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Optimal Control for Polynomial Systems using the Sum of Squares Approach

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Statement

I declare by the present that I have done the present work without the help of others and only using the sources here indicated.

Carlos Alberto Vilcarima Sabroso

Ilmenau, Germany 28. 08. 2018

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Abstract

The optimal control in linear systems is a widely known problem that leads to the solution of one or two equations of Ricatti. However, in non-linear systems is required to obtain the solution of the Hamilton-Jacobi-Bellman equation (HJB) or variations, which consist of quadratic first order and partial differential equations, that are really difficult to solve.

On the other hand, many non-linear dynamical systems can be represented as polynomial functions, where thanks to abstract algebra there are several techniques that facilitate the analysis and work with polynomials. This is where the sum-of-squares approach can be used as a sufficient condition to determine the positivity of a polynomial, a tool that is used in the search for suboptimal solutions of the HJB equation for the synthesis of a controller.

The main objective of this thesis is the analysis, improvement and/or extension of an optimal control algorithm for polynomial systems by using the sum of squares approach (SOS).

To do this, I will explain the theory and advantages of the sum-of-squares approach and then present a controller, which will serve as the basis for our proposal. Next, improvements will be added in its performance criteria and the scope of the controller will be extended, so that rational systems can be controlled. Finally an alternative will be presented for its implementation, when it is not possible to measure or estimate the state-space variables of the system. Additionally, some examples that validated the results are also presented.

Kurzfassung

Die optimale Steuerung in linearen Systemen ist ein weithin bekanntes Problem, das zur Lösung von einer oder zwei Gleichungen von Ricatti führt. In nichtlinearen Systemen ist es jedoch erforderlich, die Lösung der Hamilton-Jacobi-Bellman-Gleichung (HJB) oder Variationen, die aus quadratischen ersten und partiellen Differentialgleichungen bestehen, zu erhalten, die wirklich schwierig zu lösen sind.

Andererseits können viele nichtlineare dynamische Systeme als Polynomfunktionen dargestellt werden, wobei es dank der abstrakten Algebra mehrere Techniken gibt, die die Analyse erleichtern und mit Polynomen arbeiten. Hier kann der Ansatz der Summe der Quadrate als hinreichende Bedingung zur Bestimmung der Positivität eines Polynoms verwendet werden, ein Werkzeug, das bei der Suche nach suboptimalen Lösungen der HJB-Gleichung für die Synthese eines Reglers verwendet wird.

Das Hauptziel dieser Arbeit ist die Analyse, Verbesserung und / oder Erweiterung eines optimalen Regelalgorithmus für polynomische Systeme unter Verwendung des Summe-Quadrate-Ansatzes (SOS).

Um dies zu tun, erkläre ich die Theorie und die Vorteile des Quadrats und führe dann einen Controller ein, der als Grundlage für unseren Vorschlag dienen soll. Als nächstes werden Verbesserungen in seinen Leistungskriterien hinzugefügt und der Umfang des Controllers wird erweitert, um zu ermöglichen, dass rationale Systeme gesteuert werden. Schließlich wird eine Alternative für seine Implementierung präsentiert, wenn es nicht möglich ist, die Zustandsraumvariablen des Systems zu messen oder zu schätzen. Darüber hinaus werden einige Beispiele vorgestellt, die die Ergebnisse validieren.

List of notations

Acronyms

LMI	Linear Matrix Inequalities
SOS	Sum of Squares
SDP	Semi-definite program
HJ's	Hamilton-Jacobian inequality

Functions

$H_e\{P\}$	Function that sums its input with its transpose
$x \otimes y$	Kronecker product

Contents

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List of Figures

List of Tables

1 Introduction

1.1 Motivation

With the industrialization and growth of the technological boom, safer and more effective controllers are needed to control large processes that supply the requirements of our society; however, these processes are usually very complex and their mathematical models correspond to non-linear systems, which, due to their complexity, represent a great challenge when selecting and proposing a controller that not only allows their stabilization in an area of work, but also to make the solution optimal.

Many of the current methods focus on system linearization around a point of operation, such that the resulting mathematical model simplifies the complexity of the problem and the controller is focused on a specific working area; such is the case of systems that use linear models for Pole placement, LQR, H_∞, PID, etc. In other cases it is necessary to work with the entirely non-linear model, because the linearization could lost important and determining information, which complicates the control and its stabilization. For this reason, it is usual to resort to more sophisticated and complicated methods such as the use of non-linear controllers based on Backstepping, Sliding Mode, Neuronal networks among others.

However, there is an alternative to these methods, which take advantage of the polynomial decomposition of nonlinear systems represented as polynomial differential equations, in other words, thanks to this, the Lyapunov or storage function can be proposed as a SOS and be solved computationally with SDP solvers.

Hence the motivation to exploit the advantages of the polynomial systems to propose an optimal controller with greater scope by the SOS approach.

1.2 Previous Work

The main advantage of the representation of non-linear systems as polynomial systems is found in the construction of the Lyapunov function in the form of sum of squares (SOS) [1], which reduces the problem of finding a Lyapunov function that guarantees stability to a set of LMIs, in which it is sought to find a coefficient matrix (P) that satisfies the conditions of definite positiveness as can be seen in Pablo Parrillo's research [2, 3, 4, 5].

The problem can then be solved by computational tools directly, through the use of semi-definite programs such as CVX [6] or SOSTools [7], that, in turn, allow to set objectives to introduce the optimization criteria.

The method that has had the greatest impact and that has aroused our interest, due to its simplicity in construction and efficiency: The Ichihara's controller [8], which proposes an alternative method for the construction of an optimal controller taking advantage of the characteristics obtained when modeling a non-linear system in a polynomial representation.

However, the current control theory, which focuses on the design of controllers using SOS [8, 9], require that all the state-space variables can be measured or at least estimated in order to be able to carry out its control loop, which in many cases is very difficult to obtain. So this controller can not be fully applied, despite having great potential. Therefore, other research works (see [10]) get more relevance, since they perform their control loop directly from the outputs of their systems, which makes them feasible to be used, but not optimal. And this is because their algorithms do not currently does not include performance specifications or do not have the scope and flexibility to emulate a desired behavior, compared to the controllers that have total feedback of its variables.

1.3 Objectives

The central objective of the thesis is the analysis, improvement and/or extension of an optimal control algorithm by using the Sum-of-Squares approach to later analyze the advantages and characteristics of this new theory.

1. Simulate existing algorithms for optimal control in polynomial systems.

2. Implement improvements or extensions in the selected algorithm.
3. Carry out a comparison between the controllers and analyze the effect of the improvements.

1.4 Outline of the Chapters

The content of the following chapters is described as follows.

- Chapter 2: Describes the essential concepts needed to understand how, by means of SOS and SDP, an optimal controller can be synthesized for non-linear systems represented in a polynomial form.
- Chapter 3: An extension of the Ichihara's controller for non-linear systems using the SOS approach is proposed, such as an optimized desired performance criteria, which is used to find the optimal controller for an area of interest. At the same time the scope of the controller will be extended, such that it will be able to control rational systems.
- Chapter 4: A dynamic output feedback controller with the ability to emulate the behavior of the Ichihara's controller will be presented as an alternative for the case in which the system variables can not be measured or estimated.
- Chapter 5: It presents the main conclusions and a perspective of the future work that can be derived from this thesis.

2 Theory and background

In this chapter I present the definitions and concepts of the stability criteria, control techniques and optimization proposals that are described later in the following chapters.

2.1 Stability in the sense of Lyapunov

The Lyapunov stability theory is the basic analysis tool to determine the stability of a non-linear system, that can be described as the following [11]:

$$\dot{x} = f(x), \ x(t_0) = x_0 \tag{1}$$

where $x \in D \subseteq R^n$, $f : D \to R^n$ continuous on D with $x_0 \in D$ as the initial value.

Similarly, the equilibrium point of the system are defined as

$$\bar{x} \in X = \{\bar{x} \in D / f(\bar{x}) = 0\} \tag{2}$$

where X includes the origin thanks to a change of variable.

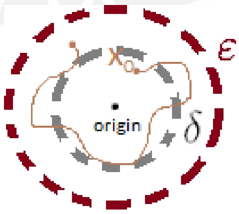

Figure 2.1: Stable equilibrium point

Theorem 1 (Lyapunov Function). [12, 13]
Consider the system (1) and assume there exist a continuously differentiable function $V : D \to R \ \forall x \in R^n$, such that the following statements are satisfied:

1. $V(0) = 0$.
2. $V(x) > 0, \forall x \in D - \{0\}$.
3. $-\dot{V}(x) \geq 0, \forall x \in D - \{0\}$.

The system (1) is stable in the origin. In the case that the third statement is positive definite (greater than zero), then the system (1) is asymptotically stable.

Proof. See [12, 13]. □

2.2 Polynomial systems

The main advantage of the representation of non-linear systems as polynomial systems is the decomposition, which means, if a polynomial function, such as $p(x) \in \mathbb{R}^n$ can be represented like a SOS, then it is a non-negative function [1]. In case it can not be represented directly as SOS, we can make an equivalent conversion by multiplying and dividing the function by a positive one, such that the numerator is a sum of squares:

$$p(x) = \sum_i f_i^2(x) > 0, \tag{3}$$

where $f_i(x)$ represents a polynomial function from x.

Definition 1 (Monomial Function).
A monomial function is a function that has just one element and is represented as [10]

$$m(x) = \prod_{i=1}^{n} x_i^{a_i}, \quad a_i \in \mathbb{Z}$$

Definition 2 (Polynomial Function).
A polynomial function, on the other hand, is a sum of monomials functions that is represented as [10]

$$f(x) = \sum_{i=1}^{m} c_i \cdot m_i(x), \quad c_i \in \mathbb{R},$$

where $c_i \in \mathbb{R}$ are the coefficients and d is the total degree.

Definition 3 (Rational Function).
A rational function is conformed by a polynomial function in the numerator and denominator.

This approach, that can be derivated from the polynomials functions, helps to determine if a function is non-negative by analysing at first if it is SOS.

2.3 Sum of squares

In this section we no longer have the premise that a function is non-negative, but rather, it is the objective to be determined. And from there comes the relevance of this approach and its matrix decomposition representation.

Theorem 2 (Sum of Squares).
A polynomial function is called SOS if there exist a vector of monomials (or polynomials) $z(x)$ and a constant symmetric positive semidefinite matrix Q, such that [5]

$$f(x) = z^T(x)\ Q\ z(x) > 0 \tag{4}$$

Proof. See [2]. □

Example 1.
For example, let's consider the following representation of a function as

$$\begin{aligned} p(x) &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= p_{11} \cdot x_1^2 + (p_{12} + p_{21}) \cdot x_1 \cdot x_2 + p_{22} \cdot x_2^2 \end{aligned}$$

Then, to determine if the function is non-negative, we must determine at first if the function is SOS, which means that the intermediate matrix must be symmetric positive (semi)defined, i.e., all its eigenvalues are greater (or equal) than zero.

Example 2.
The Motskin Polynomial is a non-negative function, but not SoS [3]:

$$p(x) = 1 + x_1^2 \cdot x_2^4 + x_1^4 \cdot x_2^2 - 3 \cdot x_1^2 \cdot x_2^2$$

Therefore, after introducing the function $m(x) = (1 + x_1^2 + x_2^2)$:

$$\begin{aligned} p'(x) = \frac{1}{m(x)} [&(x_1^2 \cdot x_2 - x_2)^2 + (x_1 \cdot x_2^2 - x_1)^2 \\ &+ (x_1^2 \cdot x_2^2 - 1)^2 + \frac{1}{4} \cdot (x_1 \cdot x_2^3 - x_1^3 \cdot x_2)^2 \\ &+ \frac{3}{4} \cdot (x_1 \cdot x_2^3 + x_1^3 \cdot x_2 - 2 \cdot x_1 \cdot x_2)^2], \end{aligned}$$

we will have built a new rational function $p'(x)$, whose numerator is a SOS.

2.3.1 SOS and its relation to stability

As it was previously described in Theorem 1: it is necessary to satisfy $V(x) \geq 0$, $\dot{V}(x) \leq 0$ to determine if the non-linear system is stable.

Therefore, we propose the following SOS Lyapunov function:

$$V(x) = x^T Px, \; P \leq 0. \tag{5}$$

And we transform the non-linear system to its equivalent in the state-space representation form:

$$\dot{x} = A(x)\, x, \;\; x(t_0) = x_0. \tag{6}$$

Then, we build the first derivative of the Lyapunov function.

