebra as just a process of memorising rules and procedures (Tall & Razali, 1993).

Mark Saul (2001) classified algebra by three stages of understanding: firstly algebra is seen as generalised arithmetic, which concurs with Wagner and Kieran (1989) and von Ameron (2003). Secondly it becomes a study of binary operations (e.g. addition/subtraction, or multiplication/division) acting on numbers (that can be represented by letters). Studies on this aspect of algebra include David Slavit's (1998) look at the role of operation sense in moving from arithmetic to algebra. Finally algebraic form is recognised, e.g. noticing the similarity between $(\cos x)^2 - (\sin x)^2$ and $A^2 - B^2$. (This is also linked to the dual nature of mathematical concepts as described by Anna Sfard (1991) as process and object to be discussed later.) Saul's classification can help to determine where learning algebra breaks down. If a student had a teacher whose understanding of algebra was limited to Saul's first or second stages, it will be harder for them to attain the third stage than a student who is helped to recognize algebraic form. The low qualification level of many school mathematics teachers (Most maths teachers 'not experts'.2008; Bonnet, 2008) makes this breakdown more probable.

The next two descriptions of algebra focus more on activities associated with algebra. Alan Bell (1995 p.50) claimed that specific misconceptions could be more easily dealt with if algebra was seen as being defined by three main activities: (a) generalizing, (b) forming and solving equations, and (c) working with functions and formulae. Eric Love (1986) defines algebra as being aware and in control of: (a) handling an unknown (i.e. being able to recognize, express and manipulate operations on variables, rather than just numbers), (b) inverting and reversing operations, and (c) seeing the general in the particular. Basic algebra skills such as simplifying expressions could fall into all categories in both of these descriptions. For example, in Bell's description, the addition of algebraic fractions can be generalized, used in solving equations, and be necessary when evaluating a function that is the sum of fractions. In Love's description, factorizing and multiplying out would involve operating on variables, using reversible operations and applying general rules for factorizing.

The three algebraic activities identified by Carolyn Kieran (Kieran, 2004 pp. 22-24; Kieran, 2007 p. 713) are quite different from all the other descriptions because they include activities that go beyond algebra. Her activities are (a) generational activities (such as forming equations to represent problem situations - not relevant for this study); (b) transformational activities (e.g. maintaining equivalence when transforming an expression or equation, and rule-based activities including collecting like terms, expanding, substituting, solving equations, exponentiation); and (c) global or meta-level activities that are not exclusive to algebra (e.g. noticing structure, generalizing, problem-solving). It is easy to imagine that meta-level activities can be improved without directly teaching algebra, as a side effect from other learning, but can the same happen to transformational activities? This is something that this study may be able to help answer.

In three of the six descriptions discussed above, algebra was described as generalised arithmetic. Much research on early algebra learning tends to take this view of algebra. The focus is either on *arithmetic* and the link between algebraic symbols (letters) and the numbers they represent, or on *generalising* the structural aspects of the number system, the latter view being less common (Wagner and Kieran, 1989, p. 221). Studies on algebra from the 'generalised arithmetic' perspective have looked at students' concepts of variable (Radford, 1996), function (Carraher, Schliemann, Brizuela, & Earnest, 2006; Thompson, 1994), understanding of procedures such as simplifying expressions using associative and commutative laws (Warren, 2003), solving equations (Knuth, Stephens, McNeil, & Alibali, 2006; Williams & Cooper, 2002) and symbolic representations (Arcavi, 2003; Hiebert, 1988; Pape & Tchoshanov, 2001).

Algebra seen as a set of rules was the basis for research on processes of solving equations, analysing errors, and building rule-based computer models that mimic learner and teacher behaviour e.g. Menucha Birenbaum *et al.* (1993) and Susan Pirie and L. Martin (1997).

Word problems and computer software relating to visual representation were the main areas of research that focused on the representational aspects of algebra (e.g. Koedinger & Nathan, 2004; Yerushalmy, 1991), as were limited views of variables (Usiskin, 1999), e.g. as single-valued unknowns (as in $2x + 4 = 14$) but not as relational variables (e.g. $y = 2x + 4$), and also difficulties with letters representing physical quantities, e.g. seeing the letter '*s*' as representing 'students' rather than 'number of students' (J. Clement, 1982).

The perspectives described above all show that there is much more to the understanding of algebra than manipulating expressions and solving equations. These different descriptions helped to limit the focus of this study to the testing of the usefulness of easy-to-implement remediation strategies. The broader views of algebra may be helpful in investigating the depth of algebraic understanding that students have or how to give tertiary students a fuller understanding of algebra (if that is what is necessary for success in tertiary mathematics), and these views could be revisited when devising remediation strategies.

Research on algebra teaching and learning

Looking at the development of the field of algebra education research will give an insight into the different ways lecturers and students might view algebra due to their past education. This is done with the intention of finding out how teaching interventions can help students who struggle with basic algebra. First I will consider the main trends in algebra education research and what influenced them, starting from pre-1950 to 2007, largely based on a comprehensive literature review by Kieran (2007). This leads into a four-part description of how meaning is constructed in learning algebra. This section is concluded by pointing out areas of algebra understanding where remediation could be effective.

The historical development of teaching and learning algebra shows how research on learning algebra has moved through different phases. Up to the 1950's, the dominant position was that algebra was a tool for manipulating symbols and solving problems (Kieran, 2007). This resulted in research that focused on the difficulties students had with the procedures and notation, for example, Thorndike *et al.* (1923) looked at the role of practice in learning algebra, and Breslich (1939) looked at errors in applying algorithms. In contrast, researchers investigating algebra learning in the 1950s and 1960s were mostly behaviourist psychologists who used the field of learning algebra to study memory and skills development. With the growth of the community of algebra education researchers, the focus shifted in the late 1970s, towards making algebra learning meaningful for students and determining the meaning that students made of algebra.

Kieran (2007) lists four influences that have shaped the trend of more recent research to focus on developing algebraic meaning:

1. The influence of Piaget's cognitive development theory (Piaget, 1972) and the theory of constructivism (Fosnot, 1996) in which students are understood to actively construct knowledge rather than passively absorb it. (See the Theoretical Framework chapter for more on constructivism.)

2. Studies (e.g. Jacob, 2001; Kilpatrick, 2001) showing that an exclusively skills-based approach to teaching algebra did not produce algebraically skilled students.

3. The influence of socio-democratic policies such as the "Algebra for All" movement in America (Gamoran & Hannigan, 2000).

4. The use of technology in schools (e.g. Mayer, 2001)

The source of meaning in algebra has been described by Luis Radford (2004) in three categories: the algebraic structure itself, the problem context, and the exterior of the problem context (for example the meaning constructed by the individual learner). Kieran (2006; 2007) divided the first category into two sub-sections to give the classification described in Figure 2.1. This classification gives a structure from which to understand the important issues in teaching and learning algebra. The points mentioned in Figure 2.1 will be elaborated on below.

1. Meaning from within mathematics:

1(a). Meaning from the algebraic structure itself, involving the letter-symbolic form.

1(b). Meaning from other mathematical representations, including multiple representations.

2. Meaning from the problem context.

3. Meaning derived from that which is exterior to the mathematics/problem context (e.g. linguistic activity, gestures and body language, metaphors, lived experience, image building).

Figure 2.1 Sources of meaning in algebra (Kieran, 2007, p. 711).

Meaning from the algebraic structure itself

The skill of manipulating algebraic symbols correctly requires an understanding of the "structural properties of mathematical operations and relations which distinguish allowable transformations from those that are not" (Booth, 1989). An implication of this is that if a student lacks an understanding of numerical operations, they may transfer their misconceptions to algebra, assuming that learning occurs as constructivism describes it, using previous knowledge to gain more knowledge. Gaps in understanding arithmetic would then be a source of errors in algebra learning.

Meaning from other mathematical representations

Many researchers have concluded that creating meaning in algebra needs learning opportunities to "coordinate objects and actions within two different representations, such as the graphical and the letter-symbolic" (Kieran, 2007). James Kaput (1989) rated highly the translation of binary information, such as values in a table, into a single entity as a curve on a graph. That view correlates with ideas in reformed mathematics where an emphasis is placed on learning through graphical representations (Lloyd & Wilson, 1998).

Meaning from the problem context

Problem-solving contexts are strongly believed to be foundational to the formation of algebraic reasoning (Kieran, 2007). Alan Bell (1996) noted that problem solving is not limited to the forming and solving of equations (as in many word problems) but can be more broadly seen as the open exploration for more and more general results, which is the essential activity of all mathematics. Insight into general results from a specific problem can provide students a deeper understanding of the use of variables and avoid mistakes such as substituting $x = 3$ into an expression because a previous problem had a solution of $x = 3$. Another consequence of algebraic meaning being based on problem exploration, as Bell (1996) claims, is that teachers who have not developed their own understanding of the purpose of algebra (due to their lack of mathematics qualifications, as is common in South African schools) and consequently deemphasize the use of problem-solving contexts in teaching, will be leading their students into an incomplete understanding of algebra.

Meaning derived from that which is exterior to the mathematics/problem context

Past studies of errors in algebra learning suggested that the theories available at the time were insufficient to fully explain student learning of algebra. More recent studies have looked at the role of gestures, bodily movements, metaphors and artifacts in building meaning in mathematical activities (Kieran, 2007 p. 712; 2006 p.30). This relates to Sfard's (2006) communicational approach to learning, elaborated on in the theoretical framework.

The four 'sources of meaning in algebra' identified by Kieran (2007) point to areas where remediation might help to improve algebra skills. Firstly, considering meaning from the algebraic structure itself, gaps in knowledge may be repaired by focusing on underlying structures, e.g. understanding multiplication as repeated addition may help to overcome errors with simplifying exponential expressions. Secondly, considering meaning from other mathematical representations can itself be a remediation strategy, e.g. when solving the inequality $x^2 -3x +2 >0$, drawing the graph of the function $y = x^2 -3x +2$ and looking for where the y-values are positive could lead a student away from making errors. Thirdly, considering meaning from the problem context, remediation could be in the form of helping students to identify whether they are answering a question (say, to factorise an expression) or doing what they think is required (perhaps solving for x when asked to factorise because this strategy has led to the right answer previous in a different question). Finally, considering meaning from sources outside the mathematics or problem context, many remediation strategies could be considered, e.g. investigating students preferred learning styles (Felder & Spurlin, 2005; G. M. Johnson & Johnson, 2006), their beliefs about how mathematics should be taught and learnt, the time of day that learning takes place, etc. Strategies on these external factors have been left out of this study but would add depth to future studies.

2.2. Are manipulative algebra skills necessary for students to cope with tertiary mathematics?

Basic algebra has long been considered a prerequisite for virtually all mathematics-related tertiary courses (Trillan, 1980). The reason for the high status of algebra is that it teaches generalization and mathematics technique, which are essential building blocks for higher mathematics learning. This is what makes elementary algebra the most important subject in remedial mathematics (Hecht & Akst, 1980, p. 255).

Weak ability in using algebraic expressions is a major cause of difficulty in a skill needed in many aspects of mathematics, particularly solving equations (Huntley, Marcus, Kahan, & Miller, 2007; Rittle-Johnson & Star, 2007). Barbara von Ameron (2003 p. 63) lists eight research projects related to solving algebraic equations where the key difficulties involved the interpretation, rewriting and simplifying of algebraic expressions.

Like most similar courses, the first-year mathematics course considered in this study assumes well-practiced and fluent algebra skills. With the availability of increasingly affordable and user-friendly computer packages and calculators to work out algebraic manipulation and to show the steps, it might seem as though the need for sophisticated manipulation skills has passed. However, researchers have shown that the use of computer algebra systems does not necessarily hide students' misconceptions with algebra (Pierce & Stacey, 2004; Tobin, 2002) and teaching needs to make careful use of the available technology so that its benefits do not negatively impact learning (Tall, Smith, & Piez, 2008). The reformed mathematics movement, discussed next, has moved tertiary mathematics towards more integration with technology.

Traditional versus reformed mathematics

The answer, "No" to the question 'Are manipulative skills important?' lies at the heart of the reformed mathematics movement which emphasizes the development of conceptual understanding over procedural fluency. This contrasts with courses that produce students able to, for example, integrate a wide range of expressions but who don't understand concepts of function and area as related to integrals. The reformed view places emphasis on using calculating tools to perform simplifications and solutions to equations, thereby freeing time to develop other aspects of mathematics. Traditional mathematics focuses on polynomial and rational expressions and their manipulation. Reformed mathematics emphasizes functions and the solution of real-world problems with technology-supported methods (Kieran, 2007, p. 709).

The 'reformed mathematics' movement has influenced tertiary-level syllabi and textbooks. Math 130 uses a 'reformed mathematics' textbook, "Calculus: Concepts and Contexts" (Stewart, 2001), but the book is not used as intended as most students do not have graphical calculators or working knowledge of computer packages. Douglas Windham (2009) reported on the difficulties of changing a course to one with a reformed approach and showed that simply adopting a reformed Calculus text did not bring about the changes in students' understanding promised by the reformed movement. The introduction of computers to perform long or complex calculations is another reform-inspired move that may not necessarily yield the desired effects of focusing students on concepts rather than procedures. Attempts to get students in Math 130 at UKZN to use the computer algebra package Mathematica were so negatively received by the students that they were dropped prior to this study.

Sheldon Gordon (2000 p.89) is a proponent of reformed mathematics who claimed that future students will first turn to graphical tools to make sense of a problem and use pen-and-paper algebraic techniques only for verification. However, he admits that students studying mathematics, physics and computer science will need more manipulative skills than other students. He suggests that advanced manipulative algebra courses be offered to these students while other students spend their time on fundamental mathematical reasoning and realistic applications. The Math 130 students in this study fall into the group needing more manipulative skills (for the way the course and later ones are currently taught). Some questions arise: How long does it take to master algebraic techniques? Is that time being used well, or can we accept lower ability with algebra manipulation?

Kathleen Heid and Glendon Blume (2008) cite studies that showed that high school students taking Computer-Intensive Algebra (CIA) for all but the last 6 – 8 weeks of an algebra course did as well as non-CIA students on final exams but had deeper conceptual understanding of fundamental algebraic ideas, such as function and variable. Other studies (Heid, 1988; Park, 1993) showed that students using computer algebra systems for Calculus performed as well as students studying the same material in a traditional way. Heid's more recent work with Rose Zbiek (Zbiek & Heid, 2009) gives more strong support to the use of computer algebra systems for high school algebra, describing them as becoming essential tools for learning and teaching.

The reformed Calculus movement has been encouraged by studies that show how computer algebra systems can help students with weak algebra skills to understand concepts in Calculus without being held back by their weak algebra skills. For example, Cheryl Hawker (1987) found that the achievement levels of two groups of Calculus students were similar, although one group had significantly lower scores in an algebra test given at the beginning of the semester. The group with lower algebra scores used the computer algebra system muMATH. Her study showed that with the use of a computer algebra system, students can give attention to concepts rather than manipulative skills.

There may be problems in courses beyond first-year if students haven't strengthened their algebra skills and so, even when using computers, algebra skills cannot be ignored. Studies by Phoebe Judson (1990) and Jeanette Palmiter (1991) support the idea that calculus courses that provide student access to computer algebra systems can be designed to focus attention on concepts and applications without a loss of manipulative skills. They suggested that algebra manipulation skills may be learned more quickly after students have developed conceptual understanding, which can be facilitated by the use of a computer algebra system. No studies have been found from tertiary South African institutions that focus on the use of computer algebra systems to improve algebra skills in a Calculus course.

2.3. How can errors be classified?

Why is it helpful to classify errors? Patterns of common errors can help to identify the types of errors that are most difficult to remediate. That would inform remediation strategies by showing what errors should be targeted.

There are many specific examples of common errors in mathematics, for example Thelma Perso (1993) identified 19 common misconceptions from 1400 high school students and Tony Barnard (2002a; 2002b; 2002c) gave groups of difficulties with remediation suggestions from experienced teachers.

Three error classifications are considered below: In the first two, Hendrik Radatz (1979) and Nitsa Movshovitz-Hadar, Orit Zaslavsky and Shlomo Inbar (1987) presented broad classifications that apply to all branches of school mathematics, while Liora Linchevski and Nicolas Herscovics (1994) classified five errors that they described as cognitive obstacles observed in students starting to learn algebra. These classifications are followed by a list of particular errors drawn from various sources, including many from Tony Barnard's (2002a; 2002b; 2002c) list of specific algebra errors generated from teachers' observations.

Classification 1: Radatz

Hendrik Radatz (1979) classified errors based on how the information would be processed. This provided a cognitive model of the cause of errors. His categories are:

1. Errors due to deficient mastery of prerequisite skills, facts, and concepts (including ignorance of algorithms, lack of mastery of basic facts, incorrect procedures in applying mathematical techniques, and insufficient knowledge of necessary concepts and symbols.)

2. Errors due to incorrect associations or rigidity of thinking, e.g. substituting, say, $x = 2$ into any expression involving x because $x = 2$ was the solution to an equation solved earlier in a lesson.

3. Errors due to the application of irrelevant rules or strategies e.g. multiplying out brackets when solving $(x + 1)^2 = 16$.

4. Errors due to difficulties in obtaining spatial information e.g. not being able to ignore irrelevant lines in geometry problems.

The first category, errors due to deficient knowledge, seemed to me to be the most common error with regard to our university students' basic algebra. If students made the errors described in categories 2 and 3 (errors due to incorrect associations or applying irrelevant strategies), different remediation strategies would be more effective, but it would appear to be difficult to distinguish between these categories based on written work alone.

I also questioned the exclusivity of these categories but could not find an example of an error that would overlap categories. It may be possible that more than one error is operating at the same time, e.g. a student may make the error of simplifying $(a + b)^2$ as $a^2 + b^2$ although they have been taught how to square a binomial. Would this be classified as a lack of mastery of a basic fact or a careless error that a student might quickly correct if prompted?

Classification 2: Movshovitz-Hadar, Zaslavsky and Inbar

Another broad classification of errors was done by Nitsa Movshovitz-Hadar, Orit Zaslavsky and Shlomo Inbar (1987). Their easy-to-use classification contains six categories of errors in high school mathematics:

1. Misused data (neglecting given data, adding irrelevant data or incorrectly copying data).

2. Misinterpreted language (incorrect translation of facts from one language to another, where the languages are possibly symbolic).

3. Logically invalid inference, e.g. given that $x^2 = 1$, concluding that $x = 1$ instead of $x = 1$ or $x = -1$.

4. Distorted theorem or definition, e.g. incorrectly applying a distributive property, e.g. $(a + b)^n = a^n + b^n$.

5. Unverified solution (each step is correct but the final result is not a solution to the original question).

6. Technical error (including computational errors e.g. $7 \times 8 = 54$, manipulation errors in algebra e.g. leaving out closing brackets).

Compared to the four categories described by Radatz (1979), this classification scheme offers more descriptive power. Categories 1, misused data, 4, 'distorted theorem or definition', and 6, 'technical error', in this classification scheme could all fall under the first category in Radatz's classification: 'Errors due to deficient mastery of prerequisite skills, facts, and concepts'. But I anticipated that both of these general classifications would still be too broad for this study on algebra skills.

Classification 3: Linchevski and Herscovics

Liora Linchevski and Nicolas Herscovics (1994) identified five cognitive obstacles in 12-13 year olds starting to learn algebra. As their subjects had just begun algebra, the errors identified are based on arithmetic. I felt that these errors would not be common in students who had studied algebra for at least 5 years so these errors were not specifically looked for in the test of university students. The list is included here to give a sense of the wide variety of possible errors, particularly in early algebra learning.

1. Problems with order of operations – giving preference to addition over subtraction and to subtraction over multiplication.

2. Not seeing the canceling effects of binary operations e.g. not seeing a quick way to work out 329 – 67 + 67.

3. Problems with factoring out –1. (Later Liora Linchevski and Drora Livneh (1999) reported that only 2 out of 27 learners thought that 926 – 167 – 167 was equivalent to 926 – (167 + 167).)

