Inventory and pricing management in probabilistic selling

PhD Thesis

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ABSTRACT

Context: Probabilistic selling is the strategy that the seller creates an additional probabilistic product using existing products. The exact information is unknown to customers until they receive the probabilistic products. This strategy is still a relatively new area for both researchers and practitioners. Many of the corresponding operations problems need to be solved to take full advantage of the opportunity of this innovative marketing strategy. However, limited attention has been paid to examining the inventory management of probabilistic selling from the perspective of Operations Management, which cannot meet the needs of decision-making in reality.

Objectives: Considering different characteristics of the probabilistic product, the buyer, and the seller involved in probabilistic selling, i.e., the probabilistic product form, the buyers‚Äô behaviours of demand switch and barter exchange, and the seller's product allocation behaviour, we establish models and solve the decision problems of pricing, inventory, joint decision of pricing-inventory, and product allocation, etc. Based on the analysis of optimal decisions and strategy comparison results, we shed some lights on the effectiveness of probabilistic selling on managing uncertainty, and its profitability.

Method: First, we analyze the practice scenarios of probabilistic selling. Next we mainly use newsvendor inventory model, hotelling model, and optimization theory to model, solve, and analyze the operational problems. Then we give some analytical results. Next we conduct the numerical analysis using softwares of Matlab and Mathematica. Finally, we provide insightful managerial implications for the practice of probabilistic selling.

Results: The thesis derives the optimal operational decisions of inventory order, pricing, inventory allocation, and product line design in probabilistic selling. Overall, the analysis of the results show that probabilistic selling can benefit the seller with higher expected profit by reducing demand/supply uncertainty and improving inventory efficiency. The performance of probabilistic selling is closely dependent on customers' price sensitivity, product similarity, and uncertainty level, etc. Main results considering different research scenarios are as follows:

1) When the price for the probabilistic product is independent on demand reshape, a proper cannibalization can benefit the retailer in terms of yielding a higher expected profit. Probabilistic selling is more profitable with relatively lower product similarity and higher price-sensitive customers, while inventory substitution strategy outperforms probabilistic selling with higher product similarity.

2) When the price for the probabilistic product is dependent on demand reshape, probabilistic selling can benefit the seller with higher expected profit and lower inventory. Probabilistic selling is more profitable with lower product differentiation, higher customers' price sensitivity, and higher demand uncertainty. Improper pricing would undermine the seller's profit.

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3) When the seller offers physical probabilistic product, he can benefit from two effects, namely the risk pooling effect due to demand reshape and the risk diversification effect due to inventory flexibility.

4) When the seller offers barter choice in probabilistic selling, he may benefit from the marketing effect in the barter process. Offering barter choice can broaden the application range of probabilistic selling, which will increase with successful barter probability.

Conclusions/Implications: First, the thesis helps sellers understand how to manage their inventory, pricing and related implementation issues to take full advantage of probabilistic selling. Second, this thesis explores the mechanism of this innovative marketing strategy as an inventory management tool to combat uncertainty which also riches the literature on Operations Management, especially inventory management.

I declare that the work in this PhD thesis was carried out in accordance with the regulations of the Universitat Politècnica de Catalunya - BarcelonaTech and the requirements of the Ph.D. program in Business Administration and Management in the Department of Management. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

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INTRODUCTION

1.1 Background

Opaque selling is becoming an emerging marketing strategy which first appeared in travel industry. There are three prominent examples in travel industry to execute opaque selling: Priceline, Hotwire and Germanwings. They also represent three different types of opaque selling in literature: Name-Your-Own-Price (NYOP), fixed opaque product (FOP), and variable opaque products (VOP). NYOP is one type of opaque selling for some information about products (such as suppliers, specific address, etc.) are unknown before customers confirm their purchases. Priceline.com is the first executor of NYOP, which integrates service products from many providers and then offers the consumers a right to bid the price of the service products including hotels, flights and rental cars (Bai et al., 2015). Priceline will look for a product to match customers' offer and charge the credit card once accept their offers. The second selling form is that some other attributes except for price are concealed from customers. For example, intermediary like Hotwire.com as shown in Fig. 1.1 would withhold information of some attributes of the service (e.g., specific address of hotels or name of airline company) until customers finish the purchases (Jiang, 2007). The third method named as variable opaque product (VOP) gives the customers the right to choose the level of uncertainty (Post, 2010; Post and Spann, 2012). For example, if you want to travel, the Germanwings.com allows you to select your departure airport and the theme you prefer for the trip by choosing one among those offered products. The destination is unknown and you can exclude certain destinations for charge. This method gives more flexibility to customers and decrease the uncertainty. To facilitate comparison, we define the second form of opaque selling as fixed opaque product (FOP).

The successful achievement in travel industry has inspired the interest of practitioners of retailing industry (e.g., Tmall.com, jd.com, AgonSwim.com, Littleblackbag.com). From the
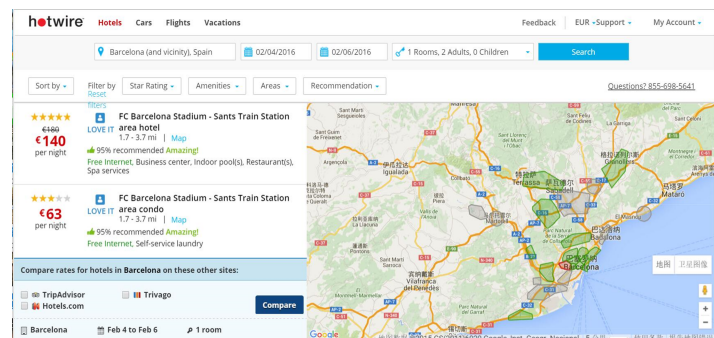
Figure 1.1: Hotwire.com website (The names of hotels are hidden)

perspective of an online retailer, Fay and Xie (2008) firstly defined probabilistic selling as "a seller who creates probabilistic goods using existing distinct products or services and offers the probabilistic product as customer's additional choice" and examined why, when, and how a seller can benefit from offering the probabilistic product. Fig. 1.2 gives an example of probabilistic selling.

Most of existing literature consider probabilistic selling the same as opaque selling because both strategies encourage the sellers to offer opaque products. However, the great difference between probabilistic selling scenario and the above three opaque selling forms is the creation method of opaque products. Opaque intermediaries which are common in travel industry generates the opaque product by mixing the products from different service providers. While in retailing industry, the sellers usually generate the opaque product by mixing their existing products. Furthermore, the capacity of service products in travel industry is usually fixed. In contrast, the seller has to make inventory decision of the products in retailing industry.

Considering the significant difference of operations management between opaque product in travel industry and opaque product in retailing industry, we consider probabilistic selling as one form of opaque selling and focus on probabilistic selling strategy in the thesis. In order to avoid ambiguity, we define the products sold with unknown information as "probabilistic product" or "opaque product", and the products sold with full information are named as "specific product" or "component product" throughout the thesis.

Although motivated by marketing tools to price discriminate among customers, expand market and improve margin price, opaque selling has also been proved by limited research to benefit sellers on inventory management (Fay and Xie, 2011, 2014; Wu and Wu, 2015). However, most of the rare operational research about opaque selling focus on the revenue management solving control mechanism and pricing issues with fixed capacity (Gallego and Phillips, 2004; Petrick et al., 2012; Gönsch and Steinhardt, 2013). The research that considers retailer's inventory decision with stochastic demand when offering probabilistic products is rare. The effect of this selling strategy on inventory management, supply chain issues are confusing but interesting.

Overall, in terms of probabilistic selling, this strategy is still a relatively new area for both
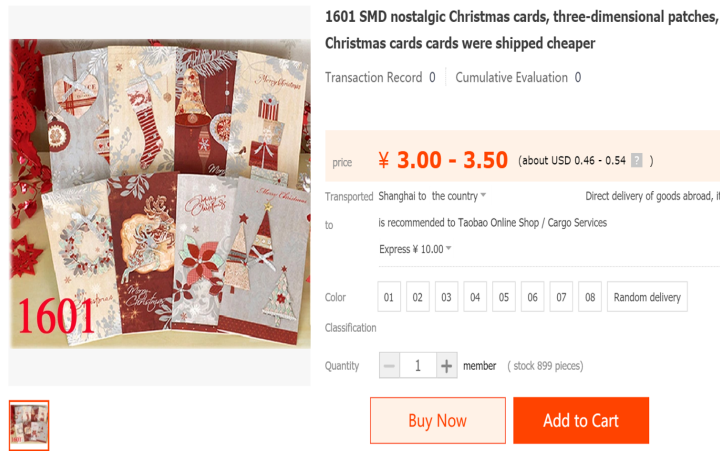
Figure 1.2: Probabilistic selling in Taobao.com

researchers and practitioners, though an emerging literature suggests that probabilistic selling is promising and has broad potential applications. Many of the corresponding operations problems need to be solved to take full advantage of the opportunity of this innovative marketing strategy.

1.2 Research objectives

Based on the urgent needs, research gaps and significance in practice application, we focus on the inventory mechanism of probabilistic selling under different practical circumstances. We seek to provide further insights into the probabilistic selling strategy in order to help sellers better understand how to manage their inventory, pricing and related implementation issues when introducing probabilistic selling. This part we will introduce the specific research motivations, research lines, and results.

1.2.1 Managing demand uncertainty: Probabilistic selling versus inventory substitution

Demand variability is prevailing in the current rapidly changing business environment, which makes it difficult for a retailer that sells multiple substitutable products to determine the optimal inventory. To combat demand uncertainty, both strategies of inventory substitution and probabilistic selling can be used. Although the two strategies differ in operation, we believe that they share a common feature in combating demand uncertainty by encouraging some customers to give up some specific demand for the product to enable demand substitution. It is interesting to explore which strategy is more advantageous to the retailer. We endogenize the inventory decision and demonstrate the efficiency of probabilistic selling through demand substitution. Then we analyze some special cases without cannibalization, and computationally evaluate the profitability and inventory decisions of the two strategies in a more general case

to generate managerial insights. The results show that the retailer should adjust inventory decisions depending on products' substitution possibility. The interesting computational result is that probabilistic selling is more profitable with relatively lower product similarity and higher price-sensitive customers, while inventory substitution outperforms probabilistic selling with higher product similarity. Higher demand uncertainty will increase the profitability advantage of probabilistic selling over inventory substitution.

1.2.2 Inventory decision in probabilistic selling with price-dependent demand reshape

By considering that the demand switch from the higher-priced specific products to the lower-priced probabilistic product depends on the price gap, we examine the optimal inventory decision and expected profit in probabilistic selling. We investigate how probabilistic selling benefit the seller through demand reshape and demand substitution in this circumstance. We perform a simulation study to extensively explore the effects of demand uncertainty, demand correlation, price sensitivity, and price discount on inventory decisions and profitability of probabilistic selling. The results show that probabilistic selling can benefit the seller with higher expected profit and lower inventory by reducing demand uncertainty and improving inventory efficiency, even without considering the increased demand due to offering the low-priced probabilistic product. Moreover, the effect of probabilistic selling is more significant with lower product differentiation, higher customers' price sensitivity, and higher demand uncertainty. It is noted that the optimal selection of the price discount is necessary to secure good performance of probabilistic selling, given that improper pricing will undermine seller's profit.

1.2.3 Inventory-pricing policy in "physical" probabilistic selling

We investigates the impacts of a new type of probabilistic selling (PS) where the retailer orders specific products and package some as a discounted physical probabilistic product (PPP) rather than merely a virtual choice. We call this PS strategy as physical probability selling (PPS). The price gap between the specific products and the probabilistic product result in demand reshape, i.e., some customers who originally buy specific products will switch to buying the probabilistic product, which decreases aggregate demand uncertainty. However the price discount decreases the profit margin. Considering this trade off, we develop a three-product newsvendor model to address the question of how to set the price for the PPP and make inventory allocation decisions. We prove that there are two effects under PPS, namely the risk pooling effect due to demand reshape and the risk diversification effect due to inventory flexibility. With demand uncertainty, PPS can improve the retailer's profit at lower inventory levels with proper demand reshape induced by the optimal price discount. The optimal price discount increases with demand uncertainty. PPS is more profitable with smaller product differentiation and higher customer

price sensitivity. With supply uncertainty, we demonstrate through numerical studies that PPS is a viable strategy to combat asymmetrical supply risk that yields higher profits and service levels.

1.2.4 Pricing-product allocation policy in probabilistic selling with barter choice

After buying the probabilistic product, the customer who is not satisfied with the allocated product may wish to barter with other customers. Moreover, there are online shops offering customers the option to barter their allocated products before confirming their orders. Exploring the seller's motivation to offer the probabilistic product with barter choice, we consider two questions: 1) How does barter affect the seller's optimal decisions in probabilistic selling? 2) Can and when does barter make probabilistic selling more advantageous to the seller? Considering the key factors of product cost, successful barter probability, and the marketing benefit brought by barter in probabilistic selling, we use the Hotelling model to address the questions. We show that barter can broaden the application range of probabilistic selling, which will increase with successful barter probability. When the marketing benefit is sufficiently large, barter can increase the profit of probabilistic selling to the seller. When the marketing benefit is low while the barter probability is high, barter will not benefit the seller in probabilistic selling. Our findings help the seller make optimal decisions on barter choice, pricing, allocation probability, and product line design, i.e., the seller merely offers the component products, merely the probabilistic product, or both.

1.3 Research innovations

The thesis has the following innovative points:

1) We construct the newsvendor model to characterize the demand reshape and demand substitution pattern in probabilistic selling. This is the first study that captures the inventory decision in probabilistic selling considering demand reshape that is independent and dependent with the price discount for the probabilistic product, respectively. We derive some management suggestions for pricing and inventory management in probabilistic selling.

On the one hand, rare literature endogenize inventory decision in probabilistic selling and majority literature set the demand for the specific product or the probabilistic product as deterministic. While we consider stochastic demand for both the specific products and the probabilistic product in our work except for the study on probabilistic selling with barter choice. This assumption enables us to explore the performance of probabilistic selling on managing stochastic demand, and improving inventory efficiency. On the other hand, the demand substitution and demand reshape pattern in probabilistic selling is unique, which also riches the study on demand reshape and demand substitution.

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2) We are the first study that compares the performance of probabilistic selling and inventory substitution in managing demand uncertainty through demand substitution. Our work enriches the research about probabilistic selling as an inventory management tool. The analytical approach and research findings may help practitioners gain more insight on the capacity of probabilistic selling on combating demand uncertainty, and facilitate their inventory related decision-making.

3) We are the first to explore the profitability and pricing-inventory policy of the retailer that offers the physical probabilistic product (PPS). We analytically examine the risk-pooling effect of PPS through demand reshape, and also find the risk diversification effect of PPS that helps alleviate asymmetrical supply risk through numerical studies.

4) We are the first to examine the decisions on pricing, allocation probability, and product line design, i.e., the seller merely offers the component products, merely the probabilistic product or both, in probabilistic selling with the barter choice. The analysis helps the seller understand when and how to offer the barter choice in probabilistic selling to achieve the maximum profit with consideration of product cost, successful barter probability, marketing benefit brought by per barter unit.

THEORETICAL FRAMEWORK

S ome literature regards probabilistic selling the same as opaque selling. However, probabilistic selling is based on the retailing industry, and the operations management in retailing industry is very different from that in the travel industry. For example, inventory is usually constrained in the travel industry while in retailing industry the seller has to make inventory decision. Furthermore, the intermediary retailer in travel industry creates the opaque product by mixing the products from different service providers, while in the retailing industry the seller usually creates the opaque product by mixing his own existing products. Therefore, we consider probabilistic selling as one form of opaque selling in our thesis. Although we focus on the probabilistic selling form in our research, literature about other forms of opaque selling is also necessary for they share the same spirit of opaque products. Some results or observations in related research on other forms of opaque selling still give us some important reference value. Then we will give a comprehensive review about opaque selling. Considering different motivations and methods in related research, we divide the literature into two streams. One investigates opaque selling in economics and marketing literature, and the other one considers this selling strategy from the aspect of operations management.

