EMBEDDING THEOREMS AND FINITENESS PROPERTIES FOR RESIDUATED STRUCTURES AND SUBSTRUCTURAL LOGICS

by

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As the candidate's supervisor I have/have not approved this thesis for submission.

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ABSTRACT

Paper 1. This paper establishes several algebraic embedding theorems, each of which asserts that a certain kind of residuated structure can be embedded into a richer one. In almost all cases, the original structure has a compatible involution, which must be preserved by the embedding. The results, in conjunction with previous findings, yield *separative* axiomatizations of the *deducibility relations* of various substructural formal systems having double negation and contraposition axioms. The separation theorems go somewhat further than earlier ones in the literature, which either treated fewer subsignatures or focussed on the conservation of theorems only.

Paper 2. It is proved that the variety of relevant disjunction lattices has the finite embeddability property (FEP). It follows that Avron's relevance logic $\mathbf{RMI}_{\text{min}}$ has a strong form of the finite model property, so it has a solvable deducibility problem. This strengthens Avron's result that $\mathbf{RMI}_{\text{min}}$ is decidable.

Paper 3. An idempotent residuated po-monoid is *semiconic* if it is a subdirect product of algebras in which the monoid identity t is comparable with all other elements. It is proved that the quasivariety SCIP of all semiconic idempotent commutative residuated po-monoids is locally finite. The lattice-ordered members of this class form a variety SCIL, which is not locally finite, but it is proved that SCIL has the FEP. More generally, for every relative subvariety K of SCIP, the lattice-ordered members of K have the FEP. This gives a unified explanation of the strong finite model property for a range of logical systems. It is also proved that SCIL has continuously many semisimple subvarieties, and that the involutive algebras in SCIL are subdirect products of chains.

Paper 4. Anderson and Belnap's implicational system $\mathbf{RMO}_{\rightarrow}$ can be extended conservatively by the usual axioms for fusion and for the Ackermann truth constant t. The resulting system $\mathbf{RMO}^*$ is algebraized by the quasivariety IP of all idempotent commutative residuated po-monoids. Thus, the axiomatic extensions of $\mathbf{RMO}^*$ are in one-to-one correspondence with the relative subvarieties of IP. It is proved here that a relative subvariety of IP consists of semiconic algebras if and only if it satisfies $x \approx (x \rightarrow \mathsf{t}) \rightarrow x$. Since the semiconic algebras in IP are locally finite, it follows that when an axiomatic extension of $\mathbf{RMO}^*$ has $((p \rightarrow \mathsf{t}) \rightarrow p) \rightarrow p$ among its theorems, then it is locally tabular. In particular, such an extension is strongly decidable, provided that it is finitely axiomatized.

PREFACE

The work described in this thesis was carried out in the School of Mathematical Sciences, University of KwaZulu-Natal, Durban, from January 2005 to December 2008, under the supervision of Professor James G. Raftery.

These studies represent original work by the author and have not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others it is duly acknowledged in the text.

Signed:

DECLARATION 1 - PLAGIARISM

I, Ai-Ni Hsieh, declare that

(1) The research reported in this thesis, except where otherwise indicated, is my original research.

(2) This thesis has not been submitted for any degree or examination at any other university.

(3) This thesis does not contain other persons' data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.

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   (a) Their words have been re-written but the general information attributed to them has been referenced
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DECLARATION 2 - PUBLICATIONS

DETAILS OF CONTRIBUTION TO PUBLICATIONS that form part and/or include research presented in this thesis (include publications in preparation, submitted, *in press* and published and give details of the contributions of each author to the experimental work and writing of each publication)

(1) A. Hsieh, J.G. Raftery, *Conserving involution in residuated structures*, Mathematical Logic Quarterly **53**(6) (2007), 583–609.

(2) A. Hsieh, J.G. Raftery, *A finite model property for* $\mathbf{RMI}_{\min}$, Mathematical Logic Quarterly **52**(6) (2006), 602–612.

(3) A. Hsieh, J.G. Raftery, *Semiconic idempotent residuated structures*, Algebra Universalis, to appear.

(4) A. Hsieh, *Some locally tabular logics with contraction and mingle*, submitted, 2008.

CONTRIBUTIONS: In the joint papers (1)–(3), most of the problems addressed were posed by the supervisor, but proofs of the main results were obtained by the candidate. The first drafts of the papers were written by the candidate. They were edited after the supervisor made suggestions concerning (i) ways of shortening or simplifying a few of the arguments, and (ii) style, including standard English usage.

Signed: aHsh

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INTRODUCTION

Residuated structures.

The general definition of a residuated structure can be phrased as follows. Consider a binary operation $\cdot$ on a partially ordered set $\langle A; \leq \rangle$, and assume that $\cdot$ preserves the order, i.e., for all $a, b, c, d \in A$,

$$\text{if } a \leq b \text{ and } c \leq d \text{ then } a \cdot c \leq b \cdot d.$$

We use $a \backslash b$ to denote the largest $c \in A$ such that $a \cdot c \leq b$, if this exists. Similarly, b/a denotes the largest $c \in A$ for which $c \cdot a \leq b$, if this exists.

We say that $\cdot$ is *residuated* (with respect to $\leq$) if $a \backslash b$ and b/a both exist for all $a, b \in A$. In this case, we also refer to $\langle A; \cdot, \backslash, /, \leq \rangle$ as a *residuated structure.*

Equivalently, $\langle A; \cdot, \backslash, /, \leq \rangle$ is a residuated structure iff $\langle A; \leq \rangle$ is a poset and $\cdot, \backslash, /$ are binary operations on A such that the *law of residuation*

$$a \cdot c \leq b \text{ iff } c \leq a \backslash b \qquad ; \qquad c \cdot a \leq b \text{ iff } c \leq b/a$$

holds for all $a, b, c \in A$. The compatibility of $\leq$ with $\cdot$ follows from this law. When $\cdot$ is commutative, we have $a \backslash b = b/a$, and it is customary to replace both expressions by $a \to b$.

Some history.

The theory of residuated structures descends from three essentially independent sources: the ideal theory of rings, the calculus of binary relations, and the model theory of non-classical logics.

In a unital ring R, the operation of ideal multiplication

$$A \cdot B := \{\Sigma_{i=1}^{n} a_i b_i : n \in \mathbb{N} \text{ and } a_i \in A \text{ and } b_i \in B \text{ for each } i\}$$

induces two division-like residual operations on ideals, viz.

$$A \backslash B := \{r \in R : ar \in B \text{ for all } a \in A\} = \max_{\subseteq} \{I \lhd R : A \cdot I \subseteq B\};$$

$$B/A := \{r \in R : ra \in B \text{ for all } a \in A\} = \max_{\subseteq} \{I \lhd R : I \cdot A \subseteq B\}.$$

The ideals of R instantiate the law of residuation, i.e., we have

$$A \cdot C \subseteq B \text{ iff } C \subseteq A \backslash B \qquad ; \qquad C \cdot A \subseteq B \text{ iff } C \subseteq B/A.$$

The role of residuation in ring and module theory is already evident in Dedekind's work. It turns out that much of the theory of commutative Noetherian rings can be obtained in the abstract setting of residuated *lattices.* This abstraction probably begins with Krull [24] (1924); it was continued by Ward and Dilworth [37] (1939), who established the notation $\backslash, /$. A much more recent example in the same spirit is the commutator operation on the congruences of a universal algebra, which gives rise to a diversity of natural residuated structures [13].

Independently, residuation comes up in the work of the Boolean school of logicians, especially De Morgan, Peirce and Schröder (see [3, 28, 29, 31]). The intended interpretation of a first order predicate is a *relation* on a set, so mathematical logic must deal with the ways in which relations interact when they are subjected to natural *operations*. For example, the identities

$$(x \circ y)^{\smile} \approx y^{\smile} \circ x^{\smile} \quad \text{and} \quad (x \cap y)^{-} \approx x^{-} \cup y^{-}$$

hold when x and y range over the *binary* relations on a set, provided that $\circ$, ${}^{\smile}$, ${}^{-}$, $\cap$ and $\cup$ are interpreted as composition, conversion, complementation, intersection and union, respectively. Implicit in De Morgan's paper [11] of 1860 is another instance of the law of residuation, viz.

$$x \circ z \subseteq y \ \text{ iff } \ z \subseteq x\backslash y \qquad ; \qquad z \circ x \subseteq y \ \text{ iff } \ z \subseteq y/x,$$

where $x\backslash y$ and y/x are $(x^{\smile} \circ y^{-})^{-}$ and $(y^{-} \circ x^{\smile})^{-}$, respectively. Abstracting from this example, Tarski [32] invented *relation algebras* [22] and showed, in joint work with Givant [33], that their equational theory has the same expressive power as first order logic in three variables.

In classical propositional logic, $p \to q$ is defined as $(\neg p) \vee q$, but most non-classical logics reject this 'material' implication. Intuitionistic logic [35] is motivated by a desire to distinguish *constructive* proofs from non-constructive ones. It rejects the Boolean principles $(\neg\neg p) \to p$ and $p \vee \neg p$. Relevance logic [1, 30] rejects the 'paradoxes' of material implication, such as $p \to (q \to p)$ and $(p \wedge \neg p) \to q$, which allow us to draw conclusions from irrelevant premisses. Linear logic [16, 6, 34] is motivated in part by computer science; it treats the premisses in a derivation as resources and is sensitive to the number of times that they are used. In particular, it rejects the contraction principle $(p \to (p \to q)) \to (p \to q)$.

The models of intuitionistic logic (i.e., Heyting algebras) are distributive lattices in which conjunction is represented by the greatest lower bound, $\wedge$. Implication can then be *characterized* by the law of residuation

$$c \leq a \to b \ \text{ iff } \ a \wedge c \leq b.$$

In the models of the principal relevance logic **R** (i.e., De Morgan monoids), a similar law holds, viz.

$$c \leq a \to b \ \text{ iff } \ a \cdot c \leq b,$$

where $\cdot$ is *co-tenability*, i.e., $p \cdot q := \neg (p \to \neg q)$. The same applies to linear logic (except that the algebras are different). In certain logics, however, $\cdot$ is a primitive connective, not definable in terms of the other symbols. This is true of several many-valued logics [17]. It is also true of the Lambek calculus [23], where $\cdot$ represents the juxtaposition of grammatical categories (such as noun or verb); it is therefore noncommutative, but the law of residuation still holds in the algebraic models.

Substructural logics.

Since so many differently motivated logics are modeled by classes of residuated structures, general studies of such systems began during the 1980s. They are often called *substructural logics* [14, 26, 27], because they may lack some of the so-called *structural rules*

(C)	$\vdash (p \to (q \to r)) \to (q \to (p \to r))$	(*exchange*)
(W)	$\vdash (p \to (p \to q)) \to (p \to q)$	(*contraction*)
(K)	$\vdash p \to (q \to p)$	(*weakening*).

For instance, linear logic satisfies (C) but violates both (W) and (K). These three laws have a natural significance in proof theory [15, 25]. In residuated structures, they take the following algebraic form, where t is an identity element for $\cdot$:

(C)	$x \cdot y \approx y \cdot x$	(*commutativity*)
(W)	$x \leq x \cdot x$	(*square-increasing law*)
(K)	$x \leq \mathsf{t}$	(*integrality*).

There are also significant logics that are not modeled by residuated structures at all, e.g., quantum logics [12] and sub-intuitionistic logics [36, 4]. But in this dissertation, we shall be concerned only with substructural logics. Moreover, in the corresponding residuated structures, $\cdot$ will always be associative and commutative, so $\to$ shall replace $\backslash, /$ throughout.

Problems and Methodology.

Logics are usually specified formally by postulates that are either *axioms* (like (C), (W), etc.) or *inference rules*, of which

$$p\,,\ p \to q \ \vdash \ q \qquad\qquad (\textit{modus ponens})$$

is a common example. A formula α is a *theorem* if it has a *proof*, i.e., if there is a finite sequence of formulas, terminating with α, where each item in the sequence is an instance of an axiom or can be obtained from previous items by application of an inference rule. It may happen that a non-theorem β becomes provable in this way when we add a set of formulas Γ to the stock of axioms (without also adding the *instances* of the formulas in Γ). In this case, we call $\Gamma \vdash \beta$ a *derivable rule* of the system. Theorems can be thought of as derivable rules $\emptyset \vdash \beta$ with no premisses.

In the context of substructural logics, some natural questions then arise. Typically, the language (a.k.a. signature) will consist of some—possibly all—of the symbols $\to, \cdot, \wedge, \vee, \neg, \mathsf{t}$. For a subset S of this language and a derivable rule $\Gamma \vdash \alpha$ in which only symbols from S occur, can $\Gamma \vdash \alpha$ already be derived from the *postulates* that use only the symbols in $S \cup \{\to\}$? To demonstrate this is to prove a *deductive separation theorem* for the system, and it signifies that the system is 'well-axiomatized'. This property is philosophically important when the system is motivated primarily by its postulates in a limited subsignature (such as $\to$ alone).

Two further questions that arise for every logic are the *decision problem* and the *deducibility problem.* The decision problem asks whether there is an *algorithm* that tests formulas for *theoremhood.* Equivalently (modulo Church's thesis), it asks whether the set of theorems of the logic is recursive. If this is the case, the system is said to be *decidable.* The deducibility problem asks whether the set of derivable rules $\Gamma \vdash \alpha$ (Γ finite) is recursive, in which case this problem is said to be *solvable* and the logic *strongly decidable.*

One way to prove a deductive separation theorem for a formal system is to establish a suite of *embedding theorems*, showing that each model of the postulates in a subset of the signature can be embedded into a model of the full system—this for all (or all indicated) subsets of the signature. And one way to prove that a finitely axiomatized system has a solvable deducibility problem is to prove that it is modeled by a class K of algebras enjoying the *finite embeddability property* (FEP): this means that every finite subset of an algebra in K can be extended to a finite algebra in K, with preservation of all partial operations.

Embedding theorems and finiteness properties (such as the FEP) are therefore related topics, and they shall be the main preoccupations of this dissertation.

Although we tend, where possible, to explain things from a general perspective (using in particular the Blok-Pigozzi theory of 'algebraizable' logics developed in [9]), the results here shall all concern substructural logics and the classes of residuated structures that model them. Where universal algebraic methods are used, we assume only a very rudimentary background on the part of the reader—no more than can be found, for instance, in the early chapters of [10].

The results.

In Paper 1 (i.e., [18]), several algebraic embedding theorems are established, each of which asserts that a certain kind of residuated structure can be embedded into a richer one. In almost all cases, the original structure has a compatible *involution* $\neg$, which must be preserved by the embedding. The required properties of $\neg$ are

$$\neg\neg\, x \approx x$$
$$x \leq y \implies \neg\, y \leq \neg\, x$$
$$x \cdot z \leq y \iff (\neg\, y) \cdot z \leq \neg\, x.$$

Using these results, we obtain deductive separation theorems for various substructural formal systems having the *double negation* and *contraposition* axioms

$$\vdash (\neg\neg p) \to p \qquad \text{and} \qquad \vdash (p \to \neg q) \to (q \to \neg p).$$

The systems covered here include exponential-free classical linear logic (a.k.a. $\mathbf{CFL_e}$), its contractive extension **LR** (a.k.a. $\mathbf{CFL_{ec}}$), its affine extension (i.e., BCK-logic or $\mathbf{CFL_{ew}}$), and to a limited extent the distributive systems **R** and **RW**.

Because the results in Paper 1 separate *rules* (as opposed to theorems), they go further than earlier work on separation in the literature, which either treated fewer subsignatures or focussed on the conservation of theorems only. This earlier work is summarized in Paper 1, with references.

Relevance logics are usually distinguished by their satisfaction of the contraction axiom (W), and sometimes other properties as well. Anderson and Belnap's system **R** and its extension **RM** by the *mingle axiom*

$$(\mathsf{M}) \qquad \vdash p \to (p \to p)$$

have been studied by relevance logicians for several decades. It is known that **R** is undecidable but, thanks to the mingle axiom, its extension **RM** is decidable. However, **RM** lacks the *variable sharing property*, which is regarded by the Anderson-Belnap school as a necessary condition for relevant implication. For instance $(\neg (p \to p)) \to (q \to q)$ is a theorem of **RM** (not of **R**), but no variable occurs both in $\neg (p \to p)$ and in $q \to q$.

To overcome this problem, Avron [5, 7, 8] introduced, among other systems, a simply axiomatized relevance logic $\mathbf{RMI}_{\min}$, which has many of the desirable features of **RM**, including the mingle axiom. He proved that $\mathbf{RMI}_{\min}$ is decidable and has the variable sharing property. But the deducibility problem and the finite model property for $\mathbf{RMI}_{\min}$ were not addressed in his work.

Paper 2 ([19]) strengthens Avron's result by showing that the set of finite derivable rules of $\mathbf{RMI}_{\min}$ is also recursive, not merely the set of theorems. As $\mathbf{RMI}_{\min}$ has (demonstrably) no deduction theorem, this stronger result does not follow from the fact that $\mathbf{RMI}_{\min}$ is decidable. The algebraic counterpart of $\mathbf{RMI}_{\min}$ is the variety RDL of all *relevant disjunction lattices*. These are idempotent commutative residuated lattice-ordered semigroups—not monoids—with a compatible involution. The approach of Paper 2 is to prove that RDL has the FEP. This implies that $\mathbf{RMI}_{\min}$ has the *strong finite model property* (i.e., every finite non-derivable rule is refutable in some *finite* relevant disjunction lattice), and it follows that $\mathbf{RMI}_{\min}$ has a solvable deducibility problem. Paper 2 also provides an analysis of RDL that is slightly simpler than Avron's.

Besides RDL, an investigation of *semiconic* residuated structures is carried out in Papers 3 and 4 ([20, 21]). These are subdirect products of residuated ordered monoids in which the identity element t is comparable with all other elements. The class IP of all idempotent commutative residuated partially ordered monoids is a quasivariety, and so is the class SCIP of all semiconic algebras in IP. Neither IP nor SCIP is a variety. The lattice-ordered members of SCIP (enriched with the operations $\wedge, \vee$) form a variety SCIL which is not locally finite. In other words, SCIL contains some finitely generated algebras that are not finite.

Nevertheless, it is proved in Paper 3 that SCIP *is* locally finite. This result facilitates a proof that for every relative subvariety K of SCIP, the lattice-ordered members of K have the FEP. (A *relative subvariety* of K is a subquasivariety that is the intersection of K with some variety.) In particular, SCIL itself has the FEP. This is of interest, since SCIL contains all Brouwerian lattices (i.e., the algebraic models of positive intuitionistic logic) as well as all positive Sugihara monoids (these model the positive fragment of $\mathbf{RM}$). The results in Paper 3 therefore give a unified explanation of the strong finite model property for many extensions of these and other systems. It is also proved in Paper 3 that SCIL has continuously many semisimple subvarieties, and that the involutive algebras in SCIL are subdirect products of chains.

Another logical system introduced by Anderson and Belnap is $\mathbf{RMO}_{\rightarrow}$ which is axiomatized in the signature $\{\rightarrow\}$ by

$$\vdash\ (p \rightarrow q) \rightarrow ((r \rightarrow p) \rightarrow (r \rightarrow q)) \qquad (\textit{prefixing}),$$

exchange (C), mingle (M) and contraction (W), with modus ponens as the sole inference rule. If we add the usual axioms for fusion $(\cdot)$ and for the Ackermann truth constant t to $\mathbf{RMO}_{\rightarrow}$, the resulting system $\mathbf{RMO}^{*}$ is a *conservative extension* of $\mathbf{RMO}_{\rightarrow}$, i.e., the deductive separation theorem holds for $\mathbf{RMO}^{*}$. (This is partly due to the absence of negation.) Now $\mathbf{RMO}^{*}$ is algebraized by the quasivariety IP. It follows from the general theory of algebraization that the axiomatic extensions of $\mathbf{RMO}^{*}$ are algebraized by the relative subvarieties of IP, and this correspondence is a bijection. Thus, by Paper 3, if the algebraic counterpart of an axiomatic extension of $\mathbf{RMO}^{*}$ consists of semiconic algebras, it will be locally finite. Consequently, such extensions are decidable provided they are finitely axiomatized.

It is therefore useful to have a characterization of the semiconic relative subvarieties of IP. In Paper 4, it is proved that a relative subvariety of IP consists of semiconic algebras if and only if it satisfies the equation

$$(x \rightarrow \mathsf{t}) \rightarrow x \approx x.$$

As SCIP does not satisfy this equation, it follows that SCIP is not itself a relative subvariety of IP.

In logical terms, it follows from Paper 4 that when an axiomatic extension of $\mathbf{RMO}^*$ has

$$((p \to \mathsf{t}) \to p) \to p$$

among its theorems then it is *locally tabular*, i.e., for each finite number n, there are only finitely many logically inequivalent formulas in n variables. The finitely axiomatized axiomatic extensions of $\mathbf{RMO}^*$ include the $\wedge, \to$ fragment of intuitionistic logic and the $\cdot, \to, \mathsf{t}$ fragment of $\mathbf{RM}^{\mathrm{t}}$. [1] It is well known that these two incomparable systems are locally tabular (and have solvable deducibility problems), but Paper 4 gives a common explanation of these phenomena, that applies to many other systems as well.

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[1] $\mathbf{RM}^{\mathrm{t}}$ is the extension of $\mathbf{RM}$ by the constant t and the axioms t and $\mathsf{t} \to (p \to p)$.

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CONSERVING INVOLUTION IN RESIDUATED STRUCTURES

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Abstract. This paper establishes several algebraic embedding theorems, each of which asserts that a certain kind of residuated structure can be embedded into a richer one. In almost all cases, the original structure has a compatible involution, which must be preserved by the embedding. The results, in conjunction with previous findings, yield *separative* axiomatizations of the *deducibility relations* of various substructural formal systems having double negation and contraposition axioms. The separation theorems go somewhat further than earlier ones in the literature, which either treated fewer subsignatures or focussed on the conservation of theorems only.

1. Introduction

Most of the results in this paper are embedding theorems for ordered structures with a residuated commutative semigroup operation $\cdot$ and a *compatible involution*, i.e., an involution $\neg$ for which the law

$$x \cdot z \leq y \iff \neg\, y \cdot z \leq \neg\, x$$

holds. The principal results are stated precisely below, for ease of subsequent reference, but the details need not be absorbed on first reading.

(i) *Every compatibly involutive residuated commutative posemigroup satisfying* $x \leq x \cdot (y \to y)$ *can be embedded into one that is a* monoid, *in such a way that the square increasing law* $x \leq x \cdot x$ *is preserved.* ($\to$ denotes residuation.)

(ii) *Every compatibly involutive residuated commutative* lattice-*ordered semigroup satisfying* $((x \to x) \wedge (y \to y)) \to z \leq z$ *can be embedded into one that is a monoid, by a construction that preserves both distributivity and the square increasing law.*

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(iii) *Every compatibly involutive residuated commutative po-monoid can be embedded into one that is* distributive lattice-ordered*, by a construction that preserves integrality.*

(*Integrality* is the demand that the monoid identity be the greatest element.)

(iv) *Every* square increasing *compatibly involutive commutative residuated po-monoid can be embedded into a* de Morgan monoid.

Finally, structures without $\cdot$ and $\neg$ are considered, and it is proved that

(v) *If a* distributive *lattice-based algebra* $\langle A; \rightarrow, \wedge, \vee \rangle$ (*with possibly a distinguished element* t) *can be embedded into a residuated lattice-ordered commutative monoid* (*with identity* t), *then it can be embedded into one that is* still *distributive. The construction can be made to preserve integrality and the square increasing law.*

The proofs of (iv) and (v) occurred to us after reading Meyer and Routley's papers [30, 36] on relational structures.

These five results can obviously be composed. They contribute to a body of recent algebraic work on residuation (see [23]), but they also have a logical interpretation: when taken in conjunction with [19, 26, 27, 34, 41], they imply *separation* theorems for the *deducibility relations* $\vdash_{\mathbf{F}}$ of various Hilbert-style formal systems **F**. In each case, **F** specifies a *substructural* logic with the *double negation* law.

The sense of deducibility in this paper is the most simple-minded one: $\Gamma \vdash \alpha$ is *deducible* in **F** iff α becomes provable in **F** once we treat each formula in Γ as an extra axiom. (These extra axioms may be used arbitrarily often in a proof of α and they need not be used at all; their instances are *not* treated as extra axioms.) So theoremhood coincides here with deducibility from an empty set of premisses. The reader is warned that in the literature, $\vdash$ sometimes denotes other forms of derivability. The separation theorems say that when $\Gamma \vdash \alpha$ is deducible in **F**, then it is already deducible in the subsystem of **F** determined by the postulates involving only $\rightarrow$ and the connectives of $\Gamma \cup \{\alpha\}$. The systems **F** include exponential-free classical linear logic (a.k.a. $\mathbf{CFL_e}$), its contractive extension **LR** (a.k.a. $\mathbf{CFL_{ec}}$), its affine extension (i.e., BCK–logic or $\mathbf{CFL_{ew}}$) and to a limited extent the distributive systems **R** and **R-W**.

Partial separation theorems for most of these systems can already be found in the literature, particularly the work of Meyer and Routley [26, 27, 30, 36], to which detailed reference will be made later. Further sources include Maksimova [25], and Ono and Komori [34]. These papers axiomatize the *theorems* of formal systems **F** in certain subsignatures. Fragments of the full deducibility relation

$\vdash_{\mathbf{F}}$ are not discussed, although separative postulates for $\vdash_{\mathbf{F}}$ can be inferred when the proofs amount to embedding constructions—see below. Also, for some negation-less substructural logics, deducibility was axiomatized separatively in [41]. Here we extend the scope of [41] by incorporating classical negation.

Deduction theorems can sometimes be used to show that an axiomatization of the theorems of a system in a restricted vocabulary also axiomatizes the corresponding fragment of the deducibility relation. But this applies only when the deduction theorem is expressed in the restricted vocabulary. For all systems considered here, with the exception of BCK–logic, the pertinent deduction theorems involve $\wedge$ and t (or alternatively, exponentials); see [4, 18]. Thus, they do not help us to axiomatize fragments of $\vdash_{\mathbf{F}}$ that have negation but which lack $\wedge$ or t. Items (i)–(v) above cover these gaps. (For **R** and **R-W**, the lingering effect of *distributivity* on signatures between $\{\rightarrow, \vee\}$ and $\{\cdot, \rightarrow, \vee, \mathsf{t}\}$ remains unknown.)

Substructural logics often have a Gentzen-style presentation **G** with the *subformula property*, e.g., Girard's full linear logic [20, 40]. To pre-empt possible confusion, let us note that this offers no short-cut to the results of this paper, even when we have good ways of translating back into the Hilbert formalism. Subformula properties yield theorem-separation only, as they apply to sequents $\vdash_{\mathbf{G}} \alpha_1, \ldots, \alpha_n \Rightarrow \beta$ that are provable without extra axioms. In all cases under discussion here, the correct interpretation of $\Gamma \vdash_{\mathbf{F}} \alpha$ is: $\Rightarrow \alpha$ is provable in **G** with the help of the new axioms $\Rightarrow \gamma$ ($\gamma \in \Gamma$).

On the other hand, *embedding* results induce deductive separation theorems in general, because $\vdash_{\mathbf{F}}$ can always be translated into a set of *universal* sentences in a first order language with just one predicate symbol. Thus $\vdash_{\mathbf{F}}$ has a semantics consisting of algebras with one indicated relation, and this semantics can be honed down systematically to a universal Horn class of 'reduced' models (see Section 7). For the systems considered in this paper, the sole predicate is interchangeable with a *partial order* in the reduced models, which include the residuated structures featuring in (i)–(v). When we restrict attention to the subsystem $\mathbf{F}_S$ determined by the postulates of **F** in a subsignature S (with $\rightarrow$), the translation is unaffected, but the reduced models have fewer operations and characteristic properties. Now suppose each reduced model of $\vdash_{\mathbf{F}_S}$ can be embedded into some reduced model of $\vdash_{\mathbf{F}}$, where the embedding preserves *and reflects* the indicated order. Of course universal sentences persist in substructures. So a universal sentence in the vocabulary of the models of $\vdash_{\mathbf{F}_S}$ will be true in these models if it holds in the richer models of $\vdash_{\mathbf{F}}$. Consequently, an expression $\Gamma \vdash \alpha$ in the vocabulary of S will be deducible in $\mathbf{F}_S$ if it is deducible in **F**, by the soundness of $\vdash_{\mathbf{F}}$, the embedding theorem, and the completeness of $\vdash_{\mathbf{F}_S}$. That is to say, $\mathbf{F}_S$ axiomatizes the S–fragment of $\vdash_{\mathbf{F}}$.

2. Residuated Structures

A (commutative) *po-semigroup* is a structure $\langle A; \cdot, \leq\rangle$ where $\langle A; \cdot\rangle$ is a commutative non-empty semigroup and $\leq$ is a partial order of A that is *compatible with* $\cdot$ in the sense that for all $a, b, c \in A$,

$$\textit{if} \quad a \leq b \quad \textit{then} \quad c \cdot a \leq c \cdot b.$$

If, in addition, $\mathsf{t} \in A$ is an identity element for $\cdot$ then we call $\langle A; \cdot, \mathsf{t}, \leq\rangle$ a (commutative) *po-monoid*.

Although we omit the word 'commutative' for the sake of brevity, commutativity is always assumed when we speak of po-semigroups or richer structures. We shall generally use t to denote the identity element of a monoid.

Definition 2.1. By a *residuated* (commutative) *po-semigroup* we mean a structure $\langle A; \cdot, \rightarrow, \leq\rangle$ such that $\langle A; \cdot\rangle$ is a commutative non-empty semigroup, $\leq$ is a partial order of A, and for all $a, b, c \in A$,

$$c \leq a \rightarrow b \iff a \cdot c \leq b \quad (\textit{residuation}).$$

If, in addition, $\mathsf{t} \in A$ is an identity element for $\cdot$ then we call $\langle A; \cdot, \rightarrow, \mathsf{t}, \leq\rangle$ a *residuated* (commutative) *po-monoid*.

Proposition 2.2 lists some well known consequences of these definitions. Item (ii) confirms that the $\{\cdot, \leq\}$–reduct of a residuated po-semigroup is indeed a po-semigroup, i.e., $\leq$ is compatible with $\cdot$. For construction purposes the following fact is more useful than the definitions:

> *A structure $\langle A; \cdot, \leq\rangle$ is the $\{\cdot, \leq\}$–reduct of a residuated po-semigroup iff it is a po-semigroup and for any pair $a, b \in A$, there is a* largest *$c \in A$ such that $a \cdot c \leq b$. (The largest c with this property becomes $a \rightarrow b$.)*

Proposition 2.2. *Every residuated po-semigroup satisfies*:

(i) $x \leq y \rightarrow (y \cdot x)$

(ii) $x \leq y \implies z \cdot x \leq z \cdot y$

(iii) $x \leq y \implies z \rightarrow x \leq z \rightarrow y \;\;\&\;\; y \rightarrow z \leq x \rightarrow z$

(iv) $x \leq y \rightarrow z \iff y \leq x \rightarrow z$

(v) $x \leq (x \rightarrow y) \rightarrow y$

(vi) $(x \cdot y) \rightarrow z \approx y \rightarrow (x \rightarrow z) \approx x \rightarrow (y \rightarrow z)$

Notation. We define $|\,x\,| := x \rightarrow x$.

Definition 2.3. The *designated* elements of a residuated po-semigroup are the elements a such that $|\,a\,| \leq a$.

In a residuated po-*monoid*, it turns out that the designated elements are just the upper bounds of the identity, i.e., for all elements a,

$$a \text{ is designated iff } \mathsf{t} \leq a.$$

This is property (3) in the proposition below. It also turns out that all elements of the form $|\,a\,|$ are themselves designated. This follows from property (6).

Proposition 2.4. *Every residuated po-*monoid *satisfies*

$$\begin{array}{ll}
(1) & \mathsf{t} \leq |\,x\,| \\
(2) & x \approx \mathsf{t} \to x \approx x \cdot |\,x\,| \approx |\,x\,| \to x \\
(3) & \mathsf{t} \leq x \iff |\,x\,| \leq x \\
(4) & x \leq y \iff \mathsf{t} \leq x \to y \iff |\,x \to y\,| \leq x \to y \\
(5) & |\,x\,| \leq y \implies |\,y\,| \leq y \\
(6) & |\,x\,| \approx ||\,x\,|| \approx |\,x\,| \cdot |\,x\,|.
\end{array}$$

Each of these laws follows easily from its predecessors and the definition of residuation.

For any partially ordered set $\langle A; \leq \rangle$, a subset X of A is said to be *upward closed* provided that whenever $a \in X$ and $a \leq b \in A$ then $b \in X$.

Definition 2.5. An *implicative po-semigroup* is a residuated po-semigroup in which the set of designated elements is upward closed and for any elements a, b,

$$a \leq b \quad \text{iff} \quad a \to b \text{ is designated.}$$

It follows from Proposition 2.4 (4),(5) that

*every residuated po-*monoid *is implicative (as a po-semigroup)*

and, as the next result makes clear, this is almost a characterization of implicativity. The equivalence of (iii) and (iv) below comes from Meyer [26]. A proof of the equivalence of (i)–(iii) can be found in [22]; a variant of this result for structures with an involution appears in Avron [4, 6].

Theorem 2.6. *For any residuated po-semigroup* $\boldsymbol{A} = \langle A; \cdot, \to, \leq \rangle$, *the following conditions are equivalent*:

(i) $\boldsymbol{A}$ *is implicative*;
(ii) $|\,a\,| \to b \leq b$ *for all* $a, b \in A$;
(iii) $a \leq a \cdot |\,b\,|$ *for all* $a, b \in A$;
(iv) $\boldsymbol{A}$ *may be embedded into a residuated po-*monoid.

At this point we should clarify the sense of 'embedded' in (iv) and in the sequel. An *algebra* is understood in this paper to be a non-empty set on which operations (and no relations) have been indicated. An *ordered algebra* is an algebra with an indicated *partial* order. An *embedding* between *algebras* is, as usual, an injection that preserves all the indicated operations. An *embedding* between *ordered* algebras is understood to be an order preserving *and order reflecting* map (hence an injection) that preserves all the indicated operations. By an *embedding* of a structure into a *richer* one we mean an embedding into the appropriate *reduct* of the richer structure. The main point of these definitions is that when an embedding $\boldsymbol{A} \to \boldsymbol{B}$ exists, if a universal first order sentence of the language of $\boldsymbol{A}$ is true in $\boldsymbol{B}$, then it is also true in $\boldsymbol{A}$.

It follows from Theorem 2.6 that implicative po-semigroups satisfy all of the conditions among (1)–(6) that do not mention t, because these are universal first order sentences. (Here and subsequently we tend to suppress quantifiers when writing universal sentences.)

Definition 2.7. A po-semigroup is said to be *square increasing* if it satisfies $x \leq x \cdot x$.

Meyer's proof of the equivalence of (iii) and (iv) in Theorem 2.6 actually shows the following (which he stated explicitly):

Theorem 2.8. *Every square increasing implicative po-semigroup can be embedded into a square increasing residuated po-monoid.*

3. Involution Properties

For any set A, a function $\neg : A \to A$ is said to be *self-inverting* if $\neg\neg a = a$ for all $a \in A$. In this case $\neg$ is obviously a bijection.

An *involution* of a partially ordered set $\langle A; \leq \rangle$ is a self-inverting function $\neg : A \to A$ that is *order reversing* in the sense that whenever $a, b \in A$ with $a \leq b$ then $\neg b \leq \neg a$.

When denoting involution, we adopt the convention that $\neg$ binds more strongly than any other basic operation, e.g., $\neg a \to b$ abbreviates $(\neg a) \to b$.

Definition 3.1. Given a po-semigroup $\langle A; \cdot, \leq \rangle$, we say that a self-inverting function $\neg : A \to A$ is *compatible* (with $\cdot$) provided that for all $a, b, c \in A$,

$$a \cdot c \leq b \text{ iff } \neg b \cdot c \leq \neg a.$$

Lemma 3.2. *Let $\boldsymbol{A} = \langle A; \cdot, \to, \leq \rangle$ be a residuated po-semigroup and $\neg$ a self-inverting unary operation on A. Consider the following conditions*:

(i) *$\neg$ is an involution of $\langle A; \leq \rangle$ and $a \cdot b = \neg(a \to \neg b)$ for all $a, b \in A$*;
(ii) *$a \to \neg b = b \to \neg a$ for all $a, b \in A$*;
(iii) *$\neg$ is compatible with $\cdot$.*

In general, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). *If $\langle A; \cdot, \to, \leq \rangle$ satisfies $x \cdot |\, x \,| \approx x$ — in particular, if $\langle A; \cdot, \to, \leq \rangle$ is* implicative *— then all three conditions are equivalent.*

A proof of this lemma can be found for instance in [22]. Note that the equation in (i) could have been replaced by

$$a \to b \;=\; \neg\,(a \cdot \neg\, b) \;\text{ for all } a, b \in A. \tag{7}$$

Indeed, each of these two equations is a substitution instance of the other, modulo the self-inversion law.

Definition 3.3. A structure $\langle A; \cdot, \to, \neg, \leq\rangle$ will be called a *c-involutive residuated po-semigroup* if $\langle A; \cdot, \to, \leq\rangle$ is a residuated po-semigroup and $\neg$ is a *compatible* involution of $\langle A; \cdot, \leq\rangle$.

c-Involutive residuated po-monoids are defined analogously.

Note. In many texts, c–involutive structures are simply called 'involutive'.

It follows that every c-involutive *implicative* po-semigroup—and in particular every c-involutive residuated po-*monoid*—has all of the properties of $\neg$ listed in Lemma 3.2. Such a structure is definitionally equivalent to its $\{\cdot, \neg, \leq\}$–reduct, by (7). And for each element a, we have $|\,\neg\, a\,| = |\,a\,|$, by part (ii) of Lemma 3.2 and the self-inversion law. Therefore,

$$\neg\, a \;\textit{ is designated } \;\textit{iff }\; |\,a\,| \leq \neg\, a \;\;\textit{iff }\; a \leq \neg\,|\,a\,|.$$

Notation. In a c-involutive residuated po-monoid, we define $\mathsf{f} \,=\, \neg\,\mathsf{t}$.