4. Detachment of a term from its operation e.g. simplifying $4 + n - 2 + 5 = 19$ to $4 + n - 7 = 19$.

5. 'Jumping off with the posterior operation' e.g. in solving $115 - n + 9 = 61$, the 9 is subtracted rather than added to 115 to give $106 - n = 61$.

All these categories could fit under Radatz's first category: 'Errors due to deficient mastery of prerequisite skills, facts, and concepts'. The first three could also fit in Movshovitz-Hadar, Zaslavsky and Inbar's sixth category 'Technical error', while the last two of Linchevski and Herscovics' categories could fit under Movshovitz-Hadar, Zaslavsky and Inbar's second category: 'Misinterpreted language.' If it was found that university students were making these mistakes, this could indicate that their foundations in algebra have been weak from the start of their algebra learning. They would have had five years of the errors being reinforced (possibly by not being convincingly shown why the errors are wrong) and this could make the errors more difficult to remediate.

Particular errors in algebra

The descriptions of algebra considered earlier show that it is a complex subject. It is no surprise then that there are many ways to get it wrong. This section discusses some specific errors relating to algebra learning identified in the literature. They will be considered together with the three error classifications just discussed and together these ideas will help the development of an error classification that can be used to inform the design of tests on basic algebra, discussed in the methods chapter.

Tony Barnard (2002a; 2002b; 2002c) lists categories of algebra errors generated from mathematics teachers in the United Kingdom. John Mason (2002) also cited some of these categories in a guide for tertiary mathematics teaching. A summary from both sources is included in the list below.

Barnard's classification was drawn from the comments of experienced teachers rather than studying students empirically. He did not say how the teachers obtained their conclusions, leaving one to assume that they came through experience rather than formal research. Although not based on traditional research, this is the most useful classification for this study as it most closely resembles the common errors seen by Math 130 students. The errors listed below are summarized in Table 2.1.

MISUNDERSTANDING THE MEANING OF VARIABLES
A letter stands for a specific value
A letter stands for the name of an object
Letters are objects that can be 'gathered' e.g. $3n + 4 = 7n$
Expressions with letters should be a single term
Letters have number value according to their place in the alphabet, e.g. $a = 1$
Not knowing how to operate on 'unknowns' e.g. adding $3x + 4x$ to get $7x$
ERRORS REGARDING EQUIVALENT FORMS AND THE MEANING OF THE EQUALS SIGN
Knowing when to change form e.g. $(a + b)^2$ or $a^2 + 2ab + b^2$
Using equals signs to link unequal steps e.g. $x^2 = 2x$
Seeing the equals sign as a 'find the answer' command rather than a sign of equivalence
Being unaware of equivalence e.g. between $2a + 5a$ and $7a$
MISINTERPRETATION OF QUESTIONS
Creating and solving an equation when asked to simplify an expression
Not being aware of mathematics communication skills other than learning vocabulary, such as using mathematical reasoning, discussing mathematics
Problems with not understanding questions given in words
OVER-GENERALISING
Over-generalising distributive laws e.g. $\sin(a + b) = \sin a + \sin b$
False linearity e.g. $(a + b)^2 = a^2 + b^2$

OPERATING ON ONE PART OF A COMPOUND TERM E.g. $(2x)^2 = 2x^2$ E.g. $x(y + z) = xy + z$
CONFUSION BETWEEN OPERATIONS E.g. $2 + a^2 = 2a^2$
ERRORS WITH MINUS SIGNS E.g. $(-x)^2 = -x^2$
MISAPPLIED RULES E.g. $2^5.a^4 = (2a)^9$
CONFUSING SIMILAR NOTATION E.g. $\sin^{-1}x = (\sin x)^{-1}$
ERRORS WITH SIMPLIFYING FRACTIONS E.g. $\frac{2h^2}{h} = 2^2$
ERRORS WITH THE WRONG APPLICATION OF ORDER OF OPERATIONS E.g. $2xy + 3xy \times \frac{z}{xy} = 5xyz$

Table 2.1 Errors in Algebra

MISUNDERSTANDING THE MEANING OF VARIABLES

Variables in algebra can have different meanings, for example they can represent a general number, as in $2x + 3x$, a specific unknown, as in $2x + 3x = 10$, and an indicator of a variable quantity in a functional relationship, as in $A = xy$ (Selden, 2002). The misunderstanding of the different meanings of algebraic variables is one reason for errors with simplifications (Booth, 1988; Linchevski & Herscovics, 1996; Stacey & McGregor, 1997). Realising that a single variable can represent many quantities at the same time is a cognitively difficult step (Wagner & Kieran, 1989). Kaye Stacey and Mollie McGregor (1997) traced the error of not recognising the different meanings that variables have in different contexts to four origins, including misleading teaching materials and students using intuition and guessing.

An understanding of the different meanings of variables does not necessarily grow with time and study. Maria Trigueros and Sonia Ursini (2003) found that many incoming students at a Mexican university were unable to distinguish different ways of understanding the variable concept. Similarly, Carmel Coady and John Pegg (1991) found that out of 116 students about to start a Science degree in Australia, about a third of the students had not progressed beyond an object view of seeing algebraic letters, meant to signify variables, as specific values or as standing for the names of objects. Lynn Bowie (1998 p. 127) looked at students in a first year calculus course at the University of Cape Town and found that some of the difficulties students had with the manipulation of formulae were a result of the incorrect meaning that they attributed to symbols and expressions.

A view that variables are objects that can be gathered together is what Lesley Booth (2001 p.109) offers as an explanation of errors such as $2a + 5b = 7ab$ (made by 45% of 13 year olds in her study), and simplifying $3n + 4$ as $7n$ or as $3n4$. These errors are also related to the belief that an answer, if not numeric, should contain a single term (Wagner & Kieran, 1989 p. 222). For $3n + 4$ to be seen as an answer, a student has to see it as an object, namely the number that is 4 greater than $3n$, as well as a process that can be performed for a given numerical value of n. This is a common problem in early algebra learning and relates to the process-object duality that will be discussed further in the theoretical framework.

A famous misconception involves seeing letters as standing for abbreviations of names rather than numerical values. If there are 6 students to each professor at a university, where S and P are the numbers of students and professors, the common error is to express this relationship as 6S = P (Bell, 1995; J. Clement, 1982; J. Kaput & Sims-Knight, 1983; Siegler, 2003b).

Another incorrect strategy is to substitute values in place of letters to transform an algebraic expression into a number, for example let $a = 1$, $b = 2$, …, $z = 26$ (Kuchemann, 1981). This error has been referred to as *letter evaluated* (French, 2002), meaning that the letter is given a numerical value instead of being treated as an unknown or a generalised number.

Connected to the misunderstanding of variables is the difficulty that students have with solving equations where the unknown must be operated on, e.g. in $3x + 4x = 27 + 69$, the terms involving x have to be added together in order to solve the equation. Students have been found to be unable to perform such operations without instruction (Linchevski & Herscovics, 1996).

Errors regarding equivalent forms and the meaning of the equals sign

Knowing how to and when to change an algebraic expression into an equivalent form by factorizing, multiplying out brackets or applying operations is an essential skill in algebra. Many expressions have multiple alternative simplifications, for example, $(a + b)^2$ can be written as $a^2 + 2ab + b^2$ or as $a^2 + b^2 + 2ab$, and a student must be able to recognize equivalent forms and know which is the most appropriate form to use in a given situation. These errors are also seen in solving equations.

Many students are unable to interpret an algebraic expression that is not attached to an equals sign (Seo & Ginsburg, 2003; Vlassis, 2002). They see an equals sign as a command to give an answer rather than representing equality. This belief also leads to the use of equal signs to link steps that do not represent equality, e.g. writing $x^3 = 3x^2$ when differentiating the function $f(x) = x^3$ to get $f'(x) = 3x^2$, as Kieran (1981) found in a study of college students. Gillian Boulton-Lewis *et al.* (1997) confirmed in a study on Grade 7 pupils that the equals sign is interpreted to mean 'find an answer' rather than an expression of equality between objects.

Booth (1988; 2001) reported that the emphasis on getting a numerical answer in arithmetic reinforces seeing the equals sign as a command to operate rather than as indicating a relation in algebra. The emphasis on getting a numerical answer in arithmetic (Booth 2001, p.109) encourages students to 'simplify' by combining any available numbers, e.g. $11x + 5 = 16x$, and to perform operations once, e.g. $2(3x + 5) = 6x + 5$. Booth (2001, p. 113) also found that early algebra learners were not aware that the same quantity can be expressed in different ways, e.g. $5a + 2a$ and $7a$.

Eric Knuth *et al.* (2006) investigated whether students' understanding of the equals sign affects their equation-solving abilities in a study of 117 grade 5 – 8 learners. They found a strong relation between understanding the equals sign and success in solving equations. This supports remediation strategies that aim to do more than solve the mechanics of algebraic problems.

MISINTERPRETATION OF QUESTIONS

A difficulty noticed by Matz (1982) was the misinterpretation of questions, e.g. creating an equation and solving when asked to simplify an algebraic expression. This may be more pronounced in second-language students (Mestre & Gerace, 1986). Judit Moschkovich (2002) looked at bilingual learners and noticed that learning mathematics is not just about vocabulary acquisition (which first-language learners may find easier) but includes other parts of communicating mathematically, such as using mathematical reasoning, participating in mathematical discussions and developing sociomathematical norms (Cobb & Bauersfeld, 1995; Yackel, Rasmussen, & King, 2000). Comparing the misunderstandings made in science, mathematics and computer science, Perkins and Simmons (1988) argued that causes of misunderstandings can be traced to four levels: (a) content, (b) problem-solving, (c) epistemic, and (d) inquiry. Highly domain-specific misconceptions occur mostly at the content level and most teaching is directed at this, at the expense of the other levels. This supports the use of remediation strategies that go beyond the content of algebra difficulties.

There is also a large body of literature on difficulties associated with word problems e.g. (T. Craig, 2002; Cummins, 1988; Gerofsky, 1996; Little, 2008) but these will not be considered in this study.

OVER-GENERALISING

Some errors in algebra may be an unavoidable part of the learning process. The findings of a study by Agnieszka Demby (1997) supported the findings of Kirshner (1985) that before achieving fluency in manipulative skill, successful students tend to go through a phase of over-generalising distributivity. This is evident in mistakes such as $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$.

Kitchener *et al.* (1993) attributed some mistakes of false linearity to over-generalising the visual coherence of rules such as $\frac{a}{b}.\frac{c}{d} = \frac{ac}{bd}$ and $(ab)^2 = a^2b^2$ to reach conclusions such as

$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$ and $(a + b)^2 = a^2 + b^2$. False linearity is also seen in errors such as

$$2\left(\frac{a}{b}\right) = \frac{2a}{2b},$$

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}$$

and $\sin(a + b) = \sin a + \sin b$.

OPERATING ON ONE PART OF A COMPOUND TERM

Tony Barnard (2002a) gives examples where part of an expression has been neglected when performing an operation e.g. $(2x)^2 = 2x^2$, $x(y + z) = xy + z$,

$$\frac{3}{2x} = 3.2x^{-1} = 6x^{-1}.$$

He suggests that these errors are unlikely to be resolved easily because they require a conceptually more challenging object view, e.g. seeing $(2x)$ as a single object. This explains why students persist in these types of errors even when they are convinced by methods like substituting numbers that show the expressions are not equivalent.

Confusion between operations

These types of errors are also called 'binary reversion errors' and the answer is correct for a different operator (Barnard, 2002b). Examples of these kinds of errors are $2 + a^2 = 2a^2$, and $4a \times 4a = 8a$. Robert Davis (1984) discusses how this type of error can be easily corrected by asking students the question for which their answer is correct.

Errors with minus signs

Tony Barnard (2002b) lists common errors relating to minus signs e.g. $(-2)^2 = -2^2$, $2 - (3x - 4y) = 2 - 3x - 4y$ and solving $5 - x = 7$ to get $x = 2$.

Misapplied rules

Robert Davis (1984) offers an explanation for the persistence of errors of this type, also identified by Tony Barnard (2002b), such as "When multiplying, add the indices, as in $2^5.a^4 = (2a)^9$" and "If $x.y = 12$ then $x = 12$ or $y = 12$." When solving an equation such as $(x - a)(x - b) = 0$, a solution for x can be found regardless of the value of a and b. The idea that specific numbers are not important can become a deeper-level rule to the student than the rule that says that to get a zero, one of the factors must be zero, and this deeper rule dominates.

Confusing similar notation

This type of error can be the result of not knowing the convention with rules that do not follow the normal use, e.g. $\sin^{-1}x = (\sin x)^{-1}$ is wrong although $\sin^2 x = (\sin x)^2$ is right (Mason, 2002).

ERRORS WITH SIMPLIFYING FRACTIONS

Tony Barnard (2002b) identified errors from inappropriate cancelling e.g.

$\frac{x+y}{y} = x+1$, $\frac{x+y}{y} = x$.

John Mason (2002) identified another type of error when simplifying fractions that don't involve plus or minus signs, e.g. $\frac{2h^2}{h} = 2^2$.

ERRORS WITH THE WRONG APPLICATION OF ORDER OF OPERATIONS

This was the first error in Linchevski and Herscovics's (1994) classification. Some errors with simplifying fractions fall into this category, e.g.

$$2xy + 3xy \times \frac{z}{xy} = 5xyz$$

instead of $2xy + 3z$.

This list of errors and the error classifications will be referred back to in the method chapter where an error classification for this study will be devised to guide a response to the third research question, Which errors are the most resistant to change?'

2.4. Addressing research question 2: Which remediation strategy is best?

Remediation at tertiary level

Fewer studies of algebra learning have been done at the upper secondary and tertiary level than with the grades below grade 10 (approximately 15 years of age) (Kieran, 2007, p. 729). Some remediation strategies have been suggested and tested with school students but there is little evidence in the literature of the methods used on school children also being effective for tertiary learners. The increased attention to functional approaches in reform curricula have given rise to more studies that look at algebraic objects from a multiple-representation perspective than from a primarily variable-symbolic manipulation focus (Kieran & Saldanha, 2005; Tabach & Friedlander, 2008). By combining these two under-researched areas of mathematics research, this study is helping to fill a gap in the body of mathematics education research.

As universities have become more accessible, there has been an increase in the diversity of preparedness levels of first-year students (Bonham & Boylan, 2007; de Guzman, Hodgson, Robert, & Villani, 1998 p. 754), a trend perhaps even more apparent in South Africa with its Apartheid legacy. Remedial mathematics refers to high-school level courses that are taught in tertiary institutions. Also called 'developmental' or 'skills' courses, they were designed to help students who might otherwise fail because of a lack of basic skills that would not be taught in a course. The popularity of remedial courses increased rapidly from 1965 to 1980, rising from 6% to 15% of total mathematics enrolments in American tertiary institutions (Albers, Elliott, Kettler, & Roach, 2005), and they are still needed now (Hagopian, 2009; M. Johnson & Kuennen, 2004).

The Research and Planning Group for Californian Community Colleges (2007) compiled an extensive literature review on developmental education, drawing from over 250 references spanning 30 years. The report covered 4 categories found in the literature to be effective practice for basic skills. The categories were:

(1) Organisational and Administrative Practices,

(2) Program Components,

(3) Staff Development and

(4) Instructional Practices.

Two general conclusions from the report were:

A single type of program is not likely to fit all institutions and a mix of programs probably best fits students' needs.

Developmental programs are not as expensive as commonly conceived. A mix of more- and less-expensive programs would generally be most beneficial.

At the University of KwaZulu-Natal, Pietermaritzburg Campus, where this study was made, no remedial mathematics courses for students taking first-year mathematics was available at the time of the study, other than a full-time, pre-first-year bridging course called the Science Foundation Programme, and extra lectures for Access students (closed to other Math 130 students). In 2009, a series of computer tests on basic algebra, analytical geometry and trigonometry, called the Booster Programme, was implemented for the first time at the University of KwaZulu-Natal. Part of the reason for the Booster Programme was to address the different skills that students matriculating in 2008 under a new South African mathematics school syllabus might have, particularly since manipulative algebra skills are less featured in the new school mathematics syllabus.

General recommendations for overcoming errors

Mistakes can be seen as a stimulus to learning because they create cognitive conflict within the learner and can show the sense-making processes that learners are using (Mason, 2002), which can help a teacher decide on suitable remediation strategies. Some errors may be due to chance or carelessness although Bouvier (1987) says this is generally not the case, for example, in the simplification of $3^2 3^4$, the answer of 3^{76} is never given. Errors seem to follow patterns. Observations of patterns of errors have given rise to classifications of common errors, as reported on earlier, and some researchers have investigated ways to overcome these errors e.g. (Barnard, 2002a; Barnard, 2002b; Koedinger & Nathan, 2004; Nardi & Jaworski, 2002; Rittle-Johnson, Siegler, & Alibali, 2001).

John Mason (2002) warns that common confusions are likely to persist unless they are addressed. This was somewhat confirmed in a study by Campbell, Anderson and Ewer (2003) in which one fifth of students in a first-year university mathematics course were found to be making basic algebra errors after their first semester of mathematics, when no specific help on basic errors was given.

To deal with the dilemma of having insufficient time to spend on these errors as well as to cover the syllabus, John Mason (2002) suggests short activities that expose the errors to the students, hopefully making them more alert to them in the future. Alan Bell (1995) on the other hand recommended alternating between focused activities aimed at acquiring specific skills and broad activities. He reported on earlier studies that found focused activities mainly involving cognitive conflict and discussion resulted in better long-term retention than more traditional methods. An example confirming this (Saenz-Ludlow & Walgamuth, 1998) showed that discussion among learners and between learners and teachers helped to build the meaning of the equals sign as sameness rather than just a command to operate. These studies suggest that approaches giving the best promise of lasting changes in students' misconceptions are also the most time-consuming. It seems likely that deep learning may occur amongst the students studied at UKZN only if there was more time or shorter syllabi in first year mathematics courses.

Students find it harder to grasp concepts that are introduced at a general level (e.g. teaching general rules before suitable practice has been given) because this increases the level of abstraction of the concepts (Tall & Vinner, 1981; Vinner, 1991). The increased abstraction prevents connections with existing knowledge being formed, a phenomenon termed by Liora Linchevski and Nicolas Herscovics (1996) as a "cognitive gap". They suggest introducing topics with examples rather than generalisations to avoid the cognitive gap. Zhu and Simon (1987) found that examples provided effective learning, even when students worked on examples and problems without lectures or any other direct instruction. This idea may clash with traditional approaches to learning which tend to introduce a topic by teaching generality and following up with examples.

When revising previously-taught concepts, however, learning by following examples is not always the best method. Nguyen-Xuan, Nicaud, Bastide and Sander (2002) found that, for non-novice learners, learning factorizing through observation of worked examples was less effective than doing the factorization themselves. This suggests that a remediation method consisting of practice could be effective for university students who have learnt algebra previously. A possible reason for worked examples being less effective for non-novice learners is suggested by Bowie's (1998) conclusion that university students' difficulties in algebra are due to the meaning that they attribute to symbols and expressions rather than the manipulation of formulae. This view supports remediation strategies that require students to do more than simply drill and practice, such as questionnaires that probe learners' interpretation of variables and expressions. Bowie (1998, p. 126) offered two such strategies: (1) to focus on the process through which algebraic formulae are developed, and (2) to emphasize the interpretation of algebraic symbols and formulae.

Another general (and time-consuming) teaching strategy, advised by Friedlander and Tabach (2001 pp. 173-174), is using multiple representation (verbal, graphical, numerical) where possible to cater for different learning styles. Apart from requiring more time, this approach requires more mental effort from students who may resent having to think about the same thing in different ways.

To overcome the problem of seeing letters as representing names rather than numerical values, as in the students and professors problem (where $6S = P$ is the common error for describing six students to every professor), Siegler (2003a p. 293) recommends practice in representing problem situations with equations and analysing why some equations do not reflect the problem. Booth (2001 p. 110) suggests writing products in full (as $6 \times s$ rather than 6s) until students have developed their understanding of what letters mean. This also have the added benefit of reducing confusion with co-joining symbols in arithmetic denoting addition, e.g. $2\frac{1}{2} = 2 + \frac{1}{2}$, 43 = 4 tens and 3 units.