The economics and marketing literature considers different issues with respect to different implementation forms in practice: NYOP, FOP, VOP and PS. Table 2.1 shows us the basic characteristics of different forms of opaque selling. The research on NYOP, FOP, VOP is usually based on the scenario of a service provider-intermediary system. And the research on probabilistic selling is based on the scenario of a service/product seller-customer system.

Table 2.1: Different forms of opaque selling

Form	Posted price	Opaqueness	Creation of probabilistic product/opaque product	Supply chain system
NYOP	No	Fixed	Product/service integrator	service provider-intermediary system
FOP	Yes	Fixed	Product/service integrator	service provider-intermediary system
VOP	Yes	Flexible	Product/service integrator	service provider-intermediary system
PS	Yes	Fixed	Product/service owner	service/product seller-customer

2.1 Opaque selling in economics and management

2.1.1 Name-Your-Own-Price

The NYOP modes opened by service integrators have been fully discussed (Hann and Terwiesch 2003; Fay 2004; Terwiesch et al. 2005). The research focus on multi aspects: pricing especially the optimal threshold price, bidding patterns (repeat bidding or single bidding) and effectiveness analysis (Bai et al., 2015). Because we focus on opaque selling strategy with posted price, we won't give a detailed review about the NYOP mode research.

2.1.2 Fixed opaque product

Some literature explore the optimal strategy and profit ability of fixed opaque product in a monopolist market (i.e., one service provider in the service provider-intermediary system). Some papers focus on investigating the effect of fixed opaque product on pricing competition, market share, etc. in competitive environments (i.e., multiple service providers in the service provider-intermediary system). And some research devote to comparing effectiveness of opaque selling with other selling strategies. Most of the research address the decisions on pricing, competition, channel selection, and optimal transparency level.

2.1.2.1 Fixed opaque product in a monopolist market

Granados et al. (2005) develop an economic model of a supplier who distributes products across two channels with different levels of market transparency. The model provides guidelines for firms to set optimal transparency levels and prices with profit maximization. And the market transparency is related with willing to pay. Jiang (2007) considers a monopoly firm selling multiple flights a day. The airlines decide whether to offer both the full-information tickets with regular price and opaque tickets at a discounted price or just one of them. They find that opaque selling can increase a firm's sales and profits depending on customer heterogeneity in terms of their willingness to pay for a particular flight. Especially when customers are too heterogeneous,

the firm will prefer to offer only full-information products. The study of Anderson and Xie (2012) use a nested logit model along with logistic regression and dynamic programming to set the optimal choice-based price for a service provider to post on the opaque intermediary. Cai et al. (2013) investigated probabilistic selling under which the retailer generates the probabilistic good by mixing products from different suppliers. They studied how probabilistic selling endogenously influences suppliers' channel selections.

2.1.2.2 Fixed opaque product in a competitive market

From the perspectives of opaque intermediary in competitive environments, Fay (2008) publishes the first paper to model an intermediary selling an opaque product. He considers two symmetric service providers who share a common opaque intermediary. Customers are divided into the "brand-loyals" and the "searchers" which are represented by a Hotelling model. The paper analyzes the pricing equilibrium and the profitability with firm competition. The results show that there is fierce competition if there is little brand-loyalty in the marketplace. However, if brand-loyalty amount is moderate, entry of an intermediary would enable service providers to raise prices. Shapiro and Shi (2008) and Tappata (2012) extend the two service provider scenario into multiple service providers by using the circular city model in Salop (1979). Shapiro and Shi (2008) attempt to explore the effect of opaque intermediary on the competition of service providers. The results show that although the opaque intermediary intensifies competition for less sensitive (non-loyals) customers, it can segment the market and allow the service providers charge a higher price for more sensitive customers(loyals). Tappata (2012) shares the same setting while allowing for elastic aggregate demand in their model. Further they also study the welfare effect of opaque intermediary. The results show that opaque intermediary can create welfare by increasing price competition and expanding market sales compared with no opaque intermediary model. Service providers can use the opaque intermediary to increase profits with intermediate product differentiation value. Other related literature also consider competing firms selling opaque products through an intermediary (Jerath et al., 2010, 2009). Granados et al. (2017) investigate the demand and cannibalization effects of the opaque channel through empirical study of an international airline. They find that airlines can benefit from opaque selling in markets with high levels of competition.

2.1.2.3 Strategy comparison

Jerath et al. (2010)constructed a stylized economic model in which two firms with fixed capacities sell products to consumers in two periods. They attempt to explain and compare the benefits of using either transparent last-minute sales or opaque sales through an intermediary. They find that sales through opaque intermediary are preferred when consumer valuations for travel are lower or there is higher service differentiation between competing service providers. Gal-Or (2011) compares the profit of a monopoly service provider who can use either a price-posted or a NYOP

opaque selling channel. They show that NYOP is preferable for a monopoly service provider. However, in the presence of competition within service providers, opaque selling with posted price is more profitable with price competition (Chen et al., 2014). Anderson and Xie (2014) considers a monopolist using three selling channels: a full information channel, an opaque channel with posted price and NYOP. The paper illustrates how opaque channels segment consumers and compares optimal revenues and prices for sellers using full information channels with the three channels situation.

2.1.3 Variable opaque product

VOP method gives more flexibility to customers who can set the opaqueness rather than the service provider (Post, 2010). Research about VOP mode investigate on the design problems of price structure, opacity design and opaque packages, etc.

For example, Post (2010) proposed the term "variable opaque product" through the case of airline industry and develop a pricing heuristic to maximize the incremental revenue from opaque selling with varied opaqueness level. This product-price pricing structure provides a useful basis for price optimization models for VOP. Lee et al. (2012) discuss how to design opaque destination packages for airline carrier. They use a multidimensional binary logit model to predict the purchase probability which is influenced by distance, city attractiveness and length of stay. Post and Spann (2012) take the "Blind Booking" at Germanwings as the case and analyze some significant results from the implementation of VOP. The encouraging results of"Blind Booking" show that, with reasonable product price and opacity level (Germanwings requires no less than 3 destinations), the revenues generated by VOP would be predominantly incremental without cannibalization. Bai et al. (2015) investigate the design problems of pricing, opacity design and depict how customers choose marketing channels based on "Blind Booking" at Germanwings.

2.1.4 Probabilistic selling

The research on probabilistic selling is based on a retailer-customer system. The economics and marketing literature related to probabilistic selling demonstrate the profit ability of probabilistic selling in conditions concerning different characteristics of subjects (e.g., market, customer, product) as shown in Table 2.2. They explore the benefits of probabilistic selling in terms of price discrimination, market segmentation and expansion, and product line extension etc.

Fay and Xie (2008) firstly define probabilistic selling strategy and attempts to use Hotelling model and Circle model to explore the fundamental conditions required for offering "probabilistic products". The results show that offering probabilistic products can combat demand uncertainty and enhance inventory efficiency. Huang and Yu (2014) explores the importance of consumer bounded rationality on the adoption of probabilistic selling and demonstrates that consumer bounded rationality in probabilistic selling may soften price competition and increase the industry profits. Rice et al. (2014) make comparison of markdown selling and probabilistic selling

Table 2.2: Probabilistic selling on economics and marketing.

Subject	Focus	Literature
Seller	Monopolist	Fay and Xie (2008); Huang and Yu (2014) Rice et al. (2014); Zhang et al. (2014)
	Market Competition	None
Customer	Preference heterogeneity	Fay and Xie (2008); Rice et al. (2014) Fay et al. (2015)
	Loyalty heterogeneity	Fay (2008); Shapiro and Shi (2008)
	Bounded rationality	Huang and Yu (2014)
Product	Horizontal product	Fay and Xie (2008); Huang and Yu (2014) Fay and Xie (2008); Rice et al. (2014) Fay et al. (2015)
	Vertical product	Zhang et al. (2014)
	Horizontal or vertical product	Huang and Yu (2014)

strategies. As price discrimination tools, markdown selling strategy depends on buyer patience and probabilistic selling segments market based on buyer preference. They identify the conditions required for probabilistic selling to be more advantageous through analytical model. The results show that probabilistic selling can improve margin management and inventory utilization. Zhang et al. (2014) investigate probabilistic selling in quality-differentiated markets rather than horizontal markets and explore whether probabilistic selling can be profitable in this situation.

The above literature focus on the profit ability of probabilistic selling concerning different characteristics of markets, customers and products. They consider how probabilistic selling influence seller's decision and customers' purchase choice. Another direction is to explore the effect of probabilistic selling on product design. For example, Fay et al. (2015) find that introducing probabilistic products by mixing component products can not get the full potential of probabilistic selling. The retailer should also adjust its product mix (e.g., optimal number and types of products) when introducing probabilistic products. The paper reveals that, when facing several consumer segments with diverse preferences, a seller should produce more differentiated products when it moves to probabilistic selling from TS. Otherwise, when there are few consumers with moderate tastes, the retailer should produce less differentiated products when switching to probabilistic selling.

2.2 Opaque selling in operations management

One stream of operational research focus on the revenue management of opaque selling which solve control mechanism and pricing issues with fixed capacity (as shown in Table 2.3). Flexible products belongs to this stream. Flexible product is firstly defined as a menu of two or more alternative products offered by a supply chain issues constrained supplier in Gallego and Phillips (2004). And the supplier reserves some information of flexible product until a time near the end

of the booking process. Therefore, we assume flexible product is the same as probabilistic product. Gallego and Phillips (2004) analyze the two-flight case for an airline with fixed capacity offering flexible products and specific products simultaneously. Considering demand induction and cannibalization, they use simulation to compare results under various control structures and different pricing scenarios. Different from Gallego and Phillips (2004), Petrick et al. (2012) consider an arbitrary notification date within the booking horizon in a similar problem setting. They present several revenue management models and control mechanisms for offering flexible products and reveal that flexible products can increase revenue with fixed capacity and unpredictable demand. Gönsch and Steinhardt (2013) extend dynamic programming decomposition techniques to develop a new approach for service provider to control capacity in situation of offering both opaque products and traditional ones simultaneously. They show that their approach outperforms other well known capacity control approaches used in the opaque product setting with a simulation study.

The second stream focus on exploring the inventory mechanism of probabilistic selling. Fay and Xie (2011) regard probabilistic selling as a new mechanism for inventory management in the presence of demand uncertainty although the seller commits to buyers before it has the opportunity to acquire more information. Fay and Xie (2014) extend the novel strategy from a marketing tool to an inventory-management mechanism, which focus on the impact of timing of probabilistic product assignment and demonstrate the advantage of probabilistic selling to improve inventory utilization. As shown in Table 2.3, the above research dealing with inventory management consider the "scenario" uncertainty that one product is more popular than the other with a probability. Rare literature consider stochastic demand rather than "scenario" uncertainty except for Wu and Wu (2015) and Fu et al. (2017). Endogenizing both capacity and pricing decisions in a single-product system, Wu and Wu (2015) considers the stochastic demand in their single-product inventory model, integrating demand postponement and opaque selling from the perspective of travel intermediary. The result demonstrate that the postponement of delivery allows the firm to use less safety stock to hedge against demand uncertainty. Fu et al. (2017) analytically demonstrated that offering a flexible product can improve the seller's profit.

The third stream of literature attempt to expand probabilistic selling into supply chain. For example, Li and Ma (2016) developed a non-cooperative dynamic price Stackelberg game model to study the dynamic characteristics of a supply chain under probabilistic selling with risk-averse customer.

2.3 Comments on the literature

Great majority of the research focus on analyzing the rationality of the mechanism, the implementation issues and the profit ability from the aspect of marketing tool with economic models. Limited literature consider opaque selling in increasing inventory efficiency with constrained

Table 2.3: Distinguishing characteristics of related literature on operational management.

Subject	Focus	Literature
Capacity	Constrained	Gallego and Phillips (2004); Petrick et al. (2012) Gönsch and Steinhardt (2013)
	Non-constrained	Fay and Xie (2014); Wu and Wu (2015)
Demand	Scenario uncertainty	Fay and Xie (2014)
	Stochastic uncertainty	Gallego and Phillips (2004); Petrick et al. (2012) Gönsch and Steinhardt (2013); Wu and Wu (2015)

capacity on revenue management. However, rare literature consider opaque selling in an operational management setting and explore the effect of opaque selling on operational issues such as capacity planning, procurement, supply chain coordination, etc. Just as Wu and Wu (2015) refers that future research should include how probabilistic selling affect inventory decisions and supply-chain dynamics.

Overall, opaque selling is still a relatively new market practice and has aroused interests from service providers, commerce retailers, customers and researchers. It is urgent to help firms to develop an optimal strategy for adopting this selling strategy. We will focus on the form of probabilistic selling in this thesis and shed light on the logic of this strategy considering more practical settings.

Managing demand uncertainty: Probabilistic selling

versus inventory substitution

3.1 Introduction

The prevailing variability of the business environment, rapidly evolving technologies, fierce competition, and sophisticated customer demands are increasing the difficulty for firms to determine the optimal inventory under demand uncertainty. For example, many retailers try to capture market share and meet customers' various demands by carrying a wide variety of products. Usually the products are similar and may be substitutable, e.g., clothes in different colour, beverages with different flavours, and bags in different patterns. Although increasing product variety can increase the retailer's market size, it would also increase its total inventory, leading to longer inventory cycles and higher safety stock (Rajagopalan, 2013). In addition, any mis-match between inventory and demand, even for a single product, would reduce profit due to the inventory cost or stock-out cost. Uncertain demand for multiple products makes it more arduous to match supply and demand for improving inventory efficiency. Therefore, it is important to effectively manage demand uncertainty when firms seek to benefit from market expansion through increasing product variety.

To address the problem of managing demand uncertainty with multiple substitutable products, the retailer can consider two strategies, namely inventory substitution, which is well known in Operations Research, and probabilistic selling, which is popular in Marketing. Being an effective tool to minimize the mis-match between capacity and demand, inventory substitution uses substitute products to meet demand when stock-out occurs (Mcgillivray and Silver, 1978; Parlar and Goyal, 1984; Ernst and Kouvelis, 1999). Probabilistic selling means the retailer creates an additional probabilistic product with hidden information using existing products (Fay and Xie,

2008). For instance, travel agencies offer probabilistic service products (e.g., hotel rooms, air tickets, package tours etc.) with some information concealed from customers until customers confirm their orders. Online retailers like Tmall.com, Amazon.com, and AgonSwim.com (Fay and Xie, 2010) offer discounted probabilistic products with some attributes, e.g., colour, style, brand etc, unknown to customers until they receive the products. The price-sensitive customers who are indifferent to the attributes would choose to buy discounted probabilistic products.

Although the two strategies are triggered by the need to address different problems in different research fields, we see that both strategies share the same spirit of demand substitution. Specifically, in applying inventory substitution, the retailer substitutes the remaining inventory of one product for another. The customer whose required product is sold out can choose to accept the substitute or not. In applying probabilistic selling, the retailer offers customers an additional lower-priced choice to enhance the demand substitution of specific products with full information by the probabilistic product. Inventory substitution induces insensitive customers and makes use of available inventory (of a substitute product) at the end stage of selling, while probabilistic selling induces insensitive customers at the beginning stage and then uses available inventory (of either the requisite product or a substitute product) during the selling stage. Consequently, the retailer can substitute products through the demand of insensitive customers to minimize the mis-match between inventory and demand. Although the two strategies differ in operation to hedge against demand uncertainty, they share a common characteristic in combating demand uncertainty by encouraging some customers to give up some specific demand, e.g., colour, pattern etc, for the product to enable demand substitution. Therefore, it is interesting to explore which strategy is more advantageous for the retailer that sells substitutable products with demand uncertainty.

However, despite the popularity of probabilistic selling in marketing research for the purposes of market expansion and price discrimination (Fay and Xie, 2008, 2014), little is known about probabilistic selling as an inventory tool. There is little research on using economic models to analyze the inventory mechanism of probabilistic selling (Fay and Xie, 2011, 2014). The first study that endogenizes the capacity decision is Wu and Wu (2015), which explores opaque selling as a strategy to induce demand postponement. They considered the one-product scenario with stochastic demand from the perspective of an intermediary. While we also study probabilistic selling in the newsvendor setting, we focus on exploring the inventory ability of probabilistic selling from the perspective of a retailer that can manage demand uncertainty through demand substitution.

