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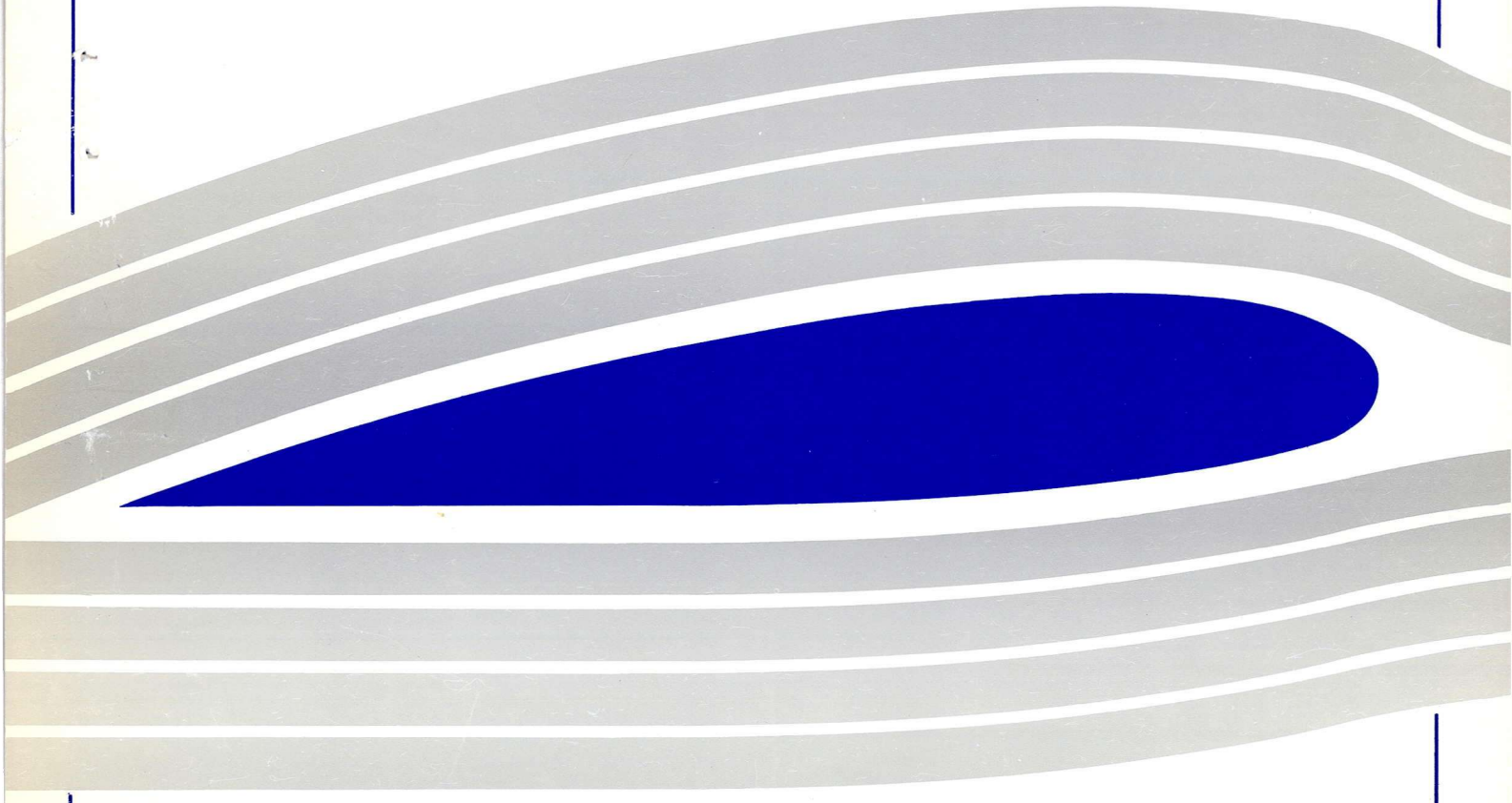

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NEWTON-LIKE METHODS FOR FAST HIGH RESOLUTION SIMULATION OF HYPERSONIC VISCOUS FLOWS

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ABSTRACT

Two Newton-like methods, i.e. the sparse finite difference Newton method and the sparse quasi-Newton method, are applied to the Navier-Stokes solutions of hypersonic flows using the Osher flux difference splitting high resolution scheme. The resulting large block structured sparse linear system is solved by a new multilevel iterative solver, the α-GMRES method, which includes a preconditioner and a damping factor. The algorithm is demonstrated to provide fast, accurate solutions of the hypersonic flow over a cone at high angle of attack. Being parallelisable on distributed memory multiprocessors and having an ability to tackle highly non-linear problems, it has great promise in tackling more complex practical air vehicle configurations. As a by-product of using the GMRES method, in which Hessenberg matrices are generated, the eigenvalues of the linear system can be estimated using the Arnoldi method. The spectra produced provide some insight into the behaviour of the GMRES method for different linear systems corresponding to different preconditioning and damping.