Recommendations for overcoming errors with algebra specifically

Marilyn Nickson (2000 p. 144-145) summarised points relevant to teaching that she found in her review of research related to algebra at primary and secondary school level. Her recommendations related to fluency in algebraic skills are:

1. Get students to recognise the equals sign as a requirement for balance rather than just a command to give a numerical answer.

2. Point out that unknowns can represent either a fixed value e.g. $2 + x = 5$ or variables e.g. $2l + 2b$ in the formula for perimeter of a rectangle.

3. Firmly establish the order of operations to help in the difficult task of seeing algebraic expressions as objects in their own right, and not just as a left-to-right set of operations.

4. Remind students of the connections and the changes between arithmetic and algebra.

5. Avoid solving equations by mechanical substitutions as this does not promote meaningful learning.

6. Show that solving equations by using only procedural rather than structural methods has limited success.

These points may be too general for university lecturers to incorporate in their teaching, for example, *how* can you get students to recognize the equals sign as a balance requirement, or introduce structural methods for solving equations? The 'mechanical substitutions' referred to in point 5 above would only be possible for very simple equations. Point 5 also implies that students are interested in meaningful learning rather than simply getting an answer. How to make students willing to make the extra effort demanded by meaningful learning demands is the topic of much research (Cavallo, 1996; L. Clement, 1997; Mayer, 2002; Morrone & Pintrich, 2006; Novak, 1990). The lack of detail and general nature of these recommendations makes their applicability to university students seem limited. More useful, in my opinion, were specific suggestions by Tony Barnard (2002a; 2002b) for two common types of errors.

Firstly, to counteract the error of operating on one piece of a compound term, he suggests:

- Separating numbers and variables, e.g. $\frac{2}{3}.\frac{1}{x}$ instead of $\frac{2}{3x}$.
- Using numerical counterexamples, e.g. 6(m – 5) and 6m - 5 are not equivalent forms.
- Using more brackets than needed.

Secondly, his suggestions on how to deal with 'false linearity' errors included:

- Relating to functions (algebraically, numerically or graphically) to show that (for example) the square root function is not linear;

- Writing numbers with denominators (e.g. $\frac{2}{1}$) when multiplying fractions;
- Asking learners to decide on the validity of a list of true and false statements by substituting numbers.

These strategies are ones that could be used during Math 130 lectures when algebra is used in class examples, or when giving one-to-one help to a student. They are quick and easy for lecturers to learn and implement. In order for this to happen, lecturers need to be aware of the common mistakes that students may have and suggestions such as these listed above. One difficulty preventing lecturers from doing so is the time pressure to complete a crowded syllabus, resulting in the sacrifice of algebra revision during class in favour of new Calculus ideas. For this reason, remediation strategies that did not need lecture time but could take place during tutorials were considered more suitable for this study. The suggestions from Barnard (2002a; 2002b) could be incorporated into other remediation strategies, such as a strategy involving cognitive conflict.

2.5. Chapter conclusion

In seeking answers to the research questions, this chapter first reviewed different perspectives on algebra, finding that algebra is more than a set of manipulation skills and techniques for solving equations. Next, arguments for and against manual algebraic manipulation skills rather than using computers were given. Increasingly, the set of algebra skills needed at tertiary level will need to be reexamined as computerized help becomes cheaper and more available to students[3].

Three error classifications and a list of common algebraic errors showed the great variety of errors that have been documented. An error classification scheme, based on the findings here

[3] For example, complex integrals that may have needed algebraic manipulation if worked out manually can be worked out for free at the website www.wolframalpha.com,

and suitable to this study will be developed in Chapter 4, Method, and this will help to answer the third research question, 'Which errors are the most resistant to change?'

The review of literature on remedial programmes for mathematics in tertiary institutions showed that more than one kind of intervention is likely to be most effective. There appears to be a gap in the literature on remedial programmes relating to mathematics at South African tertiary institutions[4]. General and specific remediation strategies to improve basic algebra skills were gathered. These will be drawn on to devise remediation strategies after the theoretical framework is clarified in the next chapter.

This chapter has raised the following questions: What error classification would be most suitable for basic algebra errors made at university, and how would knowing the types of errors made help the learning process? What are the theories behind the teaching strategies suggested in the literature? These questions will be addressed in the next chapter, which develops a theoretical framework from which to answer the research questions.

[4] The nearest research was on the evaluation of remedial programmes for Economics at the University of Cape Town, South Africa (Smith & Edwards, 2007).

3. THEORETICAL FRAMEWORK

The ideas considered in the literature review may give rise to different work-paths depending on the theoretical lens through which they are viewed. In this chapter, different learning theories are considered, some of which inform the design of remediation strategies and others of which inform the analysis of the results of remediation.

There are two main sections to this chapter. The first section addresses the first research question, "Is remediation worthwhile?" in which three related theories on the learning of mathematical concepts are considered, namely reification, process-object duality and the pseudo-structural approach. Deductions on the possibility of the errors being reduced or removed by remediation are described. These theories underpin the design of remediation strategies. Theories relating to research question two, "Which remediation strategy is best?" are described in section two. Three learning theories are considered: behaviourism, constructivism, and participationism. These theories support the analysis of the remediation.

The next chapter will describe the development and implementation of remediation strategies that arose from the literature review and the theoretical framework, including the development of an error classification scheme that will help to address the third research question, "Which errors are the hardest to remediate?"

Before discussing any learning theories, let me describe how my theoretical standpoint has evolved. My perspective at the start of the project was as an acquisitionist with respect to algebra learning, believing that skill in algebra is learnt by accepting rules and practicing until mastery is achieved. I placed emphasis on the syntactic use of algebra rather than on multiple-representations. I have gradually changed my views as I have read more about learning and now believe some viewpoints that I didn't hold before: people learn in different ways and a match between learning and teaching styles is the optimal condition for learning (Felder & Brent, 2005; G. M. Johnson & Johnson, 2006); and that instead of learning simply being the acquisition of knowledge (like a destination), the process of participating – the journey – seems to me a better description of learning (Sfard, 2006; 2008).

3.1. Is remediation worthwhile?

Learning theories and measuring learning

To answer the first research question, the term 'worthwhile' is taken to mean that significant learning has resulted from the remediation strategies, and that the time and effort spent on the intervention brought about greater improvement than having no intervention. The question of how to tell if significant learning has occurred will require an exploration of theories on learning, particularly theories on learning mathematics. These will provide a lens through which to analyze the data.

A dominant view of learning in this decade is that simple theories are inadequate to explain something so complex (Sfard, 2001). The trend is towards meta-theories that encompass a variety of theories (Duit & Treagust, 2003 p. 680; Jaworski, 2002 p. 75; Kieran *et al.*, 2001; Laurillard, 2002; Treagust & Duit, 2008), or to supplement a learning theory found to have shortcomings (Sfard, 2001). The synergetic effect from using multiple theoretical outlooks (Sfard, 2003) has inspired this study, and in this section I will consider three related learning theories in forming a theoretical framework with the aim of finding out how, if at all, teaching interventions can help tertiary students who have not acquired basic algebra skills during high school. More general learning theories are considered in the next section, which investigates the research question, "Which remediation strategy is best?"

The theories[5] considered here are: Reification (Sfard, 1991; Sfard & Linchevski, 1994), which describes how concepts are constructed; the Process-object Duality (Sfard, 1994), which

[5] It could be argued that these are not distinct theories but parts of the same theory of learning, for example, Reification deals with something becoming an object (from the Latin 'rei' meaning 'of a thing'), so it is part of Process-object Duality.

describes stages of conceptual understanding that may be necessary for fluency with algebraic manipulations; and the Pseudo-Structural approach (Vinner, 1997), which warns about mis-interpreting students' responses. These theories are closely linked. Reification, dealing with the formation of objects, is part of the Process-object Duality theory, and the Psuedo-structural approach is a way of avoiding reification. Each of these three theories is described below and then analysed in terms of providing explanations for learning failure and the possibility of remediation to reverse the learning failure.

Reification – an explanation of concept construction

Anna Sfard (1991; 2003) identifies three stages of concept construction:

- interiorisation (skillfully performing processes on known mathematical entities, for example[6], writing $x^2 + 6x + 11$ as $(x^2 + 6x + 9) + 2$, in preparation for completing the square),
- condensation (analyzing the process into manageable units through structuring and symbolizing, for instance thinking of completing the square as a sequence of steps) and
- reification (to think and speak of a process in the way we think about an object – not having to remember its inner structure each time we use it, for instance, completing the square in one step).

A possible reason for students failing to achieve interiorisation of algebra concepts from high school is John Mason's (Mason, 2002 p. 169) claim that "interiorisation is only possible if the same process is encountered in the behaviour of relative experts". This echoes the idea of the zone of proximal development (Vygotsky, 1978), in which learning occurs most effectively if students are challenged slightly above their current level of understanding, since a "relative

[6] This example was not given by Sfard.

expert" is able to provide appropriate challenges. The large numbers of unqualified teachers in many South African schools (Bertram, Appleton, Muthukrishna, & Wedekind, 2006) may have left many secondary students deprived of expert role models to provide them with the types of experiences that would foster interiorisation. As a result of interiorisation failing to occur, many students are unable to progress to condensation (where a process can be thought of as a whole and different representations of the concept can be accessed with ease) and finally to reification (where there is a sudden ability to see a familiar procedure in a new light).

Even once interiorisation is achieved, there may be other obstacles to concept construction. Anna Sfard (1995 p. 35) talks of the 'vicious cycle of reification' in which a deep understanding of a concept and familiarity with the processes underlying the concept seem to be prerequisites for each other. She admits that sometimes practice is needed before understanding is achieved. The idea that practice is a necessary step towards concept construction gave rise to a remediation strategy of weekly practice with algebraic manipulations as a means to improving students' concept formation, understanding and execution of algebraic manipulations.

With the perspective that learning happens according to Reification, successful remediation would help students to move through the three stages of concept construction. An obstacle to this may be the necessity to first reverse the interiorisation of incorrect concepts. The method of cognitive conflict or cognitive dissonance (Festinger, 1957, Bell, 1993) may be effective in this regard, as may be repeated opportunities for learners to interiorise, condense and reify the new concepts brought about by the cognitive conflict.

The Process-Object or Procedural-Structural view of algebra

Carolyn Kieran (1992) and Anna Sfard (1995) looked at the historical development of algebra to help explain how it is learnt. Both concluded that the transition from arithmetic to algebra involved a shift from procedural operations on numbers, e.g. $12 – 4 = 8$, where the answer is seen as an object, to process-oriented operations on algebraic expressions that yield other algebraic expressions, rather than numbers, as answers, e.g. $2x + 5y + x = 3x + 5y$. The expression $3x + 5y$ can be seen as both an object (the answer to the simplification) as well as a process that could be carried out for given values of x and y. The presence of the + sign makes it even more difficult for learners to appreciate the object nature of the two-termed expression $3x + 5y$, compared to a single-termed expression.

Anna Sfard (1992) describes these stages using the terms 'operational' (e.g. seeing $4x + 3$ as an action which could be performed for a given x-value) and 'structural' (e.g. seeing $4x + 3$ as an object that could itself be operated on by, say, multiplying it by 2). The same ideas have been called 'dynamic' and 'static' by Goldenberg, Lewis and O'Keefe (1992). Gray and Tall (1994) used the term ‘procept’ to indicate the duality of an expression, e.g. $2(4x + 3)$ representing the process of multiplying $4x + 3$ by 2 and the concept of multiplication of a binomial with a monomial.

Kieran (1992) gave a comprehensive review of empirical studies on the teaching and learning of school algebra and concluded that the majority of high school students do not acquire any real sense of the structural aspects of algebra. Algebra teaching usually starts with substitution exercises that reinforce the process nature of algebra (Nickson, 2000, p. 114) but the transition to operations that require a structural view, such as simplifying and solving equations, is not made clear and students are often not aware that they have moved to a level where less tangible rules apply. Sfard (1994) noted that experts are easily able to move between thinking of mathematical expressions as processes or as objects, seeing them as complementary modes of thinking, whereas novices seem to work on a mostly operational level.

Gray and Tall (1994) suggest that students fall into two groups: those that have a flexible use of a symbol as object or process, and those for which processes do not become objects. Those falling into the ‘process-only’ category of this ‘proceptual divide’ are more likely to fail at mathematical tasks. Gray and Tall showed that proceptual thinking is required at all levels of mathematics. This raises some questions: How can you identify proceptual thinking? Is it possible to succeed at a certain level with only a partial understanding of procepts? How can teaching promote proceptual thinking? There do not appear to be quick-fix teaching methods to develop this type of thinking.

Along the same lines as Sfard (1994) and Gray and Tall (1994) is the APOS theory developed by Ed Dubinsky (2001) which has been used to inform teaching and develop curriculum at tertiary level. APOS is an acronym standing for actions, processes, objects, schemas and it describes a hierarchical description of increasingly deeper levels of understanding. The problem with using this theory in teaching (as opposed to using it as a research tool) is that the transition from action to object in the context of algebraic manipulations is difficult to distinguish from procedural content that has been memorized.

Sfard (1991) stated that we must accept sometimes working with new processes on 'pseudo-objects', mathematical objects that are not yet understood. This can be seen to provide justification for the remediation strategy of practice, particularly in the case of students using algebraic objects in the study of calculus. This can also explain how some students can score better in calculus than an algebra course that they are taking concurrently (Hawker, 1987). Even though the algebra that is not well understood is used in the calculus course, by treating algebraic expressions as pseudo-objects, progress in calculus can be made.

The dual nature of algebraic expressions as processes or objects gives a framework for analyzing students' errors, but it would require an in-depth look at each student to determine their position on the continuum between procedural and structural views, which is beyond the scope of this project, although it would be an interesting topic for a future study. Instead of this theory being used in this study to analyse students' responses to determine whether or not remediation was worthwhile, it helped in the design of questions and incorrect options that could be reached from an object-only view, e.g. the error of 'factorising' $18x^3 - 8xy^2$ as $10x^2y^2$ might be explained by an object view that holds to the idea that an 'answer' should be a single term. This may be an improvement for a student who before only held a process view of algebraic expression.

The idea of something pretending to be something it isn't is also behind the pseudo-structural approach to learning algebra.

The Pseudo-structural approach

Anna Sfard and Liora Linchevski (1994, p. 223) explain how unsuccessful students develop a pseudo-structural approach to algebra, in which algebra becomes a collection of meaningless symbol manipulations. For example, a formula may be seen as equivalent to a function, so two formulae, like $2x + 3$ and $2p + 3$, are not seen as representing the same function even if they differ only in the variable used. From a constructivist perspective, a pseudo-structural approach occurs when skills and concepts are compartmentalised rather than being linked in conceptual structures.

This idea is reinforced by Tall and Thomas (1991) who state that when children are unable to give meaning to concepts, they hide their difficulties in routine activities that give them correct answers and approval. Lynn Bowie (1998) showed that a pseudo-structural approach to algebra was evident in tertiary mathematics students at a South African university. This, she claimed, can be attributed in part to the poor mathematics teaching they received at high school.

Shlomo Vinner (1997) summarizes the problem presented by the theory of pseudo-structural approaches, in which learners may take easier alternatives to the more conceptually challenging tasks involved in reification:

> **"The difficulty lies in the fact that one of our educational goals is to 'install' certain mental processes in the mind of students. In order to know whether this goal has been achieved we have to observe student behaviour. But if the expected behaviour can also be produced by alternative (undesirable) thought processes, then firstly, you might have missed our educational goal, and secondly, we might not even know that we did miss it." (page 100)**

To avoid, or at least limit, students taking a pseudo-structural approach, test questions should be carefully designed to expose common misconceptions.

Looking at learning as described by the theories above is helpful for understanding how learning progresses and suggests that there are stages of understanding through which learners move. These theories also give suggestions for the types of errors that could be expected, which would help the design of pre- and post-tests to determine if remediation is worthwhile. The challenge is how to facilitate the move from, say, a process-view to an object-view, or through interiorisation and condensation to reification. In the next section more learning theories are considered, with the aim of finding support for remediation strategies that can help the learning process as described above.

3.2. Which remediation strategy is best?

In this section, more general learning theories are considered compared to those in the previous section. Morris Bigge and Samuel Shermis (2004, p. 10) identify two broad families of learning theories: stimulus-response conditioning theories of the behaviourist family and interactionist theories of the cognitive family. This could be seen to exclude the social learning theories which are presenting an increasingly popular alternative (Lave, 1988; Lerman, 2000; Sfard, 2001, 2005; Wenger, 1998). The following sub-sections will consider the strengths and weaknesses of four learning theories that could offer explanations for how learning basic skills at university occurs. The four theories considered are:

1. behaviouristic theories including practice and rote learning
2. conceptual change, an interactionist theory
3. Bruner's cognitive-interactionist theory of learning, an interactionist theory, as the name suggests (See Bigge and Shermis, 2004, p. 133 – 134, 137, 139)
4. the communicational approach to cognition, closer to a social learning theory than an interactionist theory

I wanted to include behaviourism because I felt it was a theory that many people may have been exposed to during their mathematics education at school, and it may be an approach that could work for learning skills that had been first seen at school. It also promised to be a theory with clear support for implementing a remediation strategy.

Bearing in mind the limitations of behaviouristic theories (discussed below) interactionist theories seemed worth exploring for alternatives to behaviouristic theories. Notions in conceptual change provided ideas for remediation but I felt that there were also other ways to consider learning and this led to the inclusion of the last two theories.

Behaviouristic theories, practice and rote learning

Practice, underpinned by the learning perspectives of behaviourism and associationism, was traditionally seen as the best way to learn manipulative skill and competence with techniques (Mason, 2002 p. 174). The underlying assumption is that learning is largely a process of habit formation and that more repetition results in longer retention of a skill. Mathematics learning depends heavily on previously learnt concepts and for this reason a certain amount of memorization may serve as a foundation for achieving understanding over a period of time (An, 2004; Siegler, 2003b; Leung, 2001). Sfard (2003) promotes the idea that some proficiency in basic skills is necessary for the learning process to be possible, citing five empirical studies that support this view (p. 365). She summarises the paradox with a metaphor:

"…the emphasis on understanding may have deprived children of something to understand. The claim about the possibility of learning mathematics meaningfully without some mastery of basic procedures may thus be compared to the claim about the possibility of successfully building a brick house without bricks."

Practice and repetition have been of interest to education theorists for a long time. Thorndike (1922) used a theory of mental bonds (e.g. between '3 × 5' and '15'), in which bonds get strengthened or weakened with the presence or absence of practice. Wertheimer (1945/1959) noted the usefulness of repetition but also found that continued use of repetition induced habits of blindness and non-thinking. The dangers of an overuse of drill led to a Gestalt perspective[7] that was concerned with understanding and problem solving more than facts and processes. The Gestalt perspective changed the view of learning from a process of receiving, combining and reproducing to one of selecting, grouping and reconstructing (Schoenfeld, 2002). However, research methods used by Gestaltists, such as subjects' reports on their own mental processes, proved unreliable, and from the 1950's, other methods had replaced the Gestalt perspective.

Using practice as a learning strategy can bring positive or negative results. John Mason (2002 p. 180) distinguishes two ways to use practice: (1) helpful practice should divert attention away from the details and make actions become automatic; (2) unhelpful practice involves getting the answers while paying as little attention as possible. The difficulty is that the same classroom activity can result in either helpful or unhelpful practice depending on other factors, such as students' beliefs of how learning occurs based on their past experiences, students' and teachers' motivation towards the activity, students' level of expertise on the topic, or on their concentration levels due to their mood, when last they ate or how sleepy they are feeling (Mason, 2002, Dweck, 1999).

[7] Gestalt is a German term meaning that the whole is greater than the sum of the parts, e.g. a triangle is more the sum of the line segments that make it because if its Gestalt.