In this chapter we develop a single-period newsvendor model with three products to analyze probabilistic selling with a view to generating insights into using probabilistic selling to manage demand uncertainty. We then compare probabilistic selling with inventory substitution in the special cases without cannibalization. To gain additional insights into the normal situation, we use computational examples to compare the two strategies in terms of overall profit and

inventory with considerations of customer transition (reflected by the cannibalization index under probabilistic selling and by the substitution fraction under inventory substitution) and demand uncertainty.

We make two main contributions under this chapter: First, this is the first study that captures the inventory decision in probabilistic selling considering the cannibalization effect. Second, this is the first study that compares the performance of probabilistic selling and inventory substitution in managing demand uncertainty through demand substitution. The results show that the retailer's inventory decisions depends on products' substitution possibility. The comparison results show that probabilistic selling outperforms inventory substitution with relatively lower product similarity and higher price-sensitive customers, while inventory substitution is more profitable than probabilistic selling when product similarity is higher. Besides, higher demand uncertainty will increase the profitability advantage of probabilistic selling over inventory substitution. Our work enriches the research about probabilistic selling as an inventory management tool. The analytical approach and research findings may help practitioners gain more insight on the capacity of probabilistic selling on combating demand uncertainty, and facilitate their inventory related decision-making.

3.2 Literature Review

3.2.1 *Probabilistic selling*

Limited attention has been paid to examining the inventory mechanism of probabilistic selling. Fay and Xie (2011) regarded probabilistic selling as a new mechanism for inventory management in the presence of demand uncertainty despite that the seller is committed to buyers before it has the opportunity to acquire more information. Focusing on the impact of the timing of the assignment of the probabilistic product, Fay and Xie (2014) demonstrated the advantage of probabilistic selling in improving inventory utilization. Nevertheless, the uncertainty in the above studies concerns the probability that one product is more popular than another, and they use the "scenario-based" approach to represent uncertainty (Gupta and Maranas, 2003) rather than the "distribution-based" approach. Just as Rice et al. (2014) pointed out that little research has shown the effectiveness of probabilistic selling when the seller is uncertain about the total category demand rather than the relative popularity of a specific item.

Different from the above literature, we model the demand as normally distributed with a mean and a standard deviation, which is widely used in OM research. Some studies have considered demand uncertainty and recognized the benefit of probabilistic products in increasing inventory efficiency with fixed capacity in the study field of revenue management (Gallego and Phillips, 2004; Gönsch and Steinhardt, 2013). However, they don't endogenize inventory decision in their research. Then Wu and Wu (2015) considered stochastic demand in their single-product inventory model, and integrated demand postponement and opaque selling from the perspective

of a travel intermediary. They showed that postponement of delivery allows the firm to use less
safety stock to combat demand uncertainty. Different from Wu and Wu (2015), the demand for
the probabilistic product is also stochastic and we explore probabilistic selling as an inventory
mechanism from the perspective of a retailer selling multiple alternative products. Furthermore,
we focus on comparing probabilistic selling and inventory substitution, both of which use demand
substitution to hedge against demand uncertainty.

3.2.2 *Inventory substitution*

There is a large body of work on inventory management with substitutable demand. The
substitution phenomenon has been widely investigated considering various substitution patterns.
The substitution can be led by the supplier, which is common in the airline industry (Vulcano
et al., 2012). It can also be led by the customer that is willing to buy a substitute product
when their preferred product is out of stock (Parlar and Goyal, 1984; Ernst and Kouvelis, 1999;
Baris and Selcuk, 2013; Ye, 2014). The substitution scenarios considered in existing research
include two products with one-way or two-way substitution (Mcgillivray and Silver, 1978; Parlar
and Goyal, 1984), three products with partial substitution (Ernst and Kouvelis, 1999), and an
arbitrary number of products with demand substitution (Netessine and Rudi, 2003; Wang and
Parlar, 1994). According to the probability of customers willing to accept substitution, some
research considers total substitution (i.e., the probability is equal to 1) (Mcgillivray and Silver,
1978; Pasternack and Drezner, 1991) or constant substitution (i.e., the probability is between
0 and 1) (Parlar and Goyal, 1984; Ernst and Kouvelis, 1999). Some studies assume that the
revenue received for a product is independent of the substitution, while others assume that the
substitution will incur a performance-related cost (Pasternack and Drezner, 1991). Shah and
Avittathur (2007) examined cannibalization considering the downward substitution pattern with
a standard product and its customized extensions. We consider partial substitution with cost as
Parlar and Goyal (1984) and Pasternack and Drezner (1991) in our study.

3.3 Inventory decision under inventory substitution

3.3.1 *Notation and Assumption*

We consider a retailer that sells two specific products, indexed i, j=1,2 (it is assumed that
$i \neq j$). The retailer purchases a quantity Q_i^t of product i and a quantity Q_j^t of product j at the
same fixed unit cost $c > 0$, and sells them at price p. The clearance price is s. The stock-out
penalty is 0. We assume that the demand D_i (D_j) is normally distributed with mean u_i (u_j) and
standard deviation σ_i (σ_j). Let $f(x_i)$ ($f(x_j)$) and $F(x_i)$ ($F(x_j)$) be the probability density function
and cumulative density function of D_i (D_j), respectively. In addition, let $f(x_i, x_j)$ be the joint
probability density function of the demand for the products. When the retailer adopts neither
probabilistic selling nor inventory substitution, the optimal inventory decision for each product is

just the optimal inventory decision for the single-product newsvendor model, i.e., the optimal order quantities $Q_i^{s^*}$ and $Q_j^{s^*}$ are determined by the following equations:

$$F(Q_i^{t^*}) = \frac{p-c}{p-s},$$
(3.1)

$$F(Q_j^{t^*}) = \frac{p-c}{p-s}.$$
(3.2)

Now we consider the case where the retailer adopts inventory substitution and assume that only a fraction r_s of the unsatisfied customers that face stock-out will accept the substitution (Parlar and Goyal, 1984). Assume that substitution incurs a cost t per unit (Pasternack and Drezner, 1991). We also suppose that $p - t > s$ to make sure that the retailer can benefit from substitution. Substitution occurs when the demand for product i (j) exceeds its supply while the demand for product j (i) is less than its supply (i.e., the substitution paths in Fig. 3.1. After substitution, the total demand for product i may or may not be satisfied.
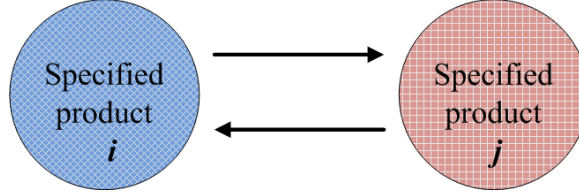
Figure 3.1: Substitution paths in adopting inventory substitution.

3.3.2 *The optimal inventory solution*

The expected profit is given in Eq.(3.3), which comprises the revenue, the savage cost, and the acquisition cost. (Q_i^s, Q_j^s) are the two inventory decisions that jointly maximize the expected profit. The demand for product $i(j)$ comes from the original demand and the substitution demand when demand of product $j(i)$ excess its supply. Therefore, the revenue under inventory substitution comes from satisfying both the original demand (i.e. $\min(D_j, Q_j^s)$ and $\min(D_i, Q_i^s)$ and the substitution demand (i.e. $\min\left[(Q_i^s - D_i)^+, r_s(D_j - Q_j^s)^+\right]$ and $\min\left[r_s(D_i - Q_i^s)^+, (Q_j^s - D_j)^+\right]$).

$$E(Q_i^s, Q_j^s) = E \begin{cases} p\min(D_i, Q_i^s) + p\min(D_j, Q_j^s) - c(Q_i^s + Q_j^s) \\ +(p-t)\min\left[(Q_i^s - D_i)^+, r_s(D_j - Q_j^s)^+\right] \\ +(p-t)\min\left[r_s(D_i - Q_i^s)^+, (Q_j^s - D_j)^+\right] \\ +s\left[(Q_i^s - D_i)^+ + (Q_j^s - D_j)^+ - r_s(D_i - Q_i^s)^+ - r_s(D_j - Q_j^s)^+\right]. \end{cases}$$
(3.3)

Pasternack and Drezner (1991) have shown the concave property of the expected profit. So the optimal inventory decisions can be determined by applying the first-order condition to the expected total profit function. We characterize the optimal order quantities $(Q_i^{s^*}, Q_j^{s^*})$ in Eq.(3.4),

which is similar to the results in Rudi et al. (2001).

$$\begin{cases} F(Q_i^{s^*}) &= \frac{p-c+(p-t-s)R(Q_i^{s^*},Q_j^{s^*})}{p-s} \\ F(Q_j^{s^*}) &= \frac{p-c+(p-t-s)T(R_i^{s^*},Q_j^{s^*})}{p-s}, \end{cases} \tag{3.4}$$

where

$$R(Q_i^{s^*},Q_j^{s^*}) = \int_0^{Q_i^s}\int_{Q_j^{s^*}+(Q_i^{s^*}-D_i^t)/r_s}^{\infty} f(x_i,x_j)dx_jdx_i - r_s\int_0^{Q_j^{s^*}}\int_{Q_i^{s^*}}^{Q_i^{s^*}+(Q_j^{s^*}-D_j^t)/r_s} f(x_i,x_j)dx_idx_j,$$

$$T(Q_i^{s^*},Q_j^{s^*}) = \int_0^{Q_j^{s^*}}\int_{Q_i^{s^*}+(Q_j^{s^*}-D_j^t)/r_s}^{\infty} f(x_i,x_j)dx_jdx_i - r_s\int_0^{Q_i^{s^*}}\int_{Q_j^{s^*}}^{Q_j^{s^*}+(Q_i^{s^*}-D_i^t)/r_s} f(x_i,x_j)dx_idx_j.$$

The first term of $R(Q_i^{s^*},Q_j^{s^*})$ raises Q_i^s due to the possibility that the excess inventory of product i may not meet the substitution demand for product j, while the second term lowers Q_i^s because the substitution demand for product Q_i^s can be satisfied with the excess inventory of product Q_j^s. The same observation holds for $T(Q_i^{s^*},Q_j^{s^*})$.

3.4 Inventory decision under probabilistic selling

3.4.1 *Notation and assumption*

Under probabilistic selling, the offer of the probabilistic product indexed k may cannibalize the specific product market (Granados et al., 2010; Post and Spann, 2012). So, given the cannibalization effect, the observed demand distribution needs to be revised as (D_i^p, D_j^p, D_k^p) in Section 3.4.2. As before, the retailer has to purchase quantities Q_j^d, Q_i^d, and Q_k^d to meet the demands for the specific products i, j, and the probabilistic product k, respectively. The quantity Q_k^d is a mix of products i and j. If we assume that the proportion of product i in the mix is r, then the retailer has to order $Q_i^p = Q_i^d + rQ_k^d$ of product i and $Q_j^p = Q_j^d + (1-r)Q_k^d$ of product j.

Following Fay and Xie (2014), Jerath et al. (2010), and Wu and Wu (2015) in operationalizing probabilistic selling, we assume that probabilistic selling postpones the delivery of the probabilistic product with regular price to meet the substitution demand for a specific product sold at a higher price. The retailer obtains revenue p for each specific product and p_0 ($p > p_0 > s$) for each probabilistic product sold. Since the consumer of the probabilistic product pays a lower price, they would accept uncertainty about product availability and postponement of product delivery. Fig. 3.2 shows the sequence of events.

3.4.2 *Revised demand distribution*

We consider the cannibalization effect on the specific products, which means that the demand for the probabilistic product D_k^p consists of two parts: the demand that switches from the specific
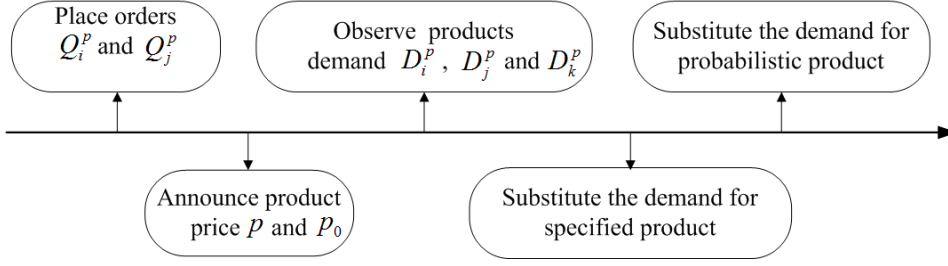
Figure 3.2: The sequence of events.

products to the probabilistic product, and the new market expansion demand D_k induced by the low-priced probabilistic product. We assume that D_k is normally distributed with mean u_k and standard deviation σ_k. The demand D_i and D_j, and the demand $D_i(D_j)$ and the new market expansion demand D_k are correlated with ρ_{ij} and $\rho_{ik}(\rho_{jk})$, respectively. Let a_i (a_j) be the cannibalization index of the demand for the specific product i (j) ($0 \le a_i(a_j) \le 1$), which is independent of D_k. The observed demand for the probabilistic product is given by $D_k^p = a_i D_i + a_j D_j + D_k$.

The observed demands D_i^p and D_j^p are different from the original demands D_i and D_j in traditional selling. It is important to define the demand relationships between traditional selling and probabilistic selling, for we will compare the two selling strategies with respect to the inventory decision and expected profit. Following Eynan and Fouque (2003), and Hsieh (2011), we characterize the distribution parameters of the observed demands D_i^p, D_j^p, and D_k^p under probabilistic selling as follows:

$$u_i^p = (1 - a_i)u_i,$$
$$u_j^p = (1 - a_j)u_j,$$
$$\sigma_i^p = (1 - a_i)\sigma_i,$$
$$\sigma_j^p = (1 - a_j)\sigma_j,$$
$$u_k^p = a_i u_i + a_j u_j + u_k,$$
$$\sigma_k^p = \sqrt{a_i^2 \sigma_i^2 + a_j^2 \sigma_j^2 + \sigma_k^2 + 2a_i \rho_{ik}\sigma_i \sigma_k + 2a_j \rho_{jk}\sigma_j \sigma_k + 2a_i a_j \rho_{ij}\sigma_i \sigma_j}.$$

Because the demand for the probabilistic product includes a part of the original demands for the specific products, the demand correlation after cannibalization should be updated as follows:

$$\rho_{ik}^* = \frac{a_i \sigma_i + a_j \rho_{ij}\sigma_j + \rho_{ik}\sigma_k}{\sigma_k^p}, \tag{3.5}$$

$$\rho_{jk}^* = \frac{a_j \sigma_j + a_i \rho_{ij}\sigma_i + \rho_{jk}\sigma_j \sigma_k}{\sigma_k^p}. \tag{3.6}$$

So we define the joint probability density function of the demand for the products under probabilistic selling $f(x_i, x_j, x_k)$ as $f^*(x_i, x_j, x_k)$ after cannibalization.
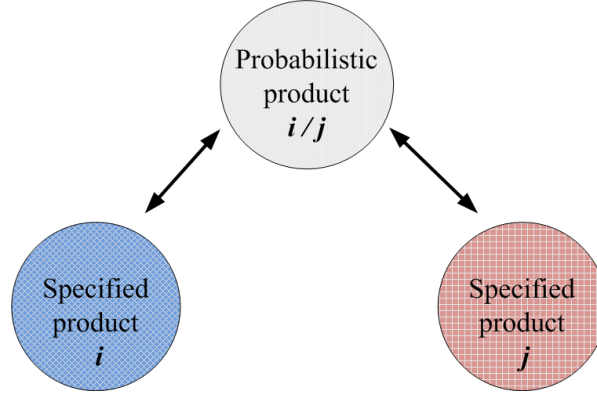
Figure 3.3: The substitution pattern under probabilistic selling.

3.4.3 *Substitution pattern*

Probabilistic selling encourages substitution between the specific products and the probabilistic
product when stock-out occurs. As shown in Fig. 3.3, different from inventory substitution, there
is no direct substitution between the specific products. The substitution between the specific
products occurs through the probabilistic product. Besides, the offer of probabilistic selling may
increase total product sales.