NOMENCLATURE

A	matrix of the linear system
E_t	total energy per unit volume (non-dimensionalised as defined in the text)
$\mathbf{E}_i, \mathbf{F}_i, \mathbf{G}_i$	inviscid flux vectors in Cartesian coordinates
$\mathbf{E}_v, \mathbf{F}_v, \mathbf{G}_v$	viscous flux vectors in Cartesian coordinates

$\overline{\mathbf{E}}, \overline{\mathbf{F}}, \overline{\mathbf{G}}$	flux vectors in transformed coordinates
$\overline{\mathbf{E}}_i, \overline{\mathbf{F}}_i, \overline{\mathbf{G}}_i$	inviscid flux vectors in transformed coordinates
$\overline{\mathbf{E}}_v, \overline{\mathbf{F}}_v, \overline{\mathbf{G}}_v$	viscous flux vectors in transformed coordinates
$\widetilde{\mathbf{E}}, \widetilde{\mathbf{F}}, \widetilde{\mathbf{G}}$	numerical flux vectors at cell interfaces
$\mathbf{e}$	unit vector in the SFDN method
$\mathbf{H}$	source term vector
h	increment in the SFDN method
J	determinant of the Jacobian of the coordinate transformation
J	Jacobian matrix of the nonlinear system
L	characteristic length
M	Mach number
$\mathbf{P}$	matrix projection operator in the SQN method
p	static pressure (non-dimensionalised as defined in the text)
$\mathbf{Q}$	conservative state vector
$\mathbf{q}$	physical state vector
q	heat transfer rate
$\mathbf{R}$	a vector function of $\mathbf{Q}$ derived from the spatial discretisation
Re	Reynolds number
r, θ, φ	spherical coordinates
s	limiter
T	static temperature (non-dimensionalised as defined in the text)
t	time
u, v, w	velocity components in Cartesian coordinates (non-dimensionalised as defined in the text)
x, y, z	Cartesian coordinates (non-dimensionalised as defined in the text)
μ	viscosity (non-dimensionalised as defined in the text)
ρ	density (non-dimensionalised as defined in the text)
τ	stress tensor

κ	interpolation parameter
ε	parameter in the limiter
$\Delta+$, $\Delta-$	forward and backward finite difference operators
γ	ratio of specific heats
ξ, η, ζ	transformed coordinates

Superscripts

L, R	on the left or right side of the cell interface
n	time level or iteration number
T	transpose

Subscripts

i	inviscid
j, k	grid corresponding to η and ζ
v	viscous
w	flow property on the wall
∞	free stream conditions

1. INTRODUCTION

To numerically model the viscous high speed aerodynamics of aircraft configurations cheaply enough to attract design managers will require advances on many fronts. Rapid advances are happening in computer hardware and its accompanying software utilities, yet there is scope for advances in algorithms to contribute to improvement in performance/cost as well as to embracing the new computer architectures becoming available. The CFD team at Glasgow has been working on these latter tasks whilst making use of such machines.

The existence of strong shock waves, thin shear layers and their interaction in hypersonic viscous flows requires the use of a high resolution scheme for an accurate numerical simulation. Through an extensive study[1] of different flux formulae on their capabilities of capturing both shock waves and shear layers, the Osher flux difference splitting scheme has been found to be satisfactory. However, high resolution schemes usually involve more complicated formulation and thus longer computation time per iteration as compared to the simpler central differencing scheme. Therefore, the acceleration of the convergence for high resolution schemes becomes an increasingly important issue.

In this paper, we will present a new iterative approach for fast steady state solution of Navier-Stokes equations and investigate the reasons for its favourable convergence characteristics. The performance of the approach is illustrated by applying it to the prediction of the hypersonic viscous flow over a cone at high angle of attack in which the high resolution Osher scheme is used.

2. THE GOVERNING EQUATIONS

Corresponding to the test case presented in Section 5, the governing equations are the locally conical Navier-Stokes equations, which can be derived through the general coordinate transformation

$$\begin{aligned} x &= x(\xi,\eta,\zeta) = r(\xi)\sin\theta(\eta,\zeta)\cos\varphi(\eta,\zeta) \\ y &= y(\xi,\eta,\zeta) = r(\xi)\sin\theta(\eta,\zeta)\sin\varphi(\eta,\zeta) \\ z &= z(\xi,\eta,\zeta) = r(\xi)\cos\theta(\eta,\zeta) \end{aligned} \tag{2.1}$$

to the three dimensional Navier-Stokes equations in Cartesian coordinates (x,y,z). Here $r(\xi)$ is the transformation of the radial coordinate. Functions $\theta(\eta,\zeta)$ and $\varphi(\eta,\zeta)$ represent general two dimensional transformations to fit different conical shapes and control the clustering of grid points. After applying the locally conical approximation,

i.e. the derivatives of flow properties to r are neglected, the following governing equations are obtained.

$$\frac{\partial \overline{\mathbf{F}}(\mathbf{Q})}{\partial \eta} + \frac{\partial \overline{\mathbf{G}}(\mathbf{Q})}{\partial \zeta} + \mathbf{H}(\mathbf{Q}) = 0 \qquad (2.2)$$

where

$$\mathbf{Q} = [\ \rho,\ \rho u, \rho v,\ \rho w,\ E_t\ ]^T$$

$$\overline{\mathbf{E}} = \overline{\mathbf{E}}_i - \overline{\mathbf{E}}_v$$
$$\overline{\mathbf{F}} = \overline{\mathbf{F}}_i - \overline{\mathbf{F}}_v$$
$$\overline{\mathbf{G}} = \overline{\mathbf{G}}_i - \overline{\mathbf{G}}_v$$

$$\overline{\mathbf{E}}_i = (\xi_x/J)\mathbf{E}_i + (\xi_y/J)\mathbf{F}_i + (\xi_z/J)\mathbf{G}_i$$
$$\overline{\mathbf{F}}_i = (\eta_x/J)\mathbf{E}_i + (\eta_y/J)\mathbf{F}_i + (\eta_z/J)\mathbf{G}_i$$
$$\overline{\mathbf{G}}_i = (\zeta_x/J)\mathbf{E}_i + (\zeta_y/J)\mathbf{F}_i + (\zeta_z/J)\mathbf{G}_i$$