Rote learning is a term for learning by repeated practice and has long been criticized for promoting memorization without understanding (Spencer, 1878, in Mason, 2002) but Mason (2002 p. 152) claims that it is not by itself harmful or unproductive. The usefulness of rote learning depends on whether the memorized fact is put to further use. Bowie (1998) investigated university students' Calculus learning and warned that learners who have achieved success with rote learning (by being admitted to university) may resist attempts to replace it with an alternative learning style that may be more demanding for them but which is necessary to achieve the outcomes of university courses. This reliance on rote learning can hinder their progress with university mathematics when there are demands on them for productively using their memorized facts.

Strong support for rote learning was given by Dick Tahta (1972 p. 15 in Mason 2002 p. 175) who advises allowing most work in mathematics to be done in a mechanical way without being directly conscious of its meaning. He adds, "…teaching involves preventing mechanicalness from reaching the degree fatal to progress". Criticisms against rote learning could be a result of poor teaching training and practice rather than problems with the learning strategy of rote learning. For example, Ma's (1999) in-depth study of American and Chinese primary school teachers showed that Chinese teacher training involves much more content knowledge than in America, where there is a greater emphasis on pedagogical issues. Comparing rote learning in Chinese and American schools may give difference due to the ways teachers work around rote learning sessions rather than the practice of rote learning alone.

CRITICISMS OF THE THEORY

Behaviourist learning theories have been criticized for being unable to account for some students' persistent failure to progress in mathematics (Sfard, 1991). In control/experiment and treatment A/treatment B studies based on behaviourism (popular in the 1950's to 1970's), unreliable conclusions were made from studies with too many uncontrolled variables, such as ill-defined treatments, different teachers for control and experiment groups, teachers' preferences for a treatment, and time of day of treatments, e.g. first period or after lunch (Schoenfeld, 2002, p. 440).

Diana Laurillard (2002) warns against trying to fix errors without considering their context. This gives support to the strategies that are more than drill-and-practice. She writes:

> **"It makes no sense to remediate a faulty procedural skill with reference to the procedure alone; we have to appeal to the conceptual apparatus that supports it as well." (p. 30) "… 'buggy' behaviour manifests an underlying conceptualisation that itself needs remediation." (p. 31)**

Weighing up the criticisms of drill and practice with the arguments supporting practice, I felt that it would still be of interest to test a behaviourist-based approach but that other remediation strategies should also be considered for this study. An alternative approach came from Jan van den Brink (Van den Brink, 1993), who found that practice increased fluency but striking surprises, i.e. cognitive conflict, were important for developing students' understanding. This gave rise to remediation based on the idea of changing students understanding by presenting them with a situation that presents a cognitive conflict with their incorrect conceptions. These ideas are founded in the learning theory of constructivism.

Conceptual Change and Constructivism

Constructivism is regarded as the basis of virtually all cognitive science theories because of its central claim that knowledge is actively constructed from experiences and interactions with the world (Lesh and Doerr, 2003, p. 532), rather than passively absorbed (the empiricist view) or deduced logically (the rationalist view). The constructivist claim that learning is an active process contrasts with behaviourist learning theories that emphasize remembering and imply the familiar metaphor of filling students minds as though they were 'empty vessels' (Kivinen and Ristela, 2003).

The term conceptual change refers to the process by which a person's central, organizing concepts change from one set of concepts to another set, incompatible with the first (Posner, Strike, Hewson, & Gertzog, 1982). Pre-instructional conceptual sets must be fundamentally restructured to allow for understanding of new knowledge (Duit and Treagus, 2003 p. 673). From this theoretical perspective, remediation 'instruction' (e.g. worksheets, discussions with peers and tutor explanations) could be designed to bring about conceptual change whereby old, faulty concepts about algebra are replaced with more correct ones. According to Lev Vygotsky's theory of the Zone of Proximal Development, learning will be most effective if the ideas shared are slightly more difficult than those currently held by the students (Watson, 2002, p. 228, Vygotsky , 1978). Teaching methods informed by conceptual change are usually superior to more traditional forms of teaching (e.g. drill and practice) (Duit and Treagust, 2003, p. 683). A choice of remediation strategies for this study based on conceptual change as well as practice would put this claim to the test at university level, although in a limited setting of tutorials but not in lectures too.

The mechanism of conceptual change can be described in the Piagetian terms of assimilation and accommodation (Duit and Treagust, 1999 p. 9). Assimilation is the fitting of new ideas into the existing cognitive structure or schema. If the existing schema has to be modified to assimilate an idea, accommodation takes place. The disturbance in mental balance when new ideas do not fit existing schema is called cognitive conflict (ref), or cognitive dissonance (Posner *et al.*, 1982). Dina Tirosh and Pessia Tsamir (2004) argued that the conceptual change approach was applicable to mathematics education since it had the explanatory and predictive power needed from a theory. Stella Vosniadou and Lieven Verschaffel (2004) distinguish conceptual change from most other learning, stating that whereas most learning involves adding to existing knowledge, conceptual change deals with changing existing knowledge.

Theories on conceptual change fall into two theoretical perspectives: (1) knowledge-as-theory perspectives, where knowledge is seen as a hierarchical framework of theory-like knowledge, and (2) knowledge-as-elements perspectives, where knowledge is seen as an "ecology of quasiindependent elements" (diSessa, 2006; Ozdemir & Clark, 2007). The first perspective has been dominant in research literature, for example, saying that conceptual change involves gaining more abstract, general and powerful explanatory frameworks (Vosniadou and Verschaffel, 2004, p. 448). The remediation strategies in this project were designed before this separation of perspectives was reported and so my ideas were positioned in the knowledge-as-theory perspective. It could be of interest for further studies to compare how a knowledge-as-elements perspective would have changed the remediation design.

The conceptual change framework offers an explanation of why teaching intervention may not be successful. If students are not dissatisfied with the new concept, it can be assimilated and used with the old concept (Duit and Treagust, 2003 p. 676). Dissatisfaction between old and new concepts causes the one with higher status, as determined by the student, to dominate. However, replaced concepts are not forgotten and may surface at a later date (*ibid.* p. 677). This could explain how students repeat errors that they have previously been able to avoid, or how students can give wrong answers to tasks they were once able to do correctly. If knowledge has to be reorganized, this may also explain why fixing some mistakes may lead to new ones. Nesher (1987) also found that students could give right answers but for the wrong reasons, thereby making the misconception hard to detect. Sfard (1991) suggests that students can hold more than one conception of a phenomenon simultaneously, externalizing whichever conception seems to fit a given problem.

Criticisms of the theory

According to the conceptual change model first developed from Posner, Strike, Hewson and Gertzog (1982), the degree of change will depend on if the student has found the concept intelligible (non-contradictory and something they can understand), initially plausible (believable) and ultimately fruitful (able to be used to solve problems). Good levels of change require learners to be self-responsible, co-operative and self-reflective (Duit and Treagust, 2003, p. 681) and to see cognitive conflict as more than marginal (Duit and Treagust, 1999 p. 12). This can be hard to determine since what counts as conflict to one learner may not mean the same to others. Watson (2002, p. 228) warns that there is no guarantee that interactions between learners will bring about the required level of conflict. To overcome this, she used research interviews in which learners could express their beliefs clearly and allow the interviewer to form a conflicting opinion with which to confront the learner. However, this does not offer a sustainable remediation strategy given the time and staff constraints that could realistically be spent on basic skills in a first-year mathematics course.

Strike and Posner (1982) criticized constructivism for neglecting social and affective issues, as did Pintrich *et al.* (1993) who called the theory 'cold' (Duit and Treagust, 1999 p. 14). Social-constructivists worked on improving the conceptual change theory to deal with the shortcomings that had been raised (Duit and Treagust, 1999 p. 18). More recent multi-perspective views on conceptual change see the learner's role as depending not only on concepts but also affective factors such as one's belief about how learning happens (Philipp, 2007). The need to extend the basic ideas of constructivism was highlighted in a literature review by Duit and Treagust (2003 p. 673) that could find no studies in which a student's existing ideas get completely extinguished and replaced with the correct scientific view.

Duit and Treagust (1999 p. 13) criticize conceptual change studies for not explaining the knowledge construction process. They found that learning pathways are very complex and often include backwards and forwards movements, suggesting that repeatedly making errors may be a necessary part of the learning process, and not a sign that a learning strategy has been unsuccessful. This suggests that once-off learning interventions may be less successful than long-term, repeated interventions and that temporary or partial conceptual change may signify progress in learning although it may not be detectable on a test.

Some critics of constructivism argue that the theory implies that remembering is not important, and that understanding concepts is all there is to learning, neither of which is true (Fox, 2001). For example learning the names and symbols of numbers requires memorization rather than understanding.

For more criticisms of the conceptual change theoretical framework described by Posner *et al.* (1982), see Vosniadou and Verschaffel (2004 p. 446). Conceptual change has also been criticized from the perspective of more social learning theories (Lerman, 1996), but I will not enter this debate here.

Bruner's cognitive-interactionist theory of learning

Jerome Bruner (1990) considered the idea of culture in learning in his cognitive-interactionist theory of learning, proposing that most learning is a sharing of human culture. Culture is constantly being recreated through negotiation of meaning, as opposed to being transmitted by experts. Learning, claims Bruner, is about making meaning, not about behavioral training. Bruner's theory has two central themes, which echo constructivist ideas: (1) knowledge acquisition is an active process; (2) knowledge is actively constructed by relating new information to a previously acquired internal model. These themes offer a perspective to view the learning of basic skills. If this is how learning happens, then to maximise learning we should investigate how the relation of new information to old can be made more efficient, and what kind of activities can enhance knowledge acquisition. The difficulty in putting this into practice with a class is that each student has a different set of 'old' information. This suggests that learning may be more successful when students are given individual tasks.

In describing learning in more detail, Bruner identified three processes that he claimed happen almost simultaneously:

1. Acquisition of new information, which may be a refinement of previous knowledge or which may run counter to previous knowledge;

2. Transformation of knowledge;

3. Checking the relevance and adequacy of knowledge.

When learning has not been successful, as in the case of weak basic algebra skills in university students, there may have been a breakdown in one or more of these processes. For example, if new information does not cause a transformation of knowledge or is perceived to be irrelevant, learning may not occur. Without motivation for the learning of basic skills, which may be seen by some students as boring or unnecessary, students may not move beyond acquiring new information (e.g. through the correction of their work by a teacher) to transforming their knowledge (e.g. by not making the same mistake again).

CRITICISMS OF THE THEORY

Sfard (2003 p.372) warns that interactivity in learning mathematics is complex and that solitary work and substantial interventions by the teacher may be vital for effective learning. She substantiates this claim by referring to a study (Sfard *et al*, 1998) that reported mathematicians' preferences for solitary rather than collaborative work. This allows a focused effort on problems that demand concentration and intellectual effort to be carried out in silence, without the distraction of the strain of communicating with others.

On this basis, remediation strategies were devised that allowed students to work individually either completely or before discussing ideas with their peers or their tutor. The strategies were called "Cognitive Conflict Self-guided" and "Cognitive Conflict Tutor-led". The students would be allowed to choose whether to work collaboratively or alone. In the 'tutor-led' strategy, the tutor would explain solutions after students had recorded their single or group answers.

The communicational approach to cognition

Anna Sfard (2001) has two main criticisms of acquisition-based learning theories (including behaviourism and constructivism): firstly, they separate cognitive activities from their context (Sfard, 2001, p. 22), leaving out elements that could radically change our view of learning, and secondly, the implied view of knowledge in these theories is that knowledge is something to be given and acquired rather than something fundamentally social in nature (Sfard, 2006; Sfard, 2000). To overcome these limitations, Sfard (2001) puts forward the communicational approach, based on the idea that communication[8] and thinking are practically the same thing, and that learning mathematics means being initiated into a well defined discourse. From this perspective, the context in which learning occurs also defines the actual learning and this should be remembered when designing remediation strategies.

Discourse is made up of two main elements: the symbolic tools of communication such as words, algebraic expressions, graphs, which Sfard calls mediating tools, and the meta-discursive rules that regulate the communication such as tone of voice, body language, gestures, intentions and feelings. Both of these elements carry cultural heritage (Sfard, 2001 p. 28). For Sfard, 'discourse' includes 'anything that goes into communication and influences its effectiveness', including body movements and situational clues. Learning is seen as increasing enculturation into a practice, with its ways of speaking, its ways of acting, its goals, ways of relating, and so forth (cf. Wenger, 1998). By considering discourse defined in this way, analysis of learning situations becomes quite complex.

According to Sfard, acquisition-based learning theories, including cognitive conflict, assume that the learner is after the 'truth of the world' (Sfard, 2001 p. 49). This is seen by the use of terms that imply a difference between learners' conceptions and expert (or official) concepts, e.g. concept image versus concept definition (Tall and Vinner, 1981), and in terms such as 'misconception studies'. These theories imply that the acquisition of new knowledge is a result of trying to understand given facts and ideas, which does not necessarily need mediation from other people. This contrast with a participationist view (Sfard, 2001) in which learning is seen as the changing of our discourse. The desire to adjust our use of discourse to that of others (in other words, the need for communication) provides the main motivation for learning, in this theory.

[8] Sfard points out that communicating can be the writing of rule-bound symbols, as in algebra.

Criticisms of the theory

The inclusion of unobservable features like intentions and feelings is both a strength and weakness with Sfard's theory. Including these features makes the attempt to understand human actions more realistic. However, the researcher has to make interpretations of other peoples' intentions and this will always be subject to questioning. The best that a researcher can do is to provide interpretations that are as convincing and trustworthy as possible.

The cultural heritage that Sfard (2001, p. 28) identified as being present in discourse makes it more difficult to convey meaning effectively to the whole class in a multi-cultural classroom, such as in this study. Students whose culture is similar to the teacher will have an advantage over other students. It is difficult to overcome this difference, particularly when teaching is not designed to take this into account.

I will look at applying this framework in a smaller scale than presented by Sfard's case studies (2001). Instead of interviewing or recording individual students while they work, I will consider the meta-discursive background to be the class environment and of the interactions between tutors and students, and between students themselves. This should make it easier to identify areas where improvements can be made, should the remediation strategies not be effective

3.3. Conclusion

In this chapter, some of the common learning theories were described and assessed in term of their applicability to an investigation of basic skills. Two main theoretical approaches will be used to underpin the development of remediation strategies for improving the basic algebra skills of students in our first-year mathematics course. The theories to be applied are Behaviourism and Cognitive Conflict. Chapter 4, which follows, will describe how the remediation strategies were designed and executed.

4. METHOD

This chapter describes the steps taken to find answers to the three research questions introduced in the rationale. First, general issues relating to the methods used are discussed, including the selection of remediation strategies, the design of the questions and the validity and reliability of the chosen methods. Next, the research questions are addressed one by one, with descriptions of the tools used to find answers to the questions. In the following chapter, the results from the methods described in this chapter and examined and discussed.

4.1. General method issues

Theories underpinning the methods chosen

Let me first explain how my own research orientation affected the choice of remediation strategies. I started out with an acquisitionist view, believing that basic algebra skills would best learnt the way I had learnt them – by accepting rules that a teacher gave and becoming able to use them fluently through practice. I recall finding the process of practicing algebra skills satisfying, but not very challenging or stimulating. Although I could see that this method didn't work for all my school-mates, I still believed that practice was the best way to remediate problems with Mathematics. My experience as a university student should have changed this view as I experienced learning through discussions with tutors, lecturers and fellow students. As a student demonstrator I was trained to ask leading questions to help students move along the path to understanding. Despite my experiences at university, which aligned more with a participationist view of learning, I still started this project from the perspective that by giving students remediation they would acquire the knowledge they were lacking, without much consideration of the social factors that I might have focused on if I had taken a participationist view at the start.

From my acquisitionist perspective, I looked to the literature to find learning theories that suggested remediation approaches we could implement. My reading led me to discover that many learning theories and research studies on algebra have focused on early algebra learning. This study is different because it looks at young adults and algebra skills they (should) have been learning for five years at high school. For this reason, choosing a learning theory to frame this study in was not straightforward. Although a theory like behaviourism is seen to be old-fashioned and has been subjected to criticism (Kieran, Forman & Sfard 2001, p.3), it may have relevance in a study of basic skills. I felt that the strategy of practice would still be useful to test – even if to discard it as unsuccessful later. Perhaps university students had missed out on sufficient practice at school and we could fill that gap now.

I was drawn to the approach of creating surprises (cognitive conflict) - through which incorrect concepts could be shown to be wrong and could be replaced by correct ones - after having experienced success in my own learning and when helping others, as well as reading about it in the literature e.g. (Tanner & Jones, 2000; Tyson, Venville, Harrison, & Treagust, 1997). The ideas of learning as a complex process of participating in a community of practice also struck a chord with me but I felt that to follow this way of viewing learning would lead to a much bigger project than I was able to cope with[9]. It has made me more aware of how others could be seeing learning and how that could motivate or de-motivate them, for example, seeing mathematics as a collection of meaningless practice would not make them want to be part of that community and their mathematics skills will probably be weak as a result.

Question design and data analysis

The research design for this project is outlined in Figure 4.1, which shows the sequence and timing of testing, intervention and data analysis that I undertook for this project.

[9] PhD studies by Mellony Graven (2004) and Murthi Maistry (2005) have taken a participationist approach.

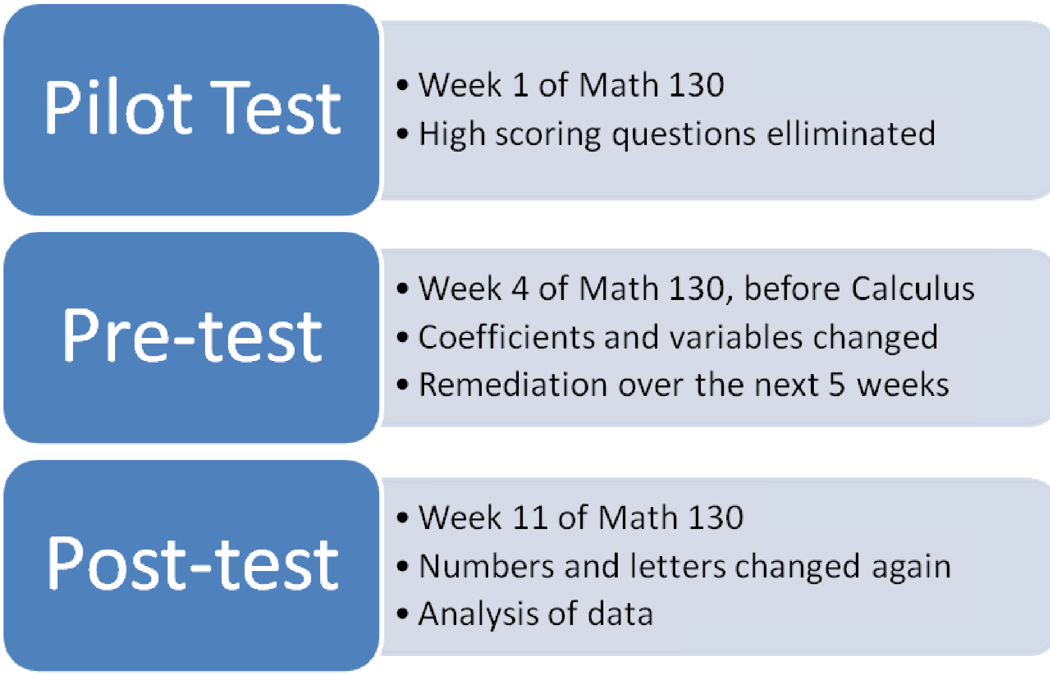

Figure 4.1 Sequence and Timing of Testing, Intervention and Data Analysis

The choice of which errors to test for was based on the literature, the error classification scheme described below, as well as common errors from experience. Table 4.1 lists the questions used in the tests and describes the errors they were testing for. The topics tested include using the order of operations, simplifying fractions, expanding, factorizing, recognizing equivalent forms, ratio word problems, logarithms and exponents.

The Error Classification detailed in section 4 below is included in this chapter because it guided the design of the multiple choice questions used in the pre-and post-tests. A literature search exposed a gap in existing error classifications: no classification was found that would be suitable for the types of algebraic errors that might be expected from tertiary students. This led me to compile a classification suitable for this study that combined elements from three error classifications and other errors from the literature.