Furthermore, there are two stages of substitution owing to postponement of product delivery
under probabilistic selling. In the first stage of substitution, substitution occurs to meet the
demands for the higher-priced specific products. For instance, if the demand realization of either
specific product i (j) exceeds its available inventory Q_i^d (Q_j^d), the retailer can select the popular
product i (j) from the probabilistic product inventory Q_k^d to meet the high-priced demand first. In
the second stage of substitution, if the demand for the probabilistic product exceeds its remaining
inventory after the first stage of substitution while the specific products are available, either of
the specific products can serve as a substitute. There is no possibility that both specific products
i and j are out of stock when the demand for the probabilistic product can be fully satisfied.

3.4.4 *The optimal inventory decision*

The decision variables in the inventory model are (Q_i^p, Q_j^p) rather than (Q_i^d, Q_j^d, Q_j^k). Thus
it suffices to characterize the second stage of substitution. Specifically, we present these cases
and their corresponding probabilities of occurrence in Table 3.1. For instance, Case 3 means
that product j is out of stock, while product i has excess inventory, and the excess inventory is
sufficient to cover the demand for the probabilistic product. The expected profit, which includes
the revenue, the savage cost, and the acquisition cost, is given as follows:

$$E(Q_i^p, Q_j^p) = E \begin{cases} p\min(D_i^p, Q_i^p) + p\min(D_j^p, Q_j^p) - c(Q_i^p + Q_j^p) \\ + p_0 \min\left[D_k, [(Q_i^p - D_i^p)^+ + (Q_j^p - D_j^p)^+]\right] \\ + s\left[(Q_i^p - D_i^p)^+ + (Q_j^p - D_j^p)^+ - D_k\right]^+. \end{cases} \tag{3.7}$$

Table 3.1: Classification scheme for all the possible cases and their corresponding occurring probabilities

Case	i	j	k	Probability
1	Yes	Yes	Yes	$Pr(D_i^p < Q_i^p, D_j^p < Q_j^p, D_k^p < (Q_i^p + Q_j^p - D_i^p - D_j^p))$
2	Yes	Yes	No	$Pr(D_i^p < Q_i^p, D_j^p < Q_j^p, D_k^p > (Q_i^p + Q_j^p - D_i^p - D_j^p))$
3	Yes	No	Yes	$Pr(D_i^p < Q_i^p, D_j^p > Q_j^p, D_k^p < (Q_i^p - D_i^p))$
4	Yes	No	No	$Pr(D_i^p < Q_i^p, D_j^p > Q_j^p, D_k^p > (Q_i^p - D_i^p))$
5	No	Yes	Yes	$Pr(D_i^p > Q_i^p, D_j^p < Q_j^p, D_k^p < (Q_i^p - D_i^p))$
6	No	Yes	No	$Pr(D_i^p > Q_i^p, D_j^p < Q_j^p, D_k^p > (Q_i^p - D_i^p))$
7	No	No	No	$Pr(D_i^p > Q_i^p, D_j^p > Q_j^p, D_k^p > (Q_i^p - D_i^p))$

Proposition 3.1. If the distribution function of the demand is continuous and differentiable, then the expected profit function is concave in (Q_i^p, Q_j^p).

Proof. See the Appendix A.

It can be recognized from Eq.(3.7) and Table 3.1 that the modelling of product i and product j are symmetrical. We just analyze the inventory decision of one product and the analysis of the other product is similar. Differentiating the expected total profit once, we obtain the expected value of a marginal unit of product i as follows:

$$
\begin{aligned}
\frac{\partial E(Q_i^p, Q_j^p)}{\partial Q_i^p} = {} & p(1 - Pr(D_i^p < Q_i^p)) \\
& + p_0[Pr(D_i^p < Q_i^p, D_j^p > Q_j^p, D_k^p > Q_i^p - D_i^p) \\
& + Pr(D_i^p < Q_i^p, D_j^p < Q_j^p, D_k^p > Q_i^p + Q_j^p - D_i^p - D_j^p)] \\
& + s[Pr(D_i^p < Q_i^p) - Pr(D_i^p < Q_i^p, D_j^p > Q_j^p, D_k^p > Q_i^p - D_i^p) \\
& - Pr(D_i^p < Q_i^p, D_j^p < Q_j^p, D_k^p > Q_i^p + Q_j^p - D_i^p - D_j^p)] - c.
\end{aligned}
\tag{3.8}
$$

The first term of Eq.(3.8) means that any additional inventory of product i will result in an incremental sales except when there is excess inventory of product i ($D_i^p < Q_i^p$). The retailer can still benefit from the marginal unit of product Q_i^p by satisfying the demand for the probabilistic product (which may happen whenever the demand for product j can be satisfied), yielding p_0. The third term means that, if the inventory of product i exceeds its demand and the demand for the probabilistic product can also be satisfied, the marginal unit of product i is only worth its salvage value s. To simplify the notation, we re-arrange Eq.(3.8) and characterize the optimal order quantities.

Proposition 3.2. The optimal order quantities $(Q_i^{p^*}, Q_j^{p^*})$ under probabilistic selling can be

expressed as

$$\begin{cases} F(Q_i^{p*}) &= \frac{p-c+(p_0-s)G(Q_i^{p*},Q_j^{p*})}{p-s} \\ F(Q_j^{p*}) &= \frac{p-c+(p_0-s)N(Q_i^{p*},Q_j^{p*})}{p-s}, \end{cases} \tag{3.9}$$

where

$$G(Q_i^{p*},Q_j^{p*}) = \int_0^{Q_i^{p*}} \int_0^{Q_j^{p*}} \int_{Q_i^{p*}+Q_j^{p*}-D_i^p-D_j^p}^{\infty} f^*(x_i,x_j,x_k) dx_i dx_j dx_k$$
$$+ \int_0^{Q_i^{p*}} \int_{Q_j^{p*}}^{\infty} \int_{Q_i^{p*}-D_i^p}^{\infty} f^*(x_i,x_j,x_k) dx_i dx_j dx_k,$$

$$N(Q_i^{p*},Q_j^{p*}) = \int_0^{Q_i^{p*}} \int_0^{Q_j^{p*}} \int_{Q_i^{p*}+Q_j^{p*}-D_i^p-D_j^p}^{\infty} f^*(x_i,x_j,x_k) dx_i dx_j dx_k$$
$$+ \int_{Q_i^p}^{\infty} \int_0^{Q_j^{p*}} \int_{Q_j^{p*}-D_j^p}^{\infty} f^*(x_i,x_j,x_k) dx_i dx_j dx_k.$$

We see that the optimal order quantities are adjustments of the solution for the newsvendor model given in Eq.(3.1) and Eq.(3.2). Specifically, $G(Q_i^{p*},Q_j^{p*})$ raises Q_i^p due to the possibility of substituting for the probabilistic product. Substitution occurs in two cases: one is the case where product i has excess inventory while product j is out of stock, the other is the case where both specific products i and j have excess inventory, while the probabilistic product is out of stock. Similarly, $N(Q_i^{p*},Q_j^{p*})$ raises Q_j^p due to the substitution ability of product j. The implication of this proposition is that the retailer should hold more inventory of the products that have greater possibilities to substitute the other products. Thus the retailer can make incremental profit from the substitute product.

Compared with inventory substitution, the inventory decision under probabilistic selling is influenced by the price of the probabilistic product. Next, we analytically derive some properties of the effect of p_0 on the optimal inventory decision.

Proposition 3.3. When the price of the probabilistic product increases, the retailer should adjust its inventory decision depending on the substitution possibility. Specifically,

a) The retailer should order more product j and less product i ($\frac{\partial E(Q_i^{p*})}{\partial p_0} < 0$, $\frac{\partial E(Q_j^{p*})}{\partial p_0} > 0$) if the substitution possibility of product i is sufficiently small ($G < \frac{b^*}{d^*}N$).

b) The retailer should order more product i and less product j ($\frac{\partial E(Q_i^{p*})}{\partial p_0} > 0$, $\frac{\partial E(Q_j^{p*})}{\partial p_0} < 0$) if the substitution possibility of product i is sufficiently large ($G > \frac{a^*}{c^*}N$).

c) The retailer should order more of both products i and j ($\frac{\partial E(Q_i^{p*})}{\partial p_0} > 0$, $\frac{\partial E(Q_j^{p*})}{\partial p_0} > 0$) if the substitution possibility is moderate ($\frac{b^*}{d^*}N < G < \frac{a^*}{c^*}N$),

where a^*, b^*, c^*, and d^* denote the value of $\frac{\partial E(Q_i^p,Q_j^p)^2}{\partial^2 Q_i^p}$, $\frac{\partial E(Q_i^p,Q_j^p)^2}{\partial Q_i^p \partial Q_j^p}$, $\frac{\partial E(Q_i^p,Q_j^p)^2}{\partial Q_j^p \partial Q_i^p}$, and $\frac{\partial E(Q_i^p,Q_j^p)^2}{\partial^2 Q_j^p}$ at (Q_i^{p*},Q_j^{p*}), respectively. Besides, $\frac{a^*}{c^*} > 1$ and $\frac{b^*}{d^*} < 1$.

Proof. See the Appendix A.

In the classic single-product newsvendor model, the optimal inventory (i.e. Eq.(3.1) and Eq.(3.2)) increases with the price. However, when it comes to the two-product setting of probabilistic selling. How does the retailer adjust the optimal inventory decision when the price of the probabilistic product increases? Proposition 3.3 states that, if there is an increase in the price of the probabilistic product, the retailer should order more of one product and less of the other when the difference of their substitution possibilities is large (e.g., $G < \frac{b^*}{d^*}N$, $G > \frac{a^*}{c^*}N$). And, when the difference is not large, the retailer should increase the inventory of both products (e.g., $\frac{b^*}{d^*}N < G < \frac{a^*}{c^*}N$). That means that the retailer should always order more product with higher substitution possibility. However, whether increase the inventory of product with lower substitution possibility or not depends on the difference of the two product's substitution possibility.

3.5 Comparisons for some special cases

Comparing the two strategies is difficult when the substitution fraction r_s, and the cannibalization indices a_i and a_j are non-zero. Therefore, we first compare the two strategies under some special cases, and then conduct computational studies in the next part to compare the two strategies in general. In practice, the cannibalization index a in probabilistic selling may become zero when customers' price sensitivity is low or product differentiation is very large (Granados et al., 2010; Post and Spann, 2012). The substitution fraction r_s may become zero when product differentiation is very large.

Case 1: $a_i = a_j = 0$, $D_k = 0$ and $r_s = 0$. Inventory substitution is equivalent to probabilistic selling as they have the same optimal order quantity and expected profit, and neither strategy can improve the profit of the retailer.

This case may arises when the product differentiation is too large in a saturated market. In this case, both inventory substitution and probabilistic selling fail to generate additional profit from substitutable demand, and the introduction of low-priced products cannot attract new demand in this market. Thus, both strategies reduce to the classical newsvendor model.

Case 2: $a_i = a_j = 0$, $D_k \neq 0$ and $r_s = 0$. Probabilistic selling outperforms inventory substitution as it yields a higher profit with a higher inventory level.

In this case, neither the customers in probabilistic selling nor in inventory substitution accept a substitute when the product differentiation is too large. However, there are some new customers enticed to buy discounted products under probabilistic selling. This case can be explained by similar arguments in Post and Spann (2012), and Anderson (2009). Therefore, inventory substitution reduces to the classical newsvendor model and probabilistic selling can increase the profit by market expansion. However, the optimal inventory level under probabilistic selling is higher than that under inventory substitution because both $Q_i^{p*} > \frac{p-c}{p-s}$ and $Q_j^{p*} > \frac{p-c}{p-s}$ when $G(Q_i^{p*}, Q_j^{p*}) > 0$ and $N(Q_i^{p*}, Q_j^{p*}) > 0$.

Case 3: $a_i = a_j = 0$, $D_k \neq 0$ and $r_s \neq 0$. Inventory substitution requires more customers willing
to accept the substitute product than probabilistic selling to achieve the same profit at the same
marginal profit from the substitute product.

In this case, the comparison of the two strategies depends on the demand and marginal profit
obtained from the substitute product. When the size of the new market expansion demand under
probabilistic selling is the same as the number of customers that face a stock-out situation and
accept another product, the retailer has the same amount of discounted product sales under
the two selling strategies. However, probabilistic selling can increase the high-priced product
sales through substitution while inventory substitution cannot. Therefore, probabilistic selling
outperforms inventory substitution when it can attract an equal number of customers willing
to accept a substitute. Besides, because the retailer offers the probabilistic product in the first
selling stage rather than the second under inventory substitution, it has the potential to secure
more demand for the low-priced products.

3.6 Computational studies

In this section we consider a more general case where the cannibalization index $a_i = a_j \neq 0$
and $r_s \neq 0$. The main question that drives the design of the computational studies is under what
conditions probabilistic selling outperforms inventory substitution, and vice versa. Specifically,
we explore the effects of the customer transfer coefficient and demand uncertainty on the optimal
profit and inventory under both strategies. The customer transfer coefficient is reflected by the
substitution fraction r_s under inventory substitution and the cannibalization index $a_i(a_j)$ under
probabilistic selling. The difference is that one is positive transfer induced by price and the other
is negative transfer forced by the stock-out of products.

We assume that the original demands D_i^t and D_j^t are equal, which are normally distributed
with parameters that satisfy the assumptions: mean $u_i(u_j) = 100$, standard deivation $\sigma_i(\sigma_j) = \sigma =$
$[20, 30, 40, 50]$, the initial correlation coefficient $\rho_{ij} = 0$, $p = 40$, $c = 20$, $s = 10$, $a_i = a_j = a \in [0, 1]$,
$t = 2$, and $r_s \in [0, 1]$. In order to focus on the substitution effect of the two strategies, we assume
that there is no new market expansion demand under probabilistic selling (e.g., $D_k = 0$ when the
original market is saturated). For simplicity, we use "PS" and "IS" to denote probabilistic selling
and inventory substitution, respectively.

3.6.1 *The effect of the customer transfer coefficient*

As shown in Fig. 3.4, with different price discounts (i.e., 95%, 90%, and 85%), the results reveal
the same trend that probabilistic selling achieves the highest expected profit at a relatively small
customer transfer coefficient, while the expected profit under inventory substitution increases
with the customer transfer coefficient. These two observations are consistent with the results in
Zhang et al. (2016) and Rajaram and Tang (2001), respectively.
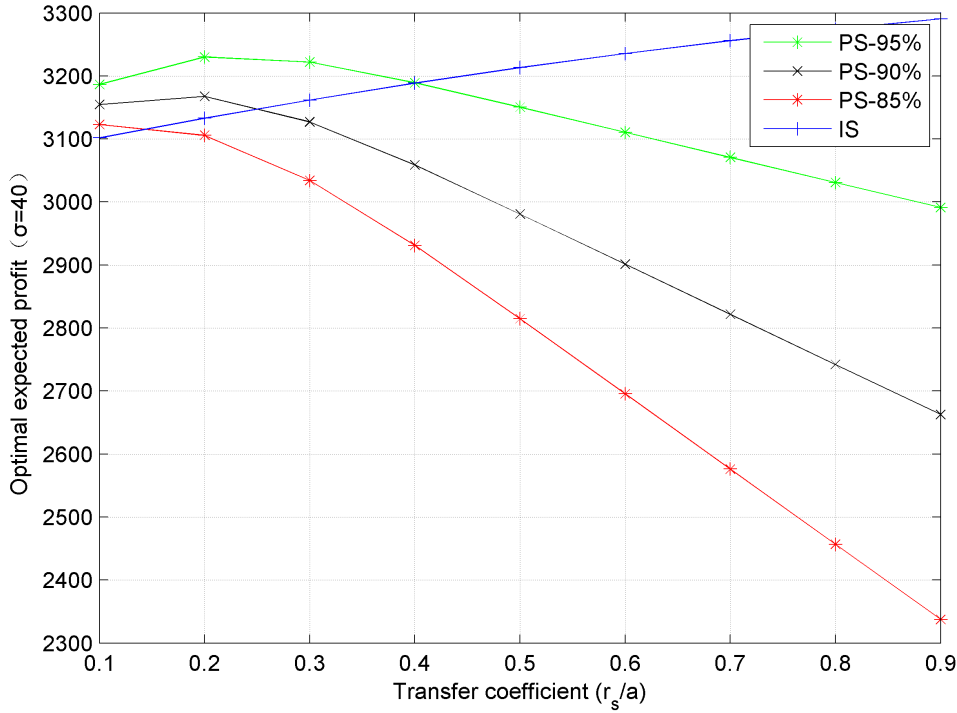
Figure 3.4: Expected profit of inventory substitution and probabilistic selling with different price discounts.

