One-step-ahead implementation

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Abstract

In many situations, agents are involved in an allocation problem that is followed by another allocation problem whose optimal solution depends on how the former problem has been solved. In this paper, we take this dynamic structure of allocation problems as an institutional constraint. By assuming a finite number of allocation problems, one for each period/stage, and by assuming that all agents in society are involved in each allocation problem, a dynamic mechanism is a period-by-period process. This process generates at any period-t history a period-t mechanism with observable actions and simultaneous moves. We also assume that the objectives that a planner wants to achieve are summarized in a social choice function (SCF), which maps each state (of the world) into a period-by-period outcome process. In each period t, this process selects for each state a period-t socially optimal outcome conditional on the complete outcome history realized up to period $t-1$. Heuristically, the SCF is one-step-ahead implementable if there exists a dynamic mechanism such that for each state and each realized period-t history, each of its subgame perfect Nash equilibria generates a period-by-period outcome process that coincides with the period-by-period outcome process that the SCF generates at that state from period t onwards. We identify a necessary condition for SCFs to be one-step-ahead implemented, one-step-ahead Maskin monotonicity, and show that it is also sufficient under a variant of the condition of no veto-power when there are three or more agents. Finally, we provide an account of welfare implications of one-step-ahead implementability in the contexts of trading decisions and voting problems.

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1. Introduction

The theory of implementation investigates the goals that a planner can achieve when these goals depend on private information held by various agents. The problem of the planner is to design a mechanism or game form in which the agents' incentives dovetail to an equilibrium outcome that coincides with the planner's goal. When such a mechanism exists, his goal is fully implementable. This paper studies full implementation problems in a dynamic environment, in which:

- A finite number of agents interacts for a finite number of periods/stages and they commonly observe the state (of the world) before starting interacting.[1]
- The central planner delegates the achievement of period-t goal to period-t planner, who designs a period-t mechanism (or game form) with observed actions and simultaneous moves.
- A period-t environment consists of a set of agents, a set of period-t outcomes and agents' preferences over those outcomes. An agent's preferences over period-t outcomes endogenously depend on planners' future goals as well as planners' decisions taken in the past.
- The period-t planner aims to solve his implementation problem by devising a period-t mechanism (one after each outcome history), which asks agents to report only the information pertaining to his problem and which gives the agents the appropriate incentives so that a period-t "socially desirable" outcome results from the strategic behavior of the agents. The period-t planner aims to solve his implementation problem in a way that makes the implementation problems of future planners solvable.
- The period-t planner aims to implement a socially desirable period-t outcome after any outcome history - *even* after off-equilibrium histories - and cannot punish agents over periods.

Many real-world allocation problems have the above dynamic structure. For example, in democratic societies, the identity of governments may change over time due to periodic elections, the policy variables chosen by the current government affect the optimal decisions of future governments and, moreover, the current government handles its decision problem without being able to commit to future policy variables (see, e.g., Persson and Tabellini, 2000; Krusell et al., 1997).[2] Also, in a market, today trading affects future trading activities, and the role of the market maker is to facilitate trade period-by-period but not to make a commitment related to future trading activities, or to enforce them over time (see, e.g., Radner, 1972, 1982; and Prescott and Mehra; 1980). More generally, the above set-up is justified by the fact that in many real-life situations, agents are involved in an allocation problem that is followed by another allocation problem, whose optimal solution depends on

[1] We assume finite periods for the sake of simplicity. Our framework extends to infinite periods by imposing a Markovian kind of refinement condition: Strategies depend only on outcome histories.

[2] We have in mind a situation in which governments share the same agenda.

how the former problem has been solved. This dependence is evident when agents have non-separable preferences.[3]

In this paper, we take this dynamic structure of allocation problems as an institutional constraint. Given that the goal of implementation theory is to study the relationship between outcomes in a society and the mechanisms under which those outcomes arise, it is important to throw light on how such an institutional constraint affects outcomes in society. In this paper, we ask the following question: If we take the described dynamic structure as an institutional constraint, can one describe the requirements on social choice functions (SCFs) that are equivalent to subgame-perfect Nash implementability by a sequence of period-t mechanisms?

This paper answers the above question by assuming that every agent in society is involved in each period-t mechanism and that there are T periods. Moreover, it does it by assuming that the SCF is a (complete) contingent plan of action: In every period t, the SCF specifies a period-t socially desirable outcome conditional on the outcome history realized up to period $t-1$. We make this assumption due to our institutional constraint and due to the notion of subgame perfection. More precisely, given that the sequence of allocation problems can only be solved period-by-period, given that the optimal solution to period-t allocation problem depends on how the previous allocation problems have been solved, and given that there is a positive, albeit small, probability that agents make mistakes when they carry out their intended actions - this is one of the assumptions on which the notion of subgame perfect equilibrium is based, it is compelling to assume that the SCF is a period-by-period process that assigns an optimal solution to period-t allocation problem that depends on the complete outcome history realized up to period $t-1$. Therefore, the SCF f is defined as a list of period-t SCFs, $\left(f^1\left[\cdot\right], ..., f^T\left[\cdot|\cdot\right]\right)$, one for each period t, such that each period-t SCF f^t depends on the state θ and on the outcome history realized up to period $t-1$. Observe that period-1 SCF f^1 depends only on the state θ.

Given that the sequence of period-t mechanisms can be thought of as one "large" dynamic mechanism, let us denote this dynamic mechanism by Γ. Before introducing our notion of implementation, it will be useful to develop some terminology. A period-t history h^t is a sequence of choices made by agents from period 1 to an intermediate period $t-1$. A period-τ history h^τ is consistent with a period-t history h^t when h^t is the initial part of h^τ. Since by assumption every agent is involved in each period-t mechanism, an agent's (pure) strategy s_i assigns a feasible action to every period-t history h^t. This paper uses subgame-perfect Nash equilibrium as the equilibrium concept for solving, after every period-t history h^t, the dynamic game that the dynamic mechanism $\Gamma\left(h^t\right)$ may lead to. Since each agent's strategy is a complete contingent plan of action, a subgame-perfect Nash equilibrium strategy profile s of the dynamic game $\left(\Gamma\left(h^t\right), \theta\right)$ generates a period-by-period outcome process, in the sense that s specifies an equilibrium outcome for every period-τ history that is consistent with h^t.

The following notion of one-step-ahead implementation is adopted. A SCF f is *one-step-ahead implementable* if there exists a dynamic mechanism Γ such that for each state θ and each period-t history h^t, the following two requirements hold. (a) The period-by-

[3] Note that even when preferences over *consumption sequences* are separable across periods, like in the standard discounted utility preferences, the corresponding preferences defined over *sequences of outcomes* such as trades are generically non-separable across periods.

period outcome process generated by f at state θ from period t onwards coincides with the period-by-period outcome process generated by at least one (pure) subgame-perfect equilibrium strategy profile of the dynamic subgame $(\Gamma(h^t), \theta)$. (b) Every subgame-perfect Nash equilibrium strategy profile of $(\Gamma(h^t), \theta)$ generates a period-by-period outcome process that coincides with the period-by-period outcome process generated by SCF f at state θ from period t onwards.

Under our notion of implementation, we provide a necessary condition, called *one-step-ahead Maskin monotonicity*. This condition is an adaptation to our framework of the fundamental property for Nash implementation, now widely referred to as Maskin monotonicity (Maskin, 1999). Maskin monotonicity says that if x is socially optimal at θ but not at θ', then the outcome x must have fallen strictly in someone's ordering at the state θ'. To introduce our variant of Maskin monotonicity, note that in every period-t environment each agent ranks period-t outcomes according to her period-t reduced preferences, which are induced by means of backward induction. This means that a period-t reduced preference ordering over the set of period-t outcomes depends on past decisions as well as on the socially optimal path that the dynamic process f will bring about in the future. Thus, one-step-ahead Maskin monotonicity requires that every period-t feasible SCF must be Maskin monotonic with respect to period-t reduced preferences. A period-t SCF is feasible when it is Maskin monotonic, after every feasible outcome history.[4]

Furthermore, if for every period t, the period-t SCF satisfies the condition of no veto-power with respect to period-t reduced preferences, we show that the necessary conditions are also sufficient. The dynamic mechanism we construct to achieve the full implementation uses the Maskin mechanism in each period, not only on the equilibrium path but also out-of-equilibrium path.[5] The reason is that period-t planner can neither punish agents over time nor compensate agents in the future when they deviate from a socially undesirable period-t outcome. Though the implementing dynamic mechanism may look complicated, the idea behind it is very simple. Indeed, it can be thought of as a tree (finite directed graph) in which a Maskin mechanism corresponds to each node and in which each node corresponds to a history. Each branch emanating from a node can be thought of as a possible outcome that players can achieve via the mechanism. Then, given a node, the corresponding Maskin mechanism associated with this node simply asks players to report their ranking of branches (plus some tie-breaking information). Note that when a player points out his best branch, she reveals only a partial information about which sequence of outcomes she wants to achieve. For instance, in a consumption-saving model, the period-t planner asks agents to report how much they would like to save/consume in the current period but not to report the full time sequence of consumption/saving. The implementability of a SCF is determined by whether such a one-step-ahead manner is enough for extracting information necessary for it.

Finally, we provide an account of welfare implications of our sufficiency result in the context of trading decisions and voting problems.

Firstly, we consider a borrowing-lending model with no liquidity constraints, in which agents trade in spot markets and transfer wealth between any two periods by borrowing and

[4] There can be (infeasible) histories that make the period-t SCF not Maskin monotonic.

[5] It is worth emphasizing here that we do not intend to seek a contribution to this static-game part. However, once the feasibility condition and one-step-ahead Maskin monotonicity are met, we are forced to accept the conclusions of static implementation.

lending. In this set-up, intertemporal pecuniary externalities arise because today's trade changes tomorrow's spot price, which, in turn, affects its associated equilibrium allocation. The quantitative implication of this is that every agent's reduced preference concerns not only her own consumption/saving behavior but also the consumption/saving behavior of all other agents. Under such a pecuniary externality, we show that the standard sequential competitive equilibrium (or Radner) solution is not one-step-ahead implementable because it is not one-step-ahead Maskin monotonic (see Claim 1 below). We have also identified preference domains - which involve no pecuniary externalities - for which the sequential competitive equilibrium solution is definable and one-step-ahead implementable. It is worth emphasizing that when we focus on non-contingent SCFs (or correspondences), the sequential competitive solution reduces to the Walrasian solution under certainty, and this solution is implementable in Nash equilibrium, and so in subgame-perfect Nash equilibrium, when Walrasian equilibrium allocations on the boundary of the feasible set are ruled out.[6] The reason is that in this case the Walrasian solution satisfies Maskin monotonicity, which is a necessary condition for implementation in Nash equilibrium (Hurwicz et al, 1995). This means that the additional requirement of the paper that implementation should be achieved in a one-step-ahead manner is indeed important and binding (see section 1.1. for a more elaborated discussion).

Secondly, we consider a bi-dimensional policy space where an odd number of agents vote sequentially on each dimension and where an ordering of the dimensions is exogenously given. We assume that each voter's type space is unidimensional, that a majority vote is organized around each policy dimension and that the outcome of the first majority vote is known to the voters at the beginning of the second voting stage. This dynamic resolution is common in political economy models (see, e.g., Persson and Tabellini, 2000). In this environment, we show that the simple majority solution, which selects the Condorcet winner in each voting stage, is one-step-ahead implementable. In this process, we explicitly state the conditions on the utility function of each voter that are needed for this SCF to be well-defined and show that this is the case. As established by De Donder et al (2012) for the case where there is a continuum of voters, the assumption that both dimensions are strategic complements, as well as the requirement that the induced utility of both dimensions is increasing in the type of the voter, are particularly important for guaranteeing the existence of a Condorcet winner in each voting stage.

1.1 Related Literature

The fundamental paper on implementation is thanks to Maskin (1999; circulated since 1977), who proves that any choice rule that can be Nash implemented satisfies a remarkably strong invariance condition, now widely referred to as Maskin monotonicity. Moreover, he shows that when the mechanism designer faces at least three agents, a choice rule is Nash implementable if it is Maskin monotonic and satisfies the condition of no veto-power.[7]

Since Maskin's result, economists have also been interested in understanding how to circumvent the limitations imposed by Maskin monotonicity by exploring the possibilities

[6] For a discussion of the so-called *boundary problem* see, for instance, Lombardi and Yoshihara (2017) and references therein.

[7] For a full characterization see, for instance, Lombardi and Yoshihara (2013).

offered by approximate (as opposed to exact) implementation (Matsushima, 1988; Abreu and Sen, 1991), as well as by implementation in refinements of Nash equilibrium (Moore and Repullo, 1988; Abreu and Sen, 1990; Palfrey and Srivastava, 1991; Herrero and Srivastava, 1992; Jackson, 1992) and by repeated implementation (Kalai and Ledyard, 1998; Lee and Sabourian, 2011; Mezzetti and Renou, 2017).

Moore and Repullo (1988)'s result and Abreu and Sen (1990)'s result say that for many non-contingent SCFs, one can design an extensive game form that yields unique implementation in subgame-perfect Nash equilibria.[8] They find that the class of implementable non-contingent SCFs is dramatically expanded by the use of extensive game forms. Our implementation model differs from the standard model of implementation in subgame-perfect Nash equilibria for two reasons. First, whereas the Moore-Repullo mechanism allows each player to change her mind in later stages so as to overturn outcome decisions made in the previous stages (because decisions are finalized only when the game reaches a terminal node), this is not allowed in our dynamic mechanism. The reason is that in our model a social outcome is chosen and finalized in each period. Thus, though in our dynamic mechanism each player can always change his mind as events unfolds, he cannot change finalized outcomes (i.e., past outcome decisions), but he can change future outcome decisions. For instance, let us consider the problem of consumption/saving allocations over time, in which agents care about consumption/saving streams. In the Moore-Repullo mechanism, the object of choice is a consumption/saving stream, and agents can deviate in each period from their plans so as to change the entire consumption/saving stream. In our mechanism, agents can deviate in period t from their plans but they can only affect consumption/saving decisions that will be made from period t onwards. Second, we are interested in implementing a social contingent plan rather than sequences of social outcomes. In terms of results, we find that we cannot escape the limitations imposed by Maskin monotonicity. Thus, and in contrast to implementation of non-contingent SCFs in subgame-perfect Nash equilibrium, we have that our notion of implementation can be robust to small perturbations from complete information. The reason is that if a choice rule is not Maskin monotonic but is implementable in subgame-perfect Nash equilibrium, then there are small perturbations from complete information under which an undesirable perfect Bayesian equilibrium appears (see Aghion et al., 2012).

The paper on dynamic implementation, which is closest to ours, in particular because it allows for non-separable preferences and outcomes are chosen on a stage-by-stage basis, is Penta (2015). This author extends the belief-free approach to robust mechanism design in dynamic environments, in which agents obtain information over time. In contrast to previous research, agents do not know their own payoff types at the outset: payoff-types are disclosed over time, and known only in the last period. By modelling agents' beliefs separately from agents' information - which is encoded in the payoff types, Penta finds that robust full implementation imposes stronger condition that, for all possible beliefs, all the Perfect Bayesian Equilibria induce outcomes consistent with the SCF. More importantly, he also shows that, for the weaker notion of interim perfect equilibrium, the set of all such equilibria can be computed by means of a recursive procedure which combines the logic of

[8] For a full characterization of the class of social choice rules that are implementable in subgame perferct Nash equilibria see Vartiainen (2007).

rationalizability and backward induction reasoning. Therefore, and similar to our result, by transforming the original dynamic problem into an artificial sequence of static problems, the backwards procedure enables Penta to build on the insights of the static literature to obtain sufficient conditions that guarantee that all the strategies consistent with this solution concept are truthful. However, and in contrast to our implementation model, Penta considers only non-contingent SCFs.

Further, our dynamic problems contrast with the repeated implementation problems studied by Lee and Sabourian (2011) and Mezzetti and Renou (2017), in which agents' period-t preferences are time-separable and they change randomly from one period to the next one. Indeed, in our setup, the evolution of agents' period-t preferences are established as an endogenous process, because they depend on past social decisions and on planners' future goals.

The endogenous evolution of agents' information is a common feature in the most recent literature on dynamic mechanism design (see, e.g., Bergemann and Välimäki, 2010; Athey and Segal, 2013; Pavan et al., 2014; Eső and Szentes, 2017), though this literature maintains the assumption of separability of preferences and focuses on partial implementation.

Finally, our implementation problems also contrast with the static implementation problems studied by Hayashi and Lombardi (2017), in which planners solve their implementation problems simultaneously and do not communicate with each other. Indeed, in our setup, a period-t planner observes the outcome history and this history affects his implementation problem.