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T\, P\, x + x^T\, P\, \dot{x} \\ &= (A\, x)^T P\, x + x^T\, P\, (A\;\; x) \\ &= x^T\, (A^T\, P + P\, A)\; x \\ &= -\; x^T\, Q_p\, x,\, Q_p \leq 0. \end{aligned} \tag{7}$$

Finally the stability analysis is reduced to finding a symmetric positive semidefinite matrix Q_p and constant symmetric positive definite matrix P.

2.3.2 SOS decomposition of polynomial matrices

The polynomial matrix Q_p depends on the state-space variables x. This makes the task of determining wheter it is positive definite more complex. For this reason, a new approach must be used in order to reduce the complexity of the problem.

Lemma 1 (SOS Representation).
A polynomial matrix $Q_p(x)$ of dimension n is SOS with respect to the monomial basis $x_{[N]}$ if there exists a symmetric constant matrix Q_1, such that:

$$Q_p = (I_n \otimes x_{[N]})^T\; Q_1\; (I_n \otimes x_{[N]}),\; Q_1 \leq 0. \tag{8}$$

where $\otimes$ represents the Kronecker product and $I_n \in \mathrm{R}^{n \times n}$ is the identity matrix.

Proof. See [14]. □

To understand this in more detail, let's analyze the following statement:

$$Q_p = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} \right)^T \begin{bmatrix} q_{11} & \cdots & q_{16} \\ \vdots & \ddots & \vdots \\ q_{16} & \cdots & q_{66} \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} \right)$$

which after being resolved is transformed to the following

$$Q_p = \begin{bmatrix} 1 & 0 \\ x_1 & 0 \\ x_2 & 0 \\ 0 & 1 \\ 0 & x_1 \\ 0 & x_2 \end{bmatrix}^T Q_1 \begin{bmatrix} 1 & 0 \\ x_1 & 0 \\ x_2 & 0 \\ 0 & 1 \\ 0 & x_1 \\ 0 & x_2 \end{bmatrix} = \left[\begin{array}{c|c} q_{11} + 2\, q_{12} \cdot x_1 + 2\, q_{13} \cdot x_2 + 2\, q_{32} \cdot x_1 \cdot x_2 + q_{22} \cdot x_1^2 + q_{33} \cdot x_2^2 & * \\ \hline * & * \end{array} \right]$$

The next step is to perform equalities on the polynomial coefficients to construct the equivalent matrix Q_1. Note that the equivalent degree must be the same as in Q_p from equation (7), so it should be noted that d and N are the degrees of Q_p and $x_{[N]}$ respectively, where:

1. if d is even, then $2N = d$ else
2. d is odd and $2N = d + 1$.

The problem here is reduced to find matrices P and Q_1 that satisfy the definite positiveness condition, which can be easily solve by the help of some semidefinite programming (SDP) solvers, such as CVx [6].

2.3.3 Schur complement

Another very useful tool to work with LMIs is the Schur Complement, which provides many advantages when it comes to restrict and manipulate matrices [15], such as:

Inverse relationship:

$$\begin{aligned} Q_2 = \begin{bmatrix} P_1 & I \\ I & P_2 \end{bmatrix} \leq 0 &\Leftrightarrow P_2 > 0 \wedge P_1 \leq P_2^{-1} \leq 0 \\ &\Leftrightarrow P_1 > 0 \wedge P_2 \leq P_1^{-1} \leq 0 \end{aligned} \tag{9}$$

Square relationship:

$$Q_2 = \begin{bmatrix} P_1 & P_2 \\ P_2^T & I \end{bmatrix} \leq 0 \Leftrightarrow P_1 \leq P_2^T P_2 \leq 0 \tag{10}$$

It can be applied also like a root relationship with an upper bound P_1.

Throughout this document, improvements and new proposals will be proposed, which use the concepts described here to add to their algorithms a better description of the desired behavior, thanks to the addition of desired performance constraints as optimization criteria.

2.4 Optimal non-linear controller

For the description of this section we take as a basis the work of Dr. Ichihara [8] and represent the full state-space system as a state-dependent linear-like form [16]:

$$\dot{x} = A(x)z(x) + B(x)u, \; x(t_0) = x_0, \; u \in \mathbb{R}^q, \tag{11}$$

where $z(x)$ is a vector of polynomials that is equivalent to the state-space vector x and satisfies the condition of $z(x) = 0$ if and only if $x = 0$ (see [17]). Although this equivalence may or not have the same dimension as x; for the purposes of this thesis we will only work with transformations, where there exists invertible matrices $W(x)$ and $M(x) \in \mathbb{R}^{n\times n}$, such that

$$z(x) = W(x)\, x, \; \dot{z}(x) = \frac{\partial z(x)}{\partial x}\, \dot{x} = M(x)\, \dot{x}, \; z(x) \in \mathbb{R}^n. \tag{12}$$

The cost function $J(x)$ is a very useful and widely used tool, since it assigns numerical values to the performance and behavior of the system; therefore, minimization of $J(x)$ yields

$$J(x) = \int_0^{\infty} (\underbrace{x^T Q x}_{(1)} + \underbrace{u^T R u}_{(2)})\, dt; \; Q > 0 \in \mathbb{R}^{n\times n}, \; R > 0 \in \mathbb{R}^{q\times q}, \tag{13}$$

it is described as the sum of the energy delivered by the controller (1) plus a penalty to the system for moving away from the origin (2) [18].

Later, a Lyapunov function $V(z)$ is proposed as sum of squares form, such that it has a symmetric positive definite matrix P, that will be the incognita to find:

$$V(z) = z^T\ P^{-1}\ z > 0,\ P > 0. \tag{14}$$

Likewise, the Hamilton-Jacobian's inequality is built as the sum of the first derivative of the Lyapunov function and the cost function in order to guarantee stability and add optimization conditions:

$$H(x) = \dot{V} + x^T\ Q\, x + u^T\ R\, u \leq 0. \tag{15}$$

The state feedback that minimizes $J(x)$ is given by

$$u(x) = -R^{-1}\ B^T\ M^T\ P^{-1}\ z(x). \tag{16}$$

After replacing the equations (11), (13), (14) and (16) into (15), the HJ's inequality is rewritten as

$$\begin{aligned} H(x) &= \dot{z}^T\ P^{-1}\ z + z^T\ P^{-1}\ \dot{z} + z^T\ Q\, z + u^T\ R\ u \\ &= z^T\ (H_e\ P^{-1}\ M\ A + Q - P^{-1}\ M\ B\ R^{-1}\ B^T\ M^T\ P^{-1})\ z \leq 0. \end{aligned} \tag{17}$$

As it has been explained in the previous section, the variable z can be omitted from the analysis to focus on the intermediate equivalent matrix. However, since the matrix P is inversed in the equation, we have to multiply P to both sides to eliminate the inverse from the equation and therefore eliminate the problem of bilinearity.

$$\begin{aligned} -H_p(x) &= P\ (H_e\ P^{-1}\ M\ A + Q - P^{-1}\ M\ B\ R^{-1}\ B^T\ M^T\ P^{-1})\ P \\ &= H_e\{M\ A\ P\} + P\ Q\ P - M\ B\ R^{-1}\ B^T\ M^T \leq 0 \end{aligned} \tag{18}$$

Finally, 18 can be formulated using Lemma 1. Numerical results can be obtained with a SDP solver.

However, and as is often the case, non-linear systems can not be controlled globally (for all values of x). For this reason the design resorts to the theory of Positivestellensatz [19], which adds a function $h(x)$ to the first derivative of the Lyapunov function, such that when the system moves away from the area of interest the condition of positiveness is still satisfied:

$$h(x) = 1 - z^T\ S_x\ z,\ S_x > 0 \in \mathbb{R}^{n\times n}. \tag{19}$$

Thereby the algorithm can focus on finding the best fit for the area of interest X :

$$X = \{x \in R^n \mid h(x) \geq 0\}. \quad (20)$$

It also assumes an invariant set with p as upper bound:

$$\varepsilon(P^{-1}, p) = \{x \in R^n \mid x^T P^{-1} x \leq p, p > 0\} \subseteq X, \quad (21)$$

Consequently, the HJ's inequality is rewritten as

$$-H_p(x) = H_e\{M A P\} + P Q P - M B R^{-1} B^T M^T + h(x) \tilde{S}_R(x) \leq 0 \quad (22)$$

with

$$\tilde{S}_R(x) = (I \otimes x_{[N_R]})^T Q_R (I \otimes x_{[N_R]}), \quad (23)$$

where $S_R(x)$ is a symmetric polynomial matrix $R^{n\times n}$, that is introduced, so that the function $h(x)$ has the appropriate dimensions to be added to the HJ's function.

Finally, if the system starts from an initial position $x_0 \in s(P^{-1}, p) \subset X$ and the HJ's inequality is satisfied, then the system response will remain in $\varepsilon(P^{-1}, p)$ and converge to the origin.

Theorem 3 (Ichihara's Optimal Controller).
A system described given by (11) with the control law 16 and $x_0 \in s(P^{-1}, p) \subset X$ is asymptotically stable in the origin if there exist a symmetric positive definite matrix P; symmetric positive semidefinite matrices Q_R, Q_1; and positive scalar values β and ξ; such that the following matrices are positive semidefinite:

$$S_1(x) = \begin{bmatrix} MBR^{-1}B^T M^T - H_e\{M A P\} - h(x)\tilde{S}_R(x) & PQ^{0.5} \\ Q^{0.5}P & I \end{bmatrix} - \xi I \leq 0,$$

$$Q_2 = \beta S_x^{-1} - P \leq 0,$$

$$Q_3 = \begin{bmatrix} \beta & \beta z_0^T \\ \beta z_0 & P \end{bmatrix} \leq 0,$$

$$Q_4 = P - \xi I \leq 0,$$

$$Q_5 = \beta - \xi \leq 0.$$

Proof. See [8]. □

$S_1(x)$ is decomposed to a SOS form:

$$S_{1(x)} = (I \otimes x_{[N]})^T Q_1 \, (I \otimes x_{[N]}), \; I \in \mathrm{R}^{2n\times 2n}.$$

And the objective of the algorithm is to minimize the value of the cost function: $J \leq x_0^T P^{-1} x_0$ for the intial conditions. But, since the SD program can not work directly with the inverse, it is necessary to propose a change of variable, such that there exist a defined positive matrix Y, which replaces P^{-1} in the upper bound of $J(x)$.

$$Q_6 = \begin{bmatrix} Y & I \\ I & P \end{bmatrix} \leq 0 \Leftrightarrow P \leq 0 \wedge Y \leq P^{-1} \leq 0 \quad (24)$$

Finally β is used as the inverse of p and ξ as the tolerance for the definite positiveness.

2.5 Conservativeness reduction

While the optimal control strategy allows us to directly find the optimal controller that will be needed for given conditions, it still has a high degree of conservativeness, which greatly restricts the response of the controller and requires a higher computational cost in its execution.

Therefore, in this section, I propose some tools that reduces the degree of conservativeness while expanding the range of solutions to find the best fit for the controller. For this purpose, we will use the theory presented in [8], which is explained in detail in the following subsections.

2.5.1 Structural reducion of LMI

Until now, and thanks to the fact that the current theory only resorts to constant matrices P, Q and R; the folowing transformation can be used, instead of the kronecker product, to reduce the dimensions of Q_1, and hence, reduce the computational cost:

$$S_1(x) = \underbrace{\begin{bmatrix} \begin{matrix} I_n \otimes v_{[N]} \\ I_n \end{matrix} & 0 \\ 0 & I_n \end{bmatrix}^T}_{s_{(1)}x} Q_1 \begin{bmatrix} \begin{matrix} I_n \otimes v_{[N]} \\ I_n \end{matrix} & 0 \\ 0 & I_n \end{bmatrix}, \; Q_1 > 0, \; I \in \mathrm{R}^{n\times n} \quad (25)$$

where $v_{[N]}$ is the vector of monomials that does not include the unit value, O is a matrix of zeros and I_n is the identity matrix.

This reduction is made posible, because the matrix $S_1(x)$ has constant values in its last three quadrants as seen in the theorem 3:

$$S_1(x) = \left[\begin{array}{c|c} * & P\,Q^{0.5} \\ \hline Q^{0.5}\,P & I \end{array}\right] - \xi\, I$$

Proof. See [8]. □

To appreciate the reduction of dimensions of Q_1 in better detail, the following example is proposed.