The efficiency of demand substitution under probabilistic selling increases with a smaller customer transfer coefficient (if the index is too small, the buffering effect of the probabilistic product for demand substitution becomes insignificant), while being restricted at larger customer transfer coefficient. One reason is that profit decreases with demand correlation when demand is multivariate normal (Netessine and Rudi, 2003). It can be obtained from Eqs 3.5-3.6 that the correlation between the newly revised demand would increase as more customers switch from product i (j) to k. The positively correlated demands of products i, j, and k result in high possibilities of large and small substitution demands simultaneously. The probability that the retailer substitutes specific products for the probabilistic product, or vice versa when stock-out occurs is relatively small. Another reason is that more customers will switch to buying the probabilistic product that yields a lower profit margin, which can harm the retailer's profit. When the profit that demand substitution brings cannot offset the lower sales of the specific products, profit improvement will decrease. Therefore, probabilistic selling is most advantageous when the customer transfer coefficient is large enough to enable substitution, but not so large that very few consumers will buy the high-priced products.

As shown in Fig. 3.4, the expected profit under probabilistic selling with a fixed customer transfer coefficient would decrease with price discount. This means that probabilistic selling requires some customers that have high price sensitivity to be attracted by the product with a small discount. Otherwise, when only few customers are attracted by a large discount, inventory

substitution would be more advantageous than probabilistic selling.

3.6.2 *Comparison of the expected profit*

In this section we compare the performance of the two strategies when customers are price-sensitive. We take the fixed discount 95% as an example and define $PS_e - IS_e$ as the difference in the optimal expected profit under the two strategies. A positive value means that probabilistic selling is more advantageous; otherwise, inventory substitution strategy is more advantageous. We draw a colour map as shown in Fig. 3.5 to facilitate analysis of strategy selection. We colour a positive value in red and a negative one in blue. From the computational results, we make the following observations:
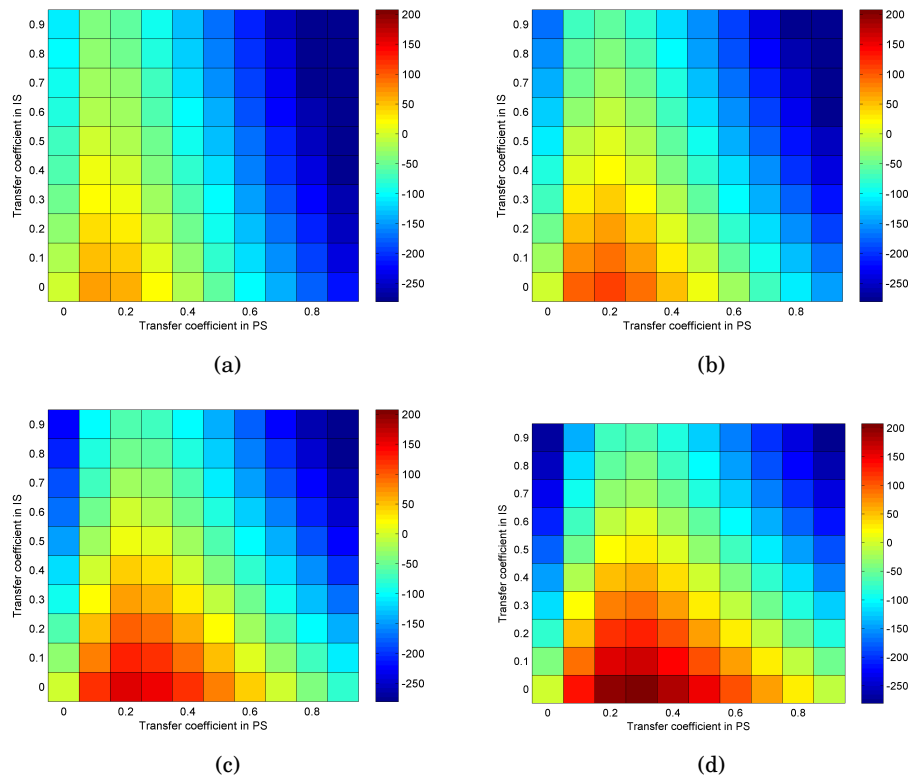
Figure 3.5: Expected profit comparison with respect to different initial demand uncertainty

Observation 3.1: With a relatively small customer transfer coefficient under probabilistic selling and inventory substitution, the former is more profitable. Inventory substitution outperforms probabilistic selling when the transfer coefficients under the two strategies are very high.

Observation 3.2: Probabilistic selling is more advantageous than inventory substitution at higher demand uncertainty.

From Fig. 3.4 and Fig. 3.5, we see that probabilistic selling can greatly improve profit with a smaller customer transfer coefficient than inventory substitution. For example, probabilistic selling can achieve a higher profit with $a = 0.1$ when $\sigma = 20$, while inventory substitution requires $r_s = 0.4$ to achieve the same profit. When $\sigma = 30$, probabilistic selling can achieve a higher profit with $a = 0.2$, while inventory substitution requires $r_s = 0.5$. However, when the customer transfer coefficient is larger, the advantage of probabilistic selling diminishes while inventory substitution can still bring more profit to the retailer. The customer transfer coefficient a under probabilistic selling mainly depends on customers' price sensitivity and product differentiation. Probabilistic selling requires that some customers are sensitive to price to be attracted to buy the probabilistic product (Zhang et al., 2016; Fay and Xie, 2008). At the same time, product differentiation should be large enough to avoid too much transfer (Post, 2010). Inventory substitution mainly depends on product differentiation. And the more customers that will accept another product are, the more sales the retailer can get in the second selling stage.

Therefore, lower product differentiation is necessary for inventory substitution to be advantageous. Just as shown in Fig. 3.5, when product differentiation is very low, and the customer transfer coefficient a and r_s are high, inventory substitution can bring more profit to the retailer. Therefore, the implication for the retailer is as follows: Adopting a proper selling strategy to manage demand uncertainty depends on customer characteristics and product differentiation. If the specific products have great similarity, inventory substitution is more advantageous, while relatively lower product similarity and higher price-sensitive customers can bring more profit to the retailer that adopts probabilistic selling. This observation is consistent with reality that probabilistic selling is common in third-party intermediary platforms which sells various products from different vendors, e.g., a seller may use inventory substitution to sell double-bed rooms and twin-bed rooms in one specific hotel, and may use probabilistic selling to sell rooms belonging to different hotels (e.g., Hotwire.com).

Observation 3.2 is obvious. The red-coloured area increases with demand uncertainty. The application range for adopting probabilistic selling is much wider when demand uncertainty is larger. Therefore, probabilistic selling is a more promising strategy to combat demand uncertainty.

3.6.3 *Comparison of the optimal inventory decision*

We define $PS_v - IS_v$ as the difference between the optimal total inventory under the two strategies. A negative value means that the retailer would hold less inventory when implementing probabilistic selling. We draw a colour map as shown in Fig. 3.6, in which we colour the positive values in red and the negative values in blue. From the results, we make the following observation.

Observation 3.3: Probabilistic selling is more advantageous than inventory substitution in reducing inventory under most circumstances.

As shown in Fig. 3.6, the blue-coloured area is very large. The inventory level under proba-
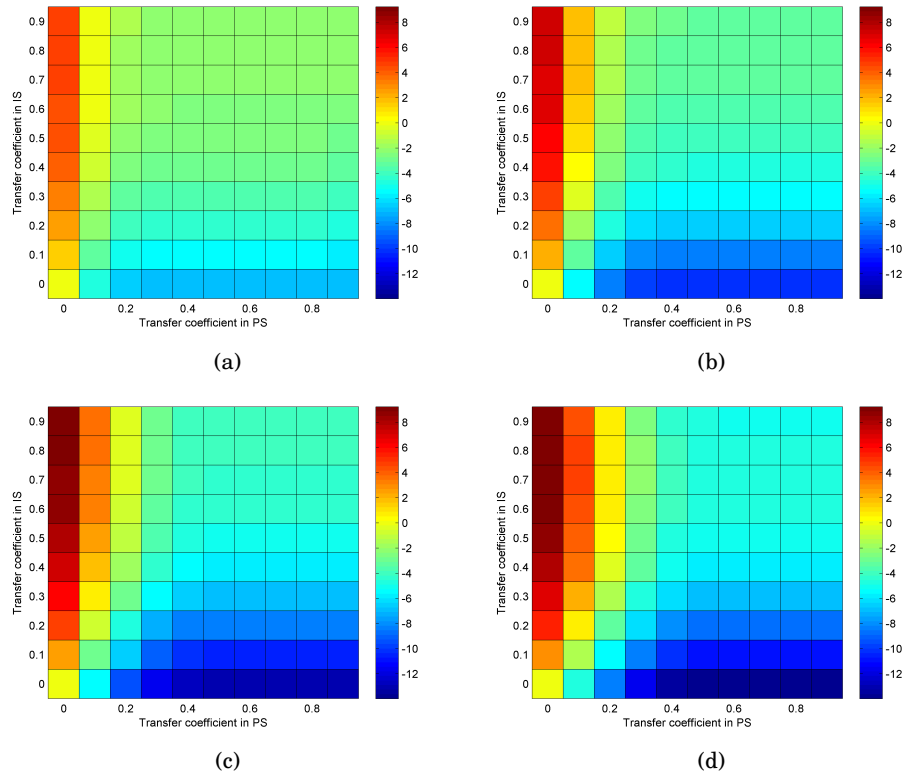
Figure 3.6: Optimal inventory comparison with respect to different initial demand uncertainty

bilistic selling is usually lower than that under inventory substitution. Combined with Fig. 3.5, we find that when probabilistic selling outperforms inventory substitution in terms of yielding a higher profit, its optimal inventory is always lower than that under inventory substitution. The only exception is when $a = 0.1$ and $r_s = 0.3$ with $\sigma=50$. In contrast, when inventory substitution is more profitable, its optimal inventory is usually higher than that under probabilistic selling. Therefore, if it is optimal for the retailer to adopt probabilistic selling in a specific environment, it can usually obtain a higher profit with lower inventory than inventory substitution.

3.7 Conclusions

By offering the low priced probabilistic product to induce some customers to buy a flexible product, the retailer can substitute demand when stock out occurs to hedge against the demand uncertainty. Both probabilistic selling and inventory substitution strategy share the common feature in combating demand uncertainty through demand substitution. Therefore, our study focuses on analyzing and comparing the efficiency of the two strategies. In this chapter we first develop a single-period newsvendor model with three products to analyze probabilistic selling with a view to generating insights into using probabilistic selling to manage demand uncertainty. We then compare probabilistic selling with inventory substitution in the special

cases without cannibalization. To gain additional insights into the normal situation, we use computational examples to compare the two strategies in terms of overall profit and inventory with considerations of customer transition and demand uncertainty.

While both inventory substitution and probabilistic selling can induce demand from product-insensitive customers to achieve demand substitution, they differ in that probabilistic selling allows customers to accept uncertainty voluntarily at a discounted price rather than forcing them to accept another product like that under inventory substitution. The computational results show that probabilistic selling will bring more profit to the retailer when selling products with relatively lower similarity to higher price-sensitive customers, and it is more profitable to use inventory substitution to sell products with high similarity. Besides, higher demand uncertainty increases the profitability of probabilistic selling over inventory substitution.

The research of exploring the inventory mechanism of probabilistic selling compared with inventory substitution has theoretical and practical significance. The study enriches the research about probabilistic selling as an inventory management tool to combat demand uncertainty. According to the conclusions of this manuscript, the retailer can choose the efficient strategy considering product differentiation, customer characteristics, and level of demand uncertainty. This has significant practical implications for the retailer that sells multiple products as follows: First, under probabilistic selling, the retailer should not be afraid of cannibalization because a proper degree of cannibalization can benefit the retailer in terms of yielding a higher expected profit. When the price of the probabilistic product increases, the retailer should always order more inventory of the product with higher substitution possibility. Second, if the retailer sells the substitute product with lower product similarity to price-sensitive customers, it is advised to use probabilistic selling to achieve a higher profit, and order less inventory than inventory substitution in most cases. On the other hand, inventory substitution is the better choice for the retailer when the product similarity is higher.

We assume in this study that the price of the probabilistic product is an exogenous variable. Future research may extend our work by combining the pricing and inventory decisions. It is also worth considering PS in a supply chain setting (Shen et al., 2017; Minner, 2003). For example, it is interesting to explore the conditions under which a retailer's probabilistic selling will benefit the supplier, the retailer, and both.

4

INVENTORY DECISION OF PROBABILISTIC SELLING WITH PRICE-DEPENDENT DEMAND RESHAPE

4.1 Introduction

When the retailer offers probabilistic goods, the retailer usually charges a lower price for the probabilistic products than their source products. According to Meredith and Maki, a low-priced brand would cannibalize sales of a higher-priced brand (upward cannibalization) with enough price gap between the two brands (Meredith and Maki, 2001). Therefore, the price gap between the specific product and probabilistic product would reshape the demand, i.e., demand for specific products switch to the probabilistic product. And it is necessary to consider the price-dependent demand reshape in probabilistic selling.

Therefore, this chapter analyzes the effect of probabilistic selling on inventory management through demand substitution and price-dependent demand reshape. The study of this chapter is based on Chapter 3. And the difference of this chapter is: In order to concentrate on the effect of probabilistic selling on inventory management, we ignore the increased demand due to the offer of the lower-priced products, i.e.,the demand for the probabilistic products comes from the switched demand. Furthermore, we assume that the demand switch is mainly determined by the price discount for the probabilistic product(Meredith and Maki, 2001; Eynan and Fouque, 2003).

It is notable that the extant studies on demand reshape rarely consider the switching cost (Eynan and Fouque, 2003, 2005). However, the demand switch in probabilistic selling is cost, i.e., price discount, driven. Therefore, we employ a switch pattern similar to Eynan and Fouque (2005) but consider the switching cost. Moreover, the switching rate is determined by the switching cost, i.e., the price gap between the probabilistic product and the specific products.

The main purpose of this study is to analyze the effect of probabilistic selling on inventory management through demand substitution and price-dependent demand reshape. We construct a newsvendor-type model to characterize the price-dependent demand substitution pattern. We perform a simulation study to extensively explore the effects of demand uncertainty, demand correlation, price sensitivity, and price discount on inventory decisions and profitability of probabilistic selling. The results show that probabilistic selling can benefit the seller with higher expected profit and lower inventory by reducing demand uncertainty and improving inventory efficiency, even without considering the increased demand due to offering the low-priced probabilistic product. Moreover, the effect of probabilistic selling is more significant with lower product differentiation, higher customers' price sensitivity, and higher demand uncertainty. It is noted that the optimal selection of the price discount is necessary to secure good performance of probabilistic selling, given that improper pricing will undermine seller's profit.

4.2 The Model

4.2.1 *Definitions and assumptions*

We consider the scenario where an online retailer of two homogeneous products, i and j, creates the virtual probabilistic product k, and sells i, j, and k simultaneously. In probabilistic selling, the specific products i and j achieve substitution through the probabilistic products k instead of being direct substitutes for each other. The event sequence of probabilistic selling is as follows: First, the retailer purchases a quantity Q_i^p of product i and a quantity Q_j^p of product j. Second, it announces the prices of the specific products and the price discount of the probabilistic product. All the demands for the products arrive at the same time. Then the retailer would substitute the available products for the stock-out products. The substitution policy is that the demands for specific products i and j have priority to be satisfied. The retailer aims to make the optimal inventory decisions with different price discounts.