Then, for any element a, we have $\neg\, a \,=\, a \to \mathsf{f}$ and $(a \to \mathsf{f}) \to \mathsf{f} \,=\, a$. Indeed, $a \to \mathsf{f} \,=\, \neg\,(a \cdot \neg\,\mathsf{f}) \,=\, \neg\,(a \cdot \mathsf{t}) \,=\, \neg\, a$, whence the second equation is just the self-inversion law. Up to definitional equivalence, the c-involutive residuated po-monoids are just the residuated pomonoids with a distinguished element f satisfying $(x \to \mathsf{f}) \to \mathsf{f} \,\approx\, x$. Any structure of the latter kind is implicative, so once we define $\neg\, x \,=\, x \to \mathsf{f}$, the identity $x \to \neg\, y \approx y \to \neg\, x$ follows from Proposition 2.2(vi). Then c–involutivity follows, by Lemma 3.2.

We can now prove the embedding theorem labeled (i) in the Introduction.

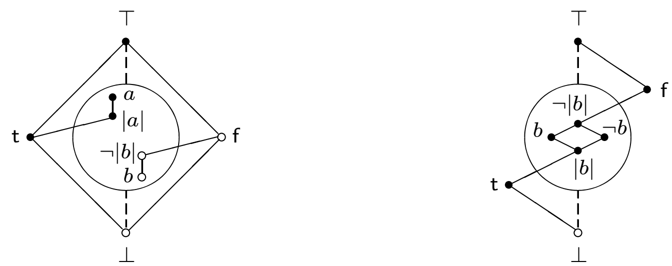
Theorem 3.4. *Every c-involutive implicative po-semigroup can be embedded into a c-involutive residuated po-monoid.*

If the original structure is square increasing then the containing structure can be chosen square increasing also.

Proof. Suppose $\boldsymbol{A} = \langle A; \cdot, \to, \neg, \leq'\rangle$ is a c-involutive implicative po-semigroup. Let $\bot, \top, \mathsf{f}$ and t be four distinct non-elements of A and let

$$B \;=\; A \cup \{\bot, \top, \mathsf{f}, \mathsf{t}\}.$$

We extend $\leq'$ to a partial order $\leq$ of B, as follows. It is easy to see that there is just one binary relation $\leq$ on B having the following itemized properties for all $a \in A$ (where of course $x < y$ means '$x \leq y$ and $x \neq y$').



Hasse diagram of $\langle B; \leq \rangle$. The circle represents $\langle A; \leq' \rangle$.
There are two possibilities: $\mathsf{t} \not\leq \mathsf{f}$ (left) and $\mathsf{t} < \mathsf{f}$ (right).
Darkened vertices indicate designated elements.

- $\leq$ is reflexive and anti-symmetric on B ;
- the restriction of $\leq$ to A is $\leq'$;
- $\bot \;<\; a, \mathsf{f}, \mathsf{t}$ and $a, \mathsf{f}, \mathsf{t} \;<\; \top$ and $\bot \;<\; \top$;
- $\mathsf{f} \not\leq \mathsf{t}$ and $\mathsf{f} \not\leq a$ and $a \not\leq \mathsf{t}$;
- $\mathsf{t} < a$ iff a is designated ;
- $a < \mathsf{f}$ iff $\neg\, a$ is designated (i.e., iff $a \leq' \neg\, |\, a\, |$) ;
- $\mathsf{t} < \mathsf{f}$ iff $\exists\, b \in A$ such that $b, \neg\, b$ are both designated.

(See the accompanying Hasse diagram.) We need to satisfy ourselves that $\leq$ is transitive, and therefore a partial order.

Condition (5) of Proposition 2.4 says that the designated elements of $\boldsymbol{A}$ form an upward closed set, so if $\mathsf{t} < a \in A$ and $a \leq' b \in A$ then $\mathsf{t} < b$. By a dual argument (using involution properties), if $a, b \in A$ with $a \leq' b$ and $b < \mathsf{f}$ then $a < \mathsf{f}$. The only conceivable problem is this: if there is no $b \in A$ for which $b, \neg\, b$ are both designated, then $\mathsf{t} \not< \mathsf{f}$, by the seventh postulate. In this case, might f nevertheless dominate t in the transitive closure of $\leq$? Since $\leq'$ (and therefore $\leq$) is transitive on A, this can happen *only* when there are elements $a, b \in A$ with $a \leq' b$, where a and $\neg\, b$ are both designated in $\boldsymbol{A}$. But since $\boldsymbol{A}$'s set of designated elements is upward closed, b must be designated in this case, so b and $\neg\, b$ are both designated, a contradiction. Thus, $\leq$ is indeed a partial order, and we may drop all reference to $\leq'$ without fear of confusion.

We extend $\neg$ to a unary operation on B by defining

$$\neg\,\top = \bot, \quad \neg\,\bot = \top, \quad \neg\,\mathsf{t} = \mathsf{f}, \quad \text{and} \quad \neg\,\mathsf{f} = \mathsf{t}.$$

Clearly, $\neg$ is an involution of $\langle B; \leq \rangle$. We extend $\cdot$ to a commutative binary operation on B by defining, for all $b \in B$,

$$\begin{array}{ll} \bot \cdot b \;=\; \bot \;=\; b \cdot \bot\,; & \qquad \top \cdot b \;=\; \top \;=\; b \cdot \top \;\text{ whenever }\; b \neq \bot\,; \\ \mathsf{t} \cdot b \;=\; b \;=\; b \cdot \mathsf{t}\,; & \qquad \mathsf{f} \cdot b \;=\; \top \;=\; b \cdot \mathsf{f} \;\text{ whenever }\; b \notin \{\bot, \mathsf{t}\}\,. \end{array}$$

Since $\boldsymbol{A}$ was a po-semigroup, it is easy to see that $\cdot$ is associative on B and that $\langle B; \cdot, \mathsf{t}, \leq \rangle$ is a po-monoid. Note that $x \cdot x = x$ for $x \in \{\bot, \top, \mathsf{t}\}$, while $\mathsf{f} \cdot \mathsf{f} = \top$. Therefore, $\langle B; \cdot, \leq \rangle$ is square increasing if $\boldsymbol{A}$ is. Now we define

$$a \to b \;=\; \neg\,(a \cdot \neg\, b) \quad \text{for all } \; a, b \in B.$$

Thus, $\to$ extends the residuation of $\boldsymbol{A}$. For all $b \in B$, we have $\mathsf{t} \to b = b$, because $\neg\,(\mathsf{t} \cdot \neg\, b) \;=\; \neg\,\neg\, b$, while $b \to \mathsf{f} = \neg\, b$, just as in the remarks preceding the statement of this theorem. To show that $\langle B; \cdot, \to, \mathsf{t}, \leq \rangle$ is a residuated po-monoid, we first verify that for all $x, y \in B$,

$$(\dagger) \qquad\qquad x \leq y \quad \text{iff} \quad \mathsf{t} \leq x \to y.$$

Let $x, y \in B$. If $x, y \in A$ then $(\dagger)$ follows from Proposition 2.4(4) and Theorem 2.6, so we may assume that x, y are not both in A. Since $\bot \to y = \neg\,(\bot \cdot \neg\, y) \;=\; \neg\,\bot \;=\; \top$ for all $y \in B$, $(\dagger)$ holds when $x = \bot$. Also, since $\mathsf{t} \to y = y$ for all $y \in B$, $(\dagger)$ holds when $x = \mathsf{t}$. So we may assume that $x \notin \{\bot, \mathsf{t}\}$.

Consequently, $x \leq \mathsf{f}$ iff $\neg\, x$ is t or a designated element of $\boldsymbol{A}$, iff $\mathsf{t} \leq \neg\, x = x \to \mathsf{f}$. This shows that $(\dagger)$ holds when $y = \mathsf{f}$. Further, since $x \to \top = \neg\,(x \cdot \neg\,\top) \;=\; \neg\,(x \cdot \bot) \;=\; \neg\,\bot = \top$, $(\dagger)$ holds when $y = \top$. So we may assume that $y \notin \{\mathsf{f}, \top\}$.

Since $x \neq \bot$, we have $x \to \bot \;=\; \neg\,(x \cdot \neg\,\bot) \;=\; \neg\,(x \cdot \top) \;=\; \neg\,\top = \bot$, so both sides of $(\dagger)$ are false for $y = \bot$. And since $y \neq \top$, i.e., $\neg\, y \neq \bot$, we have $\top \to y \;=\; \neg\,(\top \cdot \neg\, y) \;=\; \neg\,\top = \bot$, so both sides of $(\dagger)$ are false for $x = \top$. We may therefore assume that $y \neq \bot$ and $x \neq \top$.

Since $x \notin \{\bot, \mathsf{t}\}$, we have $x \to \mathsf{t} \;=\; \neg\,(x \cdot \neg\,\mathsf{t}) \;=\; \neg\,(x \cdot \mathsf{f}) \;=\; \neg\,\top = \bot$, so both sides of $(\dagger)$ are false for $y = \mathsf{t}$. We may now assume that $y \neq \mathsf{t}$, which means that $y \in A$. Then $x \notin A$, which forces $x = \mathsf{f}$. Note that $\neg\, y \in A$, so $\mathsf{f} \cdot \neg\, y = \top$. Thus, $\mathsf{f} \to y \;=\; \neg\,(\mathsf{f} \cdot \neg\, y) \;=\; \neg\,\top = \bot$, so both sides of $(\dagger)$ are false for $x = \mathsf{f}$. This completes the proof of $(\dagger)$. Now we claim that

$$(\dagger\dagger) \qquad\qquad (x \cdot z) \to y \;=\; z \to (x \to y) \;\; \text{for all } \; x, y, z \in B.$$

Indeed, let $x, y, z \in B$. By the commutativity and associativity of $\cdot$ in B,

$$\begin{aligned}
(x \cdot z) \to y \;&=\; \neg\,((x \cdot z) \cdot \neg\, y) \;&&=\; \neg\,((z \cdot x) \cdot \neg\, y) \\
&=\; \neg\,(z \cdot (x \cdot \neg\, y)) \;&&=\; \neg\,(z \cdot \neg\,\neg\,(x \cdot \neg\, y)) \\
&=\; \neg\,(z \cdot \neg\,(x \to y)) \;&&=\; z \to (x \to y).
\end{aligned}$$

Then, by $(\dagger)$ and $(\dagger\dagger)$,

$$x \cdot z \;\leq\; y \quad \text{iff} \quad \mathsf{t} \;\leq\; (x \cdot z) \to y \;=\; z \to (x \to y) \quad \text{iff} \quad z \;\leq\; x \to y.$$

Thus, $\langle B; \cdot, \to, \mathsf{t}, \leq\rangle$ is a residuated po-monoid. By design, $\neg$ is a compatible involution and the inclusion map is an embedding. □

4. Lattice-Ordered Residuated Structures

A (commutative) *lattice-ordered semigroup*, or briefly a *lo-semigroup*, is an algebra $\langle A; \cdot, \wedge, \vee\rangle$ such that $\langle A; \wedge, \vee\rangle$ is a lattice, $\langle A; \cdot\rangle$ is a commutative semigroup, and

$$a \cdot (b \vee c) \;=\; (a \cdot b) \vee (a \cdot c)$$

for all $a, b, c \in A$. In this case, clearly, $\langle A; \cdot, \leq\rangle$ is a po-semigroup, where $\leq$ denotes the lattice order.

(Commutative) *lo-monoids* $\langle A; \cdot, \wedge, \vee, \mathsf{t}\rangle$ are defined analogously.

Definition 4.1. An algebra $\langle A; \cdot, \to, \wedge, \vee\rangle$ will be called a *residuated* (commutative) *lo-semigroup* if $\langle A; \wedge, \vee\rangle$ is a lattice and $\langle A; \cdot, \to, \leq\rangle$ is a residuated po-semigroup, where $\leq$ denotes the lattice order. In this case, it is easily shown that $\langle A; \cdot, \wedge, \vee\rangle$ is indeed a lo-semigroup.

Residuated (commutative) *lo-monoids* $\langle A; \cdot, \to, \wedge, \vee, \mathsf{t}\rangle$ are defined in an analogous way. (In the literature they are often called 'commutative residuated lattices'; see [21].)

These structures will be called *distributive*, *square increasing*, or *implicative* (respectively) if their lattice reducts, their po-semigroup reducts, or their residuated po-semigroup reducts have the indicated property.

Theorem 4.2. *The following conditions on a residuated lo-semigroup* $\boldsymbol{A}$ *are equivalent*:

(i) $\boldsymbol{A}$ *may be embedded into a residuated lo-monoid*;
(ii) $\boldsymbol{A}$ *satisfies* $(|\,x\,| \wedge |\,y\,|) \to z \;\leq\; z$;
(iii) $\boldsymbol{A}$ *is implicative and* $|\,x\,| \wedge |\,y\,|$ *is designated for all elements* x, y *of* $\boldsymbol{A}$.

Proof. The equivalence of (i) and (ii) follows from axiomatizations in [41, Sec. 7]. Assuming (ii) and equating the variables x, y, we can infer that $\boldsymbol{A}$ is implicative. Also, if we substitute $|\,x\,| \wedge |\,y\,|$ for z in (ii), we get the remaining statement of (iii). Conversely, given (iii), if $||\,x\,| \wedge |\,y\,|| \;\leq\; |\,x\,| \wedge |\,y\,|$ then $(|\,x\,| \wedge |\,y\,|) \to z \;\leq\; ||\,x\,| \wedge |\,y\,|| \to z \;\leq\; z$, by Proposition 2.2(iii) and implicativity. □

Condition (ii) was isolated in [17], in a narrower context.

With every residuated lo-monoid $\boldsymbol{A} = \langle A; \cdot, \to, \wedge, \vee, \mathsf{t}\rangle$ we can associate a residuated po-monoid $\boldsymbol{A}_{\mathrm{ord}} \;=\; \langle A; \cdot, \to, \mathsf{t}, \leq\rangle$, where $\leq$ is the order induced by the lattice operations. When we say that a residuated po-monoid can be *embedded* into $\boldsymbol{A}$, we mean that it can be embedded into $\boldsymbol{A}_{\mathrm{ord}}$. Similarly for other lattice-based structures.

The following result was proved by Meyer in [26, Sec. III].

Theorem 4.3. *Every residuated po-monoid may be embedded into a residuated distributive lo-monoid by a construction that preserves the square increasing law.*

It follows that every implicative po-semigroup may be embedded into a residuated distributive lo-monoid, with preservation of the square increasing law. (See Theorems 2.6 and 2.8.) But the proofs of these theorems do not make provision for the existence and preservation of involution operators.

Definition 4.4. An algebra $\langle A; \cdot, \rightarrow, \wedge, \vee, \neg\rangle$ will be called a *c-involutive residuated lo-semigroup* if $\langle A; \cdot, \rightarrow, \wedge, \vee\rangle$ is a residuated lo-semigroup and $\neg$ is a compatible involution of its residuated po-semigroup reduct.

c-Involutive residuated lo-monoids are defined analogously.

If $\neg$ is an involution of a lo-semigroup, it is easy to see that *de Morgan's laws* $\neg\,(x \wedge y) \approx \neg\, x \vee \neg\, y$ and $\neg\,(x \vee y) \approx \neg\, x \wedge \neg\, y$ hold universally.

The next theorem is similar to a result of Meyer [27], which dealt with semigroup-based structures. In its present form, it is a specialization of [19, Thm. 9.1(ii),(iii)], where the proof can be found.

Theorem 4.5. *Every residuated lo-monoid can be embedded into a c-involutive residuated lo-monoid by a construction that preserves both distributivity and the square increasing law.*

For any algebra $\boldsymbol{A} = \langle A; \langle o^{\boldsymbol{A}} : o \in O\rangle\rangle$ and any $S \subseteq O$, the *S–subreducts* of $\boldsymbol{A}$ are just the subalgebras of the S–reduct $\langle A; \langle o^{\boldsymbol{A}} : o \in S\rangle\rangle$ of $\boldsymbol{A}$.

The following result gives algebraic form to the technique of 't–surrogates', which was invented by Anderson and Belnap in their analysis of entailment and relevance logic (see [1], [2, p. 343]).

Theorem 4.6. *Let $\boldsymbol{A}$ be a finitely generated subreduct of a c-involutive residuated lo-monoid $\boldsymbol{C}$, and assume that the signature of $\boldsymbol{A}$ includes $\rightarrow$ and $\wedge$ but* excludes t. *Let $G = \{g_1, \ldots, g_n\}$ be a finite generating set for $\boldsymbol{A}$ and define $\mathsf{t}_G = |\,g_1| \wedge \ldots \wedge |\,g_n|$.*

Then $\mathsf{t}_G \leq |\,a\,|$ and $\mathsf{t}_G \rightarrow a = a$ for all $a \in A$, and t_G is independent of the choice of G. If the signature of $\boldsymbol{A}$ includes $\cdot$ as well, then $a \cdot \mathsf{t}_G = a$ for all $a \in A$.

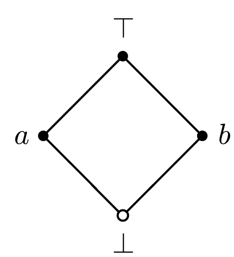
The reference to $\leq$ is unambiguous, since A is closed under $\wedge^{\boldsymbol{C}}$. Note that $\mathsf{t}_G \in A$ and $\mathsf{t}^{\boldsymbol{C}} \leq \mathsf{t}_G$, by Proposition 2.4(1) (applied in $\boldsymbol{C}$). It can happen that

$\mathsf{t}_G > \mathsf{t}^{\boldsymbol{C}} \notin A$, even when $\cdot$ is present. A detailed proof of Theorem 4.6 can be found in [31], except for the claim that $\mathsf{t}_G \to a = a$. From $\mathsf{t}_G \leq |\,a\,|$ and Proposition 2.2(iv) (applied in $\boldsymbol{C}$), we get $a \leq \mathsf{t}_G \to a$. Also, $\mathsf{t}_G \leq (\mathsf{t}_G \to a) \to a$, by a similar application of Proposition 2.2(v), so $\mathsf{t}^{\boldsymbol{C}} \leq (\mathsf{t}_G \to a) \to a$. Then $\mathsf{t}_G \to a \leq a$, by Proposition 2.4(4) (again applied in $\boldsymbol{C}$).

Remark 4.7. If we omit the term 'c-involutive' from the statement of Theorem 4.6, the result is still true, in view of Theorem 4.5.

On its own, implicativity does not entail the equivalent conditions of Theorem 4.2. This point was made in [17] and it is borne out by the next example.

Example 4.8. Let $\langle A; \wedge, \vee\rangle$ be the lattice depicted below.



Let $\cdot$ be the idempotent commutative binary operation on A such that $\bot \cdot x = \bot$ for all $x \in A$ and $x \cdot y = \top$ for distinct $x, y \in A \setminus \{\bot\}$. This is an associative operation and $\langle A; \cdot, \leq\rangle$ is a po-semigroup. It is easy to see that it is residuated. If we set $\neg\bot = \top$, $\neg\top = \bot$, $\neg a = a$ and $\neg b = b$ then the residuation is definable by $x \to y = \neg(x \cdot \neg y)$, hence $\neg$ is a compatible involution, by Lemma 3.2. Observe that $|\bot| = |\top| = \top$ and $|\,a\,| = a$ and $|\,b\,| = b$, so $\boldsymbol{A} = \langle A; \cdot, \to, \wedge, \vee, \neg\rangle$ is implicative. But $|\,a\,| \wedge |\,b\,| = a \wedge b = \bot$ is not designated; equivalently, $(|\,a\,| \wedge |\,b\,|) \to \bot = \top \not\leq \bot$. Consequently, although $\langle A; \cdot, \to, \neg, \leq\rangle$ can be embedded into a c-involutive residuated po-monoid (Theorem 3.4), no such embedding can preserve $\wedge$ and $\vee$. Even if we ignore the involution, the $\neg$–free reduct of $\boldsymbol{A}$ cannot be embedded into a residuated lo-monoid, by Theorem 4.2.

This motivates the theorem labeled (ii) in the Introduction, which we can now prove.

Theorem 4.9. *Let $\boldsymbol{A}$ be a residuated lo-semigroup or a c-involutive residuated lo-semigroup. If $\boldsymbol{A}$ satisfies the identity $(|\,x\,| \wedge |\,y\,|) \to z \leq z$ then $\boldsymbol{A}$ can be embedded into a c-involutive residuated lo-monoid.*

If, moreover, $\boldsymbol{A}$ is distributive or square increasing or both then the containing algebra can be chosen to have the same attributes.

Proof. We deal first with the case where $\boldsymbol{A}$ is c-involutive. Let $\boldsymbol{A}$ satisfy the given inequality.

First suppose $\boldsymbol{A}$ is finitely generated and let H be a finite generating set for $\boldsymbol{A}$. Then $G = H \cup \{\neg\, h : h \in H\}$ is a generating set for the $\{\cdot, \to, \wedge, \vee\}$–reduct $\boldsymbol{A}^+$ of $\boldsymbol{A}$. This follows from de Morgan's laws and the identities $\neg\,(x \cdot y) \approx x \to \neg\, y$ and $\neg\,(x \to y) \approx x \cdot \neg\, y$, which jointly show that all expressions built up from H using $\cdot, \to, \wedge, \vee$ and $\neg$ can be built up from G without using $\neg$. So $\boldsymbol{A}^+$ is also finitely generated.

Let $\mathsf{t}_G \in A$ be defined as in Theorem 4.6. By assumption, $\boldsymbol{A}^+$ satisfies $(|\, x\,| \wedge |\, y\,|) \to z \leq z$, so Theorem 4.2 tells us that $\boldsymbol{A}^+$ is a subreduct of a residuated lo-*monoid*. Now Remark 4.7 permits us to apply Theorem 4.6 to $\boldsymbol{A}^+$, so t_G is an identity for $\cdot$ on A, i.e., $\boldsymbol{A}^+$ is already the $\{\cdot, \to, \wedge, \vee\}$–*reduct* of a residuated lo-monoid. And since no new elements need be added to $\boldsymbol{A}^+$ to get a monoid, $\boldsymbol{A}$ itself is the $\{\cdot, \to, \wedge, \vee, \neg\}$–*reduct* of a *c-involutive* residuated lo-monoid $\langle A; \cdot, \to, \wedge, \vee, \neg, \mathsf{t}_G\rangle$. So the result holds for finitely generated algebras $\boldsymbol{A}$ (the preservation claims being trivial for this construction).

Next, suppose $\boldsymbol{A}$ is not finitely generated. The class of c-involutive residuated lo-semigroups satisfying $(|\, x\,| \wedge |\, y\,|) \to z \leq z$ is closed under ultraproducts and subalgebras, since it is defined by universal first order sentences. Now $\boldsymbol{A}$, like every algebra, may be embedded into an ultraproduct of finitely generated subalgebras of itself: see for instance [11, Thm. V.2.14]. We have already shown that each finitely generated subalgebra $\boldsymbol{A}_i$ of $\boldsymbol{A}$ is the reduct of a c-involutive residuated lo-monoid $\boldsymbol{A}_i^{\mathsf{t}}$, so $\boldsymbol{A}$ is a $\{\cdot, \to, \wedge, \vee, \neg\}$–subreduct of the corresponding ultraproduct $\boldsymbol{B}$ of the algebras $\boldsymbol{A}_i^{\mathsf{t}}$, and this ultraproduct is itself a c-involutive residuated lo-monoid. Distributivity and the square increasing law are preserved in the formation of subalgebras, expansions by constants, and ultraproducts, since they too are universal first order sentences. So these properties (and their conjunction) persist in the passage from $\boldsymbol{A}$ to $\boldsymbol{B}$.

Finally, suppose $\boldsymbol{A}$ is merely a residuated lo-semigroup (without $\neg$) satisfying the given inequality. Carrying out the above argument but omitting all reference to G and working with H instead, we get a simpler proof that $\boldsymbol{A}$ may be embedded into a residuated lo-monoid (without $\neg$) in such a way that distributivity and the square increasing law are preserved. (We can't simply use Theorem 4.2 here, because its proof in [41] does not establish preservation of distributivity.) Then applying Theorem 4.5, we deduce the result. $\square$

Definition 4.10. A *relevant algebra* is a square increasing c-involutive residuated distributive lo-semigroup satisfying $(|\, x\,| \wedge |\, y\,|) \to z \leq z$.

A *de Morgan monoid* is a square increasing c-involutive distributive residuated lo-monoid. (These names come from [17] and [2], respectively.)

As an instance of Theorem 4.9, we have:

Corollary 4.11. *Every relevant algebra may be embedded into a de Morgan monoid.*

5. Adding Distributive Lattice Operations

Our aim in the present section is to prove Claim (iii) from the Introduction:

Theorem 5.1. *Every c-involutive residuated po-monoid may be embedded into a c-involutive residuated* distributive lo*-monoid (which may be chosen complete).*

As we noted earlier, the proof of Theorem 4.3 is not helpful here; it can't be adapted to cater for an involution.

Definition 5.2. By an *involutive* $\to$ *po-monoid* we shall mean a structure $\boldsymbol{A} = \langle A; \cdot, \to, \neg, \mathsf{t}, \leq \rangle$ such that $\langle A; \cdot, \mathsf{t}, \leq \rangle$ is a po-monoid, $\neg$ is an involution of $\langle A; \leq \rangle$ (*not* necessarily a compatible involution), and $\to$ is an arbitrary binary operation on A.

If in addition, $\langle A; \cdot, \to, \mathsf{t}, \leq \rangle$ is a residuated po-monoid, we call $\boldsymbol{A}$ an *involutive residuated po-monoid.*

Here we continue to take commutativity of $\cdot$ for granted. The first part of the definition looks rather strange, in that $\to$ is essentially redundant, but by recognizing this category of objects we will be able to manage our notational conventions more efficiently. We repeat our caution to the reader that in the context of residuated po-monoids, the term 'involutive' is used in many texts—including some cited here—to mean 'c-involutive'.

For the remainder of this section, let $\boldsymbol{A} = \langle A; \cdot, \to, \neg, \mathsf{t}, \leq \rangle$ be an involutive $\to$ po-monoid. Let $U(\boldsymbol{A})$ be the set of all upward closed subsets of $\boldsymbol{A}$. Thus, $\emptyset$ and A itself belong to $U(\boldsymbol{A})$, and so do all sets of the form

$$[\,x\,) := \{a \in A : a \geq x\} \quad (x \in A).$$

The smallest upward closed set containing a subset S of A will be denoted by $[\,S\,)$. Thus, $[\,S\,) = \bigcup_{x \in S} [\,x\,) = \{a \in A : a \geq x \text{ for some } x \in S\}$. For $X, Y \subseteq A$ and $a \in A$, we define

$$X * Y := \{x \cdot y : x \in X \text{ and } y \in Y\}.$$

For $X, Y \in U(\boldsymbol{A})$, we define:

$$\begin{aligned} X \cdot Y &:= [\,X * Y\,) = \{a \in A : a \geq x \cdot y \text{ for some } x \in X \text{ and some } y \in Y\} \\ X \to Y &:= \{a \in A : X * \{a\} \subseteq Y\} \\ \neg\, X &:= \{a \in A : \neg\, a \notin X\} \end{aligned}$$

Lemma 5.3. *$U(\boldsymbol{A})$ is closed under $\cdot$, $\to$ and $\neg$.*

This follows easily from the definitions. Using the compatibility of $\leq$ with $\cdot$, we see that whenever $x \in A$ and $Z \in U(\boldsymbol{A})$, then

(8) $$[\,x\,) \cdot Z \;=\; [\,\{x\} * Z\,).$$

Now we define

$$\boldsymbol{U}(\boldsymbol{A}) := \langle\, U(\boldsymbol{A}); \cdot, \rightarrow, \neg, [\,\mathsf{t}\,), \subseteq \rangle.$$

With respect to present assumptions and notational conventions, we have:

Lemma 5.4. $\boldsymbol{U}(\boldsymbol{A})$ *is an involutive residuated po-monoid whose order* $\subseteq$ *is a distributive complete lattice order, with* $\cap$ *and* $\cup$ *as binary meet and join, and for all* $x, y \in A$,

$$[\,x\,) \cdot [\,y\,) \;=\; [\,x \cdot y\,) \quad \textit{and} \quad [\,x\,) \rightarrow [\,y\,) \;=\; \{z \in A : x \cdot z \geq y\}.$$

Note. $\neg\,[\,x\,) \neq [\,\neg\,x\,)$ and, usually, $[\,x\,) \rightarrow [\,y\,) \neq [\,x \rightarrow y\,)$.

The proof of Lemma 5.4 is straightforward. The order-theoretic claims are justified because upward closure persists under arbitrary intersections and unions, and of course $\cap$ distributes over $\cup$.

It follows that we may repeat the construction on $\boldsymbol{U}(\boldsymbol{A})$. Thus we define $U^2(\boldsymbol{A}) := U(\boldsymbol{U}(\boldsymbol{A}))$ and

$$\boldsymbol{U}^2(\boldsymbol{A}) := \boldsymbol{U}(\boldsymbol{U}(\boldsymbol{A})) \;=\; \langle\, U^2(\boldsymbol{A}); \cdot, \rightarrow, \neg, [[\,\mathsf{t}\,)), \subseteq \rangle$$

and we conclude from Lemma 5.4 that $\boldsymbol{U}^2(\boldsymbol{A})$ is also an involutive residuated po-monoid, whose order $\subseteq$ is a distributive complete lattice order, with $\cap$ and $\cup$ as lattice operations. If $x \in A$ and $X \in U(\boldsymbol{A})$ then

(9) $$x \in X \quad \text{iff} \quad [\,x\,) \subseteq X \quad \text{iff} \quad X \in [[\,x\,)).$$

Lemma 5.5. *If* $x, y \in A$ *then, in* $\boldsymbol{U}^2(\boldsymbol{A})$,

$$[[\,x\,)) \rightarrow [[\,y\,)) \;=\; \{Z \in U(\boldsymbol{A}) : x \cdot z \leq y \textit{ for some } z \in Z\}.$$

Proof. Let $Z \in U(\boldsymbol{A})$. By Lemma 5.4, $Z \in [[\,x\,)) \rightarrow [[\,y\,))$ iff $[\,y\,) \subseteq [\,x\,) \cdot Z = [\,\{x\} * Z\,)$ (by (8)), iff $y \in [\,\{x\} * Z\,)$ (because $[\,\{x\} * Z\,)$ is upward closed) iff $y \geq x \cdot z$ for some $z \in Z$. □

We define a function $h_{\boldsymbol{A}}$ from A into $U^2(\boldsymbol{A})$ by:

$$h_{\boldsymbol{A}} \,:\, x \mapsto [[\,x\,)) \;=\; \{X \in U(\boldsymbol{A}) : x \in X\}\,,$$

where the equality follows from (9). We have noted that the function $x \mapsto [\,x\,)$ from A to $U(\boldsymbol{A})$ fails to preserve $\neg$ and $\rightarrow$. Nevertheless, it is easy to prove

Lemma 5.6. $h_{\boldsymbol{A}}$ *is an order preserving and order reflecting homomorphism between the monoid reducts of* $\boldsymbol{A}$ *and* $\boldsymbol{U}^2(\boldsymbol{A})$ *which preserves* $\neg$.

Proof. The claims about order are obvious, while two applications of Lemma 5.4 show that $\cdot$ is preserved. For any $x \in A$ and $X \in U(\boldsymbol{A})$, we have $X \in \neg\, [[\, x\,))$ iff $\neg\, X \notin [[\, x\,))$ iff $x \notin \neg\, X$ (by (9)) iff $\neg\, x \in X$ iff $X \in [[\, \neg\, x\,))$ (by (9)). So $\neg\, [[\, x\,)) = [[\, \neg\, x\,))$. ☐

Now we strengthen the requirements on $\boldsymbol{A}$, requiring first that it be an involutive *residuated* po-monoid, not merely an involutive $\to$ po-monoid.

Lemma 5.7. *For any involutive residuated po-monoid* $\boldsymbol{A}$, *the function* $h_{\boldsymbol{A}}$ *preserves* $\to$ *and is therefore an embedding* $\boldsymbol{A} \to \boldsymbol{U}^2(\boldsymbol{A})$ *of involutive residuated po-monoids.*

Proof. Let $x, y \in A$. We need to show that $[[\, x\,)) \to [[\, y\,)) = [[\, x \to y\,))$. Let $Z \in U(\boldsymbol{A})$. By Lemma 5.5, we have $Z \in [[\, x\,)) \to [[\, y\,))$ iff $x \cdot z \leq y$ for some $z \in Z$, iff $z \leq x \to y$ for some $z \in Z$, iff $x \to y \in Z$ (since Z is upward closed), iff $Z \in [[\, x \to y\,))$ (by (9)). ☐

Finally, we show that when $\boldsymbol{A}$ is a c-involutive residuated po-monoid, as opposed to an involutive one, then $\boldsymbol{U}^2(\boldsymbol{A})$ is also c-involutive.

Theorem 5.8. *For any c-involutive residuated po-monoid* $\boldsymbol{A}$, *the structure* $\boldsymbol{U}^2(\boldsymbol{A})$ *is also a c-involutive residuated po-monoid.*

Proof. Recall that we define $\mathsf{f} = \neg\, \mathsf{t}$ in $\boldsymbol{A}$. Since $\boldsymbol{A}$ is c-involutive, $\neg\, x = x \to \mathsf{f}$ for all $x \in A$. Similarly, to show that $\boldsymbol{U}^2(\boldsymbol{A})$ is c-involutive, it suffices to show that $\neg\, C = C \to [[\, \mathsf{f}\,))$ for all $C \in U^2(\boldsymbol{A})$ (see the remarks preceding Theorem 3.4). So let $C \in U^2(\boldsymbol{A})$ and $Z \in U(\boldsymbol{A})$. Now

$$
\begin{array}{rll}
Z \in \neg\, C & \text{iff} & \neg\, Z \notin C \\
& \text{iff} & \forall V \in C,\ V \not\subseteq \neg\, Z \qquad \text{(since } C \text{ is upward closed)} \\
& \text{iff} & \forall V \in C,\ \exists v \in V \text{ such that } v \notin \neg\, Z \\
& \text{iff} & \forall V \in C,\ \exists v \in V \text{ such that } \neg\, v \in Z \\
& \text{iff} & \forall V \in C,\ \exists v \in V \text{ such that } v \to \mathsf{f} \in Z \\
& \text{iff} & \forall V \in C,\ \exists v \in V,\ \exists z \in Z \text{ such that } z \leq v \to \mathsf{f} \\
& & \qquad\qquad\qquad \text{(since } Z \text{ is upward closed)} \\
& \text{iff} & \forall V \in C,\ \exists v \in V,\ \exists z \in Z \text{ such that } v \cdot z \leq \mathsf{f} \\
& \text{iff} & \forall V \in C,\ \mathsf{f} \in [\, V * Z\,) = V \cdot Z \\
& \text{iff} & \forall V \in C,\ V \cdot Z \in [[\, \mathsf{f}\,)) \quad \text{(by (9))} \\
& \text{iff} & C * \{Z\} \subseteq [[\, \mathsf{f}\,)) \\
& \text{iff} & Z \in C \to [[\, \mathsf{f}\,)).
\end{array}
$$

This shows that $\neg\, C = C \to [[\, \mathsf{f}\,))$, as required. ☐

Now the algebra $\langle U^2(\boldsymbol{A}); \cdot, \to, \cap, \cup, \neg, [\,[\,\mathsf{t}\,))\rangle$ is a c-involutive residuated distributive complete lo-monoid, and Theorem 5.1 follows from Theorem 5.8 and Lemma 5.7.

Definition 5.9. A po-monoid is said to be *integral* if its identity t is its greatest element.

The construction for Theorem 5.1 does not preserve integrality, i.e., when $\boldsymbol{A}$ is integral, $\boldsymbol{U}^2(\boldsymbol{A})$ need not be. Indeed, if t is the greatest element of $\boldsymbol{A}$ then $h_{\boldsymbol{A}}(\mathsf{t})$ is the set of all *non-empty* elements of $U(\boldsymbol{A})$. In this case, there is one (and only one) element of $\langle U^2(\boldsymbol{A}); \subseteq\rangle$ *not* dominated by $h_{\boldsymbol{A}}(\mathsf{t})$, namely $U(\boldsymbol{A})$. On the other hand, one may check that $U^2(\boldsymbol{A}) \setminus \{\emptyset, U(\boldsymbol{A})\}$ is closed under all of the operations $\cdot, \to, \cap, \cup, \neg$, and the integral algebra on this subuniverse may take the place of $\boldsymbol{U}^2(\boldsymbol{A})$ in the theorem. Note that we do *not* also throw out the extreme elements of $U(\boldsymbol{A})$ in the amended construction. They are needed to ensure that the extrema of $U^2(\boldsymbol{A})$ are indeed superfluous. The integral algebra is still a complete lattice, as the top element of $U^2(\boldsymbol{A})$ was completely join-irreducible in $\boldsymbol{U}^2(\boldsymbol{A})$, while the involution ensures dual behaviour at the bottom. Thus we have:

Corollary 5.10. *Every* integral *c-involutive residuated po-monoid may be embedded into an* integral *c-involutive residuated* distributive lo-*monoid (which can be chosen complete).*

The construction that establishes Theorem 5.1 also fails to preserve the square increasing law. In the next section we shall present a more complex construction which does preserve this law. But to prove that the construction works, we shall need to *use* the fact that the original po-monoid $\boldsymbol{A}$ is square increasing. So that argument cannot replace our present proof of Theorem 5.1.

The clause $\neg X := \{a \in A : \neg a \notin X\}$ defining involution in $\boldsymbol{U}(\boldsymbol{A})$ appeared originally in [7] in a related but different context. It is known as the 'Routley star', owing to its appearance in [38], where $\neg a$ is denoted as a^*.