Example 3.
If we consider $x = [x_1\ x_2]^T$, $N = 1$ and $n = 3$, then we have $x_{[N]} = [1\ x_1\ x_2]^T$, $v_{[N]} = [x_1\ x_2]^T$.

$$S_1(x)_{kronecker} = (I \otimes x_{[N]})^T \cdots \Rightarrow \begin{bmatrix} I_6 \\ \mathbb{R}^{6\times 6} \\ \end{bmatrix} \in \mathbb{R}^{18\times 6} \Rightarrow Q_1 \in \mathbb{R}^{18\times 18}$$

$$S_1(x)_{reduced} = \left[\begin{array}{c|c} I_n \otimes v_{[N]} & 0 \\ I_n & \\ \hline 0 & I_n \end{array}\right]^T \cdots \Rightarrow \left[\begin{array}{c|c} \mathbb{R}^{6\times 3} & 0_{9\times 3} \\ I_3 & \\ \hline 0_{3\times 3} & I_3 \end{array}\right] \in \mathbb{R}^{12\times 6} \Rightarrow Q_1 \in \mathbb{R}^{12\times 12}$$

As can be seen in the previous equality, the matrix Q_1 corresponding to the kronecker product is much larger than the one proposed by this reduction.

In the same way, it can be seen the remarkable reduction of computational costs that this approach will have when it is used for higher order systems.

2.5.2 Polynomial annihilators

Thanks to the representation of the problem as in the lemma 1, the main objective that must be met to guarantee an asymptotically stable solution is that $-H_p(x) \leq 0$.

However, the fact that this is not fulfilled, does not mean that the system can not be stabilized, which add some degree of conservativeness, limiting the range of the solutions. This effect can be clearly seen in the following example. Consider:

$$z(x) = \begin{bmatrix} x_1 \\ x_2^2 \end{bmatrix},\; P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \; and - H_p(x) = \begin{bmatrix} x_2^4 & -\frac{1}{2}\cdot x_1 \\ -\frac{1}{2}\cdot x_1 & -x_1^2 \end{bmatrix}, \text{then}$$

$$H(x) = x_1^2\cdot x_2^4 - \frac{1}{2}\cdot x_1^2\cdot x_2^2 - \frac{1}{2}\cdot x_1^2\cdot x_2^2 - x_1^2\cdot x_2^4$$

$$H(x) = -x_1^2\cdot x_2^2 \leq 0,\; \forall x \in \mathbb{R}$$

When we start analysing the eigenvalues of $-H_p(x)$, it can be clearly seen that it does not accomplish the negative semidefiniteness condition, which is mandatory to guarantee the stability criteria, but, on the other hand, when the equation (17), described as $H(x)$, is analyzed, we find that the system is stable under those conditions.

However, the bilinearity does not allow the problem to be resolved directly, so it is proposed the use of a two step approach, where in the first step we find P by applying the Theorem 3, and in the second step we scale P by reducing the conservativeness and obtaining a better value of $J(x)$. From there and with the addition of polynomial annihilators, that are proposed in the following theorem, a better value of P can be found [8, 20, 21].

Theorem 4 (Ichihara's Optimal Controller with Annihilator).
Assume Theorem 3 is satisfied and there exist symmetric positive semidefinite matrices Q_r and Q_1, and polynomial matrices $N(x) \in \mathbb{R}^{n\times n}$ and $N_A(x) \in \mathbb{R}^{n\times n}$, such that

$$N(x)\, P^{-1} z(x) = 0,\; N_A(x)\, z(x) = 0,\; \forall x \in \mathbb{R}^n.$$

If $\alpha \in \mathbb{R}$ is maximize and the following inequalities are satisfied

$$S_{1(x)} = \left[\begin{array}{c|c} MBR^{-1}B^TM^T - h(x)\tilde{S}_R(x) & \\ -H_e\{M\,[\alpha A + N_A(x)]\,P - N(x)\} & (\alpha P)\,Q_1^{\frac{1}{2}} \\ \hline Q_1^{\frac{1}{2}}(\alpha P) & I \end{array}\right] - \varepsilon\cdot I \leq 0,$$

$$Q_2 = \beta S_x^{-1} - \alpha P \leq 0,$$

$$Q_3 = \begin{bmatrix} \beta & \beta x_0^T \\ \beta x_0 & \alpha P \end{bmatrix} \leq 0,$$

$$Q_4 = \beta - \varepsilon \leq 0,$$

$$Q_5 = \alpha - \varepsilon \leq 0.$$

Then, the asymptotic stability will be guaranteed and the new performance will be better or equal to the one obtained with Theorem 3. In addition the new control law is

$$u = -R^{-1} B^T M^T (\alpha P)^{-1} z(x).$$

Proof. See [8]. □

3 Improvements and new proposals to the optimal control algorithms

In this section, I will present some numerical examples of the current theory, that have been solved by using CVx on Matlab, and then I propose improvements to the optimal controller

3.1 Numerical examples of the current theory

Example 4 (Ichihara's Controller).
Let's consider the non-linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ x_1 + x_2 & x_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

The conditions for the development of the controller are in the table below.

Table 3.1: Example 4 - Conditions

ξ	Q	R	S_x	x_0	W
0.001	$\begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{bmatrix}$	1	$\begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 0.3 \\ -0.4 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

After adding this information to the Theorem 3 the following results are obtained (rounded to two digits).

- Lyapunov function: $V_{(x)} = 0.52\, x_1^2 - 0.34\, x_1 x_2 + 0.42\, x_2^2$.
- Control variable: $u_{(x)} = -0.91\, x_1 - 2.7\, x_2$.
- Upper bound: $p = 0.42$.
- $P = \begin{bmatrix} 0.52 & 0.17 \\ -0.17 & 0.42 \end{bmatrix}$

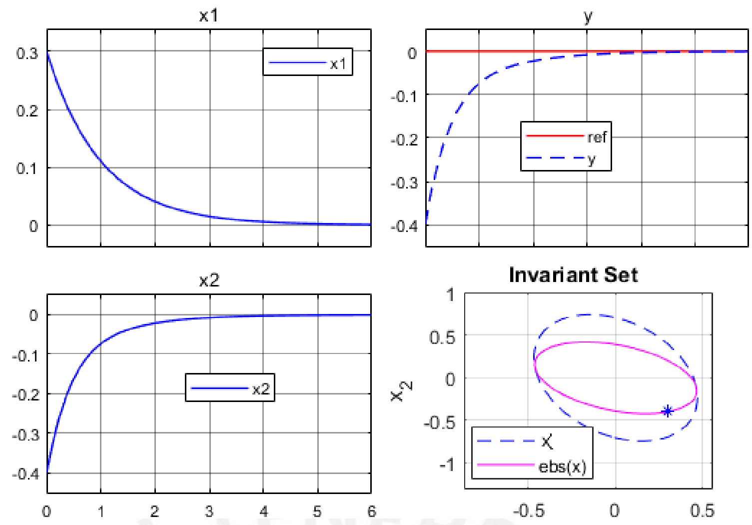

Figure 3.1: Example 4 - System response

The following observations can be obtained through the figure 3.1:

- Since $\varepsilon(P^{-1}, p) \subseteq \mathsf{X}$, if we want to have a larger area in which the controller can work and be optimal at the same time, the X defined by $h(x)$ must grow.
- The initial condition x_0 delimits the minimum volume in which the controller should be optimal.
- The gains matrices (Q, R) of the cost function $J(x)$ determines the behavior of the system in terms of time of stabilization and response speed.

In the table 3.2 we can see the effect of changing the invariant set X and the initial condition x_0 for the previous conditions.

Table 3.2: Comparison of the invariant set effect

	$h(x)$ Gain Matrix (S_x)	Initial Condition x_0	Upper Bound p	Control Variable $u_{(x)}$
1	$\begin{bmatrix} 0.5 & 0.1 \end{bmatrix}$	$\begin{bmatrix} 0.8 \\ -0.4 \end{bmatrix}$	7.62	$1.7\,10^{-8}\,x_1 - 20\,x_2$
2	$\begin{bmatrix} 0.1 & 0.2 \\ 1.8^{-2} & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0.8 \\ -0.4 \end{bmatrix}$	2.97	$3.3\,10^{-7}\,x_1 - 4.6\,x_2$
3	$\begin{bmatrix} 1.2 & 0 \\ 0 & 1.2^{-2} \end{bmatrix}$	$\begin{bmatrix} 0.2 \\ 0.65 \end{bmatrix}$	2.77	$2.9\,10^{[illegible]}\,x_1 - 6.1\,x_2$

Example 5 (Ichihara's Annihilator).
As an example, we will use the system described in the Ichihara's paper [8, first numerical example].

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1^2 & 1 \\ 0 & x_2 + x_2 \cdot x_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

The conditions for the analysis are described in the table below, while the results of the algorithm for both theorems 3 (optimal controller) and 4 (optimal controller with polynomial annihilators) are shown in the table 3.4 and figures 3.2, 3.3, 3.4.

Table 3.3: Example 5 - Conditions

ξ	Q	R	S_x	x_0	W
0.001	$\begin{bmatrix} \frac{1}{10} & 0 \\ 0 & 1 \end{bmatrix}$	1	$\begin{bmatrix} 1.05^{-2} & 0 \\ 0 & 1.05^{-2} \end{bmatrix}$	$\begin{bmatrix} 0.4 \\ -0.2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Table 3.4: Comparison of the addition of the polynomial annihilator

	Optimal Control	Annihilator
Control Variable $u_{(x)}$	$12\,x_1 - 9\,x_2$	$6.8\,x_1 - 5.2\,x_2$
Matrix P	$\begin{bmatrix} 0.12 & -0.16 \\ -0.16 & 0.32 \end{bmatrix}$	$\begin{bmatrix} 0.21 & -0.27 \\ -0.27 & 0.55 \end{bmatrix}$
Upper bound p	2.54	1.36

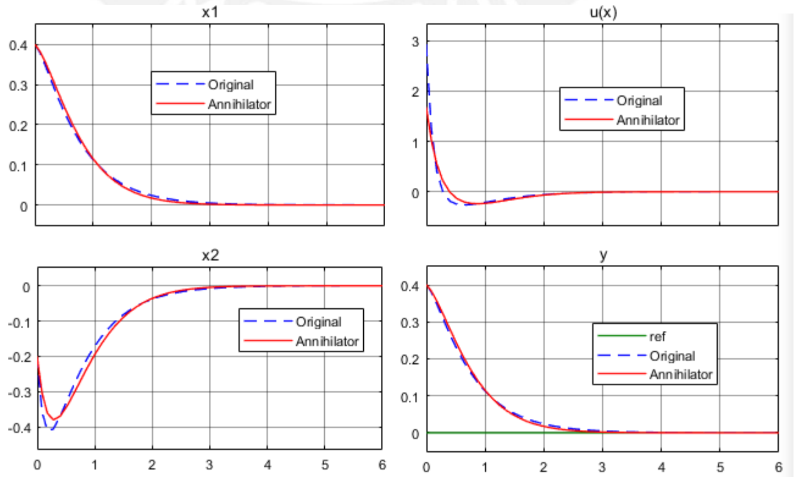

Figure 3.2: Comparison of the system response with the addition of the polynomial annihilator

Since the value of the cost function in Theorem 4 is lower, it is confirmed that the performance of the system has been improved thanks to the addition of the polynomial annihilator and how it can be seen in the figure 3.2, the system response is faster and with less overshoot.

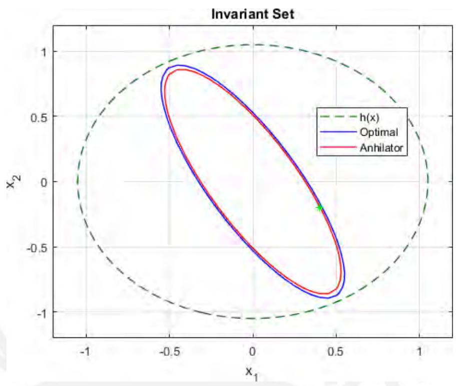

Figure 3.3: Comparison of the invariant set with the addition of the polynomial annihilator

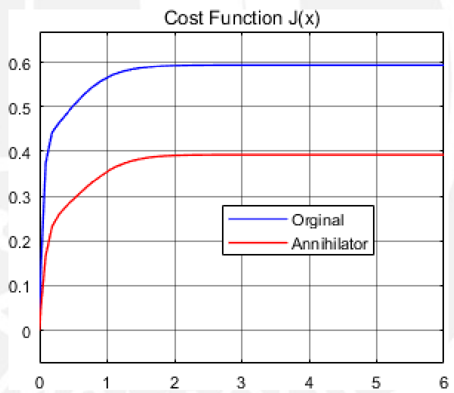

Figure 3.4: Comparison of the cost function with the addition of the polynomial annihilator

3.2 Variable gain matrix of the cost function

To find the best controller for a system, it is necessary that the desired performance specifications are accurately described in the cost function. In other words, the weight matices $Q(x)$ and $R(x)$ can be variable, such as, the penalty is not constant, but, rather, variable with respect to the area of interest.

