Bounds on Entanglement Assisted Source-channel Coding Via the Lovász ϑ Number and Its Variants*

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Abstract

We study zero-error entanglement assisted source-channel coding (communication in the presence of side information). Adapting a technique of Beigi, we show that such coding requires existence of a set of vectors satisfying orthogonality conditions related to suitably defined graphs G and H. Such vectors exist if and only if $\vartheta(\overline{G}) \leq \vartheta(\overline{H})$ where ϑ represents the Lovász number. We also obtain similar inequalities for the related Schrijver ϑ^- and Szegedy ϑ^+ numbers.

These inequalities reproduce several known bounds and also lead to new results. We provide a lower bound on the entanglement assisted cost rate. We show that the entanglement assisted independence number is bounded by the Schrijver number: $\alpha^*(G) \leq \vartheta^-(G)$. Therefore, we are able to disprove the conjecture that the one-shot entanglement-assisted zero-error capacity is equal to the integer part of the Lovász number. Beigi introduced a quantity β as an upper bound on α^* and posed the question of whether $\beta(G) = \lfloor \vartheta(G) \rfloor$. We answer this in the affirmative and show that a related quantity is equal to $\lceil \vartheta(G) \rceil$. We show that a quantity $\chi_{\text{vect}}(G)$ recently introduced in the context of Tsirelson's conjecture is equal to $\lceil \vartheta^+(\overline{G}) \rceil$.

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1 Introduction

The *zero-error source-channel coding* problem is as follows. Suppose Alice wishes to send a message $x \in X$ to Bob through a noisy classical channel $\mathcal{N} : S \to V$ in such a way that Bob may deduce Alice's message with zero probability of error. Alice encodes her message via some function $f : X \to S$ before sending it through the channel. Bob is aided by some side

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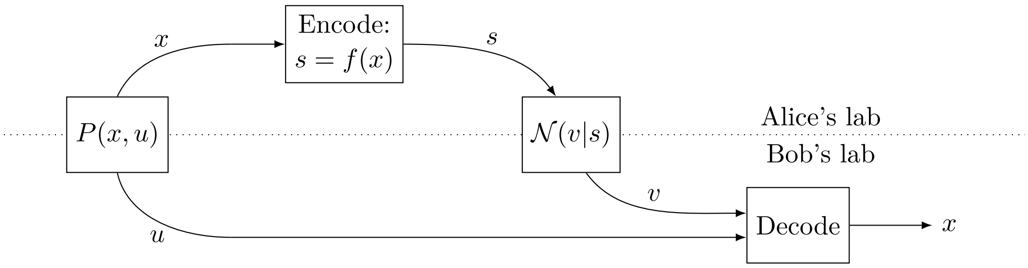

Figure 1 A zero-error source-channel coding scheme.

information $u \in U$ regarding Alice's message. Formally, we can imagine that the symbols x and u originate from a *dual source* with probability $P(x, u)$. See Fig. 1.

The success of this protocol can be analyzed using a pair of graphs: G with vertices from X and H with vertices from V, having edges

$$x \sim_G y \iff \exists u \in U \text{ such that } P(x,u)P(y,u) \neq 0 \tag{1}$$

$$s \sim_H t \iff \mathcal{N}(v|s)\mathcal{N}(v|t) = 0 \text{ for all } v \in V, \tag{2}$$

where $P(x, u)$ is the probability of input pair x, u and $\mathcal{N}(v|s)$ is the probability that the channel outputs v given input s. G is the *characteristic graph* of P and H is the complement of the *confusability graph* of $\mathcal{N}$. Intuitively, G represents the information that needs to be sent and H represents the information that survives the channel. Bob is able to decode x (with zero chance of error) if and only if Alice's encoding satisfies $x \sim_G y \implies f(x) \sim_H f(y)$ [11]. Such a function is called a *homomorphism* from G to H. If such a function exists then G is *homomorphic* to H, written $G \to H$.

