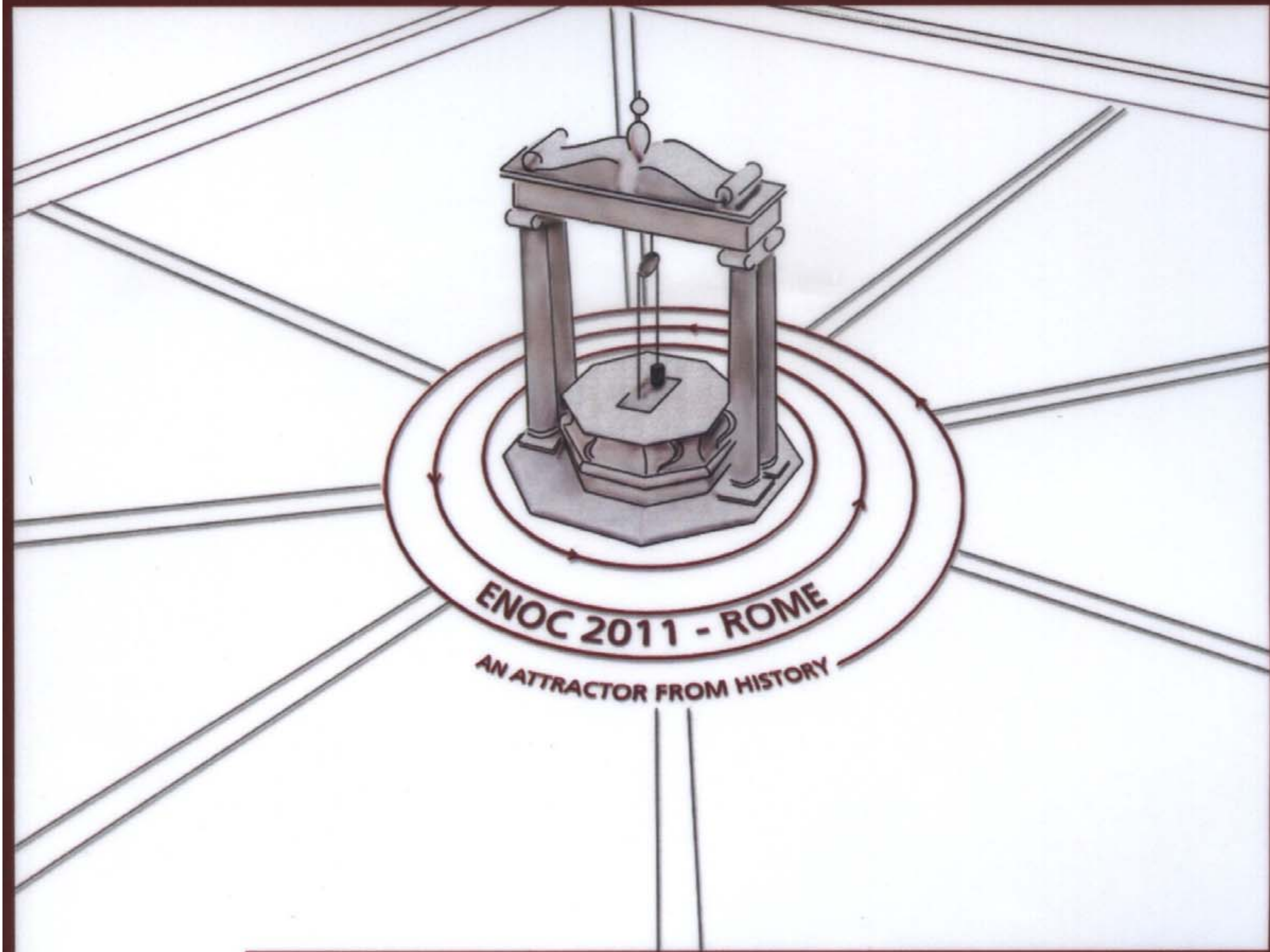

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Forced van der Pol oscillator of complex order

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Summary. In this paper it is considered a complex order forced van der Pol oscillator. The complex derivative $D^{\alpha\pm\jmath\beta}$, with $\alpha,\beta\in\mathbf{R}^+$ is a generalization of the concept of integer derivative, where $\alpha=1,\ \beta=0$. We compute amplitude and period values of the periodic solutions of the complex order forced van der Pol oscillator, for variation of distinct parameters such as forcing frequency, forcing amplitude and parameters α and β. We find interesting quasi-periodic motion for certain values of the forcing frequency. This type of behaviour is seen in the continuous forced van der Pol oscillator. **Keywords** — *forced van der Pol oscillator, complex order derivative, dynamical behavior*

