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# Universal N-partite d-level pure-state entanglement witness based on realistic measurement settings

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Entanglement witnesses are operators that are crucial for confirming the generation of specific quantum systems, such as multipartite and high-dimensional states. For this reason, many witnesses have been theoretically derived, which commonly focus on establishing tight bounds, and exhibit mathematical compactness as well as symmetry properties similar to that of the quantum state. However, for increasingly complex quantum systems, established witnesses have lacked experimental achievability, as it has become progressively more challenging to design the corresponding experiments. Here, we present a universal approach to derive entanglement witnesses that are capable of detecting the presence of any targeted complex pure quantum system and that can be customized towards experimental restrictions or accessible measurement settings. Using this technique, we derive experimentally-optimized witnesses that are able to detect multipartite $d$-level cluster states, and that require only two measurement settings. We present explicit examples for customizing the witness operators given different realistic experimental restrictions, including witnesses for high-dimensional entanglement that use only two-dimensional projection measurements. Our work enables to confirm the presence of probed quantum states using methods that are compatible with practical experimental realizations in different quantum platforms.

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Entanglement is an essential resource in quantum information science [1], playing a fundamental role in many tasks such as measurement-based quantum computation [2], error correction [3], quantum cryptography [4], and dense coding [5]. It is thus crucial to derive efficient methods that allow to experimentally detect and/or quantify the presence of entanglement in quantum systems. However, completely characterizing entanglement is still an open issue [1], especially for complex quantum systems consisting of multiple parties and/or higher dimensionality. Several approaches, for example concurrence and entanglement of formation methods [6], as well as quantum Fisher information [7, 8] allow to determine entanglement. Such strategies are based on extensive quantum tomography techniques that are extremely challenging when experimentally applied to complex quantum systems and/or work only for special quantum states that allow phase sensitivity estimation [9, 10]. Therefore, these are clearly not universal. Entanglement witnesses are one of the most appropriate and efficient approaches to verify entanglement in a system, as well as the presence of a specific state [11] since they can be measured directly via single party projectors and require a lower number of measurements than what is needed for tomography (i.e. $d^{2N}$, where $d$ is the state dimensionality the number of levels per single party and $N$ the number of parties). An entanglement witness is an operator that is used to detect the generation of a targeted quantum state and to confirm the realization of genuine multipartite [12, 13] and/or high-dimensional ($d$-level, i.e. qudit) entanglement [14, 15]. A witness $\mathcal{W}$ is conventionally designed such that, if a negative expectation value $\langle\mathcal{W}\rangle = Tr\ (\mathcal{W}\rho) < 0$ is measured (where $\rho$ is the density matrix of a generic quantum system), the probed quantum state cannot be separable or lower-dimensional [6, 12]. When a non-negative witness expectation value is measured, the result is typically ambiguous and does not indicate either the presence or the absence of entanglement. Since the witness tests for a specif state, a successful measurement of the operator also provides information about the state structure and phase, rather than only confirming the presence of entanglement. For example, a witness specifically designed for a four-qubit compact cluster state [16] confirms, when its expectation value is negative, the presence of that particular state having a very specific density function, while a positive measured expectation value of that operator only provides information that the tested state is not a compact cluster state. Indeed, the same witness, if applied to a four-qubit linear cluster or Greenberger-Horne-Zeilinger (GHZ) [17] states, would result in a positive measured expectation value, even though these two states are both highly entangled [17, 18]. Hence, a witness is a threshold test that can only detect the presence of a specific state. In contrast to an entanglement monotone (e.g. the entanglement entropy [6]), which determines the amount of entanglement, a witness cannot be used to quantify entanglement. The necessary and sufficient condition to confirm the generation of a quantum state close to the targeted one is thus to measure with high statistical confidence a negative expectation value of the witness. Specifically, such a witness is appositely formulated for the considered target.

From a measurement standpoint, the best witness demands the least effort to experimentally detect a quantum state. Ideally, it should only include measurements on single qudits and have as high a noise tolerance as possible to detect the probed state with large confidence. This is often in contrast to the theoretical perspective that rather focuses on defining the tightest bound of the witness [11–13], which in turn can result in a very complex form. In recent years, the experimental generation of more complex quantum states has intensified the need for witnesses that can detect such states and simultaneously are straightforward to measure. While this goal has been partially achieved for GHZ, graph, and cluster states of qubit (two-dimensional) based systems [18–20], extending such 'experimentally-friendly' witnesses to higher dimensional systems is still a challenge, and a universal witness capable of entanglement detection of any complex qudit state has not been demonstrated yet. Moreover, deriving witnesses customized to experimental restrictions and available measurement settings is desirable for a relatively straightforward experimental realization.