6. De Morgan Monoids and the Square Increasing Law

Recall that a de Morgan monoid is a c-involutive *square increasing* residuated *distributive* lo-monoid. Our aim in this section is to prove Claim (iv) from the Introduction, i.e.:

Theorem 6.1. *Every c-involutive square increasing residuated po-monoid can be embedded into a de Morgan monoid (which can be chosen complete).*

Although no embedding theorem of this kind was claimed by Meyer and Routley in [30], the key ingredients of our proof are suggested by that paper.

For the rest of this section, let $\boldsymbol{A} = \langle A; \cdot, \to, \neg, \mathsf{t}, \leq \rangle$ be a c-involutive *square increasing* residuated po-monoid. We define

$$x + y := (\neg\, x) \to y\,.$$

It is well known and easy to verify that $+$ is associative, commutative and has identity f $(= \neg\, \mathsf{t})$.

Lemma 6.2. $\boldsymbol{A}$ *satisfies*

(10) $x + x \;\leq\; x$

(11) $x \;\leq\; y \;\Longrightarrow\; z + x \;\leq\; z + y$

(12) $\neg\,(x + y) \;\approx\; \neg\, x \cdot \neg\, y$ *and* $\neg\,(x \cdot y) \;\approx\; \neg\, x + \neg\, y$

(13) $x \cdot (y + z) \;\leq\; (x \cdot y) + z$

(14) $x \cdot y \leq w \;\&\; x' \cdot y' \leq w' \;\Longrightarrow\; x \cdot x' \cdot (y + y') \cdot z \cdot z' \;\leq\; (z \cdot w) + (z' \cdot w')$

(15) $x \cdot y \leq w \;\&\; x' \cdot y' \leq w \;\Longrightarrow\; x \cdot x' \cdot (y + y') \;\leq\; w.$

[Of these properties, only (10) and (15) rely on the square increasing law.]

Proof. Let $x, y, z \in A$. By the square increasing law and c-involution properties, $\neg\, x \leq \neg\, x \cdot \neg\, x \;=\; \neg\,(\neg\, x \to x) \;=\; \neg\,(x + x)$, so $x + x \leq x$, proving (10). (11) follows from Proposition 2.2(iii) and the definition of $+$, while (12) follows from Lemma 3.2 and the definitions.

For (13), we need to prove that $x \cdot (\neg\, y \to z) \;\leq\; \neg\,(x \cdot y) \to z$. This is equivalent to $\neg\,(x \cdot y) \cdot x \cdot (\neg\, y \to z) \;\leq\; z$, by definition of $\to$. Now

$$\begin{array}{rcll} \neg\,(x \cdot y) \cdot x \cdot (\neg\, y \to z) & = & (x \to \neg\, y) \cdot x \cdot (\neg\, y \to z) & \\ & = & x \cdot (x \to \neg\, y) \cdot (\neg\, y \to z) & \text{(by commutativity)} \\ & \leq & \neg\, y \cdot (\neg\, y \to z) \;\leq\; z & \text{(by definition of } \to\text{).} \end{array}$$

(14): Using (13) and the commutativity and associativity of $\cdot$ and $+$, we calculate that

$$\begin{array}{rl} & x \cdot x' \cdot (y + y') \;=\; x' \cdot x \cdot (y + y') \;\leq\; x' \cdot ((x \cdot y) + y') \\ = & x' \cdot (y' + (x \cdot y)) \;\leq\; (x' \cdot y') + (x \cdot y) \;=\; (x \cdot y) + (x' \cdot y'), \end{array}$$

and similarly, $z \cdot z' \cdot (w + w') \;\leq\; (z \cdot w) + (z' \cdot w')$. So if $x \cdot y \leq w$ and $x' \cdot y' \leq w'$ then, by (11),

$$\begin{array}{rl} & x \cdot x' \cdot (y + y') \cdot z \cdot z' \;\leq\; ((x \cdot y) + (x' \cdot y')) \cdot z \cdot z' \\ \leq & (w + w') \cdot z \cdot z' \;=\; z \cdot z' \cdot (w + w') \;\leq\; (z \cdot w) + (z' \cdot w'). \end{array}$$

(15): Set $z = z' = \mathsf{t}$ and $w' = w$ in (14), then invoke (10) for $w + w \leq w$. $\square$

Definition 6.3. An *intensional filter* of $\boldsymbol{A}$ shall mean a subset F of A that is either empty or an upward closed subsemigroup of $\langle A; \cdot \rangle$, i.e., whenever $x \in F$ and $x \leq y \in A$ then $y \in F$, and whenever $x, y \in F$ then $x \cdot y \in F$.

Lemma 6.4. *For each $x \in A$, the set $[\,x\,) := \{a \in A : a \geq x\}$ is an intensional filter of $\boldsymbol{A}$.*

Proof. If $x \leq y \in A$ and $x \leq z \in A$ then $x \leq x \cdot x \leq y \cdot z$. □

Arbitrary intersections of intensional filters are intensional filters again. So the set $\mathrm{IFil}(\boldsymbol{A})$ of all intensional filters of $\boldsymbol{A}$ becomes a complete lattice with respect to set inclusion, whose extrema are $\emptyset$ and A. For each $X \subseteq A$, let $\mathrm{IFg}(X)$ denote the smallest intensional filter of $\boldsymbol{A}$ containing X, i.e.,

$$\mathrm{IFg}(X) \;=\; \bigcap \{F : X \subseteq F \in \mathrm{IFil}(\boldsymbol{A})\}.$$

Lemma 6.5.

$$\mathrm{IFg}(X) = \{a \in A : a \geq x_1 \cdot \ \ldots \ \cdot x_n \textit{ for some } x_1, \ldots, x_n \in X, \textit{ where } n > 0\}.$$

In particular, $\mathrm{IFg}(\{x\}) = [\,x\,)$ *for all $x \in A$.*

Proof. For the first claim, it suffices to note that the set on the right of the equality is an intensional filter containing X, and it is also contained in any intensional filter that contains X. The second claim follows from Lemma 6.4. □

Corollary 6.6. *If F is an intensional filter of $\boldsymbol{A}$ and $x \in A$ then*

$$\mathrm{IFg}(F \cup \{x\}) \;=\; \{a \in A : a \in F \textit{ or } a \geq x \textit{ or } \exists f \in F \textit{ such that } a \geq f \cdot x\}.$$

This follows easily from Lemma 6.5, in view of commutativity, the square increasing law and the closure properties of F.

Definition 6.7. For $F, G \in \mathrm{IFil}(\boldsymbol{A})$, we define

$$F \circ G := \{a \in A : a \geq f \cdot g \text{ for some } f \in F \text{ and some } g \in G\}.$$

Lemma 6.8. $\langle\, \mathrm{IFil}(\boldsymbol{A}); \circ, [\,\mathsf{t}\,), \subseteq \,\rangle$ *is a po-monoid, and $F \circ F \subseteq F$ for all intensional filters F of $\boldsymbol{A}$.*

Proof. Closure of $\mathrm{IFil}(\boldsymbol{A})$ under $\circ$ follows easily from the commutativity of $\cdot$ (along with associativity and order compatibility). Then $\langle \mathrm{IFil}(\boldsymbol{A}); \circ, [\,\mathsf{t}\,), \subseteq \rangle$ is a (commutative) po-monoid, by Lemma 5.4. If F is an intensional filter of $\boldsymbol{A}$ and $a \in F \circ F$ then $a \geq f_1 \cdot f_2$ for some $f_1, f_2 \in F$, hence $a \in F$. □

Definition 6.9. An intensional filter F of $\boldsymbol{A}$ is said to be *prime* provided that whenever $a, b \in A$ with $a + b \in F$ then $a \in F$ or $b \in F$.

The next lemma is analogous to the prime filter theorem for distributive lattices—but here $\cdot, +$ take over the roles of $\wedge, \vee$. Analogues of Lemmas 6.10 and 6.11 appear in [30].

Lemma 6.10. *Let $X \cup \{y\} \subseteq A$, where $y \notin \mathrm{IFg}(X)$. Then there is an intensional filter M of $\boldsymbol{A}$ that is a $\subseteq$–maximal element of the set*

$$\{F \in \mathrm{IFil}(\boldsymbol{A}) : X \subseteq F \ \textit{ and } \ y \notin F\}.$$

Further, any such M is a prime intensional filter of $\boldsymbol{A}$.

Proof. The first assertion is a straightforward application of Zorn's Lemma. Let M be an intensional filter of $\boldsymbol{A}$ that is maximal in the indicated sense. We must show that M is prime. Suppose not. Then there exist $a, b \in A \setminus M$ with $a{+}b \in M$. So M is a proper subset of each of the intensional filters $\mathrm{IFg}(M{\cup}\{a\})$ and $\mathrm{IFg}(M \cup \{b\})$. By the maximality of M, we have $y \in \mathrm{IFg}(M \cup \{a\})$ and $y \in \mathrm{IFg}(M \cup \{b\})$. By Corollary 6.6, there exist $f_1, f_2 \in M$ such that

$$(\, y \geq a \quad \text{or} \quad y \geq f_1 \cdot a\,) \quad \text{and} \quad (\, y \geq b \quad \text{or} \quad y \geq f_2 \cdot b\,).$$

Now by (10) and (15),

- if $y \geq a$ and $y \geq b$ then $y \geq a + b$
- if $y \geq f_1 \cdot a$ and $y \geq b$ then $y \geq f_1 \cdot (a + b)$
- if $y \geq a$ and $y \geq f_2 \cdot b$ then $y \geq f_2 \cdot (a + b)$
- if $y \geq f_1 \cdot a$ and $y \geq f_2 \cdot b$ then $y \geq f_1 \cdot f_2 \cdot (a + b)$.

(In the second and third items, set $x' = \mathsf{t}$ in (15).) Then, since $f_1, f_2, a{+}b \in M$, we have $y \in M$, contradicting our choice of M. ☐

Lemma 6.11. (Primeness Lemma) *Let F, G be intensional filters of $\boldsymbol{A}$ and let P be a prime intensional filter of $\boldsymbol{A}$.*

(i) *If $F \circ G \subseteq P$ then there exists a prime intensional filter F' of $\boldsymbol{A}$ such that $F \subseteq F'$ and $F' \circ G \subseteq P$.*
(ii) *If $F \circ G \subseteq P$ then there exist prime intensional filters F', G' of $\boldsymbol{A}$ such that $F \subseteq F'$ and $G \subseteq G'$ and $F' \circ G' \subseteq P$.*

Proof. Since $\circ$ is commutative on intensional filters, two applications of (i) will yield (ii). To prove (i), we apply Zorn's Lemma to the set

$$\Sigma \;=\; \{X \in \mathrm{IFil}(\boldsymbol{A}) : X \circ G \subseteq P \ \text{ and } \ F \subseteq X\}$$

and deduce that Σ has a $\subseteq$–maximal element. Let F' be any $\subseteq$–maximal element of Σ. We need to show that the intensional filter F' is prime. Suppose not. Then there exist $a, b \in A \setminus F'$ with $a + b \in F'$. So by the maximality of

F' we must have $\mathrm{IFg}(F' \cup \{a\}) \circ G \nsubseteq P$ and $\mathrm{IFg}(F' \cup \{b\}) \circ G \nsubseteq P$, i.e., there exist $e \in \mathrm{IFg}(F' \cup \{a\})$, $e' \in \mathrm{IFg}(F' \cup \{b\})$ and $g, g' \in G$ such that

$$e \cdot g \notin P \quad \text{and} \quad e' \cdot g' \notin P. \tag{16}$$

Since $e \in \mathrm{IFg}(F' \cup \{a\})$ and $e' \in \mathrm{IFg}(F' \cup \{b\})$, Corollary 6.6 shows that there exist $f, f' \in F'$ such that

$$(\, e \geq a \quad \text{or} \quad e \geq f \cdot a \,) \quad \text{and} \quad (\, e' \geq b \quad \text{or} \quad e' \geq f' \cdot b \,).$$

Now the next four assertions follow from (14), where in some cases we substitute t for suitable variables.

- if $e \geq a$ and $e' \geq b$ then $(e \cdot g) + (e' \cdot g') \geq (a + b) \cdot g \cdot g'$
- if $e \geq f \cdot a$ and $e' \geq b$ then $(e \cdot g) + (e' \cdot g') \geq f \cdot (a + b) \cdot g \cdot g'$
- if $e \geq a$ and $e' \geq f' \cdot b$ then $(e \cdot g) + (e' \cdot g') \geq f' \cdot (a + b) \cdot g \cdot g'$
- if $e \geq f \cdot a$ and $e' \geq f' \cdot b$ then $(e \cdot g) + (e' \cdot g') \geq f \cdot f' \cdot (a + b) \cdot g \cdot g'$.

Then, since F' and G are closed under $\cdot$ and $f, f', a + b \in F'$ and $g, g' \in G$ and $F' \circ G$ is upward closed and contained in P, it follows in all four cases that $(e \cdot g) + (e' \cdot g') \in P$. This, with (16), contradicts the primeness of P. □

Notation. $\mathrm{PIFil}(\boldsymbol{A})$ shall denote the set of all prime intensional filters of $\boldsymbol{A}$.

Definition 6.12. For each $P \in \mathrm{PIFil}(\boldsymbol{A})$ we define $\neg\, P := \{a \in A : \neg\, a \notin P\}$.

Lemma 6.13. $\mathrm{PIFil}(\boldsymbol{A})$ *is closed under* $\neg$.

Proof. Let $P \in \mathrm{PIFil}(\boldsymbol{A})$. Upward closure of $\neg\, P$ follows from the fact that $\neg$ is an involution. Closure under $\cdot$ follows from the primeness of P, in view of (12). Dually, primeness of $\neg\, P$ follows from (12) and the fact that $\langle P; \cdot \rangle$ is empty or a subsemigroup of $\langle A; \cdot \rangle$. □

Lemma 6.13 is the main reason for confining attention to the prime filters, since $\mathrm{IFil}(\boldsymbol{A})$ is not closed under $\neg$.

Lemma 6.14. *For all* $P, Q \in \mathrm{PIFil}(\boldsymbol{A})$ *and all* $W \in \mathrm{IFil}(\boldsymbol{A})$,

(i) $\neg\,\neg\, P = P$,
(ii) *if* $P \subseteq Q$ *then* $\neg\, Q \subseteq \neg\, P$,
(iii) *if* $P \circ W \subseteq Q$ *then* $(\neg\, Q) \circ W \subseteq \neg\, P$.

Proof. (i) and (ii) follow from Lemma 5.4.

To prove (iii), suppose $P \circ W \subseteq Q$. Let $a \in (\neg\, Q) \circ W$. Then $a \geq b \cdot w$ for some $w \in W$ and some $b \in \neg\, Q$, so $\neg\, b \notin Q$. Now $\neg\, b \geq \neg\, a \cdot w$, by compatibility of $\neg$. Since $\neg\, b \notin Q$ and Q is upward closed, we must have $\neg\, a \cdot w \notin Q$. It then follows from $P \circ W \subseteq Q$ that $\neg\, a \cdot w \notin P \circ W$. So $\neg\, a \notin P$, i.e., $a \in \neg\, P$. This shows that $(\neg\, Q) \circ W \subseteq \neg\, P$. □

For any $\alpha, \beta \subseteq \mathrm{PIFil}(\boldsymbol{A})$, we define $\alpha * \beta := \{F \circ G : F \in \alpha \text{ and } G \in \beta\}$. This is not a binary operation on the power set of $\mathrm{PIFil}(\boldsymbol{A})$, since $\mathrm{PIFil}(\boldsymbol{A})$ is not generally closed under $\circ$.

Definition 6.15. $D(\boldsymbol{A})$ shall denote the set of all upward closed subsets of $\langle \mathrm{PIFil}(\boldsymbol{A}); \subseteq \rangle$. So $\alpha \in D(\boldsymbol{A})$ iff α is a set of prime intensional filters of $\boldsymbol{A}$ such that for any prime intensional filters P, Q of $\boldsymbol{A}$, if $P \in \alpha$ and $P \subseteq Q$ then $Q \in \alpha$.

Definition 6.16. Let $\alpha, \beta \in D(\boldsymbol{A})$ and $x \in A$.

(i) We define $\alpha \cdot \beta$ to be the set of all prime intensional filters of $\boldsymbol{A}$ that contain an element of $\alpha * \beta$, i.e.,

$$\alpha \cdot \beta := \{P \in \mathrm{PIFil}(\boldsymbol{A}) : P \supseteq P_\alpha \circ P_\beta \text{ for some } P_\alpha \in \alpha \text{ and some } P_\beta \in \beta\}.$$

(ii) We define $\alpha \to \beta$ to be the set of all prime intensional filters W of $\boldsymbol{A}$ such that every prime intensional filter containing an element of $\alpha * \{W\}$ belongs to β, i.e.,

$$\alpha \to \beta := \{W \in \mathrm{PIFil}(\boldsymbol{A}) : \forall Q \in \mathrm{PIFil}(\boldsymbol{A})\ \forall P_\alpha \in \alpha\ (Q \supseteq P_\alpha \circ W \implies Q \in \beta)\}.$$

(iii) We define $\neg\, \alpha := \{P \in \mathrm{PIFil}(\boldsymbol{A}) : \neg\, P \notin \alpha\}$.

(iv) We define $h(x) := \{P \in \mathrm{PIFil}(\boldsymbol{A}) : x \in P\}$.

It is easy to see that

Lemma 6.17. *$D(\boldsymbol{A})$ is closed under $\cdot, \to$ and $\neg$, as well as the intersections and unions of arbitrary subsets. If $x \in A$ then $h(x) \in D(\boldsymbol{A})$.*

Let $\boldsymbol{D}(\boldsymbol{A}) = \langle D(\boldsymbol{A}); \cdot, \to, \cap, \cup, \neg, h(\mathsf{t}) \rangle$.

Lemma 6.18. *$\boldsymbol{D}(\boldsymbol{A})$ is a complete de Morgan monoid.*

Proof. It follows from Lemmas 6.17 and 6.8 that $\langle D(\boldsymbol{A}); \cdot, \to, \subseteq \rangle$ is a residuated lo-*semigroup*, whose order is a distributive complete lattice order, with $\cap$ and $\cup$ as meet and join. To see that this structure is square increasing, let $\alpha \in D(\boldsymbol{A})$ and let $P \in \alpha$. By Lemma 6.8, $P \supseteq P \circ P$, so $P \in \alpha \cdot \alpha$. Thus, $\alpha \subseteq \alpha \cdot \alpha$.

We need to show that $h(\mathsf{t})$ is an identity for $\cdot$ in $D(\boldsymbol{A})$. Let $P \in \alpha \cdot h(\mathsf{t})$, so $P \supseteq P_\alpha \circ P_\mathsf{t}$ for some $P_\alpha \in \alpha$ and some $P_\mathsf{t} \in h(\mathsf{t})$. Then $\mathsf{t} \in P_\mathsf{t}$, so $P_\mathsf{t} \supseteq [\,\mathsf{t}\,)$, as P_t is upward closed. Thus, $P_\alpha \circ P_\mathsf{t} \supseteq P_\alpha \circ [\,\mathsf{t}\,)$, whence $P \supseteq P_\alpha \circ [\,\mathsf{t}\,) = P_\alpha$ (by Lemma 6.8). Then $P \in \alpha$, as α is upward closed. This shows that $\alpha \cdot h(\mathsf{t}) \subseteq \alpha$.

For the reverse inclusion, let $P \in \alpha$. By Lemma 6.8, $P \supseteq P \circ [\,\mathsf{t}\,)$. So it follows from the Primeness Lemma 6.11 that there is a prime intensional filter F' of $\boldsymbol{A}$ such that $P \supseteq P \circ F'$ and $[\,\mathsf{t}\,) \subseteq F'$, i.e., $\mathsf{t} \in F'$, i.e., $F' \in h(\mathsf{t})$. This shows that $P \in \alpha \cdot h(\mathsf{t})$. Thus, $\alpha \subseteq \alpha \cdot h(\mathsf{t})$, as required.

It remains only to prove that $\neg$ is a compatible involution. Let $\alpha, \beta \in D(\boldsymbol{A})$. For any prime intensional filter P of $\boldsymbol{A}$, we have $\neg\neg P = P$, by Lemma 6.14(i). So in this case, $P \in \neg\neg\alpha$ iff $\neg P \notin \neg\alpha$ iff $\neg\neg P \in \alpha$ iff $P \in \alpha$. Thus, $\neg\neg\alpha = \alpha$.

Now $\langle D(\boldsymbol{A}); \cdot, \to, \subseteq \rangle$ is implicative, because $\cdot$ is a monoid operation on $D(\boldsymbol{A})$. So by Lemma 3.2, it remains only to show that $\alpha \to \neg\beta = \beta \to \neg\alpha$. By symmetry, we need only show that $\alpha \to \neg\beta \subseteq \beta \to \neg\alpha$. Let $W \in \alpha \to \neg\beta$. So

$$\forall Q \in \mathrm{PIFil}(\boldsymbol{A})\ \forall P_\alpha \in \alpha,\ \text{ if } Q \supseteq P_\alpha \circ W \text{ then } Q \in \neg\beta. \tag{17}$$

Now let Q' be any prime intensional filter of $\boldsymbol{A}$ and suppose $P_\beta \in \beta$ with $Q' \supseteq P_\beta \circ W$. We need to prove that $Q' \in \neg\alpha$. Suppose $Q' \notin \neg\alpha$. Then

$$\neg Q' \in \alpha. \tag{18}$$

From $Q' \supseteq P_\beta \circ W$ and Lemma 6.14(iii), we infer that

$$\neg P_\beta \supseteq (\neg Q') \circ W. \tag{19}$$

It follows from (17), (18) and (19) that $\neg P_\beta \in \neg\beta$. So $\neg\neg P_\beta \notin \beta$, i.e., $P_\beta \notin \beta$. This contradicts our assumption that $P_\beta \in \beta$, so $Q' \in \neg\alpha$. □

Lemma 6.19. *The function $x \mapsto h(x)$ from $\boldsymbol{A}$ into $\boldsymbol{D}(\boldsymbol{A})$ is order preserving and order reflecting, hence injective. It also preserves $\cdot, \neg, \to$ and the identity.*

Proof. Since (prime) intensional filters are upward closed, h preserves order. Let $x, y \in A$. Suppose $x \nleqslant y$, i.e., $y \notin [\,x\,) = \mathrm{IFg}(\{x\})$ (by Lemma 6.5). So by Lemma 6.10, there exists a prime intensional filter M of $\boldsymbol{A}$ with $y \notin M$ such that $[\,x\,) \subseteq M$, i.e., $x \in M$. Then $M \in h(x)$. Now $h(x) \nsubseteq h(y)$, otherwise we would have $M \in h(y)$, hence $y \in M$, a contradiction.

It is easily shown that $h(x) \cdot h(y) \subseteq h(x \cdot y)$. For the reverse inclusion, let $P \in h(x \cdot y)$, so $x \cdot y \in P$. Since P is upward closed and $\leq$ is compatible with $\cdot$ in $\boldsymbol{A}$, it follows that $[\,x\,) \circ [\,y\,) \subseteq P$. Then by the Primeness Lemma 6.11(ii), there are prime intensional filters F', G' of $\boldsymbol{A}$ such that $F' \circ G' \subseteq P$ and $[\,x\,) \subseteq F'$ and $[\,y\,) \subseteq G'$, i.e., $x \in F'$ and $y \in G'$, i.e., $F' \in h(x)$ and $G' \in h(y)$. It follows that $P \in h(x) \cdot h(y)$. This proves that $h(x \cdot y) \subseteq h(x) \cdot h(y)$, so h preserves $\cdot$.

Preservation of $\neg$ follows straightforwardly from the definitions. Lemma 6.18 shows that $h(\mathsf{t})$ is the identity for $\cdot$ on $D(\boldsymbol{A})$. It also shows that $\boldsymbol{A}$ and $\boldsymbol{D}(\boldsymbol{A})$ both satisfy $x \to y \approx \neg(x \cdot \neg y)$, so h preserves $\to$ as well. □

The last two lemmas establish Theorem 6.1.

There are other ways of embedding c-involutive residuated po-monoids into lattice-ordered ones, while preserving the square increasing law and integrality. One of these is an adaptation of the *Dedekind-MacNeille completion*, discussed in [4, 5, 32, 33] and [40, Ch. 8]. But the construction preserves all existent meets and joins from the original po-monoid, whence the lattice-based structure that it produces is not generally *distributive*. So we could not have used this method to prove Theorem 6.1 or Theorem 5.1. Alternative constructions involving a 'perp-style' involution feature in [14] and subsequent papers of Dunn. They derive from Goldblatt's semantics for orthologic and have been adapted to linear logic. They too are designed to produce non-distributive lattices.

7. Some Formal Systems

The embedding results established thus far will help us to prove separation theorems for the deducibility relations of various substructural logics. This will be done in Section 8. First, we need a unified account of the precise relationships between the residuated algebraic structures and the logical systems.

Positive linear (propositional) *logic* is specified by the following formal system in the signature $\{\cdot, \to, \wedge, \vee, \mathsf{t}\}$. We denote this system by $\mathbf{LL}^{+}$. Our t is often denoted as $\mathbf{1}$ in linear logic, e.g., in [40].

(P1)	$\vdash\ p \to p$	(*identity*)
(P2)	$\vdash\ (p \to q)\ \to\ ((r \to p) \to (r \to q))$	(*prefixing*)
(P3)	$\vdash\ (p \to (q \to r)) \to (q \to (p \to r))$	(*exchange*)
(P4)	$p,\ p \to q\ \vdash\ q$	(*modus ponens*)
(P5)	$\vdash\ (p \wedge q) \to p$	
(P6)	$\vdash\ (p \wedge q) \to q$	
(P7)	$\vdash\ ((p \to q)\ \wedge\ (p \to r))\ \to\ (p \to (q \wedge r))$	
(P8)	$p,\, q\ \vdash\ p \wedge q$	(*adjunction*)
(P9)	$\vdash\ p \to (p \vee q)$	
(P10)	$\vdash\ q \to (p \vee q)$	
(P11)	$p \to r,\ q \to r\ \vdash\ (p \vee q) \to r$	
(P12)	$\vdash\ p \to (q \to (q \cdot p))$	
(P13)	$\vdash\ (p \to (q \to r))\ \to\ ((q \cdot p) \to r)$	
(P14)	$\vdash\ \mathsf{t}$	
(P15)	$\vdash\ \mathsf{t} \to (p \to p)$	

We refer to (P1)–(P15) as the *postulates* of $\mathbf{LL}^{+}$, and we call modus ponens, adjunction and (P11) the *inference rules*. The remaining postulates are called the *axioms* of $\mathbf{LL}^{+}$. In an inference rule, the formulas on the left of $\vdash$ are called the *premisses*, and those on the right the *conclusion*. In this paper, when we speak of formal systems in general, it is understood that an inference rule always has a single conclusion and only finitely many premisses. A formal system may have infinitely many postulates; it is assumed to come with an infinite supply of (propositional) *variables*, which may as well be considered fixed for all systems.

The *deducibility relation* $\vdash_{\mathbf{F}}$ of a formal system $\mathbf{F}$ is defined as the relation from sets of formulas to single formulas that contains a pair $\langle \Gamma, \alpha \rangle$ just when there is a *proof of* α *from* Γ *in* $\mathbf{F}$. A proof of this kind is any finite sequence of formulas terminating with α, such that every item in the sequence belongs to Γ or is a substitution instance of a formula that is either an axiom of $\mathbf{F}$ or the conclusion of an inference rule of $\mathbf{F}$, where in the last case, the same substitution turns the premisses of the rule into previous items in the sequence. To signify that such a proof exists, we write $\Gamma \vdash_{\mathbf{F}} \alpha$, omitting Γ when it is empty. The *theorems* of $\mathbf{F}$ are the formulas α such that $\vdash_{\mathbf{F}} \alpha$.[1]

Definition 7.1. Let $\mathbf{F}$ be any formal system and S any subset of its signature.

The *S–fragment* of $\vdash_{\mathbf{F}}$ is the set of all pairs $\langle \Gamma, \alpha \rangle$ in $\vdash_{\mathbf{F}}$ such that all connectives occurring in $\Gamma \cup \{\alpha\}$ belong to S. (Sentential constants count as connectives of rank 0.)

The *S–postulates* of $\mathbf{F}$ are the postulates of $\mathbf{F}$ in which all occurring connectives belong to S.

As a consequence of general results in [24], the S–fragment of $\vdash_{\mathbf{F}}$ is itself the deducibility relation of a formal system whose signature is S. Of course the latter system is not unique, but any such system will be called an *axiomatization of the S–fragment* of $\vdash_{\mathbf{F}}$. Finite axiomatizability need not persist in fragments.

The following *separation theorem* for $\vdash_{\mathbf{LL}^+}$ was proved in [41].

Theorem 7.2. *The S–fragment of $\vdash_{\mathbf{LL}^+}$ is axiomatized by the S–postulates among* (P1)–(P15), *provided that S includes $\to$.*

Hilbert-style presentations of linear logic usually include an axiom

$$\vdash \ ((p \to r) \wedge (q \to r)) \to ((p \vee q) \to r) \tag{20}$$

and exclude the inference rule (P11). The formula (20) is a theorem of $\mathbf{LL}^+$ but to include it as an axiom in preference to (P11) would be to destroy the above separation theorem for fragments of $\vdash_{\mathbf{LL}^+}$ with $\to$ and $\vee$ but not $\wedge$: see [41, Ex. 1]. The theorems of $\mathbf{LL}^+$ do not include

$$\vdash \ (p \to (p \to q)) \to (p \to q) \qquad (\textit{contraction}) \tag{W}$$

$$\vdash \ (p \wedge (q \vee r)) \to ((p \wedge q) \vee (p \wedge r)) \qquad (\textit{distribution}). \tag{D}$$

[1] Our definition of 'deducibility' is a common one, but it is at odds, for instance, with the usage of [2]. There, 'deducibility' is taken to be the converse of 'entailment', so the meaning of $\alpha \vdash \beta$ is constrained by the principle that it must coincide with the meaning of $\vdash \alpha \to \beta$.

The following abbreviations are standard:

$$\begin{array}{lll} \mathbf{LR}^+ & := & \mathbf{LL}^+ \cup \{(\mathsf{W})\} \\ \mathbf{R\text{-}W}^+ & := & \mathbf{LL}^+ \cup \{(\mathsf{D})\} \\ \mathbf{R}^+ & := & \mathbf{LL}^+ \cup \{(\mathsf{W}), (\mathsf{D})\} \end{array}$$

(The names will be clarified later.)

Theorem 7.3. *The S–fragment of $\vdash_{\mathbf{LR}^+}$ is axiomatized by contraction and the S–postulates among* (P1)–(P15), *provided that S includes $\to$.*

This was proved in [41, Sec. 9]. Some parts of the result can be inferred from [25, 26]; see Remark 8.5.

Separation theorems in the literature have tended to focus on the axiomatization of *theorems*, rather than of deducibility relations. To be precise about this: suppose $\mathbf{F}$ and $\mathbf{F}'$ are formal systems whose respective signatures are S and S', where $S' \subseteq S$ (with preservation of the ranks of symbols). If (i) $\mathbf{F}$ and $\mathbf{F}'$ have the same *theorems* in the vocabulary of S' and (ii) every postulated inference rule of $\mathbf{F}'$, considered as a schematic rule in the vocabulary of S, preserves the full set of theorems of $\mathbf{F}$, then we say that $\mathbf{F}'$ *axiomatizes the S'–theorems* of $\mathbf{F}$. In fact, whenever we make this claim in the present paper, a stronger version of (ii) will hold: the postulated rules of $\mathbf{F}'$ will actually *belong* to $\vdash_{\mathbf{F}}$. Even in this situation, it need not follow that $\mathbf{F}'$ axiomatizes the S'–fragment of $\vdash_{\mathbf{F}}$.

Negation.

The formal system $\mathbf{CL}$ is got by adding to $\mathbf{LL}^+$ a unary connective $\neg$ (called *negation*) and the axioms

(P16)	$\vdash (p \to \neg\, q) \to (q \to \neg\, p)$	(*contraposition*)
(P17)	$\vdash \neg\,\neg\, p \to p$	(*double negation*)
(P18)	$\vdash \neg\,(\neg\,(p \to p) \vee \neg\,(q \to q))$.	

It specifies *classical linear logic* (without bounds and exponentials). We are deviating from [40] and other texts in taking $\neg$ to be a *primitive* connective. The reasons for doing this will be explained in Remark 8.6. We are also carrying over to $\neg$ our notational conventions for involution, e.g., $\neg$ binds more strongly than any other connective to be discussed. Likewise, $|\,p\,|$ shall continue to abbreviate $p \to p$. The theorems of $\mathbf{CL}$ include the converse of double negation.

Remark 7.4. The theorem $(p \cdot q) \to \neg\,(p \to \neg\, q)$ and its converse can both be proved using only the $\{\cdot, \to, \neg\}$–postulates of $\mathbf{CL}$. Moreover, if we had *defined*

$p \cdot q$ as $\neg(p \to \neg q)$ then we could have proved (P12) and (P13) using only the $\{\to, \neg\}$–postulates of **CL**. We express the conjunction of these two facts by saying that $\cdot$ is *definable over* **CL** in terms of $\to$ and $\neg$ only.

In the definition of $\vdash_{\mathbf{CL}}$, the axiom (P18) is redundant. However, when we want a *separative* axiomatization of $\vdash_{\mathbf{CL}}$, (or even just the *theorems* of **CL**) then (P18) turns out to be indispensable. The next remark explains this.

Remark 7.5. De Morgan's laws are reflected in the well known fact that

$$(p \vee q) \to \neg(\neg p \wedge \neg q) \quad \text{and} \quad (p \wedge q) \to \neg(\neg p \vee \neg q)$$

and their converses are theorems of **CL**; the proofs use only the $\{\to, \wedge, \vee, \neg\}$–postulates of **CL**. If we had defined $p \vee q$ as $\neg(\neg p \wedge \neg q)$ then we could have proved all of the $\{\to, \vee, \neg\}$–postulates of **CL** using only the $\{\to, \wedge, \neg\}$–postulates. Dually, if we had defined $p \wedge q$ as $\neg(\neg p \vee \neg q)$ then we could have proved all of the $\{\to, \wedge, \neg\}$–postulates of **CL** using only the $\{\to, \vee, \neg\}$–postulates. But it is here that we need (P18): without it we cannot derive adjunction, whence we cannot prove theorems like $(p \to p) \wedge (q \to q)$. (This is demonstrated rigorously in Remark 8.3 below.)

Extending each of $\mathbf{LR}^+$, $\mathbf{R\text{-}W}^+$ and $\mathbf{R}^+$ by the axioms (P16)–(P18), we get systems denoted respectively by **LR**, **R-W** and **R**. In other words,

$$\begin{array}{lcl} \mathbf{LR} & = & \mathbf{CL} \cup \{(\mathsf{W})\} \\ \mathbf{R\text{-}W} & = & \mathbf{CL} \cup \{(\mathsf{D})\} \\ \mathbf{R} & = & \mathbf{CL} \cup \{(\mathsf{W}), (\mathsf{D})\}. \end{array}$$

R specifies the principal relevance logic of [2], except that we follow [3] in including t in its signature. The name **R-W** (often written as **RW**) signifies '**R** minus (W)'. **LR** stands for 'Lattice–**R**', where 'lattice' is to be contrasted with 'distributive lattice'. For information about **LR**, see [39, 28]. For **R-W** see papers of Brady, Giambrone and Meyer, particularly [9].

Definition 7.6. For any formal system **F** and any subset S of its signature, let $\mathbf{F}_S$ denote the formal system with signature S consisting of just the S–postulates of **F**.

Consider the array of logical systems and algebraic structures in Table 1. On each line of the table, the algebraic structures of the indicated kind form a strict universal Horn class (in the sense of first order logic with equality). In particular, this class is closed under ultraproducts. The following soundness and completeness theorems are essentially well known. A sketch of the proof will be given presently.

TABLE 1. Some formal systems and their reduced models
(Recall that all semigroup operations are assumed commutative.)

$\mathbf{CL}$	c-involutive residuated lo-monoids
$\mathbf{LL}^+$	residuated lo-monoids
$\mathbf{CL}_{\cdot,\to,\wedge,\vee,\neg}$	c-involutive residuated lo-semigroups satisfying $(\lvert x\rvert \wedge \lvert y\rvert) \to z \leq z$
$\mathbf{CL}_{\cdot,\to,\neg,\mathsf{t}} = \mathbf{R\text{-}W}_{\cdot,\to,\neg,\mathsf{t}}$	c-involutive residuated po-monoids
$\mathbf{CL}_{\cdot,\to,\neg} = \mathbf{R\text{-}W}_{\cdot,\to,\neg}$	c-involutive implicative po-semigroups
$\mathbf{R\text{-}W}$	c-involutive residuated distributive lo-monoids
$\mathbf{R\text{-}W}^+$	residuated distributive lo-monoids
$\mathbf{R\text{-}W}_{\cdot,\to,\wedge,\vee}$	distributive residuated lo-semigroups satisfying $(\lvert x\rvert \wedge \lvert y\rvert) \to z \leq z$
$\mathbf{R\text{-}W}_{\cdot,\to,\wedge,\vee,\neg}$	c-involutive residuated distributive lo-semigroups satisfying $(\lvert x\rvert \wedge \lvert y\rvert) \to z \leq z$
$\mathbf{LR}$	square increasing c-involutive residuated lo-monoids
$\mathbf{LR}^+$	square increasing residuated lo-monoids
$\mathbf{LR}_{\cdot,\to,\wedge,\vee,\neg}$	square increasing c-involutive residuated lo-semigroups satisfying $(\lvert x\rvert \wedge \lvert y\rvert) \to z \leq z$
$\mathbf{R}$	de Morgan monoids
$\mathbf{R}^+$	square increasing distributive residuated lo-monoids
$\mathbf{R}_{\cdot,\to,\wedge,\vee}$	square increasing distributive residuated lo-semigroups satisfying $(\lvert x\rvert \wedge \lvert y\rvert) \to z \leq z$
$\mathbf{R}_{\cdot,\to,\wedge,\vee,\neg}$	relevant algebras
$\mathbf{R}_{\cdot,\to,\neg,\mathsf{t}} = \mathbf{LR}_{\cdot,\to,\neg,\mathsf{t}}$	square increasing c-involutive residuated po-monoids
$\mathbf{R}_{\cdot,\to,\neg} = \mathbf{LR}_{\cdot,\to,\neg}$	square increasing c-involutive implicative po-semigroups

---

Theorem 7.7. *Let* $\mathbf{L}$ *be any one of the formal systems in Table* 1 *and let* C *be the class of all algebraic structures of the kind indicated on its right in the table. Then for any set* $\Gamma \cup \{\alpha\}$ *of formulas of* $\mathbf{L}$*,*

$$\Gamma \vdash_{\mathbf{L}} \alpha \quad \textit{iff} \quad \{\lvert\gamma\rvert \leq \gamma : \gamma \in \Gamma\} \models_{\mathsf{C}} \lvert\alpha\rvert \leq \alpha. \tag{21}$$

If t *belongs to the signature then*

$$\Gamma \vdash_{\mathbf{L}} \alpha \quad \textit{iff} \quad \{\mathsf{t} \leq \gamma : \gamma \in \Gamma\} \models_{\mathsf{C}} \mathsf{t} \leq \alpha. \tag{22}$$

Let $\boldsymbol{Fm}$ denote the formula algebra of $\mathbf{L}$, i.e., the absolutely free algebra in the signature of $\mathbf{L}$, freely generated by the variables. The right hand side of (21) means that for any homomorphism h from $\boldsymbol{Fm}$ into the algebra reduct $\boldsymbol{A}$ of a structure $\langle \boldsymbol{A}, \leq \rangle \in \mathsf{C}$, if $h(|\,\gamma\,|) \leq h(\gamma)$ for all $\gamma \in \Gamma$ then $h(|\,\alpha\,|) \leq h(\alpha)$. The last clause really means: if $h[\Gamma]$ consists of designated elements then $h(\alpha)$ is designated (because $h(|\,\beta\,|) = |\,h(\beta)\,|$ for any formula β).