The remainder of the paper is organized as follows. Section 2 sets out the theoretical framework and outlines the basic implementation model. Section 3 presents our necessary and sufficient conditions. Section 4 covers one-step-ahead implementable SCFs in the context of trading and voting problems. Section 5 concludes. Appendix includes proofs not in the main body.

2. Basic framework

Let us imagine that a set of agents indexed by $i \in \mathcal{I} \equiv \{1, \cdots, I\}$ have to decide what outcome is best in each time period/stage indexed by $t \in \mathcal{T} \equiv \{1, 2, \cdots, T\}$. Let us denote the universal set of period-t outcomes by X^t, with x^t as a typical outcome. Thus, the universal set of outcome paths available to agents is the space:

$$\mathcal{X} \subseteq \prod_{t \in \mathcal{T}} X^t,$$

with x as a typical outcome path. The t-head x^{-t} is obtained from the path $x \in \mathcal{X}$ by omitting the last t components, that is, $x^{-t} \equiv (x^1, \cdots, x^{t-1})$, the t-tail is obtained from x by omitting the first $t-1$ components, that is, $x^{+t} \equiv \left(x^t, \cdots, x^T\right)$, and we identify (x^{-t}, x^{+t}) with x. The same notational convention will be followed for any profile of outcomes. We will refer to the t-head x^{-t} as the past outcome history x^{-t}.

The feasible set of period-$t+1$ outcomes available to agents depends upon past outcome history $x^{-(t+1)}$, that is, $X^{t+1}\left(x^{-(t+1)}\right) \subseteq X^{t+1}$ for every period $t \neq T$.

We write $\mathcal{F}^t$ for the collection of functions defined as follows:

$$\mathcal{F}^t \equiv \left\{f^t | f^t : \mathcal{X}^{-t} \to X^t \text{ such that } f^t\left[x^{-t}\right] \in X^t\left(x^{-t}\right)\right\}, \quad \text{for all } t \neq 1.$$

We also write $\mathcal{F}$ for the product space $X^1 \times \mathcal{F}^2 \times \cdots \times \mathcal{F}^T$.

The information held by the agents is summarized in the concept of a state, which is a complete description of the variable characterizing the environment. Write Θ for the domain of possible states, with θ as a typical state. For every period $t \geq 2$, the description of the variable characterizing the environment after the outcome history x^{-t} is denoted by $\theta|x^{-t}$. Moreover, for every $t \geq 2$ we write $\theta|x^{-t}, x^{+(t+1)}$ for a complete description of the variable characterizing the environment in period t after the outcome history x^{-t} and the future sure outcome path $x^{+(t+1)}$.

In the usual fashion, we assume that agent i's preference ordering in state θ admits a utility representation $U_i(\cdot, \theta) : \mathcal{X} \to \mathbb{R}$.

2.1 Implementation model

Dynamic social objectives

The goal of the central planner is to implement a social choice function (SCF) $f : \Theta \to \mathcal{F}$ that assigns to each state θ a dynamic "socially optimal period-by-period outcome" process

$$f[\theta] = \left(f^1[\theta], f^2[\theta|\cdot], \cdots, f^T[\theta|\cdot]\right),$$

where:

- $f^1[\theta] \in X^1$ is the period-1 socially optimal outcome and
- $f^t[\theta|\cdot] \in \mathcal{F}^t$ is the period-t socially optimal process that selects the socially optimal outcome $f^t[\theta|x^{-t}]$ in period $t \geq 2$ at the state θ after the past outcome history $x^{-t} \in \mathcal{X}^{-t}$.

To save writing, for every period $t \neq 1$ and every past outcome history x^{-t}, we write $f^{+t}[\theta|x^{-t}]$ for the t-tail path of socially optimal outcomes in state θ that follows the past outcome history x^{-t}, whose period-τ element is the value of the composition $f^\tau \circ f^{\tau-1} \circ \cdots \circ f^t$ at $\theta|x^{-t}$; that is:

$$f^{+t}\left[\theta|x^{-t}\right] \equiv \left(f^\tau\left[\theta|x^{-t}\right]\right)_{\tau \geq t}$$

where $f^\tau[\theta|x^{-t}] \equiv (f^\tau \circ f^{\tau-1} \circ \cdots \circ f^t)[\theta|x^{-t}]$ for every period $\tau \geq t$. The image or range of the period-t function f^t of the SCF f at the past outcome history x^{-t} is the set:

$$f^t\left[\Theta|x^{-t}\right] \equiv \left\{f^t\left[\theta|x^{-t}\right] | \theta \in \Theta\right\}, \quad \text{for every } x^{-t} \in \mathcal{X}^{-t} \text{ with } t \neq 1.$$

The image or range of the period-1 function f^1 of the SCF f is the set $f^1[\Theta] \equiv \{f^1[\theta] | \theta \in \Theta\}$.

Dynamic mechanism

The central planner delegates the achievement of period-t goal, denoted by $f^t[\cdot|\cdot]$ if $t \neq 1$ and by $f^1[\cdot]$ if $t = 1$, to period-t planner, who designs a period-t mechanism (or game form) with observed actions and simultaneous moves. We assume that the actions of every agent are perfectly monitored by every other agent as well as that every agent chooses an action in period t without knowing the period t action of any other agent.

More formally, in the first period all agents $i \in \mathcal{I}$ choose actions from nonempty choice sets $A_i(h^1)$, where $h^1 \equiv \varnothing$ denotes the initial history. Thus, the period-1 action space is the product space:

$$A\left(h^1\right) \equiv \prod_{i\in\mathcal{I}} A_i\left(h^1\right),$$

with $a(h^1) \equiv (a_1(h^1), \cdots, a_I(h^1))$ as a typical period-1 action profile.

Suppose that in period 1 agents have played the action profile $a(h^1) \equiv a^1$. In the second period, agents know the history $h^2 \equiv a^1$, and the actions that every agent $i \in \mathcal{I}$ has available in period 2 depends on what has happened previously. Then, let $A_i(h^2)$ denote the period-2 nonempty action space of agent i when the history is h^2 and let $A(h^2)$ denote the corresponding period-2 nonempty action space, which is defined by:

$$A\left(h^2\right) \equiv \prod_{i\in\mathcal{I}} A_i\left(h^2\right),$$

with $a(h^2) \equiv (a_1(h^2), \cdots, a_I(h^2))$ as a typical period-2 action profile.

Continuing iteratively, we can define h^t, the (nontrivial) history at the beginning of period $t > 1$, to be the list of $t-1$ action profiles,

$$h^t \equiv \left(a^1, a^2, \cdots, a^{t-1}\right),$$

identifying actions played by agents in periods 1 through $t-1$. We let $A_i(h^t)$ be agent i's nonempty action set in period t when the history is h^t and let $A(h^t)$ be the corresponding period-t action space, which is defined by:

$$A\left(h^t\right) \equiv \prod_{i\in\mathcal{I}} A_i\left(h^t\right),$$

with $a(h^t) \equiv (a_1(h^t), \cdots, a_I(h^t))$ as a typical profile of actions.

We assume that in each period t, every agent and period-t planner know the history h^t, this history is common knowledge at the beginning of period t, and that every agent $i \in I$ chooses an action from the action set $A_i(h^t)$. We also assume that in each period t, all agents $i \in I$ choose actions simultaneously.

We let H^t be the set of all period-t histories, where we define H^1 to be the null set, and let

$$H \equiv \bigcup_{t\in\mathcal{T}} H^t$$

be the set of all possible histories.

For any nontrivial history $h^t \equiv (a^1, a^2, \cdots, a^{t-1}) \in H$, define a subhistory of h^t to be a sequence of the form $(a^1, \cdots, a^m)$ with $1 \leq m \leq t-1$, and the trivial history consisting of no actions is denoted by $\varnothing$.

Thus, the implementation of the goals is achieved by means of a dynamic mechanism $\Gamma \equiv (\mathcal{I}, H, A(H), g)$, where H is the set of all possible histories, $A(H)$ is the set of all profiles of actions available to agents, defined by

$$A(H) \equiv \bigcup_{h \in H} A(h),$$

and $g \equiv (g^1, \cdots, g^T)$ is a sequence of outcome functions, one for each period $t \in \mathcal{T}$, with the property that: a) the outcome function g^1 assigns to period-1 action profile $a(h^1) \in A(h^1)$ a unique outcome in X^1, and b) for every period $t \neq 1$ and every nontrivial history $h \equiv (a^1, a^2, \cdots, a^{t-1}) \in H^t$, the outcome function g^t assigns to each period-t action profile $a(h) \in A(h)$ a unique outcome in $X^t(g^{-t}(h))$.

The submechanism of a dynamic mechanism Γ that follows the history h^t is the dynamic mechanism

$$\Gamma(h^t) \equiv (\mathcal{I}, H|h^t, A(H|h^t), g^{+t}),$$

where $H|h^t$ is the set of histories for which h^t is a subhistory of $h \in H|h^t$,

$$A(H|h^t) \equiv \bigcup_{h \in H|h^t} A(h)$$

is the set of all profiles of actions available to agents from period t to period T, and g^{+t} is t-tail of the sequence g that begins with period t after the history h^t such that for every $h^T \equiv (a^1, \cdots, a^{T-1}) \in H^T|h^t$ and every $a(h^T) \in A(h^T)$ it holds that $g(h^T, a(h^T)) = (g^{-t}(h^T, a(h^T)), g^{+t}(h^T, a(h^T)))$.

One-step-ahead implementation

A dynamic mechanism Γ and a state θ induce a dynamic game (Γ, θ) (with observed actions and simultaneous moves). The subgame of the dynamic game (Γ, θ) that follows the history $h^t \in H$ is the dynamic game $(\Gamma(h^t), \theta)$.

Let $A_i \equiv \bigcup_{h \in H} A_i(h)$ be the set of all actions for agent $i \in \mathcal{I}$. A (pure) strategy for agent i is a map $s_i : H \to A_i$ with $s_i(h) \in A_i(h)$ for every history $h \in H$. Individual i's space of strategies, S_i, is simply the space of all such s_i.

A strategy profile $s \equiv (s_1, \cdots, s_I)$ is a list of strategies, one for each agent $i \in \mathcal{I}$. The strategy profile s_{-i} is obtained from s by omitting the ith component, that is, $s_{-i} = (s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_I)$, and we identify (s_i, s_{-i}) with s.

For any strategy s_i of agent i and any history h^t in the dynamic mechanism Γ, the strategy that s_i induces in the dynamic subgame $(\Gamma(h^t), \theta)$ is denoted by $s_i|h^t$. Individual i's space of strategies that follows history h^t is denoted by $S_i|h^t$. The period-t strategy of agent i is sometimes denoted by s_i^t.

For every dynamic game (Γ, θ), the strategy profile s^* is a Nash equilibrium of (Γ, θ) if for every agent $i \in \mathcal{I}$ it holds that:

$$U_i\left(g\left(s_i^*, s_{-i}^*\right), \theta\right) \geq U_i\left(g\left(s_i, s_{-i}^*\right), \theta\right) \text{ for every } s_i \in S_i.$$

Let $NE(\Gamma, \theta)$ denote the set of Nash equilibrium strategy profiles of (Γ, θ).

Moreover, for every dynamic game (Γ, θ) and every nontrivial history $h^t \in H$, the strategy profile $s^*|h^t$ is a Nash equilibrium of $(\Gamma(h^t), \theta)$ if for every agent $i \in \mathcal{I}$ and a given past outcome history $g^{-t}(h^t) \in \mathcal{X}^{-t}$ it holds that:

$$U_i\left(\left(g^{-t}\left(h^t\right), g^{+t}\left(s_i^*|h^t, s_{-i}^*|h^t\right)\right), \theta\right) \geq U_i\left(\left(g^{-t}\left(h^t\right), g^{+t}\left(s_i|h^t, s_{-i}^*|h^t\right)\right), \theta\right)$$

for every $s_i|h^t \in S_i|h^t$. Let $NE(\Gamma(h^t), \theta)$ denote the set of Nash equilibrium strategy profiles of $(\Gamma(h^t), \theta)$.

A strategy profile s^* is a *subgame perfect equilibrium* (SPE) of a dynamic game (Γ, θ) if it holds that:

$$s^*|h^t \in NE\left(\Gamma\left(h^t\right), \theta\right), \qquad \text{for every history } h^t \in H.$$

Let $SPE(\Gamma, \theta)$ denote the set of SPE strategy profiles of (Γ, θ), with s^θ as a typical element.

Definition 1 A dynamic mechanism $\Gamma \equiv (\mathcal{I}, H, A(H), g)$ implements the SCF $f : \Theta \rightarrow \mathcal{F}$ in SPE if for every $\theta \in \Theta$ and every history $h^t \in H$, the following two conditions hold:

(a) There exists $s^\theta \in SPE(\Gamma(h^t), \theta)$ such that $f^\tau[\theta|g^{-\tau}(h^\tau)] = g^\tau\left(s^\theta(h^\tau)\right)$ for every $h^\tau \in H|h^t$ if $h^t \neq h^1$ is not a trivial history, otherwise, $g^1\left(s^\theta(h^1)\right) = f^1[\theta]$ and $f^\tau[\theta|g^{-\tau}(h^\tau)] = g^\tau\left(s^\theta(h^\tau)\right)$ for every $h^\tau \in H \backslash H^1$.[9]

(b) For every $s^\theta \in SPE(\Gamma(h^t), \theta)$, $f^\tau[\theta|g^{-\tau}(h^\tau)] = g^\tau\left(s^\theta(h^\tau)\right)$ for every $h^\tau \in H|h^t$ if $h^t \neq h^1$ is not a trivial history, otherwise, $g^1\left(s^\theta(h^1)\right) = f^1[\theta]$ and $f^\tau[\theta|g^{-\tau}(h^\tau)] = g^\tau\left(s^\theta(h^\tau)\right)$ for every $h^\tau \in H \backslash H^1$.

If such a mechanism exists, the SCF f is said to be *one-step-ahead implementable.*

In other words, part (a) requires that for each state θ and each period-t history h^t, the socially optimal period-by-period outcome process generated by f at state θ, after the outcome history $g^{-t}(h^t)$, coincides with the period-by-period outcome process generated by at least one subgame-perfect equilibrium strategy profile of the dynamic subgame $(\Gamma(h^t), \theta)$. Part (b) requires that for each state θ and each period-t history h^t, every subgame-perfect equilibrium strategy profile of the dynamic subgame $(\Gamma(h^t), \theta)$ generates a period-by-period outcome process that coincides with the socially optimal period-by-period outcome process generated by f from period t onwards, conditional to the outcome history associated with h^t, that is, conditional on $g^{-t}(h^t)$.

The above definition captures the idea that under our institutional constraint, the central planner is forced to delegate the achievement of social objectives to several planners, who have to decide and execute them in a one-step-ahead manner. However, the central planner wishes

[9] Recall that $H^1 = \{\varnothing\}$ and that $H|h^t$ is the set of histories for which h^t is a subhistory of $h^\tau \in H|h^t$.

to be mindful of what can happen along the way because there is a positive, albeit small, probability that players make mistakes.[10] For this reason, he requires that for each state θ, each period t and every period-t history h^t, that is, after every out-of-equilibrium period-t history as well as every period-t equilibrium history, every subgame-perfect equilibrium of the dynamic subgame $(\Gamma(h^t), \theta)$ generates a period-by-period outcome process that coincides with the socially optimal period-by-period outcome process that the SCF f generates at state θ after the realization of the outcome history associated with h^t.

The above definition is based on two tacit assumptions. The first one is that after each out-of-equilibrium history as well as each equilibrium history, the period-t designer assumes that agents are sequentially rational and that he expects that agents will play equilibrium strategies from period t onwards. The second assumption is that every history h^t matters. This implies that after an outcome history x^{-t}, when the period-t designer devises the period-t action space and the period-t outcome function, he cannot limit his attention only to equilibrium histories resulting in x^{-t}. Indeed, if he behaves in this way, he may fail to one-step-ahead implement f. This point is made clear in example 2 below. Note that these two assumptions are not in contradiction with the equilibrium concept of subgame perfect equilibrium, which, indeed, is based on them.