In this letter, we provide a compact and universal method to construct entangled pure-state witness operators that are capable of detecting the presence of any arbitrarily-complex quantum state and can be customized to account for experimental restrictions. As an example, we here apply this method to derive a universal entanglement witness for multipartite $d$-level cluster states which requires only two measurement settings, allowing to investigate their tolerance to white noise. Starting from theoretically-optimal witnesses, we significantly simplify them to derive experimentally-optimized pure-state witness operators.

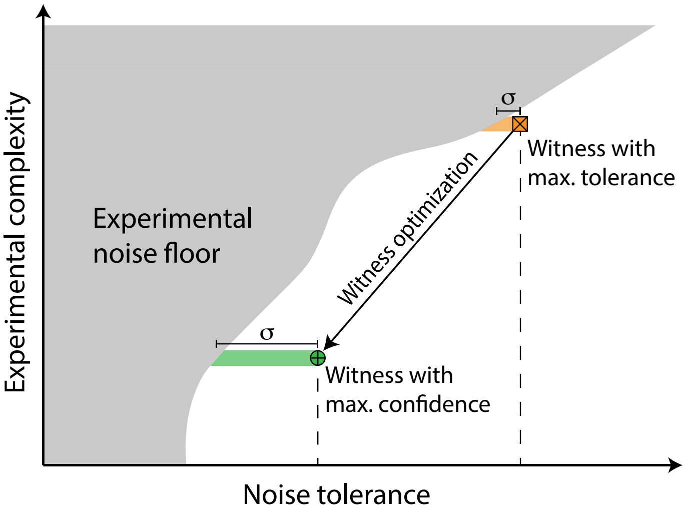

Figure 1. Complexity of the measurement settings vs. noise tolerance of the witness. In any experiment, the measurement settings introduce an experimental noise floor (grey area), which increases with the measurement complexity. The optimal theoretical witness (orange square) has the highest noise tolerance, but demands intricate and often unfeasible measurements. The optimal experimental witness (green circle) requires reduced measurement complexity, at the cost of having a lower noise tolerance. The goal is to maximize $\sigma$ (the shaded green or orange area), which represents the measurement confidence, given by the distance between the noise floor and the witness noise tolerance. Such maximization provides the highest confidence to measure a negative expectation value of the witness.

A theoretically-optimal witness provides the highest selectivity (i.e. tolerance towards noise), but requires highly intricate measurements. With such increasing measurement complexity, the experimental noise floor rises and it is very likely that it is not possible to actually measure the witness, despite its large bound. Since a witness comprises many measurements, when these (and in turn their related errors) are reduced in number and complexity, the bound is reduced (i.e. the robustness to noise) but also the experimental noise floor decreases as well (see Fig. 1 and Supplemental material [21]). A judiciously chosen measurement setting thus allows for a trade-off between these effects, where an experimentally-optimized witness is one that has a maximum separation between measurement noise floor and witness bound. Our approach is general and applicable to any witness that has been derived in the literature. Here, we consider a standard theoretically-optimal witness, which is used to detect the presence of a pure multipartite quantum state $|\psi\rangle$ and those states close to it (e.g. states affected by white noise) [22]:

$$\mathcal{W}_{theor}^{opt} = \frac{1}{1-\alpha}\left(\alpha \cdot \mathbb{I} - \rho\right), \tag{1}$$

with $\mathbb{I}$ being the identity operator, and $\rho$ the density matrix of the state. The factor $\alpha$ must be chosen such that all separable systems lead to non-negative expectation values (a method for determining $\alpha$ is presented in [22]). For convention, we normalize the witness such that its expectation value for the ideal state is minus one. Despite its elegant and compact mathematical form, the witness in Eq. (1) is not practical to measure, as it requires quantum state tomography to determine $\rho$ [23, 24]. Consequently, its characterization becomes quickly challenging for states with large $N$ and $d$ [14, 25]. A judicious modification and simplification of the witness in Eq. (1) allows to experimentally confirm the generation of a targeted state by making use of a significantly lower number of projection measurements, which in turn can be implemented through a reduced number of experimental settings. Without loss of generality, we first introduce the operator $\mathcal{W}_{meas}$ containing the specific measurement settings that can be implemented experimentally (see Supplemental material [21] for details on how to measure an operator by means of multiple measurement settings). The only restriction for this operator is that it must be Hermitian, thus having real eigenvalues and being measurable. We then add and subtract $\mathcal{W}_{meas}$ from the witness of Eq. (1), so that we keep the overall expression unchanged:

$$\mathcal{W} = \frac{1}{1-\alpha}\left(\alpha \cdot \mathbb{I} - \mathcal{W}_{meas} + \Theta\right), \tag{2}$$

where we introduce the Hermitian operator $\Theta = \mathcal{W}_{meas} - \rho$. This is allowed since the analysis is based on entanglement detection via witness measurements, not on entanglement quantification via monotones. The goal is to exchange $\Theta$ with a scalar number that keeps the witness bounds unchanged, such that a measurement of the expectation value of $\Theta$ is not experimentally required. To determine this scalar number, we need to consider the worst possible measurement outcome, corresponding to the maximal expectation value of $\Theta$ for any probed state. This concept can be summarized by the inequality

$$\langle \Theta \rangle \leq \lambda_{max}\left(\Theta\right) = \langle \lambda_{max}\left(\Theta\right) \cdot \mathbb{I} \rangle \tag{3}$$

where $\lambda_{max}\left(\Theta\right)$ is the largest eigenvalue of $\Theta$. Replacing $\Theta$ in Eq. (2) with $\lambda_{max}\left(\Theta\right) \cdot \mathbb{I}$ results in a experimentally-optimized entangled pure-state witness:

$$\mathcal{W}_{exp}^{opt} = \frac{1}{1-\alpha}\left[\left(\alpha + \lambda_{max}\right) \cdot \mathbb{I} - \mathcal{W}_{meas}\right], \tag{4}$$

which is valid to detect the realization of any quantum state for which Eq. (1) is a witness and eventually, the presence of entanglement. In particular, any $\mathcal{W}_{meas}$ with real eigenvalues can be selected. Reducing the number and complexity of measurement settings decreases the sensitivity of an entanglement witness, which means that less experimental noise can be tolerated. However, since the experimental goal is to measure a negative expectation value, this reduction is justified by the significantly decreased number of measurements, as well as by the experimentally feasible form that the operator assumes (see Fig. 1).

The task is to find measurement settings that are optimized for given experimental restrictions, and still provide sufficient tolerance to noise and/or experimental imperfections. Towards this purpose, stabilizers can be used [3, 18] to construct $\mathcal{W}_{meas}$. A set of observables $\mathcal{S}_k$ are the stabilizers of an $N$-qudit state $|\psi\rangle$ if they satisfy the eigenvalue equations $\mathcal{S}_k\,|\psi\rangle = |\psi\rangle$. The full set of $d^N$ stabilizers uniquely describes the quantum state, and allows to reconstruct the density matrix of the probed state as [18]:

$$\rho = \frac{1}{d^N}\sum_{n=1}^{d^N} \mathcal{S}_n. \tag{5}$$

Stabilizer operators are commonly used to derive the witness for GHZ, graph [26], and cluster [16, 18] states. From an experimental standpoint, the most important feature of stabilizers is that they are described by generalized Pauli matrices, which make them measurable through projections on the single parties of the quantum system [18, 27] (see Supplemental material [21] and Ref. [28] for the definition of projection measurements).

We here provide an example of this method for multipartite $d$-level cluster states $\rho_{N,d}$ [29]. An entanglement witness for two-level (qubit) cluster states has previously been derived [18]. We use the technique presented here to derive a generalized witness for cluster states with $N$ parties and $d$ levels. The first step is to determine the coefficient $\alpha$ in Eq. (1). To do so, we exploit a unique property of cluster states, i.e. their maximal connectedness [16]: it is possible to project any subset of cluster state qudits into maximally entangled bipartite states by only performing local operations on the other qudits. This means that, for any bipartite sub-system that is formed by any combination of two parties of the initial cluster state, the maximal Schmidt coefficient is $1/\sqrt{d}$. This implies that, in case of multipartite cluster states, $\alpha = 1/d$ [22]. With this in mind, the operator in Eq. (1) becomes:

$$\mathcal{W}_{theor}^{opt} = \frac{d}{d-1}\left(\frac{1}{d} \cdot \mathbb{I} - \rho_{N,d}\right). \tag{6}$$