Note that the relation $\models_{\mathsf{C}}$ is finitary, because C is closed under ultraproducts. Since $\vdash_{\mathbf{L}}$ is also finitary, it would not have weakened Theorem 7.7 had we restricted (21) to the case where Γ is finite. In that case the right hand side of (21) asserts that C satisfies a strict universal Horn sentence, viz.,

$$\mathsf{C} \models (\&_{\gamma \in \Gamma} \ |\,\gamma\,| \leq \gamma) \implies |\,\alpha\,| \leq \alpha.$$

Of course (22) follows from (21) when t is expressed, in view of Proposition 2.4(3), and in this case the universal Horn sentence above becomes

$$\mathsf{C} \models (\&_{\gamma \in \Gamma} \ \mathsf{t} \leq \gamma) \implies \mathsf{t} \leq \alpha.$$

Proof sketch for Theorem 7.7: Following Bloom [8], we may consider any formal system $\mathbf{F}$ as a first order system $\mathbf{F}^{\forall}$, without equality, having just one unary predicate, P say. The postulates $\Gamma \vdash \alpha$ of $\mathbf{F}$ induce the proper axioms $\forall \overline{x}\,((\&_{\gamma \in \Gamma} P(\gamma)) \implies P(\alpha))$ of $\mathbf{F}^{\forall}$, where $\overline{x}$ lists the apparent variables of $\Gamma \cup \{\alpha\}$. (The proper axiom is understood to be $\forall \overline{x}\, P(\alpha)$ when $\Gamma = \emptyset$.) A *matrix model* of $\mathbf{F}$ is just a model of $\mathbf{F}^{\forall}$, i.e., it is a structure $\langle \boldsymbol{A}, X \rangle$, where $\boldsymbol{A}$ is an algebra in the signature of $\mathbf{F}$ and X is a subset of the universe of $\boldsymbol{A}$ that contains all $\boldsymbol{A}$–instances of the axioms of $\mathbf{F}$ and is closed under the inference rules of $\mathbf{F}$. Such a model is said to be *reduced* if no non-identity congruence of $\boldsymbol{A}$ makes X a union of congruence classes. Now every formal system $\mathbf{F}$ is sound and complete with respect to its class of *reduced* matrix models—where the sense of '$\langle \boldsymbol{A}, X \rangle$ *satisfies* $\Gamma \vdash \alpha$' is

(23) for every homomorphism $h : \boldsymbol{Fm} \to \boldsymbol{A}$, if $h[\Gamma] \subseteq X$ then $h(\alpha) \in X$.

This result can be found in [42]. But we can simplify the meaning of 'reduced'.

If $\mathbf{F}$ has a binary connective $\to$ such that

$$\vdash_{\mathbf{F}} \ p \to p$$

$$p,\, p \to q,\, q \to p \ \vdash_{\mathbf{F}} \ q$$

$$p_1 \to q_1,\, q_1 \to p_1, \ldots, p_n \to q_n,\, q_n \to p_n \ \vdash_{\mathbf{F}} \ \alpha(p_1, \ldots, p_n) \to \alpha(q_1, \ldots, q_n)$$

for every connective α of $\mathbf{F}$, where n is the rank of α, then the *reduced* matrix models of $\mathbf{F}$ are just the matrix models $\langle \boldsymbol{A}, X \rangle$ of $\mathbf{F}$ such that for all $a, b \in A$,

$$\text{if } \ a \to b \in X \ \text{ and } \ b \to a \in X \ \text{ then } \ a = b.$$

(Formal systems with the properties just postulated belong to the class of 'equivalential' systems of [12, 16], whose definition is more general. An analogous characterization of reduced models holds for all such systems.)

Now let $\mathbf{L}$ and C be as in Theorem 7.7. The above discussion applies to $\mathbf{L}$. Up to first order definitional equivalence, the class of all reduced matrix models $\langle \boldsymbol{A}, X\rangle$ of $\mathbf{L}$ turns out to be just C where, for any $x, y \in A$, we define that $x \leq y$ iff $x \to y \in X$, and conversely that $x \in X$ iff $|\,x\,| \leq x$ (iff $\mathsf{t} \leq x$ when t is expressed). One has to verify that the axioms and inference rules of $\mathbf{L}$ induce in the *reduced* matrix models exactly the defining postulates of C. This is an exercise that has been carried out by many researchers for many of the values of $\mathbf{L}$ and C, so we shall omit the calculations. (Much of the detail can be found in [41, Sec. 6,7].) Thus the 'reduced' soundness and completeness theorem involving (23) becomes (21) in the case of $\mathbf{L}$. □

8. Deductive Separation Theorems

We can now prove some separation theorems that incorporate negation. For the remainder of this section, we assume that $S \subseteq \{\cdot, \to, \wedge, \vee, \neg, \mathsf{t}\}$.

Theorem 8.1. *If S includes $\to$ then the S–fragment of $\vdash_{\mathbf{CL}}$ is axiomatized by the S–postulates of* $\mathbf{CL}$, *i.e., by the S–postulates among* (P1)–(P18).

Proof. Let S be a *proper* subset of $\{\cdot, \to, \wedge, \vee, \neg, \mathsf{t}\}$, containing $\to$. Suppose $\Gamma \vdash_{\mathbf{CL}} \alpha$ where $\Gamma \cup \{\alpha\}$ consists of S–formulas. We must show that $\Gamma \vdash_{\mathbf{CL}_S} \alpha$. We may assume without loss of generality that Γ is finite, since $\vdash_{\mathbf{CL}}$ is finitary and $\vdash_{\mathbf{CL}_S}$ is monotonic.

Let RL and $\mathsf{c\text{-}IRL}$ denote the class of all residuated lo-monoids and the class of all c-involutive residuated lo-monoids, respectively. Since $\Gamma \vdash_{\mathbf{CL}} \alpha$, it follows from Theorem 7.7 that

$$\mathsf{c\text{-}IRL} \models (\&_{\gamma\in\Gamma}\ |\,\gamma\,| \leq \gamma) \implies |\,\alpha\,| \leq \alpha. \tag{24}$$

Suppose first that S excludes $\neg$. By Theorem 4.5, every residuated lo-monoid can be embedded into one that is c-involutive. So

$$\mathsf{RL} \models (\&_{\gamma\in\Gamma}\ |\,\gamma\,| \leq \gamma) \implies |\,\alpha\,| \leq \alpha, \tag{25}$$

because universal sentences persist in substructures. Then by Theorem 7.7, $\Gamma \vdash_{\mathbf{LL}^+} \alpha$, so $\Gamma \vdash_{\mathbf{LL}^+{}_S} \alpha$, by the Separation Theorem 7.2 for $\vdash_{\mathbf{LL}^+}$. But $\mathbf{LL}^+{}_S = \mathbf{CL}_S$ (as $\neg \notin S$), so $\Gamma \vdash_{\mathbf{CL}_S} \alpha$.

We may now assume that S contains $\neg$. Then by Remark 7.4, we may assume that S also contains $\cdot$. Likewise, by Remark 7.5, we may assume that S contains $\wedge$ iff it contains $\vee$.

Let C be the class of all structures of the kind alongside $\mathbf{CL}_S$ in Table 1. In view of Theorem 7.7, we need only show that

$$\mathsf{C} \models (\&_{\gamma\in\Gamma}\ |\,\gamma\,| \leq \gamma) \implies |\,\alpha\,| \leq \alpha. \tag{26}$$

Just as above, (26) will follow from (24) if every structure in C can be embedded into an algebra in $\mathsf{c\text{-}IRL}$. So it remains only to show that this is the case.

Now the possible values of S are

$$S_1 = \{\cdot, \to, \neg, \mathsf{t}\}, \qquad S_2 = \{\cdot, \to, \wedge, \vee, \neg\}, \qquad S_3 = \{\cdot, \to, \neg\}.$$

If $S = S_1$ then by Theorem 5.1, every structure in C embeds into one in $\mathsf{c\text{-}IRL}$. If $S = S_2$, the same conclusion follows from Theorem 4.9. If $S = S_3$ then every structure in C embeds into a c-involutive residuated po-monoid, by Theorem 3.4, and therefore into an algebra from $\mathsf{c\text{-}IRL}$, by Theorem 5.1. □

A slight modification of the argument gives:

Theorem 8.2. *If S includes $\to$ then the S–fragment of $\vdash_{\mathbf{LR}}$ is axiomatized by the S–postulates of* **LR**, *i.e., by contraction and the S–postulates among* (P1)–(P18).

Proof. Where we invoked Theorems 7.2 and 5.1 in the previous proof, we rely instead on Theorems 7.3 and 6.1. Then all embedding constructions used in the proof preserve the square increasing law, so the result follows. □

Remark 8.3. (Irredundance of (P18)) We claimed earlier that if (P18) is omitted from the definition of **CL** then the separation theorem fails. To demonstrate this, let

$$\mathbf{F} \;=\; \mathbf{CL}_{\cdot,\to,\vee,\neg} \setminus \{(\mathsf{P18})\}.$$

It is enough to show that $\vdash_{\mathbf{F}} \;\neq\; \vdash_{\mathbf{CL}_{\cdot,\to,\vee,\neg}}$, because the $\{\cdot, \to, \vee, \neg\}$–fragment of $\vdash_{\mathbf{CL}}$ is just $\vdash_{\mathbf{CL}_{\cdot,\to,\vee,\neg}}$, by Theorem 8.1. Let $\boldsymbol{A}$ be the algebra in Example 4.8, let $\boldsymbol{A}^-$ be its $\wedge$–free reduct $\langle A; \cdot, \to, \vee, \neg\rangle$ and let $X = \{a, b, \top\}$. Then $\langle \boldsymbol{A}^-, X\rangle$ is a matrix model of $\mathbf{F}$, and it is reduced, because for all $x, y \in A$, if $x \to y \in X$ and $y \to x \in X$ then $x = y$. In $\langle \boldsymbol{A}^-, X\rangle$, we have

$$\neg\,(\neg\,|\,a\,| \vee \neg\,|\,b\,|) \;=\; \neg\,(\neg\, a \vee \neg\, b) \;=\; \neg\,(a \vee b) \;=\; \neg\,\top \;=\; \bot \notin X.$$

Since $\langle \boldsymbol{A}^-, X\rangle$ does not satisfy (P18), it is not a matrix model of $\vdash_{\mathbf{CL}_{\cdot,\to,\vee,\neg}}$. Therefore $\vdash_{\mathbf{F}} \;\neq\; \vdash_{\mathbf{CL}_{\cdot,\to,\vee,\neg}}$. Of course this reflects the fact that adjunction is not satisfied by $\langle \boldsymbol{A}, X\rangle$. Similar remarks apply to **LR**, because the algebra $\boldsymbol{A}$ is square increasing (in fact idempotent).

The next theorem incorporates the distributivity axiom, and a large part of its content is due to Meyer and Routley; see Remark 8.5 below.

Theorem 8.4. *Let* **F** *be* **R-W** *or* **R** *and let S contain $\to$. If S includes $\wedge$ or* excludes $\vee$ *then the S–fragment of $\vdash_{\mathbf{F}}$ is axiomatized by the S–postulates of* **F**.

Proof. In the case of **R-W**, we use the proof of Theorem 8.1, omitting the excluded signatures from consideration. Where we invoked Theorem 7.2, we need instead a distributive version of the result. If $S \subseteq \{\cdot, \to, \wedge, \mathsf{t}\}$ then the S–fragment of $\vdash_{\mathbf{R\text{-}W}_{\cdot,\to,\wedge,\mathsf{t}}} = \vdash_{\mathbf{CL}_{\cdot,\to,\wedge,\mathsf{t}}}$ is axiomatized by the common S–postulates of **R-W** and **CL**, by Theorem 8.1. Moreover, the common $\{\cdot, \to, \wedge, \mathsf{t}\}$–postulates axiomatize the $\{\cdot, \to, \wedge, \mathsf{t}\}$–fragment of $\vdash_{\mathbf{R\text{-}W}^+}$. This follows from Theorem 4.3. And $\vdash_{\mathbf{R\text{-}W}^+}$ is a fragment of $\vdash_{\mathbf{R\text{-}W}}$, by Theorem 4.5.

With these modifications, the proof strategy of Theorem 8.1 takes care of almost all the subsignatures. But we must also show that when S is $\{\cdot, \to, \wedge, \vee\}$ or $\{\to, \wedge, \vee\}$ or $\{\to, \wedge, \vee, \mathsf{t}\}$, then the S–fragment of $\vdash_{\mathbf{R\text{-}W}}$ is axiomatized by the S–postulates for **R-W**. The embedding theorem needed in the first case is supplied by Theorem 4.9. In the latter two cases, we use Theorem 9.13 below, which we have postponed because of its more complex statement.

In the case of **R**, we make similar modifications to the proof of Theorem 8.2 (rather than that of 8.1) noting that the square increasing law is preserved in all of the constructions now employed—including that of 9.13. □

Remark 8.5. (Antecedents) The *theorems* of **R** in most subsignatures of $\{\cdot, \to, \wedge, \vee, \neg, \mathsf{t}\}$ were axiomatized in the early 1970s, mostly by Meyer and Routley. See [26, 27, 36, 30] and Maksimova's paper [25]. In some (but not all) cases the arguments can be applied to systems weaker than **R**. Analogous results for logics weaker than **CL** and $\mathbf{LL}^+$ can be found in [37] and [10]; see Dunn [13] also.

Theorem 8.4 makes a mathematically stronger statement in that it deals with the full relation $\vdash_{\mathbf{R}}$, as opposed to the theorems only. Many of the proofs in [25, 26, 27, 29] supply algebraic embedding constructions, so the conservation results in these cases already apply to the deducibility relations. The exceptions have been covered in the present paper. In [30, 36], Meyer and Routley used their ternary frame semantics to obtain further conservation results for the theorems of **R**. The proofs in these papers would need to be supplemented before they could be applied to deducibility relations. Meyer and Routley had the tools to supplement them, but they also had reasons for not emphasizing deducibility relations. In relevance logic (and elsewhere), a 'logic' is usually identified with its set of theorems, and not with the deducibility relation of the formal system that initially specifies it.

Remark 8.6. (Inferential Negation) In many expositions of linear logic, negation is embodied by the sentential constant f (often denoted $\mathbf{0}$) that corresponds intuitively to $\neg\,\mathsf{t}$; see for instance [40]. Then $\neg\, p$ can be defined as $p \to \mathsf{f}$. Observe that t is then definable in terms of $\to$ and f alone, as $\mathsf{f} \to \mathsf{f}$. So in this approach, any subsignature that expresses implication and negation *must* express t. This would eliminate several fragments considered above, making any separation theorem less informative. Some of the fragments at issue have significant *extensions* in which it is not possible to add t conservatively (along with (P14), (P15)); see [5, 6].

Remark 8.7. (Adding Bounds) The formal systems considered here are often augmented by sentential constants $\bot, \top$ (called 'bounds') together with the axioms $\vdash \bot \to p$ and $\vdash p \to \top$. This move preserves the separation theorem for $\mathbf{LL}^+$, as can be shown by an easy modification of [41, Prop. 5.4]. But it destroys the separation theorem for $\mathbf{CL}$, since the bounds interact with double negation to produce negation-free theorems that are not provable from the negation-free postulates (including the postulates for the bounds). For instance, as we verify below, the bounded involutive algebraic models satisfy

$$\top \to \mathsf{t} \leq ((x \to \bot) \to \bot) \to x \tag{27}$$

but the bounded non-involutive models (which include Heyting algebras) clearly do not. So by the obvious extension of Theorem 7.7, the formula

$$(\top \to \mathsf{t}) \to (((p \to \bot) \to \bot) \to p) \tag{28}$$

is a theorem of the bounded extension of $\mathbf{CL}$ but not that of $\mathbf{LL}^+$. To verify (27), observe that in any bounded involutive model, we have $\top \to \mathsf{t} = \mathsf{f} \to \bot$, by contraposition. Thus, using the principle that $(x \to y) \cdot (y \to z) \leq x \to z$, we infer that $(x \to \mathsf{f}) \cdot (\top \to \mathsf{t}) \leq x \to \bot$, i.e., $\top \to \mathsf{t} \leq (x \to \mathsf{f}) \to (x \to \bot)$. Then

$$\begin{aligned}
&(\top \to \mathsf{t}) \cdot ((x \to \bot) \to \bot) \leq ((x \to \mathsf{f}) \to (x \to \bot)) \cdot ((x \to \bot) \to \bot) \\
&\leq (x \to \mathsf{f}) \to \bot \leq (x \to \mathsf{f}) \to \mathsf{f} = x
\end{aligned}$$

and (27) follows. We could eliminate t from (28), replacing it by $p \to p$.

In the usual Gentzen systems for $\mathbf{CL}$ and $\mathbf{LL}^+$, a similar phenomenon arises: see [40, p. 39].

For the case of $\mathbf{LR}$, we can give analogous examples in which $\bot$ is also eliminated. Let φ be any negation-free tautology of classical propositional logic that is not provable in intuitionistic propositional logic, and let t_φ denote the formal conjunction of all $|\,p\,|$ where p is a variable occurring in φ. For instance, we could take Peirce's Law $((p \to q) \to p) \to p$ for φ, with $\mathsf{t}_\varphi = |\,p\,| \wedge |\,q\,|$. It can be shown that the formulas $(\top \to \mathsf{t}) \to \varphi$ and $(\top \to \mathsf{t}_\varphi) \to \varphi$ are theorems of the bounded extension of $\mathbf{LR}$, but they are not provable in the bounded extension of $\mathbf{LR}^+$ (nor that of $\mathbf{R}^+$).

9. Product-free Algebras

Theorem 9.13 below is needed to complete the proof of Theorem 8.4. We postponed this material until now because the algebras involved are less intuitive than those discussed thus far.

Definition 9.1. By a *pre-residuated lattice* we shall mean a $\{\to, \wedge, \vee\}$–subreduct of a residuated lo-monoid, i.e., an algebra $\boldsymbol{A} = \langle A; \to, \wedge, \vee\rangle$ that can be embedded into (the $\{\to, \wedge, \vee\}$–reduct of) some residuated lo-monoid.

Similarly, a *pre-residuated* t*–lattice* is a $\{\to, \wedge, \vee, \mathsf{t}\}$–subreduct of a residuated lo-monoid.

Note that pre-residuated t–lattices satisfy all of the identities and quasi-identities in Propositions 2.2 and 2.4 whose statements omit $\cdot$, and pre-residuated lattices satisfy those that omit both $\cdot$ and t. In particular, pre-residuated t–lattices satisfy $\mathsf{t} \to x \approx x$. On the other hand, it follows directly from [41, Thm. 6.6] that an algebra $\boldsymbol{A} = \langle A; \to, \wedge, \vee, \mathsf{t}\rangle$ is a pre-residuated t–lattice if (and only if) $\langle A; \to, \wedge, \vee\rangle$ is a pre-residuated lattice and $\boldsymbol{A}$ satisfies $|\,\mathsf{t}\,| \leq \mathsf{t}$ and $|\,\mathsf{t} \to |\,x\,||\leq \mathsf{t} \to |\,x\,|$. Now these two laws follow easily from the axiom $\forall\, x\, (\mathsf{t} \to x \approx x)$ over the class of pre-residuated lattices with a distinguished element t. (In the case of the second law, use the fact that all pre-residuated lo-monoids satisfy $||\,x\,|| \approx |\,x\,|$, by Proposition 2.4(6).) Therefore we have

Lemma 9.2. *An algebra* $\boldsymbol{A} = \langle A; \to, \wedge, \vee, \mathsf{t}\rangle$ *is a pre-residuated* t*–lattice iff* $\langle A; \to, \wedge, \vee\rangle$ *is a pre-residuated lattice and* $\mathsf{t} \to a = a$ *for all* $a \in A$.

For the sake of concreteness, we add that the class of pre-residuated lattices is axiomatized by the lattice identities for $\wedge$ and $\vee$ together with the following laws (where $\alpha \leq \beta$ formally abbreviates $\alpha \wedge \beta \approx \alpha$):

$$
\begin{aligned}
& x \to y \,\leq\, (y \to z) \,\to\, (x \to z)\\
& x \to (y \to z) \,\approx\, y \to (x \to z)\\
& (x \to y) \,\wedge\, (x \to z) \,\approx\, x \to (y \wedge z)\\
& y \to z \,\leq\, (x \wedge y) \to z\\
& x \,\leq\, ((x \to y) \wedge z) \to y\\
& (|\,x\,| \,\wedge\, |\,y\,|) \to z \,\leq\, z
\end{aligned}
$$

This can be inferred from [41, Sec. 7], but we shall not rely on this axiomatization in the sequel.

Definition 9.3. A pre-residuated lattice (or t–lattice) is said to be *contractive* if it satisfies the law $x \to (x \to y) \leq x \to y$.

Lemma 9.4. *A pre-residuated lattice (or* t*–lattice) is contractive iff it can be embedded into a square increasing residuated lo-monoid.*

Proof. Use the proof of [41, Thm. 6.6], but replace the appeal to [41, Thm. 5.6] by an appeal to [41, Thm. 9.3]. $\square$

Taking Theorem 4.5 into account we have:

Corollary 9.5. *Every pre-residuated lattice (or* t*–lattice) can be embedded into a c-involutive residuated lo-monoid, and if the original algebra is contractive then the containing algebra can be chosen square increasing.*

Theorem 7.7 and its proof remain intact if we extend Table 1 as follows.

$\mathbf{R\text{-}W}_{\to,\,\wedge,\,\vee,\,\mathsf{t}}$	distributive pre-residuated t–lattices
$\mathbf{R\text{-}W}_{\to,\,\wedge,\,\vee}$	distributive pre-residuated lattices
$\mathbf{R}_{\to,\,\wedge,\,\vee,\,\mathsf{t}}$	contractive distributive pre-residuated t–lattices
$\mathbf{R}_{\to,\,\wedge,\,\vee}$	contractive distributive pre-residuated lattices

Although the construction in [19] that establishes Theorem 4.5 preserves distributivity, the one used in [41] does not. Therefore, a different argument will be needed to show that

Theorem 9.6. *Every distributive pre-residuated lattice (or* t*–lattice) can be embedded into a distributive residuated lo-monoid in such a way that if the original algebra is contractive then the containing algebra is square increasing.*

The proof of Theorem 9.6 requires some work. The next result allows us to confine the problem to the case of t–lattices.

Theorem 9.7. *Every distributive pre-residuated lattice can be embedded into a distributive pre-residuated* t*–lattice in such a way that contraction is preserved.*

Proof. The proof is similar to that of Theorem 4.9. Let $\boldsymbol{A}$ be a distributive pre-residuated lattice. If $\boldsymbol{A}$ is finitely generated then, by Theorem 4.6, $\boldsymbol{A}$ has an element t_G such that $\mathsf{t}_G \to a = a$ for all $a \in A$. In this case, by Lemma 9.2, $\langle A; \to, \wedge, \vee, \mathsf{t}_G \rangle$ is already a distributive pre-residuated t–lattice, trivially contractive if $\boldsymbol{A}$ was. And if $\boldsymbol{A}$ is not finitely generated, we can imitate the ultraproduct argument in the proof of Theorem 4.9. ☐

So, for the rest of this section, $\boldsymbol{A} = \langle A; \to, \wedge, \vee, \mathsf{t} \rangle$ is assumed to be a distributive pre-residuated t–lattice and $\boldsymbol{B} = \langle B; \cdot, \to, \wedge, \vee, \mathsf{t} \rangle$ a residuated lo-monoid (not assumed distributive) into which $\boldsymbol{A}$ can be embedded. By Corollary 9.5, we may assume that $\boldsymbol{B}$ is square increasing if $\boldsymbol{A}$ is contractive.

The proof of Theorem 9.6 given below is suggested by results of Routley and Meyer from [36]. The distributivity of $\boldsymbol{A}$ will not be used until Lemma 9.10.

By a *filter* of $\boldsymbol{A}$ we shall mean a lattice-filter of the $\{\wedge, \vee\}$–reduct of $\boldsymbol{A}$, i.e., a subset F of A that is upward closed and closed under meets. (We allow filters to be empty.) The set $\mathrm{Fil}(\boldsymbol{A})$ of all filters of $\boldsymbol{A}$ is closed under arbitrary intersections and so becomes a complete lattice when ordered by set inclusion. The least filter containing a subset X of A is denoted by $\mathrm{Fg}(X)$. Thus,

$$\mathrm{Fg}(X) \;=\; \{a \in A : a \geq x_1 \wedge \cdots \wedge x_n \text{ for some } x_1, \ldots, x_n \in X, \text{ where } n > 0\}.$$

In particular, if $x \in A$ then $\mathrm{Fg}(\{x\}) = [\,x\,)$. We may conclude:

Lemma 9.8. *If $\emptyset \neq F \in \mathrm{Fil}(\boldsymbol{A})$ and $x \in A$ then*

$$\mathrm{Fg}(F \cup \{x\}) \;=\; \{a \in A : a \geq f \wedge x \textit{ for some } f \in F\}.$$

For $F, G \in \mathrm{Fil}(\boldsymbol{A})$, we define

$$F \circ G := \{a \in A : f \leq g \to a \text{ for some } f \in F \text{ and some } g \in G\}.$$

Notice that Definition 6.7 could have been phrased this way, in view of the definition of residuation. Indeed, for any $a \in A$, we have $a \in F \circ G$ iff there exist $f \in F$ and $g \in G$ such that $a \geq g \cdot f$ $(= f \cdot g)$ *in* $\boldsymbol{B}$. Working in $\boldsymbol{B}$ we see immediately that $F \circ G$ is upward closed in A. Also, if $f \cdot g \leq a$ and $f' \cdot g' \leq b$, where $f, f' \in F$, $g, g' \in G$ and $a, b \in A$, then in $\boldsymbol{B}$,

$$(f \wedge f') \cdot (g \wedge g') \;\leq\; (f \cdot g) \wedge (f' \cdot g') \;\leq\; a \wedge b.$$

So $F \circ G$ is closed under $\wedge$, whence $\circ$ is a binary operation on $\mathrm{Fil}(\boldsymbol{A})$.

Lemma 9.9. *$\langle \mathrm{Fil}(\boldsymbol{A}); \circ, [\,\mathsf{t}\,), \subseteq \rangle$ is a po-monoid, and if $\boldsymbol{A}$ is contractive then $F \circ F \subseteq F$ for all $F \in \mathrm{Fil}(\boldsymbol{A})$.*

Proof. Evidently, $\circ$ is commutative and $\subseteq$–preserving on $\mathrm{Fil}(\boldsymbol{A})$ and, working again in $\boldsymbol{B}$, we see easily that it is associative, with identity $[\,\mathsf{t}\,)$. If $\boldsymbol{A}$ is contractive then $\boldsymbol{B}$ may be assumed square increasing, hence $\boldsymbol{B}$ satisfies

$$x \wedge y \;\leq\; (x \wedge y) \cdot (x \wedge y) \;\leq\; x \cdot y.$$

Now let $a \in F \circ F$, where $F \in \mathrm{Fil}(\boldsymbol{A})$. Then for some $f, f' \in F$, we have $f \cdot f' \leq a$ in $\boldsymbol{B}$, hence $f \wedge f' \leq a$, so $a \in F$. This shows that $F \circ F \subseteq F$. $\square$

A filter F of $\boldsymbol{A}$ is said to be *prime* if whenever $a, b \in A$ with $a \vee b \in F$ then $a \in F$ or $b \in F$. We shall need the following variant of Lemma 6.11 (which has analogues in [15, 35]):

Lemma 9.10. (Second Primeness Lemma) *Let F, G be filters of $\boldsymbol{A}$ and let P be a prime filter of $\boldsymbol{A}$. If $F \circ G \subseteq P$ then there exists a prime filter F' of $\boldsymbol{A}$ such that $F \subseteq F'$ and $F' \circ G \subseteq P$.*

Proof. By Zorn's Lemma, we may choose an F' that is $\subseteq$–maximal in the set

$$\Sigma \;=\; \{X \in \mathrm{Fil}(\boldsymbol{A}) : X \circ G \subseteq P \;\text{ and }\; F \subseteq X\}.$$

We show that F' is prime. If not then there exist $a, b \in A \setminus F'$ with $a \vee b \in F'$. So $\mathrm{Fg}(F' \cup \{a\}) \circ G \not\subseteq P$ and $\mathrm{Fg}(F' \cup \{b\}) \circ G \not\subseteq P$, by the maximality of F', i.e., there exist $e \in \mathrm{Fg}(F' \cup \{a\})$, $e' \in \mathrm{Fg}(F' \cup \{b\})$, $g, g' \in G$ and $k, k' \in A \setminus P$ with $e \leq g \to k$ and $e' \leq g' \to k'$. In $\boldsymbol{B}$ we have $g \cdot e \leq k$ and $g' \cdot e' \leq k'$, so

$$(g \wedge g') \cdot (e \vee e') = ((g \wedge g') \cdot e) \vee ((g \wedge g') \cdot e') \leq (g \cdot e) \vee (g' \cdot e') \leq k \vee k',$$

whence, in $\boldsymbol{A}$,

$$e \vee e' \leq (g \wedge g') \to (k \vee k'). \tag{29}$$

Recall that $F' \neq \emptyset$, as $a \vee b \in F'$. Since $e \in \mathrm{Fg}(F' \cup \{a\})$ and $e' \in \mathrm{Fg}(F' \cup \{b\})$, Lemma 9.8 shows that there exist $f, f' \in F'$ with $e \geq f \wedge a$ and $e' \geq f' \wedge b$. Then, by distributivity,

$$f \wedge f' \wedge (a \vee b) = (f \wedge f' \wedge a) \vee (f \wedge f' \wedge b) \leq (f \wedge a) \vee (f' \wedge b) \leq e \vee e',$$

whence $f \wedge f' \wedge (a \vee b) \leq (g \wedge g') \to (k \vee k')$, by (29). Since $f \wedge f' \wedge (a \vee b) \in F'$ and $g \wedge g' \in G$, it follows that $k \vee k' \in F' \circ G$. Thus $k \vee k' \in P$, because $F' \circ G \subseteq P$. But $k, k' \notin P$, so this contradicts the primeness of P. ☐

We use $\mathrm{PFil}(\boldsymbol{A})$ to denote the set of all prime filters of $\boldsymbol{A}$, and $M(\boldsymbol{A})$ to denote the set of all upward closed subsets of $\langle \mathrm{PFil}(\boldsymbol{A}); \subseteq \rangle$. For $\alpha, \beta \in M(\boldsymbol{A})$ and $x \in A$, we define $\alpha \cdot \beta$, $\alpha \to \beta$ and $h(x)$ just as in Definition 6.16, except that we replace $\mathrm{PIFil}(\boldsymbol{A})$ by $\mathrm{PFil}(\boldsymbol{A})$ throughout. It is easy to see that $M(\boldsymbol{A})$ is closed under $\cdot$, $\to$ and the intersections and unions of arbitrary subsets, and that if $x \in A$ then $h(x) \in M(\boldsymbol{A})$.

Let $\boldsymbol{M}(\boldsymbol{A}) = \langle M(\boldsymbol{A}); \cdot, \to, \cap, \cup, h(\mathsf{t}) \rangle$.

Lemma 9.11. *$\boldsymbol{M}(\boldsymbol{A})$ is a distributive residuated lo-monoid. If $\boldsymbol{A}$ is contractive then $\boldsymbol{M}(\boldsymbol{A})$ is square increasing.*

The proof is similar to that of Lemma 6.18, but $\neg$ is not involved, and we use the second Primeness Lemma 9.10 in the proof, instead of the first.

Lemma 9.12. *An injective homomorphism from $\boldsymbol{A}$ into the $\{\to, \wedge, \vee, \mathsf{t}\}$–reduct of $\boldsymbol{M}(\boldsymbol{A})$ is defined by $x \mapsto h(x) = \{P \in \mathrm{PFil}(\boldsymbol{A}) : x \in P\}$.*

Proof. The argument that h reflects order (and is therefore injective) is similar to the proof of Lemma 6.19. Where we previously invoked Lemma 6.10, we use the prime filter theorem for distributive lattices instead.

Let $x, y \in A$. Clearly, a filter of $\boldsymbol{A}$ contains $x \wedge y$ iff it contains both x and y, while a prime filter of $\boldsymbol{A}$ contains $x \vee y$ iff it contains at least one of x, y. It follows that $h(x \wedge y) = h(x) \cap h(y)$ and $h(x \vee y) = h(x) \cup h(y)$. By Lemma 9.11, $h(\mathsf{t})$ is the identity of $\boldsymbol{M}(\boldsymbol{A})$, i.e., h preserves t.

It remains only to show that $h(x \to y) = h(x) \to h(y)$. Let $P \in h(x \to y)$, so $x \to y \in P$. Let $Q \in \mathrm{PFil}(\boldsymbol{A})$ and $P_x \in h(x)$, i.e., $x \in P_x$. If $Q \supseteq P_x \circ P$

then, since $x \leq (x \to y) \to y$, we have $y \in P_x \circ P$, hence $y \in Q$, i.e., $Q \in h(y)$. Thus, $P \in h(x) \to h(y)$. Conversely, suppose $P \in h(x) \to h(y)$, so

$$\forall\, Q \in \mathrm{PFil}(\boldsymbol{A}),\ \forall\, P_x \in h(x),\ \text{ if } Q \supseteq P_x \circ P \text{ then } Q \in h(y). \tag{30}$$

We must show that $P \in h(x \to y)$, i.e., that $x \to y \in P$. We claim that $y \in [\,x\,) \circ P$. Indeed, suppose $y \notin [\,x\,) \circ P$. Since $\langle A; \wedge, \vee \rangle$ is distributive, there is a prime filter Q of $\boldsymbol{A}$ with $[\,x\,) \circ P \subseteq Q$ and $y \notin Q$. By the second Primeness Lemma 9.10, there is a prime filter F of $\boldsymbol{A}$ such that $F \circ P \subseteq Q$ and $[\,x\,) \subseteq F$. Then $x \in F$, i.e., $F \in h(x)$. Since $Q \supseteq F \circ P$, it follows from (30) that $Q \in h(y)$, i.e., $y \in Q$, a contradiction.

This vindicates the claim that $y \in [\,x\,) \circ P$. Thus there exist $z \in P$ and $x' \in [\,x\,)$ such that $z \leq x' \to y$. In this case $x \leq x'$, so $x' \to y \leq x \to y$, hence $z \leq x \to y$. Since $z \in P$, it follows that $x \to y \in P$, as required. □

In view of Theorem 9.7, the last two lemmas deliver Theorem 9.6. Applying Theorem 4.5, we get the next result, finishing the proof of Theorem 8.4.

Theorem 9.13. *Every distributive pre-residuated lattice (or* t*–lattice) can be embedded into a c–involutive distributive residuated lo-monoid in such a way that if the original algebra is contractive then the containing algebra is square increasing.*

Remark 9.14. Suppose $\boldsymbol{A}$ is an *integral* distributive pre-residuated t–lattice. Just as in the remarks preceding Corollary 5.10, the function h of Lemma 9.12 embeds $\boldsymbol{A}$ into the integral algebra that results from removing the two extreme elements of $\boldsymbol{M}(\boldsymbol{A})$. So Theorem 9.6 remains true if we add the demand that integrality (or integrality and contraction) be preserved by the construction. This completes the proof of item (v) from the introduction.

The following result is proved in [19, Sec. 6]:

Theorem 9.15. *Every residuated lo-monoid can be embedded into a c-involutive residuated lo-monoid that is bounded (as a lattice), by a construction that preserves distributivity as well as the greatest element of the algebra, if this exists.*

The construction mentioned here does not preserve the square increasing law, unlike the one in Theorem 4.5. Also, it does not preserve least elements, where these exist. But from Remark 9.14 and Theorem 9.15 we obtain

Corollary 9.16. *Every integral distributive pre-residuated* t*–lattice can be embedded into an integral distributive c-involutive residuated lo-monoid.*

Remark 9.17. (BCK-logic) If we add to **CL** the *weakening* axiom $p \to (q \to p)$, we get *BCK-logic*, or *affine* classical linear logic (without exponentials). The models that would correspond to this extension in Table 1 are just the *integral* c-involutive residuated lo-monoids. In these algebras, t is definable as $x \to x$.