First and since the controller will be designed to work within an area, where the solution has to be optimal, the state-space penalty function $Q(x)$ should only be constructed to describe the desired conditions in it. Then, as a solution it is propose that the further away the states of the system are to the origin, then the penalty values should increase. In other words, it is penalized mored when the solution approaches the limits of the controller. Finally, the remaining values of the penalty matrix must define the desired characteristics.

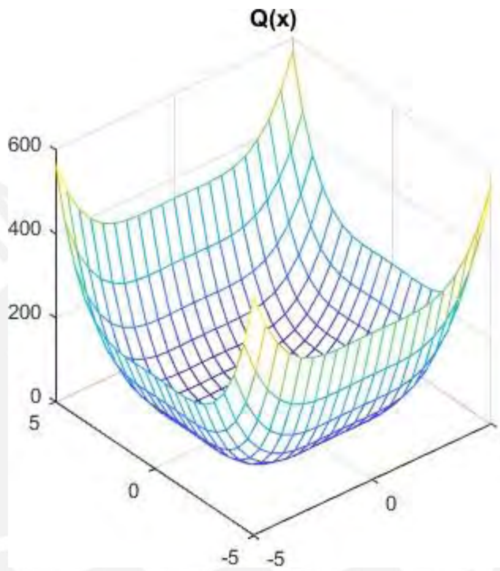

Figure 3.5: Variable gain matrix $Q_{(x)}$

The figure 3.5 shows how these three conditions fulfill this concept while describing a desired performance, which is based on an exponential penalty function for the state-space variables.

On the other hand, it should be noted that the matrix $Q(x)$ can not have a freely arbitrary value, but rather has a couple of conditions to satisfy and which in turn are a bit restrictive, such as

$$S_1(x) = \begin{bmatrix} F & P\,Q^{\frac{1}{2}} \\ Q^{\frac{1}{2}}\,P & I \end{bmatrix} \leq 0 \Rightarrow F \leq P\;Q\;P \leq 0,\;\; Q > 0. \qquad (26)$$

To save some space we will refer to F as the equivalent matrix that goes in the first quadrant of $S_1(x)$.

For this reason the Schur Complement is used to transform the matrix $S_1(x)$ into an equivalent, which gives the variable $Q(x)$ greater flexibility:

$$S_1(x) = \begin{bmatrix} F & P \\ -1P & \end{bmatrix} \leq 0. \tag{27}$$

Although the polynomial matrix $Q_{(x)}$ is no longer represented as a root, we do need to represent the matrix $S_1(x)$ as a polynomial matrix, therefore the following substitution is proposed

$$\bar{J}(x) = \int_0^{\infty} (z^T Q^{-1} z + u^T R u)dt; \; Q > 0, R > 0 \tag{28}$$

Hereinafter we can focus on finding a polynomial matrix $Q(x)$ that must be positive defined and invertible in the invariant set X .

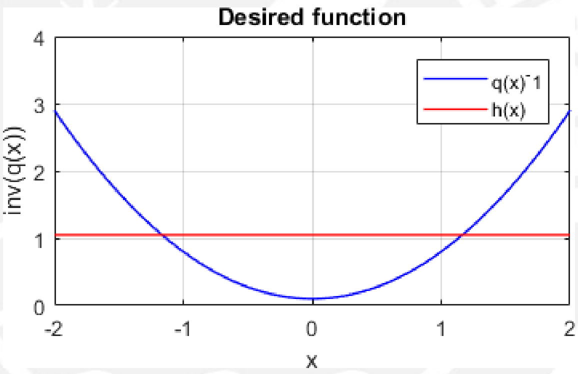

Figure 3.6: Desired Gain Function

However, obtaining a polynomial matrix whose inverse satisfies the condition shown in the figure 3.6 (exponential penalty) is not easy, much less simple, because it requires a high degree function. So, instead of looking for a function that fully satisfy this condition, we look for a function that satisfies the condition locally and contains a local maximum (the origin) and two local minimums (according to the desired specifications), such that, the curve drawn in this area describes a parabola or exponential function:

$$\begin{aligned} \frac{\partial q_{(x)}}{\partial x} &= a \cdot (x - x_{min}) \cdot (x - x_0) \cdot (x - x_{max}) > 0 \\ &= a \cdot (x^3 - (x_{min} + x_{max}) \cdot x^2 + x_{min} \cdot x_{max} \cdot x), \\ q_{(x)} &= \frac{a}{4} \cdot x^4 - \frac{a}{3} \cdot (x_{min} + x_{max}) \cdot x^3 + \frac{a}{2} \cdot x_{min} \cdot x_{max} \cdot x^2 + c > 0 \end{aligned} \tag{29}$$

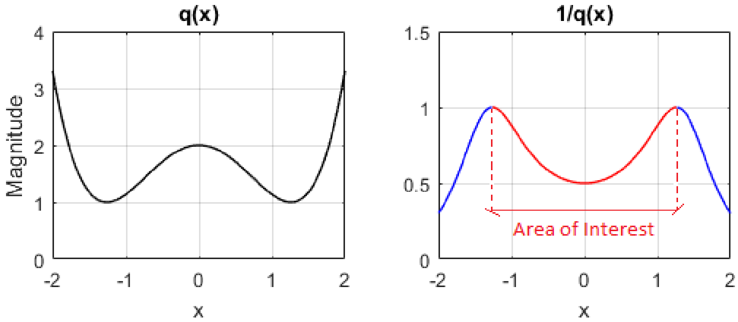

Figure 3.7: Proposed gain function

Theorem 5 (Optimal Controller $Q_{(x)}$).
Assume all condition of Theorem 3 are satisfied with S_1 replaced by

$$\bar{S}_1 = \begin{bmatrix} MBR^{-1}B^TM^T - H_e\{M\ A\ P\} - h(x)\tilde{S}_R(x) & P \\ P & Q(x) \end{bmatrix} \leq 0, \quad (30)$$

Proof. Given the new cost function $\bar{J}(x)$, described in the equation (28), the HJ's inequality is rewritten as

$$-H_p(x) = H_e\{M\ A\ P\} + P\ Q^{-1}\ P - M\ B\ R^{-1}\ B^T\ M^T + h(x)\ \tilde{S}_R(x) \leq 0$$

which, thanks to the Schur complement, can be transformed to an equivalent matrix $S_1(x)$ as

$$\bar{S}_1(x) = \begin{bmatrix} F & P \\ P & Q(x) \end{bmatrix} \leq 0 \Rightarrow F - P\ Q^{-1}\ P \leq 0$$

where $F = -H_e\{M\ A\ P\} + M\ B\ R^{-1}\ B^T\ M^T - h(x)\ \tilde{S}_R(x)$ □

Theorem 5 allows the addition of a variable gain matrix *Q(x)* to the algorithm and the following lemmas present the proposals for its value.

Lemma 2 ($Q(x)$ as a cuadratic function).
In this case it is desired that the matrix $Q(x)$ be described as (29) and satisfy the following conditions:

1. The extreme points of the area of interest (see figure 3.7) contain the invariant

set X and $x_{lim} = x_{max} = -x_{min}$ with

$$x_{ilim} = (1+\theta)\ eig(S_x)_i^{-\frac{1}{2}},\ S_x \in \mathbb{R}^{n\times n},\ 0 < i \leq n,$$
$$q(x) = \frac{a}{4}x^4 - \frac{a}{2}{x_{lim}}^2 x^2 + c > 0, \tag{31}$$

for $\theta \in [1.2, 1.5]$ the percentage, whose usefulness is to extend the curve to emphasize the limits of the area of interest.

2. The polinomial matrix $Q(x)$ must be symmetric positive definite, therefore the constant value: $c \geq 1$.
3. The equation, that described the distance d between the points of inflection of the curve, is ruled by:

$$d = \frac{q(x_0)}{q(x_{lim})},\ a = \frac{4 \cdot c \cdot (1 - \frac{1}{d})}{x_{lim}^4}$$

4. The matrix $Q_{(x)}$ is built as

$$Q(x) = (\sum_i^n q_i(x_i)) \cdot I_n,\ I_n \in \mathbb{R}^{n\times n},$$

Then, the closed-loop system is asymptotically stable in the origin with performance $\bar{J}(x)$.

Similarly, it should be mentioned that in this case the structural reduction of LMI can not be used, because the structure of $S_1(x)$ now contains variables of space state in its last quadrant.

Example 6.

As an example, we will use the same system exposed in the example 5 employing the same conditions for the design of the optimal controller with the state dependent the matrix Q, in order to be able to compare the results and understand the advantages of this improvement proposal.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1^2 & 1 \\ 0 & x_2 + x_2 \cdot x_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

The specifications for the design of the matrix *Q(x)* are shown in the table below.

Table 3.5: Example 6 - Specifications of $Q(x)$

	x_1	x_2
Constant c	2	2
Distance d	5	5
Gain percentage θ	1.2	1.2

Subsequently, the functions that conforms the matrix *Q(x)* are

$$q_1(x) = 2 + 0.4\,x_1^4 - 1.26\,x_1^2,$$
$$q_2(x) = 2 + 0.63\,x_2^4 - 2.02\,x_2^2,$$
$$Q(x) = (q_1(x) + q_2(x)) \cdot \begin{bmatrix} 1 & 0 \\ 0 & \end{bmatrix}.$$

The graphics of *Q(x)* and its inverse are shown in the figure 3.8, where it can be clearly seen the region of interest as the concave surface of the inverse matrix of *Q(x)*

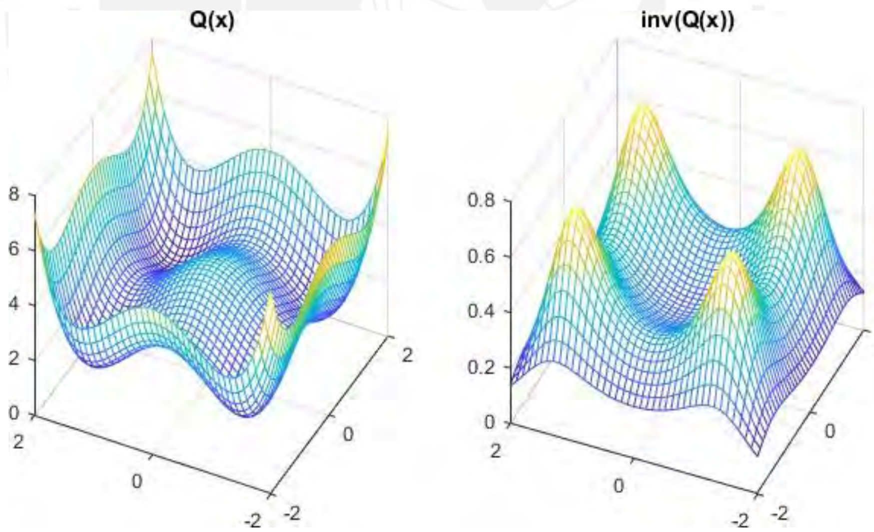

Figure 3.8: Matrix $Q_{(x)}$ in 3D

Finally the results are shown in the figures 3.10, 3.9 and table 3.6, where it can be seen that the upper bound on $\bar{J}(x)$ has been reduced with success.