1 Introduction

The van der Pol (VDP) oscillator is a well-known model for the variation of voltage and current intensity of electrical circuits containing vacuum tubes [8]. The forced van der Pol system is given by the following second order differential equation:

$$\ddot{x}+\mu\left(x^2-1\right)\dot{x}+x-b\cos(\omega_0 t)=0 \tag{1.1}$$

This forced equation has brought major developments for the nonlinear dynamical systems theory, due to the richness of it dynamical behavior [2, 7]. For small values of μ, the fundamental frequency ω is close to one. As μ increases, ω decreases to values away from one. For values of $b\neq 0$, the forced van der Pol system exhibits a variety of dynamical behavior, from periodic motion, to quasi-periodic states, depending on the values of $\mu,\ \omega$ and b.
Fractional calculus (FC) has been an important research issue in the last few decades. FC is a generalization of the ordinary integer differentiation and integration to an arbitrary, real or complex, order [5, 6, 4].
The Grünwald - Letnikov definition of a fractional derivative of order $\alpha\in\mathbf{R}$ given by:

$$D_t^\alpha f(t)=\lim_{h\to 0}\frac{1}{h^\alpha}\sum_{h=0}^{\left[\frac{t-a}{h}\right]}(-1)^k\left(\begin{array}{c}\alpha\\ k\end{array}\right)f(t-kh) \tag{1.2}$$

inspires the numerical calculation of the fractional derivative based on the approximation of the time increment h through the sampling period T and the series truncation at the r^{th} term. This method is often denoted as Power Series Expansion (PSE) yielding the equation in the z - domain:

$$Z\{D^\alpha x(t)\}\approx\left[\frac{1}{T^\alpha}\sum_{k=0}^{r}\frac{(-1)^k\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)}z^{-k}\right]X(z) \tag{1.3}$$

where $X(z)=Z\{x(t)\}$ and z and Z represent the z-transform variable and operator, respectively. In fact, expression (1.2) represents the Euler (or first backward difference) approximation in the $s\to z$ discretization scheme, being the Tustin approximation another possibility. The most often adopted generalization of the generalized derivative operator consists in $\alpha\in\mathbf{R}$. The case of having fractional derivative of complex-order $\alpha\pm\jmath\beta\in\mathbf{C}$ leads to complex output valued results and imposes some restrictions before a practical application. To overcome this problem, it was proposed recently [1] the association of two complex-order derivatives. For example, with the real part of two complex conjugate derivatives $D^{\alpha\pm\jmath\beta}$ we get:

$$\begin{aligned}Z\left\{\ \tfrac{1}{2}\left[D^{\alpha-\jmath\beta}x(t)+D^{\alpha+\jmath\beta}x(t)\right]\ \right\}\approx\tfrac{1}{T^\alpha}\Big\{\ &\sin\left[\beta\ln(\tfrac{1}{T})\right]\left[\beta z^{-1}+\tfrac{1}{2}\beta(1-2\alpha)z^{-2}+\cdots\right]+\\ &+\cos\left[\beta\ln(\tfrac{1}{T})\right]\left[-1+\alpha z^{-1}-\tfrac{1}{2}\beta(\alpha^2-\alpha-\beta^2+\cdots)\right]\ \Big\}X(z)\end{aligned} \tag{1.4}$$

2 Complex order forced van der Pol oscillator

In this paper, we consider the following complex order state-space models of the VDP oscillator (CVDP):

$$\begin{array}{lcl}\frac{1}{2}\left(D^{\alpha+\jmath\beta}+D^{\alpha-\jmath\beta}\right)x_1&=&x_2\\ \dot{x_2}&=&-x_1-\mu(x_1^2-1)x_2+b\cos\left(\omega_0 t\right)\end{array} \tag{2.5}$$

where $D^{\alpha\pm\jmath\beta}$, $\alpha,\beta\in\mathbf{R}^+$, is a generalization of the concept of the integer derivative, that corresponds to $\alpha=1$ and $\beta=0$, b is the forcing amplitude and ω_0 is the forcing frequency.
We adopt the PSE method for the approximation of the complex-order derivative in the discrete time numerical integration.
The discretisation of the CVDP oscillator (2.5) leads to:

$$\begin{array}{l}x_1(k+1)=\frac{1}{\psi(\beta,\Delta t)}\left\{\ H[x_1(k)]+(\Delta t)^\alpha x_2(k)\ \right\}\\ x_2(k+1)=x_2(k)+\Delta t\left\{\ -x_1(k)-\mu[x_1^2(k)-1]x_2(k)+b\cos(\omega x_3(k))\ \right\}\\ x_3(k+1)=x_3(k)+\delta_t\end{array} \tag{2.6}$$

where $\Delta t=0.0005$ is the time increment, $\delta_t=0.001$, $\psi(\beta,\Delta t)=\cos\left[b\log\left(\frac{1}{\Delta t}\right)\right]$, function $H(x_i)$, $i=1,2$, results from the Taylor series expansion truncation.