The Separation Theorem 8.1 remains true if we replace **CL** by its affine extension. (Adjunction can be replaced by the axiom $\vdash\ p \to (q \to (p \wedge q))$, while (20) could replace (P11).) Much of the content of this result was established by Ono and Komori [34], but the only negation dealt with in [34] is the intuitionistic one, i.e., inferential negation without the double negation axiom. The proof of the separation theorem remains the same, except that Theorems 3.4 and 4.9 are not needed, since a monoid identity is always definable. When using Theorem 5.1, we need the amendment in Corollary 5.10. This amendment and the one in Remark 9.14 (yielding Corollary 9.16) give us a deductive separation theorem for the extension of **R-W** by weakening, excluding signatures with $\vee$ but not $\wedge$ from consideration. (The use of Theorem 9.7 falls away.) No such result holds for **LR**, nor for **R**, because weakening collapses these systems to classical logic, while it collapses their positive fragments only to positive intuitionistic logic.

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A FINITE MODEL PROPERTY FOR $\mathbf{RMI}_{\min}$

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Abstract. It is proved that the variety of relevant disjunction lattices has the finite embeddability property. It follows that Avron's relevance logic $\mathbf{RMI}_{\min}$ has a strong form of the finite model property, so it has a solvable deducibility problem. This strengthens Avron's result that $\mathbf{RMI}_{\min}$ is decidable.

1. Introduction

A formal system is said to have the *variable sharing property* for a connective $\to$ provided that in all theorems of the form $\varphi \to \psi$, at least one variable occurs both in φ and in ψ. This property serves as a measure of reliability in relevance logic. Anderson and Belnap's system $\mathbf{R}$ satisfies this criterion [1, pp. 254, 417] [1], but it turns out that $\mathbf{R}$ is undecidable [20]. Its extension $\mathbf{RM}$ is much more intelligible—in particular its deducibility problem is solvable [12]—but $\mathbf{RM}$ lacks the variable sharing property [1, pp. 148, 397].

In [2, 5] Avron introduced, among other systems, a simply axiomatized relevance logic $\mathbf{RMI}_{\min}$, which has many of the desirable features of $\mathbf{RM}$, including the *mingle* axiom $p \to (p \to p)$. He proved that $\mathbf{RMI}_{\min}$ is decidable *and* has the variable sharing property for $\to$.

The proof of decidability for $\mathbf{RMI}_{\min}$ is given in [6], and it proceeds by cut elimination in a suitable Gentzen system. As usual with such arguments, no light is shed on the effectiveness of *deducibility*. Moreover, $\mathbf{RMI}_{\min}$ has (demonstrably) no deduction theorem of the kind that transforms deducibility problems into decision problems. Another question not addressed in [2]–[7] is

Key words. Relevance logic, mingle, residuation, relevant disjunction lattice, finite embeddability property, finite model property, deducibility problems.

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[1] Here we refer to the original formulation of $\mathbf{R}$, without the constant $\mathtt{t}$, i.e., we mean the system called $\mathbf{R}_{\boldsymbol{\cdot},\to,\wedge,\vee,\neg}$ in Paper 1.

whether $\mathbf{RMI}_{\min}$ has the *finite model property* with respect to its equivalent algebraic semantics—the variety RDL of all *relevant disjunction lattices.* In other words, can every non-theorem of $\mathbf{RMI}_{\min}$ be refuted in a *finite* relevant disjunction lattice?

Building on Avron's work, we shall prove here that RDL has the *finite embeddability property*—i.e., every finite subset of a relevant disjunction lattice may be extended to a *finite* relevant disjunction lattice with preservation of all partial operations. This implies a *strong* form of the finite model property for $\mathbf{RMI}_{\min}$, which allows us to conclude that $\mathbf{RMI}_{\min}$ has a solvable deducibility problem, i.e., its set of finite derivable rules is recursive.

2. The Formal Systems

We use $\mathbf{RMI}_{\to,\neg}$ to denote the following formal system, whose signature is $\{\to,\neg\}$. We adopt the convention that $\neg$ binds more strongly than any other connective to be discussed, e.g., $\neg p \to q$ abbreviates $(\neg p) \to q$.

(B)	$\vdash (p \to q) \to ((r \to p) \to (r \to q))$	(*prefixing*)
(C)	$\vdash (p \to (q \to r)) \to (q \to (p \to r))$	(*exchange*)
(W)	$\vdash (p \to (p \to q)) \to (p \to q)$	(*contraction*)
(M)	$\vdash p \to (p \to p)$	(*mingle*)
(CP)	$\vdash (p \to \neg q) \to (q \to \neg p)$	(*contraposition*)
(DN)	$\vdash \neg\neg p \to p$	(*double negation*)
(MP)	$p,\ p \to q \vdash q$	(*modus ponens*)

Using (M), (W) and (MP), we can prove that $\vdash_{\mathbf{RMI}_{\to,\neg}} p \to p$. So we may view $\mathbf{RMI}_{\to,\neg}$ as the system got by adding the *mingle* axiom (M) to the implication-negation fragment of the principal relevance logic $\mathbf{R}$. Like all relevance logics, $\mathbf{RMI}_{\to,\neg}$ rejects the *weakening* postulate, i.e.,

$$\nvdash_{\mathbf{RMI}_{\to,\neg}} p \to (q \to p).$$

We denote by $\mathbf{RMI}_{\min}$ the formal system with signature $\{\to,\wedge,\neg\}$ got by adding the following postulates to $\mathbf{RMI}_{\to,\neg}$.

($\wedge$1)	$\vdash (p \wedge q) \to p$	
($\wedge$2)	$\vdash (p \wedge q) \to q$	
(RAd)	$p \to q,\ p \to r \vdash p \to (q \wedge r)$	(*rule addition*)

This system was introduced by Avron, who proved in [4, 5] that $\mathbf{RMI}_{\min}$ is a *strongly conservative* extension of $\mathbf{RMI}_{\to,\neg}$. In other words, every derivable rule of $\mathbf{RMI}_{\text{min}}$ whose statement uses nothing but $\to$, $\neg$ and variables is already derivable in $\mathbf{RMI}_{\to,\neg}$.

We are using 'derivable' in its traditional sense here. To be clear about this: for any formal system $\mathbf{F}$, the notation $\Gamma \vdash_{\mathbf{F}} \alpha$ signifies that there is a *proof* of α from Γ in $\mathbf{F}$. A proof of this kind is understood to be any finite sequence of formulas terminating with α, such that each item in the sequence belongs to Γ or is a substitution instance of a formula that is either an axiom of $\mathbf{F}$ or the conclusion of an inference rule of $\mathbf{F}$, where, in the last case, the same substitution turns the premisses of the rule into previous items in the sequence. When this is true we call the string of symbols $\Gamma \vdash \alpha$ a *derivable rule* of $\mathbf{F}$. So the *theorems* of $\mathbf{F}$ are just the derivable rules with an empty set of premisses.

Avron proved in [6] that $\mathbf{RMI}_{\text{min}}$ has the *interpolation property*, i.e., whenever $\vdash_{\mathbf{RMI}_{\text{min}}} \varphi \to \psi$ then $\vdash_{\mathbf{RMI}_{\text{min}}} \varphi \to \eta$ and $\vdash_{\mathbf{RMI}_{\text{min}}} \eta \to \psi$ for some formula η whose variables all occur both in φ and in ψ. This implies the variable sharing property for $\to$, defined in the Introduction.

As in [5], it can be shown that:

$$\vdash_{\mathbf{RMI}_{\text{min}}} ((p \to q) \wedge (p \to r)) \to (p \to (q \wedge r)) \qquad (\textit{addition})$$
$$p \to q,\, q \to p \ \vdash_{\mathbf{RMI}_{\text{min}}} (p \to q) \wedge (q \to p)$$

The converse of *addition* is also a theorem of $\mathbf{RMI}_{\text{min}}$. However, the full *adjunction* rule is *not* derivable in $\mathbf{RMI}_{\text{min}}$, i.e.,

$$p,\, q \ \nvdash_{\mathbf{RMI}_{\text{min}}} p \wedge q\,,$$

and this is one of the system's most important features. A related fact is that $\mathbf{RMI}_{\text{min}}$ cannot be extended conservatively by the Ackermann truth constant t with its characteristic axioms $\vdash \mathsf{t}$ and $\vdash \mathsf{t} \to (p \to p)$, as this would make adjunction derivable. Even in $\mathbf{RMI}_{\to,\neg}$, the t–postulates would induce new *theorems* in the t–free vocabulary, such as $\neg(p \to p) \to (q \to q)$. (This formula is a theorem of $\mathbf{RM}$ but not of $\mathbf{RMI}_{\text{min}}$, nor of $\mathbf{R}$. It shows that $\mathbf{RM}$ lacks the variable sharing property for $\to$.)

If we define disjunction and *fusion* $(\cdot)$ by

$$p \vee q := \neg(\neg p \wedge \neg q) \qquad \text{and} \qquad p \cdot q := \neg(p \to \neg q)$$

then, as shown in [5],

$$\vdash_{\mathbf{RMI}_{\text{min}}} p \to (p \vee q)$$
$$\vdash_{\mathbf{RMI}_{\text{min}}} q \to (p \vee q)$$
$$\vdash_{\mathbf{RMI}_{\text{min}}} ((p \to r) \wedge (q \to r)) \to ((p \vee q) \to r)$$

(F1) $\quad \vdash_{\mathbf{RMI}_{\text{min}}} p \to (q \to (q \cdot p))$

(F2) $\vdash_{\mathbf{RMI}_{\min}} \; (p \to (q \to r)) \;\to\; ((q \cdot p) \to r)\,.$

We could have formulated $\mathbf{RMI}_{\min}$ with $\vee$ and $\cdot$ as primitive connectives, provided the above five formulas were added as axioms. Similarly, fusion can be taken primitive in $\mathbf{RMI}_{\to,\neg}$, with axioms (F1) and (F2). In the algebraic analysis of these systems it is convenient to take the signatures of $\mathbf{RMI}_{\to,\neg}$ and $\mathbf{RMI}_{\min}$ to be $\{\cdot, \to, \neg\}$ and $\{\cdot, \to, \wedge, \vee, \neg\}$, respectively.

3. Residuated Structures

The equivalent algebraic models for $\mathbf{RMI}_{\to,\neg}$ and $\mathbf{RMI}_{\min}$ were identified in [4, 5]. They are special residuated structures with a unary operator $\neg$ that represents negation. We shall define these structures in a slightly more economical way, using a formalism dual to [4, 5] and giving more emphasis to residuation than to properties of $\neg$. This makes it easier to compare the structures with the models of neighbouring systems like intuitionistic linear logic.

Definition 3.1. By a (commutative) *residuated po-semigroup* we shall mean a structure $\langle A; \cdot, \to, \leq \rangle$ such that

(a1) $\langle A; \leq \rangle$ is a partially ordered set, with $A \neq \emptyset$;
(a2) $\cdot$ is a binary operation on A that is associative and commutative ;
(a3) $\to$ is a binary operation on A such that for all $a, b, c \in A$,

$$c \leq a \to b \quad \text{iff} \quad a \cdot c \leq b\,;$$

in particular, $a \cdot (a \to b) \leq b\,.$

Note that the semigroup $\langle A; \cdot \rangle$ in the definition is not assumed to be a monoid, i.e., $\cdot$ need not have an identity element in A. This is essential for the applications to follow. We summarize some well known consequences of the definition.

Proposition 3.2. *Every residuated po-semigroup satisfies*:

(i) $x \leq y \to (y \cdot x)$
(ii) $x \leq y \implies z \cdot x \leq z \cdot y$
(iii) $x \leq y \implies z \to x \leq z \to y \;\;\&\;\; y \to z \leq x \to z$
(iv) $x \leq y \to z \iff y \leq x \to z$
(v) $(x \cdot y) \to z \approx y \to (x \to z) \approx x \to (y \to z)$

Condition (ii) says that $\leq$ is *compatible* with $\cdot$ in a residuated po-semigroup. To show that a commutative semigroup with a compatible partial order is the $\{\cdot, \leq\}$–reduct of a residuated po-semigroup, it is enough to verify that it is *residuated* in the sense that for any x, y, there is always a *largest* z with the property that $x \cdot z \leq y$. (The largest such z becomes $x \to y$.)

An element a of a residuated po-semigroup $\boldsymbol{A}$ is said to be *idempotent* if $a \cdot a = a$. We say that $\boldsymbol{A}$ is *idempotent* if all of its elements are.

Lemma 3.3. *Let $\langle A; \cdot, \to, \leq \rangle$ be an idempotent residuated po-semigroup. Then $a \cdot (a \to a) = a$ for all $a \in A$.*

Proof. Let $a \in A$. Applying (a3) to $a \cdot a = a$, we get $a \leq a \to a$. Then $a = a \cdot a \leq a \cdot (a \to a) \leq a$, so $a \cdot (a \to a) = a$. $\square$

4. Negation and Involution

For any set A, a function $\neg : A \to A$ is said to be *self-inverting* if $\neg\neg a = a$ for all $a \in A$. In this case $\neg$ is obviously a bijection.

Lemma 4.1. *Let $\langle A; \cdot, \to, \leq \rangle$ be a residuated po-semigroup and $\neg : A \to A$ a function. Then the following conditions are equivalent.*

(i) *For all $a, b, c \in A$, we have $a \cdot c \leq b$ iff $\neg b \cdot c \leq \neg a$.*
(ii) *$a \to b = \neg b \to \neg a$ for all $a, b \in A$.*

If $\neg$ is also self-inverting then these conditions are equivalent to

(iii) *$a \to \neg b = b \to \neg a$ for all $a, b \in A$.*

Proof. By (a3), the quasi-equation $x \cdot z \leq y \iff \neg y \cdot z \leq \neg x$ is equivalent to $z \leq x \to y \iff z \leq \neg y \to \neg x$ in any residuated po-semigroup. This in turn is equivalent to the law $x \to y \approx \neg y \to \neg x$, because $\leq$ is reflexive and anti-symmetric. Now (ii) and (iii) are interchangeable when $\neg$ is self-inverting. $\square$

Definition 4.2. A self-inverting unary operation on (the universe of) a residuated po-semigroup will be called a *negation* if it satisfies the equivalent conditions of Lemma 4.1[2].

Definition 4.3. An *involution* of a partially ordered set $\langle A; \leq \rangle$ is a self-inverting function $\neg : A \to A$ that is *order reversing* in the sense that whenever $a, b \in A$ with $a \leq b$ then $\neg b \leq \neg a$.

Lemma 4.4. *Let $\boldsymbol{A} = \langle A; \cdot, \to, \leq \rangle$ be a residuated po-semigroup satisfying $x \cdot (x \to x) \approx x$. Then any negation on $\boldsymbol{A}$ is an involution of $\langle A; \leq \rangle$.*

Proof. Let $\neg$ be a negation on $\boldsymbol{A}$, and suppose $a, b \in A$ with $a \leq b$. Then $a \cdot (\neg b \to \neg b) = a \cdot (b \to b) \leq b \cdot (b \to b) \leq b$, so $\neg b \cdot (\neg b \to \neg b) \leq \neg a$ (as $\neg$ is a negation). But by assumption, $\neg b = \neg b \cdot (\neg b \to \neg b)$, so $\neg b \leq \neg a$. $\square$

Combining Lemmas 3.3 and 4.4, we have:

[2] i.e., if it is compatible with $\cdot$ in the sense of Paper 1.

Corollary 4.5. *On an idempotent residuated po-semigroup, any negation is an involution.*

Lemma 4.6. *Let $\langle A; \cdot, \rightarrow, \leq \rangle$ be a residuated po-semigroup and $\neg$ an involution of $\langle A; \leq \rangle$. Then the following conditions are equivalent.*

(i) $\neg$ *is a negation.*
(ii) $a \cdot b \;=\; \neg\,(a \rightarrow \neg\,b)$ *for all* $a, b \in A$.
(iii) $a \rightarrow b \;=\; \neg\,(a \cdot \neg\,b)$ *for all* $a, b \in A$.

Proof. Suppose that (i) holds and let $a, b \in A$. Since $\neg\,\neg\,(a \rightarrow \neg\,b) \cdot a = a \cdot (a \rightarrow \neg\,b) \leq \neg\,b$ and $\neg$ is a negation, $b \cdot a \leq \neg\,(a \rightarrow \neg\,b)$. Also, from $b \cdot a \leq b \cdot a$ and negation properties, we get $\neg\,(b \cdot a) \cdot a \leq \neg\,b$, i.e., $\neg\,(b \cdot a) \leq a \rightarrow \neg\,b$. Since $\neg$ is also an involution, we may infer that $\neg\,(a \rightarrow \neg\,b) \leq b \cdot a$, hence $\neg\,(a \rightarrow \neg\,b) = b \cdot a = a \cdot b$. We have shown that (i) implies (ii).

If (ii) holds then, by the commutativity of $\cdot$ and the self-inversion law, $a \rightarrow \neg\,b = b \rightarrow \neg\,a$ for all $a, b \in A$, i.e., (i) is true. Finally, each of the equations in (ii) and (iii) is an instance of the other, modulo the self-inversion law. ☐

Lemma 4.7. *Let $\neg$ be a negation on an idempotent residuated po-semigroup $\langle A; \cdot, \rightarrow, \leq \rangle$. Then* $\neg\,a \rightarrow a \;=\; a$ *for all* $a \in A$.

Proof. By Corollary 4.5, $\neg$ is an involution of $\langle A; \leq \rangle$. Let $a \in A$. By idempotence and Lemma 4.6, $\neg\,a = \neg\,a \cdot \neg\,a = \neg\,(\neg\,a \rightarrow a)$, hence $a = \neg\,a \rightarrow a$. ☐

5. Implicative Po-Semigroups

Notation. From now on, $|\,x\,|$ shall abbreviate $x \rightarrow x$.

Definition 5.1. An *implicative po-semigroup* is a residuated po-semigroup $\langle A; \cdot, \rightarrow, \leq \rangle$ such that $|\,a\,| \rightarrow b \leq b$ for all $a, b \in A$.

The significance of this notion will emerge in Theorem 5.5. First, we need:

Lemma 5.2. *A residuated po-semigroup $\boldsymbol{A} = \langle A; \cdot, \rightarrow, \leq \rangle$ is implicative iff $a \leq a \cdot |\,b\,|$ for all $a, b \in A$. In this case, $a \cdot |\,a\,| = a$ for all $a \in A$.*

Proof. Let $a, b \in A$. Then $a \leq |\,b\,| \rightarrow (|\,b\,| \cdot a)$, by Proposition 3.2(i). If $\boldsymbol{A}$ is implicative then $|\,b\,| \rightarrow (|\,b\,| \cdot a) \leq |\,b\,| \cdot a = a \cdot |\,b\,|$, so $a \leq a \cdot |\,b\,|$. Conversely, if $\boldsymbol{A}$ satisfies $x \leq x \cdot |\,y\,|$, i.e., $x \leq |\,y\,| \cdot x$, then $|\,a\,| \rightarrow b \leq |\,a\,| \cdot (|\,a\,| \rightarrow b) \leq b$, so $\boldsymbol{A}$ is implicative.

If $\boldsymbol{A}$ is implicative then, as we have just shown, $a \leq a \cdot |\,a\,|$. But the reverse inequality $a \cdot |\,a\,| \leq a$ follows from (a3), so $a \cdot |\,a\,| = a$. ☐

The last assertion of Lemma 5.2 combines with Lemma 4.4 to give:

Corollary 5.3. *On an implicative po-semigroup, any negation is an involution.*

Definition 5.4. An element a of a residuated po-semigroup will be called a *designated* element if $|\,a\,| \leq a$.

The next result serves to motivate Definition 5.1. It is a $\neg$–free formulation of [4, Prop. I.3]. (See [3] also.)

Theorem 5.5. *A residuated po-semigroup $\langle A; \cdot, \rightarrow, \leq \rangle$ is implicative iff both of the following two conditions hold for all $a, b \in A$.*

(i) *$a \leq b$ iff $a \rightarrow b$ is designated.*
(ii) *If a is designated and $a \leq b$ then b is designated.*

In this case, $|\,a\,|$ is designated for all $a \in A$.

Proof. $(\Rightarrow)$: Suppose $\langle A; \cdot, \rightarrow, \leq \rangle$ is implicative, and let $a, b \in A$. We must prove statements (i) and (ii).

(i) Suppose $a \leq b$. We must show that $|\,a \rightarrow b\,| \leq a \rightarrow b$. From $a \leq b$ we infer $|\,a\,| = a \rightarrow a \leq a \rightarrow b$, because $\rightarrow$ preserves order in its second argument (Proposition 3.2(iii)). Then, because $\rightarrow$ reverses order in its first argument,

$$|\,a \rightarrow b\,| = (a \rightarrow b) \rightarrow (a \rightarrow b) \leq |\,a\,| \rightarrow (a \rightarrow b) \leq a \rightarrow b\,,$$

using implicativity for the last inequality. Conversely, if $|\,a \rightarrow b\,| \leq a \rightarrow b$ then, by Lemma 5.2, commutativity and order compatibility,

$$a \leq a \cdot |\,a \rightarrow b\,| \leq a \cdot (a \rightarrow b) \leq b.$$

(ii) Assume $|\,a\,| \leq a$ and $a \leq b$, hence $|\,a\,| \leq b$. We must prove that $|\,b\,| \leq b$. Using Lemma 5.2, order compatibility and commutativity, we may verify this as follows: $|\,b\,| \leq |\,b\,| \cdot |\,a\,| \leq |\,b\,| \cdot b = b \cdot |\,b\,| = b$.

$(\Leftarrow)$: Assume that (i) and (ii) hold, and let $a, b \in A$. Since $b \leq b$, it follows from (i) that $|\,b\,|$ is designated. Now $|\,b\,| \leq a \rightarrow (a \cdot |\,b\,|)$, by Proposition 3.2(i). So by condition (ii) of the present theorem, $a \rightarrow (a \cdot |\,b\,|)$ is designated. Then by condition (i), $a \leq a \cdot |\,b\,|$. Thus $\langle A; \cdot, \rightarrow, \leq \rangle$ is implicative, by Lemma 5.2. $\square$

In [16], Meyer proved that a residuated po-semigroup is implicative iff it can be embedded into a residuated po-*monoid*, i.e., into a residuated po-semigroup that has an identity element for $\cdot$. An 'embedding' is understood here to preserve all operations and to preserve and reflect the relation $\leq$. Meyer's result would fail if we added lattice operations to the signature of residuated po-semigroups; see [14]. (The additional axiom $x \leq x \cdot x$ is assumed throughout [16], but it is not needed in the proof of the embedding theorem.)

Proposition 5.6. *Let $\langle A; \cdot, \rightarrow, \leq \rangle$ be an implicative po-semigroup in which every designated element is idempotent. Then $\leq$ is equationally definable by $x \leq y \iff |\, x \rightarrow y \,| \approx x \rightarrow y$.*

Proof. Let $a \in A$. We claim that a is designated iff $|\, a \,| = a$. The implication from right to left is immediate. Conversely, if a is designated, i.e., if $|\, a \,| \leq a$, then by assumption, $a \cdot a = a$, whence $a \leq a \rightarrow a = |\, a \,|$. So in this case $|\, a \,| = a$, and the claim is true. Now the result follows from Theorem 5.5(i). $\square$

6. Relevant Disjunction Algebras

Definition 6.1. A *relevant disjunction structure* is an idempotent residuated po-semigroup with a negation.

The negation in the definition is necessarily an involution, by Corollary 4.5. Therefore, the structures are definitionally equivalent to ones introduced under the same name by Avron in [4]. The name refers to the derived operation $x + y := \neg x \rightarrow y$, which is often called *relevant disjunction*. In [4], $+$ was taken as primitive in preference to $\cdot$, and properties of $\rightarrow$ were mainly expressed in terms of $+, \neg$. Up to definitional equivalence, the following result is [4, Prop. I.5(1)] (minus the redundant hypothesis that $\neg$ is an involution). The proof below is modeled on Avron's argument.

Theorem 6.2. (Avron) *Every relevant disjunction structure is implicative.*

Proof. Let $\boldsymbol{A} = \langle A; \cdot, \rightarrow, \neg, \leq \rangle$ be a relevant disjunction structure. We have observed that $\boldsymbol{A}$ satisfies $x \leq |\, x \,|$, as a consequence of idempotence. Since $\neg$ is a negation, $|\, \neg x \,| = |\, x \,|$ holds in $\boldsymbol{A}$, so $\boldsymbol{A}$ satisfies

$$x \leq |\, x \,| \quad \text{and} \quad \neg x \leq |\, x \,|. \tag{1}$$

Let $a, b \in A$ and define $c = |\, a \,| \rightarrow b$. We must show that $c \leq b$. By (1), $\neg a \leq |\, a \,|$. Therefore,

$$\neg a \cdot c = \neg a \cdot (|\, a \,| \rightarrow b) \leq |\, a \,| \cdot (|\, a \,| \rightarrow b) \leq b,$$

so $c \leq \neg a \rightarrow b$, by (a3). We may re-write this as

$$c \leq \neg b \rightarrow a\,, \tag{2}$$

since $\neg$ is a negation. Using commutativity and (1), we also get

$$c \cdot a = a \cdot c = a \cdot (|\, a \,| \rightarrow b) \leq |\, a \,| \cdot (|\, a \,| \rightarrow b) \leq b,$$

so $a \leq c \rightarrow b$, by (a3). Since $\rightarrow$ preserves order in its second argument, it follows that $\neg b \rightarrow a \leq \neg b \rightarrow (c \rightarrow b)$. This together with (2) yields $c \leq \neg b \rightarrow (c \rightarrow b)$. By Proposition 3.2(v), this becomes $c \leq c \rightarrow (\neg b \rightarrow b)$.

Then by (a3), $c \cdot c \leq \neg\, b \to b$. But $c \cdot c = c$, by assumption, and $\neg\, b \to b = b$, by Lemma 4.7, so $c \leq b$, as required. □

We could have defined relevant disjunction structures as idempotent residuated po-semigroups with an *involution*, satisfying $x \cdot y \approx \neg\,(x \to \neg\, y)$: see Lemma 4.6. From Theorem 6.2 and Proposition 5.6, we infer:

Corollary 6.3. *The order on a relevant disjunction structure is equationally definable by* $x \leq y \iff |\, x \to y \,| \approx x \to y\,.$

It follows that these structures may be treated as pure algebras. To be precise, we make the following definition.

Definition 6.4. The algebra reduct $\langle A; \cdot, \to, \neg \rangle$ of a relevant disjunction structure $\langle A; \cdot, \to, \neg, \leq \rangle$ will be called a *relevant disjunction algebra.*

The class of all relevant disjunction algebras will be denoted by RDA.

When $\boldsymbol{A}$ denotes an algebra, it will be understood that A denotes its universe. When dealing with algebras, we use the class operator symbols I, H, S, P, $\mathrm{P_S}$ and $\mathrm{P_U}$ to stand, respectively, for closure under isomorphic and homomorphic images, subalgebras, direct and subdirect products, and ultraproducts. Recall that a *variety* is the class of all models of a set of formal *equations* $\alpha \approx \beta$ in a fixed algebraic signature. A *quasivariety* is the class of all models of a set of formal *quasi-equations* $(\,\&_{i<k}\, \alpha_i \approx \beta_i\,) \implies \alpha \approx \beta$ (k finite). Recall also that for any class X of similar algebras, the variety *generated* by X, i.e., the smallest variety containing X, is $\mathrm{HSP}(\mathsf{X})$, while the quasivariety generated by X is $\mathrm{ISPP_U}(\mathsf{X})$ (see for instance [10, Thms. II.9.5, II.11.9, V.2.23, V.2.25]). The variety generated by a quasivariety Q is just its homomorphic closure $\mathrm{H}(\mathsf{Q})$. We abbreviate $\mathrm{HSP}(\{\boldsymbol{A}\})$ as $\mathrm{HSP}(\boldsymbol{A})$, etc.

Using Corollary 6.3, we may express all the defining properties of relevant disjunction structures as quasi-equations in $\cdot, \to, \neg\,$; in particular, the anti-symmetric law for $\leq$ becomes

$$(3) \qquad |\, x \to y \,| \approx x \to y \quad \& \quad |\, y \to x \,| \approx y \to x \quad \implies \quad x \approx y.$$

Therefore, RDA is a quasivariety. We shall see presently that it is not a variety.

Let $\boldsymbol{Fm}$ denote the absolutely free algebra in the signature $\{\cdot, \to, \neg\}$, freely generated by an infinite set of variables. The following strong soundness and completeness theorem for $\mathbf{RMI}_{\to,\neg}$ was proved in [4, 5].

Theorem 6.5. (Avron) *For any set* $\Gamma \cup \{\alpha\}$ *of* $\{\cdot, \to, \neg\}$*–formulas,*

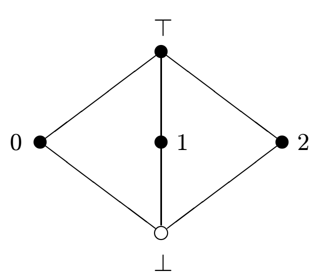
$$(4) \qquad \Gamma \vdash_{\mathbf{RMI}_{\to,\neg}} \alpha \quad \textit{iff} \quad \{|\,\gamma\,| \approx \gamma : \gamma \in \Gamma\} \models_{\mathsf{RDA}} |\,\alpha\,| \approx \alpha.$$

The right hand side of (4) means that for any homomorphism h from $\boldsymbol{Fm}$ into a relevant disjunction algebra, if $h(|\,\gamma\,|) = h(\gamma)$ for all $\gamma \in \Gamma$ then $h(|\,\alpha\,|) = h(\alpha)$. The last clause really means: if $h[\Gamma]$ consists of designated elements then $h(\alpha)$ is designated (because $h(|\,\varphi\,|) = |\,h(\varphi)\,|$ for any formula φ). When Γ is finite, the right hand side of (4) asserts that RDA satisfies a quasi-equation, viz.

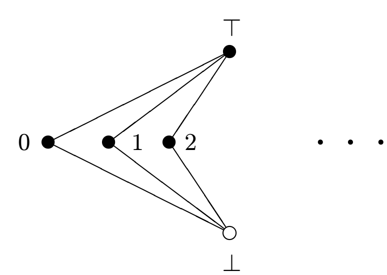
$$\mathsf{RDA} \models\ (\&_{\gamma\in\Gamma}\, |\,\gamma\,| \approx \gamma\,) \implies |\,\alpha\,| \approx \alpha.$$

Example 6.6. Let n be an ordinal, $n \leq \omega$. We identify n with its set of predecessors, i.e., $n = \{m \in \omega : m < n\}$. Let $\bot, \top$ be two distinct non-elements of ω and let $A_n = n \cup \{\bot, \top\}$. We impose a partial order $\leq$ on A_n, defining that $\bot < m < \top$ for all $m \in n$, while distinct elements of n are incomparable. In $A_0 = \{\bot, \top\}$, we define $\bot < \top$.

Let $\cdot$ be the idempotent commutative binary operation on A_n such that $\bot \cdot a = \bot$ for all $a \in A_n$ and $a \cdot b = \top$ whenever a, b are distinct elements of $A_n \setminus \{\bot\}$. This is an associative operation, with which $\leq$ is compatible. We define $\neg\bot = \top$ and $\neg\top = \bot$ and $\neg\, a = a$ for all $a \in n$, so $\neg$ is an involution of $\langle A_n; \leq\rangle$. It is easily verified that $\langle A_n; \cdot, \leq\rangle$ is residuated, and that $a \to \neg\, b = b \to \neg\, a$ for all $a, b \in A_n$, so $\boldsymbol{A}_n = \langle A_n; \cdot, \to, \neg\rangle$ belongs to RDA. Note that $|\,\bot\,| = \top$ and that $|\,a\,| = a$ for all $a \in A_n \setminus \{\bot\}$. The Hasse diagrams of $\boldsymbol{A}_3$ and $\boldsymbol{A}_\omega$ are depicted below. Darkened circles indicate the designated elements.



$\boldsymbol{A}_3$



$\boldsymbol{A}_\omega$

Because $x \to y$ is definable as $\neg\,(x \cdot \neg\, y)$, subalgebras of relevant disjunction algebras are just subsemigroups closed under $\neg$. For each $m < n$, the set $\{m\}$ is a subuniverse of $\boldsymbol{A}_n$. If $m \leq n$ then $\boldsymbol{A}_m$ is a subalgebra of $\boldsymbol{A}_n$. Conversely, every nontrivial subalgebra of $\boldsymbol{A}_n$ is isomorphic to $\boldsymbol{A}_m$ for some $m \leq n$.

In $\boldsymbol{A}_1$, the equivalence relation θ that identifies $\bot$ with $\top$ but not with 0 is a congruence, and $\boldsymbol{A}_1/\theta$ violates (3), so $\boldsymbol{A}_1/\theta \notin \mathsf{RDA}$. Thus RDA is not closed under homomorphic images, so RDA is not a variety.

Avron proved in [4, I.11, II.24] that if a formal equation $\alpha \approx \beta$ is valid in $\boldsymbol{A}_\omega$ then it holds in all relevant disjunction algebras (see [2] also). Since varieties are determined by the equations that they satisfy, this amounts to:

Theorem 6.7. *The variety generated by all relevant disjunction algebras is generated by* $\boldsymbol{A}_\omega$, *i.e.,* $\mathrm{H}(\mathsf{RDA}) = \mathrm{HSP}(\boldsymbol{A}_\omega)$.

On the other hand, $\mathsf{RDA} \neq \mathrm{ISPP_U}(\boldsymbol{A}_\omega)$, i.e., $\boldsymbol{A}_\omega$ does not generate the *quasivariety* RDA; see [4, p. 729].

By an *n–generated* algebra we mean one that is generated by a set having at most n elements. For signatures with no constant symbols, we make the convention that no algebra is '0–generated'. A variety V is said to be *locally finite* if every finitely generated algebra in V is a finite algebra.

In [2, II.7], a finite model property for $\mathbf{RMI}_{\rightarrow,\neg}$ was inferred from Theorems 6.5 and 6.7. The strictly stronger assertion that $\mathrm{H}(\mathsf{RDA})$ is a locally finite variety was not claimed or proved in [2]–[7]. It is essential for the main result of the present paper, so we shall confirm it here. A key fact is the next lemma, which is readily verified.

Lemma 6.8. *Every n–generated subalgebra of* $\boldsymbol{A}_\omega$ *is isomorphic to a subalgebra of* $\boldsymbol{A}_n$.

Now we employ a standard argument. For any variety V, the V–free n–generated algebra *in* V shall be denoted by $\boldsymbol{F}_{\mathsf{V}}(n)$. If $\boldsymbol{F}_{\mathsf{V}}(n)$ is finite for all finite n then V is locally finite, because every n–generated algebra in V is a homomorphic image of $\boldsymbol{F}_{\mathsf{V}}(n)$. Recall that for any class X of similar algebras, $\boldsymbol{F}_{\mathrm{HSP}(\mathsf{X})}(n) \in \mathrm{ISP}(\mathsf{X})$. (See [10, § II.10–11].)

Theorem 6.9. *The variety* $\mathrm{H}(\mathsf{RDA})$ *is locally finite. In particular, every finitely generated relevant disjunction algebra is finite.*

Proof. Let $\boldsymbol{F} = \boldsymbol{F}_{\mathrm{H}(\mathsf{RDA})}(n)$, where n is finite. It suffices to show that $\boldsymbol{F}$ is finite. By Theorem 6.7, $\boldsymbol{F} = \boldsymbol{F}_{\mathrm{HSP}(\boldsymbol{A}_\omega)}(n)$ so, as noted above, $\boldsymbol{F} \in \mathrm{ISP}(\boldsymbol{A}_\omega)$. For every class X of similar algebras, we have $\mathrm{ISP}(\mathsf{X}) = \mathrm{IP_S S}(\mathsf{X})$, so we may assume that $\boldsymbol{F}$ is a subdirect product of subalgebras of $\boldsymbol{A}_\omega$. These subalgebras are n–generated, because they are homomorphic images of $\boldsymbol{F}$. Then by Lemma 6.8, $\boldsymbol{F}$ is a subdirect product of subalgebras of $\boldsymbol{A}_n$.

Let Y be an n–element free generating set for $\boldsymbol{F}$. Every homomorphism from $\boldsymbol{F}$ into $\boldsymbol{A}_n$ is determined by its restriction to Y, but there are only $|\,A_n\,|^{|\,Y\,|} = (n+2)^n$ distinct functions from Y into A_n. Therefore $\boldsymbol{F}$ may be represented as an irredundant subdirect product of subalgebras of $\boldsymbol{A}_n$, indexed by a set I with $|\,I\,| \leq (n+2)^n$. Now $\boldsymbol{F}$ embeds into the direct power ${\boldsymbol{A}_n}^I$, so $\boldsymbol{F}$ is finite, with $|\,F\,| \leq |\,A_n\,|^{|\,I\,|} \leq (n+2)^{((n+2)^n)}$. $\square$

7. Relevant Disjunction Lattices

In general, the order reduct of a relevant disjunction algebra need not be lattice-ordered; see [4, Prop. I.29].

Definition 7.1. An algebra $\langle A; \boldsymbol{\cdot}, \rightarrow, \wedge, \vee, \neg \rangle$ is called a *relevant disjunction lattice* if $\langle A; \wedge, \vee \rangle$ is a lattice and $\langle A; \boldsymbol{\cdot}, \rightarrow, \neg, \leq \rangle$ is a relevant disjunction structure, where $\leq$ denotes the lattice order.

The class of all relevant disjunction lattices will be denoted by RDL.

It was pointed out in [4] that RDL is a variety. The following strong soundness and completeness theorem for $\mathbf{RMI}_{\min}$ was proved in [5].