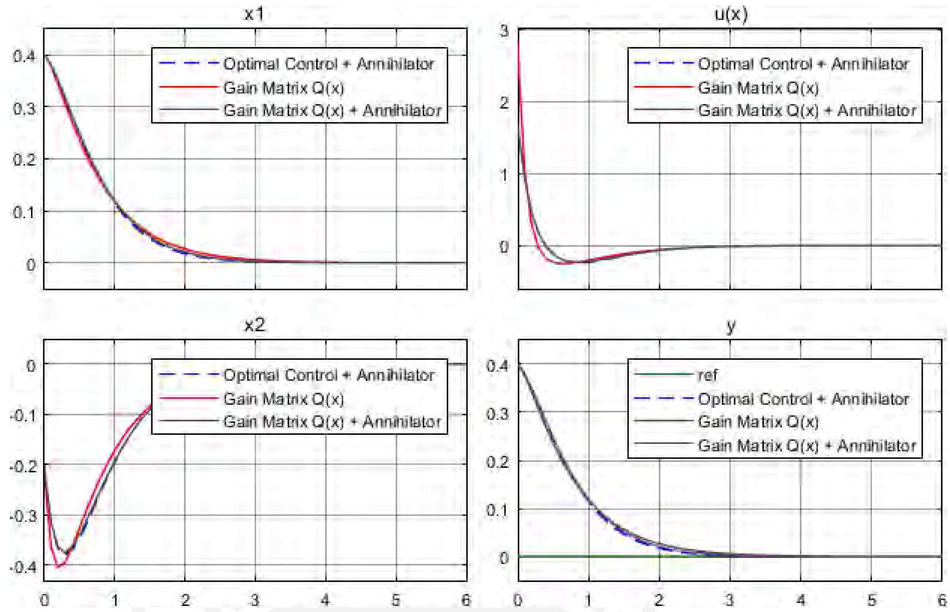

Figure 3.9: Comparison of the system response with $Q_{(x)}$

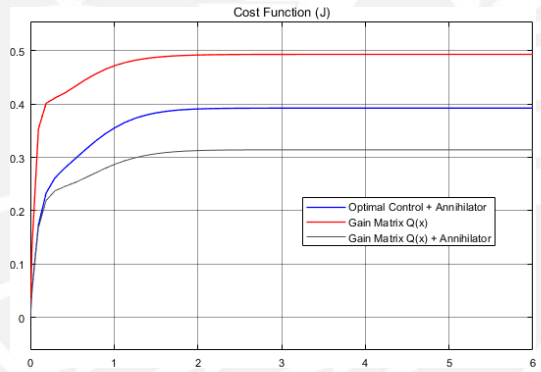

Figure 3.10: Comparison of the cost function with $Q_{(x)}$

Table 3.6: Comparison of the addition of $Q_{(x)}$

	Optimal Control with $Q_{(x)}$	Optimal Control with $Q_{(x)}$ and Annihilator
Control Variable $u_{(x)}$	$-11.69 \cdot x_1 - 9.18 \cdot x_2$	$-6.93 \cdot x_1 - 5.44 \cdot x_2$
Matrix P	$\begin{pmatrix} 0.13 & -0.17 \\ -0.17 & 0.33 \end{pmatrix}$	$\begin{pmatrix} 0.23 & -0.29 \\ -0.29 & 0.55 \end{pmatrix}$
Upper bound p	2.32	1.32
Cost function J	0.5	0.31

Example 7.

In the same way, during the development of this research, the following matrices were also proposed as alternatives:

$$Q_{1_a}(x) = \begin{bmatrix} q_1(x) & 0 \\ 0 & q_2(x) \end{bmatrix} \wedge Q_{1_b}(x) = \begin{bmatrix} q_2(x) & 0 \\ 0 & q_1(x) \end{bmatrix}$$

However, those matrices did not present the best results compared to the Lemma 2, as it can be seen in the tables 3.7 and 3.8, and this is due to the fact that the surface of the matrix $Q_1(x)$ does not fully contain $\mathcal{X}$, how it is shown in the figure 3.11.

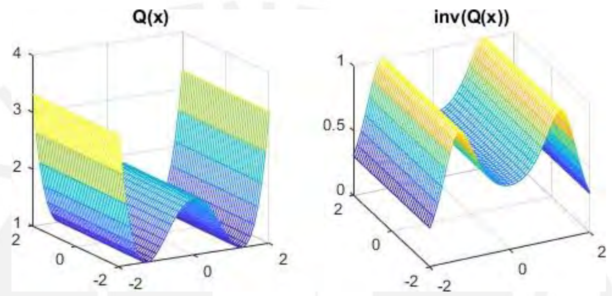

Figure 3.11: Matrix $Q_1(x)$ in 3D

Table 3.7: Comparison of the addition of $Q_{1_a}(x)$

	Optimal Control with $Q_{1_{a(x)}}$	Optimal Control with $Q_{1_{a(x)}}$ and Annihilator
Control Variable $u_{(x)}$	$-12.35\ x_1 - 8.87\ x_2$	$-8.508\ x_1 - 6.11\ x_2$
Matrix P	$\begin{bmatrix} 0.09 & -0.12 \\ -0.12 & 0.28 \end{bmatrix}$	$\begin{bmatrix} 0.13 & -0.18 \\ -0.18 & 0.41 \end{bmatrix}$
Upper bound p	3.14	2.13
Cost function J	0.67	0.49

Table 3.8: Comparison of the addition of $Q_{1_b}(x)$

	Optimal Control with $Q_{1_{b(x)}}$	Optimal Control with $Q_{1_{b(x)}}$ and Annihilator
Control Variable $u_{(x)}$	$-12.09\ x_1 - 8.78\ x_2$	$-7.86\ x_1 - 5.71\ x_2$
Matrix P	$\begin{bmatrix} 0.10 & -0.14 \\ -0.14 & 0.31 \end{bmatrix}$	$\begin{bmatrix} 0.16 & -0.22 \\ -0.22 & 0.47 \end{bmatrix}$
Upper bound p	2.82	1.79
Cost function J	0.64	0.45

Lemma 3 ($Q(x)$ as $h(x)$).
Consider $Q(x)$ as a function $h(x)$ that includes the invariant set X :

$$Q(x) = (c + z^T S_x z)\ I_n, \quad I_n \in \mathbb{R}^{n\times n},$$

where the constant value $c > 0$ and I_n is an identity matrix.

Example 8.
Consider the system exposed in the example 5 with its same conditions and

$$Q_2(x) = \begin{bmatrix} 2.05 + x^T S_x x & 0 \\ 0 & 2.05 + x^T S_x x \end{bmatrix}$$

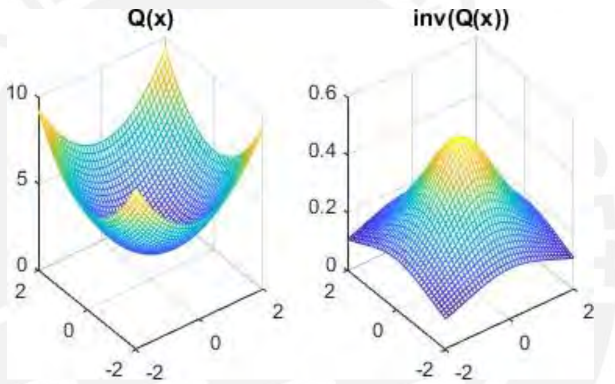

Figure 3.12: Matrix $Q_{2(x)}$ in 3D

How it can be seen in the table 3.9 the cost function is considerably reduced. And its main advantage is the simplicity of its design compared to Theorem 5.

Table 3.9: Comparison of the addition of $Q_2(x)$

	Optimal Control with $Q_{2(x)}$	Optimal Control with $Q_{2(x)}$ and Annihilator
Control Variable $u_{(x)}$	$-11.59 \cdot x_1 - 9.34 \cdot x_2$	$-6.59 \cdot x_1 - 5.3 \cdot x_2$
Matrix P	$\begin{bmatrix} 0.15 & -0.18 \\ -0.18 & 0.33 \end{bmatrix}$	$\begin{bmatrix} 0.26 & -0.32 \\ -0.32 & 0.59 \end{bmatrix}$
Upper bound p	2.17	1.15
Cost function J	0.45	0.27

At the same time, another interesting approach is to let the algorithm find the best performance criteria for the controller, which means, to assign $Q(x)$ as a variable with just a specification of a desired behavior in the origin.

Lemma 4 ($Q(x)$ as a polynomial matrix to be find).
As it has been done in the case of the polynomial annihilator, this time a polynomial matrix $Q(x)$ will be proposed, such that

$$Q(x) = Q_x + Q_0,\ Q_{(x)} > 0, \tag{32}$$

where Q_x is a symmetric polynomial matrix of x with even absolute degree and Q_0 a constant positive definite matrix, which defines the desired performance at the origin.

Example 9.
Let's consider the system shown in the example 6 with the addition of the matrix $Q(x)$ according to Lemma 4.

$$Q_0 = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix},\ max_{degree}(Q_x) = 4$$

The surface of the matrix $Q(x)$ is shown in the figure 3.13, while the results of this theorem are shown in the table 3.10 and figure 3.14.

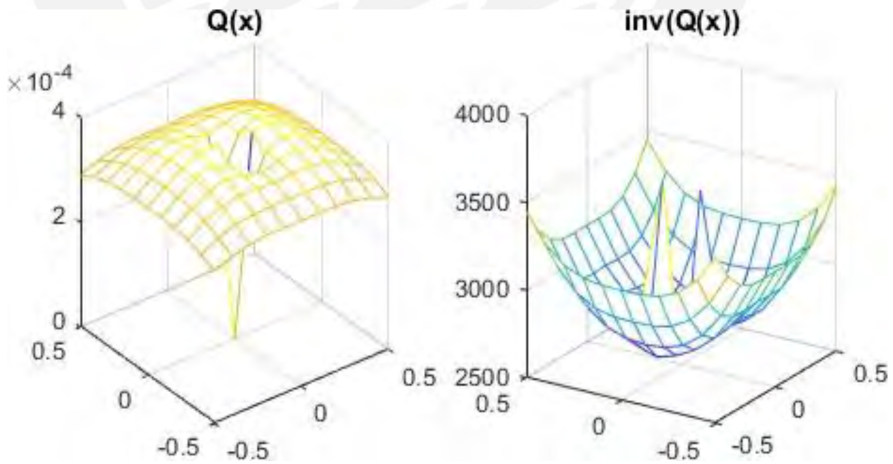

Figure 3.13: Matrix $Q_{(x)}$ in 3D

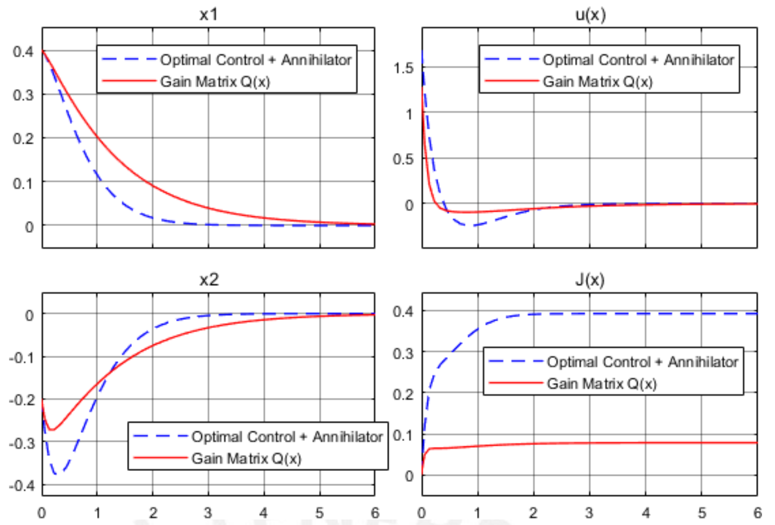

Figure 3.14: Comparison of the system response with Lemma 4

Table 3.10: Comparison of the results of Lemma 2 and 4

	Lemma 2	Lemma 3
Control Variable $u_{(x)}$	$-11.69 \cdot x_1 - 9.18 \cdot x_2$	$-11.6 \cdot x_1 - 9 \cdot x_2$
Matrix P	$\begin{bmatrix} 0.13 & -0.17 \\ -0.17 & 0.33 \end{bmatrix}$	$\begin{bmatrix} 0.14 & -0.18 \\ -0.18 & 0.34 \end{bmatrix}$
Upper bound p	2.32	2.24
Cost function J	0.49	0.08

On the other hand, the proposals for the matrix $Q(x)$ can also be applied to the matrix $R(x)$, as it is shown in the following example.

Example 10.

For this example, consider the system of example 6 and for the design of the controller a matrix $R(x)$ and $Q(x)$ applying the theory of Lemma 2:

$$q_1(x) = 2 + 0.41 x_1^4 - 1.26 x_1^2,$$
$$q_2(x) = 2 + 0.63 x_2^4 - 2.02 x_2^2,$$

$$Q(x) = \left(\sum_i^n q_i(x_i)\right) \cdot I_n, \; I_n \in \mathbb{R}^{n\times n},$$

$$R^{-1}(x) = q_1(x) + q_2(x).$$

Its curve is shown in the figure 3.15.

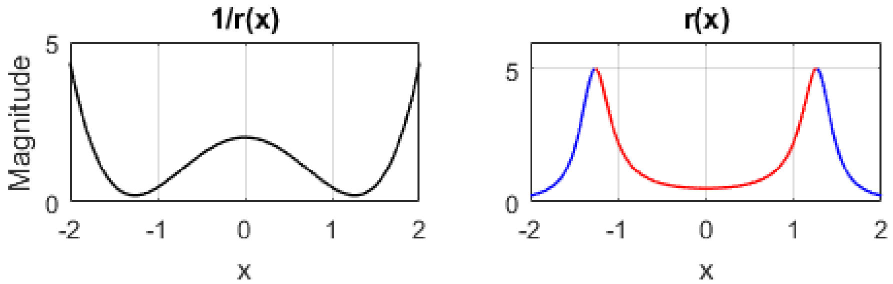

Figure 3.15: Desired matrix $R_{(x)}$

As it can be seen in the table 3.11 and figure 3.16, the value of the cost function is considerably reduced.