Table 4.1 Pre- and Post-test Questions and the Errors Tested by Each Question

Question No.	Question (Pre-test)	Question (Post-test)	Errors Tested
1	$4x^2y + 2x^2y \times \frac{3z}{x^2y} =$	$2ab + 3ab \times \frac{c}{ab} =$	Simplifying fractions Distorted algorithm
2	$\frac{4ab + 2b^2c}{8abc} =$	$\frac{6a^2b + 4b^2}{12a^2b^2} =$	Simplifying fractions Over-generalising
3	$(a^2 + a^4)^{\frac{1}{2}} =$	$(x^2 + 9)^{\frac{1}{2}} =$	Distorted algorithm Over-generalising
4	If $y^x = 3$, then $x =$	If $a^x = 5$, then $x =$	Distorted algorithm
5	If $\log_5 x = 10y$, then	If $\log_3 x = 6y$, then $x =$	Distorted algorithm
6	$(7x^2)^3(7x)^2 =$	$(3x^2)^3(3x)^2 =$	Distorted algorithm
7	$(3x^2 - 2)(2x^2 - 3) =$	$(3x - 2)(2x - 3) =$	Distorted algorithm Technical/Careless Minus signs
8	If $-y + 8 < 10 - 2y$, solve for y.	If $-2y + 6 < 8 - 3y$, solve for y.	Distorted algorithm Technical/Careless
9	$9x^2 - 4y^4 =$	$9y^2 - x^4 =$	Minus signs Equals signs Technical/Careless
10	$(2a - 2b - 7) - (a + 2b - 6) =$	$(4a - 2b - 7) - (a + 2b - 6) =$	Distorted algorithm Minus signs

Table 4.1 (continued) Pre- and Post-test Questions and the Errors Tested by Each Question

Question No.	Question (Pre-test)	Question (Post-test)	Errors Tested
11	$\frac{(x+7)(2x^2-5)+x^2}{(x+7)(x^3-4)} =$	$\frac{(x+7)(2x^2-5)+x^2}{(x+7)(x^3-4)} =$	Simplifying fractions
12	$(3x^2y)(-2xy)^2(-x) =$	$(3x^2y)(-2xy)^2(-x) =$	Distorted algorithm Technical/Careless
13	$\sqrt[3]{-(8x^3)} =$	$\sqrt[3]{-(3x^3)} =$	Omitted
14	Factorise fully: $4a^3b^4 + 2a^2b^5$	Factorise fully: $4a^3b^4 + 2a^2b^5$	Distorted algorithm Technical/Careless
15	If $2(x-10) = 4(3x+5)$ then $x =$	If $2(x-8) = 4(3x+4)$ then $x =$	Omitted
16	$\frac{\lvert x-4\rvert}{\lvert 4-x\rvert} =$	$\frac{\lvert x-2\rvert}{\lvert 2-x\rvert} =$	Distorted algorithm
17	If $\frac{1}{5}$ of a number is 20, what is $\frac{1}{4}$ of the number?	If $\frac{1}{4}$ of a number is 20, what is $\frac{1}{5}$ of the number?	Distorted algorithm
18	$\frac{5}{a} + \frac{3}{b} =$	$\frac{4}{a} + \frac{3}{b} =$	Distorted algorithm Over-generalising
19	If 20 bags of seed are needed to seed 8 hectares of land, how many bags are needed to seed 12 hectares of land?	If 20 bags of seed are needed to seed 8 hectares of land, how many bags are needed to seed 12 hectares of land?	Distorted algorithm
20	$\tan^{-1}x =$	$\sin^{-1}x =$	Notation
21	$\log_a a^3 + \log_a a =$	$\log_a a^4 + \log_a a =$	Distorted algorithm
22	$\frac{\log_a 6}{\log_a 3} =$	$\frac{\log_a 6}{\log_a 3} =$	Distorted algorithm Technical/Careless

The first basic skills test (referred to as the pilot test) was given to all students in their first tutorial of the course. The questions for the test were chosen to expose common errors from the error classification drawn up in the theoretical framework and from past basic skills tests used at UKZN. The results of this 36-question test were analyzed and questions that 95% or more of the class could correctly do were discarded. The remaining 22 questions made up the second test, referred to as the pre-test. This test was administered to the students in the fifth week of term, once they had completed the Discrete Mathematics part of the course and were about to start Calculus, but the name pre-test refers to it being prior to remediation.

The teaching interventions were made during the weekly tutorials rather than during lectures. This was beneficial for two reasons: Firstly, if the interventions were successful, we could be sure that it was not the result of a highly effective lecturer who might not take the course in the future. Secondly, the logistics favoured the use of tutorial groups. More time was available in the three-hour tutorials compared to the 45-minute lectures, and the class was divided into 5 tutorial groups, making it convenient to test a variety of remediation strategies.

In the second-last week of term, six weeks after the pre-test, a post-test was administered during tutorials. Experience had shown that attendance in the last week was usually low. The same types of questions (with changes to coefficients and variables) were used for the pre- and post-tests. This made it possible to compare questions and responses from test to test. However, it could have the effect of making the subsequent tests easier because the questions were familiar, so higher marks could be expected in the later tests. Since the average marks for the tests were close to each other, this appeared not to be a significant factor. The gap of 6 weeks between tests also helped to lessen the impact of memory (of previously-seen similar questions) on the results (Squire, 1986).

The format of the tests was a compromise between the convenience of multiple choice answers and the value of free-response questions. Each question had five multiple choice answers (a to e) as well as option (f) which had space for the students' own answers if they felt that none of the listed options was correct. Since the correct answers were within the given options, most of the (f) responses were incorrect although there were a few responses that were alternative forms of the given correct answer. This usually occurred when their written answer in (f) was very similar to one of the given answers. Correct alternative results were coded as the correct answer for the purpose of analyzing the data. The answer sheets were scanned and marked by a computer using the software OpScan and Scan Tools.

The limitations of using multiple choice questions (see for example (Haladyna, 2004)) were partly eliminated by having the option of a free response. However, having a list of answers to choose from could have guided students compared to completely free-response questions. For example, a student who might have set an expression equal to 0 and solved for x when asked only to factorise might change their mind if none of the given options are an equation like $x = 3$. Also, when faced with a question that they cannot do, a student can guess an answer. There would be no way to know that the answer was guessed rather than represented a deliberate choice.

Written responses from students explaining the reasons for their choices could have added more depth to this study but this was avoided for two reasons. Firstly, it would have made the analyses of results complex, given the variety of possible reasons that students might give. Alternatively the responses may have been of low value, as in "this is how fractions are simplified," accompanying an answer where a fraction had been incorrectly simplified. Interviews would have allowed deeper probing into students' reasoning but that would have limited the study to a small subsection of the class. While it is a limitation of this study that students' reasoning was not investigated more deeply, this is something that future studies could take up.

Validity and reliability

If the research tools used in this study were valid, the results should give an appropriate judgement of what they intended to measure (White & Gunstone, 1992 p. 177). I shall consider two types of validity: Content validity and mode validity, and three types of reliability: test-retest reliability, internal consistency and reader reliability.

Content validity for this study refers to whether the pilot, pre- and post-tests included sufficient questions on the basic algebra relevant to first-year students. I believe the questions used in the pilot study did cover the common types of mistakes made by first-year students as many of them had been collected from years of observations of students' mistakes. Relating to the categories in the error classification that was devised, I feel that the number of questions from each category may have been better balanced as the category "Distorted algorithm, definition or theory" had many more questions than the other categories. The elimination of the 'easy' questions from the pilot test further skewed the balance of error types tested, but it seemed sensible to discontinue questions that most students got right.

Mode validity considers whether the tests used were adequate for students to express their understanding. Apart from a written test, students' understanding could have been measured with interviews, observations of their written answers to tutorial questions, or video recordings of students working on problems. The main reason for the choice of multiple choice tests with space for students' own answers was because other methods were too time consuming, unless only a sample of students were observed or interviewed and then a whole-class study would not be possible. Many of the multiple choice options had emerged from answers to free-response questions used in previous years, and the open option for all questions where students could write their own answer if it didn't match any of the given answers, negates somewhat the concerns associated with multiple choice questions. Writing rather than speaking algebra was also be the common way that these students expressed their understanding of mathematics at high school, and that strengthens the mode validity of the tests.

The use of questions that differed only slightly (with coefficients and variables changed) in the pilot, pre- and post-tests allows for test-retest reliability to be assessed. Improvement between tests could be expected (White & Gunstone, 1992), either as a result of remediation, or from familiarity with the test questions, or because algebra skills developed as they used them in the Math 130 course. The improvement in test marks recorded between the pre- and post- tests (particularly for the group that received no remediation) supports the claim for good test-retest reliability in this study.

The question-by-question analysis in Table 5.2 can be used to determine internal consistency. Table 5.2 showed that some questions testing the same error had high numbers of wrong answers in the post-test compared to other questions testing the same error that were well done. This is particularly true for the largest error category, distorted algorithm, definition or theorem. On this basis, the internal consistency for the errors tested could have been improved.

The statistical nature of the results obtained from the multiple-choice question design means that the same results would likely be obtained by anyone else analyzing them. This means that this study can claim high reader reliability.

Ethical considerations regarding data collection

The pre-test given in the first tutorial was written by 88 students. Due to absenteeism, 10 students did not write the pre-test that was given in the week before the interventions. A further 12 students did not sign the consent forms. All data from the absent and non-consenting students (22 in total) was separated and not used in data analysis. The sample size was then 66 students unevenly distributed across the five groups.

Reflecting on the ethical correctness of not using the data from the non-consenting students, I struggled to see how using the results from these 12 students could possibly harm them, as I would not be using any written answers that they could identify as their own work. Omitting their results would give less reliable conclusions for this study especially since the groups were relatively small and the non-consenting students were not always representative of their group. The results from non-consenting students could be inaccurate as they might not have taken the study seriously, but the pattern of responses gives me confidence that the students were not simply guessing at responses but answering honestly. In the end, I decided to err on the side of caution and not include their results.

The remediation strategies

After synthesizing the learning theories and remediation strategies described in the theoretical framework and literature review, two main remediation strategies were chosen based on two learning theories: practice based on behaviourism and cognitive conflict based on constructivism. These theories supported strategies that I felt were feasible (given the time constraints for developing basic skills) and reproducible in future years with different tutors.

For the five tutorial groups, three groups had variations of the cognitive conflict approach, one tested the strategy of practice and one group had no remediation apart from writing pre- and post-tests. The strategies are described in more detail below.

It is a weakness that the same strategy wasn't used with more than one group to minimize the potential influence from the tutor but this wasn't possible given the small class size and my desire to test different versions of cognitive conflict. Future studies could do this. Other validity issues include the small group sizes (10 to 22 students per group), and variations of students in the groups regarding different schooling and/or university backgrounds, ages, gender, having friends in their tutorial group who they can work with, and the role of the tutor's personality in influencing the students during remediation.

The strategies for each of the five groups are described below:

No remediation

Having one group receiving no remediation would allow for comparisons to be made between this 'control' group and the groups receiving intervention. However, caution would have to be taken when interpreting the results because there could be many factors that could invalidate the results. Care was taken to minimize the differences between the control group and the other groups, for example, all tutorials were held at the same time of day (2 – 5 p.m.), students and tutors were randomly assigned to tutorial groups (controlling the potential mis-match of learning style preferences), students were not told about the tasks given to the other groups and all the remediation worksheets that the students used were handed back to the tutors during the tutorials to limit the chance of students getting extra remediation from a friend in another group. Of course, this may have happened anyway and is a limitation of this study.

COGNITIVE CONFLICT ONCE

Cognitive conflict theory (Posner *et al.*, 1982) suggests that when a student gets a question wrong, they may have a misconception. If the misconception can be shown to be in conflict with existing correct knowledge, the student should react to this conflict by adjusting their misconception. In this strategy, questions similar to some of the worst-answered pre-test questions were given to the students, along with 'clues' designed to create cognitive conflict for those students who held misconceptions. After giving the students time to work through the questions by themselves or in self-formed groups, the tutor wrote the answers on the board and explained some mistakes, by referring to the definition of the mathematics involved, or a similar example, or by substituting numerical values into the original expression and the simplification and showing that they are not the same. This was the only strategy to be implemented once-off.

Weaknesses with this approach are that students may not have the misconception that was targeted by the 'clues' or they may have made a careless mistake when writing. It is assumed that with a large enough sample size, the number of non-misconception errors is likely to be a small percentage of the group. The relatively small group sizes in this study compounds the potential error from this assumption and is a limitation of this study. Another validity issue arising from this approach is that seeing the pre-test questions (or similar questions) may give this group an advantage over groups that didn't see the questions again before the post-test.

This was controlled somewhat by the post-test questions using different numbers or letters so that direct memory recall of the answers could not be used in the post-test.

Self-led Cognitive Conflict

As for strategy 1, but over 5 weeks and incorporating the idea from constructivist theories that learning is more effective if students are actively participating rather than listening and absorbing information from a teacher. Students were arranged into groups of 2 or 3 and each group was given a sheet with questions and a selection of possible answers. The students' task was to identify the correct answers and to provide convincing reasons for why the wrong answers were wrong. The correct answers and sample reasons were given to the students on the back side of the sheet to allow students to check their reasoning and review incorrect answers. The tutor was on hand to answer questions from individual students if they asked any.

The self-led cognitive conflict method required students to be active rather than passive and it may have taken some time before students learnt to behave in the manner expected of them. Unfortunately, it seems that many students are passive learners and changing them to active learners may require more time and input than was given in this study. Guidance from the tutor would play an important role in the implementation of this strategy.

Tutor-led Cognitive Conflict

A similar procedure as in strategy 1 was followed but over 5 weeks. Each week the students would get 4 – 6 questions directly from or similar to pre-test questions, with 'clues' to induce cognitive conflict. The idea behind repeating the same strategy each week (although on different topics) was to allow students to get used to following up a cognitive conflict with correct reasoning, in the hope that they would learn to do this in the future by themselves.

PRACTICE

Behaviourism (Mergel, 1998; Skinner, 1974) suggests that students learn from positive reinforcement on their correct behaviour (which they will then repeat) and negative reinforcement on undesirable behaviour (which should make them change that behaviour). For example, marking students' answers to worksheet questions as right or wrong gives them positive or negative reinforcement. Furthermore, giving them model solutions of the questions they attempted should give them an example of correct behaviour that they can model. Each week for five weeks, students were given basic algebra questions to complete and hand in. During the following tutorial their marked quizzes, showing ticks or crosses and with errors circled in red pen, were returned with worked solutions on a handout.

A weakness of this strategy is that what counts as positive or negative reinforcement for one student may not be the same for another. For example, a script with many marks in red pen may be seen as negative even if the writing is praise. For this study, it was assumed that sufficient positive reinforcement would be in the form of ticks for correct answers and sufficient negative reinforcement would be crosses next to wrong answers with errors indicated with a circle or short phrase near the mistake. The page with correct solutions would show how to correctly work out the answer.

4.2. Research question 1: Is remediation worthwhile?

The first sub-question considered in this section was if remediation improves algebra skills significantly more than having no remediation. The means for assessing this was a comparison of the percentage improvement between pre- and post-test for the remediated and non-remediated groups, to account for different-sized groups and different pre-test scores between the groups.

The second sub-question asked if good algebra skills are necessary for success in Math 130. Correlations between the pre- and post-tests and the final results for Math 130 were made. Scatter plots of pre-test vs. post-test and Math 130 final marks vs. Pre- and Post-test marks were made to see if weak algebra skills corresponded to low Math 130 marks.

The final sub-question in this section was 'Do the changes in the types of answers to individual questions support the use of remediation?' The motivation for this was to see if remediation had worked well for individual students but their improvement had not been detected due to the poor performance of other students in their groups. It may also determine whether a certain remediation might change the types of answers students gave, even if the answers were not changed to the correct answers. It may be possible that after remediation, students had partially given up certain misconceptions although more improvement was still necessary. To facilitate the analysis, the incorrect post-test responses were colour-coded: red for an error that was the same as in the pre-test; pink for a different error from the pre-test; and yellow for an error to a question answered correctly in the pre-test. Corrections to wrongly answered pre-test questions were coded in green. Responses that were correct in both pre- and post-tests were not coloured. The proportions of wrong and corrected answers per group were also compared in a table.

4.3. Research question 2: Which remediation strategy is best?

A statistical analysis of pre-, post- and post-post-tests for students in each of the five different groups was performed in order to answer the question of which strategy worked best to improve basic algebra skills. The analysis was done using comparisons of the groups' average scores as well as t-tests on the spreadsheet package, Microsoft Excel.

I also considered the proportions of students in each group that did worse, better or the same in the pre- and post-tests to test whether any of the strategies had made improvements in some students that were outweighed by declines in other students and so were not picked up by comparing the average pre- and post-test marks per group.

4.4. Research question 3: Which errors are the hardest to remediate?

The three error classifications described in chapter 2 (see p. 29) (Radatz, 1979; Movshovitz-Hadar, Zaslavsky and Inbar, 1987; and Linchevski and Herscovics, 1994) have limitations when considering them for this study. The first two classifications considered a broad scope of mathematical errors, not just errors with basic algebra. This resulted in categories that would probably not be used in a focused study of basic algebra, e.g. 'errors due to difficulties in obtaining spatial information' (Radatz's last category) is more relevant to questions involving diagrams, which were not considered in this study. The categories of 'technical error' (Movshovitz-Hadar, Zaslavsky and Inbar's final category) and 'errors due to deficient mastery of prerequisite skills, facts, and concepts' (Radatz's first category) were too broad to distinguish between the many types of errors that could fall into these categories.

The third classification considered errors made by students learning algebra for the first time but although this focused on errors with algebra, it again missed many possible errors that tertiary students might make, particularly after years of positive or negative reinforcement of concepts. It also excluded errors from more advanced sections of high school algebra, such as simplifying complex fractions.[10]

Thinking about the deficiencies of these error classifications, I thought of three questions to guide the choice of error categories in a classification that would be suitable to this study.

1. Are the categories distinct?

2. Do the categories reflect expected errors at a tertiary level?

3. Could the error be classified on the basis of students' written responses to allow for an analysis of the whole class?

Drawing from the error classifications and other reported errors discussed in chapter 2, I devised the classification detailed below.

The Error Classification

The categories considered were errors due to:

1. Use of a distorted theorem, definition or algorithm e.g. $(a + b)^{1/2} = a^{1/2} + ½\, ab + b^{1/2}$; $2^5a^4 = (2a)^9$, including

   a. errors from confusion between operations, e.g. $2 + a^2 = 2a^2$

   b. operating on one part of a compound term, e.g. $(2x)^2 = 2x^2$.

   c. wrong application of order of operations, e.g. favouring addition over multiplication.

2. Over-generalising, including false-linearity, e.g. $(a + b)^2 = a^2 + b^2$; $\sin(a + b) = \sin a + \sin b$; $(a/b) + (c/d) = (a + c)/(b + d)$; $2(a/b) = 2a/2b$.

3. Other errors simplifying fractions e.g. $\frac{x+y}{y} = x+1$, $\frac{x+y}{y} = x$, $\frac{2h^2}{h} = 2^2$.

4. Misuse of the equals sign e.g. setting an expression equal to 0 and solving for x when asked to factorise; not recognizing $5a + 2a$ is the same as $7a$.

[10] An error classification by Demby (1997) was considered but to correctly classify errors using this classification would require interviews with learners, which, even if limited to a few students, would take too long for the diversity of questions intended in this study.

5. Errors with distributing or factoring minus signs, e.g. $2 - (3x - 4y) = 2 - 3x - 4y$; $(-2)^2 = -2^2$.

6. Confusing similar notation, e.g. $\sin^{-1}x = (\sin x)^{-1}$

7. Misunderstanding the meaning of variables (and consequently joining algebraic 'objects' as a new 'object' e.g. $2a + 5b = 7ab$; $3n + 4 = 7n$ or $3n4$), including 'letter evaluated' expressions where a is assumed to equal 1, b to equal 2, etc.