$$\overline{\mathbf{E}}_v = (\xi_x/J)\mathbf{E}_v + (\xi_y/J)\mathbf{F}_v + (\xi_z/J)\mathbf{G}_v$$
$$\overline{\mathbf{F}}_v = (\eta_x/J)\mathbf{E}_v + (\eta_y/J)\mathbf{F}_v + (\eta_z/J)\mathbf{G}_v$$
$$\overline{\mathbf{G}}_v = (\zeta_x/J)\mathbf{E}_v + (\zeta_y/J)\mathbf{F}_v + (\zeta_z/J)\mathbf{G}_v$$

$$\mathbf{E}_i = [\ \rho u,\ \rho u^2 + p,\ \rho uv,\ \rho uw,\ (E_t + p)u\ ]^T$$
$$\mathbf{F}_i = [\ \rho v,\ \rho uv,\ \rho v^2 + p,\ \rho vw,\ (E_t + p)v\ ]^T$$
$$\mathbf{G}_i = [\ \rho w,\ \rho uw,\ \rho vw,\ \rho w^2 + p,\ (E_t + p)w\ ]^T$$

$$\mathbf{E}_v = [\ 0,\ \tau_{xx},\ \tau_{xy},\ \tau_{xz},\ u\tau_{xx} + v\tau_{xy} + w\tau_{xz} - q_x]^T$$
$$\mathbf{F}_v = [\ 0,\ \tau_{xy},\ \tau_{yy},\ \tau_{yz},\ u\tau_{xy} + v\tau_{yy} + w\tau_{yz} - q_y]^T$$
$$\mathbf{G}_v = [\ 0,\ \tau_{xz},\ \tau_{yz},\ \tau_{zz},\ u\tau_{xz} + v\tau_{yz} + w\tau_{zz} - q_z]^T$$

$$\mathbf{H} = (\ 2\overline{\mathbf{E}}_i - \overline{\mathbf{E}}_v\ )/\xi$$

In the above equations, the source term **H** is introduced from the term $\partial\overline{\mathbf{E}}/\partial\xi$ in the 3D NS equations after the locally conical approximation, J is the determinant of the Jacobian of the transformation, and the (x, y, z), (u, v, w), ρ, T, μ, p and E_t are non-dimensionalised using L, V_∞, ρ_∞, T_∞, μ_∞, $\rho_\infty V_\infty^2$ and $\rho_\infty V_\infty^2$, respectively. Then

$$p = (\gamma - 1) [E_t - \frac{1}{2} \rho (u^2 + v^2 + w^2)] \tag{2.3}$$

$$T = \gamma M_\infty^2 \, p / \rho \tag{2.4}$$

The viscosity is calculated from temperature T through the Sutherland formula.

3. THE HIGH RESOLUTION SCHEME

3.1. The Osher Flux Difference Splitting Scheme

In the cell centred finite difference or finite volume formulation, the state variables are evaluated at cell centres and represent cell-averaged values. The fluxes are evaluated at cell interfaces. The spatial derivatives are then represented as a flux balance across a cell. The diffusive fluxes are calculated at cell interfaces using a central differencing scheme. The convective interface flux is determined from a local one-dimensional model of wave interactions normal to the cell interfaces. With the flux difference splitting (FDS) model developed by Osher and Chakravarthy[2], the convective interface flux can be written as

$$\tilde{\mathbf{F}}_i = \frac{1}{2} [\overline{\mathbf{F}}_i(\mathbf{Q}^L) + \overline{\mathbf{F}}_i(\mathbf{Q}^R) - \int_{\mathbf{Q}^L}^{\mathbf{Q}^R} \left| \frac{\partial \overline{\mathbf{F}}_i}{\partial \mathbf{Q}} \right| d\mathbf{Q}] \tag{3.1}$$

where the integral in the state variable domain is carried out along a path piecewise parallel to the eigenvectors of $\partial \overline{\mathbf{F}}_i / \partial \mathbf{Q}$.

3.2. The MUSCL Approach for Higher Order Accuracy

The state-variable interpolations determine the resulting accuracy of the scheme. A κ-parameter family of higher-order schemes[3] can be written as

$$\begin{aligned} \mathbf{q}^L_{j+\frac{1}{2},\,k} &= \mathbf{q}_{j,\,k} + \{ (\tfrac{s}{4}) [(1-\kappa s)\Delta_- + (1+\kappa s)\Delta_+] \mathbf{q} \}_{j,\,k} \\ \mathbf{q}^R_{j+\frac{1}{2},\,k} &= \mathbf{q}_{j+1,\,k} - \{ (\tfrac{s}{4}) [(1+\kappa s)\Delta_- + (1-\kappa s)\Delta_+] \mathbf{q} \}_{j+1,\,k} \end{aligned} \tag{3.2}$$

where Δ_+ and Δ_- denote forward and backward difference operators, respectively, in the η direction. The parameter κ determines the spatial accuracy of the difference approximation. We choose $\kappa = 1/3$ for a third order upwind-biased scheme. The parameter s serves to limit higher-order terms in the interpolation in order to avoid oscillations at discontinuities such as shock waves in the solutions. The limiting is implemented by locally modifying the difference values in the interpolation to ensure monotone interpolation as

$$s = \frac{2\Delta_+\mathbf{q}\Delta_-\mathbf{q} + \varepsilon}{(\Delta_+\mathbf{q})^2 + (\Delta_-\mathbf{q})^2 + \varepsilon} \tag{3.3}$$

where ε is a small number preventing division by zero in regions of null gradients.