Many graph quantities can be defined in terms of homomorphisms [8, 9], and the above protocol puts these in an operational context. If there is no side information then $G = K_n$, the complete graph on $n = |X|$ vertices. The largest n such that $K_n \to H$ is the clique number $\omega(H)$. Thus the largest number of error-free messages that can be sent through $\mathcal{N}$ is $\omega(H)$ (equivalently, $\alpha(\overline{H})$), the independence number of the complementary graph). If $\mathcal{N}$ is the perfect channel then $H = K_n$ with $n = |S|$. The smallest n such that $G \to K_n$ is the chromatic number $\chi(G)$. This is the size of the smallest channel that suffices to communicate inputs from a dual source with characteristic graph G.

Source-channel coding may also be considered in the case where Alice and Bob make use of an entanglement resource, Fig. 2 [3]. Now Alice's encoding operation consists of a POVM $\{M_s^x\}_{s \in S}$ depending on her input x and producing a value s to be sent to Bob through the channel. Bob can successfully decode if and only if

$$\rho_s^x \perp \rho_t^y \text{ for all } x \sim_G y \text{ and } s \not\sim_H t, \tag{3}$$

where ρ_s^x is Bob's share of the post-measurement entanglement resource after POVM outcome M_s^x. By analogy to the above, a successful protocol is called an *entanglement assisted homomorphism* from G to H. If such a thing exists, one writes $G \xrightarrow{*} H$. Also by way of analogy, the *entanglement assisted independence number* $\alpha^*(\overline{H})$ is the largest n such that $K_n \xrightarrow{*} H$ and the *entanglement assisted chromatic number* $\chi^*(G)$ is the smallest n such that $G \xrightarrow{*} K_n$. These have similar operational interpretations as $\alpha(\overline{H})$ and $\chi(G)$ discussed above.

We consider two relaxations of condition (3) for $G \xrightarrow{*} H$. The first we denote $G \xrightarrow{B} H$ since it reduces to a construction of Beigi [2] when $G = K_n$. We say $G \xrightarrow{B} H$ if there are vectors $|w\rangle$ and $|w_s^x\rangle$ such that

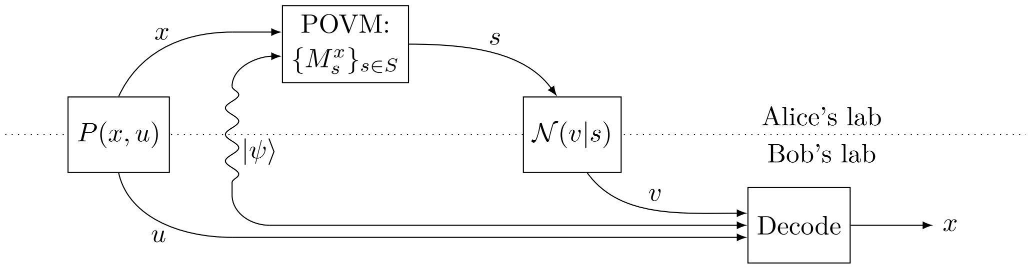

Figure 2 An entanglement assisted zero-error source-channel coding scheme.

1. $\langle w|w\rangle = 1$
2. $\sum_s |w_s^x\rangle = |w\rangle$
3. $\langle w_s^x|w_t^y\rangle = 0$ for all $x \sim_G y$, $s \not\sim_H t$
4. $\langle w_s^x|w_t^x\rangle = 0$ for all $s \neq t$.

Another relaxation $G \overset{+}{\to} H$ is defined similarly, except that the last condition is replaced by

4. $\langle w_s^x|w_t^y\rangle \geq 0$.