3. Necessary and sufficient conditions

3.1 One-step-ahead Maskin monotonicity

A condition that is central to the Nash implementation thanks to Maskin (1999) is an invariance condition, now widely referred to as Maskin monotonicity. This condition says that if an outcome x is socially optimal at the state θ and this x does not strictly fall in preference for anyone when the state is changed to θ', then x must remain a socially optimal outcome at θ'. An equivalent statement of Maskin monotonicity follows the reasoning that if x is socially optimal at θ but not socially optimal at θ', then the outcome x must have fallen strictly in someone's ordering at the state θ' in order to break the Nash equilibrium via some deviation. Therefore, there must exist some (outcome-)preference reversal if a Nash equilibrium strategy profile at θ is to be broken at θ'. Let us formalize that condition as follows: For any state θ and any agent i and any outcome $x \in X$, the lower contour set of $U_i(x, \theta)$ is defined by $L^{\theta}(x, U_i) \equiv \{y \in X | U_i(x, \theta) \geq U_i(y, \theta)\}$. For any set $Z \subseteq X$ and any $x \in Z$, the lower contour set of $U_i(x, \theta)$ restricted to Z is defined by $L_Z^{\theta}(x, U_i) \equiv \{y \in Z | U_i(x, \theta) \geq U_i(y, \theta)\}$. Therefore:

Definition 2 The SCF $F : \Theta \to X$ is *Maskin monotonic* with respect to $Z \subseteq X$ provided that for all $\bar{\theta}, \theta \in \Theta$, if $L_Z^{\bar{\theta}}(F(\bar{\theta}), U_i) \subseteq L_Z^{\theta}(F(\bar{\theta}), U_i)$ for every $i \in \mathcal{I}$, then $F(\bar{\theta}) = F(\theta)$.

We basically require an adaptation of Maskin monotonicity to each period-t implementation problem. In other words, one-step-ahead Maskin monotonicity requires that every period-t social choice function f^t is Maskin monotonic. To formalize this condition, we need additional notation and requirements.

[10] This is one of the assumptions on which the notion of subgame perfect equilibrium is based.

For a SCF to be one-step-ahead Maskin monotonic it must be *feasible*. This requirement is derived by using the approach developed by Moore and Repullo (1990) and thus it is stated in terms of the existence of certain sets. These sets are denoted by $\mathcal{Y}^{-t}$, Y^1 and $Y^t(y^{-t})$ and represent respectively the set of feasible past outcome histories up to period $t \neq 1$, the set of period-1 attainable outcomes and the set of period-t attainable outcomes after the past outcome history y^{-t}. Formally, this first part of the condition can be stated as follows:

(i) The SCF $f : \Theta \to \mathcal{F}$ is *feasible* if there is a collection of spaces of sequences of past outcomes $\{\mathcal{Y}^{-t}\}_{t\in\mathcal{T}\setminus\{1\}}$, there is a period-1 outcome space $Y^1 \equiv \mathcal{Y}^{-2}$ and there is a collection of period-t outcome spaces $\left\{\{Y^t(y^{-t})\}_{y^{-t}\in\mathcal{Y}^{-t}}\right\}_{t\in\mathcal{T}\setminus\{1\}}$ such that

a) $f^1[\Theta] \subseteq Y^1$ and $f^t[\Theta|y^{-t}] \subseteq Y^t(y^{-t})$ for every $t \neq 1$;

b) for every $t \neq 1$, it holds that

$$y^{-t} \in \mathcal{Y}^{-t} \iff y^1 \in Y^1 \text{ and } y^\tau \in Y^\tau\left(y^{-\tau}\right) \text{ for every } 2 \leq \tau \leq t.$$

A SCF satisfying the above requirement is said to be a *feasible* SCF. This condition requires the existence of some feasibility constraints represented by the sets $\mathcal{Y}^{-t}$, Y^1 and $Y^t(y^{-t})$. As the proof of Theorem 1 below will show, the set Y^1 depends on the range of g^1, whereas the set $Y^t(y^{-t})$ depends on the range of g^t, where $y^{-t} = g^{-t}(h)$ for some history $h \in H^t$. This means that the set Y^1 can be a proper subset of X^1 and that $Y^t(y^{-t})$ can be a proper subset of X^t. Moreover, the reason of the existence of the set $\mathcal{Y}^{-t}$ is that there can be infeasible outcome histories x^{-t} after which the SCF f is not one-step-ahead implementable. The existence of the set $\mathcal{Y}^{-t}$ is guaranteed by defining it by $\mathcal{Y}^{-t} \equiv \{g^{-t}(h) \in \mathcal{X}^{-t} | \text{for some } h \in H^t\}$, for every $t \neq 1$. This set entails that in order to implement a SCF in one step-ahead manner there must exist some degree of coordination between the current designer and the future designers, in the *sense* that the current designer takes into account not just his goal by also the future goals in his design. To see it clearly, see example 1 below. This is in line with our definition of implementation. The reason is that if Γ one-step-ahead implements f, then after every history, the period-t action space and the period-t outcome function are such that they make the implementation problems of future designers solvable, as well as solving the period-t implementation problem. The example below also shows that the requirement of feasibility is *not* trivial.

Example 1 Let $\mathcal{I} \equiv \{1,2\}$, $\mathcal{T} \equiv \{1,2\}$ and $\Theta \equiv \{\theta, \theta', \theta'', \theta'''\}$. Suppose that the set X^t consists of three distinct period-t outcomes, $X^t = \{x^t, y^t, z^t\}$, for each $t \in \mathcal{T}$, and that the set $\mathcal{X}$ is $\mathcal{X} \equiv X^1 \times X^2$. Agent i's utility function is $U_i(a^1, a^2, \eta) = u_i^1(a^1, \eta) + u_i^2(a^2, \eta)$ for all $(a^1, a^2) \in \mathcal{X}$ and all $\eta \in \Theta$, where $u_i^t(\cdot, \eta)$ is agent i's period-t utility function. The per-period utilities are summarized in the tables below.

	$u_1^1(\cdot,\theta) = u_2^1(\cdot,\theta')$	$u_1^1(\cdot,\theta') = u_2^1(\cdot,\theta)$	$u_1^1(\cdot,\theta'') = u_2^1(\cdot,\theta'')$	$u_1^1(\cdot,\theta''') = u_2^1(\cdot,\theta''')$
x^1	1	1	0	0
y^1	1	0	1	-1
z^1	0	2	-1	1

	$u_1^2(\cdot,\theta)=u_2^2(\cdot,\theta)$	$u_1^2(\cdot,\theta')=u_2^2(\cdot,\theta')$	$u_1^2(\cdot,\theta'')=u_2^2(\cdot,\theta'')$	$u_1^2(\cdot,\theta''')=u_2^2(\cdot,\theta''')$
x^2	0	0	0	0
y^2	0	0	1	-1
z^2	0	0	-1	1

Finally, let the SCF f be defined as follows: $f^1[\theta]=f^1[\theta']=x^1$, $f^1[\theta'']=y^1$, $f^1[\theta''']=z^1$, $f^2[\theta|x^1]=f^2[\theta'|x^1]=f^2[\theta''|x^1]=f^2[\theta'''|x^1]=x^2$, $f^2[\theta|y^1]=f^2[\theta|z^1]=f^2[\theta''|y^1]=f^2[\theta''|z^2]=y^2$ and $f^2[\theta'|y^1]=f^2[\theta'|z^1]=f^2[\theta'''|y^1]=f^2[\theta'''|z^2]=z^2$.

We will show that this SCF cannot be one-step-ahead implemented if period-1 designer neglects period-2 SCF f^2. To see it, suppose that period-1 designer designs agent i's period-1 action space by setting $A_i^1(\varnothing)\equiv\{a_i^1,\hat{a}_i^1\}$, for each agent $i\in\mathcal{I}$, and he defines the period-1 outcome function g^1 as illustrated in the table below. In this table, the two columns correspond to the two the possible actions of agent 2, the two rows correspond to the two possible actions of agent 1, and the outcomes in each box are the outcomes to the action profile to which the box corresponds.

	a_2^1	$\hat{a}_2^1$
a_1^1	x^1	y^1
$\hat{a}_1^1$	z^1	x^1

Whatever the agents believe will happen in period 2, the unique period-1 Nash equilibrium of the period-1 mechanism is (a_1^1,a_2^1) in state θ, $(\hat{a}_1^1,\hat{a}_2^1)$ in state θ', $(a_1^1,\hat{a}_2^1)$ in state θ'' and $(\hat{a}_1^1,a_2^1)$ in state θ'''. These equilibria result in socially optimal outcomes, that is, $x^1=g^1(a_1^1,a_2^1)=f^1[\theta]$, $x^1=g^1(\hat{a}_1^1,\hat{a}_2^1)=f^1[\theta']$, $y^1=g^1(a_1^1,\hat{a}_2^1)=f^1[\theta'']$ and $z^1=g^1(\hat{a}_1^1,a_2^1)=f^1[\theta''']$. Therefore, period-1 designer solves his implementation problem. However, this makes impossible for the period-2 designer to solve his.

To see it, assume, to the contrary, that there is a dynamic mechanism Γ that one-step-ahead implements f and that it is such that its period-1 action space is $A(\varnothing)=\{a_1^1,\hat{a}_1^1\}\times\{a_2^1,\hat{a}_2^1\}$ and its period-1 outcome function is g^1. Thus, by implementability of f, we can define $Y^1=X^1=\{g^1(h)\in X^1|h\in A(\varnothing)\}$. Part (b) of requirement (i) implies that $\mathcal{Y}^{-2}=Y^1$, whereas its part (a) implies that $f^2[\Theta|x^1]=\{x^2\}\subseteq Y^2(x^1)$ and $f^2[\Theta|b]=\{y^2,z^2\}\subseteq Y^2(b)$ for $b\in\{y^1,z^1\}$. The fact that $f^2[\Theta|y^1]=\{y^2,z^2\}\subseteq Y^2(y^1)$ implies that the range of period-2 outcome function g^2 of Γ, after the outcome history y^1, must contain at least y^2 and z^2. Given that, by our initial assumption, agents are indifferent between period-2 outcomes at the states θ and θ', it follows that y^2 and z^2 are two Nash equilibrium outcomes of the period-2 game $(A(a_1^1,\hat{a}_2^1),g^2,u_1^2(\cdot),u_2^2(\cdot))$, which contradicts the assumption that Γ one-step-ahead implements f. Note that the same conclusion is reached when one considers the outcome history z^1.[11]

Next, let us introduce the other requirements. Solving backward, for any feasible past outcome history y^{-T}, agent i's *period-T reduced utility* in state θ at y^{-T}, denoted

[11] When $\Theta=\{\theta,\theta'\}$, one can see that Γ can one-step-ahead implement f if the range of period-1 outcome function is $\{x^1\}$, so that this outcome is the only possible period-2 outcome history. Indeed, it is not difficult to see that when $\Theta=\{\theta,\theta'\}$ any dynamic mechanism in which the range of period-t outcome function g^t is equal to the set $\{x^t\}$, for each $t\in\mathcal{T}$, one-step-ahead implements f.

by $U_i\left[\cdot,\theta|y^{-T}\right]$, is equal to:

$$U_i\left[y^T,\theta|y^{-T}\right] \equiv U_i\left(y^{-T},y^T,\theta\right) \quad (1)$$

for every $y^T \in Y^T\left(y^{-T}\right)$. We denote by $U\left[\theta|y^{-T}\right]$ the profile of period-T reduced utilities at $\theta|y^{-T}$ and by $\mathcal{U}\left[\Theta|y^{-T}\right]$ the period-T domain of reduced utilities at $\Theta|y^{-T}$.[12] Therefore, the second part of the condition is represented by the requirements (ii)-(iv) below.

(ii) The period-T SCF f^T is Maskin monotonic with respect to $Y^T\left(y^{-T}\right)$ and $\mathcal{U}\left[\Theta|y^{-T}\right]$ provided that for all $\bar{\theta},\theta \in \Theta$, if for every $i \in \mathcal{I}$,

$$\begin{aligned} & L^{\theta}_{Y^T(y^{-T})}\left(f^T\left[\theta|y^{-T}\right],U_i\left[\cdot,\theta|y^{-T}\right]\right) \\ & \quad \subseteq L^{\bar{\theta}}_{Y^T(y^{-T})}\left(f^T\left[\theta|y^{-T}\right],U_i\left[\cdot,\bar{\theta}|y^{-T}\right]\right), \end{aligned} \quad (2)$$

then $f^T\left[\theta|y^{-T}\right] = f^T\left[\bar{\theta}|y^{-T}\right]$.

Next, suppose that in our way back to period 1 we have reached period $t \neq 1,T$ and that y^{-t} is a feasible past outcome history. Given that in our framework rationality is common knowledge between the players and given that the objective of the central planner is to implement a dynamic social choice process prescribed by the SCF f, every player will "look ahead" and a period-t outcome y^t will be evaluated at the past outcome history y^{-t} as well as at the future sure outcome path $f^{+(t+1)}$ prescribed by the SCF in response to the outcome history path (y^{-t},y^t). On this basis, agent i's *period-t reduced utility* in state θ at the past outcome history y^{-t} and at the future sure outcome path prescribed by the social process $f^{+(t+1)}$, denoted by $U_i\left[\cdot,\theta|y^{-t},f^{+(t+1)}\right]$, is equal to:

$$U_i\left[y^t,\theta|y^{-t},f^{+(t+1)}\right] \equiv U_i\left(y^{-t},y^t,f^{+(t+1)}\left[\theta|\left(y^{-t},y^t\right)\right],\theta\right) \quad (3)$$

for every $y^t \in Y^t\left(y^{-t}\right)$. Let us denote by $U\left[\theta|y^{-t},f^{+(t+1)}\right]$ the profile of period-t reduced utilities at $\theta|y^{-t},f^{+(t+1)}$ for $t \neq 1,T$ and by $\mathcal{U}\left[\Theta|y^{-t},f^{+(t+1)}\right]$ the period-t domain of reduced utilities at $\Theta|y^{-t},f^{+(t+1)}$. Therefore, as for the first part of the condition, the second part can be stated as follows:

(iii) The period-t SCF f^t is Maskin monotonic with respect to $Y^t\left(y^{-t}\right)$ and $\mathcal{U}\left[\Theta|y^{-t},f^{+(t+1)}\right]$ provided that for all $\bar{\theta},\theta \in \Theta$, if for every $i \in \mathcal{I}$,

$$\begin{aligned} & L^{\theta}_{Y^t(y^{-t})}\left(f^t\left[\theta|y^{-t}\right],U_i\left[\cdot,\theta|y^{-t},f^{+(t+1)}\right]\right) \\ & \subseteq L^{\bar{\theta}}_{Y^t(y^{-t})}\left(f^t\left[\theta|y^{-t}\right],U_i\left[\cdot,\bar{\theta}|y^{-t},f^{+(t+1)}\right]\right), \end{aligned} \quad (4)$$

then $f^t\left[\theta|y^{-t}\right] = f^t\left[\bar{\theta}|y^{-t}\right]$.

[12] That is, $U\left[\theta|y^{-T}\right] \equiv \left(U_i\left[\cdot,\theta|y^{-T}\right]\right)_{i\in\mathcal{I}}$ and $\mathcal{U}\left[\Theta|y^{-T}\right] \equiv \left\{U\left[\theta|y^{-T}\right]|\theta \in \Theta\right\}$.

Reasoning like that used in the preceding paragraphs, agent i's *period*-1 *reduced utility* in state θ at the outcome path prescribed by the social process f^{+2}, denoted by $U_i[\cdot,\theta|f^{+2}]$, is equal to:

$$U_i\left[y^1,\theta|f^{+2}\right] \equiv U_i\left(y^1, f^{+2}\left[\theta|y^1\right],\theta\right) \tag{5}$$

for every $y^1 \in Y^1$. Denoting the profile of period-1 reduced utilities at $\theta|f^{+2}$ by $U[\theta|f^{+2}]$, and the period-1 domain of reduced utilities at $\Theta|f^{+2}$ by $\mathcal{U}[\Theta|f^{+2}]$, the third part of the condition can be stated as follows:

(iv) The period-1 SCF f^1 is Maskin monotonic with respect to Y^1 and $\mathcal{U}[\Theta|f^{+2}]$ provided that for all $\bar{\theta},\theta \in \Theta$, if for every $i \in \mathcal{I}$,

$$L^{\theta}_{Y^1}\left(f^1[\theta], U_i\left[\cdot,\theta|f^{+2}\right]\right) \subseteq L^{\bar{\theta}}_{Y^1}\left(f^1[\theta], U_i\left[\cdot,\bar{\theta}|f^{+2}\right]\right), \tag{6}$$

then $f^1[\theta] = f^1\left[\bar{\theta}\right]$.

The condition of one-step-ahead Maskin monotonicity can be stated as follows:

Definition 3 A SCF $f:\Theta\to\mathcal{F}$ is *one-step-ahead Maskin monotonic* provided that it is a feasible SCF, that every period-t SCF f^t is Maskin monotonic with respect to Y^1 and $\mathcal{U}[\Theta|f^{+2}]$ if $t=1$, with respect to $Y^{-t}(y^{-t})$ and $\mathcal{U}\left[\Theta|y^{-t},f^{+(t+1)}\right]$, for every $y^{-t}\in\mathcal{Y}^{-t}$, if $t\neq 1,T$, and with respect to $Y^{-T}(y^{-t})$ and $\mathcal{U}\left[\Theta|y^{-T}\right]$, for every $y^{-t}\in\mathcal{Y}^{-t}$, if $t=T$.