8. Technical error probably due to carelessness (including computational errors e.g. $7 \times 8 = 54$, and small manipulation errors in algebra e.g. leaving out closing brackets).

4.5. Conclusion

The literature review, theoretical framework and the set-up of the course, with five tutorial groups, supported the choice of four remediation strategies and a control group that received no remediation. The effect of the strategies was measured using pre- and post-tests, with a follow-up test (called the post-post-test) 3 months later. The strategies selected were:

1. Cognitive conflict with explanations from the tutor (hereafter named Cognitive Conflict Once)

2. Self-led Cognitive Conflict

3. Tutor-led Cognitive Conflict

4. Practice

This chapter has described the manner in which the strategies were implemented and how the data was collected in order to address each of the three research questions. Next is a report on the findings of the five remediation strategies with regard to the three research questions.

5. RESULTS AND DISCUSSION

This chapter presents the findings from the implementation of the remediation strategies described in the Method chapter. The results are grouped according to each of the three research questions, along with discussion relevant to the results. Conclusions and limitations of this study follow in the next chapter.

5.1. Is remediation worthwhile?

Does remediation improve algebra skills significantly more than having no remediation?

A comparison of the scores for the pre-test (given before any remediation) and post-test (given six weeks later) showed that having some kind of remediation is better than none at all. The group receiving no remediation improved from an average of 66.2% in the pre-test to 72.7% in the post-test, an improvement of 6.5 percentage points or 9.9%. The improvement by all the remediation groups was higher: 9.2 percentage points (12.4%) for the Cognitive Conflict Once group, 10.7 percentage points (15.9%) for the Self-led Cognitive Conflict group, 12.5 percentage points (17.9)% for the Practice group and 13.5 percentage points (21.5%) for the Tutor-led Cognitive Conflict group. These results are displayed in Figure 5.2 and, in greater detail, in Table 5.4 in the next section. The improvement of all the remediated groups compared to the non-remediated group was significant at the 99% confidence level. $P(T<= t) = 0.093$; t critical = 1.294, assuming equal variance. (Assuming unequal variance, $P(T<= t) = 0.070$; t critical = 1.323).

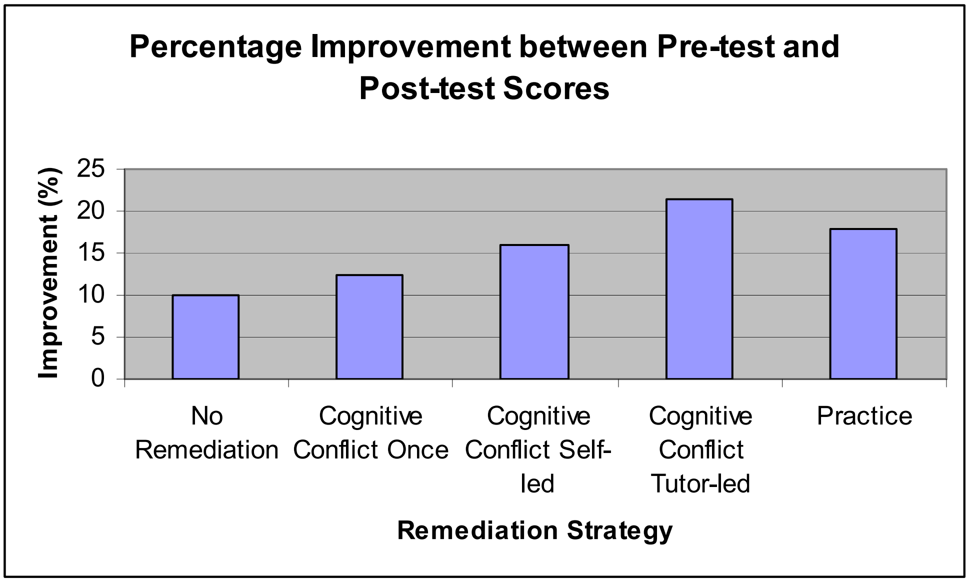

Figure 5.1 Percentage Improvement between Pre-test and Post-test Scores

When considering the improvement between pre- and post-tests, the low improvement of the No Remediation group stands out. The improvement of this group was the lowest of all the groups and less than half of the improvement of the most improved group (Tutor-led Cognitive Conflict). All groups apart from the Cognitive Conflict group had percentage improvements more than 50% better than the group that had no remediation. A possible reason for the smaller improvement in the Cognitive Conflict Once group the was that the average score in the pre-test was already high (73.8% pre-test and 82.9% post-test, see Table 5.3 in the next section). From this perspective, these results support the use of remediation to improve basic algebra skills in Math 130.

However, the same results can also be seen from a perspective that does not support remediation. Although improvements between the pre-test and the post-test were made in all groups, the final test scores could be considered too low to claim that mastery of the basic algebra skills necessary for first year mathematics had been achieved. The highest group average in the post-test, after remediation, was only 82.9% (see Table 5.4). It could be argued that even experts sometimes make careless mistakes although they know the correct way to carry out a procedure (Lewis, 1981) p. 107. Taking this into account, there may be a ceiling effect that makes a realistic top score slightly lower than 100% in large groups. My feeling is that 80% is too low a score for ceiling effects to come into question; a score higher than 90% would allow for up to 2 out of the 20 questions to be answered incorrectly, and seems a better benchmark for a claim of mastery.

It could also be argued that the improvement in average scores between pre-tests and post-tests is somewhat to be expected since the tests were very similar, with only coefficients and variables changed. In addition, some students may have been reminded of basic algebra procedures in the course of their Calculus (or other) studies leading to higher scores in the post-test. This would support the claim that remediation was not worthwhile.

Are good algebra skills necessary for success in Math 130?

The results of correlations between the pre-test, post-test and the Math 130 final results (after supplementary exams) are recorded in Table 5.1.

Table 5.1 Correlation between Math 130 Final Marks and Pre-test and Post-test Marks

	Correlation Coefficient Math 130 Final Mark	*Correlation Coefficient Post-test*
Pre-test	0.43	0.66
Post-test	0.40	

There was a positive correlation of 0.43 between basic algebra ability before remediation and the Math 130 final marks. The correlation between the post-test marks and the final Math 130 marks is slightly lower at 0.40. Greater correlation (0.66) was found between the pre-test and post-test marks. In Figure 5.2, a scatter plot shows that the post-test marks were generally better than pre-test marks. The greatest differences between pre- and post-test marks were for students who attained much higher post-test marks compared to their pre-test marks, rather than the other way round. Similar findings are shown in Figure 5.3, a scatter plot of Math 130 marks vs. pre- and post-test marks. Very few post-test marks were lower than Math 130 marks and students with very low Math 130 marks were able to achieve high scores in the post-test. Students with high per-test marks (greater than 80%) all passed Math 130. The same is not true for the post-test.

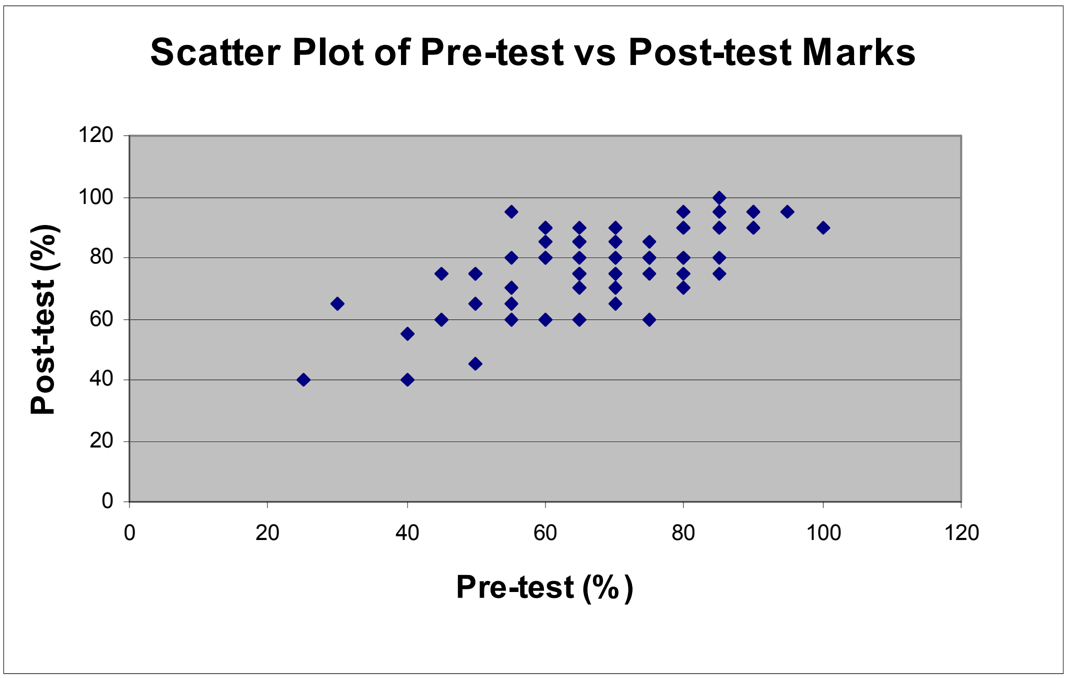

Figure 5.2 Scatter Plot of Pre-test vs. Post-test Marks

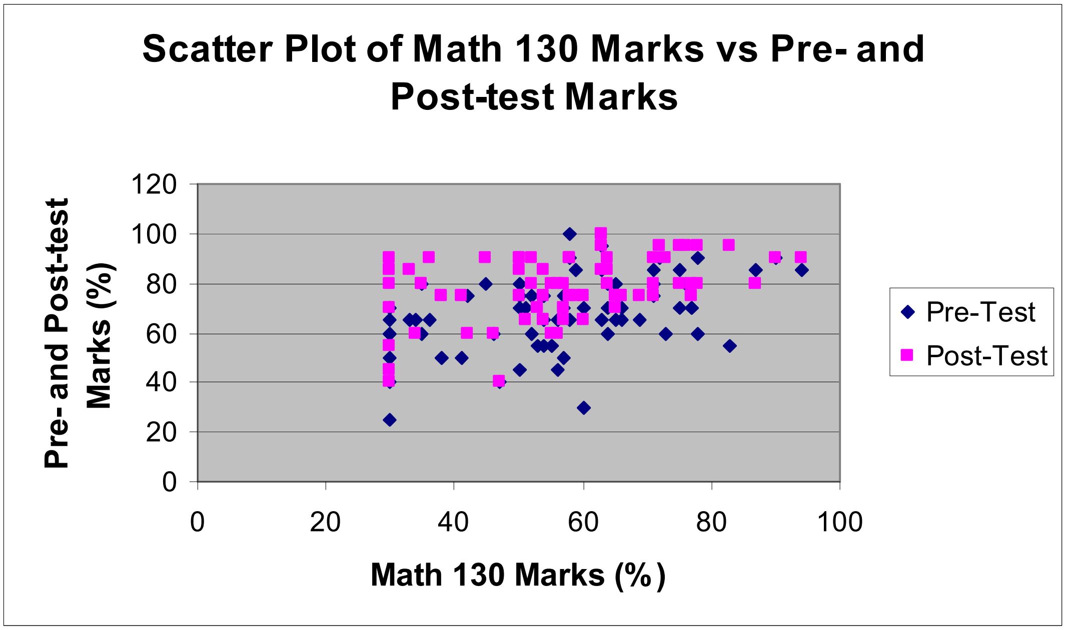

Figure 5.3 Scatter Plot of Math 130 Marks vs. Pre-test and Post-test Marks

It is interesting to note from Table 5.1 that the correlation between pre-test marks and Math 130 final marks was slightly stronger than the correlation between post-test marks and Math 130 final marks (0.43 compared to 0.40). The fairly good correlation between the pre-test and Math 130 marks of 0.43 may be because many students who have high (or low) achievement in basic algebra are generally high (or low) achievers in Mathematics. But this could be an indication that factors other than basic algebra skills are more important for determining success in Math 130 and more effort should be directed at finding and remediating them rather than on basic algebra. Some suggestions are listed at the end of the next chapter.

A reason for the slightly lower correlation between the post-test and Math 130 results (0.40) may be explained by the fact that Math 130 requires many more abilities than basic algebra. By the end of Math 130, students who have good algebra skills but have not grasped the concepts in Calculus may achieve a good score in the post-test but not for Math 130. Alternatively, some students who are weak in algebra skills may get through Math 130 because they engage more with the new concepts in the course. The positive correlation means that more students had high (or low) scores in both the post-test and Math 130 than in only one of them. The scatter plot in Figure 5.3 shows that some students with low algebra post-test scores did pass Math 130, but low Math 130 marks were obtained by students with algebra marks ranging from low to high.

The fairly low positive correlation between Math 130 marks and Pre-test marks (0.43) and Post-test marks (0.40) suggest that the pre- and post-tests are not good predictors of Math 130 results. Clearly (and unsurprisingly) there are other factors that determine success in Math 130.

Do the changes in the types of answers to individual questions support the use of remediation?

Students with the same score in pre- and post-tests may have had unchanged marks because their marks were already high, or if they improved in some questions but regressed in others. A question-by-question analysis would help to show the distribution of types of answers across the different questions. This information is shown in Table 5.2, which gives a detailed comparison of the pre- and post-test answers per group. Four colour-coded categories were used: **improved answers** (coded green) were those wrong in the pre-test and correct in the post-test; two types of **still incorrect answers** (coded red for the same error as the pre-test and pink for a different error); and **choosing an incorrect answer when they had previously chosen a correct one** (coded yellow). To make comparisons between the groups easier, the numbers of improved responses per group were converted to percentages.[11]

In Table 5.2, the sum of wrong responses per question gives an indication of the changes in the answers given for each question. Of the six questions for which less than 80% of the class gave correct answers, four of them (Q11, Q16, Q18 and Q22) involved fractions. Only one

[11] When analyzing the results, errors with two of the questions in the post-test, Q13 and Q15, were discovered. Neither of these had the correct answer as one of the listed options. These questions were disregarded in the analysis of the results, including those of the previous section.

other question in the test involved fractions, showing that this is an area of difficulty. The other two questions involved multiplying out an expression with exponents and simplifying the square-root of a sum of squares (which cannot be simplified).

The information in Table 5.2 is also displayed in Figures 5.4, 5.5, 5.6 and 5.7, which show the proportions of answer types for each question according to the categories: Correct in Pre- and Post-tests; Wrong in Pre-test, Right in Post-test; Different Wrong Answer in Post-test; Same Wrong Answer in Pre- and Post-tests. Figures 5.4 and 5.5 detail responses for the first 10 questions; Figures 5.6 and 5.7 for the remaining questions. Figures 5.4 and 5.6 show the combined results for the remediated groups while Figures 5.5 and 5.7 show results for the No Remediation group only. Grouping the remediated groups together allowed for a quick comparison with the No Remediation group to ascertain if having some kind of remediation affected results in certain questions compared to having no remediation at all.

Table 5.2 Classification of Students' Responses per Question in Pre-and Post-Tests

No Remediation (13 students)	**Totals**	**Q1**	**Q2**	**Q3**	**Q4**	**Q5**	**Q6**	**Q7**	**Q8**	**Q9**	**Q10**	**Q11**	**Q12**	**Q14**	**Q16**	**Q17**	**Q18**	**Q19**	**Q20**	**Q21**	**Q22**
Same Wrong Answer in Pre- and Post-tests	21		3	5		1						4	2	1	2		1		1		1
Different Wrong Answer in Post-test	29	2		1		1	3		1						6	1		1	3	3	7
Right in Pre-test, Wrong in Post-test	20		3	1	1		1		1		1	4	1	2		1	2	1		1	
Wrong in Pre-test, Right in Post-test	37			3	1	2	1	1	2	3	3	3	1	1	2	3	2	1	3	2	3
Correct in Pre- and Post-tests	153	11	7	3	11	9	8	12	9	10	9	2	9	9	3	8	8	10	6	7	2
TOTALS	***260***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***

Cognitive Conflict Once (12 students)	**Totals**	**Q1**	**Q2**	**Q3**	**Q4**	**Q5**	**Q6**	**Q7**	**Q8**	**Q9**	**Q10**	**Q11**	**Q12**	**Q14**	**Q16**	**Q17**	**Q18**	**Q19**	**Q20**	**Q21**	**Q22**
Same Wrong Answer in Pre- and Post-tests	12			2				1				1			2		2				4
Different Wrong Answer in Post-test	9								1					1	2		1			1	3
Right in Pre-test, Wrong in Post-test	18	2	1	3			3				2		1	2	1	1	1				1
Wrong in Pre-test, Right in Post-test	38	1	2	4	2	2	3	1			3	4	2		4				8	1	1
Correct in Pre- and Post-tests	163	9	9	3	10	10	6	10	11	12	7	7	9	9	3	11	8	12	4	10	3
TOTALS	***240***	***12***	***12***	***12***	***12***	***12***	***12***	***12***	***12***	***12***	***12***	***12***	***12***	***12***	***12***	***12***	***12***	***12***	***12***	***12***	***12***

Self-led Cognitive Conflict (22 students)	**Totals**	**Q1**	**Q2**	**Q3**	**Q4**	**Q5**	**Q6**	**Q7**	**Q8**	**Q9**	**Q10**	**Q11**	**Q12**	**Q14**	**Q16**	**Q17**	**Q18**	**Q19**	**Q20**	**Q21**	**Q22**
Same Wrong Answer in Pre- and Post-tests	37		1	8			2					4	1		5		1		4		11
Different Wrong Answer in Post-test	32	1	3	1		2	1					4	2	2	7		1		1		7
Right in Pre-test, Wrong in Post-test	25				1	3	2	1	3				1	1	5	1	4			2	1
Wrong in Pre-test, Right in Post-test	75	5	3	6	3	5	1	1		2	4	9	5	3	3	3	3	1	10	6	2
Correct in Pre- and Post-tests	271	16	15	7	18	12	16	20	19	20	18	5	13	16	2	18	13	21	7	14	1
TOTALS	***440***	***22***	***22***	***22***	***22***	***22***	***22***	***22***	***22***	***22***	***22***	***22***	***22***	***22***	***22***	***22***	***22***	***22***	***22***	***22***	***22***

Tutor-led Cognitive Conflict (13 students)	**Totals**	**Q1**	**Q2**	**Q3**	**Q4**	**Q5**	**Q6**	**Q7**	**Q8**	**Q9**	**Q10**	**Q11**	**Q12**	**Q14**	**Q16**	**Q17**	**Q18**	**Q19**	**Q20**	**Q21**	**Q22**
Same Wrong Answer in Pre- and Post-tests	10			4	1								1		1		1				2
Different Wrong Answer in Post-test	31			4		3	2		3			4		1	3	1	1	1		2	6
Right in Pre-test, Wrong in Post-test	19	1	1	1	1	2	1	1			1	1			2	4		2			1
Wrong in Pre-test, Right in Post-test	52	6	4	1	2	1	4	1	3		1	4	3	3	2	1	3		7	3	3
Correct in Pre- and Post-tests	148	6	8	3	9	7	6	11	7	13	11	4	9	9	5	7	8	10	6	8	1
TOTALS	***260***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***	***13***

Practice (10 students)	**Totals**	**Q1**	**Q2**	**Q3**	**Q4**	**Q5**	**Q6**	**Q7**	**Q8**	**Q9**	**Q10**	**Q11**	**Q12**	**Q14**	**Q16**	**Q17**	**Q18**	**Q19**	**Q20**	**Q21**	**Q22**
Same Wrong Answer in Pre- and Post-tests	12		1	2					1			1	1		5						1
Different Wrong Answer in Post-test	14			1				1	1			1			3			1		2	4
Right in Pre-test, Wrong in Post-test	10						2	1					1	2				1	1		2
Wrong in Pre-test, Right in Post-test	34	3	1	3	1	2	1			1	3	1	3	1	1	2	5		4	2	
Correct in Pre- and Post-tests	130	7	8	4	9	8	7	8	8	9	7	7	5	7	1	8	5	8	5	6	3
TOTALS	***200***	***10***	***10***	***10***	***10***	***10***	***10***	***10***	***10***	***10***	***10***	***10***	***10***	***10***	***10***	***10***	***10***	***10***	***10***	***10***	***10***

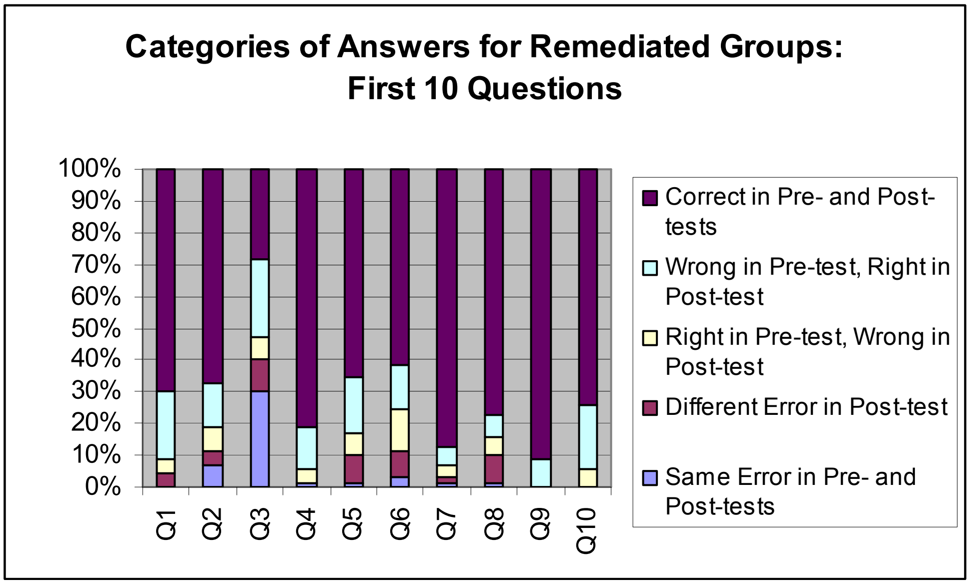

Figure 5.4 Categories of Answers for Questions 1 to 10, for Remediated Groups.