The full density matrix of the cluster state can be expressed in terms of its unique reduced set of main stabilizers (see Supplemental material [21]), as:

$$\rho_{N,d} = \prod_{k=1}^{N} \frac{1}{d}\sum_{l=1}^{d} \mathcal{S}_k^{(l)}, \tag{7}$$

where *(l)* denotes the power degree of the stabilizer. We also uphold the the convention that, for the properties of the generalized Pauli matrices, $S_k^{(d)} = \mathbb{I}$, independently of the label $k$. Among all such stabilizers, we can choose a subset of those consisting of only two independent measurement settings [27], which can be found by considering only even or odd main stabilizers (see the Supplemental material [21]). These subsets are included in the operator $\mathcal{W}_{meas}$ (see Supplemental material [21] for a detailed explanation about the measurement of such operators):

$$\mathcal{W}_{meas} = \prod_{odd\ k} \frac{1}{d} \sum_{l=1}^{d} \mathcal{S}_k^{(l)} + \prod_{even\ k} \frac{1}{d} \sum_{l=1}^{d} \mathcal{S}_k^{(l)}, \tag{8}$$

from which it follows

$$\Theta = \prod_{odd\ k} \frac{1}{d} \sum_{l=1}^{d} \mathcal{S}_k^{(l)} + \prod_{even\ k} \frac{1}{d} \sum_{l=1}^{d} \mathcal{S}_k^{(l)} - \prod_{k=1}^{N} \frac{1}{d} \sum_{l=1}^{d} \mathcal{S}_k^{(l)}. \tag{9}$$

Exploiting the properties of the stabilizers, we find that the maximum eigenvalue of $\Theta$ is 1 for any $N$ and $d$, i.e. $\langle\Theta\rangle \leq \lambda_{max} = 1$ (see Supplemental material [21], where we demonstrate this statement by also making use of Refs. [30–33]). This leads to an experimentally-optimized witness for $N$-partite $d$-level cluster states consisting of only two measurement settings:

$$\mathcal{W}_{exp}^{opt} = \frac{d+1}{d-1}\mathbb{I} - \frac{d}{d-1} \left\{ \prod_{odd\ k} \frac{1}{d} \sum_{l=1}^{d} \mathcal{S}_k^{(l)} + \prod_{even\ k} \frac{1}{d} \sum_{l=1}^{d} \mathcal{S}_k^{(l)} \right\}. \tag{10}$$

For $d = 2$ this operator reduces to the well-known witness for cluster states of qubits [18].

This witness can be used to investigate the sensitivity of cluster states to white noise, which reasonably models experimental noise contributions and is thus considered in many theoretical and experimental scenarios [18, 22, 34–37]. When noise affects a pure entangled state, it adds some mixture to its density matrix, thus modifying the entanglement properties. Testing the robustness of any pure entangled state towards noise is thus crucial to determine the threshold at which the state becomes a non-entangled mixture. We demonstrate here that the noise tolerance of $\mathcal{W}_{exp}^{opt}$ for multipartite states increases with growing dimensionality, similar to two-partite states [38–40]. The presence of white noise within the cluster state modifies the density matrix as [18]:

$$\rho_{noise} = \varepsilon \frac{\mathbb{I}}{d^N} + (1-\varepsilon)\, \rho_{N,d}, \tag{11}$$

with $0 \leq \varepsilon \leq 1$ being the probability that the state is affected by noise. The robustness of a witness is determined by a noise threshold $\varepsilon_{th}$ up to which it still detects the presence of entanglement, and which reads (see Supplemental material [21]):

$$\varepsilon_{th} = \begin{cases} \left[\frac{2d}{d-1}\left(1 - \frac{1}{d^{N/2}}\right)\right]^{-1}, & \text{even N} \\ \\ \left[\frac{2d}{d-1}\left(1 - \frac{1}{2d^{(N+1)/2}} - \frac{1}{2d^{(N-1)/2}}\right)\right]^{-1}, & \text{odd N.} \end{cases} \tag{12}$$

The noise tolerance of high-dimensional cluster states with respect to this witness increases with $d$, reaching a maximum of $\varepsilon_{th} \to 0.5$ for $d \to \infty$. For $d = 2$ the result is identical to the noise sensitivity of the qubit cluster states derived in [18]. Most remarkably, our result shows that different cluster states with the same Hilbert-space sizes can have distinct noise tolerances. As an example, cluster states with $N = 8$, $d = 2$ and $N = 4$, $d = 4$ have equal Hilbert-space sizes, but the noise tolerances are $\varepsilon_{th} = 0.2667$ and $\varepsilon_{th} = 0.4$ (i.e. 1.5 times higher), respectively. While here we consider white noise due to its broad relevance, in future work other noise sources could be considered.