Our second result is that only one-step-ahead Maskin monotonic SCFs are one-step-ahead implementable.

Theorem 1 If $I \geq 2$ and the SCF $f:\Theta\to\mathcal{F}$ is one-step-ahead implementable, then it is one-step-ahead Maskin monotonic.

Proof. See Appendix. ■

Before closing this subsection, let us show, by means of an example, that given an outcome history x^{-t}, if the period-t mechanism designer devises the period-t action space and the period-t outcome function by taking into account only equilibrium histories resulting in this outcome history, as well as expecting that agents will follow equilibrium strategies in the future, he may fail to one-step-ahead implements f. The reason is that according to our notion of implementation, the period-t mechanism designer needs to achieve a socially optimal decision irrespective of whether he is on equilibrium path or not. Continuing the analogy of the implementing dynamic mechanism with a tree (see introduction), the period-t mechanism designer must select the right branch and, at the same time, he needs to take into account the fact that he may be at a wrong node. Note that this is nothing but the spirit of subgame-perfection, and in this sense of our notion of implementation is not in contradiction with our equilibrium notion.

Example 2 Let $\mathcal{I} \equiv \{1,2\}$, $\mathcal{T} \equiv \{1,2\}$ and $\Theta \equiv \{\theta, \theta'\}$. Suppose that the set X^t consists of three distinct period-t outcomes, $X^t = \{x^t, y^t, z^t\}$, for each $t \in \mathcal{T}$, and that the set $\mathcal{X}$ is $\mathcal{X} \equiv X^1 \times X^2$. Agent i's state-dependent utility function is $U_i(a^1, a^2, \eta) = u_i^1(a^1, \eta) + u_i^2(a^2, \eta)$ for all $(a^1, a^2) \in \mathcal{X}$ and all $\eta \in \Theta$, where $u_i^t(\cdot, \eta)$ is agent i's period-t state-dependent utility function. The per-period utilities are summarized in the tables below.

	$u_1^1(\cdot,\theta) = u_2^1(\cdot,\theta')$	$u_1^1(\cdot,\theta') = u_2^1(\cdot,\theta)$
x^1	2	2
y^1	2	0
z^1	0	4

	$u_1^2(\cdot,\theta)$	$u_2^2(\cdot,\theta)$	$u_1^2(\cdot,\theta')$	$u_2^2(\cdot,\theta')$
x^2	2	1	0	0
y^2	0	0	1	1
z^2	1	2	2	2

Finally, let the SCF f be defined as follows: $f^1[\theta] = f^1[\theta'] = x^1$, $f^2[\theta|x^1] = x^2$, $f^2[\theta'|x^1] = y^2$, $f^2[\theta|y^1] = f^2[\theta|z^1] = f^2[\theta'|y^1] = f^2[\theta'|z^1] = z^2$. Though f is one-step-ahead Maskin monotonic, we will show that designers may fail to implement it if, in their design, they focus only on equilibrium histories.

To check that f is one-step-ahead Maskin monotonic, let $Y^1 = \{x^1, y^1, z^1\} = \mathcal{Y}^{-2}$, $Y^2(x^1) = \{x^2, y^2\}$ and $Y^2(y^1) = Y^2(z^1) = \{y^2, z^2\}$. By definition of these sets, it is clear that f is feasible. Also, one can easily check that f^t is Maskin monotonic, for each $t \in \mathcal{T}$. Thus, f is one-step-ahead Maskin monotonic.

Suppose that period-1 designer designs the period-1 action space and the period-1 outcome function as illustrated in example 1 above. Suppose that period-2 designer ignores out-of-equilibrium histories that result in the outcome history x^1 when he designs the period-2 action space and the period-2 outcome function. To this end, suppose that period-2 designer defines them as follows.

- If the period-1 play has been (a_1^1, a_2^1), then $A_1(a_1^1, a_2^1) \times A_2(a_1^1, a_2^1) = \{(a_1^2, a_2^2)\}$ and $g^2(a_1^2, a_2^2) = x^2$.
- If the period-1 play has been $(\hat{a}_1^1, \hat{a}_2^1)$, then $A_1(\hat{a}_1^1, \hat{a}_2^1) \times A_2(\hat{a}_1^1, \hat{a}_2^1) = \{(\hat{a}_1^2, \hat{a}_2^2)\}$ and $g^2(\hat{a}_1^2, \hat{a}_2^2) = y^2$.
- If the period-1 play has been $(a_1^1, \hat{a}_2^1)$ or $(\hat{a}_1^1, a_2^1)$, then $A_1(a_1^1, \hat{a}_2^1) \times A_2(a_1^1, \hat{a}_2^1) = A_1(\hat{a}_1^1, a_2^1) \times A_2(\hat{a}_1^1, a_2^1) = \{(\bar{a}_1^2, \bar{a}_1^2)\}$ and $g^2(\bar{a}_1^2, \bar{a}_2^2) = z^2$.

Note that the game form faced by agents is as follows.

	a_2^1	$\hat{a}_2^1$
a_1^1	$x^1 + x^2$	$y^1 + z^2$
$\hat{a}_1^1$	$z^1 + z^2$	$x^1 + y^2$

Whatever the agents believe will happen in period 2, the unique period-1 Nash equilibrium is (a_1^1, a_2^1) in state θ, whereas the unique period-1 Nash equilibrium is $(\hat{a}_1^1, \hat{a}_2^1)$ in state

θ'. Both result in the socially optimal outcome $x^1 = f^1[\theta] = f^1[\theta']$. Given this period-1 equilibrium behavior, the socially optimal outcome (x^2 at θ or y^2 at θ') is selected in period 2. Thus, the socially optimal outcomes are selected on the equilibrium paths. Note that if the period-1 outcome history is either y^2 or z^2, the socially optimal outcome is still selected in period 2. However, the above dynamic mechanism fails to one-step-ahead implement f because it fails to select the period-2 socially optimal outcome if in period 1, the agents have played (a^1, a^2) when the true state is θ' or played $(\hat{a}^1, \hat{a}^2)$ when the true state is θ.

Note that this dynamic mechanism makes the SCF f not feasible. To see it, suppose that it one-step-ahead implements f. Take x^1. Let us consider the case where $h^2 = (a^1, a^2)$, so that $g^1(h^2) = x^1$. In this case, let us define the set of period-2 obtainable outcomes $Y^2(g^1(h^2))$ by $Y^2(g^1(h^2)) = \{g^2(a^2(h^2)) \in X^2 | a^2(h^2) \in A(h^2)\} = \{x^2\}$. Similarly, we have that $Y^2\left(g^1\left(\hat{h}^2\right)\right) = \{y^2\}$ in the case where $\hat{h}^2 = (\hat{a}^1, \hat{a}^2)$. One can now see that both sets fail to meet part (a) of requirement (i), which is a contradiction.

As mentioned above, the SCF f is one-step-ahead Maskin monotonic. Indeed, we show below that it is implementable. To see this, consider the following dynamic mechanism. The period-1 action space and the period-1 outcome function are as before. The period-2 action spaces and the period-2 outcome function g^2 are as follows.

- If the period-1 play has been either (a_1^1, a_2^1) or $(\hat{a}_1^1, \hat{a}_2^1)$, then $A_i(h^2) = \{a_i^2, \hat{a}_i^2\}$ for each $i \in \mathcal{I}$ and the period-2 outcome function g^2 is as illustrated in the table below.

	a_2^2	$\hat{a}_2^2$
a_1^2	x^2	y^2
$\hat{a}_1^2$	y^2	x^2

- Otherwise, let $A_i(h^2) = \{\bar{a}_i^2, \tilde{a}_i^2\}$ for each $i \in \mathcal{I}$ and the period-2 outcome function g^2 is as illustrated in the table below.

	$\bar{a}_2^2$	$\tilde{a}_2^2$
$\bar{a}_1^2$	z^2	y^2
$\tilde{a}_1^2$	y^2	z^2

When x^1 is the period-1 outcome history, at state θ, both (a_1^2, a_2^2) and $(\hat{a}_1^2, \hat{a}_2^2)$ are period-2 Nash equilibria, with outcome x^2. When x^1 is the period-1 outcome history, at state θ', both $(a_1^2, \hat{a}_2^2)$ and $(\hat{a}_1^2, a_2^2)$ are period-2 Nash equilibria, with outcome y^2. Then, the socially optimal outcome is selected in period 2 after the outcome history x^1. Moreover, let $b^1 \in \{y^1, z^2\}$ be the period-1 outcome history. At both states, both $(\bar{a}_1^2, \bar{a}_2^2)$ and $(\tilde{a}_1^2, \tilde{a}_2^2)$ are period-2 Nash equilibria, with outcome z^2. Again, the socially optimal outcome is selected in period 2 after the period-1 outcome history b^1. Thus, f is one-step-ahead implementable.

3.2 The characterization theorem

In the abstract Arrovian domain, the condition of no veto-power says that if an outcome is at the top of the preferences of all agents but possibly one, then it should be chosen irrespective of the preferences of the remaining agent: that agent cannot veto it. Formally, this property can be stated as follows for an abstract outcome space X:[13]

Definition 4 A SCF $F : \Theta \twoheadrightarrow X$ satisfies *no veto-power* with respect to $Z \subseteq X$ provided that for all $\theta \in \Theta$ and all $x \in Z$, if

$$\left|\left\{i \in N | Z \subseteq L_Z^{\theta}(x, U_i)\right\}\right| \geq n - 1,$$

then $x = F(\theta)$.

As a part of sufficiency, we require an adaptation of the above definition to each period-t implementation problem. In other words, one-step-ahead no veto power requires that each of period-t function f^t defined over period-t domain of reduced utilities satisfies the condition of no veto-power. The condition can be stated as follows:

Definition 5 A feasible SCF $f : \Theta \to \mathcal{F}$ satisfies *one-step-ahead no veto-power* provided that every period-t SCF f^t satisfies the condition of no veto-power with respect to Y^1 and $\mathcal{U}[\Theta|f^{+2}]$ if $t = 1$, with respect to $Y^{-t}(y^{-t})$ and $\mathcal{U}\left[\Theta|y^{-t}, f^{+(t+1)}\right]$, for every $y^{-t} \in \mathcal{Y}^{-t}$, if $t \neq 1, T$, and with respect to $Y^{-T}(y^{-t})$ and $\mathcal{U}\left[\Theta|y^{-T}\right]$, for every $y^{-t} \in \mathcal{Y}^{-t}$, if $t = T$.

Our characterization of one-step-ahead implementable SCFs can thus be stated as follows:[14]

Theorem 2 Let $I \geq 3$. If a SCF $f : \Theta \to \mathcal{F}$ satisfies one-step-ahead Maskin monotonicity and one-step-ahead no veto-power, then it is one-step-ahead implementable.

Proof. See Appendix. ■

4. Implications

4.1 Impossibility of implementing the dynamic competitive solution

In this section, we investigate whether the trading rule as considered in the dynamic general equilibrium framework is indeed one-step-ahead implementable.

When it is literally understood, the concept of Arrow-Debreu-McKenzie (ADM) (Arrow and Debreu, 1954; McKenzie,1954) equilibrium says that all the agents meet on the first day of their life and write down a contract on all the deliveries of consumption contingent on every date-event, and simply commit to it. A more realistic description of trading over time is by Radner (1972, 1982), which considers that at each period agents can trade only between current consumption and assets to be carried over to the next period. To our knowledge, however, the Radner-type model has not been given a strategic foundation. In the Radner model prices are defined only for *on-path* situation and it is left unclear what prices should be

[13]For any finite set S, $|S|$ denotes the cardinality of S.

[14]Note that the sets with respect to which f satisfies one-step-ahead maskin monotonicity and one-step-ahead no veto-power are the same.

formed in *off-path* situations, while a strategic outcome function in a dynamic environment must specify prices and allocations even at off-path histories. In fact, as far as the markets are sequentially complete ADM equilibrium and Radner equilibrium are equivalent (Arrow, 1964). This means that from strategic viewpoints the Radner model cannot escape the problem which the ADM model has. The competitive models are silent about what prices and allocations should be formed after the society makes mistake.

Strategic implementation of competitive solutions in general involves a strange story: each agent is supposed to behave as a price-taker, despite he is aware that message he sends may affect the market price. In the static setting, these apparently contradicting natures can be made compatible by making the mechanism nicely so that agents face a kind of coordination game in which they are induced to agree on prices in equilibrium. In fact, the (feasibility-constrained version of) ADM solution is Nash-implementable.

Being a price-taker is harder in dynamic environments, however, when social decision and execution can be made only in a sequential manner. It requires that every agent perceives that he cannot affect spot price/interest rate at any period, in particular that the amount of asset to carry over to the future does not affect the spot prices/interest rates in the future, despite he is aware that messages he sends may affect the market price in both the current period and the future periods, and that equilibrium prices and allocations in the future periods are a function of *whole allocations* in the current period including his own.

Below we explain the nature of the problem and see whether the Radner-type solution can clear this bar.

For the sake of convenience, we assume that there are only *three* consumption periods (CPs), and so *two* trading periods (TPs), and that there is one perfectly divisible commodity in each CP. In TP1 agents transfer consumption between CP1 and CP2, and in TP2 they transfer consumption between CP2 and CP3.

In TP1, agents sell/buy consumption in CP1 and buy/sell consumption in CP2. In TP2, agents sell/buy consumption in CP2 and buy/sell consumption in CP3. Let q^1 be the TP1 spot price, the relative price of CP2 consumption for CP1 consumption, and q^2 be the TP2 spot price, the relative price of CP3 consumption for CP2 consumption.

Each agent i is endowed with an amount ω_i^t of the commodity in CPt. The total endowment of the commodity in CPt is denoted by ω^t. Agent i's consumption set is $\mathbb{R}^3_+$, and her consumption in CPt is denoted by c_i^t. In state θ, this agent has preference ordering over consumption sequences in her consumption set. Endowments are given once and for all, and therefore an *economy* is described by a state θ.

The domain assumption is that at each economy $\theta \in \Theta$ agent i's preference ordering $\succsim_i^\theta$ is represented by an additively separable utility function

$$U_i(c_i^1, c_i^2, c_i^3, \theta) = v_i^1(c_i^1, \theta) + v_i^2(c_i^2, \theta) + v_i^3(c_i^3, \theta).$$

This guarantees that all consumption goods in CP1, CP2 CP3 are gross-substitutes of each other and the ADM and Radner equilibrium is unique.

We describe feasible allocations by using net trade vectors. Let

$$H = \left\{ z \in \mathbb{R}^I | \sum_{i \in \mathcal{I}} z_i = 0 \right\},$$

which is the set of closed net trades. Thus, the set of closed net trade vectors for TPt can be defined by

$$Z^t = H^t \times H^{t+1}, \qquad \text{for } t = 1, 2.$$

A TPt net trade allocation is thus a vector $z^t = (z^{tt}, z^{tt+1})$ in Z^t, where the ith element z_i^{tt} of z^{tt} denotes agent i's net trade of consumption in CPt, and where the ith element z_i^{tt+1} of z^{tt+1} denotes agent i's net trade of consumption in CP$(t+1)$.

The set of feasible net trade allocations over the two trading periods is denoted by Z and defined by

$$Z = \left\{(z^1, z^2) \in Z^1 \times Z^2 | \omega_i^1 + z_i^{11} \geq 0,\ \omega_i^2 + z_i^{12} + z_i^{22} \geq 0, \omega_i^3 + z_i^{23} \geq 0,\ \forall i \in \mathcal{I}\right\}.$$

The set of feasible TP1 net trade allocations is given by

$$\bar{Z}^1 = \{z^1 \in Z^1 | (z^1, z^2) \in Z \text{ for some } z^2 \in Z^2\},$$

while the set of TP2 net trade allocation, conditional on z^1, is given by

$$\bar{Z}^2(z^1) = \{z^2 \in Z^2 | (z^1, z^2) \in Z\}, \quad \text{for all } z^1 \in \bar{Z}^1.$$

In economy $\theta \in \Theta$, agent i's utility function U_i over consumption sequences induces a utility function V_i over the set of feasible net trade allocations Z in the natural way: for all $z, \hat{z} \in Z$,

$$\begin{array}{rcl} V_i(z, \theta) & \geq & V_i(\hat{z}, \theta) \\ & \Longleftrightarrow & \\ U_i(\omega_i^1 + z_i^{11}, \omega_i^2 + z_i^{12} + z_i^{22}, \omega_i^3 + z_i^{23}, \theta) & \geq & U_i(\omega_i^1 + \hat{z}_i^{11}, \omega_i^2 + \hat{z}_i^{12} + \hat{z}_i^{22}, \omega_i^3 + \hat{z}_i^{23}, \theta). \end{array}$$

Though the utility $U_i(\cdot, \theta)$ exhibits separability over consumption sequences, the derived utility $V_i(\cdot, \theta)$ over Z is typically non-separable since consumption in CP2 depends on net trades in both TP1 and TP2.