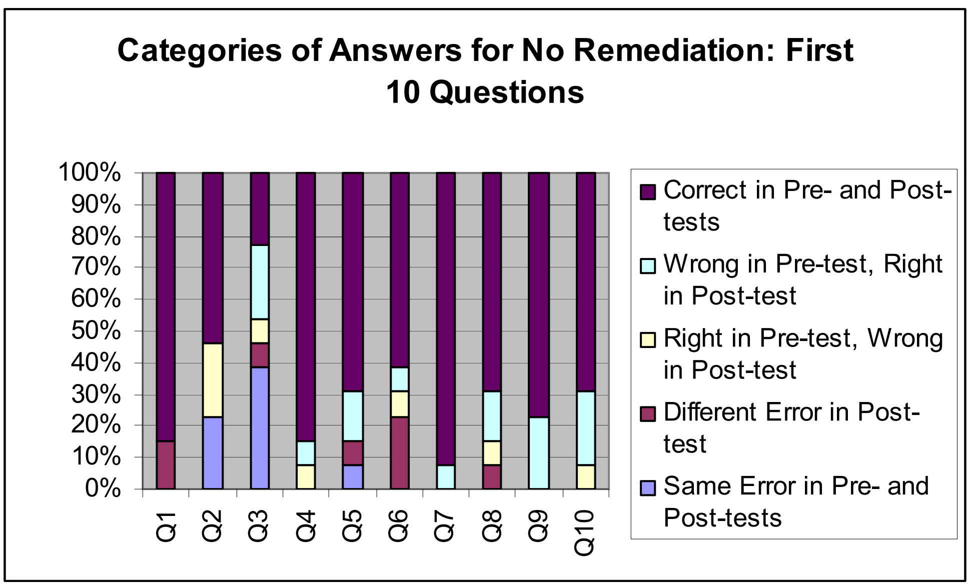

Figure 5.5 Categories of Answers for Questions 1 to 10, for No Remediation.

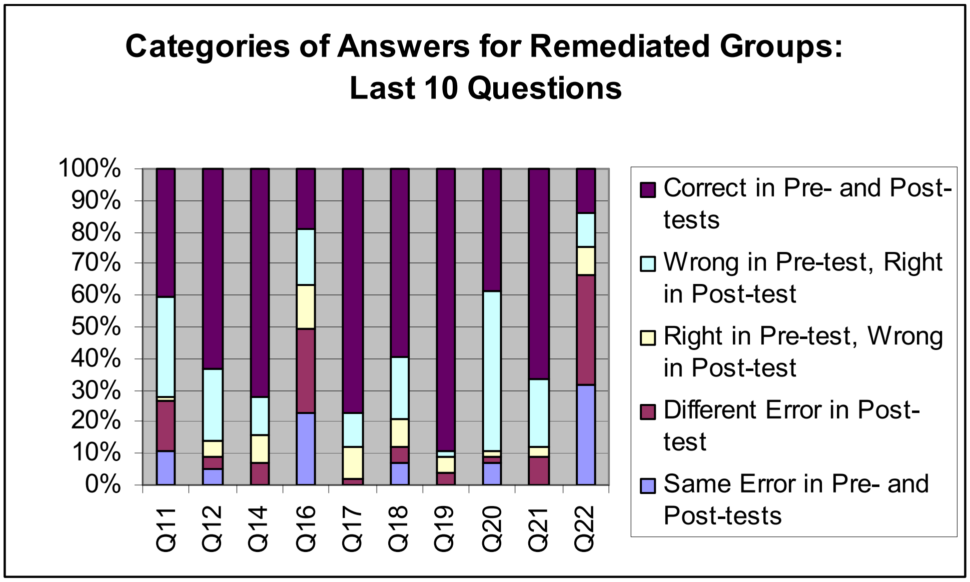

Figure 5.6 Categories of Answers for Remediated Groups for the Last Ten Questions

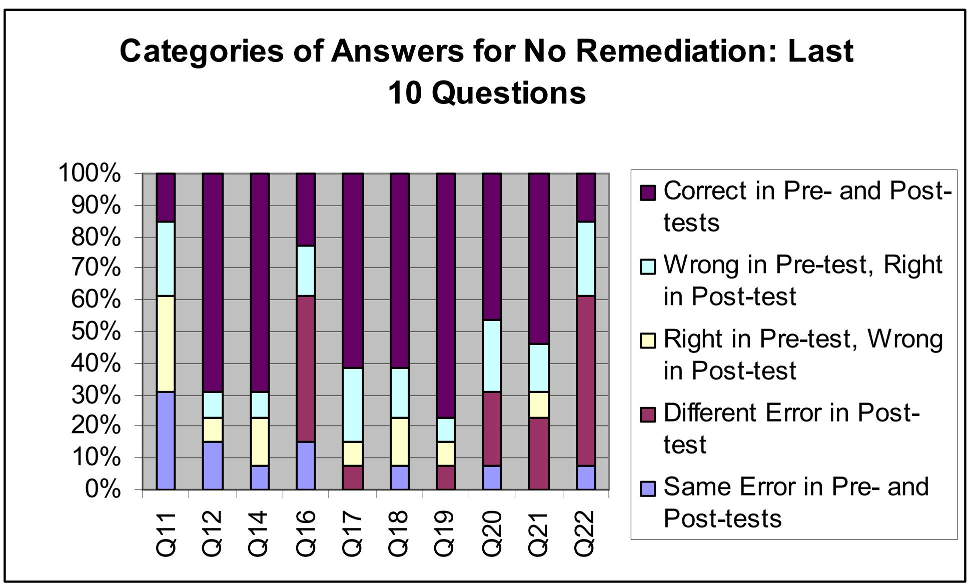

Figure 5.7 Categories of Answers for No Remediation for the Last Ten Questions

The stacked bar graphs in Figures 5.4, 5.5, 5.6 and 5.7 show which questions were easiest (those with the longest top bars, e.g. Q4, 7, 9, 19), those for which remediation seemed to work well (long second bars, e.g. Q1, 20), and those that were answered incorrectly even after remediation (long third and fourth bars, e.g. Q 16, 22). Although remediation did not result in improvement in some questions, there were questions for which remediation did make a significant change compared to having no remediation. Overall, these results suggest that investing time and effort into remediating algebra skills is worthwhile, since improvement happens more often when there has been remediation.

The low improvement without remediation is also apparent when considering the corrected and wrong answers in each group, i.e. excluding the answers that were correct in both the pre- and post-tests. Table 5.3 shows that the percentage of these answers that were correct was lowest for No Remediation (35%), while the other groups had larger proportions of corrected answers: 49% for Cognitive Conflict Once, 44% for Self-led Cognitive Conflict, 46% for Tutor led Cognitive Conflict and 49% for Practice. The higher percentages of wrong answers compared to corrected answers among all groups supports the argument that although remediation seems more effective than having no remediation, the strategies tried here could be improved on.

Table 5.3 Percentage of Corrected Answers and Wrong Answers per Group

Remediation Strategy	**No Remediation**	**Cognitive Conflict Once**	**Self-led Cognitive Conflict**	**Tutor-led Cognitive Conflict**	**Practice**
No. of answers excluding those correct in both tests	107	77	169	112	70
Corrected Answers (%)	35%	49%	44%	46%	49%
Wrong Answers (%)	65%	51%	56%	54%	51%

5.2. Which remediation strategy is best?

The improvement between the pre- and post-test averages for each group was used to compare the effectiveness of the remediation strategies. The results are displayed in Table 5.4.

Table 5.4 Pre-Test and Post-Test Averages and Percentage Improvement per Group

Remediation Strategy	**No remediation**	**Cognitive Conflict Once**	**Self-led Cognitive Conflict**	**Tutor-led Cognitive Conflict**	**Practice**
Group size (n)	13	12	22	13	10
Pre-test Averages (%)	66.2	73.8	67.3	62.7	70
Post-test Averages (%)	72.7	82.9	78.0	76.2	82.5
Improvement (percentage-points)	6.5	9.2	10.7	13.5	12.5
Percentage Improvement	9.9	12.4	15.9	21.5	17.9

The best improvement between pre-test and post-test averages was by 21.5%, which occurred in the Tutor-led Cognitive Conflict group that was given weekly cognitive-conflict-inducing worksheets with the tutor providing solutions and explanations after the students had worked by themselves[12]. The test averages for this group increased from 62.7% (the lowest pre-test average) to 76.2% (the second lowest post-test average). Thirteen students in this group wrote both the pre- and post-tests[13]. The improvement in this group was significant at the 99% confidence level ($P(T<= t) = 0.011$; t critical = 1.318).

[12] As this group had the lowest pre-test average, it could be argued that any kind of remediation would affect this group more than the other groups. The difference of 11.1% between the lowest and highest pre-test scores and the maximum score of 73.8% in the pre-test were considered sufficient to support the conclusions made from the data.

[13] The number of students in each group excludes those who did not give consent for their responses to be included in the study, as well as students that missed either the pre-test or post-test.

The strategy of Practice was used on the second-best improving group. An increase by 17.9% from 70% in the pre-test to 82.5% in the post-test maintained this group's ranking as the second-best scoring group in both pre- and post-tests. This group had the smallest sample size of only 10 students, but the improvement was statistically significant at the 99% confidence level (P(T<= t) = 0.010; t critical = 1.330). The improvement of the Tutor-led Cognitive Conflict group was found to be statistically significant when compared to the improvement of the Practice group (P(T<= t) = 0.422; t critical = 1.323).

Self-led Cognitive Conflict was the third-best performing group in the pre-test (67.3%), post-test (78.0%) and improvement between tests (15.9%) and was also the group for which the number of students that wrote both the pre- and post-tests was the highest (22 students). The remediation strategy for this group was almost the same as for the group with the greatest improvement. Students were given worksheets to induce cognitive conflict but instead of the tutor giving solutions and explanations, the solutions were given to the students with the worksheets so they could lead themselves through the questions. The improvement in this group was significant at the 99% confidence level (P(T<= t) = 0.008; t critical = 1.302), and the difference in improvement between this group and the Tutor-led Cognitive Conflict group was also found to be significant at the 99% confidence level (P(T<= t) = 0.276; t critical = 1.308).

The second-lowest improvement in Table 5.4 (12.4%) came from the Cognitive Conflict Once group that had a cognitive conflict worksheet and tutor explanations just once after the pre-test. This group had the highest averages for both the pre- and post-tests (73.8% and 82.9% respectively), and twelve students wrote both tests. The improvement in this group was significant at the 99% confidence level (P(T<= t) = 0.031; t critical = 1.321).

The group that was given no remediation started with the second-lowest pre-test average of 66.2%. The post-test average of 72.7% was an improvement of 9.9%, giving this group the lowest post-test average and the lowest improvement rate. Thirteen students in this group wrote both tests. The improvement in this group was also significant at the 99% confidence level (P(T<= t) = 0.172; t critical = 1.318).

The average percentages in the post-test were higher than the pre-test for all five tutorial groups although not all individuals improved. Table 5.5 gives the numbers and percentages of students in each group whose post-test mark was worse, better or the same compared to their pre-test mark. The group that received no remediation had the greatest percentage of students having the same results for both pre-and post-tests and the smallest percentage of students doing better in the post-test.

Table 5.5 Group-wise Comparison of Improvement between Pre- and Post-test Results

Comparison of Improvement between Pre-and Post-test Results					
	No Rem.	Cognitive Conflict Once	Self-led Cognitive Conflict	Tutor-led Cognitive Conflict	Practice
Worse (no. of students)	2	2	4	1	0
Better (no. of students)	8	9	16	10	9
Same (no. of students)	3	1	2	2	1
TOTAL no. of students	13	12	22	13	10
PERCENTAGES:					
Worse (%)	15	17	18	8	0
Better (%)	62	75	73	77	90
Same (%)	23	8	9	15	10

Table 5.4 and Table 5.5 show that the most successful remediation strategies were the more resource-demanding ones – Tutor-led Cognitive Conflict (involving more preparation time for the tutor), and Practice (involving marking students worked answers to the practice questions). The once-off strategy (Cognitive Conflict Once) was found to be less effective than any of the repeated strategies. It might have been informative to see if the strategy of Practice given once only would have been more or less effective than Cognitive Conflict Once but I would be surprised to find that less practice is better than more.

5.3. Which errors are the hardest to remediate?

Table 5.6 shows the occurrence of the seven types of errors tested in the pre- and post-tests. The remediation strategy corresponding to the greatest reduction in each error is highlighted. The No Remediation group never earned top spot for reducing errors, confirming that having remediation was more successful than having no remediation. The strategies of Tutor-led Cognitive Conflict and Practice were overall the most effective at decreasing errors, both reducing all errors tested by 52%, compared to a reduction of 50% by Cognitive Conflict Once, 47% by Self-led Cognitive Conflict and 38% for no remediation.

The most frequently-tested error, tested in sixteen questions, was 'Use of a Distorted Algorithm, Definition or Theorem' abbreviated to 'Distorted Algorithm' in Table 5.6. Remediation by Practice corresponded to the greatest decrease of errors with a 58% reduction in frequency of this error in the post-test compared to the pre-test. Cognitive Conflict Once and Tutor-led Cognitive Conflict were next-best with reductions of 47% and 48%, respectively. Self-led Cognitive Conflict was slightly less effective than No Remediation for decreasing this error, corresponding to 39% reduction compared to 40%.

Six questions tested technical or careless errors. This error category showed the lowest overall improvement, which is not too surprising since these types of mistakes are not necessarily related to a student's understanding of a concept and are made by experts too (Lewis, 1981). Still, it is interesting to note that the number of these errors increased for the group that received no remediation (from 4 to 7 occurrences, an increase of 75%) and for the least successful remediation group (Cognitive Conflict Once) (from 2 to 5 occurrences, or 150% more). The other three groups showed small reductions in these errors: 29% for Self-led Cognitive Conflict and 25% for both Tutor-led Cognitive Conflict and Practice.

Table 5.6 Frequency of Error Types in Pre- and Post-tests Sorted by Error Type

		Frequency of Error									
		No Remediation		Cog Conflict Once		Self-led C. Conflict		Tutor-led C. Conflict		Practice	
Question	Errors Tested	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
1	Distorted algorithm	1	2	0	2	3	1	3	0	0	0
3	Distorted algorithm	0	0	0	0	1	0	3	1	0	1
4	Distorted algorithm	1	1	2	0	2	1	2	2	1	0
5	Distorted algorithm	3	1	1	0	6	5	1	1	2	0
6	Distorted algorithm	3	3	2	3	4	5	3	2	2	2
7	Distorted algorithm	0	0	0	0	0	0	0	1	0	0
8	Distorted algorithm	2	2	1	1	0	3	6	2	2	1
10	Distorted algorithm	0	0	0	0	1	0	0	0	0	0
12	Distorted algorithm	0	0	0	0	0	1	1	0	0	0
14	Distorted algorithm	0	0	0	0	1	0	0	1	0	0
16	Distorted algorithm	9	8	8	5	14	16	8	6	9	7
17	Distorted algorithm	11	2	11	2	21	1	12	5	8	0
18	Distorted algorithm	3	3	3	4	5	6	5	2	5	0
19	Distorted algorithm	1	2	0	0	0	0	0	2	0	1
21	Distorted algorithm	9	1	12	1	20	2	10	1	9	1
22	Distorted algorithm	9	6	7	7	18	18	10	7	5	6
7	Technical/careless	0	0	1	1	0	1	0	0	0	0
8	Technical/careless	0	0	0	0	0	0	1	1	0	0
9	Technical/careless	0	0	0	0	1	0	0	0	0	0
12	Technical/careless	2	3	1	1	4	2	2	1	3	1
14	Technical/careless	2	3	0	3	2	2	1	0	1	2
22	Technical/careless	0	1	0	0	0	0	0	1	0	0
7	Minus signs	1	0	1	2	1	0	1	0	0	1
9	Minus signs	0	0	0	0	0	0	0	0	0	0
10	Minus signs	3	1	3	2	3	0	1	1	3	0
2	Over-generalising	3	5	1	1	4	1	3	1	2	1
3	Over-generalising	7	7	5	5	12	9	6	8	5	2
18	Over-generalising	0	0	0	0	0	0	1	0	0	0
1	Simplifying fractions	0	0	0	0	1	0	1	0	0	0
2	Simplifying fractions	0	1	0	0	0	2	1	0	0	0
11	Simplifying fractions	5	6	5	1	12	8	5	1	3	2
20	Notation	6	3	9	0	13	5	7	0	4	1
9	Equals sign	0	0	0	0	0	0	0	0	0	0
All	Omitted	20	2	9	0	28	4	21	8	7	5

Table 5.6 (cont.): Frequency of Error Types in Pre- and Post-tests Sorted by Error Type

	Frequency of Error										
	No Remediation		**Cog Conflict Once**		**Self-led C. Conflict**		**Tutor-led C. Conflict**		**Practice**		
Errors Tested	**Pre**	**Post**	**Pre**	**Post**	**Pre**	**Post**	**Pre**	**Post**	**Pre**	**Post**	**ALL GROUPS**
Distorted algorithm											
TOTAL ERRORS	52	31	47	25	96	59	64	33	43	19	
% Decrease in Errors		40%		47%		39%		48%		56%	47%
Technical/careless											
TOTAL ERRORS	4	7	2	5	7	5	4	3	4	3	
% Decrease in Errors		-75%		-150%		29%		25%		25%	-18%
Minus signs											
TOTAL ERRORS	4	1	4	4	4	0	2	1	3	1	
% Decrease in Errors		75%		0%		100%		50%		67%	54%
Over-generalising											
TOTAL ERRORS	10	12	6	6	16	10	10	9	7	3	
% Decrease in Errors		-20%		0%		38%		10%		57%	26%
Simplifying fractions											
TOTAL ERRORS	5	7	5	1	13	10	7	1	3	2	
% Decrease in Errors		-40%		80%		23%		86%		33%	56%
Notation											
TOTAL ERRORS	6	3	9	0	13	5	7	0	4	1	
% Decrease in Errors		50%		100%		62%		100%		75%	84%
Equals sign											
TOTAL ERRORS	0	0	0	0	0	0	0	0	0	0	
% Decrease in Errors		0%		0%		0%		0%		0%	0%
Omitted											
TOTAL ERRORS	20	2	9	0	28	4	21	8	7	5	
% Decrease in Errors		90%		100%		86%		62%		29%	69%
TOTALS	101	63	82	41	177	93	115	55	71	34	
% Decrease in Errors		38%		50%		47%		52%		52%	50%

Errors with the distribution of minus signs were tested in three questions. For the Cognitive Conflict Once group, the total number of minus sign errors remained the same at 4, but there was an increase by 1 for question seven and a decrease by 1 for question ten. All the other groups showed a reduction in this error. The best reduction was for the Self-led Cognitive Conflict group (from 4 error occurrences to none), followed by no remediation (75% error reduction), Practice (67% error reduction) and Tutor-led Cognitive Conflict (50% error reduction).