We use the following notation throughout this chapter:

p: the selling price of two specific products i and j,

r: discount of the regular price p (i.e., the probabilistic product k would be sold at price $(1-r)p$,

λ: price sensitivity level of customers (the customers with higher price sensitivity are more likely to switch from purchasing specific products to the probabilistic product),

α: the rate of customers switching from the specific products i and j to the k,

c: purchasing cost of the products,

v: salvage cost of the products i, j and k,

q: penalty cost of the products i, j and k,

D_i:distribution of the original demand for the specific product i, $D_i \sim (u_i, \sigma_i)$,

D_j:distribution of the original demand for the specific product j, $D_j \sim (u_j, \sigma_j)$,

D_i^p: stochastic demand of product i when offering the probabilistic product,

D_j^p: stochastic demand of product j when offering the probabilistic product,

u_k:mean of the demand for the probabilistic product k,

σ_k: standard deviation of the demand for the probabilistic product k,

ρ_{ij}: the correlation coefficient of the original demands for products i and j,

$f(x,y)$:the joint distribution function of the original demands for specific products i and j.

Without loss of generality, we assume that the demands for specific products follow the normal distribution. The retailer gives a discount of r to induce some customers to buy the probabilistic product. When introducing the discounted probabilistic product, α per cent of the customers who originally intend to buy specific products would switch to buy the probabilistic product k. The switch rate α is related to two key elements, namely the price gap and the level of product differentiation (Meredith and Maki, 2001). Therefore, give two certain specific products, we define the switch rate as

$$\alpha = \lambda r, \tag{4.1}$$

where λ is the price sensitivity level of the customers. The discount rate r satisfies $0 \le r < 1 - c/p$ to make sure that the price of the probabilistic product is higher than the cost c. Eq.(4.1) implies that the switch rate is positively related to the discount r when $\lambda > 0$. When $r = 0$, the probabilistic selling problem degenerates into the classic newsvendor problem without demand substitution and into the centralized inventory strategy when $r = 1$.

4.2.2 *Model*

We assume that the total demand for the probabilistic product comes from demand for the specific source products. Thus, the original demands for the two specific products i and j would be reshaped under probabilistic selling as follows:

$$\begin{aligned} D_i^p &= (1-\alpha)D_i, \\ D_j^p &= (1-\alpha)D_j. \end{aligned} \tag{4.2}$$

The demand distribution of the probabilistic product can be expressed as

$$D_k^p = \alpha D_i + \alpha D_j. \tag{4.3}$$

The expected profit in Eq.(4.4) includes the revenue, the salvage cost, the penalty cost, and the acquisition cost as follows:

$$E\left(Q_i^p, Q_j^p\right) = E \left\{ \begin{array}{l} pmin\left(D_i^p, Q_i^p\right) + pmin\left(D_j^p, Q_j^p\right) \\ +(1-\lambda r)pmin\left[D_k^p, \left[\left(Q_i^p - D_i^p\right)^+ + \left(Q_j^p - D_j^p\right)^+\right]\right] \\ +s\left[\left(Q_i^p - D_i^p\right)^+ + \left(Q_j^p - D_j^p\right)^+ - D_k^p\right] - c\left(Q_i^p + Q_j^p\right) \\ -q\left[\left(D_i^p - Q_i^p\right)^+ + \left(D_j^p - Q_j^p\right)^+ - \left(D_k^p - \left(Q_i^p - D_i^p\right)^+ - \left(Q_j^p - D_j^p\right)^+\right)^+\right] \end{array} \right. \tag{4.4}$$

35

The demands for the products vary over the domain $D_i \geq 0$ and $D_j \geq 0$. With a given switch rate α, the expected profit $E\left(Q_i^p, Q_j^p\right)$ depends on the relationship between α and the optimal order sizes Q_i^p and Q_j^p. Therefore, the expected profit can be calculated in four different cases, i.e., Case I: $\alpha Q_i^p + \alpha Q_j^p \leq Q_i^p$ and $\alpha Q_i^p + \alpha Q_j^p \leq Q_j^p$; Case II: $\alpha Q_i^p + \alpha Q_j^p \leq Q_i^p$ and $\alpha Q_i^p + \alpha Q_j^p \geq Q_j^p$; Case III: $\alpha Q_i^p + \alpha Q_j^p \geq Q_i^p$ and $\alpha Q_i^p + \alpha Q_j^p \leq Q_j^p$; Case IV: $\alpha Q_i^p + \alpha Q_j^p \geq Q_i^p$ and $\alpha Q_i^p + \alpha Q_j^p \geq Q_j^p$, as shown in Fig. 4.1.
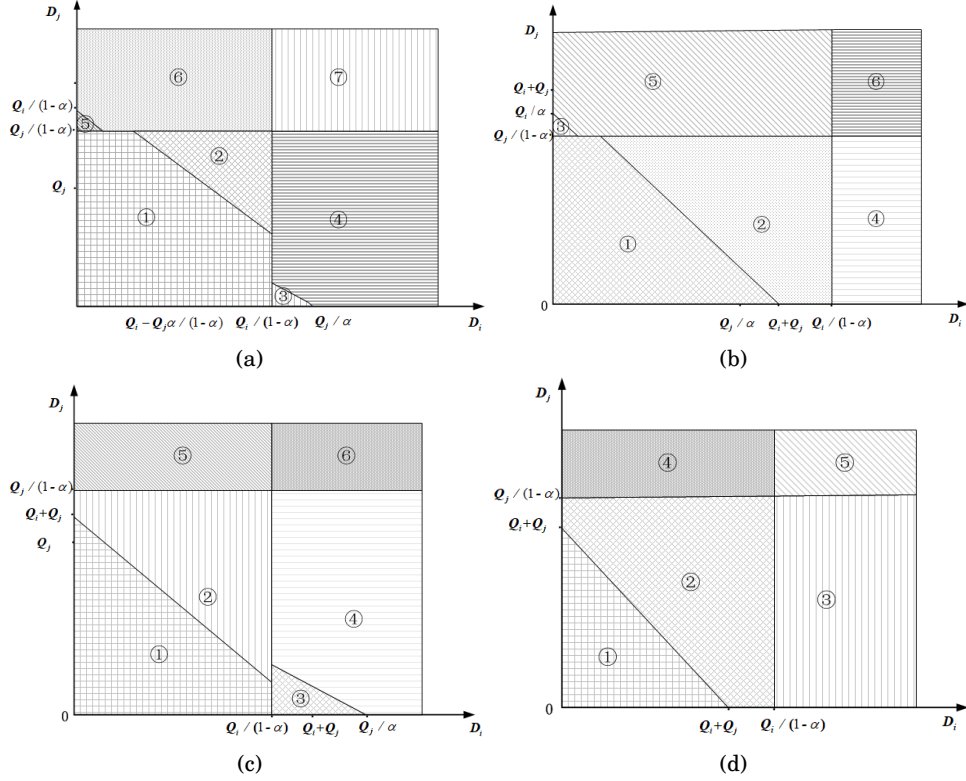
Figure 4.1: The four cases of the expected profit.

Considering the spirit of modelling the expected profit is the same as those in the above four cases, we just present in this section the details of case IV for conciseness. The expected profit functions in the other three cases can be formulated in the same way. Analysis of Case IV proceeds as follows:

(1)Under the condition of $\{Q_i \geq (1-\alpha)D_i, Q_j \geq (1-\alpha)D_j, Q_i + Q_j - D_i - D_j \geq 0\}$, the probabilistic product is out of stock. However, the generated substitute demand for product i (or product j) can be met from stock. The expected profit function is

$$\int_0^{Q_i^p + Q_j^p} \int_0^{Q_i^p + Q_j^p - x} \left[p(1-\alpha)x + p(1-\alpha)y + (1-r)p(\alpha x + \alpha y) + s\left(Q_i^p + Q_j^p - x - y\right) - c\left(Q_i^p + Q_j^p\right) \right] f(x,y) \, dx \, dy$$

(2)Under the condition of $\{Q_i \geq (1-\alpha)D_i, Q_j \geq (1-\alpha)D_j, Q_i + Q_j - D_i - D_j \leq 0\}$, the probabilistic product is stock out. Only part of the substitute demand can be met from the remaining

inventory of products i and j. The relevant expected profit function is

$$\int_0^{Q_i^p+Q_j^p} \int_{Q_i^p+Q_j^p-x}^{\frac{Q_j^p}{1-\alpha}} \left[p(1-\alpha)x + p(1-\alpha)y + (1-r)p\left(Q_i^p + Q_j^p - (1-\alpha)x - (1-\alpha)y\right) \right.$$

$$\left. - q\left(x+y-Q_i^p-Q_j^p\right) - c\left(Q_i^p+Q_j^p\right) \right] f(x,y)dxdy$$

$$+ \int_{Q_i^p+Q_j^p}^{\frac{Q_i^p}{1-\alpha}} \int_0^{\frac{Q_j^p}{1-\alpha}} \left[p(1-\alpha)x + p(1-\alpha)y + (1-r)p\left(Q_i^p + Q_j^p - (1-\alpha)x - (1-\alpha)y\right) \right.$$

$$\left. - q\left(x+y-Q_i^p-Q_j^p\right) - c\left(Q_i^p+Q_j^p\right) \right] f(x,y)dxdy$$

(3)Under the condition of $\{Q_i \leq (1-\alpha)D_i, Q_j \geq (1-\alpha)D_j, Q_j - (1-\alpha)D_j \leq \alpha D_i + \alpha D_j\}$, both the probabilistic product and specific product i are stock out. Furthermore, only part of the substitute demand can be met from the remaining inventory of product j. The relevant expected profit function is

$$\int_{\frac{Q_i^p}{1-\alpha}}^{\infty} \int_0^{\frac{Q_j^p}{1-\alpha}} \left[pQ_i^p + p(1-\alpha)y + (1-r)p\left[Q_j^p - (1-\alpha)y\right] - q\left(x+y-Q_i^p-Q_j^p\right) - c\left(Q_i^p+Q_j^p\right) \right] f(x,y)dxdy$$

(4)Under the condition of $\{Q_i \leq (1-\alpha)D_i, Q_j \leq (1-\alpha)D_j\}$, both specific products i and j are stock out. None of the substitute demand can be met. The relevant expected profit function is

$$\int_{\frac{Q_i^p}{1-\alpha}}^{\infty} \int_{\frac{Q_j^p}{1-\alpha}}^{\infty} \left[pQ_i^p + pQ_j^p - q\left(x+y-Q_i^p-Q_j^p\right) - c\left(Q_i^p+Q_j^p\right) \right] f(x,y)dxdy$$

(5) Under the condition of $\{Q_i \geq (1-\alpha)D_i, Q_j \leq (1-\alpha)D_j, Q_i - (1-\alpha)D_i \leq \alpha D_i + \alpha D_j\}$, both the probabilistic product and specific product j are stock out. Furthermore, only part of the substitute demand can be met from the remaining inventory of product i. The relevant expected profit function for this part is

$$\int_0^{\frac{Q_i^p}{1-\alpha}} \int_{\frac{Q_j^p}{1-\alpha}}^{\infty} \left[p(1-\alpha)x + pQ_j^p + (1-r)p\left[Q_i^p - (1-\alpha)x\right] - q\left(x+y-Q_i^p-Q_j^p\right) - c\left(Q_i^p+Q_j^p\right) \right] f(x,y)dxdy$$

By calculating the Hessian matrices in the four cases, we can show that the expected function $E\left(Q_i^p, Q_j^p\right)$ is concave with respect to (Q_i^p, Q_j^p) (Proof. See the Appendix B). However, we cannot derive closed-form solutions due to the high complexity of the expected profit function, which prevents us from investigating the relationships among different variables of interest. Fortunately, the expected profit function is concave, so we can search for the optimal solution using a numerical simulation approach.

4.3 Simulation insights on inventory and profit performance

4.3.1 Simulation design

This simulation aims to obtain insights into the inventory and profit performance in probabilistic selling by addressing the following two questions: (1) What is the performance of probabilistic selling in terms of total inventory level and expected profit compared with that of the classic newsvendor model? (2) What is the effect of demand uncertainty, demand correlation, price discount, and price sensitivity on the profitability of probabilistic selling?

Consistent with the above assumptions, we set the parameter values for the simulation studies as follows: $u_i = u_j = 1200, p = 40, c = 10, s = 0, q = 4, \sigma = \sigma_1 = \sigma_2 = 400, 500, 600, \rho_{ij} = -0.5, 0, 0.5, r \in [0, 0.75]$, and $\lambda \in [0, 4/3]$ [1]. With different parameter combinations, we use the Mathematica to search for the optimal orders of products i and j (Q_i^p and Q_j^p), and the corresponding expected profit $E(Q_i^p, Q_j^p)$. Then we calculate the profit increase $PI = \frac{E(Q_i^p, Q_j^p) - E(Q_i^t, Q_j^t)}{E(Q_i^t, Q_j^t)}$ and inventory decrease $DI = \frac{(Q_i^t + Q_j^t) - (Q_i^p + Q_j^p)}{Q_i^t + Q_j^t}$ with respect to those of the classical newsvendor model, where Q_i^t and Q_j^t denote the optimal orders of products i and j in the classical newsvendor model, respectively.

4.3.2 Simulation analysis

(1) Effects of price discount and demand uncertainty
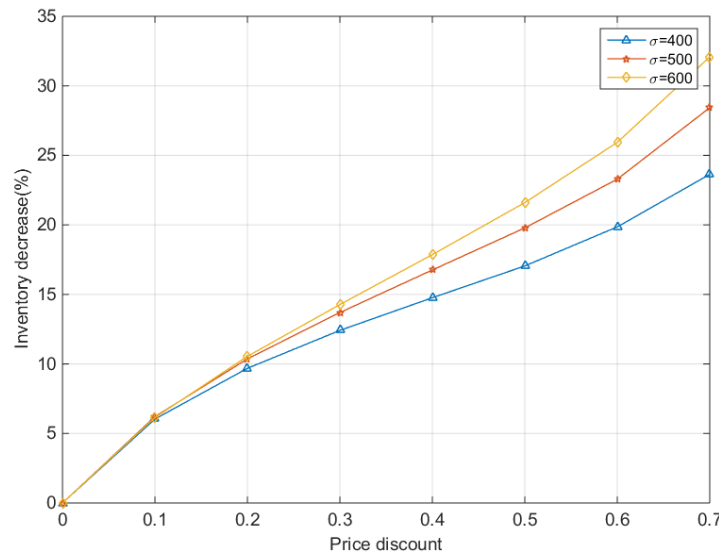
Figure 4.2: The effects of price discount and demand uncertainty on inventory level

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[1]The switch rate becomes 1 When the customer price sensitivity is larger than 4/3. Then all the customers will buy the probabilistic product, which will not affect optimal inventory level anymore but decrease the expected profit
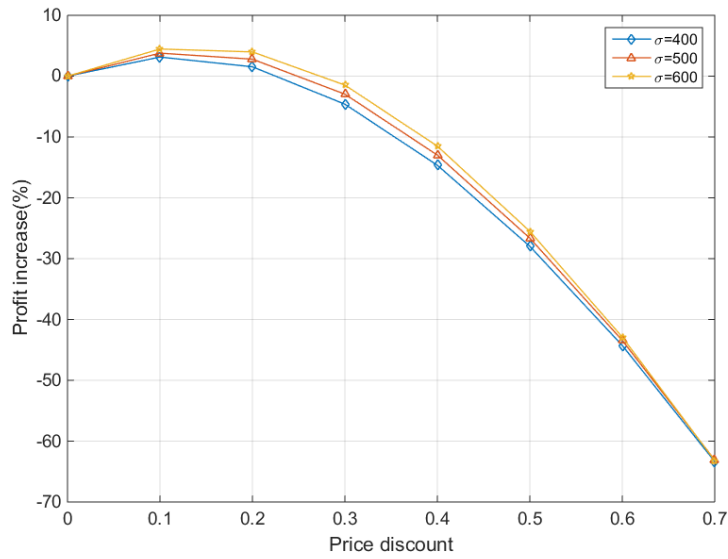
Figure 4.3: The effects of price discount and demand uncertainty on the expected profit

As shown in Fig. 4.2, the inventory level decreases with price discount. This pattern can be explained from two aspects. On the one hand, the increasing price discount induces more customers who intended to buy the two different specific products to buy single species of probabilistic product, which reshapes the demand distribution. Consequently, the aggregate demand uncertainty would decreases with the switching amount (Eynan and Fouque, 2003). Besides, the demand substitution also contributes to inventory decrease. Because with nonzero price discount, the demand of probabilistic product become a pool for substitution when stock-out occurs, which also helps to reduce demand uncertainty. However, on the other hand, the retailer would sell more of the lower-priced probabilistic product to customers with increasing price discount. Consequently, this phenomenon leads to a trade-off between the benefits of decreased inventory and profit decrease resulting from more low-priced customers. Therefore, the curves in Fig. 4.3 show that the expected profit increases with price discount first and then decreases with it. With lower price discounts between 0 and 0.2, probabilistic selling can benefit the retailer with higher profit and lower inventory. However, with larger price discounts, the benefit of lower inventory would not offset the profit decrease due to lower prices, resulting in a rapid decline. Another observation from Fig. 4.2 and Fig. 4.3 is that the efficiency of probabilistic selling would increase with respect to higher demand uncertainty. As demand reshape and demand substitution would benefit the retailer with reduced demand uncertainty, there will be more room for improvement through probabilistic selling.