We provide the definition of competitive equilibrium backward. The definition of equilibrium when we start from TP2 is straightforward.[15]

Definition 6 For every economy $\theta \in \Theta$ and every $z^1 \in \bar{Z}^1$, the net trade allocation $f^2[\theta|z^1] \in \bar{Z}^2(z^1)$ constitutes a *TP2 competitive net trade allocation*, conditional on z^1, if there is a TP2 spot price $q^2[\theta|z^1]$ such that for every agent i this allocation $f^2[\theta|z^1]$ solves the following problem:

$$\begin{array}{ll} \underset{z^2 \in \bar{Z}^2(z^1)}{\text{Maximize}} & U_i(\omega_i^1 + z_i^{11}, \omega_i^2 + z_i^{12} + z_i^{22}, \omega_i^3 + z_i^{23}, \theta) \\ \text{subject to} & z_i^{22} + q^2[\theta|z^1] z_i^{23} \leq 0. \end{array}$$

[15] Note that this is a feasibility-constrained version. As it is known that the ADM solution fails to satisfy Maskin monotonicity when it results in boundary allocations, and it is necessary to modify the solution by truncating each agent's consumption set by the set of feasible allocations. Here each individual's admissible set of trades is truncated by $\bar{Z}^2(z^1)$, although it does not matter when we can restrict attention to interior allocations.

Let $V_i[\cdot,\theta|f^2]$ denote agent i's TP1 reduced utility over the set of feasible TP1 net trade allocations, $\bar{Z}^1$, and be defined by

$$\begin{array}{rcl} V_i[x^1,\theta|f^2] & \geq & V_i[y^1,\theta|f^2] \\ & \Longleftrightarrow & \\ V_i(x^1,f^2[\theta|x^1],\theta) & \geq & V_i(y^1,f^2[\theta|y^1],\theta) \end{array} \tag{7}$$

for all $x^1,y^1\in\bar{Z}^1$.

In contrast to static pure exchange economies where each agent's preferences are defined over her own net trade vectors, in sequential trading, each agent must have preferences over *whole* TP1 net trade allocations. This is due to the presence of *intertemporal pecuniary externalities.* Indeed, an outcome of the trading rule in TP2 depends on the net trade allocation assigned in TP1, because trading in TP1 affects the values of endowments in the next trading period. Moreover, the period-1 induced preference ordering may be *non-convex.* In order for it to be a convex preference ordering, it is required that the TP2 function f^2 that maps every economy, conditional on past trades, into a TP2 net trade allocation be a concave function, but this requirement fails for any reasonable trading rule. As is known, although convexity is no more than a sufficient technical condition for things to work, it becomes extremely difficult to establish any reasonable solution once it is violated.

We may proceed in two ways. First, we can still *define* a concept of competitive equilibrium following the tradition of dynamic general equilibrium theory.

Definition 7 For every economy $\theta\in\Theta$, a TP1 net trade allocation $f^1[\theta]\in\bar{Z}^1$ constitutes a *TP1 competitive net trade allocation* if there is a TP1 spot price $q^1[\theta]$ such that for every agent i the net trade allocation profile $(f^1[\theta],f^2[\theta|f^1[\theta]])$ solves the following problem:

$$\begin{array}{ll} \underset{z\in Z}{\text{Maximize}} & U_i(\omega_i^1+z_i^{11},\omega_i^2+z_i^{12}+z_i^{22},\omega_i^3+z_i^{23},\theta) \\ \text{subject to} & z_i^{11}+q^1[\theta]z_i^{12}\leq 0, \\ & z_i^{22}+q^2[\theta|f^1[\theta]]z_i^{23}\leq 0. \end{array}$$

This is consistent with the existing dynamic general equilibrium framework, in the sense that agents take the price *path* as given. Note that it assumes that each agent perceives that her saving choice does not affect either TP1 spot price $q^1[\theta]$ or TP2 spot price $q^2[\theta|f^1[\theta]]$, despite that in the next period the spot price $q^2[\theta|z^1]$ is affected by whole z^1 which includes his own trade vector z_i^1.

The *path* of consumptions given by this solution is equivalent to the ADM solution.[16] This solution is not one-step-ahead implementable, however. We prove this by means of an example.

Claim 1 Let $I\geq 2$. Then, the Radner solution, defined over Θ, is not one-step-ahead Maskin monotonic.

[16] Note again that this is the feasibility-constrained version, while it does not matter when we can restrict attention to interior allocations.

Proof. Suppose that there are three agents, i, j and k. Assume that agents' intertemporal endowments are as follows:

$$\omega_i = (\omega_i^1, 0, 0), \qquad \omega_j = (0, \omega_j^2, 0), \qquad \omega_k = (0, 0, \omega_k^3),$$

where $\omega_i^1, \omega_j^2, \omega_k^3 > 1$. Each economy $\theta \in \Theta = (0, 1]$ specifies a preference profile over consumption paths represented by:

$$U_i(c_i^1, c_i^2, c_i^3, \theta) = c_i^1 + \theta \ln c_i^3, \quad U_j(c_j^1, c_j^2, c_j^3, \theta) = \ln c_j^1 + c_j^2, \quad U_k(c_k^1, c_k^2, c_k^3, \theta) = \ln c_k^2 + c_k^3.$$

Then, the TP2 spot price equilibrium is given by:

$$q^2[\theta|x^1] = x_i^{12},$$

for all $x^1 \in \bar{Z}^1$, and the TP2 competitive net trade allocation is given by:

$$\begin{aligned} f_i^{22}[\theta|x^1] &= -x_i^{12}, & f_i^{23}[\theta|x^1] &= 1, \\ f_j^{22}[\theta|x^1] &= 0, & f_j^{23}[\theta|x^1] &= 0, \\ f_k^{22}[\theta|x^1] &= x_i^{12}, & f_k^{23}[\theta|x^1] &= -1, \end{aligned}$$

for all $x^1 \in \bar{Z}^1$.

The TP1 utilities over $\bar{Z}^1$ induced by TP2 competitive net trade allocations are represented respectively by:

$$V_i\left[x^1, \theta|f^2\right] = \omega_i^1 + x_i^{11}, \quad V_j\left[x^1, \theta|f^2\right] = \ln x_j^{11} + \omega_j^2 + x_j^{12}, \quad V_k\left[x^1, \theta|f^2\right] = \ln x_i^{12} + \omega_k^3 - 1,$$

for all $x^1 \in \bar{Z}^1$, for all $\theta \in \Theta$.

For every economy $\theta \in \Theta$, the TP1 equilibrium spot price is:

$$q^1[\theta] = \theta,$$

which results in the following TP2 equilibrium spot price:

$$q^2[\theta|f^1[\theta]] = 1,$$

and in the following competitive equilibrium net trade allocations:

$$\begin{aligned} f_i^{11}[\theta] &= -\theta, & f_i^{12}[\theta] &= 1, & f_i^{22}[\theta|f^1[\theta]] &= -1, & f_i^{23}[\theta|f^1[\theta]] &= 1, \\ f_j^{11}[\theta] &= \theta, & f_j^{12}[\theta] &= -1, & f_j^{22}[\theta|f^1[\theta]] &= 0, & f_j^{23}[\theta|f^1[\theta]] &= 0, \\ f_k^{11}[\theta] &= 0, & f_k^{12}[\theta] &= 0, & f_k^{22}[\theta|f^1[\theta]] &= 1, & f_k^{23}[\theta|f^1[\theta]] &= -1. \end{aligned}$$

We have found that $f^1[\theta] \neq f^1[\theta']$ for all θ,$\theta' \in \Theta$ with $\theta \neq \theta'$, though TP1 reduced utility profiles are identical across economies in Θ, in violation of one-step-ahead Maskin monotonicity. ■

When an agent reveals an intention to save more, there may be different reasons to do so. It may be because he is simply patient, or it may be because he wants to manipulate the

Radner equilibrium outcome in the next period. The period-1 planner does not distinguish between those reasons. In particular, the above example is the case that *no* information is revealed to the planner after TP1.

While we can induce agents to behave as price-takers in the static setting, it is thus in general impossible to do that in the realistic dynamic setting in which social decision and execution can be done only in a sequential manner. The problem will disappear when there is a large number of traders, as each agent tends to be small and unable to manipulate through intertemporal pecuniary externalities. Then we would say that the dynamic general equilibrium model should be understood as such a limit model rather than an exact finite-person model.

The second way is to find a domain in which an exact finite-person implementation is possible. It is the domain such that there are no intertemporal pecuniary externalities.[17]

Condition 1 For all $\theta \in \Theta$, the TP2 spot price $q^2[\theta|x^1]$ is constant in $x^1 \in \bar{Z}^1$.

Note that when the above condition is met, a TP2 competitive net trade vector assigned to agent i depends only on her own past saving/borrowing behavior. For this reason, we write $f_i^{22}[\theta|z_i^{12}]$ and $f_i^{23}[\theta|z_i^{12}]$ for $f_i^{22}[\theta|z^1]$ and $f_i^{23}[\theta|z^1]$ respectively.

Here are examples of domains which satisfy Condition 1. In what follows, let us focus on economies where the quantity ω_i^t is strictly positive for every agent i and every consumption period $t = 1, 2, 3$.

Assumption 1 (Θ^1) Assume that aggregate endowment is constant over time; that is, $\omega^1 = \omega^2 = \omega^3$. Also, assume that the agents have identical discount factors, while they may exhibit different elasticities of intertemporal substitution. That is, for every economy $\theta \in \Theta^1$ it holds that $\omega^1 = \omega^2 = \omega^3$ and that there is (β^1, β^2) such that every i's preference over consumptions is represented in the form:

$$U_i(c_i^1, c_i^2, c_i^3, \theta) = v_i(c_i^1, \theta) + \beta^1 v_i(c_i^2, \theta) + \beta^1 \beta^2 v_i(c_i^3, \theta),$$

where:

- the sub-utility $v_i(\cdot, \theta)$ is twice continuously differentiable, strictly increasing and strictly concave over $\mathbb{R}_{++}$.
- the limit of the first derivative of the sub-utility $v_i(\cdot, \theta)$ is positive infinity as c_i^t approaches 0; that is, $\lim_{c_i^t \to 0} \frac{\partial v_i(c_i^t, \theta)}{\partial c_i^t} = \infty$.
- the limit of the first derivative of the sub-utility $v_i(\cdot, \theta)$ is zero as c_i^t approaches positive infinity; that is, $\lim_{c_i^t \to \infty} \frac{\partial v_i(c_i^t, \theta)}{\partial c_i^t} = 0$.
- the sub-utility $v_i(\cdot, \theta)$ satisfies the requirement that $-\left(\frac{\partial^2 v_i(c_i^t, \theta)}{\partial^2 c_i^t} c_i^t / \frac{\partial v_i(c_i^t, \theta)}{\partial c_i^t}\right) < 1$ for all $c_i^t \in \mathbb{R}_{++}$.

[17] Such situation emerges also when constant returns to scale in intertemporal production prevails, since interest rate in such economy is constant.

For this domain, we obtain that the TP2 competitive spot price, net trade allocations and consumption allocations prescribed for every $\theta \in \Theta^1$ are:

$$\begin{aligned} q^2\left[\theta|z^1\right] &= \beta^2 \\ f_i^{22}\left[\theta|z_i^{12}\right] &= -\frac{\beta^2}{1+\beta^2}\cdot\left(z_i^{12}+\omega_i^2-\omega_i^3\right) \\ f_i^{23}\left[\theta|z_i^{12}\right] &= \frac{1}{1+\beta^2}\cdot\left(z_i^{12}+\omega_i^2-\omega_i^3\right) \\ c_i^{*2}\left[\theta|z^1\right] &= c_i^{*3}\left[\theta|z^1\right] = \frac{z_i^{12}+\omega_i^2+\beta^2\omega_i^3}{1+\beta^2}, \quad \forall i \in \mathcal{I} \text{ and } \forall z^1 \in \bar{Z}^1. \end{aligned}$$

Note that period-1 reduced utility on $\bar{Z}^1$ is represented by:

$$V_i\left[z^1,\theta|f^2\right] = v_i\left(\omega_i^1+z_i^{11},\theta\right)+\beta^1\left(1+\beta^2\right)v_i\left(\frac{z_i^{12}+\omega_i^2+\beta^2\omega_i^3}{1+\beta^2},\theta\right), \quad \forall i \in \mathcal{I} \text{ and } \forall z^1 \in \bar{Z}^1.$$

Assumption 2 (Θ^2) In this domain we drop the assumption of constant aggregate endowment over time, but we assume that agents have identical CES preferences. That is, for every $\theta \in \Theta^2$ there is a triplet (β^1, β^2, ρ) such that every i's preference ordering over consumptions is represented in the form:

$$U_i(c_i^1,c_i^2,c_i^3,\theta) = \frac{(c_i^1)^{1-\rho}}{1-\rho}+\beta^1\frac{(c_i^2)^{1-\rho}}{1-\rho}+\beta^1\beta^2\frac{(c_i^3)^{1-\rho}}{1-\rho}, \qquad \text{with } \rho > 0.$$

When agents have identical CES preferences, we obtain that the TP2 competitive equilibrium spot price, net trade allocations and consumption allocations prescribed for every $\theta \in \Theta^2$ are:

$$\begin{aligned} q^2\left[\theta|z^1\right] &= \beta^2\left(\frac{\omega^2}{\omega^3}\right)^\rho \\ f_i^{22}\left[\theta|z_i^{12}\right] &= -\frac{z_i^{12}+\omega_i^2-\omega_i^3\left(\frac{\omega^2}{\omega^3}\right)}{1+\frac{1}{\beta^2}\left(\frac{\omega^2}{\omega^3}\right)^{1-\rho}} \\ f_i^{23}\left[\theta|z_i^{12}\right] &= \frac{\omega^3}{\omega^2}\cdot\frac{z_i^{12}+\omega_i^2-\omega_i^3\left(\frac{\omega^2}{\omega^3}\right)}{1+\beta^2\left(\frac{\omega^3}{\omega^2}\right)^{1-\rho}} \\ c_i^{*2}\left[\theta|z^1\right] &= \frac{z_i^{12}+\omega_i^2+\beta^2\omega_i^3\left(\frac{\omega^2}{\omega^3}\right)^\rho}{1+\beta^2\left(\frac{\omega^3}{\omega^2}\right)^{1-\rho}} \\ c_i^{*3}\left[\theta|z^1\right] &= \frac{\omega^3}{\omega^2}\cdot c_i^{*2}\left[\theta|z^1\right], \qquad \forall i \in \mathcal{I} \text{ and } \forall z^1 \in \bar{Z}^1. \end{aligned}$$

Next, let us define a TP1 competitive equilibrium when Condition 1 is satisfied.