Errors with over-generalizing were the second-hardest error type to remediate, giving an average improvement of only 17% across all groups, or 26% for the remediated groups. The best error reduction came from the Practice group (56%), followed by the Self-led Cognitive Conflict group (38%) and the Tutor-led Cognitive Conflict group (10%). The Cognitive Conflict Once group showed no improvement and the No Remediation group made 20% more errors in the post-test.

Errors with simplifying fractions are, to me, the most visually disturbing to see. The results from the three questions that tested this showed that the group with no remediation made 40% more errors in the post-test while the other groups all made fewer errors. Small reductions were made by the Self-led Cognitive Conflict group (23%) and the Practice group (33%) while Cognitive Conflict Once and Tutor-led Cognitive Conflict corresponded to high error reductions by 80% and 86% respectively.

Errors with notation, tested in one question, reduced in frequency for all groups. For Cognitive Conflict Once and Tutor-led Cognitive Conflict, no occurrence of this error appeared in the post-test, a 100% improvement. Notation errors for the Practice and Self-led Cognitive Conflict groups reduced by 75% and 62% respectively, while the group with No Remediation had 50% fewer notation errors.

No errors involving the misuse of the equals sign were found by any of the groups. This may be accounted for by the fact that only one option in one question tested this error. The pilot test had more questions involving this error but they were eliminated from the pre- and post-tests because so many students could answer the questions correctly. This result suggests that the pre- and post-tests could have been set to exclude this error.

Although not an error category in its own right, the omission of an answer suggests a lack of confidence with a question, or lack of time to complete the questions, or possibly that the question was no clearly understood. Most omissions were distributed over a large number of questions, with one, two or three students per group making an omission. The exceptions were question 6 (4 omissions from Tutor-led Cognitive Conflict), question 11 (5 omissions from Self-led Cognitive Conflict) and question 21 (4 omissions from No Remediation).

5.4. Conclusion

The results shown in this chapter have provided evidence for answers to the research questions. In response to the first research question "Is remediation worthwhile?" it was shown that the group without any remediation had the lowest improvement in algebra skills over the six weeks that the experiment was run. This would seem to confirm that remediation is worthwhile. However, if the post-test achievement of the remediated groups is considered, there remains room for improvement, suggesting that the remediation strategies tested could be improved on.

The investigations to find out "Which remediation strategy is best?" showed that the strategies of Tutor-led Cognitive Conflict and Practice resulted in the greatest improvements between pre- and post-tests. These strategies were also the most demanding in terms of implementation time and effort, as they ran for 5 weeks and required more tutor preparation (for Tutor-led Cognitive Conflict) and marking (Practice), apart from setting worksheets for each week.

The error classification developed in the Methods chapter had eight categories of errors, seven of which were tested in the pre- and post-tests. Questions on the eighth category, misunderstanding the meaning of variables, were eliminated from the pre-tests following very high levels of correct answers in the pilot test, suggesting that this error was not as important to tertiary students as it may be to early algebra learners.

The error of confusing similar notation responded best to remediation (although only one question tested this), followed by errors with simplifying fractions and errors from over-generalizing. Errors with distributing minus signs responded well to Self-led Cognitive Conflict but the other strategies were less effective than having no remediation for treating this error. Practice was most effective at reducing errors from the use of a distorted algorithm, definition or theorem. Technical or careless errors were found to be the most resistant to remediation.

The next chapter makes conclusions, notes the limitations of this study and gives recommendations for future studies.

6. CONCLUSIONS

This chapter summarizes the answers to the three remediation questions based on the results and discussion in the previous chapters. This is followed by discussion of the limitations of this study and recommendations for further research.

6.1. Is remediation worthwhile?

The results in the previous chapter showed that remediation had an effect on improving basic algebra skills but the strategies implemented in this project were not effective enough. Better results might be obtained by either spending more than five weeks on remediation using Cognitive Conflict with guidance from a tutor, or on Practice, or perhaps a combination of these or other strategies.

The high correlation between the pre-test and the Math 130 marks could be in indicator that not enough is being done in Math 130 to address the needs of low achieving students during the course, as many of the students who score low in the pre-test also scored low in Math 130. The pre-test could serve to identify students at risk of low achievement at the start of Math 130 and guide them to further help, such as listed in section 5 below.

THE CHANGING ROLE OF UNIVERSITY TEACHING

The low level of basic algebra skills compared to what most lecturers remember from their school days is to be expected given the change in focus at secondary school level in South Africa towards understanding rather than drill and practice. Although the quote from Booth below was speaking about American schools two decades ago, the same emphasis change is evident in the latest South African Curriculum statements too.

"The essential dimension underlying current views of the nature of the algebra curriculum is the move from an emphasis on manipulative skills to an emphasis on conceptual understanding and problem solving, that is, a move from doing algebra to using algebra." (Booth, 1989 p. 244)

I would suggest that university mathematics teaching is not changing adequately to cope with the changes in secondary teaching. This may be partly due to the lack of connection to high schools that most university mathematicians have in their research-demanding jobs. South African school teachers have also had to cope with frequent curriculum changes (Blignaut, 2008) and may not wish to discuss the school curriculum with university lecturers while they are not very familiar with it themselves. If lecturers were more in touch with the mathematics done in high school, their lecturing and course planning might include addressing expected problems in basic algebra and separate remediation on basic algebra might not be necessary.

Universities should have dedicated staff to focus on the transition from high school to first year university, to identify problem areas and help to devise and implement strategies to overcome them, and to inform university staff about the changes in the high school curriculum and how schools, particularly the university's main feeder schools, are coping with it. In 2008, a Teaching and Learning Manager was appointed to all Schools at the University of KwaZulu-Natal. This person became responsible for the points raised here. The Booster Programme, a series of revision notes and on-line quizzes on Algebra, Trigonometry and Analysis for first-year students at UKZN, started in 2009 and hopes to address the gap between high school and university.

Recent studies are confirming that remedial programmes at tertiary education institutions do make a big difference in students' success at university (e.g. Bahr, 2008; Brants & Struvven, 2009; Brouwer, Ekimova, Jasinska, van Gastel, & Virgailaite-Meckauskaite, 2009; Kozeracki, 2002; Rienties, Tempelaar, Dijkstra, Rehm, & Gijselaers, 2008).

The approach of making the teaching interventions during tutorials had some limitations, (as all studies do) for example, different tutors may prefer different teaching methods and somehow convey their favour or disfavour of a method to the students. I tried to control this by making sure that the tutors were clear about the purpose of the remediation activities and their role in it, and that they seemed to be positive about it.

Another limitation was that the tutorial groups met on different days and may have been influenced by the content of their mathematics lectures. For example, a tutorial group taking the post-test on Monday may have been disadvantaged if in the Tuesday lecture, an example involving a question similar to one in the post-test was used, e.g. factorizing a difference of squares.

It is also possible that students discussed the work outside of tutorials with class mates from other tutorial groups and this influenced their progress in addition to (or perhaps in place of) the intervention in tutorials. I attempted to control this by not making the intervention worksheets or pre- and post-tests available to students to take away from the tutorials.

In addition to these limitations with the set-up of the tutorial groups, the learning theories reviewed earlier suggest that students may sometimes get something right without understanding it, as in the pseudo-structural approach (Vinner, 1997). I felt that probing their understanding through interviews or their written explanations could add depth to this study but would be beyond the scope of this project. Instead I note this as a limitation of this study and something that other studies could pursue.

6.2. Which remediation strategy is best?

It would appear that challenging long-held beliefs was more successful when the strategy was reinforced using different material each week, as was done in the three highest ranked remediation strategies, Tutor-led Cognitive Conflict, Practice and Self-led Cognitive Conflict.

From the low ranking of the Cognitive Conflict Once strategy, it appears that students did not interiorise the method of cognitive conflict after experiencing it only once and so they were unable to transfer the method of thinking to other questions. The cognitive conflict strategies were designed to make students examine their beliefs about certain concepts in mathematics, something they might never have had to do before. Tanner and Jones (2000, p. 92) point out that in a constructivist paradigm, considerable mental effort is needed for new information to be assimilated into an existing mental structure. Only one session of cognitive conflict seems to be insufficient to allow for students to make the additional effort demanded by a conceptual change approach. An intervention period longer than five weeks may have brought about greater improvements in the Cognitive Conflict strategies as it would have given students more time to adjust to this type of thinking. The strategy of Practice, in which marked work was returned to the students, was probably of a style more familiar to most students and this may have influenced the results in favour of Practice compared to Cognitive Conflict strategies.

6.3. Which errors are the hardest to remediate?

Looking at the test questions, some interesting observations can be made. The low scores on four out of five questions involving fractions confirmed a well-known problem area: working with simplifying fractions. Other low-scoring questions involved operating with exponents, where rules for simplifying expressions are distorted, perhaps because of over-generalizing. For example, the over-generalization error of simplifying $(x^2 + 9)^{1/2}$ as $x + 3$, in question 3 was shown to be a persistent error with 28% of the class still over-generalizing power distribution for factors to a binomial. It may be that students are more easily drawn to making this mistake when they do not have alternative ways of thinking about an expression, for example, a binomial raised to the power of 2 may have more meaning to a student because they have studied quadratic functions, but they may not have a sense of how a binomial raised to the power of one-half can be thought of except as a collection of algebraic symbols. Strategies that get students to consider alternative representations of algebraic expressions, such as using tables of values or graphs, might be more effective at remediating such errors (Friedlander & Tabach, 2001; McCoy, Baker, & Little, 1996; Tabach & Friedlander, 2008).

The error classification devised in the Method chapter identified eight categories of errors, more than the number of categories in each of the three error classifications that were described in the Literature Review. The tests used in this study did not have an even number of questions from each error category and this is a limitation when considering which errors were the hardest to remediate. For example, only one question on confusing similar notation was included in the tests, compared to sixteen questions on the use of a distorted algorithm, definition or theorem. However, the opportunities for students to make notation errors are much fewer in first year mathematics than at school and the number of questions relating to each error category reflected what I felt was the proportions in which these errors commonly occur.

The error category 'Technical or careless error' was found to be the hardest to remediate. This may be of comfort to teachers because these are 'human' errors that generally only need pointing out in order to be corrected. The use of computer algebra systems might help eliminate some of these errors, if errors such as leaving out a closing bracket can be avoided while using them. Errors with distributing minus signs and errors from the use of a distorted algorithm, definition or theorem were also found to be hard to remediate.

6.4. Limitations

Radatz (1979) warns that errors are the result of complex processes involving many factors, for example, the teacher, the curriculum and the environment. This makes it difficult to make sharp separations between the possible causes of an error. This is pertinent to the attempts to classify the types of errors that were made by the students.

Ethics of no remediation

At the time of this experiment, I did not feel that students in the control group were being treated unfairly because it had been the case in previous years that no students received any form of help on basic algebra apart from questions that they raised with tutors or lecturers themselves. Students in this study had also covered all the basic algebra they needed in high school and could refer to their school notes or other reference material including library books if they needed to improve their algebra skills. In retrospect, I think the students who received no remediation were a little disadvantaged, as were some from the less successful intervention groups. This could have been minimized by posting all the materials used for the interventions on the course website after the post-test but before the final exam. If time had allowed, and the students had been willing, all students could have been given the most successful remediation after the post-test.

How could this study be improved?

The ideas on how this study could be improved can be grouped into two areas: those relating to the test and those relating to the students.

Test-related improvements

The remediation that was implemented once-off was less effective than any of the strategies that were implemented weekly for five weeks. Perhaps if remediation was implemented for longer than the five weeks used in this project, the results might have been even better. Including some remediation strategies in lectures when algebra is used could increase the frequency of remediation which could strengthen the positive effects from remediation.

Some questions had more than one correct form for the answer but only one version was listed in the five given options. For example, question 11 (simplifying a complex fraction) and question 22 (simplifying a fraction involving logs to the same base) have alternative simplifications that were not included in the listed options. Although there was space for students to write their own answer if it was different from any of the ones listed, this may have caused confusion for some students. It may have helped if the correct options were in forms most likely to be obtained by students. Alternatively, questions where different answer forms could create confusion could be asked only as free-response rather than multiple choice questions.

The limitations with multiple choice questions have been researched extensively (Dufresne, Leonard, & Gerace, 2002; Haladyna, 2004). A big limitation of multiple choice questions is that you cannot tell whether a student guessed an answer. The option of putting in a free-response answer may have limited guessing since the students didn't know that one of the options had to be correct. Other problems identified by Mark Simkin and Willaim Kuechler (2005) are, firstly, that multiple choice test items may test widely varying levels of student understanding but be interpreted as testing understanding homogenously, and secondly, that free-response questions test student mastery differently from multiple choice questions. The first of these concerns was somewhat addressed in this study by the question-by-question analysis of error types and student responses. The second concern was addressed by the use of a combined multiple-choice and free-response test, similar to what they described as a better alternative to pure multiple choice tests.

A limitation of error studies was pointed out by Nesher (1987) and Wagner and Kieran (1989 p. 227) who found that learners could hold misconceptions despite being able to give correct answers. This would be a limitation in a multiple choice or free-response test. Interviews may provide a way to uncover misconceptions with more reliability.

In each group, a small number of answers (5% to 7.7%) were changed from correct answers in the pre-test to incorrect ones in the post-test. This could indicate one of three things: carelessness, guessing when answering in the first test, or unstable knowledge, perhaps because students are in a process of restructuring their knowledge (Martin, Mintzes, & Clavijo, 2000). Guessing is more likely in the last questions of the test that slower-working students may not have had time to adequately complete. If there had been a large number of right-to-wrong answers, this could have suggested that students were not taking the tests seriously and could indicate that the responses may not reflect their understanding. The fact that there was one question (Question 9) for which the entire class gave correct responses strengthened my confidence that the students did work through the questions before choosing an answer. However, guessing could have been used when a student could not immediately work out an answer and this wouldn't have been picked up by the test. Methods to avoid the problem of not knowing whether students are guessing include interviews and written reasons for answers.

There have been numerous studies (Kvale, 1996; e.g. Osborne & Gilbert, 1980) that have yielded valuable in-depth knowledge as the result of interviews. The disadvantage of interviews is that they are very time-consuming, and to be practical, only a sample of students would be interviewed. Also, if held before the post-test, the interview itself would form part of the remediation, causing students to think more deeply about basic algebra and perhaps making the interview itself a successful remediation strategy but impractical to implement with large numbers of students because of time constraints. So while interviews could yield interesting results, I chose to focus instead on analysis of the results as a whole. In retrospect, I feel that interviews of students who were identified as having persistent difficulties with algebra could have added depth to this study. Interviews could be held after the post-test so as not to interfere with the testing of the remediation strategy and may even produce new ideas for remediation that could be developed for groups rather than individuals. It is a limitation of this study that interviews (and possibly other qualitative data collection instruments) were not used and these would be good to include if this study was extended.

Abraham Arcavi (1994) argues that just as arithmetic teaching endeavours to instill a 'number sense' rather than just the correct performance of operations, algebra learning should include the development of 'symbol sense' and not just symbol manipulations. This important aspect of algebra understanding could be seen as far more important as the focus on fluency with algebraic manipulation, and perhaps this project could be criticized for not looking far enough beyond the mastery of symbolic manipulations. However, interventions to develop algebraic reasoning may need much longer to implement (Yerushalmy, 1997).

STUDENT-RELATED IMPROVEMENTS

This study has not considered affective factors, such as students' goals, intentions, expectations and motivation. Ignoring affective and social factors can limit conceptual change (Linnenbrink & Pintrich, 2004; Pintrich, 2004). Strategies that incorporate a holistic approach to student learning such as counseling students may yield more positive results.

The group receiving practice had the largest number of absentees when the post-test was written and this reduced the group size from 24 to only 10. This may be because the class marks (which allow or disallow students to write the exam) were released before the Practice group met for the last time. For students who were not allowed to write the exam, there was little incentive to attend the last tutorial. Many students who would be writing the mathematics exam missed the last tutorial in order to catch up on other work. The number of absentees in the post-test may have reduced by delaying the release of the class marks, giving the post-test during a lecture, making it a required part of the course, or trying to get students to value the help they could get in their final tutorial.

Given the diverse high school backgrounds of the students in the Math 130 class, it seems likely that different students will prefer different teaching methods. It has been shown that university students across different disciplines prefer to be taught in a way that matched their learning style (Hativa & Birenbaum, 2000). It may be that the group that improved the most did so because there were a greater proportion of students matched to a method they preferred. As the students' preferred learning strategies were not investigated, we can only assume that there was a more-or-less even distribution of students preferring different learning styles in each group. Further studies could include a learning preference test, such as the one found at http://www.learning-styles-online.com.

6.5. Recommendations for further research

Other remediation strategies for improving algebra skills

As universities worldwide continue to cater for more diverse student bodies, more ways to meet the needs of modern students are being tested and used (Fry, Ketteridge, & Marshall, 2008). The most effective approach is likely to be a combination of strategies, so that students can find the one that is best suited to them. Possible strategies include:

Computer-based tests with immediate feedback, including reasons for incorrect results, such as can be created using Moodle (www.moodle.org).

A drop-in help centre with mathematics books and a tutor available to answer any questions, such as the Hot Seat (North & Zewotir, 2007).

Giving students incentives to improve their basic skills by making basic skills tests a course requirement that contributes to final marks (Brouwer *et al.*, 2009).

Self-paced instruction involving mastery learning, requiring students to achieve 80% or more (Trillan, 1980). This could be in the form of a handout or web-based collection of notes and practice questions.

Peer tutoring (Boud, Cohen, & Sampson, 2001; Trillan, 1980) including Supplemental Instruction (Arendale, 1994) in which a student that recently completed a course leads students in revision sessions. Boud (2001) warns that peer tutoring may take a few cycles before it works well but the interactions that follow are worth the effort.

Resource centres, also known as maths labs or learning centres, have been found effective at addressing the diversity of students' mathematics backgrounds and allow students to work at an individual pace (Trillan, 1980). Two problems with resource centres are how to encourage students who need help to attend and the high cost of tutors.

As the doors of universities globally have opened wider to include a larger and more diverse student body than ever before new demands are being placed on institutions to provide support to the student they admit. South Africa may feel greater pressure to deal with this problem, given the academic disadvantages imposed by apartheid policies of the past. The challenges to improving students' performance at university are many. There are still many under-resourced primary and high school, unqualified or absent teachers, a curriculum that many teachers are still not familiar with and many social problems that could affect student's ability to make good academic progress, such as crime and poverty.

This study has shown that there is a need to improve the algebra skills of first-year students and the most success is likely to come from combining different remediation strategies and to continue to address algebra difficulties during the course of first-year mathematics. This job will be made easier if lecturers are made aware of the common algebra difficulties and strategies that have been shown to be successful at addressing them. Collaboration between institutions would help to develop this knowledge.

On a deeper level, increasing students' understanding of their self-worth and their connectedness with the world and other people (Palmer, 2003), something that we have not tried to address, may be key to developing the self-motivation that appears to be missing from so many of our students, and may be what is necessary for students to make use of available remediation strategies.

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