4. THE SFDN–α-GMRES AND SQN–α-GMRES METHODS

After the above discretisation and proper treatment at the domain boundaries, a large sparse nonlinear system results, which we denote as

$$\mathbf{R}(\mathbf{Q}) = 0 \tag{4.1}$$

4.1 Discussion

For steady state problems, a time dependent approach is often employed, which can be written as

$$\frac{\partial \mathbf{Q}}{\partial t} + \mathbf{R}(\mathbf{Q}) = 0 \tag{4.2}$$

Using a fully implicit method, e.g. the backward Euler implicit method,

$$\left[\frac{1}{\Delta t} I + \left(\frac{\partial \mathbf{R}}{\partial \mathbf{Q}}\right)^n\right]\Delta\mathbf{Q}^n = -\mathbf{R}(\mathbf{Q}^n) \tag{4.3}$$

unconditional stability can be achieved and as the time step approaches infinity the method approaches the Newton method

$$\left(\frac{\partial \mathbf{R}}{\partial \mathbf{Q}}\right)^n \Delta \mathbf{Q}^n = -\mathbf{R}(\mathbf{Q}^n) \tag{4.4}$$

for the solution of the nonlinear system (4.1). In practical applications to CFD problems, however, it is very difficult (1) to get the analytical Jacobian of the nonlinear system for a high order high resolution scheme for viscous flows (it is almost impossible if turbulence or chemical reactions are involved) and (2) to solve the resulting large sparse nonsymmetric linear system efficiently. Previous researchers in CFD have tried to avoid these two difficulties in the following ways respectively: (1) to construct simplified implicit operators, e.g. to use only first order inviscid implicit operators; (2) to use approximate factorization for the multidimensional implicit operator so that the resulting linear systems can be solved easily. Both of these naturally negate the advantages of the fully implicit scheme. Therefore the time step size is still limited due to the inconsistency of the implicit operator and the right hand side (the nonlinear system) and the factorization error which increases with the time step. Simplified implicit methods will thus obviously not approach a Newton iterative method as the time step approaches infinity.

4.2 The SFDN and SQN nonlinear iterative methods

Instead of avoiding the difficulties for a fully implicit method, Qin and Richards[4,5] tackled the problem directly in order to achieve fast convergence for the steady state solution. The discretisation of the Navier-Stokes equations results in a large sparse nonlinear system to be solved, which can be considered as a fully implicit scheme with an infinite time step. Viewing the Navier-Stokes solution as the solution of a large

sparse nonlinear system, we derived a fast convergence algorithm which is general and robust.

The algorithm is based on the Newton iterative method. Due to the complexity of the nonlinear system, an analytical expression for the Jacobian matrix is usually not obtainable. Therefore, we then took the following two approaches: (1) the sparse finite difference Newton(SFDN) method[6]; and (2) the sparse quasi-Newton(SQN) method[7].

The SFDN method calculates numerically the Jacobian of the nonlinear system. Making use of its structured sparsity, Qin and Richards[4] devised a practical way of calculating the Jacobian using finite differences. If we take the present 2-D as an example, the above higher order spatial discretisation will result in a 13-point stencil (Fig.1). In the calculation of the Jacobian, we can minimize the number of calculations of $\mathbf{R}(\mathbf{Q})$ in the following way. Because the discretisation has a 13-point stencil, we can perturb one of the five state variables by a local increment $h^l_{i,j}$ at every 5 points in both coordinate directions in one evaluation of $\mathbf{R}(\mathbf{Q})$, i.e. we calculate

$$\mathbf{R}(\mathbf{Q} + \sum_{\substack{i=m,I,5 \\ j=n,J,5}} h^l_{i,j}\mathbf{e}^l_{i,j}), \quad l=1,5;\ m=1,5;\ n=1,5 \tag{4.5}$$

where $\mathbf{e}^l_{i,j}$ is the unit vector at point (i,j) for the lth component of the state. Therefore we can get the finite difference approximation of the Jacobian column by column through a total number of 125 evaluations of $\mathbf{R}(\mathbf{Q})$. If the increments are properly chosen according to machine zero and the rounding errors in calculating $\mathbf{R}(\mathbf{Q})$, the SFDN method can still give a quadratic convergence rate as has been shown by Dennis and Schnabel[8].

The SQN method updates an approximation to the Jacobian from the solution of the linear system and the value of $\mathbf{R}(\mathbf{Q})$ available. It is an extension of the quasi-Newton method to nonlinear systems with sparse Jacobians. To keep the sparsity structure of the Jacobian, only those non-zero elements are updated through a matrix projection operator $\mathbf{P}_J$, which maps a matrix M to a matrix retaining only those non-

zero elements according to the sparsity structure of the Jacobian. The updating procedure can be written as

$$
\begin{aligned}
A^n \Delta \mathbf{Q}^n &= -\mathbf{R}(\mathbf{Q}^n) \\
\mathbf{Y}^n &= \mathbf{R}(\mathbf{Q}^{n+1}) - \mathbf{R}(\mathbf{Q}^n) \\
\Delta A^n &= \mathbf{P}_J[D^+(\mathbf{Y}^n - A^n \Delta \mathbf{Q}^n)(\Delta \mathbf{Q}^n)^T] \\
A^{n+1} &= A^n + \Delta A^n
\end{aligned}
\tag{4.6}
$$

where D^+ is a diagonal matrix which is determined from the linear solution $\Delta \mathbf{Q}^n$ and the sparsity structure of the Jacobian matrix. One can see that there is no extra evaluation of $\mathbf{R}(\mathbf{Q})$ involved in updating the approximation. It has been proven that the SQN method has a superlinear convergence rate[8]. Qin and Richards[4,5] formulated its application to nonlinear systems with sparse block structured Jacobian matrices arising from Euler and Navier-Stokes solutions.

It is obvious that the SFDN method requires much more computing time in generating the Jacobian approximation as compared to the SQN method in which the computing time for generating the Jacobian approximation is negligible. On the other hand, the difference between quadratic convergence and superlinear convergence can be significant in practical applications because a large amount of computing time has to be spent in solving the large sparse nonsymmetric linear system at each iteration.