Definition 8 For every economy θ satisfying Condition 1, a TP1 net trade allocation $\hat{f}^1[\theta] \in \bar{Z}^1$ constitutes a *backward TP1 competitive net trade allocation* if there is a TP1 spot price $q^1[\theta]$ such that for every agent i the net trade allocation $\hat{f}^1[\theta]$ solves the following problem:

$$\begin{array}{ll} \underset{z^1 \in \bar{Z}^1}{\text{Maximize}} & U_i(\omega_i^1 + z_i^{11}, \omega_i^2 + z_i^{12} + f_i^{22}\left[\theta | z_i^{12}\right], \omega_i^3 + f_i^{22}\left[\theta | z_i^{12}\right], \theta) \\ \text{subject to} & z_i^{11} + q^1[\theta] z_i^{12} \leq 0. \end{array}$$

Using this definition, we obtain that the competitive equilibrium spot prices prescribed for every economy $\theta \in \Theta^1$ are:

$$q^1[\theta] = \beta^1 \text{ and } q^2\left[\theta | \hat{f}^1[\theta]\right] = \beta^2,$$

and so the competitive net trade allocations and the equilibrium consumption allocations are for every agent $i \in \mathcal{I}$ as follows:

$$\begin{aligned} \hat{f}_i^{11}[\theta] &= -\frac{\beta^1}{1+\beta^1+\beta^1\beta^2} \cdot \left(\omega_i^1\left(1+\beta^2\right) - \omega_i^2 - \beta^2\omega_i^3\right) \\ \hat{f}_i^{12}[\theta] &= \frac{1}{1+\beta^1+\beta^1\beta^2} \cdot \left(\omega_i^1\left(1+\beta^2\right) - \omega_i^2 - \beta^2\omega_i^3\right) \\ f_i^{22}\left[\theta | \hat{f}_i^{12}[\theta]\right] &= -\frac{\beta^2}{1+\beta^2} \cdot \left(\hat{f}_i^{12}[\theta] + \omega_i^2 - \omega_i^3\right) \\ f_i^{23}\left[\theta | \hat{f}_i^{12}[\theta]\right] &= \frac{1}{1+\beta^2} \cdot \left(\hat{f}_i^{12}[\theta] + \omega_i^2 - \omega_i^3\right) \\ c_i^{*1}[\theta] &= c_i^{*2}\left[\theta | \hat{f}_i^{12}[\theta]\right] = c_i^{*3}\left[\theta | \hat{f}_i^{12}[\theta]\right] = \frac{\omega_i^1 + \beta^1\omega_i^2 + \beta^1\beta^2\omega_i^3}{1+\beta^1+\beta^1\beta^2}. \end{aligned}$$

For economies in Θ^2, we obtain that the equilibrium spot prices prescribed for every $\theta \in \Theta^2$ are:

$$q^1[\theta] = \beta^1\left(\frac{\omega^1}{\omega^2}\right)^\rho \text{ and } q^2\left[\theta | \hat{f}^1[\theta]\right] = \beta^2\left(\frac{\omega^2}{\omega^3}\right)^\rho.$$

Thus, the competitive net trade allocations are:

$$\hat{f}_i^{11}[\theta] = -\frac{\omega_i^1\left(\beta^2\left(\frac{\omega^3}{\omega^2}\right)^{1-\rho}+1\right)-\left(\frac{\omega^1}{\omega^2}\right)\left(\omega_i^2+\beta^2\omega_i^3\left(\frac{\omega^2}{\omega^3}\right)^{\rho}\right)}{1+\frac{1}{\beta^1}\left(\frac{\omega^1}{\omega^2}\right)^{1-\rho}+\beta^2\left(\frac{\omega^3}{\omega^2}\right)^{1-\rho}}$$

$$\hat{f}_i^{12}[\theta] = \frac{\omega^2}{\omega^1}\cdot\frac{\omega_i^1\left(\beta^2\left(\frac{\omega^3}{\omega^2}\right)^{1-\rho}+1\right)-\left(\frac{\omega^1}{\omega^2}\right)\left(\omega_i^2+\beta^2\omega_i^3\left(\frac{\omega^2}{\omega^3}\right)^{\rho}\right)}{1+\beta^1\left(\frac{\omega^2}{\omega^1}\right)^{1-\rho}+\beta^1\beta^2\left(\frac{\omega^3}{\omega^1}\right)^{1-\rho}}$$

$$f_i^{22}\left[\theta|\hat{f}_i^{12}[\theta]\right] = -\frac{\hat{f}_i^{12}[\theta]+\omega_i^2-\omega_i^3\left(\frac{\omega^2}{\omega^3}\right)}{1+\frac{1}{\beta^2}\left(\frac{\omega^2}{\omega^3}\right)^{1-\rho}}$$

$$f_i^{23}\left[\theta|\hat{f}_i^{12}[\theta]\right] = \frac{\omega^3}{\omega^2}\cdot\frac{\hat{f}_i^{12}[\theta]+\omega_i^2-\omega_i^3\left(\frac{\omega^2}{\omega^3}\right)}{1+\beta^2\left(\frac{\omega^3}{\omega^2}\right)^{1-\rho}},$$

while the corresponding equilibrium consumption allocations are:

$$c_i^{*1}[\theta] = \frac{\omega_i^1+\beta^1\omega_i^2\left(\frac{\omega^1}{\omega^2}\right)^{\rho}+\beta^1\beta^2\omega_i^3\left(\frac{\omega^1}{\omega^3}\right)^{\rho}}{1+\beta^1\left(\frac{\omega^2}{\omega^1}\right)^{1-\rho}+\beta^1\beta^2\left(\frac{\omega^3}{\omega^1}\right)^{1-\rho}}$$

$$c_i^{*2}\left[\theta|z^{*1}[\theta]\right] = \frac{\omega^2}{\omega^1}\cdot c_i^{*1}[\theta]$$

$$c_i^{*3}\left[\theta|z^{*1}[\theta]\right] = \frac{\omega^3}{\omega^1}\cdot c_i^{*1}[\theta].$$

The *backward competitive solution* of an economy θ is a SCF $\bar{f}=\left(\bar{f}^1[\cdot],\bar{f}^2[\cdot|\cdot]\right)$ associating the period-1 function $\bar{f}^1[\theta]$ with the backward TP1 competitive net trade allocation $\hat{f}^1[\theta]$, that is, $\bar{f}^1[\theta]=\hat{f}^1[\theta]\in\bar{Z}^1$, and the period-2 function $\bar{f}^2[\theta|\cdot]$ with the TP2 competitive net trade allocation for any TP1 net trade allocation in the set $\bar{Z}^1$, that is, $\bar{f}^2[\theta|z^1]=f^2[\theta|z^1]$ for every $z^1\in\bar{Z}^1$. Thanks to Condition 1, we can now state and prove the following permissive result.

Claim 2 Assume that $I\geq 2$. Suppose that the quantity ω_i^t is strictly positive for every agent i and every consumption period $t=1,2,3$. Then, the backward competitive solution $\bar{f}$ is one-step-ahead implementable if it is defined either over Θ^1 or over Θ^2.

Proof. Let the premises hold. To show that $\bar{f}$ is one-step-ahead implementable when it is defined either over Θ^1 or over Θ^2, we need to show that this solution satisfies one-step-ahead Maskin monotonicity and one-step-ahead no veto-power.

One-step-ahead no veto-power is satisfied since agents' reduced utilities are strictly monotonic in consumption. Let $Y^1=\mathcal{Y}^{-2}=\bar{Z}^1$ and let $Y^2(z^1)=\bar{Z}^2(z^1)$ for every $z^1\in\bar{Z}^1$. Then, the sets $Y^1=\mathcal{Y}^{-2}$ and $Y^2(z^1)$ are not empty sets. One can see that parts a) and b) of requirement (i) are satisfied, too. From these definitions, one can see that $\bar{f}$ is one-step-ahead Maskin monotonic. ■

4.2 Period-by-period implementability of the Condorcet winner

In this section, we consider a bi-dimensional policy space where an *odd* number of agents vote sequentially on each dimension and where an ordering of the dimensions is exogenously given. We assume that a majority vote is organized around each policy dimension and that the outcome of the first majority vote is known to the voters at the beginning of the second voting stage. This stage-by-stage resolution is common in political economy models (see, e.g., Persson and Tabellini, 2000). We are interested in one-step-ahead implementing the simple majority solution, which selects the Condorcet winner in each voting stage.

A policy choice is an ordered pair $(x^1, x^2) \in X^1 \times X^2$, where the policy space of dimension $d = 1, 2$ is an open interval.[18] Each voter i is described by a one-dimensional type θ_i. The type space is the open interval $\left(\underline{\eta}, \bar{\eta}\right)$.

Assumption 3 The voter i's utility function $U_i : X^1 \times X^2 \times \left(\underline{\eta}, \bar{\eta}\right) \to \mathbb{R}$ is a twice-continuously differentiable satisfying, for every $(x^1, x^2) \in X^1 \times X^2$:
(a) *Strict concavity*, that is:

$$\frac{\partial^2 U_i(x^1, x^2, \theta_i)}{\partial^2 x^1} < 0 \quad \text{and} \quad \frac{\partial^2 U_i(x^1, x^2, \theta_i)}{\partial^2 x^2} < 0.$$

(b) *induced single-crossing* property, that is:

$$\frac{\partial^2 U_i(x^1, x^2, \theta_i)}{\partial \theta_i \partial x^1} > 0 \quad \text{and} \quad \frac{\partial^2 U_i(x^1, x^2, \theta_i)}{\partial \theta_i \partial x^2} > 0,$$

for every $\theta_i \in \left(\underline{\eta}, \bar{\eta}\right)$.
(c) *Strategic complementarity*, that is:

$$\frac{\partial^2 U_i(x^1, x^2, \theta_i)}{\partial x^1 \partial x^2} \geq 0.$$

Note that each voter's utility depends only on the outcomes and his own type (private values). Also, note that the induced single-crossing property simply requires that the induced utility of both dimensions is increasing in the type of voter. This property can also be found in De Donder et al. (2012).

We now introduce the definition of a Condorcet winner for an arbitrary policy space P:

Definition 9 Suppose that agents in $\mathcal{I}$ votes over the set of policies P. We say that $p \in P$ is a majority voting outcome, also known as a *Condorcet winner* (CW), if there does not exist any other distinct outcome $p' \in P$ that is strictly preferred by more than half of voters to the outcome p.

For any integer $k \geq 2$, the set of states Θ takes the structure of the Cartesian product of allowable independent types for voters, that is, $\Theta \equiv \left(\underline{\eta}, \bar{\eta}\right)^{2k-1}$, with θ as typical element. It simplifies the argument, and causes no loss of generality, to assume that $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_{2k-1}$. Therefore, the type θ_k is the median type, denoted by θ_{med}, at state θ.

[18] The choice of a bi-dimensional policy space is motivated by convenience.

At the state θ, each voter is assumed to have an ordering preference relation $R_i(\theta)$ over the policy space $X^1 \times X^2$ which is represented by $U_i(\cdot,\cdot,\theta_i)$. Below we write $U_i(\cdot,\cdot,\theta)$ for $U_i(\cdot,\cdot,\theta_i)$.

If $x^1 \in X^1$ is the outcome of the first majority voting, then the stage-2 majority voting function $f^2 : \Theta|x^1 \to X^2$ is defined as follows:

$$f^2\left[\theta|x^1\right] = CW\left(U\left[\theta|x^1\right]\right),$$

where $CW(U[\theta|x^1])$ denotes the Condorcet winner under the profile $U[\theta|x^1]$. It will be shown below that this outcome is the most-preferred outcome of the median type.

Let us suppose that the stage-2 majority voting function is well-defined for every outcome $x^1 \in X^1$. Then, the stage-1 majority voting function $f^1 : \Theta \to X^1$ is defined as follows:

$$f^1[\theta] = CW\left(U\left[\theta|f^2\right]\right), \quad \text{for every } \theta \in \Theta,$$

where $CW(U[\theta|f^2])$ denotes the Condorcet winner under the profile $U[\theta|f^2]$.

Definition 10 The SCF $f(\cdot) = (f^1[\cdot], f^2[\cdot|\cdot])$ on Θ is the *majority voting* solution if for every $\theta \in \Theta$:

$$f^1[\theta] = CW\left(U\left[\theta|f^2\right]\right) \text{ and } f^2\left[\theta|x^1\right] = CW\left(U\left[\theta|x^1\right]\right) \text{ for every } x^1 \in X^1.$$

The following lemma shows that the majority voting solution is a single-valued function. The intuition behind it is similar to that of Proposition 4 of De Donder et al. (2012) for the case where there is a continuum of voters. Firstly, the assumption of strict concavity assures the existence and unicity of the Condorcet winner in the second voting stage. This assumption, combined with the assumption of strategic complementarity and with the induced single-crossing property, assures that the stage-1 induced utility of voter i on X^1 in state θ at the majority voting function $f^2[\theta|\cdot]$ is single-crossing. This guarantees the existence and unicity of the Condorcet winner in the first voting stage.

Lemma 1 Suppose that the cardinality of $\mathcal{I}$ is $2k-1$ with $k \geq 2$. Suppose that voter $i \in \mathcal{I}$'s utility function U_i on $X^1 \times X^2 \times \Theta$ meets the requirements of Assumption 3 and depends only on her own type. Then, the majority voting SCF $f(\cdot) = (f^1[\cdot], f^2[\cdot|\cdot])$ over Θ is a single-valued function on each policy dimension.

Proof. See Appendix. ■

Thanks to the above lemma, we can now state and prove the main result of this section.

Claim 3 Suppose that the cardinality of $\mathcal{I}$ is $2k-1$ with $k \geq 2$. Suppose that voter $i \in \mathcal{I}$'s utility function U_i on $\Theta \times X^1 \times X^2$ meets the requirements of Assumption 3 and depends only on her own type. Then, the majority voting solution is one-step-ahead implementable.

Proof. Let the premises hold. By Theorem 2, it suffices to show that the majority voting solution satisfies the feasibility condition, one-step-ahead Maskin monotonicity and one-step-ahead no veto-power. Let $Y^1 = X^1$ and $Y^2(x^1) = X^2$ for every $x^1 \in X^1$. One can see that parts a) and b) of requirement (i) are satisfied. Moreover, since in each period agents have single crossing preferences, one can see that the majority voting solution is one-step-ahead Maskin monotonic. Finally, since one-step-ahead no veto-power is vacuously satisfied, we conclude that the majority voting solution on Θ is one-step-ahead implementable. ■

5. Conclusion

We have identified a necessary condition for one-step-ahead implementability, *one-step-ahead Maskin monotonicity.* First, this condition guarantees that the set of period-t attainable outcomes is never empty, after every outcome history. This allows us to derive agents' reduced preferences for each period t. Each reduced preference ordering is constructed in the manner of backward-induction. This means that a period-t reduced preference ordering over the current component set depends on past decisions as well as on the socially optimal path that the dynamic process will bring about in the future. Second, it states that every period-t SCF needs to satisfy a remarkably strong invariance condition for Nash implementation, now widely referred to as Maskin monotonicity (Maskin, 1999). We have also shown that under the auxiliary condition of no veto-power, one-step-ahead Maskin monotonicity is sufficient, as well.

We have applied our analysis to two prominent dynamic problems, voting over time and sequential trading. In the voting application, we have shown that on the domain satisfying the single-crossing property the simple majority solution, which selects the Condorcet winner in each voting stage (after every history), is one-step-ahead implementable.

In a borrowing-lending model with no liquidity constraints, in which agents trade in spot markets and transfer wealth between any two periods by borrowing and lending, we have noted that intertemporal pecuniary externalities arise because today's trade changes the spot price of tomorrow, which, in turn, affects its associated equilibrium allocation. The quantitative implication of this is that every agent's reduced utility concerns not only her own consumption/saving behavior but also the consumption/saving behavior of all other agents. In this set-up, we have shown that, under such pecuniary externalities, the standard dynamic competitive equilibrium solution is not one-step-ahead implementable. However, we have also identified preference domains – which involve no pecuniary externalities – for which the no-commitment version of the dynamic competitive equilibrium solution is definable and one-step-ahead implementable. It remains an open question how we should deal with intertemporal pecuniary externalities. We hope that this and other topics related to this paper will be investigated in future research.

The paper has focused on full implementation under complete information, and so it has ruled out dynamic uncertainty both as arrival of private information and as arrival of common shocks. Whereas the latter type of uncertainty can easily be accommodated and shocks could even be made path-dependent, it is less obvious how to model the former type of dynamic uncertainty. In this regard, the recent works on dynamic mechanism design such as Bergemann and Välimäki (2010), Athey and Segal (2013), Pavan et al. (2014) and Ező and Szentes (2017) can be helpful when integrating this type of dynamic uncertainty.

Finally, though the main motivation of one-step-ahead implementation is practical, our characterization result offers little insight on the properties that more realistic mechanisms should satisfy, in order to ensure one-step-ahead implementation. In this regard, a more pragmatic approach to one-step-ahead implementation is needed. One promising possibility seems to be the implementation model developed by Ollár and Penta (2017). These authors circumvents the complications of our mechanisms by studying full implementation via transfer schemes, under general restrictions on agents' beliefs.

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Appendix

Proof of Theorem 1. Let the premises hold. Thus, there exists $\Gamma \equiv (\mathcal{I}, H, A(H), g)$ that one-step-ahead implements f. Fix any $s^{\bar{\theta}} \in SPE\left(\Gamma, \bar{\theta}\right)$. Then, $s^{\bar{\theta}}|h^t$ is a Nash equilibrium of $\left(\Gamma\left(h^t\right), \bar{\theta}\right)$ for every history $h^t \in H$. Moreover, by one-step-ahead implementability of f, it also follows that:

$$f^{+t}\left[\bar{\theta}|g^{-t}(h)\right] = g^{+t}\left(s^{\bar{\theta}}|h\right), \quad \text{for every } h \in H^t \text{ with } 2 \leq t \leq T. \tag{8}$$

Fix any period $t \neq 1$. Let us define the sets $\mathcal{Y}^{-t}$, Y^1 and $Y^t\left(g^{-t}(h)\right)$ as follows:

$$\begin{aligned} Y^1 &\equiv \left\{g^1\left(a\left(h^1\right)\right) \in X^1 | \text{for some } a\left(h^1\right) \in A\left(h^1\right)\right\}, \\ \mathcal{Y}^{-t} &\equiv \left\{g^{-t}(h) \in \mathcal{X}^{-t} | \text{for some } h \in H^t\right\}, \\ Y^t\left(g^{-t}(h)\right) &\equiv \left\{g^t(a(h)) \in X^t | a(h) \in A(h)\right\}, \end{aligned}$$

for every $g^{-t}(h) \in \mathcal{Y}^{-t}$. By their definitions as well as by the assumption that the dynamic mechanism Γ one-step-ahead implements f, one can check that $f^t\left[\Theta | g^{-t}(h)\right] \subseteq Y^t\left(g^{-t}(h)\right)$ and that $f^1[\Theta] \subseteq Y^1$. Thus, part a) of requirement (i) is met.