(2) Effects of customer price sensitivity and price discount

As shown in Fig. 4.4, when the customers' price sensitivity is higher, the retailer can achieve higher profit increases with the optimal price discount. This is logical because with higher
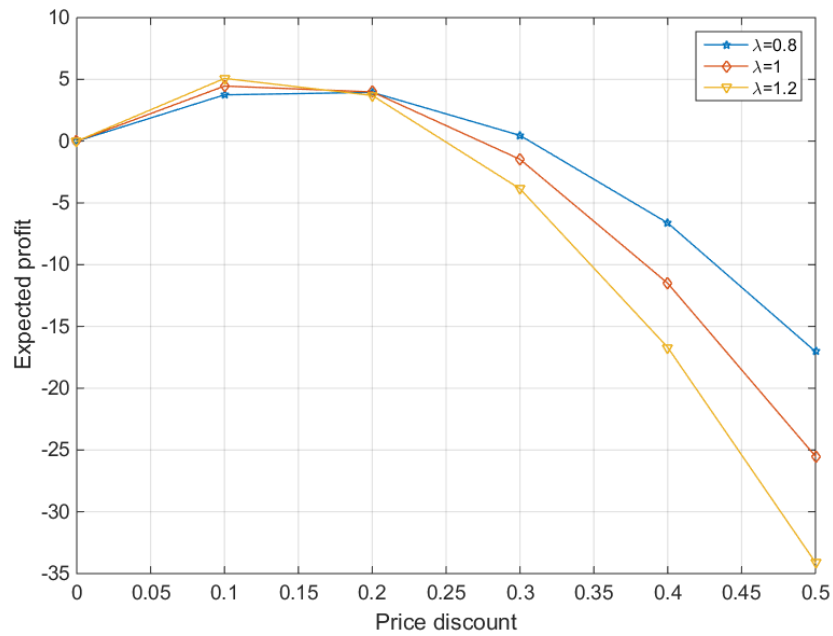
Figure 4.4: The effects of price sensitivity on the expected profit
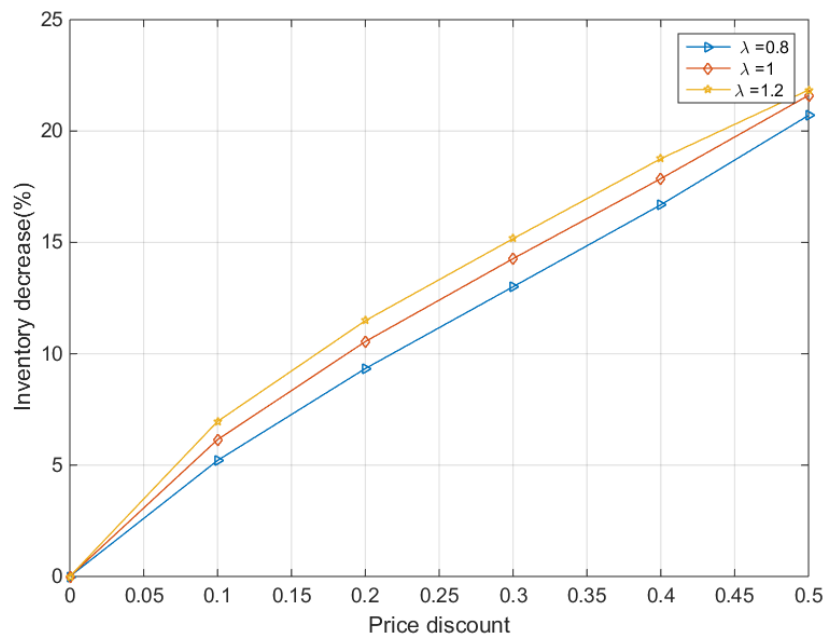
Figure 4.5: The effects of price sensitivity on the inventory level

customers' price sensitivity, there would be more demand switch, even though the retailer offers a small price discount. In other words, the retailer will benefit more from the smaller "price sacrifice". The curves in Fig. 4.5 share the same logic that the retailer can always get higher inventory decrease with higher customer price sensitivity. When the switch rate approaches 1, we can expect that probabilistic selling reduces to centralized inventory and the inventory levels would be the same. We can also interpret the observation from the perspective of product differentiation. It is reasonable that lower product differentiation leads to higher price sensitivity with some certain price discount. And according to the observation that higher price sensitivity can benefit the retailer with higher profit, we can conclude that the retailer has the potential to gain more profit by offering products with weaker product differentiation at the optimal price discount.

(3) Effects of demand correlation and price discount
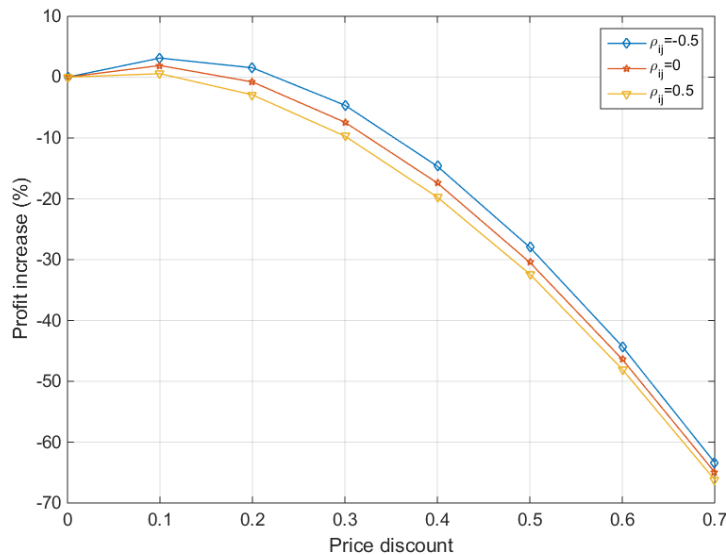
Figure 4.6: The effects of demand correlation on the expected profit

From Fig. 4.6 we can see that the expected profit decreases with demand correlation, which is consistent with Netessine and Rudi (2003). As for the aspect of demand reshape, lower demand correlation results in lower demand uncertainty. As for the aspect of demand substitution, positively correlated demands of products i and j result in simultaneous large or small substitution demands with a high possibility. That means that the probability that the retailer substitutes specific products for the probabilistic product occurs is low. Therefore, the negative correlation will result in higher expected profit.

However, the price profit gap becomes small when the price discount is very large. The possible reason is that when price discount is large, the demand for the specific products is small while the demand for the probabilistic product is large. Then the substitution demand from the customers
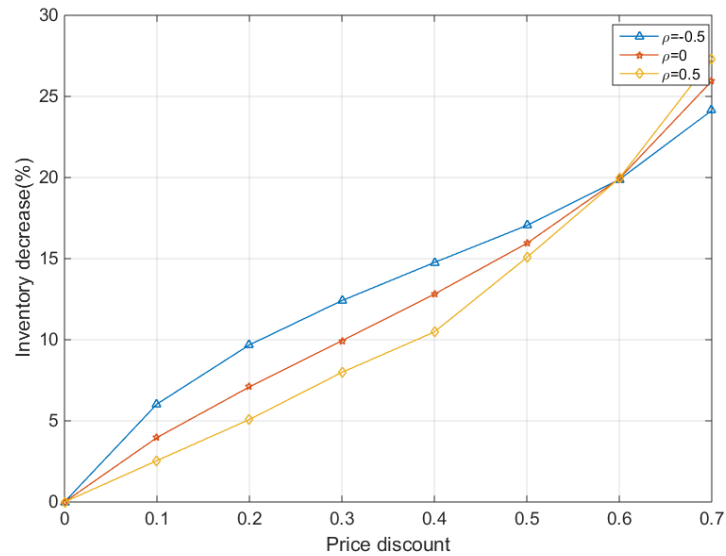
Figure 4.7: The effects of demand correlation on the expected profit

who buy specific products is small, which reduces the effect of demand substitution. Similarly, the optimal inventory level with smaller demand correlation is lower, while becomes higher when the price discount is very large. The reason is that the benefit from demand substitution would diminish when the price discount become larger. Meanwhile, the substitution probability is larger when demand correlation is negative, which increases optimal inventory (e.g., the optimal inventory increases with its substitution probability). Therefore, the inventory level with lower demand correlation maybe higher than the inventory level with higher demand correlation.

4.4 Conclusions

To investigate the effects of probabilistic selling on inventory decisions and expected profit through demand reshape and demand substitution, we first analyse the demand reshape and substitution patterns under probabilistic selling and propose a single-period newsvendor model with stochastic demands for multiple products. Subsequently, we perform a simulation experiment to figure out the effects of price discount, customer price sensitivity, demand uncertainty, and demand correlation on the retailer's optimal inventory level and expected profit. The results show that probabilistic selling can benefit the seller with higher expected profit and lower inventory by reducing demand uncertainty and improving inventory efficiency, even without considering the increased demand due to offering the low-priced probabilistic product. Besides, the efficiency of probabilistic selling with respect to managing inventory is closely dependent on customers' price sensitivity or product differentiation, demand uncertainty, and the demand correlation. Moreover, the effect of probabilistic selling is more significant with lower product differentiation, higher

customers' price sensitivity, and higher demand uncertainty. It is noted that the optimal selection of the price discount is necessary to secure good performance of probabilistic selling, given that unsuitable values of lead to lower expected profits in our simulation.

INVENTORY-PRICING POLICY IN "PHYSICAL" PROBABILISTIC SELLING

5.1 Introduction

In the retailing industry, retailers can attract customers by providing mystery products (e.g., some product information is unknown to customers until they place the orders). For example, the mystery sealed "lucky bags" of discounted products (called Fukubukuro) are very popular in Japan during the new year holiday. These lucky bags come in all shapes and sizes with various themes ranging from clothes, accessories, toys, cosmetics etc. Customers do not know what is exactly inside the bag until they buy and open them. Every year, customers in Japan and overseas fans are attracted by the random surprise and discount from the lucky bags. Another example is Ferrero that proffers the Kinder Joy or Kinder Surprise eggs containing random toys. Ferrero reported that the good performance of its range of Kinder products helped it achieve a 14% increase in annual pretax profit after sales in 2014 (Abdulla, 2015). Chocolate company Zaini also produces similar mystery chocolate toys. Yet another example is the toys proffered in a gashapon machine or an opaque box, which can be found in shopping stores or online websites, e.g., Taobao.com, Amazon.com, ToyWiz.com etc. Fig. 5.1 provides a specific example of an online retailer selling toys packaged in an opaque box (referred to as mystery products). A customer buying such an opaque product has a chance to get one of the component cartoon figures printed on the box. If the customer does not accept uncertainty, they may pay a higher price for a specific product packed in a transparent box.

Except for the benefit of demand expansion due to customer curiosity, it is a significant

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[1]Source of the specific product: https://item.taobao.com. The figure of the probabilistic product was taken in a retail store in Japan.

(a)   (b)

Figure 5.1: An example. [1]

issue to ascertain the operational benefits to retailers of offering such mystery products. Besides, when offering the specific and mystery products simultaneously, how should the retailer set its inventory-pricing policy? Specifically, we address the following questions in this chapter: What is the optimal price discount for the mystery product that yields the maximum profit to the retailer? How should the retailer allocate inventory between the specific and mystery products? What are the impacts of the key factors such as demand variability, customer price sensitivity, and product differentiation on the optimal outcomes? And, how does supply uncertainty affect the retailer's performance?

The selling strategy that offers mystery products can be regarded as a special form of probabilistic selling (PS). Fay and Xie (2008) defined PS as the selling strategy under which the retailer selling multiple products creates an additional "virtual" probabilistic product by hiding some product information to be sold at a discount. However, differing from the probability product in Fay and Xie (2008), ours is a discounted physical probabilistic product (PPP) that can also be bought in bricks-and-mortar shops rather than merely a "virtual" product bought online. To differentiate our work from Fay and Xie (2008), we refer to the PS strategy under study as "physical probabilistic selling" (PPS).

Related research has demonstrated that the offer of a lower-priced probabilistic product enables the retailer to segment the market (Shapiro and Shi, 2008; Fay and Xie, 2008) and increase the product category range to satisfy customers' personalized needs, which facilitate differential pricing to increase market size and profit (Anupindi and Jiang, 2008; Fay and Xie, 2008). Some operational studies also have explored the benefits of adopting PS for inventory management through postponing the delivery of the probabilistic product (Petrick et al., 2012; Gönsch and Steinhardt, 2013; Gallego and Phillips, 2004; Wu and Wu, 2015), dynamic allocation depending on inventory (Elmachtoub and Wei, 2015), or demand substitution (Zhang et al., 2016). However, the above literature considers "virtual" probabilistic products rather than physical ones.

In addition, the benefits for inventory management are not applicable under PPS. In our setting, the customer that buys the PPP will get the product immediately and the retailer cannot benefit from PPS through demand postponement or dynamic allocation. In other words, the retailer has to make the pricing and inventory allocation decisions in advance. The benefits of demand postponement and dynamic allocation from selling "virtual" probabilistic products do not exist and the motivation for adopting PPS is unclear.

In this study we consider a retailer ordering two specific products from a upstream supplier has the choice to allocate some inventory as a discounted PPP. All the demand for the PPP comes from the stochastic demand for the specific products, depending on the price discount and product differentiation. We construct a model to derive the retailer's optimal price-inventory decisions to maximize its expected profit. We then compare PPS with traditional selling (TS) with respect to expected profit and optimal inventory. Finally, we perform numerical studies to investigate the effect of supply risk on PPS. To address the research questions, we follow the approaches of Eynan and Fouque (2003), and Hsieh (2011) for modelling and analysis.

Our main result is that PPS can help alleviate both demand and supply uncertainty risks. First, we find that PPS can pool the uncertainty risk through demand reshape. Under PPS, the price gap that exists between the specific product and the PPP causes demand reshape, i.e., the demand for the specific product switches to the demand for the PPP. The practice of demand reshape through offering a low-priced PPP affects the retailer's profits in two ways. On the one hand, the aggregate demand uncertainty will be reduced. On the other hand, the revenue will decrease due to the product switch. We show that the retailer can improve its profit while decreasing its inventory with an optimal price discount for the PPP. The larger the demand uncertainty is, the larger discount the retailer needs to offer to induce more demand switch to pool the demand risk. In addition, we find that the advantage of PS is decreasing in product differentiation and increasing in customer price sensitivity. Second, PPS can even be more advantageous due to its inventory allocation flexibility when the supply is subject to asymmetrical uncertainty risk.

We make four contributions in this study. 1) Considering PS, we are the first to explore the profitability and pricing-inventory policy of the retailer that offers the PPP. 2) We analytically examine the risk-pooling effect of PPS through demand reshape. 3) We find the risk diversification effect of PPS that helps alleviate asymmetrical supply risk through numerical studies. 4) In terms of demand reshape, we establish that PPS is a special application of the demand switch induced by price discount in real practice. Our research findings and their management implications help practitioners gain more insight on the risk-pooling effect of PPS and facilitate their pricing-inventory related decision-making.