4.3. The α-GMRES linear solver

After the linearization of the nonlinear system, a large sparse nonsymmetric linear system results, either (4.4) for the SFDN method or (4.6) for the SQN method, which we denote as

$$
A\mathbf{x} = \mathbf{b} \tag{4.7}
$$

For a 2-D case, A is a block 13-point diagonal structured sparse matrix as shown in Fig.2.

One of the most successful methods for solving large sparse nonsymmetric linear systems is the GMRES (Generalized Minimal RESidual) method[9], which generally requires preconditioning of the matrix for practical problems. Direct use of the GMRES method to the present problem (4.7) produced nonconvergent results. A simple block diagonal preconditioning improved the results very little in convergence. Based on these observations, Xu et al.[10] proposed a new efficient multilevel iterative method, the α-GMRES method for the solution of the sparse nonsymmetric linear system. The matrix is first preconditioned by the inverse of its block diagonal matrix and a parameter α ($0<\alpha<1$) is added to the diagonal to further improve the matrix property enabling a successful application of GMRES method. Thus a multi-level iterative solver results, which is written as

$$(\alpha I + D^{-1}A)\,\mathbf{x}^{k+1} = D^{-1}\mathbf{b} + \alpha\mathbf{x}^{k} \tag{4.8}$$

where D is the block diagonal matrix of A. We have proven the existence of a value of α ($0<\alpha<1$) such that the above iterative procedure will converge[10]. In practical application, the parameter α is determined by a balanced convergence of the GMRES inner loop and the outer loop, which is found to be around 0.1 for the test cases. Another promising aspect of the α-GMRES is that it can easily be parallelised for distributed memory parallel computers, which has been demonstrated in Ref. 10.

Combining the α-GMRES linear solver with the nonlinear SQN and SFDN methods, we have thus devised fast convergent solvers for Navier-Stokes solutions, which we have named the SFDN-α-GMRES and SQN-α-GMRES methods respectively.

5. NUMERICAL EXAMPLES

The test case chosen is a hypersonic viscous flow around a sharp cone at high angle of attack. The flow is modelled by the Locally Conical Navier-Stokes equations, which is discretised using the Osher flux difference splitting scheme for the inviscid fluxes and a central differencing scheme for the viscous terms. The resulting nonlinear system is then solved by the SQN-α-GMRES method or the SFDN-α-GMRES method. In the present tests, we choose α=0.1 and the Krylov subspace dimension in the GMRES method as 30 and 50 for 33×33 or 66×66 girds respectively. To produce a starting solution suitable for an effective application, we use a time dependent approach for the initial phase, in which a Runge-Kutta method with local time stepping is employed. The computation was carried out on the IBM RISC System/6000 320H workstation.

Fig.3 shows the flow conditions and the cross sectional view of the solved flowfield, in which the strong bow shock wave on the windward side and the separated shear layer on the leeward side can clearly be seen.

Fig.4 plots the convergence against computing time for calculations using the SQN-α-GMRES method or the SFDN-α-GMRES method on a 33×33 grid. As can be seen, the convergence for the explicit scheme is typically slow even though local time stepping has already been employed for efficiency. After switching to the SFDN-α-GMRES method or the SQN-α-GMRES method, the solutions converges quadratically or superlinearly respectively and the residuals reduce to machine zero in 4 or 8 iterations. For this particular case, the two methods produce similar efficiency but the SQN-α-GMRES method is expected to be more promising for problems involving more complicated physics when the expense in evaluating $\mathbf{R}(\mathbf{Q})$ is much higher.

In Fig. 5, we show a test on a 66×66 grid using different convergence criteria for the iterative linear solver. We do not need to solve the linear systems (4.8) using the GMRES method or (4.7) using the α-GMRES method to a high accuracy as long as a reasonable convergence in the nonlinear iteration can be achieved. In Fig. 5, e1 and e2 represent the convergence criteria for the solution of (4.8) and (4.7) respectively. As can be seen, a larger convergence criterion can save computing time in the linear solver

and it will also degrade the convergence rate of the outer nonlinear iteration. An optimum choice can be made through numerical experiments.

6. APPROXIMATE EIGENVALUE ANALASIS

Spectra analysis of the Jacobian matrix or the iterative matrix can provide further understanding of the stability of the time dependent approach or the convergence of the iterative scheme. For this analysis, the eigensystem has to be solved. A practical way is to estimate a desired subset of the eigenvalues using Arnoldi's process[11]. Using such a method, Eriksson and Rizzi[12] analysed the influence of local time stepping and artificial viscosity on the convergence of a time dependent approach to the steady state solution for transonic flows. Cheer et al.[13] studied the effect of a number of numerical parameters on the convergence of a time dependent approach to the steady state solution for a nozzle problem, where they studied the iterative matrix instead of the Jacobian.

Saad and Shultz established a convergence theorem[9] which relates the convergence of the GMRES method to the spectrum of the linear system to be solved. The theorem gave an upper bound for the residual which is proportional to $(D/d)^{\nu}(R/C)^{m-\nu}$, where ν is the number of the eigenvalues of A with nonpositive real parts, R is the radius of a circle centred at C enclosing all the other eigenvalues, D is the maximum distance between any two eigenvalues with positive and negative real parts respectively and d is the minimum distance to the origin of any eigenvalues with negative real part. Based on the theorem, we can use the Arnoldi method to analyse the spectrum of the matrices in the above Newton-like linearisation of the high resolution discretisation and the influence of preconditioning and damping on the spectrum and thus the convergence of the GMRES method. As the GMRES method was derived from the Arnoldi process, we can calculate the eigenvalues of the Hessenberg matrix available in the GMRES procedure to obtain the desired approximate subset of eigenvalues.