Theorem 7.2. (Avron) *For any set* $\Gamma \cup \{\alpha\}$ *of* $\{\boldsymbol{\cdot}, \rightarrow, \wedge, \vee, \neg\}$*–formulas,*

$$\Gamma \vdash_{\mathbf{RMI}_{\min}} \alpha \quad \textit{iff} \quad \{|\,\gamma\,| \approx \gamma : \gamma \in \Gamma\} \models_{\mathsf{RDL}} |\,\alpha\,| \approx \alpha.$$

In [4], there are two proofs that every relevant disjunction algebra may be embedded into a relevant disjunction lattice. The first proof, which is sketched briefly and abstractly, relies implicitly on the Dedekind-MacNeille completion procedure, so the embedding preserves all existent meets and joins from the original algebra. This feature of the construction is critical for our application. For the monoid-based algebras of linear logic, analogues of the Dedekind-MacNeille construction have been studied in [3, 17, 18, 19]. We sketch the proof of the embedding theorem below, giving details only where the account from the monoidal case requires modification. Recall that the order $\leq$ on a relevant disjunction algebra $\boldsymbol{A}$ is understood to be defined on A as in Corollary 6.3.

Theorem 7.3. (Avron) *Every relevant disjunction algebra* $\boldsymbol{A}$ *can be embedded into a relevant disjunction lattice* $\boldsymbol{B}$ *in such a way that, for any* $a, b, c \in A$,

if $c = \inf_{\langle A\,;\,\leq\rangle} \{a, b\}$ *then* $c = a \wedge^{\boldsymbol{B}} b\,$;
if $c = \sup_{\langle A\,;\,\leq\rangle} \{a, b\}$ *then* $c = a \vee^{\boldsymbol{B}} b\,$;
if $\boldsymbol{A}$ *is finite then so is* $\boldsymbol{B}$.

Proof. Let $\boldsymbol{A} \in \mathsf{RDA}$. We apply the Dedekind-MacNeille construction to $\langle A; \leq \rangle$. Thus, for any subset X of A, we define

$X^{\rightarrow} = \{a \in A : a \geq x \text{ for all } x \in X\}$ and $X^{\leftarrow} = \{a \in A : a \leq x \text{ for all } x \in X\}$

and $C(X) = X^{\rightarrow\leftarrow}$. A subset X of A will be called *closed* if $C(X) = X$. In this case X is also *downward closed*, i.e., whenever $y \in A$ and $y \leq x \in X$ then $y \in X$. Let B denote the set of all closed subsets of A. Then $|\,B\,| \leq 2^{|\,A\,|}$, so B is finite if A is. Now C a closure operator on subsets of A (i.e., arbitrary intersections of closed sets are closed), so B is a complete lattice with respect to set inclusion, where

$$X \wedge Y = X \cap Y \quad \text{and} \quad X \vee Y = C(X \cup Y), \quad \text{provided } X, Y \in B.$$

The function $h : A \to B$ defined by

$$h(x) \;=\; C(\{x\}) \;\; (=\, (\,x\,] := \{a \in A : a \leq x\}\,)$$

is order preserving and order reflecting (hence injective) and it preserves existent suprema and existent infima from $\langle A; \leq \rangle$.

Proceeding as in [17] or [19, Ch. 8], we define, for subsets X, Y of A,

$$X * Y \;=\; \{x \cdot y : x \in X \text{ and } y \in Y\},$$

and it follows that $C(X) * C(Y) \subseteq C(X * Y)$, hence

$$C(X * Y) \;=\; C(C(X) * C(Y)). \tag{5}$$

Then for any closed sets $X, Y \in B$, we define

$$X \cdot Y \;=\; C(X * Y) \quad \text{and} \quad X \to Y \;=\; \{a \in A : X * \{a\} \subseteq Y\}.$$

Evidently, $\cdot$ is a commutative binary operation on B. It is also associative on B, as a consequence of (5). Further, B is closed under $\to$, and the lattice-ordered semigroup $\langle B; \cdot, \subseteq \rangle$ is residuated, with $\to$ as its residuation operation. Also, h preserves $\cdot$ and $\to$. All of these claims can be proved just as in [17] or [19]. In these sources it is assumed that $\boldsymbol{A}$ has an identity for $\cdot$, but no use is made of that assumption in establishing the present claims.

To see that $\cdot$ is also idempotent, let $X \in B$. For each $x \in X$, the idempotence of $\cdot$ in $\boldsymbol{A}$ gives $x = x \cdot x \in X * X \subseteq X \cdot X$. Conversely, if $x, x' \in X$ and $u \in X^{\to}$ then $x \cdot x' \leq u \cdot u = u$, so $X * X \subseteq X^{\to\leftarrow} = X$, as X is closed. Thus, $X \cdot X = C(X * X) \subseteq X$ and so $X \cdot X = X$, as required.

The treatment of $\neg$ in [17] and [19] *does* rely on the existence of an identity for $\cdot$. But following Avron [4] instead, we may define, for each $X \in B$,

$$\neg X \;=\; \textstyle\bigcap_{x \in X} (\,\neg x\,]\,,$$

i.e., for each $a \in A$, we have

$$a \in \neg X \quad \text{iff} \quad a \leq \neg x \quad \text{for all } x \in X.$$

We always have $\neg X \in B$, because $\neg X$ is an intersection of closed sets.

Let $X, Y \in B$. Since the operation $\neg$ of $\boldsymbol{A}$ is an involution of $\langle A; \leq \rangle$, it is easy to see that $X \subseteq \neg\neg X$. Conversely, let $a \in \neg\neg X$ and let $u \in X^{\to}$. For all $x \in X$, we have $x \leq u$, hence $\neg u \leq \neg x$, i.e., $\neg u \in \neg X$. Since $a \in \neg\neg X$, this implies that $a \leq \neg\neg u = u$. So $a \in X^{\to\leftarrow} = X$ (as X is closed). Thus, $\neg\neg X \subseteq X$, and so $\neg\neg X = X$.

To show that $X \to \neg Y = Y \to \neg X$, let $a \in X \to \neg Y$. This means that for all $x \in X$, we have $x \cdot a \in \neg Y$, i.e., $x \cdot a \leq \neg y$ for all $y \in Y$. Since the operation $\neg$ of $\boldsymbol{A}$ is a negation, for all $y \in Y$, we have $y \cdot a \leq \neg x$ for all $x \in X$, i.e., $Y * \{a\} \subseteq \neg X$, i.e., $a \in Y \to \neg X$. We have shown that $X \to \neg Y \subseteq Y \to \neg X$, and the converse inclusion follows by symmetry. Thus $\neg$ is a negation on $\langle B; \cdot, \to, \subseteq \rangle$, and so $\boldsymbol{B} = \langle B; \cdot, \to, \cap, \vee, \neg \rangle$ is a relevant disjunction lattice. Using only the fact that $\neg$ (on $\boldsymbol{A}$) is an involution, we can easily show that $\neg\,(\,x\,] = (\,\neg\, x\,]$ for all $x \in A$, i.e., h preserves $\neg$. $\square$

8. The Strong Finite Model Property

Definition 8.1. Let K be a class of algebras in a common signature. We say that K has the *finite embeddability property*, or briefly the *FEP*, provided that every finite subset X of an algebra $\boldsymbol{C}$ in K may be extended to a *finite* algebra $\boldsymbol{B}$ in K with preservation of all partial operations. (This means that for any basic operation symbol f of arbitrary rank m, if $a_1, \dots, a_m \in X$ and $f^{\boldsymbol{C}}(a_1, \dots, a_m) \in X$ then $f^{\boldsymbol{B}}(a_1, \dots, a_m) = f^{\boldsymbol{C}}(a_1, \dots, a_m)$.)

Let V be any variety. Clearly, if V is locally finite then it has the FEP. But varieties with the FEP need not be locally finite. Let $\mathsf{V}_{\mathrm{fin}}$ denote the class of all finite algebras in V. If V has the FEP, it is easy to see that a universal sentence of the first order theory (with equality) determined by the signature of V holds in $\mathsf{V}_{\mathrm{fin}}$ only if it holds throughout V. Since quasi-equations are universal sentences and since quasivarieties are determined by the quasi-equations that they satisfy, it follows in this case that $\mathsf{V} = \mathrm{ISPP_U}(\mathsf{V}_{\mathrm{fin}})$. That is, any variety with the FEP is generated as a *quasivariety* by its finite members. (The converse holds if the signature is finite: see for instance [9].) We can now prove the main algebraic result of the present paper:

Theorem 8.2. *The variety of relevant disjunction lattices has the finite embeddability property.*

Proof. Let $\boldsymbol{C} \in \mathsf{RDL}$ and let X be a finite subset of C. Let $\boldsymbol{C}^-$ be the RDA–reduct $\langle C; \cdot, \to, \neg \rangle$ of $\boldsymbol{C}$, and let $\boldsymbol{A}$ be the subalgebra of $\boldsymbol{C}^-$ generated by X, so $\boldsymbol{A} \in \mathsf{RDA}$. Now $\boldsymbol{A}$ is a *finite* relevant disjunction algebra, by Theorem 6.9, because X is finite. The partial operations $\cdot, \to$ and $\neg$ of X are preserved in the passage from X to $\boldsymbol{A}$, by definition of $\boldsymbol{A}$. Note that $\boldsymbol{C}$ and $\boldsymbol{A}$ induce the same order $\leq$ on X, because this order is equationally defined in terms of $\to$, as in Corollary 6.3, and $\to^{\boldsymbol{C}}$ extends $\to^{\boldsymbol{A}}$.

Suppose $x, y \in X$ with $x \vee^{\boldsymbol{C}} y \in X$. Then $x, y \leq x \vee^{\boldsymbol{C}} y \in A$, and if $x, y \leq z \in A$ then $x, y \leq z \in C$, so $x \vee^{\boldsymbol{C}} y \leq z$. This shows that $x \vee^{\boldsymbol{C}} y$ is the join of $\{x, y\}$ in $\langle A; \leq \rangle$. The same applies to meets, so all $\boldsymbol{C}$–induced meets and joins that exist in $\langle X; \leq \rangle$ are preserved in $\langle A; \leq \rangle$.

Now by Theorem 7.3, $\boldsymbol{A}$ can be embedded into a finite relevant disjunction lattice $\boldsymbol{B}$ (because $\boldsymbol{A}$ is finite). This implies that the operations $\cdot, \rightarrow$ and $\neg$ of $\boldsymbol{A}$ are preserved in the passage from $\boldsymbol{A}$ to $\boldsymbol{B}$. But the statement of Theorem 7.3 allows us to choose $\boldsymbol{B}$ so that existent meets and joins in $\langle A; \leq \rangle$ are also preserved in the passage to $\boldsymbol{B}$. So all $\boldsymbol{C}$–induced partial operations on X are preserved in the passage from X to $\boldsymbol{B}$, proving the FEP for RDL. $\square$

Corollary 8.3. $\mathsf{RDL} = \mathbb{ISPP}_{\mathrm{U}}(\mathsf{RDL}_{\mathrm{fin}})$, *i.e.,* RDL *is generated as a quasi-variety by its finite members.*

Although Avron proved in [6] that $\mathbf{RMI}_{\min}$ is decidable, no finite model property for $\mathbf{RMI}_{\min}$ was asserted or proved in [2]–[7]. Corollary 8.3 implies the following 'strong' finite model property:

Corollary 8.4. *For any finite set* $\Gamma \cup \{\alpha\}$ *of* $\{\cdot, \rightarrow, \wedge, \vee, \neg\}$*–formulas, if the rule* $\Gamma \vdash \alpha$ *is not derivable in* $\mathbf{RMI}_{\min}$ *then it is refuted in some* finite *relevant disjunction lattice* $\boldsymbol{B}$, *in the sense that* $\boldsymbol{B}$ *fails to satisfy the quasi-equation*

$$(\&_{\gamma \in \Gamma} \, |\, \gamma \,| \approx \gamma \,) \implies |\, \alpha \,| \approx \alpha \,.$$

Proof. If $\Gamma \nvdash_{\mathbf{RMI}_{\min}} \alpha$ then the displayed quasi-equation fails in some member of RDL, by Theorem 7.2. Then, by Corollary 8.3, the same quasi-equation must fail in some finite algebra in RDL. $\square$

By a well known theorem of Harrop [15], if a formal system is finitely axiomatized in a finite signature and has the strong finite model property with respect to an effective semantics then it has a solvable deducibility problem, i.e., its set of finite derivable rules is recursive. So we may infer from Corollary 8.4 that:

Corollary 8.5. $\mathbf{RMI}_{\min}$ *has a solvable deducibility problem.*

This strengthens Avron's result in [6] that $\mathbf{RMI}_{\min}$ is decidable (i.e., that its set of *theorems* is recursive). The two results are not connected by any deduction theorem. Indeed, $\mathbf{RMI}_{\min}$ demonstrably lacks even a *local* deduction-detachment theorem in the precise general sense of [8, 11, 13]. This can be shown by analogy with [21, Ex. 2].

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SEMICONIC IDEMPOTENT RESIDUATED STRUCTURES

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Abstract. An idempotent residuated po-monoid is *semiconic* if it is a subdirect product of algebras in which the monoid identity is comparable with all other elements. It is proved that the quasivariety SCIP of all semiconic idempotent commutative residuated po-monoids is locally finite. The lattice-ordered members of this class form a variety SCIL, which is not locally finite, but it is proved that SCIL has the finite embeddability property (FEP). More generally, for every relative subvariety K of SCIP, the lattice-ordered members of K have the FEP. This gives a unified explanation of the strong finite model property for a range of logical systems. It is also proved that SCIL has continuously many semisimple subvarieties, and that the involutive algebras in SCIL are subdirect products of chains.

1. Introduction

A class K of similar algebras is said to have the *finite embeddability property* (briefly, the *FEP*) if every finite subset of an algebra in K can be extended to a finite algebra in K, with preservation of all partial operations. If a finitely axiomatized variety or quasivariety of finite type has the FEP, then its universal first order theory is decidable, hence its equational and quasi-equational theories are decidable as well. Where the algebras are residuated ordered groupoids, these theories are often interchangeable with logical systems of independent interest. Partly for this reason, there has been much recent investigation of finiteness properties such as the FEP in varieties of residuated structures.

A residuated partially ordered monoid is said to be *idempotent* if its monoid operation is idempotent. In this case, the partial order is equationally definable, so the structures can be treated as pure algebras. Such an algebra is said to be *conic* if each of its elements lies above or below the monoid identity; it is *semiconic* if it is a subdirect product of conic algebras. In this paper, it is proved that the class SCIP of all semiconic idempotent commutative residuated

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po-monoids is locally finite, i.e., every finitely generated member of this class is a finite algebra. It turns out that SCIP is a quasivariety; it is not a variety.

The lattice-ordered members of SCIP form a variety SCIL, provided that we add the lattice operations $\wedge, \vee$ to the similarity type. This variety is not locally finite, but the local finiteness of SCIP facilitates a proof that SCIL has the FEP. In fact, it is shown here that for every relative subvariety K of SCIP, the lattice-ordered members of K form a variety with the FEP. It is also shown that SCIL has a continuum of semisimple subvarieties.

The variety SCIL contains all Brouwerian lattices, i.e., the algebraic models of positive intuitionistic logic. SCIL also includes all positive Sugihara monoids (cf. [27]); these algebras model the positive fragment of the system **RM**. The results here give a unified explanation of the strong finite model property for many extensions of these and other systems. They partially generalize the main theorem of [30], which showed that the variety generated by all idempotent commutative residuated *chains* is locally finite. Another generalization of that result, in a different direction, has been obtained in [28]. Finally, it is shown here that the *involutive* algebras in SCIL are subdirect products of chains.

Notation 1.1. In a given poset, $[\,a\,)$ denotes the set of all upper bounds of an element a (including a itself), and $(\,a\,]$ the set of all lower bounds. For a subset X of a given poset, $[\,X\,)$ abbreviates $\bigcup_{a\in X}[\,a\,)$, and $(\,X\,] = \bigcup_{a\in X}(\,a\,]$. We say that X is *upward closed* if $[\,X\,) = X$, and *downward closed* if $(\,X\,] = X$.

The class operator symbols H, S, P and $\mathrm{P_U}$ stand for closure under homomorphic images, subalgebras, direct products and ultraproducts, respectively.

2. Residuated Structures, Idempotence and Conicity

A structure $\langle A; \cdot, \rightarrow, \mathsf{t}, \leq \rangle$ is called a *commutative residuated po-monoid* (briefly, a *CRP*) if $\langle A; \leq \rangle$ is a poset, $\langle A; \cdot, \mathsf{t} \rangle$ is a commutative monoid, and $\rightarrow$ is a binary *residuation* operator—which means that for all $a, b, c \in A$, we have

$$c \leq a \rightarrow b \quad \text{iff} \quad a \cdot c \leq b.$$

This residuation law can be stated equivalently as follows: $\leq$ is compatible with $\cdot$ and for every $a, b \in A$, there is a *largest* $c \in A$ with $a \cdot c \leq b$; moreover, the largest such c is $a \rightarrow b$.

The following well-known properties of CRPs will be needed.

Proposition 2.1. *Every CRP satisfies*:

(1) $x \cdot (x \to y) \,\leq\, y$

(2) $x \leq y \implies z \cdot x \,\leq\, z \cdot y$

(3) $x \leq y \implies z \to x \,\leq\, z \to y$ *and* $y \to z \,\leq\, x \to z$

(4) $(x \cdot y) \to z \,\approx\, y \to (x \to z) \,\approx\, x \to (y \to z)$

(5) $x \leq (x \to y) \to y,$ *hence*

(6) $((x \to y) \to y) \to y \,\approx\, x \to y$

(7) $x \leq y \iff \mathsf{t} \leq x \to y,$ *hence*

(8) $\mathsf{t} \leq x \to x$

(9) $\mathsf{t} \leq x \iff x \to x \leq x$

(10) $x \leq y \iff (x \to y) \to (x \to y) \,\leq\, x \to y$

(11) $x \,\approx\, \mathsf{t} \to x \,\approx\, (x \to x) \to x \,\approx\, x \cdot (x \to x)$

(12) $x \to x \,\approx\, (x \to x) \to (x \to x)$

(13) $x \to y \,\leq\, (z \to x) \to (z \to y)$ *and* $x \to y \,\leq\, (y \to z) \to (x \to z)$.

A CRP is said to be *idempotent* if it satisfies $x \cdot x \approx x$. In this case, we must have $a \leq a \to a$ for every element a (because $a \cdot a \leq a$). Thus, the following result is immediate from (9), (10) and (4).

Corollary 2.2. *For any elements a, b of an idempotent CRP, we have*

(14) $\mathsf{t} \leq a$ *iff* $a = a \to a\,;$

(15) $a \leq b$ *iff* $a \to b \,=\, (a \to b) \to (a \to b)\,;$

(16) $a \to (a \to a) = a \to a.$

It follows from (15) that an idempotent CRP $\langle A; \cdot, \to, \mathsf{t}, \leq \rangle$ is definitionally equivalent to its pure algebra reduct $\boldsymbol{A} = \langle A; \cdot, \to, \mathsf{t} \rangle$. So, from now on, we treat these idempotent structures as pure algebras with a definable partial order.

Definition 2.3. A CRP is said to be *conic* if each of its elements a is comparable with t, i.e., $a \leq \mathsf{t}$ or $\mathsf{t} \leq a$.

Although an idempotent CRP need not be lattice-ordered, the next proposition shows that certain meets and joins are forced to exist.

Proposition 2.4. *Let $\boldsymbol{A}$ be an idempotent CRP. Then, for all $a, b \in A$,*

(i) *if* $\mathsf{t} \leq a, b$ *then* $a \cdot b \,=\, a \vee b\,;$
(ii) *if* $a, b \leq \mathsf{t}$ *then* $a \cdot b \,=\, a \wedge b\,;$
(iii) *if* $a \leq b$ *and* $\boldsymbol{A}$ *is conic then* $a \cdot b = a$ *or* $a \cdot b = b$.

Proof. (i) Suppose $\mathsf{t} \leq a, b$. So, $b \leq a \cdot b$ and $a \leq a \cdot b$, by (2). Let $c \in A$ be an upper bound of a, b, i.e., $a, b \leq c$. Then, by (2) and idempotence, we have $a \cdot b \leq c \cdot c = c$. Thus $a \cdot b = a \vee b$.

(ii) is proved similarly.

(iii) Suppose $a \leq b$. If $\boldsymbol{A}$ is conic then, by (i) and (ii), we may assume that $a < \mathsf{t} < b$. In this case, $a \leq a \cdot b \leq b$, by (2). If $a \cdot b \leq \mathsf{t}$, then by (ii) and idempotence, $a \cdot b = a \cdot a \cdot b = a \wedge (a \cdot b) = a$. Otherwise, $\mathsf{t} \leq a \cdot b$, by conicity, and then by (i) and idempotence, $a \cdot b = a \cdot b \cdot b = (a \cdot b) \vee b = b$. □

Notation 2.5. From now on, $|\,x\,|$ shall abbreviate the term $x \to x$.

The following result is essentially taken from [28].

Lemma 2.6. *Let $\boldsymbol{A}$ be a conic idempotent CRP and let $a, b \in A$.*

(i) $a \nleq b$ *if and only if* $a \to b < \mathsf{t}$.
(ii) *If* $|\,a\,| < |\,b\,|$ *then* $a \cdot b = b$.
(iii) *If* $|\,a\,| = |\,b\,|$ *then* $|\,a \cdot b\,| = |\,a \to b\,| = |\,a\,| = |\,b\,|$.
(iv) *If* $|\,a\,| < |\,b\,|$ *then* $a \to b = b$ *and* $b \to a = b \to \mathsf{t}$.
(v) *If* $|\,a\,| \leq b$ *then* $a \to b = b$.

Proof. Detailed proofs of (i), (ii) and (iii) can be found in [28, Lemma 4.8, Thms. 4.10, 5.3].

(iv) Suppose $|\,a\,| < |\,b\,|$. Then $b \neq \mathsf{t}$, because $|\,\mathsf{t}\,| = \mathsf{t} \leq |\,a\,|$, by (11) and (8). By (ii), $a \cdot b = b \leq b$, so $b \leq a \to b$. On the other hand, $|\,a\,| < |\,b\,| \leq |\,a \to b\,|$, by (13). So, $a \to b = a \cdot (a \to b) \leq b$, by (ii) and (1), whence $a \to b = b$. Next, we show that $b \to a = b \to \mathsf{t}$. If $a \leq \mathsf{t}$, then $b \to a \leq b \to \mathsf{t}$, by (3). On the other hand, $b \cdot (b \to \mathsf{t}) \leq \mathsf{t}$. Thus, by (ii) and (2), $b \cdot (b \to \mathsf{t}) = a \cdot b \cdot (b \to \mathsf{t})$ $\leq a \cdot \mathsf{t} = a$. So, $b \to \mathsf{t} \leq b \to a$, whence $b \to a = b \to \mathsf{t}$. We may therefore assume that $\mathsf{t} < a$. Then, by (3), $b \to \mathsf{t} \leq b \to a$. It remains to show that $b \to a \leq b \to \mathsf{t}$. Since $\boldsymbol{A}$ is conic, there are only the following two possibilities: $b < \mathsf{t} < a$ or $\mathsf{t} < a, b$.

If $b < \mathsf{t} < a$, then since $b \cdot (b \to a) \leq a$, it follows from idempotence, (2) and (ii) that $b \cdot (b \to a) = b \cdot b \cdot (b \to a) \leq b \cdot a = b < \mathsf{t}$. So, $b \to a \leq b \to \mathsf{t}$.

If $\mathsf{t} < a, b$, then $a = |\,a\,| < |\,b\,| = b$, by (14), so $b \to a < \mathsf{t}$, by (i). Since $|\,a\,| < |\,b\,| \leq |\,b \to a\,|$ (by (13)), we have $a \cdot (b \to a) = b \to a$, by (ii). It then follows from idempotence, (1) and (2) that

$$b \cdot (b \to a) \;=\; b \cdot (b \to a) \cdot (b \to a) \;\leq\; a \cdot (b \to a) \;=\; b \to a \;<\; \mathsf{t}.$$

So, $b \to a \leq b \to \mathsf{t}$.

(v) Suppose $|\,a\,| \leq b$. If $|\,a\,| < b$, then $b = |\,b\,|$, by (14), so the result is immediate from (iv). If $|\,a\,| = b$, then $a \to b = a \to |\,a\,| = |\,a\,| = b$, by (16). □

Lemma 2.7. *Let* $\boldsymbol{A}$ *be a conic idempotent CRP, and let* $a, b \in A$.

(i) *If* $a \leq \mathsf{t}$, *then* $a \to \mathsf{t} = a \to a$.
(ii) *If* $\mathsf{t} \leq a, b$ *and* $a \not\leq b$, *then* $a \to b = a \to \mathsf{t}$.
(iii) $(a \to \mathsf{t}) \cdot b \leq a \to b$.
(iv) *If* $b \leq \mathsf{t} \leq a$, *then* $a \to b = (a \to \mathsf{t}) \cdot b$ *and* $b \to a = (b \to \mathsf{t}) \cdot a$.
(v) *If* a *and* b *are incomparable, then* $a \to \mathsf{t} = b \to \mathsf{t}$.
(vi) *The elements* a *and* $b \to \mathsf{t}$ *are comparable.*
(vii) *If* $|a|$ *and* $|b|$ *are incomparable, then* $\mathsf{t} < a, b$.

Proof. (i) Suppose $a \leq \mathsf{t}$. Then $a \to a \leq a \to \mathsf{t}$, by (3). For the reverse inequality, it follows from (1) and (8) that $a \cdot (a \to \mathsf{t}) \leq \mathsf{t} \leq a \to a$. Consequently, $a \to \mathsf{t} \leq a \to (a \to a) = a \to a$, by (16).

(ii) Suppose $\mathsf{t} \leq a, b$ and $a \not\leq b$. By Lemma 2.6(i), $a \to b < \mathsf{t} \leq a$. It then follows from Proposition 2.4(iii) that $a \cdot (a \to b) = a$ or $a \cdot (a \to b) = a \to b$. But $a \cdot (a \to b) \leq b$ and $a \not\leq b$, so $a \cdot (a \to b) = a \to b < \mathsf{t}$. It follows that $a \to b \leq a \to \mathsf{t}$. For the reverse inequality, since $\mathsf{t} \leq b$, we have $a \to \mathsf{t} \leq a \to b$, by (3).

(iii) By (1), $a \cdot (a \to \mathsf{t}) \leq \mathsf{t}$. So, $a \cdot (a \to \mathsf{t}) \cdot b \leq \mathsf{t} \cdot b = b$, by (2). It follows that $(a \to \mathsf{t}) \cdot b \leq a \to b$.

(iv) Suppose $b \leq \mathsf{t} \leq a$. By (3) and (11), we have $a \to b \leq a \to \mathsf{t}$ and $a \to b \leq \mathsf{t} \to b = b$. Thus, $a \to b \leq (a \to \mathsf{t}) \cdot b$, by (2) and idempotence. It then follows from (iii) that $a \to b = (a \to \mathsf{t}) \cdot b$.

Since $b \leq \mathsf{t} \leq a$, we have $b \to \mathsf{t} \leq b \to a$ and $a = \mathsf{t} \to a \leq b \to a$, by (3) and (11). We shall show that $b \to a$ is the least upper bound of $b \to \mathsf{t}$ and a. Let c be any upper bound of $b \to \mathsf{t}$ and a in A. So, $|b| = b \to \mathsf{t} \leq c$, using (i). Thus, $b \to c = c$, by Lemma 2.6(v). It then follows from $a \leq c$ and (3) that $b \to a \leq b \to c = c$. So, $b \to a = (b \to \mathsf{t}) \vee a = (b \to \mathsf{t}) \cdot a$, by Proposition 2.4(i).

(v) Suppose a and b are incomparable. So, $a \to b < \mathsf{t}$ and $b \to a < \mathsf{t}$, by Lemma 2.6(i). It then follows from (iii) that $(a \to \mathsf{t}) \cdot b \leq a \to b < \mathsf{t}$ and $(b \to \mathsf{t}) \cdot a \leq b \to a < \mathsf{t}$. Thus, $a \to \mathsf{t} \leq b \to \mathsf{t}$ and $b \to \mathsf{t} \leq a \to \mathsf{t}$, i.e., $a \to \mathsf{t} = b \to \mathsf{t}$.

(vi) Suppose a and $b \to \mathsf{t}$ are incomparable. Then $a \to \mathsf{t} = (b \to \mathsf{t}) \to \mathsf{t}$, by (v). So $(a \to \mathsf{t}) \to \mathsf{t} = ((b \to \mathsf{t}) \to \mathsf{t}) \to \mathsf{t} = b \to \mathsf{t}$, by (6). Thus, a is incomparable with $(a \to \mathsf{t}) \to \mathsf{t}$, contradicting (5).

(vii) If $\mathsf{t} \not< b$ then $b \leq \mathsf{t}$, so $|b| = b \to \mathsf{t}$, by (i). In this case $|a|$ is comparable with $|b|$, by (vi). The same conclusion follows when $\mathsf{t} \not< a$, by symmetry. ☐

As a consequence of Lemma 2.7(v) and (3), we have:

Corollary 2.8. *If $\boldsymbol{A}$ is a conic idempotent CRP then $\{a \to \mathsf{t} : a \in A\}$ is a chain in $\langle A; \leq \rangle$.*

3. Lattice-Ordered Structures

An algebra $\langle A; \cdot, \to, \wedge, \vee, \mathsf{t} \rangle$ is called a *commutative residuated lattice* (briefly, a *CRL*) if $\langle A; \wedge, \vee \rangle$ is a lattice and $\langle A; \cdot, \to, \mathsf{t}, \leq \rangle$ is a CRP, where $\leq$ denotes the lattice order. The class of all CRLs is a finitely axiomatized variety [14].

Definition 3.1. A CRL is said to be *semiconic* if it is isomorphic to a subdirect product of conic CRLs.

Lemma 3.2. *The class of all semiconic CRLs is a variety.*

Proof. Let K be the class of all conic CRLs and V the variety generated by K. Since K is axiomatized by universal positive sentences, viz. the identities of CRLs together with $\forall x \, (x \wedge \mathsf{t} = x \text{ or } x \wedge \mathsf{t} = \mathsf{t})$, it is closed under the class operators H, S and $\mathrm{P_U}$. Now CRLs are congruence distributive, as they have lattice reducts, so the subdirectly irreducible algebras in V belong to $\mathrm{HSP_U}(\mathsf{K})$ (by Jónsson's Theorem, see [16] or [17] or [10, Thm. IV.6.8]), hence they are conic. Thus, V consists of semiconic CRLs, by Birkhoff's subdirect decomposition theorem [10, Thm. II.8.6]. The converse inclusion is obvious, so the class of semiconic CRLs coincides with V, and is therefore a variety. □

Remark 3.3. By the above proof and [12, p. 234], the variety of semiconic CRLs is axiomatized, relative to CRLs, by the identity

$$(x \wedge \mathsf{t}) \vee ((x \to \mathsf{t}) \wedge \mathsf{t}) \approx \mathsf{t}.$$

If $\boldsymbol{A} = \langle A; \cdot, \to, \wedge, \vee, \mathsf{t} \rangle$ is an idempotent CRL, then the algebra $\langle A; \cdot, \to, \mathsf{t} \rangle$ is called the $\{\cdot, \to, \mathsf{t}\}$*–reduct* of $\boldsymbol{A}$, and its subalgebras are called the $\{\cdot, \to, \mathsf{t}\}$–*subreducts* of $\boldsymbol{A}$. Evidently, these subreducts are idempotent CRPs, and they are conic if $\boldsymbol{A}$ is. Conversely:

Theorem 3.4. (cf. [26]) *Every idempotent CRP $\boldsymbol{A}$ is a $\{\cdot, \to, \mathsf{t}\}$–subreduct of an idempotent CRL, which can be chosen conic if $\boldsymbol{A}$ is conic.*

Proof. Let $\boldsymbol{A}$ be an idempotent CRP. We apply the Dedekind-MacNeille construction to $\langle A; \leq \rangle$. For each $X \subseteq A$, let $X^{\to} = \bigcap_{a \in X} [\, a \,)$ and $X^{\leftarrow} = \bigcap_{a \in X} (\, a \,]$ and $C(X) = X^{\to\leftarrow}$. Let $B = \{C(X) : X \subseteq A\}$, so B consists of downward closed subsets of $\langle A; \leq \rangle$. For $X, Y \in B$, define

$$\begin{aligned} &X \cdot Y = C(\{x \cdot y : x \in X \text{ and } y \in Y\})\,; \\ &X \to Y = \{a \in A : x \cdot a \in Y \text{ for all } x \in X\}\,; \\ &X \vee Y = C(X \cup Y). \end{aligned}$$

Now $\boldsymbol{B} = \langle B; \cdot, \to, \cap, \vee, (\, \mathsf{t} \,] \rangle$ is a CRL and the map $a \mapsto (\, a \,]$ is an injective homomorphism from $\langle A; \cdot, \to, \mathsf{t} \rangle$ into $\langle B; \cdot, \to, (\, \mathsf{t} \,] \rangle$ (see [26] or [33, Chap. 8]).

Also, $\boldsymbol{B}$ is idempotent, because $\boldsymbol{A}$ is (see the proof of [15, Thm. 7.3] if necessary). It is easily verified that $\boldsymbol{B}$ is conic if $\boldsymbol{A}$ is. □

Since we consider idempotent CRPs as pure algebras, we may define a *semiconic idempotent CRP* to be any subdirect product of conic idempotent CRPs. The obvious commutativity between taking subreducts and taking subdirect products yields the next corollary.

Corollary 3.5. *An algebra is a semiconic idempotent CRP iff it is a* $\{\cdot, \to, \mathsf{t}\}$*–subreduct of a semiconic idempotent CRL.*

4. A Locally Finite Quasivariety

Notation 4.1. From now on, SCIP shall denote the class of all semiconic idempotent CRPs, and SCIL the class of all semiconic idempotent CRLs.

Obviously, SCIL is a variety, in view of Lemma 3.2. Since SCIL is a quasivariety, it follows from Corollary 3.5 and a well known theorem of Maltsev that SCIP is also a quasivariety (see [23, p. 216]). However, SCIP is not a variety, as it contains a reduct of the 3-element Sugihara monoid. This reduct is the idempotent CRP on the chain $-1 < 0 < 1$, where 0 is the identity for $\cdot$ and $1 \cdot -1 = -1$. It has a homomorphic image that is not an idempotent CRP. (This is well known. The image is got by identifying -1 with 1 but not with 0, and it violates the quasi-identity

$$\forall x \, \forall y \; ((x \to y \; = \; |\, x \to y \,| \quad \& \quad y \to x \; = \; |\, y \to x \,|) \;\; \Longrightarrow \;\; x \; = \; y) \, .$$

This quasi-identity holds in all idempotent CRPs, as it expresses the antisymmetry of the definable order $\leq$.)

A CRP or a CRL is said to be *integral* if its monoid identity t is its greatest element. An integral idempotent CRP [resp. CRL] is called a *Brouwerian semilattice* [resp. a *Brouwerian lattice*]. Brouwerian semilattices form a variety, denoted here by BS.

A class K of similar algebras is said to be *locally finite* if every finitely generated algebra in K is finite. It is proved in [24] that BS is locally finite, but the variety of Brouwerian lattices is not: see for instance [4, Chap. IX]. So SCIL is not locally finite, as it obviously contains all Brouwerian lattices.

The *free spectrum* of BS is the function $f_{\mathsf{BS}} : \omega \to \omega$ such that each $f_{\mathsf{BS}}(n)$ is the cardinality of the free n-generated algebra in BS. So, every n-generated Brouwerian semilattice has at most $f_{\mathsf{BS}}(n)$ elements. Obviously, $f_{\mathsf{BS}}(0) = 1$ and $f_{\mathsf{BS}}(1) = 2$. It is known that $f_{\mathsf{BS}}(2) = 18$ ([3]) and that

$$f_{\mathsf{BS}}(3) \; = \; 623,662,965,552,330$$

([9, 21, 20]). More information can be found in [18].

Theorem 4.2. *Let $\boldsymbol{A}$ be an n-generated conic idempotent CRP, where $n \in \omega$. Then* $|A| \leq 2^n + n - 1 + f_{\mathsf{BS}}(2^n + n - 1)$.

Proof. Let Y be a generating set for $\boldsymbol{A}$, where $|Y| \leq n$. We may assume that $\mathsf{t} \notin Y$. Let

$$r = |Y \cap [\mathsf{t})| \quad \text{and} \quad s = |Y \cap (\mathsf{t}]|,$$

so $r + s \leq n$. Also, let

$$C = \{\mathsf{t}\} \cup Y \cup \{y \to \mathsf{t} : y \in Y\} \cup \{(y \to \mathsf{t}) \to \mathsf{t} : y \in Y\}.$$

Then C is closed under the term function of $x \to \mathsf{t}$. This follows from (6) and the fact that $\mathsf{t} \to \mathsf{t} = \mathsf{t}$. We partition C into two sets C_1 and C_2, where

$$C_1 = \{a \in C : \mathsf{t} < a\} \quad \text{and} \quad C_2 = \{a \in C : a \leq \mathsf{t}\}.$$

Let

$$D = \{a_1 \cdot \ldots \cdot a_k \ : \ a_1, \ldots, a_k \in C_1 \ \text{and} \ 0 < k \in \omega\}.$$

Obviously, D is closed under $\cdot$. By Proposition 2.4(i), the elements of D satisfy $x \cdot y = x \vee y$. Thus $\langle D; \cdot \rangle$ is a (join) semilattice generated by C_1.

By Corollary 2.8, the set

$$X := \{y \to \mathsf{t} : \mathsf{t} > y \in Y\} \cup \{(y \to \mathsf{t}) \to \mathsf{t} : \mathsf{t} < y \in Y\}$$

is a chain in $C_1 \cup \{\mathsf{t}\}$, with $|X| \leq r + s \leq n$. Now $|D|$ is the number of joins of non-empty subsets of C_1. We shall establish a certain upper bound for $|D|$. Any join of elements of X is an element of X, so we need only consider

(i) joins of non-empty subsets of $C_1 \setminus X$ and
(ii) joins of sets of the form $\{x\} \cup Z$ where $\mathsf{t} < x \in X$ and $Z \subseteq C_1 \setminus X$.