Detecting $\mathcal{W}_{exp}^{opt}$ needs significantly less measurements than the $d^{2N}$ required by full quantum tomography, thus being more advantageous from an experimental standpoint. For example, in the case of a four-partite three-level (qutrit) cluster state [41], $\mathcal{W}_{exp}^{opt}$ requires $3 * 3^4 = 243$ measurements, vs. the $3^{2*4} = 6561$ needed for tomography.

The reduced witness of Eq. (4) enables performing very specific customizations. Let us consider the example of a four-partite qutrit optical cluster state, which we have recently demonstrated in an optical system exploiting the time/frequency framework [41] (see also Supplemental material [21] and Ref. [42] for the concept of frequency-bin entanglement). We consider that it is only possible to project either qutrits 1 and 2, or 3 and 4 into superposition states (i.e., mathematically, all four measurement settings $X$, $Z$, $Y = XZ$, and $V = XZ^\dagger$ can be implemented), yet it is not feasible to project all four qutrits at the same time. Thus, if qutrits 1 and 2, or 3 and 4 were projected on $X$, $Y$, or $V$, the other qutrits had to be projected in the eigenbasis of $Z$. Measuring on the $X$-, $Y$-, or $V$-basis typically coincides with significant experimental complexity and losses, making it undesirable to perform such measurements on many qudits. Considering these very specific restrictions, only $M = 20$ out of the $3^4$ stabilizers, together with their transposed conjugates, were considered (see Supplemental material [21]). We can therefore construct a different witness, making use of all the measurement capabilities:

$$\mathcal{W}_{meas} = \frac{1}{27} \sum_{k=1}^{20} \left(\mathcal{S}_k + \mathcal{S}_k^\dagger\right)$$

$$\Theta = \frac{1}{27} \sum_{k=1}^{20} \left(\mathcal{S}_k + \mathcal{S}_k^\dagger\right) - \frac{1}{81} \sum_{k=1}^{81} \mathcal{S}_k.$$

The largest eigenvalue of $\Theta$ is $13/27$, which leads to the experimentally-optimized witness

$$\mathcal{W}_{exp}^{opt}=\frac{11}{9}-\frac{1}{18}\sum_{k=1}^{20}\left(\mathcal{S}_k+\mathcal{S}_k^{\dagger}\right).$$

This witness has a noise tolerance of $\varepsilon_{th}=0.45$. Thanks to the customization with respect to the specific experimental restriction, this value is 20% higher than that of the witness which only uses two measurement settings ($\varepsilon_{th}=0.375$). This demonstrates that different experimental restrictions lead to different optimal witnesses for the same given quantum state.

The introduced method allows to not only select any subset of stabilizers, but also to construct almost arbitrary measurement operators. One case in which these operators cannot be constructed by using stabilizers is for confirming the presence of a qudit quantum state with the experimental restriction of only two-level projection measurements being possible. For example, one can consider a three-level two-photon time-bin entangled state [43–48], i.e. $|\psi\rangle=\frac{1}{\sqrt{3}}\left(|0,0\rangle+|1,1\rangle+|2,2\rangle\right)$. In this case, performing three-dimensional projection measurements would require a stable three-arm interferometer [49], which is challenging to realize. Instead, we note that only two-arm interferometers are readily available [45]. This allows performing two-basis measurements [50], which in our case means projections on superpositions of only two time-bins at a time. This experimental restriction signifies that it is possible to measure the diagonal elements of the quantum state (i.e. $Z$ and $\mathbb{I}$), but not its off-diagonal elements (i.e. $X$, $V$, and $Y$). The goal is to construct a witness that can be measured exclusively through two-dimensional projection measurements. To this end, we first consider the optimal witness of Eq. (1):