Table 3.11: Comparison of the addition of $R_{(x)}$ together with $Q_{(x)}$

	Optimal Control with $Q_{(x)} \wedge R_{(x)}$	Optimal Control with $Q_{(x)} \wedge R_{(x)}$ and Annihilator
Matrix P	$\begin{bmatrix} 0.23 & -0.315 \\ -0.315 & 0.7223 \end{bmatrix}$	$\begin{bmatrix} 0.3431 & -0.47 \\ -0.47 & 1.0777 \end{bmatrix}$
Upper bound ρ	1.1848	0.7857
Cost function J	0.2543	0.1892

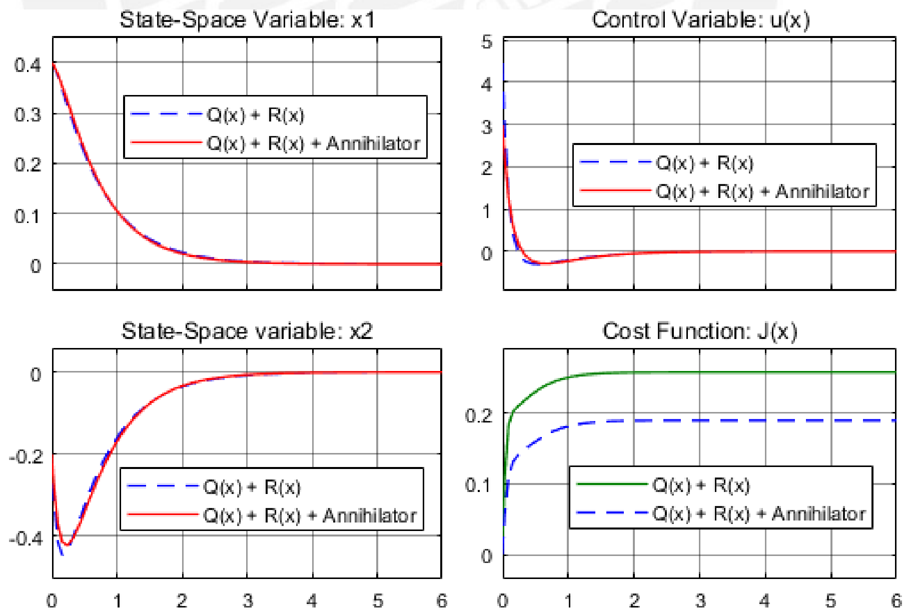

Figure 3.16: Desired matrix $R_{(x)}$

3.3 Optimal non-linear controller for rational systems

In this section, we propose a control algorithm for rational systems, whose main characteristic is that the derivative vector of the state-space variable is not free in its equality, but rather has a non-linear dependence with its state-space variables, which can be seen reflected in this well-known dynamic equation [22]:

$$H(q)\ \ddot{q} = C(q, \dot{q})\ \dot{q} + G(q) = B\, u,\ q_0 = 0,$$

where q is just another representation of the variable x and equivalent to the state-space form

$$L(x)\ \dot{x} = A(x)\ z(x) + B(x)\ u,\ \ z(x) = W(x)\ x,\ \ z(x_0) = 0, \tag{33}$$

where $W(x) \in \mathrm{R}^{n\times n}$ is invertible.

For the proposal of an optimal controller represented in a polynomial form (equation 33), we resort to the basis of Theorem 3 and add the characteristics of this type of systems.

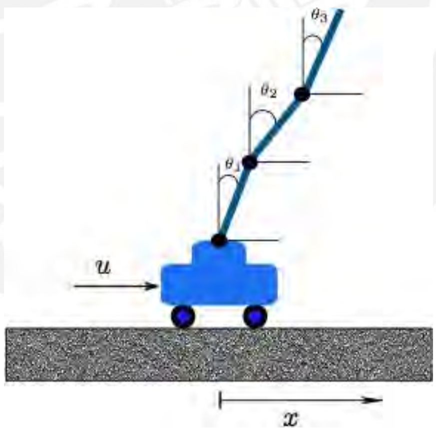

Figure 3.17: Triple inverted pendulum

Source: Control Theory III - Triple inverted pendulum [23].

Consider the cost function as in equation (28) and the Lyapunov function as $V(x) = z^T P^{-1} z$, then assume that the polynomial matrix $L(x)$ is invertible and the feedback control law is

$$u_{(x)} = -R^{-1}\ B^T\ L^{-T}\ \ M^T\ P^{-1}\ z. \tag{34}$$

The first derivative of $V(x)$ with respect to time is:

$$
\begin{aligned}
\dot{V}(x) &= \dot{z}^T P^{-1} z + z^T P^{-1} \dot{z} \\
&= (M L^{-1} A z + M L^{-1} B u)^T P^{-1} z + z^T P^{-1} (M L^{-1} A z + M L^{-1} B u) \\
&= z^T \left(H_e \left\{ P^{-1} M L^{-1} A \right\} - 2 P^{-1} M L^{-1} B R^{-1} B^T L^{-T} M^T P^{-1} \right) z
\end{aligned}
$$

Subsequently we obtain the HJ's inequality as

$$
\begin{aligned}
H(x) &= \dot{V}(x) + z^T Q^{-1} z + u^T R u + z^T h(x) S_R(x) z \prec 0 \\
&= \dot{V}(x) + z^T (Q + h(x) S_R(x) + P^{-1} M L^{-1} B R^{-1} B^T L^{-T} M^T P^{-1}) z \\
&= z^T \underbrace{\begin{bmatrix} H_e \{P^{-1} M L^{-1} A\} + Q + h(x) S_R(x) \\ - P^{-1} M L^{-1} B R^{-1} B^T L^{-T} M^T P^{-1} \end{bmatrix}}_{-H_p(x)} z \le 0
\end{aligned}
$$

Later, we multiply $H_p(x)$ by $L(x) M^{-1} P$ in both sides to obtain a result that will be free of any inverse and keep the result in a polynomial representation.

$$
H_p(x) = \begin{matrix} -H_e \left\{ A P M^{-T} L^T \right\} + B R^{-1} B^T - L M^{-1} P Q P M^{-T} L^T \\ - h(x) L M^{-1} \tilde{S}_R(x) M^{-T} L^T \end{matrix} \le 0 \qquad (35)
$$

Finally, and in this particular case the selection of the matrix $M(x)$ is very important, since its inverse or the product between $L(x) M^{-1}(x)$ must maintain the condition of being a polynomial matrix.

Theorem 6 (Optimal Controller for rational systems).
A system described as in the equation (33) with $x_0 \in s(P^{-1}, p) \subset X$ is asymptotically stable in the origin if there exist a symmetric positive definite matrix P; symmetric positive semidefinite matrices Q_R and Q_1; and positive scalar values β and ξ; such that the following matrices are positive semidefinite:

$$
S_1(x) = \left[\begin{array}{c|c} B R^{-1} B^T - H_e \left\{ A P M^{-T} L^T \right\} - h(x) L M^{-1} \tilde{S}_R(x) M^{-T} L^T & L M^{-1} P \\ \hline P M^{-T} L^T & Q \end{array} \right] - \xi I \le 0
$$

$$
Q_2 = \beta S_x^{-1} - P \le 0, \quad Q_3 = P - \xi I \le 0, \quad Q_4 = \beta - \xi \le 0
$$

$$
Q_5 = \begin{bmatrix} \beta & \beta z_0^T \\ \beta z_0 & P \end{bmatrix} \le 0
$$

$$
Q_6 = \begin{bmatrix} Y & I \\ I & P \end{bmatrix} \le 0
$$

And the objective function is to minimize the cost function: $J = z_0^T\, Y z_0$.

Proof. The HJ's inequality, described in the equation (35), has been represented as a matrix by using Schur complement:

$$S_1(x) = \begin{bmatrix} F & LM^{-1}P \\ PM^{-T}L^T & Q \end{bmatrix} - \xi I \;\Rightarrow\; F - LM^{-1}PQ^{-1}PM^{-T}L^T \leq 0,$$

where F represents the component of the first quadrant of $S_1(x)$.

On the other hand, the weight matrix Q is presented as an inverse, such that the algorithm is flexible to be used in conjunction with the Theorem 5. Finally, the proof of the other statements are in [8]. □

Example 11.
Let's consider the non-linear system

$$\begin{bmatrix} 10+x_1^2+x_2^2 & 0 \\ 0 & 10+x_1^2+x_2^2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1^2 & 1 \\ 0 & x_2\cdot(1+x_1) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

For this example we will apply the Theorem 6 in conjunction with the Theorem 5.

Table 3.12: Example 11 - Conditions

ξ	S_x	x_0	W
0.001	$\begin{bmatrix} 1.05^{-2} & 0 \\ 0 & 1.05^{-2} \end{bmatrix}$	$\begin{bmatrix} 0.4 \\ -0.2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The conditions for this design are shown in the table above, while the conditions for the weight matrices are

$$q_1(x) = q_2(x) = 2 + 0.71\,x_1^4 - 2.27\,x_1^2,$$

$$Q(x) = \begin{bmatrix} \sum_i^2 q_i(x_i)) & 0 \\ 0 & \sum_i^2 q_i(x_i)) \end{bmatrix},$$

$$R(x) = \frac{0.01}{\sum_i^2 q_i(x_i)}$$

The state feedback control law results as follows

$$u_{(x)} = \frac{\begin{array}{c} -872.61 \cdot x_1^5 - 606.36 \cdot x_1^4 \cdot x_2 + 2770.7 \cdot x_1^3 \\ +1925.3 \cdot x_1^2 \cdot x_2 - 872.61 \cdot x_1 \cdot x_2^4 + 2770.7 \cdot x_1 \cdot x_2^2 \\ -4887.6 \cdot x_1 - 606.36 \cdot x_2^5 + 1925.3 \cdot x_2^3 - 3396.2 \cdot x_2 \end{array}}{10 + x_1^2 + x_2^2}$$

The characteristics of the optimal controller can be seen in the table 3.13, while the system response in the figure 3.18, from where it can be seen that the system stabilizes in 40 seconds and the value of the control variable varies between [-111.50, 0.32].

Table 3.13: Example 11 - Optimal Controller

	Optimal Control
Matrix P:	$\begin{bmatrix} 0.0464 & -0.0667 \\ -0.0667 & 0.2138 \end{bmatrix}$
Upper bound ρ:	4.6498
Cost function J :	1.129

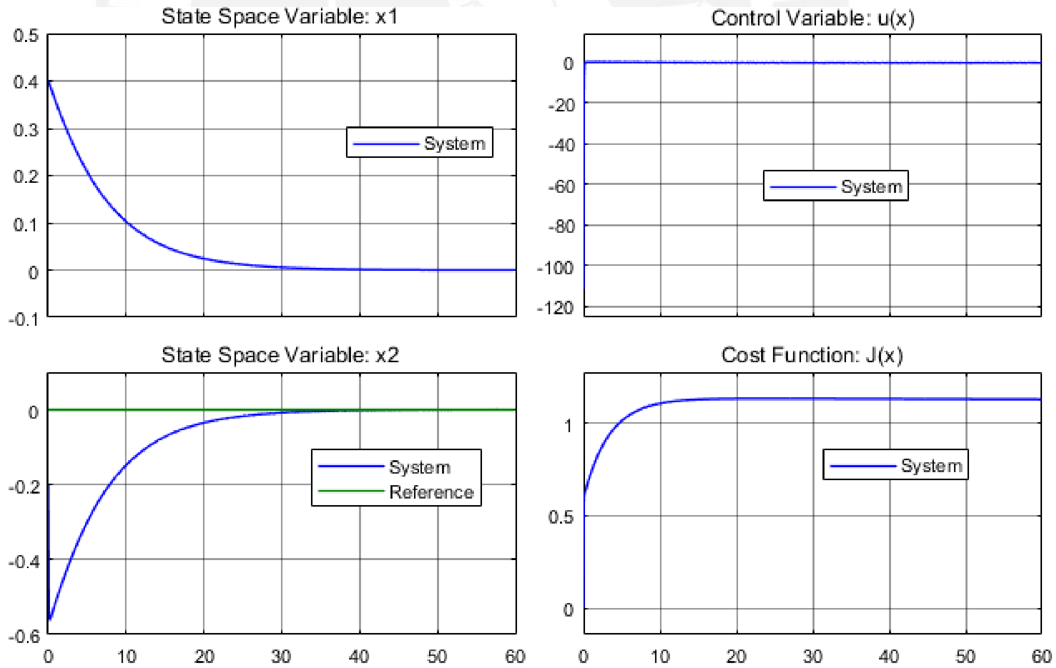

Figure 3.18: Example 11 - System response

Finally, and as an observation, it has to be noted that for this theorem the reduction of LMI can not be used, because of the addition of state-space variables to the lower quadrants of the matrix $S_1(x)$.