In Fig.6 and Fig.7, we plot out the approximate spectra of the Jacobian and the matrix after block diagonal preconditioning with a damping factor $\alpha = 0.1$. The Arnoldi

method needs a starting vector to define the Krylov subspace. We used random vectors for this and found out that the calculated eigenvalue spectra were not sensitive to this starting procedure. As indicated in Ref.14, the best accuracy is first obtained for those eigenvalues which lie on the outer part of the spectrum. So practically the dimension of the Krylov subspace can be chosen reasonably small compared to the dimension of the system if only the outer part of the spectrum is of interest. In the present estimation, we have chosen the Krylov subspace dimension as 30 and 70. Comparing Fig.6(a) and Fig.7(a) with Fig.6(b) and Fig.7(b) respectively, we find that those eigenvalues on the outer part of the spectra are very close for these two significantly different subspace dimensions, which indicates that with these subspace dimensions we obtained a reasonably good approximate subset of the eigenvalues for the outer part of the spectra.

Comparing Fig.6 and Fig.7, it can be clearly seen that the block diagonal preconditioning has a strong effect on changing the distribution of the eigenvalues and moves most of the eigenvalues around the point (1, 0) and along the line $Re(\lambda)=1$. The damping will not change the relative position of the eigenvalues but only shift all the eigenvalues to the right by a distance of α. Analysis based on the convergence theorem indicates that both the preconditioning and damping have positive effects on the convergence of the GMRES procedure, which agrees with our numerical results.

7. CONCLUSIONS

The SFDN–α-GMRES and SQN–α-GMRES methods presented in this paper have provided a new approach for fast steady state Navier-Stokes solutions, when complexity from using high resolution schemes produces slow convergence using conventional time-dependent approaches and when the analytical Jacobian is difficult to obtain. In comparison, both of the methods produce similar improvement over the corresponding explicit method in computing time for the test case. It is planned to investigate these methods further in parallel to calculate three-dimensional cases including turbulent modelling and/or real gas effects. Approximate eigenvalue analysis using Arnoldi's method revealed the effects of preconditioning and damping on the

eigenvalue spectra and thus the convergence of the GMRES method. Being also parallelisable on distributed memory multiprocessors, the methods will be well suited for the new generation of the high performance parallel computers. This work will contribute to the achievement of efficient, accurate Navier-Stokes solvers for complex aerospace applications.

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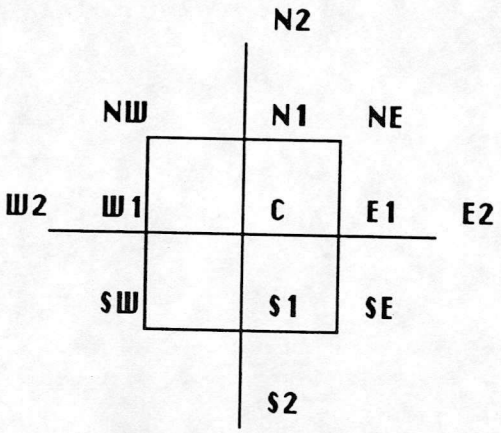