Since $|C_1 \setminus X| = r \leq n$, the number of joins as in (i) is at most

$$2^{|C_1 \setminus X|} - 1 \leq 2^n - 1.$$

In (ii), the join of $\{x\} \cup Z$ is x or the join of Z, because Lemma 2.7(vi) shows that x is comparable with all elements of Z. Thus, the number of joins arising from (ii) and *not* arising from (i) is at most $|X| \leq n$. Consequently,

$$|D| \leq 2^n - 1 + n.$$

Let

$$E = \{b_1 \cdot \ldots \cdot b_k \ : \ b_1, \ldots, b_k \in C_2 \ \text{and} \ 0 < k \in \omega\}.$$

It follows similarly from Proposition 2.4(ii) that $\langle E; \cdot \rangle$ is a (meet) semilattice generated by C_2, and by Corollary 2.8,

$$\{y \to \mathsf{t} : \mathsf{t} < y \in Y\} \cup \{(y \to \mathsf{t}) \to \mathsf{t} : \mathsf{t} > y \in Y\}$$

is a chain in C_2 with at most n elements. So, just as in the case of D, we obtain

$$|E \setminus \{\mathsf{t}\}| \leq 2^n - 1 + n.$$

We verify inductively that

(17) $$\text{if } \ a \in D \ \text{ then } \ a \to \mathsf{t} \in E.$$

Certainly, if $a \in C_1$ then $a \to \mathsf{t} \in C_2 \subseteq E$. Now let $a = a_1 \cdot \ldots \cdot a_{k+1} \in D$, where $0 < k \in \omega$ and $a_1, \ldots, a_{k+1} \in C_1$. Then

$$\begin{aligned} a \to \mathsf{t} &= (a_1 \cdot \ldots \cdot a_{k+1}) \to \mathsf{t} \\ &= (a_1 \cdot \ldots \cdot a_k) \to (a_{k+1} \to \mathsf{t}) \quad \text{(by (4))} \\ &= ((a_1 \cdot \ldots \cdot a_k) \to \mathsf{t}) \cdot (a_{k+1} \to \mathsf{t}) \quad \text{(by Lemma 2.7(iv))}. \end{aligned}$$

By an induction hypothesis, both $(a_1 \cdot \ldots \cdot a_k) \to \mathsf{t}$ and $a_{k+1} \to \mathsf{t}$ are in E, so $a \to \mathsf{t} \in E$, as required. This confirms (17). Similarly,

(18) $$\text{if } \ b \in E \ \text{ then } \ b \to \mathsf{t} \in D \cup \{\mathsf{t}\}.$$

We also need to verify that, for all $a \in A$,

(19) $$\text{if } \ |\,a\,| \in D \cup \{\mathsf{t}\} \ \text{ then } \ |\,a \to \mathsf{t}\,| \in D \cup \{\mathsf{t}\}.$$

Let $a \in A$. If $a \leq \mathsf{t}$ then $|\,a\,| = a \to \mathsf{t}$, by Lemma 2.7(i). Thus, by (12), $|\,a \to \mathsf{t}\,| = ||\,a\,|| = |\,a\,|$. Then (19) is trivial. So we may assume that $\mathsf{t} < a$. By (14), $a = |\,a\,|$. If $|\,a\,| \in D \cup \{\mathsf{t}\}$ (i.e., $a \in D$), then $a \to \mathsf{t} \in E$, by (17), hence $|\,a \to \mathsf{t}\,| = (a \to \mathsf{t}) \to \mathsf{t} \in D \cup \{\mathsf{t}\}$, by Lemma 2.7(i) and (18). This proves (19).

Let $X_0 = E$, and for each $i \in \omega$, let

$$X_{i+1} \ = \ X_i \, \cup \, \{a \cdot b \, : \, a, b \in X_i\} \, \cup \, \{a \to b \, : \, a, b \in X_i\}.$$

We claim that, for every $i \in \omega$ and $c \in A$,

(20) $$\text{if } \ c \in X_i \ \text{ then } \ |\,c\,| \in D \cup \{\mathsf{t}\}.$$

We prove (20) by induction on i. If $i = 0$, then (20) is just (18). Suppose that $i \geq 0$ and that (20) holds for i. Let $c \in X_{i+1} \setminus X_i$, so c is $a \cdot b$ or $a \to b$ for some $a, b \in X_i$. If $|\,a\,|$ and $|\,b\,|$ are comparable then $|\,a \cdot b\,| \in \{|\,a\,|, |\,b\,|\}$ and $|\,a \to b\,| \in \{|\,b\,|, |\,a \to \mathsf{t}\,|\}$, by Lemma 2.6(ii),(iii),(iv), so $|\,c\,| \in D \cup \{\mathsf{t}\}$, by the induction hypothesis and (19). If $|\,a\,|$ and $|\,b\,|$ are incomparable then $\mathsf{t} < a = |\,a\,|$ and $\mathsf{t} < b = |\,b\,|$, by Lemma 2.7(vii) and (14), so $a, b \in D \cup \{\mathsf{t}\}$, by the induction hypothesis. In this case, $\mathsf{t} < a \cdot b$, so $|\,a \cdot b\,| = a \cdot b \in D \cup \{\mathsf{t}\}$ (because $D \cup \{\mathsf{t}\}$ is closed under $\cdot$), while Lemma 2.7(ii) and (19) show that $|\,a \to b\,| = |\,a \to \mathsf{t}\,| \in D \cup \{\mathsf{t}\}$. This completes the proof of (20).

Let

$$F \ = \ \{a \in \textstyle\bigcup_{i \in \omega} X_i \, : \, a \leq \mathsf{t}\}.$$

Evidently, F is closed under $\cdot$, so $\langle F; \cdot, \mathsf{t}\rangle$ is a commutative idempotent monoid, still partially ordered by the restriction of $\leq$, which remains compatible with the restriction of $\cdot$. Note that t is the greatest element of F. For every $a, b \in F$, we define

$$a \to' b = \begin{cases} \mathsf{t} & \text{if } a \leq b\,; \\ a \to b & \text{otherwise.} \end{cases}$$

The operation $\to'$ is well defined on F, by Lemma 2.6(i) and the fact that $\mathsf{t} \in E \subseteq F$. For each $a, b \in F$, it is easy to see that $a \to' b$ is the largest $c \in F$ such that $a \cdot c \leq b$, so $\langle F; \cdot, \to', \mathsf{t}\rangle$ is a Brouwerian semilattice.

Let E^* be the subuniverse of $\langle F; \cdot, \to', \mathsf{t}\rangle$ generated by $E \setminus \{\mathsf{t}\}$. We shall show that $F = E^*$. Let $c \in F$. We need to prove that, for every $i \in \omega$,

$$\text{if } c \in X_i \text{ then } c \in E^*. \tag{21}$$

The proof is by induction on i. Since $X_0 = E \subseteq E^*$, we assume that (21) holds for some $i \geq 0$, and that $c \in X_{i+1} \setminus X_i$, so c is $a \cdot b$ or $a \to b$ for some $a, b \in X_i$. Note that $c \leq \mathsf{t}$, because $c \in F$.

Suppose $c = a \cdot b$. Since $c \leq \mathsf{t}$, the proof of Proposition 2.4(iii) shows that if $a < \mathsf{t} < b$ then $c = a \in X_i \cap F$, while if $b < \mathsf{t} < a$ then $c = b \in X_i \cap F$. In both cases, $c \in E^*$, by the induction hypothesis. If $\mathsf{t} \leq a, b$, then $c = \mathsf{t} \in E^*$. If $a, b \leq \mathsf{t}$ then $a, b \in F$, so by the induction hypothesis, $a, b \in E^*$, whence $c \in E^*$, because E^* is closed under $\cdot$.

Suppose $c = a \to b$. If $a \leq b$, then $c = \mathsf{t} \in E^*$, so we may assume that $a \not\leq b$. If $\mathsf{t} < a$ and $\mathsf{t} \leq b$ then $c = a \to \mathsf{t}$, by Lemma 2.7(ii). In this case, because $\mathsf{t} < a$, we also have $|\,a\,| = a \in D$, by (20), so it follows from (17) that $c = a \to \mathsf{t} \in E$, hence $c \in E^*$. Suppose $b \leq \mathsf{t} < a$. Then $c = (a \to \mathsf{t}) \cdot b$, by Lemma 2.7(iv). Just as in the previous case, $a \to \mathsf{t} \in E^*$, because $\mathsf{t} < a$. Since $b \leq \mathsf{t}$, we have $b \in F$, hence $b \in E^*$, by the induction hypothesis. Since E^* is closed under $\cdot$, we have $c = (a \to \mathsf{t}) \cdot b \in E^*$. Finally, suppose $a, b \leq \mathsf{t}$, so $c = a \to' b$ and $a, b \in F$. Thus $a, b \in E^*$, by the induction hypothesis. Now E^* is closed under $\to'$, so $c = a \to' b \in E^*$. This finishes the proof of (21), and so $F = E^*$. It follows that

$$|\,F\,| \,=\, |\,E^*\,| \,\leq\, f_{\mathsf{BS}}\,(\,|\,E \setminus \{\mathsf{t}\}\,|\,) \,\leq\, f_{\mathsf{BS}}(2^n + n - 1).$$

Lastly, we show that $D \cup F$ is closed under $\cdot$ and $\to$. Clearly, D and F are closed under $\cdot$. And if $a \in F$ and $b \in D$, or vice versa, then a and b are comparable, hence $a \cdot b = a$ or $a \cdot b = b$, by Proposition 2.4(iii). So $D \cup F$ is closed under $\cdot$.

If $a \in F$ then $a \leq \mathsf{t}$, so by Lemma 2.7(i) and (20), $a \to \mathsf{t} = |\,a\,| \in D \cup \{\mathsf{t}\}$. If $a \in D$ then by (17), $a \to \mathsf{t} \in E \subseteq F$. Therefore $D \cup F$ is closed under the term function of $x \to \mathsf{t}$. If $a \in F$ and $b \in D$, then by Lemma 2.7(iv), $a \to b = (a \to \mathsf{t}) \cdot b$, hence $a \to b \in D \cup F$. Similarly, if $a \in D$ and $b \in F$, then $a \to b \in D \cup F$. If $a, b \in D$ then, by Lemma 2.6(v), Lemma 2.7(ii) and (17), $a \to b = b \in D$ or $a \to b = a \to \mathsf{t} \in E$. If $a, b \in F$ and $a \leq b$, then

$a \to b = |\, a \to b \,| \in D \cup \{\mathsf{t}\}$ by (15) and (20). If $a, b \in F$ and $a \not\leq b$, then $a \to b < \mathsf{t}$, hence $a \to b \in F$, by definition of F. So, $D \cup F$ is closed under $\to$.

Since $Y \subseteq D \cup F$, it follows that $D \cup F = A$. Now $|\, A \,| = |\, D \,| + |\, F \,|$, so

$$|\, A \,| \;\leq\; 2^n + n - 1 + f_{\mathsf{BS}}(2^n + n - 1). \qquad \square$$

For any quasivariety K, an algebra $\boldsymbol{A}$ is said to be K–*subdirectly irreducible* if the following is true: whenever $h : \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{A}_i$ is a subdirect embedding with $\boldsymbol{A}_i \in \mathsf{K}$ for every i, then at least one of the projection maps $\pi_j : \prod_{i \in I} \boldsymbol{A}_i \to \boldsymbol{A}_j$ has the property that $\pi_j h : \boldsymbol{A} \cong \boldsymbol{A}_j$. Just as in Birkhoff's subdirect decomposition theorem, every algebra in a quasivariety K is isomorphic to a subdirect product of K–subdirectly irreducible algebras *in* K (see [29, Thm. 1.1]).

Corollary 4.3. SCIP *is locally finite.*

Proof. If $\boldsymbol{A} \in \mathsf{SCIP}$ is n-generated and SCIP–subdirectly irreducible then $\boldsymbol{A}$ is conic, so by Theorem 4.2, $|\, A \,|$ is bounded by a finite cardinal whose value depends only on n (and not on the choice of $\boldsymbol{A}$). Therefore, a standard argument shows that SCIP is locally finite—see for instance [30, Thm. 1]. $\square$

5. The Finite Embeddability Property

Definition 5.1. Let K be a class of similar algebras. We say that K has the *finite embeddability property* (briefly, the *FEP*) if every finite subset X of an algebra $\boldsymbol{A} \in \mathsf{K}$ can be extended to a *finite* algebra $\boldsymbol{B} \in \mathsf{K}$ with preservation of all partial $\boldsymbol{A}$–operations, i.e., for any basic operation symbol f of arbitrary rank n, if $a_1, \dots, a_n \in X$ and $f^{\boldsymbol{A}}(a_1, \dots, a_n) \in X$ then

$$f^{\boldsymbol{B}}(a_1, \dots, a_n) \;=\; f^{\boldsymbol{A}}(a_1, \dots, a_n).$$

Without weakening the FEP, we could re-define 'preservation of partial $\boldsymbol{A}$–operations' as follows: there exists an injective function $\alpha : X \to B$ such that for every basic operation symbol f, of rank n say, if $a_1, \dots, a_n \in X$ and $f^{\boldsymbol{A}}(a_1, \dots, a_n) \in X$ then

$$f^{\boldsymbol{B}}(\alpha(a_1), \dots, \alpha(a_n)) \;=\; \alpha(f^{\boldsymbol{A}}(a_1, \dots, a_n)).$$

A function α of this kind is called a *partial* $\boldsymbol{A}$*–embedding of* X *into* $\boldsymbol{B}$.

Clearly, every locally finite class has the FEP. When a variety has the FEP, it may have subvarieties that lack the FEP (unlike the case of local finiteness). Although the variety SCIL is not locally finite, we shall prove in this section that it does have the FEP.

Lemma 5.2. ([5, Lemma 3.7]) *Let* V *be a variety, and suppose that every finite subset of a subdirectly irreducible algebra in* V *can be extended to a finite algebra in* V *with preservation of partial operations. Then* V *has the FEP.*

In [5], this lemma is stated with the extra hypothesis that the finite containing algebra can be chosen subdirectly irreducible as well. We repeat the proof here, to reassure the reader that this extra demand is not necessary.

Proof. Let X be a finite subset of some $\boldsymbol{A} \in \mathsf{V}$. We may assume that $\boldsymbol{A}$ is a subdirect product of subdirectly irreducible algebras $\boldsymbol{A}_i \in \mathsf{V}$ $(i \in I)$. For each i, the finite subset $\pi_i[X]$ of $\boldsymbol{A}_i$ can be extended to a finite algebra $\boldsymbol{A}_i' \in \mathsf{V}$ with preservation of partial $\boldsymbol{A}_i$–operations, by assumption. Then the algebra $\prod_{i\in I} \boldsymbol{A}_i' \in \mathsf{V}$ is an extension of X in which all partial $\boldsymbol{A}$–operations are preserved. Since X is finite, there is a finite subset J of I such that the projection map $\pi_J : \prod_{i\in I} \boldsymbol{A}_i' \to \prod_{i\in J} \boldsymbol{A}_i'$ restricts to an injection $X \to \prod_{i\in J} A_i'$. This injection is a partial $\boldsymbol{A}$–embedding of X into the finite algebra $\prod_{i\in J} \boldsymbol{A}_i' \in \mathsf{V}$. ☐

Lemma 5.3. *Let $\boldsymbol{A}$ be a finite conic idempotent CRP. Then $\boldsymbol{A}$ is lattice-ordered.*

Proof. Let $a, b \in A$. We claim that a and b have a greatest lower bound in $\boldsymbol{A}$. We may assume that a and b are incomparable, so $a, b \leq \mathsf{t}$ or $\mathsf{t} \leq a, b$. In the first case, $a \cdot b = a \wedge b$, by Proposition 2.4(ii). In the second case, the set $S = \{c \in A : \mathsf{t} \leq c \text{ and } c \leq a, b\}$ is closed under $\cdot$, by idempotence. Since S is finite and non-empty, the product p of all elements of S exists, and $p \in S$. Clearly, p is the greatest element of S, so p is the greatest lower bound of $\{a, b\}$, because $\boldsymbol{A}$ is conic. This shows that $\langle A; \leq\rangle$ is a meet semilattice, and a dual argument shows that it is also a join semilattice. ☐

A *relative subvariety* of a quasivariety K is a subquasivariety M of K such that $\mathsf{M} = \mathsf{K} \cap \mathsf{V}$ for some variety V. Equivalently, it is a subclass of K that is axiomatized, relative to K, by some set of identities.

Notation 5.4. For any class K of idempotent CRPs, let K^L denote the class of all lattice-ordered members of K, considered as CRLs. In other words, K^L is the class of all CRLs whose $\{\cdot, \to, \mathsf{t}\}$–reducts belong to K.

Clearly, if K is a relative subvariety of SCIP then K^L is a subvariety of SCIL.

Theorem 5.5. *Let K be a relative subvariety of SCIP. Then the variety K^L has the finite embeddability property.*

Proof. Let X be a finite subset of a subdirectly irreducible algebra $\boldsymbol{A} \in \mathsf{K}^L$. Since $\mathsf{K}^L \subseteq \mathsf{SCIL}$, it follows that $\boldsymbol{A}$ is conic. The $\{\cdot, \to, \mathsf{t}\}$–reduct $\boldsymbol{A}^-$ of $\boldsymbol{A}$ belongs to K, and therefore to SCIP. Let $\boldsymbol{B}^-$ be the subalgebra of $\boldsymbol{A}^-$ generated by X. Then $\boldsymbol{B}^-$ is a finite conic algebra in K, by Corollary 4.3. By Lemma 5.3, $\boldsymbol{B}^-$ is lattice ordered, i.e., it is the $\{\cdot, \to, \mathsf{t}\}$–reduct of a (finite) CRL $\boldsymbol{B} \in \mathsf{K}^L$.

The partial $\boldsymbol{A}$–operations $\cdot, \to, \mathsf{t}$ are preserved in the passage from X to $\boldsymbol{B}$, by definition of $\boldsymbol{B}^-$. Observe that for any $b, b' \in B$, we have $b \leq^{\boldsymbol{A}} b'$ iff $b \leq^{\boldsymbol{B}} b'$, because both orders are definable in terms of $\to$ alone (by (15)), and because $\to^{\boldsymbol{A}}$ and $\to^{\boldsymbol{B}}$ coincide on B. So we can write $b \leq b'$ unambiguously in this case. Now let $c, d \in X$ and suppose $c \vee^{\boldsymbol{A}} d \in X$. Since $c, d \leq c \vee^{\boldsymbol{A}} d \in B$, we have $c \vee^{\boldsymbol{B}} d \leq c \vee^{\boldsymbol{A}} d$. Also, since $c, d \leq c \vee^{\boldsymbol{B}} d \in B \subseteq A$, we have $c \vee^{\boldsymbol{A}} d \leq c \vee^{\boldsymbol{B}} d$, whence $c \vee^{\boldsymbol{B}} d = c \vee^{\boldsymbol{A}} d$. A similar argument holds for meets, so all partial $\boldsymbol{A}$–operations are preserved in the passage from X to $\boldsymbol{B}$. This shows that K^L has the FEP, in view of Lemma 5.2 (which applies because K^L is a variety). $\square$

Remark 5.6. A variant of Lemma 5.2 holds for quasivarieties K if we replace 'subdirectly irreducible' by 'K–subdirectly irreducible'. However, in an arbitrary subquasivariety K of SCIP or of SCIL, there is no guarantee that the K–subdirectly irreducible algebras will be conic. This is why Theorem 5.5 is restricted to the relative subvarieties of SCIP. In this connection, see Remark 5.9 as well.

For any class M of similar algebras, the class of all finite algebras in M will be denoted by M_{fin}. If M has the FEP, then every universal first order sentence that fails in some member of M must also fail in some member of M_{fin}. This conclusion, restricted to quasi-identities, is often called the 'strong finite model property'. It shows that every quasivariety M with the FEP is generated by its finite members, i.e., $\mathsf{M} = \mathbb{SPP}_{\mathrm{U}}(\mathsf{M}_{\text{fin}})$. (The converse holds for quasivarieties of finite type: see [8, Thm. 3.1].) A finitely axiomatized quasivariety of finite type with the FEP has a decidable universal theory; in particular its equational and quasi-equational theories are decidable (see for instance [5, Lemma 3.13]).

Corollary 5.7. *For every relative subvariety K of SCIP, the variety K^L is generated as a quasivariety by its finite members. If K^L is also finitely axiomatized (e.g., if K is finitely axiomatized), then K^L has a decidable universal theory.*

In particular, since $\mathsf{SCIL} = \mathsf{SCIP}^L$ and since SCIL is finitely axiomatized (see Remark 3.3), we may deduce:

Corollary 5.8. SCIL *has the finite embeddability property, and it is generated as a quasivariety by its finite members and has a decidable universal theory.*

Remark 5.9. Theorem 5.5 says, in effect, that when we add to the axioms of SCIL any set of equational axioms that use only the operation symbols $\cdot, \to, \mathsf{t}$, then the resulting subvariety of SCIL has the FEP. The reader might therefore wonder whether every subvariety of SCIL has the FEP. This is not the case. There are even varieties of Brouwerian lattices that are not generated *as varieties* by their finite members. This can be deduced from the literature on Heyting algebras and super-intuitionistic logics. See for instance [31], [11, Chap. 6] and [22, Thm. 2]. (A Heyting algebra is a Brouwerian lattice with a distinguished least element.)

6. The Lattice of Varieties of Idempotent Semiconic CRLs

Definition 6.1. A (deductive) *filter* of a CRL is an upward closed submonoid that is also closed under meets.

Let $\boldsymbol{A}$ be a CRL. Arbitrary intersections of filters of $\boldsymbol{A}$ are filters again, so the filters of $\boldsymbol{A}$ form a complete lattice when ordered by set inclusion. For each $X \subseteq A$, let $\operatorname{Fg} X$ denote the smallest filter of $\boldsymbol{A}$ containing X. For any $a, b \in A$,

$$b \in \operatorname{Fg}\{a\} \quad \text{iff} \quad (a \wedge \mathsf{t})^n \leq b \ \text{ for some } n \in \omega \tag{22}$$

(see [1]). Clearly, the smallest filter of $\boldsymbol{A}$ is $[\,\mathsf{t}\,)$. Every filter distinct from $[\,\mathsf{t}\,)$ contains an element strictly below t (as it contains t and is closed under meets). It is well known that the congruence lattice of $\boldsymbol{A}$ is isomorphic, under the map $\theta \mapsto [\,\mathsf{t}/\theta\,)$, to the lattice of filters of $\boldsymbol{A}$. This, together with (22), implies that a nontrivial CRL is simple iff for any two elements $a, b < \mathsf{t}$, there is a natural number n such that $a^n \leq b$. In the idempotent case, this becomes:

Lemma 6.2. *An idempotent CRL is simple iff it has exactly one element strictly below* t.

The sole element below t is obviously the least element of a simple idempotent CRL, but there may be elements incomparable with t. Of course, if $\bot$ is the least element of a CRL, then the CRL also has a greatest element, namely $\bot \to \bot$. Therefore, every simple idempotent CRL is bounded.

The variety of idempotent CRLs has equationally definable principal congruences (EDPC) [1]. This claim amounts, in effect, to the special case of (22) in which n is always 1 (because of idempotence). Every variety with EDPC has the congruence extension property and is congruence distributive [19]. Further information about EDPC can be found in [6] and subsequent papers of Blok and Pigozzi, as well as [17].

Lemma 6.3. *In a variety with EDPC, the class of simple algebras is closed under ultraproducts and under subalgebras.*

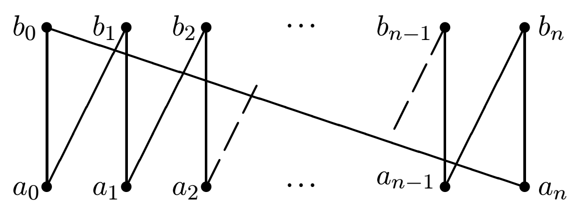
The claim about ultraproducts is pointed out in [7]. The one about subalgebras follows from the congruence extension property.

A variety is said to be *semisimple* if all of its subdirectly irreducible members are simple algebras. Galatos [13] has shown that there are just two minimal (nontrivial) varieties of idempotent CRLs. These two varieties are semisimple, and they consist of semiconic algebras. One of them is generated by the unique 2-element CRL, denoted here by $\boldsymbol{C}_2$. The other is generated by the CRL–reduct of the 3-element Sugihara monoid discussed in Section 4.

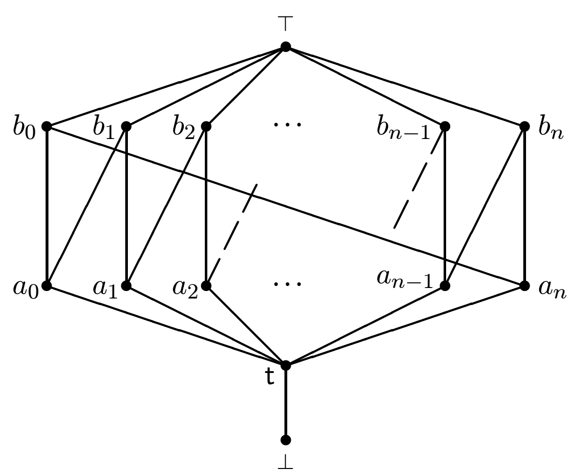
Further information about varieties of idempotent CRLs can be found in [32]. It is known that there are $2^{\aleph_0}$ such varieties, because there are already this many varieties of Brouwerian lattices [34]. However, there is only one nontrivial *semisimple* variety of Brouwerian lattices, viz. $\mathrm{HSP}(\boldsymbol{C}_2)$. So there remains the question: *how many semisimple varieties of idempotent CRLs are there?*

We shall prove in this section that there are already $2^{\aleph_0}$ semisimple varieties of *semiconic* idempotent CRLs.

We consider a denumerable sequence of finite posets $P_2, P_3, P_4, \ldots,$ none of which embeds into any other. These posets are called "crowns" and they appeared in [25], where it was first reported that there are $2^{\aleph_0}$ varieties of lattices. The n th crown P_n has the following Hasse diagram.



For each n, we extend P_n $(= \{a_0, \ldots, a_n, b_0, \ldots, b_n\})$ to the poset A_n depicted below, by adding three elements $\bot, \mathsf{t}, \top$.



This construction does not disturb the non-embeddability relations between the structures. Clearly, A_n is lattice-ordered for $n \geq 2$. For every $a, b \in A_n$, we define

$$a \cdot b = \begin{cases} a \vee b & \text{if } a, b \geq \mathsf{t}; \\ \bot & \text{otherwise.} \end{cases}$$

Then $\langle A_n\,; \cdot\,, \mathsf{t}\rangle$ is an idempotent commutative monoid, and for all $a, b, c \in A_n$, if $a \leq b$ then $c \cdot a \leq c \cdot b$. Moreover,

(23) for every $a, b \in A_n$ there is a largest $c \in A_n$ with $a \cdot c \leq b$.

So, if we denote the largest such c by $a \to b$, then $\boldsymbol{A}_n = \langle A_n; \cdot, \to, \wedge, \vee, \mathsf{t}\rangle$ is a conic idempotent CRL, which is simple, by Lemma 6.2. To establish (23), one verifies that

$$a \to b \;=\; \begin{cases} \top & \text{if } \; a = \bot\,; \\ b & \text{if } \; \mathsf{t} \leq a \leq b\,; \\ \bot & \text{if } \; a \not\leq b. \end{cases}$$

Lemma 6.4. *For any subsets* M *and* N *of* $\{\boldsymbol{A}_2, \boldsymbol{A}_3, \boldsymbol{A}_4, \dots\}$, *if* $\mathsf{M} \neq \mathsf{N}$ *then* $\mathrm{HSP}(\mathsf{M}) \neq \mathrm{HSP}(\mathsf{N})$.

Proof. If $\mathsf{M} \neq \mathsf{N}$ then we may assume (by symmetry) that there is an $i \geq 2$ such that $\boldsymbol{A}_i \in \mathsf{M}$ but $\boldsymbol{A}_i \notin \mathsf{N}$. In view of the non-embeddability relations, no algebra in N has a subalgebra isomorphic to $\boldsymbol{A}_i$. Since $\boldsymbol{A}_i$ is a finite algebra of finite type, the property of not having a subalgebra isomorphic to $\boldsymbol{A}_i$ can be expressed as a universal first order sentence (equivalent to the negation of the diagram sentence of $\boldsymbol{A}_i$). Since universal sentences persist in ultraproducts and subalgebras, $\boldsymbol{A}_i \notin \mathrm{SP}_{\mathrm{U}}(\mathsf{N})$. As the variety of idempotent CRLs has EDPC, Lemma 6.3 shows that $\mathrm{SP}_{\mathrm{U}}(\mathsf{N})$ consists of simple algebras, hence the nontrivial algebras in $\mathrm{HSP}_{\mathrm{U}}(\mathsf{N})$ are isomorphic to ones in $\mathrm{SP}_{\mathrm{U}}(\mathsf{N})$. Thus, $\boldsymbol{A}_i \notin \mathrm{HSP}_{\mathrm{U}}(\mathsf{N})$. Then, since $\boldsymbol{A}_i$ is simple (and therefore subdirectly irreducible), it follows from Jónsson's Theorem that $\boldsymbol{A}_i \notin \mathrm{HSP}(\mathsf{N})$, and so $\mathrm{HSP}(\mathsf{M}) \neq \mathrm{HSP}(\mathsf{N})$. $\square$

This proof also shows that all subdirectly irreducible algebras in $\mathrm{HSP}(\mathsf{N})$ are simple, because they are isomorphic to algebras in $\mathrm{SP}_{\mathrm{U}}(\mathsf{N})$. Thus, $\mathrm{HSP}(\mathsf{N})$ is semisimple for all $\mathsf{N} \subseteq \{\boldsymbol{A}_2, \boldsymbol{A}_3, \boldsymbol{A}_4, \dots\}$. So the lemma shows that the number of semisimple subvarieties of SCIL is at least as large as the power set of ω. On the other hand, every variety of countable type has at most $2^{\aleph_0}$ subvarieties, so we have proved:

Theorem 6.5. *There are just* $2^{\aleph_0}$ *semisimple varieties of semiconic idempotent CRLs.*

Remark 6.6. Lemma 6.4 could alternatively be proved with the aid of a result of Blok and Pigozzi [6]: in a variety V of finite type with EDPC, every finite subdirectly irreducible algebra $\boldsymbol{A}$ is 'splitting', i.e., the lattice of subvarieties of V is the disjoint union of $[\,\mathrm{HSP}(\boldsymbol{A})\,)$ and $(\,\mathsf{W}\,]$ for some variety W.

7. Distributivity and Involution

The lattice reduct of an algebra in SCIL need not be distributive, e.g., the algebras $\boldsymbol{A}_n$ constructed in Section 6 are non-distributive for $n \geq 3$. But it is well known and easily verified that every CRL satisfies

$$x \cdot (y \vee z) \approx (x \cdot y) \vee (x \cdot z).$$

In the idempotent case, Proposition 2.4(ii) shows that $\cdot$ and $\wedge$ coincide for elements below t. This establishes the following result.

Lemma 7.1. *In an idempotent CRL, the sublattice $\langle (\,\mathsf{t}\,]; \leq \rangle$ is distributive.*

A (compatible) *involution* on a CRL $\boldsymbol{A}$ is a function $\neg : A \to A$ such that $\neg\neg\, a = a$ and $a \to \neg\, b = b \to \neg\, a$ for all $a, b \in A$. In this case, De Morgan's laws for $\neg, \wedge$ and $\vee$ hold, so the map $a \mapsto \neg\, a$ is an anti-automorphism of the lattice reduct of $\boldsymbol{A}$. If we add $\neg$ to the basic operations of $\boldsymbol{A}$, the resulting algebra is called an *involutive CRL.*

It is well known that a CRL has an involution iff it has an element f such that $(a \to \mathsf{f}) \to \mathsf{f} = a$ for all elements a. Indeed, we can define $\mathsf{f} = \neg\, \mathsf{t}$, and conversely, $\neg\, a = a \to \mathsf{f}$ for all a. This shows that an involutive CRL and its CRL–reduct have the same congruences, so the former is subdirectly irreducible (or semiconic) iff the latter is.

An involutive CRL is called a *Sugihara monoid* if it is distributive and idempotent. Dunn, in his contributions to [2], proved that every Sugihara monoid is a subdirect product of chains, and that the variety of Sugihara monoids is locally finite.

Theorem 7.2. *Every semiconic idempotent involutive CRL is distributive, and therefore a Sugihara monoid.*

Proof. Since the semiconic idempotent involutive CRLs form a variety, it is enough to prove the result for subdirectly irreducible algebras. Let $\boldsymbol{A}$ be such an algebra, so $\boldsymbol{A}$ is conic.

The lattice $\langle (\,\mathsf{t}\,]; \leq \rangle$ is distributive, by Lemma 7.1, so by conicity, it suffices to show that $\langle [\,\mathsf{t}\,); \leq \rangle$ is also distributive. Let $\mathsf{f} = \neg\, \mathsf{t}$. Since $\neg$ is an anti-automorphism of $\langle A; \wedge, \vee \rangle$ that sends t to f, and since distributivity is a self-dual property, the lattice $\langle [\,\mathsf{f}\,); \leq \rangle$ is distributive. But $\mathsf{f} \leq \mathsf{t}$, by idempotence. Indeed, from $\mathsf{f} \cdot \mathsf{f} = \mathsf{f}$, we get

$$\mathsf{f} \leq \mathsf{f} \to \mathsf{f} = \mathsf{f} \to \neg\, \mathsf{t} = \mathsf{t} \to \neg\, \mathsf{f} = \neg\, \mathsf{f} = \mathsf{t}.$$

So $\langle [\,\mathsf{t}\,); \leq \rangle$, being a sublattice of $\langle [\,\mathsf{f}\,); \leq \rangle$, is distributive. $\square$

Thus, we gain no new finiteness results by imposing involution on the algebras in SCIL. Moreover, since the subdirectly irreducible Sugihara monoids are totally ordered, most of the algebras in SCIL (or in SCIP) cannot be embedded into involutive algebras in SCIL.

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SOME LOCALLY TABULAR LOGICS WITH CONTRACTION AND MINGLE

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Abstract. Anderson and Belnap's implicational system $\mathbf{RMO}_{\rightarrow}$ can be extended conservatively by the usual axioms for fusion and for the Ackermann truth constant t. The resulting system $\mathbf{RMO}^{*}$ is algebraized by the quasivariety IP of all idempotent commutative residuated po-monoids. Thus, the axiomatic extensions of $\mathbf{RMO}^{*}$ are in one-to-one correspondence with the relative subvarieties of IP. An algebra in IP is called *semiconic* if it decomposes subdirectly (in IP) into algebras where the identity element t is order-comparable with all other elements. The semiconic algebras in IP are locally finite. It is proved here that a relative subvariety of IP consists of semiconic algebras if and only if it satisfies $x \approx (x \rightarrow \mathsf{t}) \rightarrow x$. It follows that if an axiomatic extension of $\mathbf{RMO}^{*}$ has $((p \rightarrow \mathsf{t}) \rightarrow p) \rightarrow p$ among its theorems then it is locally tabular. In particular, such an extension is strongly decidable, provided that it is finitely axiomatized.

1. Introduction

There are now several different motivations for the study of logics that lack the *weakening* axiom $p \rightarrow (q \rightarrow p)$. The first systems of this kind were developed by relevance logicians, who also debated the merits of the weaker *mingle* postulate $p \rightarrow (p \rightarrow p)$. In the principal relevance logic $\mathbf{R}$, and more generally in extensions of the intensional fragments of $\mathbf{R}$, this postulate amounts to idempotence of the fusion connective $(\cdot)$, so its adoption as an axiom leads to a reduction in the number of independent formulas, improving the chances of decidability.

In [1, p. 98], Anderson and Belnap introduced the purely implicational formal system $\mathbf{RMO}_{\rightarrow}$ axiomatized by

Key words. Residuation, mingle, semiconic, locally tabular, quasivariety.
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(B)	$(p \to q) \to ((r \to p) \to (r \to q))$	(*prefixing*)
(C)	$(p \to (q \to r)) \to (q \to (p \to r))$	(*exchange*)
(I)	$p \to p$	(*identity*)
(W)	$(p \to (p \to q)) \to (p \to q)$	(*contraction*)
(M)	$p \to (p \to p)$	(*mingle*),

where the sole inference rule is *modus ponens*, viz.

(MP) $\langle \{p,\, p \to q\},\, q \rangle$.

The postulates other than (M) axiomatize the implication fragment of $\mathbf{R}$, and they are intuitionistically valid. In $\mathbf{RMO}_{\to}$, the identity axiom is redundant, since it can be derived from (W), (M) and (MP).

Information about $\mathbf{RMO}_{\to}$ can be found in [1, 13, 16]. It follows from a result of Church [8, 9] that $\mathbf{RMO}_{\to}$ enjoys a variant of the classical deduction theorem:

$$\Gamma \cup \{\varphi\} \vdash_{\mathbf{RMO}_{\to}} \psi \quad \text{iff} \quad (\,\Gamma \vdash_{\mathbf{RMO}_{\to}} \varphi \to \psi \ \text{ or } \ \Gamma \vdash_{\mathbf{RMO}_{\to}} \psi\,).$$

As Church observed (in greater generality), this result persists even when we extend $\mathbf{RMO}_{\to}$ by arbitrary new axioms, possibly involving new connectives or sentential constants, *provided* that we do not add any new inference rules.

If we add a negation to $\mathbf{RMO}_{\to}$, as well as the usual axioms of double negation and contraposition, we obtain a definable fusion $p \cdot q := \neg\,(p \to \neg\, q)$, but we also obtain new theorems in the purely implicational vocabulary [2]; systems of this kind have been analyzed in detail in [3, 4, 5, 11].

On the other hand, we might choose to omit negation and to add to $\mathbf{RMO}_{\to}$ a *primitive* fusion and the Ackermann truth constant, accompanied by the usual postulates, as follows:

Definition 1.1. $\mathbf{RMO}^*$ shall denote the formal system with language $\cdot, \to, \mathsf{t}$ that is axiomatized by the postulates of $\mathbf{RMO}_{\to}$, together with

$$p \to (q \to (q \cdot p))$$
$$(p \to (q \to r)) \to ((q \cdot p) \to r)$$
$$\mathsf{t}$$
$$\mathsf{t} \to (p \to p).$$

It turns out that the purely implicational theorems of $\mathbf{RMO}^*$ are just those of $\mathbf{RMO}_{\to}$. The same applies to derivable rules, in view of Church's deduction theorem. This conservation result is explained, for instance, in [17, Remark, p. 267].