We now simulate the ordinary differential systems given by expression (2.6) for $\beta = 0.8$, $\alpha \in \{0.0, 0.1, \ldots, 1.0\}$, $\mu = \{5.0\}$, $b = 10.0$, and we measure the amplitude and the period of the solutions for two values of ω_0, $\omega_0 = 0.5$ and $\omega_0 = 2.46$. We adopted the initial conditions $x_1(1) = 0.0$, $x_1(2) = 0.005$, $x_1(3) = 0.010$, $x_1(4) = 0.015$, $x_1(5) = 0.02$, $x_2(1) = 1.0$, $x_2(2) = 1.005$, $x_2(3) = 1.010$, $x_2(4) = 1.015$, $x_2(5) = 1.02$. Each simulation is executed until a stable periodic solution is found. The amplitude and the period of the solutions versus α are depicted in Figures 1-2. We find that for $\omega_0 = 0.5$, the period is constant and the amplitude decreases with α. For $\omega_0 = 2.46$, the amplitude decreases with α but the period is increasing with α in the following way: (i) for $\alpha \leq 0.6$ the period is held constant, (ii) for $\alpha = 0.7$, the period of solutions is 3 times the former value, (iii) for $\alpha = 0.8$ the period is 5 times the former value, (iv) for $\alpha = 0.9$ the period is 7 times the previous value, and (v) for $\alpha = 1.0$ the period is 9 times the value at $\alpha = 0.6$. It appears that we have here a period-doubling cascade for the bifurcation parameter α (see Figure 3).

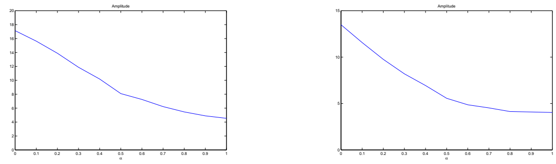

Figure 1: Amplitude of the periodic solutions $x_1(t)$ produced by the CVDP oscillator (2.6) for $\beta = 0.8$, $\alpha \in \{0.0, 0.1, \ldots, 1.0\}$, $\mu = 0.5$, $b = 10.0$, and $\omega_0 = 0.5$ (left) and $\omega_0 = 2.46$ (right).

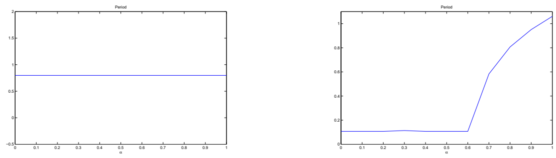

Figure 2: Log of the period of the solutions $x_1(t)$ produced by the CVDP oscillator (2.6) for $\beta = 0.8$, $\alpha \in \{0.0, 0.1, \ldots, 1.0\}$, $\mu = 5.0$, $b = 10.0$, and $\omega_0 = 0.5$ (left) and $\omega_0 = 2.46$ (right).

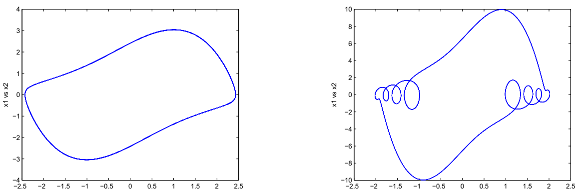

Figure 3: Phase space solution of system (2.6) for $\beta = 0.8$, $\mu = 5.0$, $b = 10.0$, $\omega_0 = 2.46$, and $\alpha = 0.6$ (left) and $\alpha = 1$ (right).

3 Conclusions

In this paper we study a complex-order fractional approximation the well-known forced van der Pol oscillator. The amplitude and the period of solutions produced by these two approximations were then measured. The imaginary part was fixed while the real component was varied, for $\mu = 5.0$, $b = 10.0$ and for two distinct values of ω_0. It was observed that the amplitude decreases for increasing values of α, for the two values of ω_0. The period is held constant for $\omega_0 = 0.5$ and shows what appears to be duplicating phenomena for $\omega_0 = 2.46$, for $\alpha \geq 0.7$. Future work will focus on the behavior of the complex-order forced system for other values of μ and ω_0.

References

[1] Hartley, T. T., Adams, J. L. and Lorenzo, C. F. Complex-order distributions. *Proceedings of the ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, Long Beach, CA, DETC2005-84952, on CD-ROM, (2005).

[2] Liu, L., Dowell, E.H., and Hall, K.C. A novel harmonic balance analysis for the van der Pol oscillator. *International Journal of Non-Linear Mechanics*, **42**, 2–12, (2006).

[3] Mainardi, F. Fractional relaxation-oscillation and fractional diffusion-wave phenomena. *Chaos, Solitons and Fractals*, **7**, 1461–1477, (1996).

[4] Miller, K. S., Ross, B. *An introduction to the fractional calculus and fractional differential equations*, John Wiley and Sons, (1993).

[5] Oldham, K. B., Spanier, J. *The fractional calculus: theory and application of differentiation and integration to arbitrary order*. Academic Press (1974).

[6] Samko, S. G., Kilbas, A. A., Marichev, O. I. *Fractional integrals and derivatives: theory and applications.* Gordon and Breach Science Publishers (1993).

[7] Storti, D., and Rand, R.H. Subharmonic entrainment of a forced relaxation oscillator. *Int. J. Non-Linear Mechanica*, **23**, No 3, 231–239 (1988)

[8] B. van der Pol. Forced oscillations in a circuit with nonlinear resistance (receptance with reactive triode), *London, Edinburgh and Dublin Phil. Mag.*, **3**, 65–80, (1927).