Since these are relaxed conditions, $G \overset{*}{\to} H$ implies $G \overset{B}{\to} H$ and $G \overset{+}{\to} H$. All of our results follow from two theorems. With $\bar{\vartheta}(G)$, $\bar{\vartheta}^-(G)$, and $\bar{\vartheta}^+(G)$ being the Lovász, Schrijver, and Szegedy numbers of the complementary graph $\overline{G}$, we have

▶ **Theorem 1.** *$G \overset{B}{\to} H$ if and only if $\bar{\vartheta}(G) \leq \bar{\vartheta}(H)$.*

▶ **Theorem 2.** *If $G \overset{+}{\to} H$ then $\bar{\vartheta}(G) \leq \bar{\vartheta}(H)$, $\bar{\vartheta}^-(G) \leq \bar{\vartheta}^-(H)$, and $\bar{\vartheta}^+(G) \leq \bar{\vartheta}^+(H)$.*

A number of original results follow as immediate corollaries:

- Entanglement assisted zero-error source-channel coding ($G \overset{*}{\to} H$) requires $\bar{\vartheta}(G) \leq \bar{\vartheta}(H)$, $\bar{\vartheta}^-(G) \leq \bar{\vartheta}^-(H)$, and $\bar{\vartheta}^+(G) \leq \bar{\vartheta}^+(H)$.
- $\alpha^*(H) \leq \vartheta^-(H)$ (previously only $\alpha^*(H) \leq \vartheta(H)$ was known [2, 4]).
- The average number of channel uses required per input, in the asymptotic limit, is known as the *entanglement assisted cost rate* $\eta^*(G, \overline{H})$. Since $\bar{\vartheta}$ is multiplicative under appropriate graph products, $\eta^*(G, \overline{H}) \geq \log \bar{\vartheta}(G) / \log \bar{\vartheta}(H)$.
- Beigi defined $\beta(\overline{H})$ to be the largest n such that $K_n \overset{B}{\to} H$ (paraphrased into our terminology) and asked whether $\beta(\overline{H}) = \lfloor \bar{\vartheta}(H) \rfloor$. The answer is "yes" – this follows directly from Theorem 1.
- By considering instead $G \overset{B}{\to} K_n$ one can define a quantity similar to Beigi's, equal to $\lceil \bar{\vartheta}(H) \rceil$.

Also as immediate corollaries, we reproduce the following known results:

- $\chi^*(G) \geq \bar{\vartheta}^+(G)$ [3].
- There is a notion of a *quantum homomorphism* $G \overset{q}{\to} H$ defined in the context of a quantum pseudo-telepathy game [14, 13]. Since $G \overset{q}{\to} H \implies G \overset{*}{\to} H \implies G \overset{+}{\to} H$, the inequalities of Theorem 2 apply to $G \overset{q}{\to} H$ as well.

These various generalized homomorphisms can be arranged in a sequence of most to least strict:

$$G \to H \implies G \overset{q}{\to} H \implies G \overset{*}{\to} H \implies G \overset{+}{\to} H \implies G \overset{B}{\to} H. \tag{4}$$

It is known that the converse of the first implication does not hold [5, 14], and we show the converse of the last does not hold. The other two are open. The second converse holds if and only if entanglement assisted source-channel coding can always be accomplished using projective measurements and a maximally entangled state. The third converse holds if, loosely speaking, it is permissible to drop all mathematical structure from (3) except for the basic properties related to inner products $\langle \rho_s^x, \rho_t^y \rangle$.

It is not known whether there can be a gap between the asymptotic entanglement assisted zero-error capacity Θ^* and ϑ. To show such a gap requires a stronger bound on α^*. Since Beigi's β is now shown to be essentially no different from ϑ, this dashes the hope that β could be used to show such a gap. Our bound $\alpha^*(H) \leq \vartheta^-(H)$ would imply a gap, unless ϑ^- regularizes to ϑ in the asymptotic limit. Haemers provided a bound on Shannon capacity which is sometimes stronger than Lovász's bound [6, 7, 1, 12]; however, this bound does not apply to the entanglement assisted case [10].

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