The finitely axiomatized extensions of $\mathbf{RMO}^*$ include two well understood systems, viz. the $\wedge, \rightarrow$ fragment of intuitionistic propositional logic and the intensional fragment of $\mathbf{RM}^{\mathbf{t}}$. These two mutually incomparable systems and all of their finitely axiomatized extensions are decidable, because both systems are *locally tabular*—this means that for each finite number n, there are only finitely many logically inequivalent formulas in n variables.

In this paper, we shall prove a simultaneous generalization of these facts by considering the equivalent algebraic semantics for $\mathbf{RMO}^*$, which is the quasivariety of *idempotent commutative residuated po-monoids*. It follows from a result in [12] that an axiomatic extension of $\mathbf{RMO}^*$ will be locally tabular whenever its algebraic counterpart consists of *semiconic* algebras (defined in Section 5). We prove here that this happens exactly when the extension includes the formula $((p \rightarrow \mathsf{t}) \rightarrow p) \rightarrow p$ among its theorems. The result encompasses the intuitionistic case and the case of $\mathbf{RM}^{\mathbf{t}}$.

2. Preliminaries

Given a fixed algebraic language (or type) and an infinite set of variables, let $\boldsymbol{Fm}$ denote the absolutely free algebra, freely generated by the variables. *Formulas* are just elements of the universe of $\boldsymbol{Fm}$, and *substitutions* are endomorphisms of $\boldsymbol{Fm}$.

A (finitary) *formal system* $\mathbf{F}$ over this language is meant here to consist of a set of formulas, called *axioms*, and a set of pairs $\langle \Phi, \varphi \rangle$, called *inference rules*, where $\Phi \cup \{\varphi\}$ is a finite set of formulas. The elements of Φ are called the *premisses* of $\langle \Phi, \varphi \rangle$, and φ is called the *conclusion*.

Given a formal system $\mathbf{F}$, the *deducibility relation* $\vdash_{\mathbf{F}}$ is the relation from sets of formulas to single formulas that contains a pair $\langle \Gamma, \alpha \rangle$ just when there is a *proof of* α *from* Γ *in* $\mathbf{F}$. A proof of this kind is any finite sequence of formulas terminating with α, such that every item in the sequence belongs to Γ or is a substitution instance of a formula that is either an axiom of $\mathbf{F}$ or the conclusion of an inference rule of $\mathbf{F}$, where in the last case, the same substitution turns the premisses of the rule into previous items in the sequence. To signify that such a proof exists, we write $\Gamma \vdash_{\mathbf{F}} \alpha$; then $\langle \Gamma, \alpha \rangle$ is called a *derivable rule* of $\mathbf{F}$. In this case, we omit Γ when it is empty. The *theorems* of $\mathbf{F}$ are the formulas α such that $\vdash_{\mathbf{F}} \alpha$.

Let K be a class of algebras in the language under discussion. The *equational consequence relation* $\models_{\mathsf{K}}$ from sets Σ of equations to single equations $\varphi \approx \psi$ is defined as follows: $\Sigma \models_{\mathsf{K}} \varphi \approx \psi$ iff for every homomorphism h from $\boldsymbol{Fm}$ into an algebra in K, if $h(\alpha) = h(\beta)$ for all $\alpha \approx \beta \in \Sigma$ then $h(\varphi) = h(\psi)$.

For sets of equations Σ and Ψ, the notation $\Sigma \models_{\mathsf{K}} \Psi$ means $\Sigma \models_{\mathsf{K}} \varphi \approx \psi$ for all $\varphi \approx \psi \in \Psi$, and similarly for $\vdash_{\mathbf{F}}$. We shall use $\Sigma \mathrel{=}\joinrel\mathrel{|}\joinrel\mathrel{\models}_{\mathsf{K}} \Psi$ as an abbreviation for the conjunction of $\Sigma \models_{\mathsf{K}} \Psi$ and $\Psi \models_{\mathsf{K}} \Sigma$, and similarly for $\dashv\vdash_{\mathbf{F}}$.

Blok and Pigozzi proposed a general notion of an algebraizable logic in [7]. In current terminology, a formal system $\mathbf{F}$ is said to be (elementarily) *algebraizable* if there exists a quasivariety K in the language of $\mathbf{F}$, as well as a finite family of unary equations $\delta_i(x) \approx \varepsilon_i(x)$, $i \in I$, and a finite family of binary formulas $\Delta_j(x, y)$, $j \in J$, such that for any set of formulas $\Gamma \cup \{\alpha\}$,

$$\Gamma \vdash_{\mathbf{F}} \alpha \quad \text{iff} \quad \{\delta_i(\gamma) \approx \varepsilon_i(\gamma) : \gamma \in \Gamma,\, i \in I\} \models_{\mathsf{K}} \{\delta_i(\alpha) \approx \varepsilon_i(\alpha) : i \in I\};$$
$$\{\delta_i(\Delta_j(x, y)) \approx \varepsilon_i(\Delta_j(x, y)) : i \in I,\, j \in J\} \mathrel{=}\joinrel\mathrel{|}\joinrel\mathrel{\models}_{\mathsf{K}} x \approx y.$$

In this case, for any set of equations $\Sigma \cup \{\varphi \approx \psi\}$, we also have

$$\Sigma \models_{\mathsf{K}} \varphi \approx \psi \quad \text{iff} \quad \{\Delta_j(\alpha, \beta) : \alpha \approx \beta \in \Sigma,\, j \in J\} \vdash_{\mathbf{F}} \{\Delta_j(\varphi, \psi) : j \in J\};$$
$$\{\Delta_j(\delta_i(p), \varepsilon_i(p)) : i \in I,\, j \in J\} \dashv\vdash_{\mathbf{F}} p.$$

Furthermore, the so-called *defining equations* $\delta_i(x) \approx \varepsilon_i(x)$, $i \in I$, and the *equivalence formulas* $\Delta_j(x, y)$, $j \in J$, are unique up to interderivability in $\models_{\mathsf{K}}$ and in $\vdash_{\mathbf{F}}$, respectively, and the quasivariety K is unique [7]. We call K the *equivalent quasivariety* of $\mathbf{F}$.

3. Residuated Po-Monoids

In this section and the next, we discuss the algebraization of $\mathbf{RMO}^*$.

Definition 3.1. A structure $\langle A; \cdot, \to, \mathsf{t}, \leq \rangle$ is called a *commutative residuated po-monoid* (briefly, a *CRP*) if $\langle A; \leq \rangle$ is a po-set, $\langle A; \cdot, \mathsf{t} \rangle$ is a commutative monoid, and $\to$ is a binary *residuation* operator—which means that for all $a, b, c \in A$, we have

$$c \leq a \to b \quad \text{iff} \quad a \cdot c \leq b.$$

This residuation law can be stated equivalently as follows: $\leq$ is *compatible with* $\cdot$ (in the sense of (2) below) and for every $a, b \in A$, there is a *largest* $c \in A$ with $a \cdot c \leq b$. (The largest such c becomes $a \to b$.)

Notation 3.2. From now on, $|x|$ shall abbreviate $x \to x$.

The following properties of CRPs are well known.

Proposition 3.3. *Every CRP satisfies*:

(1) $x \cdot (x \to y) \leq y$

(2) $x \leq y \implies z \cdot x \leq z \cdot y$

(3) $x \leq y \implies z \to x \leq z \to y$ *and* $y \to z \leq x \to z$

(4) $(x \cdot y) \to z \approx y \to (x \to z) \approx x \to (y \to z)$

(5) $x \leq (x \to y) \to y$, *hence*

(6) $((x \to y) \to y) \to y \approx x \to y$

(7) $\mathsf{t} \leq |x|$

(8) $x \leq y \iff \mathsf{t} \leq x \to y \iff |x \to y| \leq x \to y$

(9) $x \approx \mathsf{t} \to x \approx |x| \to x$

(10) $||\,x\,|| \approx |x|$.

A CRP is said to be *idempotent* if it satisfies $x \cdot x \approx x$.

Proposition 3.4. *For any elements a, b of an idempotent CRP, we have*

(11) $a \leq b$ *iff* $a \to b = |a \to b|$; *in particular,*

(12) $\mathsf{t} \leq a$ *iff* $a = |a|$;

(13) $a \to |a| = |a|$;

(14) $a \leq (a \to \mathsf{t}) \to a$.

(15) *if* $\mathsf{t} \leq a \leq b$ *then* $a \cdot b = b$.

Proof. By idempotence, we have $a \cdot a \leq a$ and thus $a \leq a \to a = |a|$. So (11) follows immediately from (8). Then (12) follows from (11), because $\mathsf{t} \to a = a$ (by (9)). Also, (13) follows from (4) and idempotence.

By (1), we have $a \cdot (a \to \mathsf{t}) \leq \mathsf{t}$, so $a \cdot a \cdot (a \to \mathsf{t}) \leq a \cdot \mathsf{t}$, by (2). Thus, $a \cdot (a \to \mathsf{t}) \leq a$, by idempotence, i.e., $a \leq (a \to \mathsf{t}) \to a$.

If $\mathsf{t} \leq a \leq b$ then, by (2) and idempotence, $b = \mathsf{t} \cdot b \leq a \cdot b \leq b \cdot b = b$, so $a \cdot b = b$. □

It follows from (11) that an idempotent CRP $\langle A; \cdot, \to, \mathsf{t}, \leq \rangle$ is definitionally equivalent to its pure algebra reduct $\boldsymbol{A} = \langle A; \cdot, \to, \mathsf{t} \rangle$. So, from now on, we treat these idempotent structures as pure algebras with an equationally definable partial order, always denoted by $\leq$.

Notation 3.5. For the remainder of the paper, IP shall denote the class of all idempotent CRPs.

Obviously, IP is a quasivariety. It is not a variety, as it contains the idempotent CRP on the 3-element chain $-1 < 0 < 1$, where 0 is the identity for $\cdot$ and

$1 \cdot -1 = -1$. It is well known that this 3-element algebra has a homomorphic image that is not an idempotent CRP (see for instance [12]).

In [12, Thm. 3.4], it is shown that every algebra in IP can be embedded into a lattice-ordered algebra in IP. This, together with [17, Cor. 9.4; Remark, p. 267] establishes that for any set of formulas $\Gamma \cup \{\alpha\}$ over the language of $\mathbf{RMO}^*$,

$$\Gamma \vdash_{\mathbf{RMO}^*} \alpha \quad \text{iff} \quad \{\gamma \approx |\gamma| : \gamma \in \Gamma\} \models_{\mathsf{IP}} \alpha \approx |\alpha| \,.$$

Since IP satisfies (10) and $x \leq y \iff x \to y \approx |x \to y|$, it is easy to see that

$$\{x \to y \approx |x \to y| \,,\ y \to x \approx |y \to x|\} \mathrel{=\!\mid\!\models}_{\mathsf{IP}} x \approx y.$$

Thus, we have

Theorem 3.6. $\mathbf{RMO}^*$ *is algebraizable with equivalence formulas* $x \to y$ *and* $y \to x$ *and defining equation* $x \approx x \to x$*, and* IP *is the equivalent quasivariety of* $\mathbf{RMO}^*$.

4. Filters, Relative Congruences and Relative Subvarieties

Definition 4.1. Let $\mathbf{F}$ be a formal system and $\boldsymbol{A}$ an algebra of the same type. A subset X of A is called an $\mathbf{F}$*–filter* of $\boldsymbol{A}$ if for every homomorphism h from $\boldsymbol{Fm}$ into $\boldsymbol{A}$, we have

$h(\varphi) \in X$, for every axiom φ of $\mathbf{F}$;
if $h[\Phi] \subseteq X$ then $h(\varphi) \in X$, for every inference rule $\langle \Phi, \varphi \rangle$ of $\mathbf{F}$.

In this case, for any set of formulas $\Gamma \cup \{\alpha\}$ over the language of $\mathbf{F}$, if $\Gamma \vdash_{\mathbf{F}} \alpha$ and h is a homomorphism from $\boldsymbol{Fm}$ into $\boldsymbol{A}$ with $h[\Gamma] \subseteq X$, then $h(\alpha) \in X$. This follows by induction on the length of a proof of α from Γ in $\mathbf{F}$. Note that arbitrary intersections of $\mathbf{F}$–filters are still $\mathbf{F}$–filters.

A subset X of an idempotent CRP $\boldsymbol{A} = \langle A; \cdot, \to, \mathsf{t} \rangle$ is said to be *upward closed* provided that whenever $a \in X$ and $a \leq b \in A$, then $b \in X$. We call X a *submonoid* of $\boldsymbol{A}$ if $\mathsf{t} \in X$ and whenever $a, b \in X$, then $a \cdot b \in X$.

Lemma 4.2. *Let* $\boldsymbol{A}$ *be an idempotent CRP. Then the* $\mathbf{RMO}^*$*–filters of* $\boldsymbol{A}$ *are exactly the upward closed submonoids of* $\boldsymbol{A}$.

Proof. Suppose X is an $\mathbf{RMO}^*$–filter of $\boldsymbol{A}$. If $a \in X$ and $a \leq b \in A$, then $a \to b = |a \to b|$, by (11), so $a \to b \in X$ (by the identity axiom of $\mathbf{RMO}^*$). In this case, it follows that $b \in X$, by modus ponens. Thus, X is upward closed. Certainly, $\mathsf{t} \in X$ because t is an axiom of $\mathbf{RMO}^*$. If $a, b \in X$ then, since $q \to (p \to (p \cdot q))$ is an axiom of $\mathbf{RMO}^*$, we have $a \cdot b \in X$, by two applications of modus ponens. So X is a submonoid of $\boldsymbol{A}$.

Conversely, suppose X is an upward closed submonoid of $\boldsymbol{A}$. Let h be a homomorphism from $\boldsymbol{Fm}$ into $\boldsymbol{A}$, and let φ be an axiom of $\mathbf{RMO}^*$. Then

$\mathsf{t} \leq |h(\varphi)| = h(\varphi)$, by (7) and Theorem 3.6. So $h(\varphi) \in X$, because $\mathsf{t} \in X$ and X is upward closed. If $a, a \to b \in X$, where $b \in A$, then $a \cdot (a \to b) \leq b$, by (1), whence $b \in X$, because X is an upward closed submonoid of $\boldsymbol{A}$. This shows that X is an $\mathbf{RMO}^*$–filter of $\boldsymbol{A}$. □

Notation 4.3. From now on, given any po-set $\langle A; \leq\rangle$ and $a \in A$, we use $[a)$ to abbreviate $\{b \in A : a \leq b\}$, and $(a]$ to abbreviate $\{b \in A : b \leq a\}$.

If $\boldsymbol{A}$ is an idempotent CRP and $X \subseteq A$, then $\mathrm{Fg}\, X$ shall denote the smallest $\mathbf{RMO}^*$–filter of $\boldsymbol{A}$ containing X. Lemma 4.2 yields:

Corollary 4.4. *For any element a of an idempotent CRP, we have*

$$\mathrm{Fg}\{a\} = [\mathsf{t}) \cup [a).$$

Clearly, the smallest $\mathbf{RMO}^*$–filter of any idempotent CRP is $[\mathsf{t})$. Thus, every $\mathbf{RMO}^*$–filter distinct from $[\mathsf{t})$ contains an element not above t.

Definition 4.5. Let K be a quasivariety and $\boldsymbol{A}$ an algebra of the same type. A congruence θ of $\boldsymbol{A}$ is called a K–*congruence* if the factor algebra $\boldsymbol{A}/\theta$ belongs to K. We refer to K–congruences as *relative congruences* when K is understood.

The K–congruences of $\boldsymbol{A}$ form an algebraic lattice under set inclusion, which coincides with the ordinary congruence lattice when K is a variety and $\boldsymbol{A} \in \mathsf{K}$.

Definition 4.6. Given a subset X of an algebra $\boldsymbol{A}$, we use $\Omega(X)$ to denote the largest congruence of $\boldsymbol{A}$ such that X is a union of congruence classes.

The congruence $\Omega(X)$ always exists. When X is a filter of an algebraizable formal system then $\Omega(X)$ has the internal characterization given in the next theorem. This result is one of several characterizations of algebraizable logics proved by Blok and Pigozzi in [7].

Theorem 4.7. *A formal system $\mathbf{F}$ is algebraizable with equivalent quasivariety K iff for every algebra $\boldsymbol{A}$ of the same type, the mapping $X \mapsto \Omega(X)$, restricted to the $\mathbf{F}$–filters X of $\boldsymbol{A}$, is an isomorphism between the lattices of $\mathbf{F}$–filters and K–congruences of $\boldsymbol{A}$.*

In this case, for every $\mathbf{F}$–filter X of an algebra $\boldsymbol{A}$, we have

$$\Omega(X) := \{\langle a, b\rangle \in A \times A \,:\, \Delta_j^{\boldsymbol{A}}(a, b) \in X \text{ for all } j \in J\},$$

where $\Delta_j(x, y)$, $j \in J$, are the equivalence formulas.

Corollary 4.8. *Let X be an $\mathbf{RMO}^*$–filter of an algebra $\boldsymbol{A} \in \mathsf{IP}$. Then*

(i) $\Omega(X) = \{\langle a, b\rangle \in A \times A : a \to b,\ b \to a \in X\}$, *and this relation is an IP–congruence of $\boldsymbol{A}$.*
(ii) *for any $a \in A$, we have $a \in X$ iff $\langle a, |a|\rangle \in \Omega(X)$.*

(iii) *for any elements* a, b *in* A, *we have* $a \to b \in X$ *iff* $a/\Omega(X) \leq b/\Omega(X)$ *in the factor algebra* $\boldsymbol{A}/\Omega(X)$.

Proof. Item (i) follows from Theorems 3.6 and 4.7. Then (ii) follows from (i). Indeed, $|a| \to a = a$, by (9), while $a \to |a| = |a| \in X$, by (13) and the identity axiom of $\mathbf{RMO}^*$. Finally, (iii) follows from (ii), using (11). □

Definition 4.9. An algebra $\boldsymbol{A}$ in a quasivariety K is said to be K–*subdirectly irreducible* (or *relatively subdirectly irreducible*) if the identity relation on A is completely meet irreducible in the K–congruence lattice of $\boldsymbol{A}$, i.e., $\boldsymbol{A}$ has a least non-identity K–congruence.

Clearly, if a K–subdirectly irreducible algebra $\boldsymbol{A} \in \mathsf{K}$ is a subdirect product of a family of algebras $\boldsymbol{A}_i \in \mathsf{K}$ $(i \in I)$, then $\boldsymbol{A} \cong \boldsymbol{A}_i$ for some $i \in I$. The following adaptation of Birkhoff's subdirect decomposition theorem to quasivarieties is well known (see [15, Thm. 1.1]).

Theorem 4.10. *Every algebra in a quasivariety* K *is isomorphic to a subdirect product of relatively subdirectly irreducible algebras in* K.

Given a po-set $\langle A; \leq \rangle$, and an element $x \in A$, we say that x *splits* $\langle A; \leq \rangle$ if

$$A = [x) \mathbin{\dot{\cup}} (a]$$

for some $a \in A$, where $\dot{\cup}$ indicates *disjoint* union (i.e., $x \not\leq a$). That is to say, x splits $\langle A; \leq \rangle$ iff $a := \max_{\leq} \{b \in A : x \not\leq b\}$ exists, i.e., iff A has a largest element not above x.

Theorem 4.11. *An idempotent CRP* $\boldsymbol{A}$ *is* IP*–subdirectly irreducible iff* t *splits the po-set* $\langle A; \leq \rangle$.

Proof. By Theorems 3.6 and 4.7, $\boldsymbol{A}$ is IP–subdirectly irreducible iff $\boldsymbol{A}$ has a least $\mathbf{RMO}^*$–filter distinct from $[\mathsf{t})$. Since $[\mathsf{t})$ is contained in every $\mathbf{RMO}^*$–filter, the latter demand means that there exists $a \in A$ such that $\mathsf{t} \not\leq a$ and $\mathrm{Fg}\{a\} \subseteq \mathrm{Fg}\{b\}$ whenever $\mathsf{t} \not\leq b \in A$. But for $a, b \in A$ with $\mathsf{t} \not\leq a$, we have

$$\mathrm{Fg}\{a\} \subseteq \mathrm{Fg}\{b\} \text{ iff } a \in \mathrm{Fg}\{b\} = [\mathsf{t}) \cup [b) \text{ (Corollary 4.4) iff } a \in [b) \text{ iff } b \leq a.$$

So $\boldsymbol{A}$ is IP–subdirectly irreducible iff $a := \max_{\leq} \{b \in A : \mathsf{t} \not\leq b\}$ exists. □

Definition 4.12. A *relative subvariety* of a quasivariety K is a subquasivariety M of K such that $\mathsf{M} = \mathsf{K} \cap \mathsf{V}$ for some variety V. Equivalently, it is a subclass of K that is axiomatized, relative to K, by some set of equations.

If M is a relative subvariety of a quasivariety K then for every $\boldsymbol{A} \in \mathsf{M}$, the M–congruences of $\boldsymbol{A}$ are exactly the K–congruences of $\boldsymbol{A}$. So in this case, an algebra in M is M–subdirectly irreducible iff it is K–subdirectly irreducible. This need not be true if M is merely a subquasivariety of K.

Definition 4.13. By a (*co-lingual*) *extension* of a formal system $\mathbf{F}$, we mean any formal system $\mathbf{F}'$ over the same language such that for any set of formulas $\Gamma \cup \{\alpha\}$, if $\Gamma \vdash_{\mathbf{F}} \alpha$ then $\Gamma \vdash_{\mathbf{F}'} \alpha$.

In this case, we call $\mathbf{F}'$ an *axiomatic extension* of $\mathbf{F}$ if there is a set Π of formulas, closed under substitution, such that for every set of formulas $\Gamma \cup \{\alpha\}$, we have $\Gamma \vdash_{\mathbf{F}'} \alpha$ iff $\Gamma \cup \Pi \vdash_{\mathbf{F}} \alpha$.

In practice, axiomatic extensions of $\mathbf{F}$ are normally produced by adjoining new axioms to $\mathbf{F}$ but leaving the inference rules fixed.

In general, the extensions of an algebraizable system are themselves algebraizable, with the same defining equations and equivalence formulas. The next result is a consequence of this. It follows directly from [7, Cor. 4.9, Thm. 2.17].

Theorem 4.14. *If we identify formal systems that have the same deducibility relation, then the extensions of* $\mathbf{RMO}^*$ *are in one-to-one correspondence with the subquasivarieties of* IP, *and the axiomatic ones with the relative subvarieties of* IP. *In the case of the axiomatic extensions, the mutually inverse correspondences are*

$$\mathbf{F} \mapsto \{\boldsymbol{A} \in \mathsf{IP} : \boldsymbol{A} \text{ satisfies } \alpha \approx |\alpha| \text{ for every theorem } \alpha \text{ of } \mathbf{F}\};$$
$$\mathsf{Q} \mapsto \mathbf{RMO}^* \cup \{\alpha : \mathsf{Q} \text{ satisfies } \alpha \approx |\alpha|\}.$$

The former function takes an axiomatic extension to its equivalent quasivariety.

The one-to-one correspondences in this theorem are in fact lattice anti-isomorphisms.

5. Semiconic Algebras

Definition 5.1. A CRP is said to be *conic* if each of its elements a is comparable with t, i.e., $a \leq \mathsf{t}$ or $\mathsf{t} \leq a$.

An idempotent CRP is said to be *semiconic* if it is isomorphic to a subdirect product of conic idempotent CRPs.

Proposition 5.2.

(i) *For any element* a *of a conic CRP, if* $a \to \mathsf{t} < \mathsf{t}$ *then* $\mathsf{t} < a$;
(ii) *Every conic CRP satisfies the quasi-equation* $x \to \mathsf{t} \leq x \implies \mathsf{t} \leq x$.

Proof. (i) If $a \leq \mathsf{t}$ then $\mathsf{t} \leq a \to \mathsf{t}$, by (8). So the result follows from conicity.

(ii) Let $\boldsymbol{A}$ be a conic CRP and $a \in A$. Suppose that $a \to \mathsf{t} \leq a$. By conicity, $a < \mathsf{t}$ or $\mathsf{t} \leq a$. If $a < \mathsf{t}$ then $\mathsf{t} \leq a \to \mathsf{t}$, by (8), and thus $a < a \to \mathsf{t}$, which contradicts $a \to \mathsf{t} \leq a$. So we must have $\mathsf{t} \leq a$, as required. $\square$

In the idempotent case, the following additional properties are known. Proofs can be found in [12, 14].

Lemma 5.3. *Let $\boldsymbol{A}$ be a conic idempotent CRP. Then, for all $a, b \in A$,*

(16) *if* $a \leq b$ *then* $a \cdot b = a$ *or* $a \cdot b = b$;

(17) *if* $a \leq \mathsf{t}$ *then* $a \to a = a \to \mathsf{t}$;

(18) *if* $\mathsf{t} \leq a \leq b$ *then* $a \to b = b$;

(19) *if* $\mathsf{t} \leq a < b$ *then* $b \to a = b \to \mathsf{t}$;

(20) *if* $b \leq \mathsf{t} \leq a$ *then* $a \to b = (a \to \mathsf{t}) \cdot b$ *and* $b \to a = (b \to \mathsf{t}) \cdot a$.

Notation 5.4. We denote the class of all semiconic idempotent CRPs by SCIP.

It is shown in [12] that SCIP is a quasivariety, but not a variety. The next theorem is also proved in [12].

Theorem 5.5. SCIP *is locally finite, i.e., every finitely generated semiconic idempotent CRP is finite.*

In the equivalent quasivariety of an algebraizable logic, finiteness results of this kind have implications for the decidability of the system and its extensions (see Section 6). So Theorem 5.5 prompts the question: which axiomatic extensions of $\mathbf{RMO}^*$ are algebraized by semiconic algebras? In view of Theorem 4.14, this problem amounts to finding a syntactic characterization of the relative subvarieties of IP that consist of semiconic algebras. The solution is given below, and this is the main algebraic result of the present paper.

Theorem 5.6. *A relative subvariety* W *of* IP *consists of semiconic algebras iff* W *satisfies* $x \approx (x \to \mathsf{t}) \to x$.

Proof. ($\Leftarrow$) Suppose W satisfies $x \approx (x \to \mathsf{t}) \to x$, and let $\boldsymbol{A}$ be a relatively subdirectly irreducible algebra in W. In view of Theorem 4.10, it suffices to show that $\boldsymbol{A}$ is conic. Since W is a relative subvariety of IP, $\boldsymbol{A}$ is IP–subdirectly irreducible. So, by Theorem 4.11, $A = (a] \cup [\mathsf{t})$ for some $a \in A$ with $\mathsf{t} \not\leq a$. In particular, $a \to \mathsf{t}$ belongs to $(a]$ or to $[\mathsf{t})$. If $a \to \mathsf{t} \in (a]$, then $\mathsf{t} \leq (a \to \mathsf{t}) \to a = a$, by (8) and the assumption. This contradicts $\mathsf{t} \not\leq a$, so we must have $a \to \mathsf{t} \in [\mathsf{t})$, i.e., $\mathsf{t} \leq a \to \mathsf{t}$. Then $a < \mathsf{t}$ and, since $A = (a] \cup [\mathsf{t})$, this shows that $\boldsymbol{A}$ is conic.

($\Rightarrow$) Conversely, let W consist of semiconic algebras, and suppose that W does not satisfy $x \approx (x \to \mathsf{t}) \to x$. Since subdirect products preserve equations, Theorem 4.10 shows that there is a relatively subdirectly irreducible algebra $\boldsymbol{B}$ in W and an element $b \in B$ such that $b \neq (b \to \mathsf{t}) \to b$. Then, by (14), we must have $b < (b \to \mathsf{t}) \to b$.

Since $\boldsymbol{B} \in \mathsf{W}$ and W is a relative subvariety of IP, $\boldsymbol{B}$ is IP–subdirectly irreducible. But, by assumption, $\boldsymbol{B}$ is a subdirect product of conic algebras from IP, so one of these algebras is isomorphic to $\boldsymbol{B}$. Thus, $\boldsymbol{B}$ is conic.

Now if $\mathsf{t} \leq b \to \mathsf{t}$, then by (3) and (9), $(b \to \mathsf{t}) \to b \leq \mathsf{t} \to b = b$, contradicting $b < (b \to \mathsf{t}) \to b$. So $b \to \mathsf{t} < \mathsf{t}$, by conicity of $\boldsymbol{B}$. It then follows from Proposition 5.2(i) that $\mathsf{t} < b$. So $b \to \mathsf{t} < \mathsf{t} < b < (b \to \mathsf{t}) \to b$. Let

$$B' = \{b \to \mathsf{t},\ \mathsf{t},\ b,\ (b \to \mathsf{t}) \to b\}.$$

We shall show that B' is a subuniverse of $\boldsymbol{B}$. Since B' is linearly ordered, it follows from (16) that B' is closed under $\cdot$. Using (20), (5) and (15), we obtain $(b \to \mathsf{t}) \to b = ((b \to \mathsf{t}) \to \mathsf{t}) \cdot b = (b \to \mathsf{t}) \to \mathsf{t}$. So B' is closed under the term function of $x \to \mathsf{t}$, by (6). Using (17)–(20), we see that for any elements $c, d \in B'$,

$$c \to d \;=\; \begin{cases} d & \text{if } \mathsf{t} \leq c \leq d\,; \\ c \to \mathsf{t} & \text{if } c = d \leq \mathsf{t} \text{ or } \mathsf{t} \leq d < c\,; \\ (c \to \mathsf{t}) \cdot d & \text{if } c \leq \mathsf{t} \leq d \text{ or } d \leq \mathsf{t} \leq c. \end{cases}$$

Therefore, B' is closed under $\to$ (since it is closed under $\cdot$ and under the term function of $x \to \mathsf{t}$). This confirms that B' is the universe of a subalgebra $\boldsymbol{B'}$ of $\boldsymbol{B}$. Let $\boldsymbol{A} = \boldsymbol{B'} \times \boldsymbol{B'}$. Then $\boldsymbol{A} \in \mathsf{W}$, because quasivarieties are closed under subalgebras and products. Let

$$a' = \langle b, b \to \mathsf{t}\rangle, \quad b' = \langle (b \to \mathsf{t}) \to b,\ b \to \mathsf{t}\rangle \quad \text{and} \quad \mathsf{t}' = \langle \mathsf{t}, \mathsf{t}\rangle,$$

so $a', b', \mathsf{t}' \in A$. Now

$$\begin{aligned} (a' \to \mathsf{t}') \to a' &= \langle b \to \mathsf{t},\ (b \to \mathsf{t}) \to \mathsf{t}\rangle \to \langle b, b \to \mathsf{t}\rangle \\ &= \langle (b \to \mathsf{t}) \to b,\ b \to \mathsf{t}\rangle \quad \text{(by (20) and (6))} \\ &= b'. \end{aligned}$$

So $(a' \to \mathsf{t}') \to a' \in \mathrm{Fg}\{b'\}$. But, $a' \notin \mathrm{Fg}\{b'\}$, by Corollary 4.4, because neither $[\mathsf{t}')$ nor $[b')$ contains a'. This, together with Corollary 4.8(iii), shows that the factor algebra $\boldsymbol{A}/\Omega(\mathrm{Fg}\{b'\})$ does not satisfy the quasi-equation

(21) $$x \to \mathsf{t} \leq x \implies \mathsf{t} \leq x$$

(as $a'/\Omega(\mathrm{Fg}\{b'\})$ violates this law). Since W is a relative subvariety of IP and $\boldsymbol{A} \in \mathsf{W}$, any IP–congruence of $\boldsymbol{A}$ is a W-congruence. So $\boldsymbol{A}/\Omega(\mathrm{Fg}\{b'\}) \in \mathsf{W}$, by Corollary 4.8(i). Thus, W does not satisfy (21).

On the other hand, because $\mathsf{W} \subseteq \mathsf{SCIP}$, every quasi-equation that holds in all conic idempotent CRPs must hold in W, and one of these is (21), by Proposition 5.2(ii). This contradiction completes the proof. $\square$

The next example shows that SCIP itself does not satisfy the equation in Theorem 5.6.

Example 5.7. The chain $-2 < 0 < 1 < 2$ is the order reduct of an idempotent CRP $\boldsymbol{A}$ with identity 0, in which

$$a \cdot b \;=\; \begin{cases} \text{the element of } \{a,b\} \text{ with the larger absolute value, if } |a| \neq |b|\,; \\ \min_{\leq}\{a,b\}, \;\; \text{otherwise.} \end{cases}$$

To see quickly that $\cdot$ is associative, note that it is also the minimum operation of a different chain on A, viz. $-2 \prec 2 \prec 1 \prec 0$. (We shall make no further use of $\preceq$.) Now $\leq$ is compatible with $\cdot$, and for all $a, b \in A = \{-2, 0, 1, 2\}$, the set $\{c \in A : a \cdot c \leq b\}$ is non-empty, as $a \cdot -2 = -2$. So this set has a $\leq$–greatest element, which becomes $a \to b$.

Clearly, $\boldsymbol{A} \in \mathsf{SCIP}$. But in $\boldsymbol{A}$, we have $(1 \to 0) \to 1 = (-2) \to 1 = 2 > 1$. This shows that SCIP does not satisfy $x \approx (x \to \mathsf{t}) \to x$.

Corollary 5.8. SCIP *is not a relative subvariety of* IP.

Proof. This follows from Theorem 5.6 and Example 5.7. □

In other words, although SCIP is axiomatizable by quasi-equations, it cannot be axiomatized relative to IP by any set of equations. In fact, because of Corollary 5.8, the following problem is open:

Problem 1. *Axiomatize* SCIP. *Is* SCIP *finitely axiomatizable*?

The analogous problem for the algebras in IP that are subdirect products of *chains* does not seem to be any easier.

6. Logical Consequences

Definition 6.1. If a formal system $\mathbf{F}$ is algebraizable with equivalence formulas $\Delta_j(x, y)$, $j \in J$, then two formulas φ and ψ of $\mathbf{F}$ are said to be *logically equivalent* provided that $\vdash_{\mathbf{F}} \Delta_j(\varphi, \psi)$ for all $j \in J$.

In this case, $\mathbf{F}$ is said to be *locally tabular* if for each integer $n \geq 0$, there are only finitely many logically inequivalent formulas in n fixed variables.

So in $\mathbf{RMO}^*$, logical equivalence of φ and ψ has the expected meaning: $\vdash_{\mathbf{RMO}^*} \varphi \to \psi$ and $\vdash_{\mathbf{RMO}^*} \psi \to \varphi$.

When a formal system $\mathbf{F}$ is algebraizable with equivalent quasivariety K, then $\mathbf{F}$ is locally tabular if and only if K is locally finite. (This follows easily from a consideration of free algebras in K.) In this case, it is clear that $\mathbf{F}$ has the *strong finite model property*, i.e., whenever $\Gamma \nvdash_{\mathbf{F}} \alpha$ (Γ finite) then some *finite* algebra in K witnesses the failure of

$$\Big(\&_{i \in I\,;\, \gamma \in \Gamma} \;\; \delta_i(\gamma) \approx \varepsilon_i(\gamma) \Big) \;\;\Longrightarrow\;\; \delta_k(\alpha) \approx \varepsilon_k(\alpha)$$

for some $k \in I$, where $\delta_i(x) \approx \varepsilon_i(x)$, $i \in I$, are the defining equations. Indeed, some algebra $\boldsymbol{A} \in \mathsf{K}$ must witnesses such a failure (by algebraizability), and then the witnessing elements generate a finite witnessing subalgebra of $\boldsymbol{A}$ (by local finiteness). Theorem 5.6 has the following consequence:

Corollary 6.2. *An axiomatic extension* $\mathbf{F}$ *of* $\mathbf{RMO}^*$ *is locally tabular (and therefore has the strong finite model property) if its theorems include the formula* $((p \to \mathsf{t}) \to p) \to p$.

Proof. Let K be the equivalent quasivariety of $\mathbf{F}$. For any formulas α and β, Theorem 4.14 and (11) show that $\vdash_{\mathbf{F}} \alpha \to \beta$ iff K satisfies $\alpha \to \beta \approx |\alpha \to \beta|$ iff K satisfies $\alpha \leq \beta$. In particular, if $\vdash_{\mathbf{F}} ((p \to \mathsf{t}) \to p) \to p$, then K satisfies $(x \to \mathsf{t}) \to x \leq x$, and therefore $x \approx (x \to \mathsf{t}) \to x$, by (14). Then, since K is a relative subvariety of IP, it follows from Theorem 5.6 that K consists of semiconic algebras. So K is locally finite, by Theorem 5.5, hence the result. $\square$

Using a variant of Harrop's theorem [10] (cf. [6, Lemma 3.13]), we infer:

Corollary 6.3. *If an axiomatic extension* $\mathbf{F}$ *of* $\mathbf{RMO}^*$ *is finitely axiomatized and if* $\vdash_{\mathbf{F}} ((p \to \mathsf{t}) \to p) \to p$*, then* $\mathbf{F}$ *has a solvable deducibility problem, i.e., its set of finite derivable rules is recursive. In particular,* $\mathbf{F}$ *is decidable.*

Recall that the semi-relevant system $\mathbf{RM}$ ('$\mathbf{R}$–*mingle*') is the extension of $\mathbf{R}$ by (M), and that $\mathbf{RM}^{\mathsf{t}}$ is the extension of $\mathbf{RM}$ by the constant t and the axioms t and $\mathsf{t} \to (p \to p)$. These systems are discussed for instance in [1]. Corollaries 6.2 and 6.3 both apply to the $\wedge, \to$ fragment of intuitionistic logic and to the $\cdot, \to, \mathsf{t}$ fragment of $\mathbf{RM}^{\mathsf{t}}$. For these two incomparable systems, the conclusions of the corollaries are of course well known, but their common explanation, via the shared theorem $((p \to \mathsf{t}) \to p) \to p$, is new.

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