4 Closed loop system

4.1 Estimator design

In many systems, the outputs are the only ones that can be measured, which mostly do not include all the state space variables. For this reason, it is necessary to add an estimator, that can feed the controller with all the variables, which are necessary for the control of non-linear systems as it was appreciated in the previous chapters.

In the figure below, it can be seen how the estimator is going to be placed in order to provide the estimated variables to the controller.

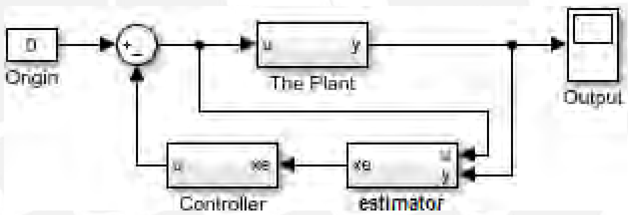

Figure 4.1: Full system block diagram

The equation of the estimator is described as

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u + \hat{E}(y - \hat{y}) \\ \dot{\hat{x}} &= (\hat{A} - \hat{B}\hat{K} - \hat{E}\hat{C})\hat{x} + \hat{E}Cx \end{aligned} \tag{36}$$

Therefore the equation of the full system is represented as

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \left[\begin{array}{c|c} A & -B\hat{K} \\ \hat{E}C & \hat{A} - \hat{B}\hat{K} - \hat{E}\hat{C} \end{array}\right] \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \left[\begin{array}{c|c} A & -B\hat{K} \\ \hat{E}C & \hat{H} \end{array}\right] \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \tag{37}$$

where $\hat{K}$ represents the gain vector of the optimal controller, which can be found by any of the methods previously described and as best corresponds; while $\hat{E}$ is the gain vector of the estimator.

The stability analysis of the global system has to include all the variables of the system (real and estimated), for this reason we introduce a new variable x_d:

$$x_d = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

which is used to build the new proposed Lyapunov function $W(x_d)$:

$$W(x_d) = x_d^T P x_d = \begin{bmatrix} x & \hat{x} \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \leq 0, \; P > 0, \tag{38}$$

where $P_i > 0, \; i \in \{1, 2, 3\}$;

$$\dot{W}(x_d) = x_d^T \underbrace{\begin{bmatrix} H_e\left(P_1 A + P_2 \hat{E} C\right) & \begin{matrix} P_2 \hat{H} + C^T \hat{E}^T P_3 \\ + A^T P_2 - P_1 B \hat{K} \end{matrix} \\ \hline \begin{matrix} \hat{H}^T P_2 + P_3 \hat{E} C \\ + P_2 A^T - \hat{K}^T B^T P_1 \end{matrix} & H_e\left(P_3 \hat{H} - P_2 B \hat{K}\right) \end{bmatrix}}_{-H_p(x)} x_d \leq 0. \tag{39}$$

However, if we take a deeply view to the equation (39), it can be seen that it has a problem of bilinearity as a result of a multiplication between unknown variables that must be found: $P_2 \hat{E}$ and $P_3 \hat{E}$. Therefore we decided to choose the following assumptions to prevent the conflict of bilinearity [24]:

$$\begin{aligned} P_x &= P_2 = P_3, \\ \hat{E}(\hat{x}) &= P_x^{-1} E_x(\hat{x}), \end{aligned} \tag{40}$$

where P_x is a symmetric positive definite matrix and E_x is a polynomial symmetric matrix to be find by the algorithm. Then, the function $H_p(x)$ is rewritten as

$$H_p(x) = \begin{bmatrix} H_{p1} & H_{p2} \\ H_{p2}^T & H_{p3} \end{bmatrix} \leq 0 \tag{41}$$

where

$$\begin{aligned} H_{p1} &= -H_e\{P_1 A + E_x C\} \\ H_{p2} &= -A^T P_x - P_x \hat{A} + P_x \hat{B} \hat{K} + E_x \hat{C} - C^T E_x^T + P_1 B \hat{K} \\ H_{p3} &= -H_e\left(P_x \hat{A} - P_x \hat{B} \hat{K} - E_x \hat{C} - P_x B \hat{K}\right) \end{aligned} \tag{42}$$

Although, it would be great to guarantee the stability of the system for all $\mathbb{R}^n$, this will not always be fulfilled, so in the same way as the previous cases, an equation representing the area of interest should be added to $H_p(x)$, such that the analysis is restricted to finding the stability of the system for the invariant set

$$\bar{X} = \left\{ x_e \in \mathbb{R}^{2n} \mid h(x_e) \geq 0 \right\}, \quad x_e = \begin{bmatrix} x \\ e \end{bmatrix} \tag{43}$$

where $e = x - \hat{x}$ is the estimation error and

$$h(x_e) = 1 - \begin{bmatrix} x \\ x - \hat{x} \end{bmatrix}^T \bar{S}_x \begin{bmatrix} x \\ x - \hat{x} \end{bmatrix}, \quad \bar{S}_x > 0, \tag{44}$$

and its gain matrix $\bar{S}_x$ is built as

$$\bar{S}_x = \left[\begin{array}{c|c} S_x & 0 \\ \hline 0 & \mathrm{diag}([b_1 \dots b_i]) \end{array} \right], \quad I \in \mathbb{R}^n, \; b_i > 0, i \in \{1, \cdots, n\}, \tag{45}$$

where S_x is the gain matrix of the area of interest of the optimal controller that will be used at first, while b_i is a scalar value assigned by the designer, such that it represents the tolerance of the estimation error.

In practical cases it can be considered that

$$b_1 = \cdots = b_n$$

however, this decision is left to the criteria of the person who uses this algorithm.

Theorem 7 (Estimator for the Optimal Controller).
To guarantee the asymptotical stability of the closed-loop system with controller given by Theorem 3 or 5, there must exist positive semidefinite matrices Q_1, Q_R, P_1, P_x; a polinomial matrix E_x; and a positive scalar value ξ; such that, the following is satisfied:

$$P = \begin{bmatrix} P_1 & P_x \\ P_x & F_x \end{bmatrix} > 0,$$

$$S_{1(x_d)} = H_{p(x_d)} - h(x_e) S_R(x_d) - \xi I \leq 0,$$

$$S_{R(x_d)} = (I \otimes x_{d[N_R]})^T Q_R (I \otimes x_{d[N_R]}),$$

$$S_{1(x_d)} = (I \otimes x_{d[N]})^T Q_1 (I \otimes x_{d[N]}), \; N_R < N$$

where N_R and N represents the higher degree of its corresponding function.

Proof. Direct application of Theorem 3 for the Lyapunov candidate function W yield the desired results. □

Example 12.
Consider the non-linear system of the example 4:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ x_1 + x_2 & x_2^2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

And its optimal controller: $u_{(x)} = -0.91\ x_1 - 2.7\ x_2$. Then, for the design of the estimator it is proposed that the gain matrix of the invariant set S_1 be

$$\bar{S}_x = \left[\begin{array}{c|c} S_x & 0\\ \hline 0 & I\end{array}\right],$$

where S_x is the same that have been used for finding the optimal controller.

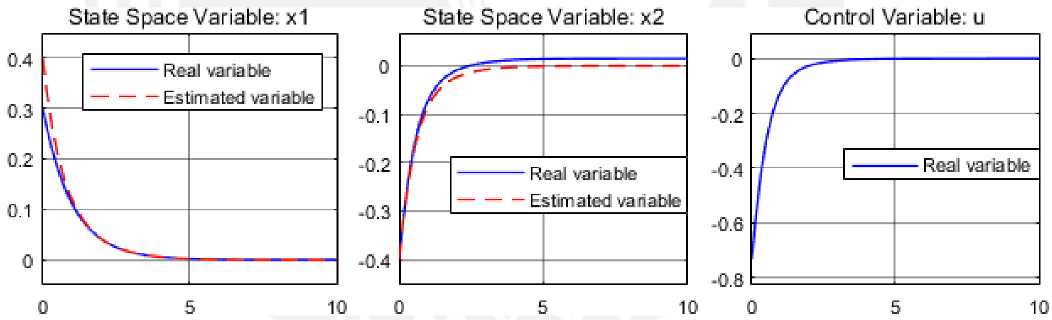

Figure 4.2: System response with estimator

The estimator vector law is (rounded to 4 digits):

$$\hat{E}(\hat{x}) = \begin{bmatrix}0.002\ \hat{x}_1 - 2.55\ 10^{-4}\hat{x}_2 + 1.55\\ 2.49\ 10^{-4}\ \hat{x}_1 + 0.526\end{bmatrix}.$$

Finally, it can be seen in figure 4.2, that the state-space variable x_2 does not converge quickly to zero, as it does x_1, and, this is, because the dynamic controller only knows the value of x_1, therefore this variable is stabilized at first and after a certain time and due to the condition of zero input, the variable x_2 converges to equilibrium.

On the other hand, given the choice of matrix $\bar{S}_x$, the tracking error must be within a tolerance, such that it is in the invariant set. Therefore, the values of the estimator should not be far from the system.

4.2 Dynamic output feedback law

As it was explained previously, for the implementation of the Ichihara controller or any of the improvement proposals explained here, it is necessary to know the value of all the state-space variables at all time. However, this is not always possible to obtain, due to the characteristics of the system and the difficulty of placing sensors to measure the variables. Therefore, we must resort to some other method that allows the control of the non-linear system through the feedback of its outputs and that, in turn, is capable of emulating the desired behavior.

In other words, in this section we present another solution that can guarantee the stability of the closed-loop system, when the other method can not, by the development of a dynamic controller that is able to emulate the behavior of another controller by its dynamic variable [10].

At first, let's consider the following polynomial dynamic output feedback law:

$$\begin{aligned} \dot{\hat{x}} &= A_c(\hat{x}, y)\hat{x} + B_c(\hat{x}, y)y, \quad \hat{x}(t_0) = \hat{x}_0, \hat{x} \in \mathrm{R}^1 \\ u &= C_c(\hat{x}, y)\hat{x} + D_c(\hat{x}, y)y \end{aligned} \tag{46}$$

where $A_c \in \mathrm{R}^{n_{\hat{x}} \times n_{\hat{x}}}$, $B_c \in \mathrm{R}^{n_{\hat{x}} \times n_y}$, $C_c \in \mathrm{R}^{n_u \times n_{\hat{x}}}$, $D_c \in \mathrm{R}^{n_u \times n_y}$ are polynomial matrices to be found by the control algorithm; u is the input variable from the system; and y, its output variable.

Then, the closed-loop system as

$$\dot{x}_f = \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} M(A + BD_cC) & MBC_c \\ B_cC & A_c \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = A_f x_f \tag{47}$$

for the state dependent linear-like form representation of the system.

Theorem 8 (Dynamic Output Feedback Law inspired from [10]).
Assume that there exist a state feedback controller (obtained by some other theorem) in the form: $k = KP_1^{-1}z(x)$. Then, the closed-loop system (47) is asymptotically stable if there exist symmetric positive definite matrices Q_{1f}, P_f and Q_{2f}; and a non-negative value ξ; such that $Q_{1f} + Q_{2f} + \xi I \leq 0$ for some polynomial matrices A_c, B_c, C_c and D_c where

$$Q_{1f} = \begin{bmatrix} H_e\left\{P_1^{-1}M(A + BD_cC)\right\} & P_1^{-1}MBC_c \\ C_c^T B^T M^T P_1^{-1} & 0 \end{bmatrix},$$

$$Q_{2f} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_{22} \end{bmatrix},$$

$$\Lambda_{11} = H_e\left\{P_1^{-1}K^T\left(\frac{\partial k}{\partial x}(A + BD_cC)\right) - B_cC\right\},$$

$$\Lambda_{12} = P_1^{-1}K^T\left(\frac{\partial k}{\partial x}BC_c - A_c\right) + \left(B_cC - \frac{\partial k}{\partial x}(A + BD_cC)\right)^T,$$

$$\Lambda_{22} = H_e\left\{\left(A_c - \frac{\partial k}{dx}B_cC\right)\right\}.$$

where ξ is a tolerance for the definite positiveness.