$$\begin{aligned}\mathcal{W}&=\frac{\mathbb{I}}{2}-\frac{3}{2}\left|\psi\right\rangle\left\langle\psi\right|\\&=\frac{\mathbb{I}}{2}-\frac{1}{6}(\mathbb{I}+Z^{\dagger}Z+XX+VY\\&\quad+YV+ZZ^{\dagger}+X^{\dagger}X^{\dagger}+V^{\dagger}Y^{\dagger}+Y^{\dagger}V^{\dagger})\\&=\frac{\mathbb{I}}{3}-\frac{1}{6}\left(Z^{\dagger}Z+ZZ^{\dagger}\right)-\Phi,\end{aligned}$$

where we enclose the part of the witness that cannot be directly measured due to the experimental restrictions in the operator $\Phi=\frac{1}{6}\left(XX+VY+YV+X^{\dagger}X^{\dagger}+V^{\dagger}Y^{\dagger}+Y^{\dagger}V^{\dagger}\right)$. In order to construct a different witness, we introduce the operators $X_p$ that correspond to two-dimensional projections, being identical to the Gell-Mann matrices [51] (see Supplemental material [21]). Furthermore, they are comparable to partial operators [52], with the subscript $p$ labeling the sub-part of the system on which the projection is performed. These operators project on two-level superpositions of only two modes at a time, while removing the third mode, which corresponds to the action of a two-arm interferometer on a qutrit time-bin state. In the computational basis $\{|0\rangle,|1\rangle,|2\rangle\}$, they are: $X_{0,1}=|0\rangle\langle1|+|1\rangle\langle0|$, $X_{0,2}=|0\rangle\langle2|+|2\rangle\langle0|$, and $X_{1,2}=|1\rangle\langle2|+|2\rangle\langle1|$. We can now construct an operator that has significant overlap with the non-zero elements of the $\Phi$ operator:

$$\begin{aligned}\mathcal{W}_{meas}&=\frac{1}{6}\left(Z^{\dagger}Z+ZZ^{\dagger}\right)\\&\quad+\frac{1}{2}\left(X_{0,1}X_{0,1}+X_{0,2}X_{0,2}+X_{1,2}X_{1,2}\right)\\\Theta&=\frac{1}{2}\left(X_{0,1}X_{0,1}+X_{0,2}X_{0,2}+X_{1,2}X_{1,2}\right)\\&\quad-\frac{1}{6}\left(XX+VY+YV+X^{\dagger}X^{\dagger}+V^{\dagger}Y^{\dagger}+Y^{\dagger}V^{\dagger}\right)\end{aligned}$$

The largest eigenvalue of $\Theta$ is 1, which results in a normalized experimentally-optimized witness for the chosen two-level projection measurements:

$$\begin{aligned}\mathcal{W}_{exp}^{opt}=\frac{5}{3}\mathbb{I}-&\left(Z^{\dagger}Z+ZZ^{\dagger}\right)\\&-\left(X_{0,1}X_{0,1}+X_{0,2}X_{0,2}+X_{1,2}X_{1,2}\right)\end{aligned}$$

This witness has a noise tolerance of 0.375, which is lower than that of the witness in Eq. (10) (i.e. $\varepsilon_{th}=0.5$). However, it allows to perform measurements on a $d$-level quantum state via measurement settings which access a lower number of levels only. Moreover, such measurement settings are significantly simplified and reduced in number (39 rather than 81), and only require projections on two-dimensional superpositions.

In conclusion, we presented a versatile and compact approach to customize entanglement pure-state witnesses that are capable of detecting any pure quantum state, as well as its eventual entanglement, and that account for experimental restrictions. We first derive experimentally-optimized entanglement witnesses for $N$-partite $d$-level cluster states that consist of only two measurement settings, finding that increasing the dimensionality of cluster states significantly decreases their white noise sensitivity. We show that it is possible to further customize such operators in the presence of certain experimental restrictions, such as limitations in the capability of performing projection measurements on $d$-level superpositions. The method used here to reduce the witness experimental complexity by adding and subtracting a measurement operator is universal and can be applied to any witness previously derived for detecting the presence of any specific quantum states:

$$\mathcal{W}_{exp}^{opt}=\left[\lambda_{max}\left(\mathcal{W}_{meas}+\mathcal{W}\right)\right]\mathbb{I}-\mathcal{W}_{meas}. \tag{13}$$

The presented technique therefore provides a powerful tool to simplify the experimental validation of quantum states. Our approach can be applied to any quantum state, such as photonic systems [41], cold atoms [53, 54], and trapped ions [55, 56].

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