Fig. 1: Discretisation stencil using the high resolution scheme

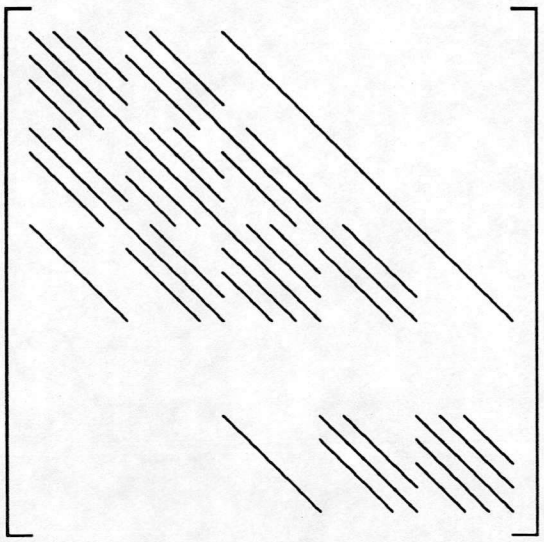


Fig. 2: Sparsity pattern of the Jacobian matrix

10° Cone

AoA = 24°

$M_\infty = 7.95$

$T_\infty = 55.4$ K

$T_W = 309.8$ K

$Re_\infty = 4.1 \times 10^6$

r = 0.1 m

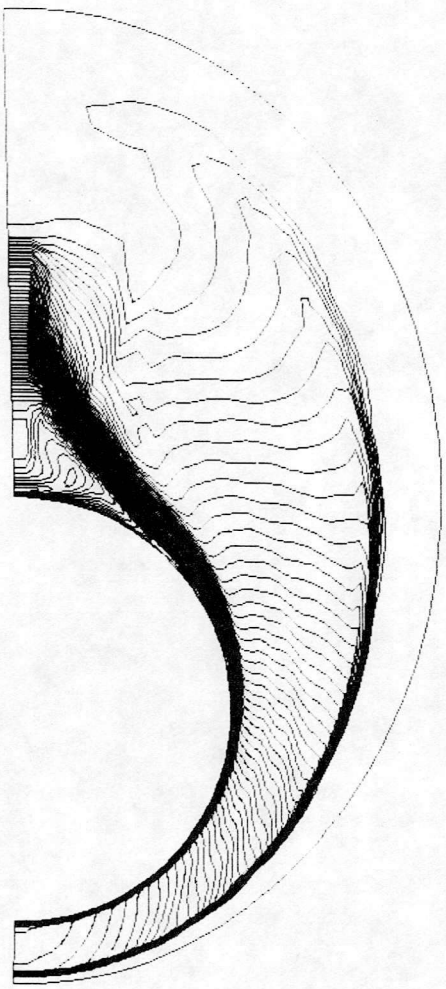


Fig. 3: Crossflow temperature contours of the test case

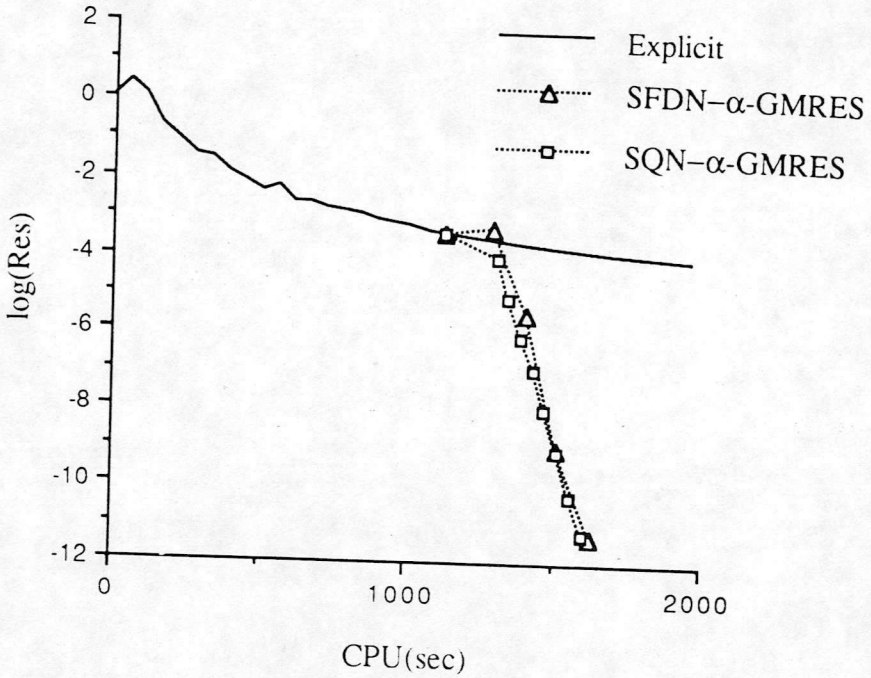


Fig. 4: Convergence of the SFDN–α-GMRES and SQN–α-GMRES methods as compared with the Runge-Kutta explicit method (grid 33×33)

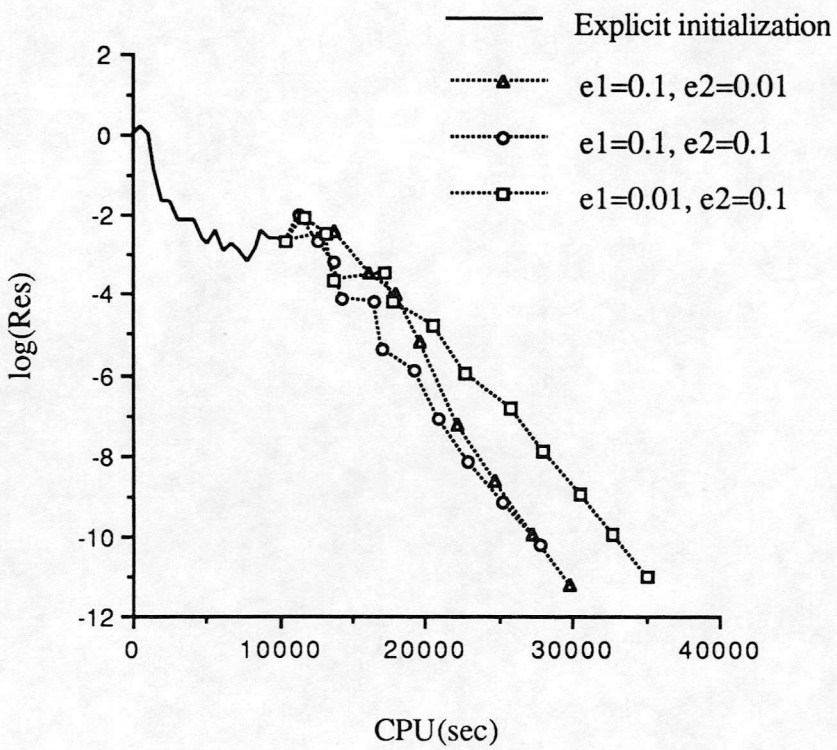
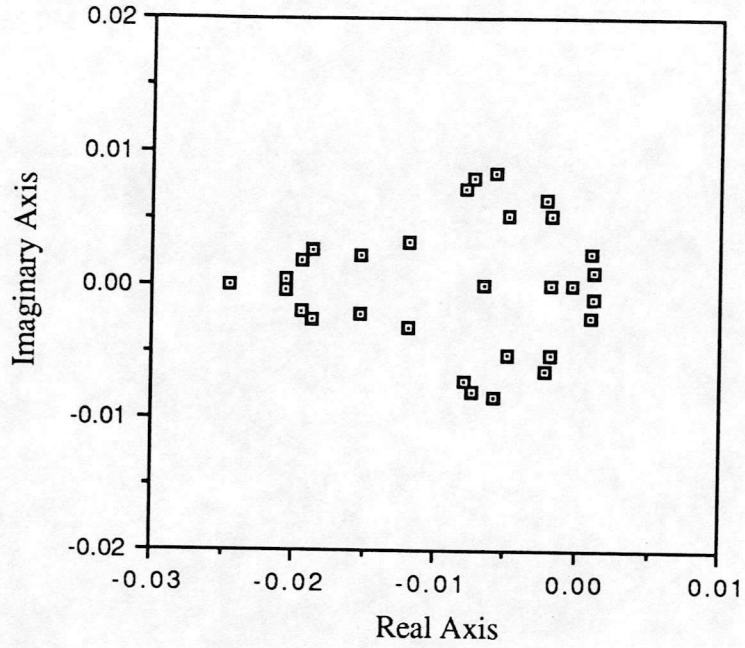
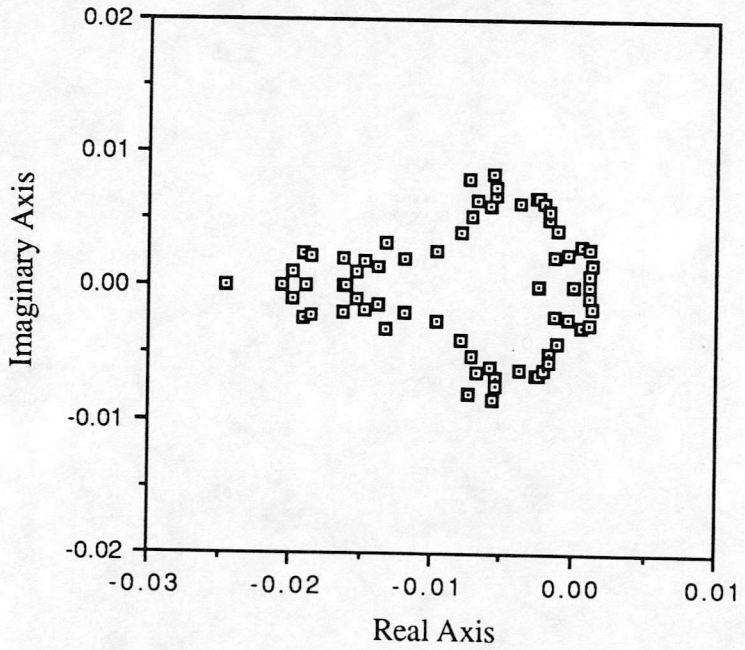