Proof. The matrices Q_{1f} and Q_{2f} are built on the proposed Lyapunov function to ensure the close-loop asymptotic stability using Theorem 3:

$$\begin{aligned} V(x_f) &= V_1(x_f) + V_2(x_f) > 0, \\ V_1(x_f) &= z^T P_1^{-1} z > 0, \\ \dot{V}_1(x_f) &= x_f^T Q_{1f} x_f \prec 0, \\ V_2(x_f) &= (k - \hat{x})^T (k - \hat{x}) > 0, \\ \dot{V}_2(x_f) &= x_f^T Q_{2f} x_f \prec 0. \end{aligned}$$

For more information see [10]. □

Finally we can replace the law of the state feedback controller to the dynamic feedback law found by this theorem.

Example 13.
Let's take as our first example the non-linear system presented by the author of this theory [10].

$$\dot{x} = \begin{bmatrix} 0.5 - 0.1 \cdot x_1^2 & 0 \\ x_1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x, \; z = x$$

The considerations assigned for the design of the optimal controller by Theorem 3 are shown in the table below.

Table 4.1: Example 13 - Conditions

ε	$Q_{(x)}$	$R_{(x)}$	S_x	x_0
0.001	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	1	$1.05 \cdot \begin{bmatrix} \text{[illegible]} \\ 0 \; 1 \end{bmatrix}$	$\begin{bmatrix} 0.4 \\ -0.2 \end{bmatrix}$

Then, the result of the Theorem 3 is as follows

Table 4.2: Example 13 - Results

$u_{(x)}$	P matrix	p	$J(x)$
$-1.62 \cdot x_1 - 6.646e^{-10} \cdot x_2$	$\begin{bmatrix} 0.6174 & -8.196e^{-10} \\ -8.196e^{-10} & 1.997 \end{bmatrix}$	0.3368	0.3252

Finally, and after constructing the conditions of the theorem 8, we obtain the following matrices for the dynamic output feedback law (rounded to 3 digits):

$$A_c = -52x_3 - 16$$
$$B_c = 0$$
$$C_c = 0.051x_3 + 0.057$$
$$D_c = 0.1y^2 - 0.049x_3^2 - 0.56,$$

which guarantee the the asymptotic stability of the closed loop system as it can be seen in the figure 4.3.

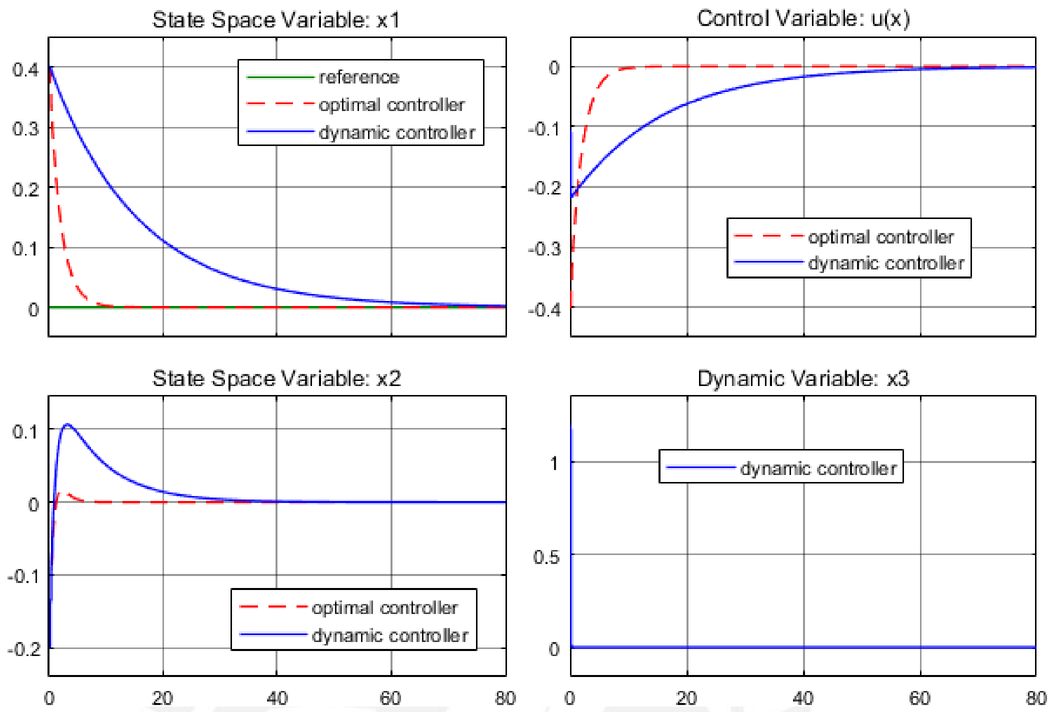

Figure 4.3: System response to the dynamic controller

Similarly, it should be noted that the dynamic controller can not always guarantee the global stability of a system, therefore this theory must also be modified, such that its resulting controller focuses on an area of interest.

An extension of this controller will be proposed below, so that it can work with a greater number of dynamic variables and, in turn, add the local action advantage, which will increase the range of solutions.

Theorem 9 (Extension of the Output Feedback Law).
Consider the polynomial dynamic output feedback as

$$\dot{\hat{x}} = A_c(\hat{x}, y)\ \hat{x} + B_c(\hat{x}, y)\ y,\ \ \hat{x}(t_0) = \hat{x}_0$$
$$u = C_c(\hat{x}, y)\ \hat{x} + D_c(\hat{x}, y)\ y$$
$$\hat{y} = E_c\ \hat{x}.$$

Then, the closed-loop system is asymptotically stable if there exist symmetric positive definite matrices Q_{1f}, P_f, Q_{2f}, Q_{3f} and Q_R; and a non-negative value ξ; such that $Q_{1f} + Q_{2f} + Q_{3f} - h(x_f)\ S_R(x_f) + \xi I \leq 0$ for some polynomial matrices A_c, B_c, C_c and D_c; where

$$Q_f = Q_{1f} + Q_{2f} + Q_{3f}$$

$$Q_{1f} = \begin{bmatrix} He\left\{P_1^{-1}M(A + BD_cC)\right\} & P_1^{-1}MBC_c \\ M^T B^T C_c^T P_1^{-1} & 0 \end{bmatrix}$$

$$Q_{2f} = \begin{bmatrix} He\left\{ P_1^{-1}K^T\left(\frac{dk}{dx}(A + BD_cC) - E_c B_c C\right)\right\} & \Delta \\ \Delta^T & He\left(E_c A_c - \frac{dk}{dx} B_c C\right)^T \end{bmatrix}$$

$$\Delta = P_1^{-1}K^T\left(\frac{dk}{dx}BC_c - E_cA_c\right) + \left(E_cB_cC - \frac{dk}{dx}(A + BD_cC)\right)$$

$$Q_{3f} = \begin{bmatrix} 0 & C^T B_c^T \\ B_cC & A_c^T + A_c \end{bmatrix}$$

and E_c must be choosen, such that $E_c\,\hat{x} = \hat{x}_1$.

Proof. In this theorem the Lyapunov function $V(x_f)$ is represented as

$$V(x) = V_1(x_f) + V_2(x_f) + V_3(x_f)$$

where

$$\begin{aligned} V_1(x_f) &= z^T P_1^{-1} z \rightarrow \dot{V}_1 = x_f^T Q_{1f}\, x_f \leq 0 \\ V_2(x_f) &= (k - \hat{y})^T (k - \hat{y}) \rightarrow \dot{V}_2 = x_f^T Q_{2f}\, x_f \leq 0 \\ V_3(x_f) &= \hat{x}^T \hat{x} \rightarrow \dot{V}_3 = x_f^T Q_{3f}\, x_f \leq 0 \end{aligned}$$

Then, the function $h(x_f)$ is added, as in the Theorem 7, to focus the algorithm on finding solutions only within the area of interest $\bar{X}$. See (44). □

Example 14.

Continuing with what was developed in example 13, a dynamic two-variable controller with a gain matrix for the function $h(x_f)$ as

$$\overline{S_x} = diag([1.05^{-2}, 1.05^{-2}, 10^{-5}, 10^{-5}])$$

will give the following results (rounded to 5 digits), which also guarantee the stability of the closed-loop system as it is shown in the figure 4.4.

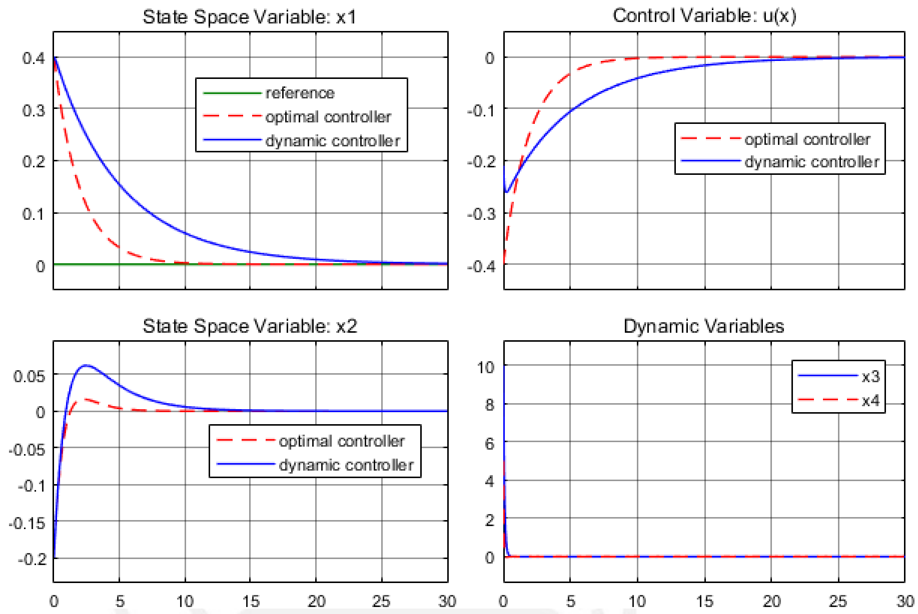

Figure 4.4: Response to 2-Variable Dynamic

$$A_c = \begin{bmatrix} 1.2e^{-3}y - 8.5e^{-5}x_3 - 8.5e^{-5}x_4 - 4 & 0 \\ 1.9e^{-5}x_3 - 4.9e^{-5}y + 1.9e^{-5}x_4 - 0.2 & 1.9e^{-5}x_3 - 1.7e^{-4}y + 1.9e^{-5}x_4 - 7.8 \end{bmatrix}$$

$$B_c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C_c = \begin{bmatrix} -4.4e^{-4}x_1 + 3.7e^{-3} & -8.9e^{-5}x_1 \end{bmatrix}$$

$$D_c = -6.7e^{-4}x_1 - 0.67$$

5 Conclusions and outlook

5.1 Summary

This thesis aims to improve the design of an optimal controller for non-linear systems represented in a polynomial form by extending its range of application and improving the selection of the desired performance criteria.

For this, we began with the presentation of the main theory, to then continue with the proposal of providing greater flexibility to the algorithm, such that it is capable of accepting matrices of variable weight. Later we extended the scope of the algorithm, so that it could be used on rational systems.

Finally, two algorithms were developed to guaranty asymptotic stability in the origin of the closed loop, when some states are not available for measurement. First it is proposed an observer approach and then a dynamic output feedback controller is improved.

5.2 Conclusions

The main advantage of LMIs is that there exist computational methods to solve them and, in turn, these methods are scalable and allow us to play with large amounts of variables, which usually appear in SOS.

The addition of the S_1 matrix in the design of the estimator provides as main advantage the ability to keep the tracking error in a tolerance range, such that stability of the closed-loop system can be guaranteed.

After comparing the different selections of $Q(x)$ from Theorem 5 represented in each Lemma (2, 3, 4), it can be seen that the main advantage that the addition of $Q(x)$ grants is the ability to define a desired behavior outside and at the origin.

5.3 Future work

The currently theory [8, 10] seeks to avoid the bilinearity generated by the optimization criteria by resorting to assumptions that can and do reduce the range of solutions and the scope of the algorithm. For this reason, the future work should focus on reducing the problem of bilinearity by adding better restrictions that allow keeping it within an acceptable margin.

Similarly, the proposal made for variable weight matrices opens a door to define and add functions that more accurately describe the desired parameters for the system, as do the fuzzy controllers, such that we divide the invariant set per zones, where the cost function can penalize with different intensity.

On the other hand, it would be very interesting to add an optimization criteria to Theorem 9, so that it is able to faithfully emulate the proposed controllers, that are used for its design, and to keep a desired performance at the origin.

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