Moreover, given that Γ is a dynamic mechanism, one can also check that for every period $t \neq 1$:

$$g^{-t}(h) \in \mathcal{Y}^{-t} \iff g^1\left(a^1\right) \in Y^1 \text{ and } g^\tau\left(a^\tau\right) \in Y^\tau\left(g^{-\tau}\left(a^1, \cdots, a^{\tau-1}\right)\right)$$

for every τ such that $2 \leq \tau \leq t-1$, for every $h \equiv \left(a^1, \cdots, a^{t-1}\right) \in H^t$. This shows part b) of requirement (i). Thus, f is a feasible SCF.

Next, fix any $g^{-T}(h) \in \mathcal{Y}^{-T}$ with $h \in H^T$ and suppose that for every $i \in \mathcal{I}$, it holds that:

$$\begin{aligned} & L^{\theta}_{Y^T\left(g^{-T}(h)\right)}\left(f^T\left[\theta | g^{-T}(h)\right], U_i\left[\cdot, \theta | g^{-T}(h)\right]\right) \\ & \qquad \subseteq L^{\bar{\theta}}_{Y^T\left(g^{-T}(h)\right)}\left(f^T\left[\theta | g^{-T}(h)\right], U_i\left[\cdot, \bar{\theta} | g^{-T}(h)\right]\right). \end{aligned} \tag{9}$$

We show that $f^T\left[\theta | g^{-T}(h)\right] = f^T\left[\bar{\theta} | g^{-T}(h)\right]$.

Since Γ one-step-ahead implements f, we have that $g^T\left(s^\theta(h)\right) = f^T\left[\theta | g^{-T}(h)\right]$ and that $s^\theta(h) \in NE\left(\tilde{\Gamma}(h), U\left[\theta | g^{-T}(h)\right]\right)$.[19] Since (9) holds, one can see that $s^\theta(h) \in NE\left(\tilde{\Gamma}(h), U\left[\bar{\theta} | g^{-T}(h)\right]\right)$. Then, from the definition of $U_i\left[\cdot, \bar{\theta} | g^{-T}(h)\right]$ in (1), one can see $s^\theta(h) \in NE\left(\Gamma(h), \bar{\theta}\right)$. By implementability, it follows that $g^T\left(s^\theta(h)\right) = f^T\left[\bar{\theta} | g^{-T}(h)\right]$, and so $f^T\left[\theta | g^{-T}(h)\right] = f^T\left[\bar{\theta} | g^{-T}(h)\right]$, as was to be proved.

Fix any $t \neq 1, T$ and consider any $g^{-t}(h) \in \mathcal{Y}^{-t}$ with $h \in H^t$. Suppose that for every $i \in \mathcal{I}$, it holds that:

$$\begin{aligned} & L^{\theta}_{Y^t\left(g^{-t}(h)\right)}\left(f^t\left[\theta | g^{-t}(h)\right], U_i\left[\cdot, \theta | g^{-t}(h), f^{+(t+1)}\right]\right) \\ & \subseteq L^{\bar{\theta}}_{Y^t\left(g^{-t}(h)\right)}\left(f^t\left[\theta | g^{-t}(h)\right], U_i\left[\cdot, \bar{\theta} | g^{-t}(h), f^{+(t+1)}\right]\right). \end{aligned} \tag{10}$$

We show that $f^t\left[\theta | g^{-t}(h)\right] = f^t\left[\bar{\theta} | g^{-t}(h)\right]$.

Since Γ one-step-ahead implements f, we have that $f^t\left[\theta | g^{-t}(h)\right] = g^t\left(s^\theta(h)\right)$. Since (10) holds, one can see that $s^\theta(h) \in NE\left(\tilde{\Gamma}(h), U\left[\bar{\theta} | g^{-t}(h), f^{+(t+1)}\right]\right)$. From the definition

[19] For any dynamic mechanism Γ and any history $h \in H^t$, $\tilde{\Gamma}(h) \equiv \left(\prod_{i \in \mathcal{I}} A_i(h), g^t\right)$ denotes the *static* period-t mechanism of Γ after the history h. A period-t mechanism $\tilde{\Gamma}(h)$ and a profile of period-t reduced utilities at $\bar{\theta} | y^{-t}$, where $y^{-t} = g^{-t}(h)$, induce a period-t game $\left(\tilde{\Gamma}(h), U\left[\bar{\theta} | y^{-t}, f^{+(t+1)}\right]\right)$. $NE\left(\tilde{\Gamma}(h), U\left[\bar{\theta} | y^{-t}, f^{+(t+1)}\right]\right)$ denotes the set of period-t Nash equilibria at $U\left[\bar{\theta} | y^{-t}, f^{+(t+1)}\right]$. It is plain that $\left(\tilde{\Gamma}(h), U\left[\bar{\theta} | y^{-t}, f^{+(t+1)}\right]\right)$ differs from the dynamic game $\left(\Gamma(h), \bar{\theta}\right)$.

of $U_i\left[\cdot,\bar{\theta}|y^{-t},f^{+(t+1)}\right]$ given in (3), and from (8), it follows that for every $i\in\mathcal{I}$ and $a_i(h)\in A_i(h)$, it holds that:

$$U_i\left(g^{-t}(h),g^t\left(s^{\theta}(h)\right),g^{+(t+1)}\left(s^{\bar{\theta}}|\left(h,s^{\theta}(h)\right)\right),\bar{\theta}\right) \geq U_i\left(g^{-t}(h),g^t\left(a_i(h),s^{\theta}_{-i}(h)\right),g^{+(t+1)}\left(s^{\bar{\theta}}|\left(h,\left(a_i(h),s^{\theta}_{-i}(h)\right)\right)\right),\bar{\theta}\right). \tag{11}$$

Since f is one-step-ahead implementable and $\bar{\theta}\in\Theta$, there exists $s^{\bar{\theta}}\in SPE\left(\Gamma,\bar{\theta}\right)$. Recall that $h\in H^t$, where $t\neq 1,T$. For each i and each $a(h)\in A(h)$, let s_i denote agent i's strategy according to which $s_i|h'=s^{\bar{\theta}}_i|h'$ for every $h'\in H|\left(h,a(h)\right)$ and $s_i(h)=s^{\theta}_i(h)$. Note that $s|h'$ is a Nash equilibrium of $\left(\Gamma(h'),\bar{\theta}\right)$ for every $h'\in H|\left(h,a(h)\right)$ since $s^{\bar{\theta}}\in SPE\left(\Gamma,\bar{\theta}\right)$. Thus, to have that $s\in SPE\left(\Gamma(h),\bar{\theta}\right)$, we need to show that $s|h\in NE\left(\Gamma(h),\bar{\theta}\right)$.

Since the action profile $s(h)\in NE\left(\tilde{\Gamma}(h),U\left[\bar{\theta}|g^{-t}(h),f^{+(t+1)}\right]\right)$, it follows that (11) holds for every $i\in I$ and every $a_i(h)\in A_i(h)$. Thus, no agent i can gain by deviating from the action $s_i(h)$ and thereafter conforming to s_i. Since the one deviation property (see, e.g., Osborne and Rubinstein, 1994; Lemma 98.2) holds for a finite-horizon multi-period game with observed actions and simultaneous moves, it follows that the strategy profile $s|h\in NE\left(\Gamma(h),\bar{\theta}\right)$, and so $s\in SPE\left(\Gamma(h),\bar{\theta}\right)$. Since Γ one-step-ahead implements f and $g^t\left(s^{\theta}(h)\right)=g^t(s(h))$, we have that $f^t\left[\bar{\theta}|g^{-t}(h)\right]=f^t\left[\theta|g^{-t}(h)\right]$, as we sought.

Finally, suppose that for every $i\in\mathcal{I}$, it holds that:

$$L^{\theta}_{Y^1}\left(f^1[\theta],U_i\left[\cdot,\theta|f^{+2}\right]\right)\subseteq L^{\bar{\theta}}_{Y^1}\left(f^1[\theta],U_i\left[\cdot,\bar{\theta}|f^{+2}\right]\right). \tag{12}$$

Since Γ one-step-ahead implements f, we have that $f^1[\theta]=g^1\left(s^{\theta}(h^1)\right)$. Since (12) holds, one can see that $s^{\theta}(h^1)\in NE\left(\tilde{\Gamma}(h^1),U\left[\bar{\theta}|f^{+2}\right]\right)$. From the definition of $U_i\left[\cdot,\bar{\theta}|f^{+2}\right]$ given in (5), and from (8), it follows that for every $i\in\mathcal{I}$ and $a_i(h^1)\in A_i(h^1)$, it holds that:

$$U_i\left(g^1\left(s^{\theta}(h^1)\right),g^{+2}\left(s^{\bar{\theta}}|\left(h^1,s^{\theta}(h^1)\right)\right),\bar{\theta}\right) \geq U_i\left(g^1\left(a_i(h^1),s^{\theta}_{-i}(h^1)\right),g^{+2}\left(s^{\bar{\theta}}|\left(h^1,\left(a_i(h^1),s^{\theta}_{-i}(h^1)\right)\right)\right),\bar{\theta}\right).$$

Since f is one-step-ahead implementable and $\bar{\theta}\in\Theta$, there exists $s^{\bar{\theta}}\in SPE\left(\Gamma,\bar{\theta}\right)$. For each i and each $a(h^1)\in A(h^1)$, let s_i denote agent i's strategy according to which $s_i|h'=s^{\bar{\theta}}_i|h'$ for every $h'\in H|a(h^1)$ and $s_i(h^1)=s^{\theta}_i(h^1)$. By arguments similar to those used above to prove that $f^t\left[\bar{\theta}|g^{-t}(h)\right]=f^t\left[\theta|g^{-t}(h)\right]$, one can see that $f^1\left[\bar{\theta}\right]=f^1[\theta]$, as we sought. ■

Proof of Theorem 2. The proof is based on the construction of a dynamic mechanism Γ, where each period-t mechanism is a canonical mechanism.

Period-1 mechanism:

Individual i's period-1 action space is defined by:

$$A_i\left(H^1\right) \equiv \mathcal{U}\left[\Theta|f^{+2}\right] \times Y^1 \times \mathcal{Z}_+,$$

where $\mathcal{Z}_+$ is the set of nonnegative integers and H^1 is the null set. Thus, a period-1 action of agent i consists of an element of the set Y^1, an element of the period-1 domain of reduced utilities induced by the set Θ at the socially optimal 2-tail outcome paths f^{+2}, and a nonnegative integer. A typical period-1 action played by agent i is denoted by $a_i\left(h^1\right) \equiv \left(\left(U\left[\bar{\theta}|f^{+2}\right]\right)^i, \left(x^1\right)^i, (z)^i\right)$.

Period-1 action space of agents is the product space:

$$A\left(H^1\right) \equiv \prod_{i \in \mathcal{I}} A_i\left(H^1\right),$$

with $a\left(h^1\right)$ as a typical period-1 action profile.

The period-1 outcome function g^1 is defined by the following three rules:

Rule 1: If $a_i\left(h^1\right) \equiv \left(U\left[\bar{\theta}|f^{+2}\right], x^1, 0\right)$ for every $i \in \mathcal{I}$ and $x^1 = f^1\left[\bar{\theta}\right]$, then $g^1\left(a\left(h\right)\right) = x^1$.

Rule 2: If $n-1$ agents play $a_j\left(h^1\right) \equiv \left(U\left[\bar{\theta}|f^{+2}\right], x^1, 0\right)$ with $x^1 = f^1\left[\bar{\theta}\right]$ but agent i plays $a_i\left(h^1\right) \equiv \left(\left(U\left[\bar{\theta}|f^{+2}\right]\right)^i, \left(x^1\right)^i, (z)^i\right) \neq a_j\left(h^1\right)$, then we can have two cases:

1. If $U_i\left[x^1, \bar{\theta}|f^{+2}\right] \geq U_i\left[\left(x^1\right)^i, \bar{\theta}|f^{+2}\right]$, then $g^1\left(a\left(h^1\right)\right) = \left(x^1\right)^i$.
2. Otherwise, $g^1\left(a\left(h^1\right)\right) = x^1$.

Rule 3: Otherwise, an integer game is played: identify the agent who plays the highest integer (if there is a tie at the top, pick the agent with the lowest index among them.) This agent is declared the winner of the game, and the alternative implemented is the one she selects.

Period-t mechanism with $t \neq 1, T$:

Individual i's period-t action space after history $h \in H^t$ such that $g^{-t}(h) \in \mathcal{Y}^{-t}$ is defined by:

$$A_i(h) \equiv \mathcal{U}\left[\Theta|g^{-t}(h), f^{+(t+1)}\right] \times Y^t\left(g^{-t}(h)\right) \times \mathcal{Z}_+.$$

Thus, a period-t action of agent i after history $h \in H^t$ consists of an element of the set $Y^t\left(g^{-t}(h)\right)$, an element of the period-t domain of reduced utilities induced by the set Θ at the t-head outcome path $g^{-t}(h)$ and at the socially optimal $t+1$-tail outcome paths $f^{+(t+1)}$, and a nonnegative integer. A typical period-t action played by agent i after history $h \in H^t$ is denoted by $a_i(h) \equiv \left(\left(U\left[\bar{\theta}|g^{-t}(h), f^{+(t+1)}\right]\right)^i, \left(x^t\right)^i, (z)^i\right)$.

Period-t action space of agents after history $h \in H^t$ is the product space:

$$A(h) \equiv \prod_{i \in \mathcal{I}} A_i(h),$$

with $a(h)$ as a typical period-t action profile after history $h \in H^t$.

The period-t outcome function g^t is defined by the following three rules for every $h \in H^t$ such that $g^{-t}(h) \in \mathcal{Y}^{-t}$:

Rule 1: If $a_i(h) \equiv \left(U\left[\bar{\theta}|g^{-t}(h), f^{+(t+1)}\right], x^t, 0\right)$ for every $i \in \mathcal{I}$ and $x^t = f^t\left[\bar{\theta}|g^{-t}(h)\right]$, then $g^t(a(h)) = x^t$.

Rule 2: If $n-1$ agents play $a_j(h) \equiv \left(U\left[\bar{\theta}|g^{-t}(h), f^{+(t+1)}\right], x^t, 0\right)$ with $x^t = f^t\left[\bar{\theta}|g^{-t}(h)\right]$ but agent i plays $a_i(h) \equiv \left(\left(U\left[\bar{\theta}|g^{-t}(h), f^{+(t+1)}\right]\right)^i, (x^t)^i, (z)^i\right) \neq a_j(h)$, then we can have two cases:

1. If $U_i\left[x^t, \bar{\theta}|g^{-t}(h), f^{+(t+1)}\right] \geq U_i\left[(x^t)^i, \bar{\theta}|g^{-t}(h), f^{+(t+1)}\right]$, then $g^t(a(h)) = (x^t)^i$.
2. Otherwise, $g^t(a(h)) = x^t$.

Rule 3: Otherwise, an integer game is played: identify the agent who plays the highest integer (if there is a tie at the top, pick the agent with the lowest index among them.) This agent is declared the winner of the game, and the alternative implemented is the one she selects.

Period-T mechanism:

Individual i's period-T action space after history $h \in H^T$ such that $g^{-T}(h) \in \mathcal{Y}^{-T}$ is defined by:

$$A_i(h) \equiv \mathcal{U}\left[\Theta|g^{-T}(h)\right] \times Y^T\left(g^{-T}(h)\right) \times \mathcal{Z}_+.$$

Thus, a period-T action of agent i after history $h \in H^T$ consists of an element of the set $Y^T\left(g^{-T}(h)\right)$, an element of the period-T domain of reduced utilities induced by the set Θ and the T-head outcome path $g^{-T}(h)$, and a nonnegative integer. A typical period-T action played by agent i after history $h \in H^T$ is denoted by $a_i(h) \equiv \left(\left(U\left[\bar{\theta}|g^{-T}(h)\right]\right)^i, \left(x^T\right)^i, (z)^i\right)$.

Period-T action space of agents after history $h \in H^T$ is the product space:

$$A(h) \equiv \prod_{i \in \mathcal{I}} A_i(h),$$

with $a(h)$ as a typical period-T action profile after history $h \in H^T$.

The period-T outcome function g^T is defined by the following three rules for every $h \in H^T$ such that $g^{-T}(h) \in \mathcal{Y}^{-T}$:

Rule 1: If $a_i(h) \equiv \left(U\left[\bar{\theta}|g^{-T}(h)\right], x^T, 0\right)$ for every $i \in \mathcal{I}$ and $x^T = f^T\left[\bar{\theta}|g^{-T}(h)\right]$, then $g^T(a(h)) = x^T$.