Fig.5 Parameter tests for the SFDN–α-GMRES method (grid 66×66).

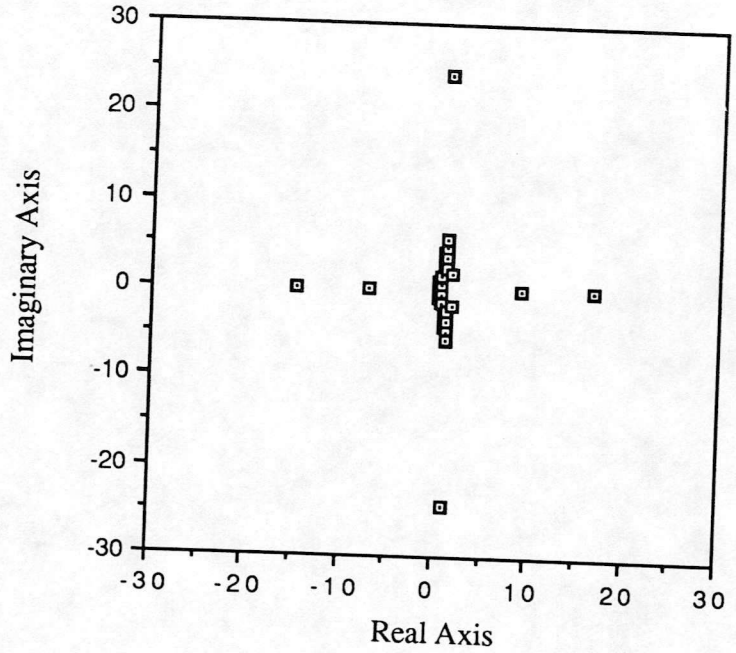


(a)

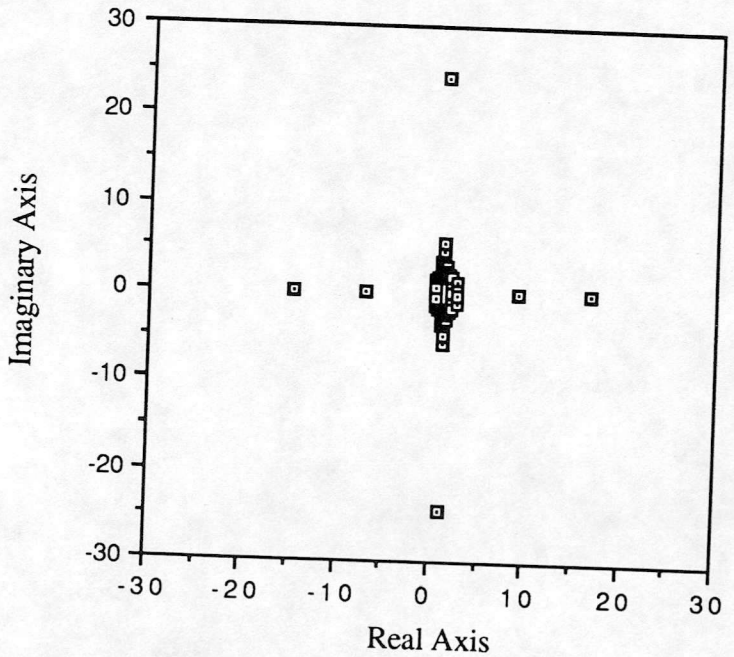


(b)

Fig. 6. Eigenvalue spectrum before preconditioning and damping (a) m=30; (b) m=70



(a)



(b)

Fig. 7. Eigenvalue spectrum after preconditioning and damping (a) m=30; (b) m=70