Rule 2: If $n-1$ agents play $a_j(h) \equiv \left(U\left[\bar{\theta}|g^{-T}(h)\right], x^T, 0\right)$ with $x^T = f^T\left[\bar{\theta}|g^{-T}(h)\right]$ but agent i plays $a_i(h) \equiv \left(\left(U\left[\bar{\theta}|g^{-T}(h)\right]\right)^i, \left(x^T\right)^i, (z)^i\right) \neq a_j(h)$, then we can have two cases:

1. If $U_i\left[x^T, \bar{\theta}|g^{-T}(h)\right] \geq U_i\left[\left(x^T\right)^i, \bar{\theta}|g^{-T}(h)\right]$, then $g^T(a(h)) = \left(x^T\right)^i$.

2. Otherwise, $g^T(a(h)) = x^T$.

Rule 3: Otherwise, an integer game is played: identify the agent who plays the highest integer (if there is a tie at the top, pick the agent with the lowest index among them.) This agent is declared the winner of the game, and the alternative implemented is the one she selects.

Let

$$H \equiv \bigcup_{t\in\mathcal{T}} H^t$$

be the set of all possible histories, let $A_i \equiv \bigcup_{h\in H} A_i(h)$ be the set of all actions for agent $i \in \mathcal{I}$, let $A(H)$ be the set of all profiles of actions available to agents, defined by

$$A(H) \equiv \bigcup_{h\in H} A(h),$$

and let $g \equiv (g^1, \cdots, g^T)$ be the sequence of outcome functions, one for each period $t \in \mathcal{T}$. Note that g satisfies the following properties: a) the outcome function g^1 assigns to period-1 action profile $a(h^1) \in A(h^1)$ a unique outcome in Y^1, and b) for every period $t \neq 1$ and every nontrivial history $h \in H^t$, the outcome function g^t assigns to each period-t action profile $a(h) \in A(h)$ a unique outcome in $Y^t(g^{-t}(h))$. Thus, by construction, $\Gamma \equiv (\mathcal{I}, H, A(H), g)$ is a dynamic mechanism.

Fix any state $\theta \in \Theta$.

Let us first prove part (a) of Definition 1. Let us define agent $i \in \mathcal{I}$'s strategy $s_i^\theta : H \to A_i$ by:

$$s_i^\theta(h^1) = \left(U\left[\theta|f^{+2}\right], f^1[\theta], 0\right),$$

$$s_i^\theta(h) = \left(U\left[\theta|g^{-t}(h), f^{+(t+1)}\right], f^t\left[\theta|g^{-t}(h)\right], 0\right),$$

for every $h \in H^t$ with $t \neq 1, T$, and

$$s_i^\theta(h) = \left(U\left[\theta|g^{-T}(h)\right], f^T\left[\theta|g^{-T}(h)\right], 0\right),$$

for every $h \in H^T$. To complete the proof of part (a), it suffices to show that $s^\theta \in SPE(\Gamma, \theta)$. To this end, by the one deviation property, it suffices to show that no agent i can gain by unilaterally deviating from s_i^θ in a single period t and conforming with s_i^θ thereafter.

To this end, note that the action profile $s^\theta(h)$ falls into *Rule 1* for every history $h \in H$, by construction. Also, for every history h, note that no unilateral deviation of agent i from $s_i^\theta(h)$ can induce *Rule 3*. Finally, note that each agent reports the true profile of reduced utilities, after every history.

Now, fix any history h. If agent i deviates from $s_i^\theta(h)$ by playing a period-t action $a_i^t \neq s_i^\theta(h)$, then *Rule 2* applies. This means that this agent i will obtain an outcome which is not better than the outcome he obtains by playing $s_i^\theta(h)$. Since the choice of history h, as well as the choice of agent i, is arbitrary, it follows that $s^\theta(h) \in NE\left(\tilde{\Gamma}(h), U\left[\theta|f^{+2}\right]\right)$ if $h \in H^1$,

$s^{\theta}(h) \in NE\left(\tilde{\Gamma}(h), U\left[\theta|g^{-T}(h)\right]\right)$ if $h \in H^T$, and $s^{\theta}(h) \in NE\left(\tilde{\Gamma}(h), U\left[\theta|g^{-t}(h), f^{+(t+1)}\right]\right)$ if $h \in H\backslash\left(H^1 \cup H^T\right)$. By definitions of reduced utilities provided in (1), (3) and (5), and by definition of s^{θ}, one can see $s^{\theta}|h \in NE(\Gamma(h), \theta)$ for every history $h \in H$, and so $s^{\theta} \in SPE(\Gamma, \theta)$. This proves part (a) of Definition 1.

Next, let us prove part (b) of Definition 1. We prove it only for the case that $h \in H$ is the trivial history. The following arguments can be suitably modified for the case where h is not a trivial history.

Assume that $s \in SPE(\Gamma, \theta)$. Moreover, fix any $h^* \in H$. Thus, the strategy profile $s|h^* \in NE(\Gamma(h^*), \theta)$. Assume, to the contrary, that there is a period $t \in \mathcal{T}$ as well as a history $h^t \in H|h^*$ such that either $f^t[\theta|g^{-t}(h^t)] \neq g^t(s(h^t))$ if $t \neq 1$, or $f^1[\theta] \neq g^1(s(h^1))$ if $t = 1$. Among all such histories, let $h^{\tau} \in H|h^*$ be one of the longest histories such that $f^{\tau}[\theta|g^{-\tau}(h^{\tau})] \neq g^{\tau}(s(h^{\tau}))$ and $f^{\hat{\tau}}\left[\theta|g^{-\hat{\tau}}\left(h^{\hat{\tau}}\right)\right] = g^{\hat{\tau}}\left(s\left(h^{\hat{\tau}}\right)\right)$ for every $h^{\hat{\tau}} \in H|(h^{\tau}, s^{\tau}(h^{\tau}))$ if $\tau \neq T$.

Let us suppose that $\tau \neq 1, T$. Then, $s(h^{\tau}) \in NE\left(\tilde{\Gamma}(h^{\tau}), U\left[\theta|g^{-\tau}(h^{\tau}), f^{+(\tau+1)}\right]\right)$. We proceed according to whether $s(h^{\tau})$ falls into *Rule 1* or not.

Suppose that $s(h^{\tau})$ falls into *Rule 1* of period-τ mechanism. Thus, $g^{\tau}(s(h^{\tau})) = f^{\tau}\left[\bar{\theta}|g^{-\tau}(h^{\tau})\right]$ for some $\bar{\theta}$, and this outcome is an element of $Y^{\tau}(g^{-\tau}(h^{\tau}))$. Then, given that $f^{\tau}[\theta|g^{-\tau}(h^{\tau})] \neq g^{\tau}(s(h^{\tau}))$, it follows that $f^{\tau}[\theta|g^{-\tau}(h^{\tau})] \neq f^{\tau}\left[\bar{\theta}|g^{-\tau}(h^{\tau})\right]$. One-step-ahead Maskin monotonicity implies that there exists an agent i and a period-τ outcome $y^{\tau} \in Y^{\tau}(g^{-\tau}(h^{\tau}))$ such that

$$U_i\left[f^{\tau}\left[\bar{\theta}|g^{-\tau}(h^{\tau})\right], \bar{\theta}|g^{-\tau}(h^{\tau}), f^{+(\tau+1)}\right] \geq U_i\left[y^{\tau}, \bar{\theta}|g^{-\tau}(h^{\tau}), f^{+(\tau+1)}\right]$$

and

$$U_i\left[y^{\tau}, \theta|g^{-\tau}(h^{\tau}), f^{+(\tau+1)}\right] > U_i\left[f^{\tau}\left[\bar{\theta}|g^{-\tau}(h^{\tau})\right], \theta|g^{-\tau}(h^{\tau}), f^{+(\tau+1)}\right].$$

By changing $s_i(h^{\tau})$ into $a_i(h^{\tau}) = \left(U\left[\theta|g^{-\tau}(h^{\tau}), f^{+(\tau+1)}\right], y^{\tau}, 1\right)$, agent i can induce *Rule 2* and obtain $g^{\tau}(a_i(h^{\tau}), s_{-i}(h^t)) = y^{\tau}$, thereby contradicting the fact that $s(h^{\tau})$ is an element of $NE\left(\tilde{\Gamma}(h^{\tau}), U\left[\theta|g^{-\tau}(h^{\tau}), f^{+(\tau+1)}\right]\right)$.

Suppose that $s(h^{\tau})$ falls into either *Rule 2* or *Rule 3* of period-τ mechanism. Thus, for every agent $j \neq i$, the period-τ outcome determined by this rule is maximal for this j in $Y^{\tau}(g^{-\tau}(h^{\tau}))$ according to her period-τ reduced utility $U_j\left[\cdot, \theta|g^{-\tau}(h^{\tau}), f^{+(\tau+1)}\right]$. Since the SCF f satisfies the one-step-ahead no veto-power, this implies that $g^{\tau}(s(h^{\tau})) = f^{\tau}[\theta|g^{-\tau}(h^{\tau})]$, which is a contradiction.

We conclude the proof by mentioning that, suitably modified, the above proof provided for the case where $\tau \neq 1, T$ applies to the case where $\tau = 1$ as well as to the case where $\tau = T$. ■

Proof of Lemma 1. Let the premises hold. To save notation, in what follows we write $U(\cdot, \cdot, \theta_i)$ for $U_i(\cdot, \cdot, \theta)$. Fix any $x^1 \in X^1$ and any $\theta \in \Theta$. Let $x^2[\eta|x^1]$ be the solution to:

$$\frac{\partial U(x^1, x^2, \eta)}{\partial x^2} = 0.$$

By the implicit function theorem, we have that:

$$\frac{\partial x^2\left[\eta|x^1\right]}{\partial \eta} = -\frac{\frac{\partial^2 U\left(x^1, x^2\left[\eta|x^1\right], \eta\right)}{\partial^2 x^2}}{\frac{\partial^2 U(x^1, x^2[\eta|x^1], \eta)}{\partial \eta \partial x^2}} > 0.$$

Therefore, the peak for the median type $\eta = \theta_{med}$ is always the peak in the second voting stage for each $x^1 \in X^1$. Write $x^2\left[\theta_{med}|x^1\right]$ for the peak of the median type in the second voting stage conditional on x^1.

Since it holds that:

$$\frac{\partial U\left(x^1, x^2\left[\theta_{med}|x^1\right], \theta_{med}\right)}{\partial x^2} = 0,$$

from the implicit function theorem we obtain that:

$$\frac{\partial x^2\left[\theta_{med}|x^1\right]}{\partial x^1} = -\frac{\frac{\partial^2 U\left(x^1, x^2\left[\theta_{med}|x^1\right], \theta_{med}\right)}{\partial x^1 \partial x^2}}{\frac{\partial^2 U(x^1, x^2[\theta_{med}|x^1], \theta_{med})}{\partial^2 x^2}} \geq 0.$$

Let us show that $x^2\left[\theta_{med}|x^1\right]$ is the Condorcet winner under $U\left[\theta|x^1\right]$ for every $x^1 \in X^1$. For every allowable type $\eta \in \left(\underline{\eta}, \bar{\eta}\right)$ and policy (x^1, x^2), let:

$$\Phi(\eta, x^1, x^2) = U\left(x^1, x^2\left[\theta_{med}|x^1\right], \eta\right) - U(x^1, x^2, \eta).$$

Then, for every $x^2 < x^2\left[\theta_{med}, x^1\right]$, we have that:

$$\Phi(\theta_{med}, x^1, x^2) = \int_{x^2}^{x^2\left[\theta_{med}|x^1\right]} \frac{\partial U(x^1, z^2, \theta_{med})}{\partial z^2} dz^2.$$

Furthermore, for every $\eta > \theta_{med}$, it holds that:

$$\Phi(\eta, x^1, x^2) - \Phi(\theta_{med}, x^1, x^2) = \int_{\theta_{med}}^{\eta} \int_{x^2}^{x^2\left[\theta_{med}|x^1\right]} \frac{\partial^2 U(x^1, z^2, \alpha)}{\partial \alpha \partial z^2} dz^2 d\alpha > 0.$$

Since

$$\Phi(\theta_{med}, x^1, x^2) = U(x^1, x^2\left[\theta_{med}|x^1\right], \theta_{med}) - U(x^1, x^2, \theta_{med}) \geq 0,$$

it follows that:

$$\Phi(\eta, x^1, x^2) > 0,$$

which, in turn, guarantees that:

$$U(x^1, x^2\left[\theta_{med}|x^1\right], \eta) > U(x^1, x^2, \eta).$$

Therefore, for every voter $j = k+1, \cdots, 2k-1$, it holds that:

$$U(x^1, x^2\left[\theta_{med}|x^1\right], \theta_j) > U(x^1, x^2, \theta_j).$$

Likewise, for every $x^2 > x^2\left[\theta_{med}|x^1\right]$, one can show that for every voter $j = 1, \cdots, k-1$

it holds that:

$$U(x^1, x^2\left[\theta_{med}|x^1\right], \theta_j) > U(x^1, x^2, \theta_j).$$

Therefore, $x^2[\theta_{med}|x^1]$ is a Condorcet winner under $U[\theta|x^1]$, that is, $CW(U[\theta|x^1]) = x^2[\theta_{med}|x^1]$, and so the majority voting function $f^2[\cdot|\cdot]$ is a single-valued function for every $\theta \in \Theta$ and every $x^1 \in X^1$.

Let $x[\theta_{med}] = (x^1[\theta_{med}], x^2[\theta_{med}])$ be the global peak for the median type θ_{med}. Next, we show that $x^1[\theta_{med}]$ is the Condorcet winner under $U[\theta|f^2]$.

Solving backward, given that the majority voting function $f^2[\theta|x^1] = x^2[\theta_{med}|x^1]$ for every $x^1 \in X^1$, we have that the reduced utility of type η is:

$$V(\eta, x^1) = U(x^1, x^2\left[\theta_{med}|x^1\right], \eta).$$

Then, we have that:

$$\frac{\partial V(\eta, x^1)}{\partial x^1} = \frac{\partial U(x^1, x^2[\theta_{med}|x^1], \eta)}{\partial x^1} + \frac{\partial U(x^1, x^2[\theta_{med}|x^1], \eta)}{\partial x^2}\frac{\partial x^2[\theta_{med}|x^1]}{\partial x^1},$$

and so, by Assumption 3, it follows that:

$$\frac{\partial^2 V(\eta, x^1)}{\partial\eta\partial x^1} = \frac{\partial^2 U(x^1, x^2[\theta_{med}|x^1], \eta)}{\partial\eta\partial x^1} + \frac{\partial^2 U(x^1, x^2[\theta_{med}|x^1], \eta)}{\partial\eta\partial x^2}\frac{\partial x^2[\theta_{med}|x^1]}{\partial x^1} > 0.$$

Then, for every $x^1 \in X^1$, let:

$$\Pi(\eta, x^1) = V(\eta, x^1[\theta_{med}]) - V(\eta, x^1)$$

Next, take any $x^1 < x^1[\theta_{med}]$. Then, it holds that:

$$\Pi(\theta_{med}, x^1) = \int_{x^1}^{x^1[\theta_{med}]} \frac{\partial V(\theta_{med}, z^1)}{\partial z^1} dz^1.$$

Moreover, for every $\eta > \theta_{med}$, it also holds that:

$$\Pi(\eta, x^1) - \Pi(\theta_{med}, x^1) = \int_{\theta_{med}}^{\eta}\int_{x^1}^{x^1[\theta_{med}]} \frac{\partial^2 V(\alpha, z^1)}{\partial\alpha\partial z^1} dz^1 d\alpha > 0.$$

Since

$$\Pi(\theta_{med}, x^1) = V(\theta_{med}, x^1[\theta_{med}]) - V(\theta_{med}, x^1) \geq 0,$$

we have that:

$$\Pi(\eta, x^1) > 0,$$

which, in turn, guarantees that:

$$V(\eta, x^1[\theta_{med}]) > V(\eta, x^1).$$

Therefore, for every voter $j = k+1, \cdots, 2k-1$, we have that:

$$V(\theta_j, x^1[\theta_{med}]) > V(\theta_j, x^1).$$

Likewise, for every $x^1 > x^1[\theta_{med}]$ one can also show that:

$$V(\theta_j, x^1[\theta_{med}]) > V(\theta_j, x^1), \quad \text{for every voter } j = 1, \cdots, k-1.$$

We conclude that $x^1[\theta_{med}]$ is a Condorcet winner under $U[\theta|f^2]$, that is, $CW(U[\theta|f^2]) = x^1[\theta_{med}]$, and so the majority voting function $f^1[\cdot]$ is a single-valued function for every $\theta \in \Theta$.
■