5.2 Literature Review

5.2.1 *Probabilistic selling*

Our work differs from this stream of literature by exploring the benefits of PS when the retailer offers a PPP rather than merely a virtual choice. Given that the retailer cannot reap the benefits postponement and dynamic allocation by offering the PPP, we analytically examine the benefit of PPS for managing demand uncertainty through demand reshape. In addition, we also evaluate by numerical studies the performance of PPS in alleviating supply uncertainty through inventory flexibility.

5.2.2 *Demand reshape*

Our model also considers the three-product scenario similar to the "merge-switching" pattern in Hsieh (2011), whereby a firm selling multiple products (e.g., A, B, and C) convinces customers that originally purchase products A and B to switch to purchasing product C without cost. The demand switching pattern under PPS in our study is different from his as follows: 1) There are two kinds of product under PPS, namely products A and B, while product C is a PPP that may become product A or B. 2) Demand switching in his research is driven by advertising rather than the price gap under PPS (Meredith and Maki, 2001). Therefore, we have to characterize the relationship between the price discount and switch rate, and then analytically derive both the price and inventory allocation decisions.

5.2.3 *Risk management strategies*

Managing inventory is difficult due to risk, which is usually subject to prevalent demand or supply uncertainty, and firms can adopt a number of strategies to manage the corresponding risk.

In the single-product system, firms can consider adopting the strategies of centralized inventory, transshipment, or postponement to combat demand uncertainty. Centralized inventory is used by a firm that sells the same product at multiple locations, whereby it consolidates inventory in one single warehouse. Pooling inventory allows a firm to take advantage of random fluctuations in demand (Eppen, 1979; Snyder and Shen, 2011). The transshipment strategy is used by a firm to re-distribute inventory of the same echelon among multiple locations (Dong and Rudi, 2004; Tai and Ching, 2014; Dehghani and Abbasi, 2018). Postponement benefits a firm by delaying its operational activities (e.g., production and delivery of products) in the supply chain until the firm receives more information about the demand (Aviv and Federgruen, 2001; Tang, 2011; Anand and Girotra, 2007; Anupindi and Jiang, 2008). For a review of the research on transshipment and postponement, the reader may refer to Paterson et al. (2011) and Van Hoek (2001).

In a multi-product system, strategies such as component commonality, inventory substitution, and demand reshape are usually considered to combat risk. Component commonality means that a firm that manufactures different end products can decrease inventory and manufacturing

cost by improving component part standardization (Collier, 1981, 1982; Gerchak et al., 1988). Inventory substitution is used to persuade the customer to buy a substitute when their required product is out of stock (Parlar, 1988; Bassok et al., 1999; Lee et al., 2015; Chen et al., 2015). Eynan and Fouque (2003), and Hsieh (2011) explored the risk polling effect of "demand reshape" by encouraging the customer to switch to buying another product. The difference between demand reshape and inventory substitution is that the former is a voluntary switch while the latter is a forced switch when stock-out occurs (Eynan and Fouque, 2003).

Supply uncertainty has also been widely studied mainly in two forms, i.e., supply disruption and yield uncertainty. Supply disruption means that the supply may be halted due to natural disasters, political events, social threats etc. Yield uncertainty usually means that the delivery quantity of the supplier is random. Regarding the strategies to combat supply uncertainty, Tomlin (2006) pointed out that in the single-product setting, faced with supply uncertainty, a firm can consider increasing the inventory level, souring from more reliable suppliers (Parlar and Wang, 1993; Wang et al., 2010; Tang and Kouvelis, 2011; Giri, 2011), or just accepting the risk passively. Schmitt et al. (2015), and Atan and Snyder (2012) showed that decentralized inventory design, i.e., stocking inventory at multiple locations, can help the firm reduce cost variance by risk diversification. Schmitt et al. (2015) also pointed out that centralization is optimal when supply is deterministic and demand is stochastic due to the risk-pooling effect.

The majority of the literature explores the use of a strategy into combat either demand uncertainty by risk pooling or supply uncertainty by risk diversification. In contrast, we study the advantage of PPS as a strategy to combat both demand and yield uncertainty. Specifically, we focus on the performance of PPS in terms of its risk-pooling effect, i.e., demand reshape by price discount, and risk-diversification ability ,i.e., inventory flexibility by offering PPP.

5.3 Model

5.3.1 *Definitions and assumptions*

1) *Definitions*

p: the selling price of two specific products i and j,

r: the PPP k is sold at a discount r, where $0 < r \le 1$, so the price of the PPP is $(1-r)p$,

α: the rate of customers switching from the specific products i and j to the PPP k,

d: level of product differentiation between the two specific products,

λ: the sensitivity of the switch rate to r/d,

c: purchasing cost of the products,

v: salvage cost of the products i, j and k,

q: penalty cost of the products i, j and k,

D_i:distribution of the original demand for the specific product i, $D_i \sim (u_i, \sigma_i)$,

D_j:distribution of the original demand for the specific product j, $D_j \sim (u_j, \sigma_j)$,

$\tilde{D}_i$:distribution of the demand for the specific product i after demand reshape, $\tilde{D}_i \sim (\tilde{u}_i, \tilde{\sigma}_i)$,

$\tilde{D}_j$:distribution of the demand for the specific product j after demand reshape, $\tilde{D}_j \sim (\tilde{u}_j, \tilde{\sigma}_j)$,

u_k:mean of the demand for the PPP k,

σ_k: standard deviation of the demand for the PPP k,

ρ_{ij}: the correlation coefficient of the original demands for products i and j,

X, Y, Z: the demands for the specific products i and j, and the PPP k, respectively,

$\tilde{f}(x, y, z)$: the joint distribution function of the demand for the specific and probabilistic products after demand reshape,

Q_i^p, Q_j^p, Q_k^p: the inventory for the specific products i, j, and the PPP k, respectively,

Q_i^t, Q_j^t: the inventory for the specific products i and j respectively, under TS,

Q_{pps}^*, Q_{ts}^*: the optimal total inventory under PPS and TS, respectively,

ϕ_{pps}, ϕ_{ts}: the expected profit under PPS and TS, respectively.

    2) *Assumptions*

a) We make the usual assumption that $p > c > v$ to make the analysis meaningful.

b) We assume that the price of the specific products p is exogenous, which is determined by the market.

c) The retailer orders the products from a upstream supplier and has the ability to package them. In order to focus on the demand reshape effect, we ignore the package cost in our study.

d) We assume that the retailer cannot open the package to substitute the PPP for the specific products, or vice versa[2]. All the unsold products will be disposed of at the salvage cost.

e) The customer who faces a stock out will leave the market rather than choosing a substitute product.

5.3.2 *Model*

The retailer creates the PPP using the existing specific products and gives a discount of r to induce some customers to buy the PPP. Given the discounted PPP, a fraction α of the customers who originally intend to buy specific products will switch to buying the PPP. The value of α depends on two key elements, namely the price gap between the specific and probabilistic products, and the level of product differentiation (Anderson, 2009; Post, 2010). Therefore, we define

$$\alpha = \begin{cases} \lambda r/d & \text{if } 0 < r \leqslant d/\lambda \\ 1 & \text{if } r > d/\lambda \end{cases} \tag{5.1}$$

where $0 < \alpha \leq 1$. d ($d_0 < d < 1$) measures the level of product differentiation. The lowest product differentiation d_0 ensures that the assumption about the original demand distributions for the two specific products is reasonable [3]. A smaller d denotes small differentiation between the two

---

[2]This is reasonable for the operation costs (e.g., the machine setup cost and labour cost to check the product information or to re-package the product).

[3]When product differentiation is very small, i.e., the extreme case when $d = 0$ and the two products reduced to one single product, the retailer may lose some customers due to lower product variety.

specific products (e.g., two T-shirts in different colours have smaller differentiation than those made of different materials). λ ($\lambda > 0$) denotes the sensitivity of the switch rate, i.e., from a specific product to the PPP k, to r/d [4]. A larger λ means that the customers are more sensitive to price, and a smaller λ means that the customers are more sensitive to product differentiation. It is evident from Eq.(5.1) that the switch rate increases with the discount rate and decreases with product differentiation. When λ is large enough to make $\lambda r/d > 1$, then all the customers buying the specific products will switch to buying the PPP, for which we define $\alpha = 1$. Thus, the original demands for the specific products i and j are as follows:

$$
\begin{aligned}
\tilde{u}_i &= (1-\alpha)u_i, \\
\tilde{u}_j &= (1-\alpha)u_j, \\
\tilde{\sigma}_i &= (1-\alpha)\sigma_i, \\
\tilde{\sigma}_j &= (1-\alpha)\sigma_j.
\end{aligned}
\tag{5.2}
$$

It follows that the mean and standard deviation of the demand distribution of the PPP are, respectively, as follows:

$$
\begin{aligned}
u_k &= au_i + au_j, \\
\sigma_k &= \alpha\sqrt{\sigma_i^2 + \sigma_j^2 + 2\rho_{ij}\sigma_i\sigma_j}.
\end{aligned}
\tag{5.3}
$$

With the discount rate $0 < r \le 1$, we can divide the demand scenario into eight cases as follows: 1)$X \le Q_i^p, Y \le Q_j^p, Z \le Q_k^p$; 2)$X > Q_i^p, Y \le Q_j^p, Z \le Q_k^p$; 3)$X \le Q_i^p, Y > Q_j^p, Z \le Q_k^p$; 4)$X \le Q_i^p, Y \le Q_j^p, Z > Q_k^p$; 5)$X > Q_i^p, Y > Q_j^p, Z \le Q_k^p$; 6)$X \le Q_i^p, Y > Q_j^p, Z > Q_k^p$; 7)$X > Q_i^p, Y \le Q_j^p, Z > Q_k^p$; and 8)$X > Q_i^p, Y > Q_j^p, Z > Q_k^p$.

We can express the retailer's expected profit function under PPS as the sum of the expected profits of the demands in each case, e.g., the expected profit in Case 4) is

$$
\int_0^{Q_i^p} \int_0^{Q_j^p} \int_{Q_k^p}^{\infty} \left[px + py + (1-r)pQ_k^p + v(Q_i + Q_j^p - x - y) - q(z - Q_k^p)) \right] \tilde{f}(x,y,z)dzdydx.
\tag{5.4}
$$

According to Eynan and Fouque (2003), and Hsieh (2011), we can express the expected profit as the sum of independent newsvendor problems as follows:

$$
\begin{aligned}
\phi_{pps} &= \int_0^{Q_i^p} \left[px + v(Q_i^p - x)) \right] \tilde{f}(x)dx + \int_{Q_i^p}^{\infty} \left[pQ_i^p - q(x - Q_i^p) \right] \tilde{f}(x)dx \\
&+ \int_0^{Q_j^p} \left[py + v(Q_j^p - y)) \right] \tilde{f}(y)dy + \int_{Q_j^p}^{\infty} \left[pQ_j^p - q(y - Q_j^p) \right] \tilde{f}(y)dy \\
&+ \int_0^{Q_k^p} \left[(1-r)pz + v(Q_k^p - z)) \right] \tilde{f}(z)dz + \int_{Q_k^p}^{\infty} \left[(1-r)pQ_k^p - q(z - Q_k^p) \right] \tilde{f}(z)dz,
\end{aligned}
$$

---

[4]The implicit assumption is that the demand switch due to the price and the product dimensiona are independent. This is similar to and supported by the prospect theory (Tversky and Kahneman, 1991). Zhou (2011) also pointed out that "customers' preference-dependent 'loss utility' occurs separately in the price dimension and product dimension".

where $\tilde{f}(x) = \int_{y=0}^{\infty} \int_{z=0}^{\infty} \tilde{f}(x, y, z) dz dy$, $\tilde{f}(y) = \int_{x=0}^{\infty} \int_{z=0}^{\infty} \tilde{f}(x, y, z) dz dx$, and $\tilde{f}(z) = \int_{x=0}^{\infty} \int_{y=0}^{\infty} \tilde{f}(x, y, z) dy dx$. Therefore, the optimal inventory for the specific products i and j $Q_i^{p^*}$, $Q_j^{p^*}$ and that for the PPP $Q_k^{p^*}$ satisfy the equations $F(Q_i^{p^*}) = \int_{-\infty}^{Q_i^{p^*}} \tilde{f}(x) dx = \int_{-\infty}^{z_i} f_s(x) dx = \frac{p-c+q}{p-v+q}$, $F(Q_j^{p^*}) = \int_{-\infty}^{Q_j^{p^*}} \tilde{f}(y) dy = \int_{-\infty}^{z_j} f_s(x) dx = \frac{p-c+q}{p-v+q}$, and $F(Q_k^{p^*}) = \int_{-\infty}^{Q_k^{p^*}} \tilde{f}(z) dz = \int_{-\infty}^{z_k} f_s(x) dx = \frac{(1-r^*)p-c+q}{(1-r^*)p-v+q}$, respectively, where $f_s(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Assuming that all the demands follow the normal distribution, we characterize the optimal inventory decisions as follows:

$$\begin{aligned} Q_i^{p^*} &= \tilde{u}_i + z_i \tilde{\sigma}_i, \\ Q_j^{p^*} &= \tilde{u}_j + z_j \tilde{\sigma}_j, \\ Q_k^{p^*} &= u_k + z_k \sigma_k. \end{aligned} \tag{5.5}$$

Then the expected profits of products i, j, and k can be characterized as $\phi_i = (p-c)\tilde{u}_i - [(c-v)z_i + (p-v+q)L(z_i)]\tilde{\sigma}_i$, $\phi_j = (p-c)\tilde{u}_j - [(c-v)z_j + (p-v+q)L(z_j)]\tilde{\sigma}_j$, and $\phi_k = ((1-r)p - c)u_k - [(c-v)z_k + ((1-r)p-v+q)L(z_k)]\sigma_k$, respectively, where $L(z) = \int_{x=z}^{\infty}(x-z)f_s(x)dx$ ($z = \{z_i, z_j, z_k\}$) (see, e.g., Silver et al., 1998). Then the expected profit is the sum of ϕ_i, ϕ_j, and ϕ_k as follows:

$$\begin{aligned} \phi_{pps} &= (p-c)u_i - rp(u_i \lambda \frac{r}{d}) - [(c-v)z_i + (p-v+q)L(z_i)](1 - \lambda \frac{r}{d})\sigma_i \\ &+ (p-c)u_j - rp(u_j \lambda \frac{r}{d}) - [(c-v)z_j + (p-v+q)L(z_j)](1 - \lambda \frac{r}{d})\sigma_j \\ &- [(c-v)z_k + ((1-r)p-v+q)L(z_k)] \lambda \frac{r}{d} \sqrt{\sigma_i^2 + \sigma_j^2 + 2\rho_{ij}\sigma_i\sigma_j}. \end{aligned} \tag{5.6}$$

5.4 Optimal pricing and inventory allocation

In this section we derive the optimal price discount for the PPP r^*, the optimal switching rate α^*, and the optimal inventory allocation, and then analyze the conditions under which PPS is more profitable than TS.

Proposition 5.1. *ϕ_{pps} is concave in r when $\lambda \frac{r}{d} \leqslant 1$ and decreases with r when $\lambda \frac{r}{d} > 1$. ϕ_{pps} is concave in α.*

Proof. See the Appendix C.

This proposition implies that PPS can achieve its highest profit with a proper switch rate induced by an optimal price discount. When $\lambda \frac{r}{d} > 1$, all the customers switch to buying the PPP and the model reduces to the single-product newsvendor problem, i.e. $\alpha = 1$. With increasing r, the aggregate demand uncertainty remains the same but the price of the PPP decreases. Then the retailer cannot gain any benefit from the discount. Therefore, ϕ_{pps} decreases with r when $\lambda \frac{r}{d} > 1$ and the optimal price discount r^* is located within the range $r^* \subseteq (0, min(d/\lambda, 1)]$. Fig. 5.2 gives an example of the profit change under PPS with respect to price discount, which also verifies Proposition 5.1.
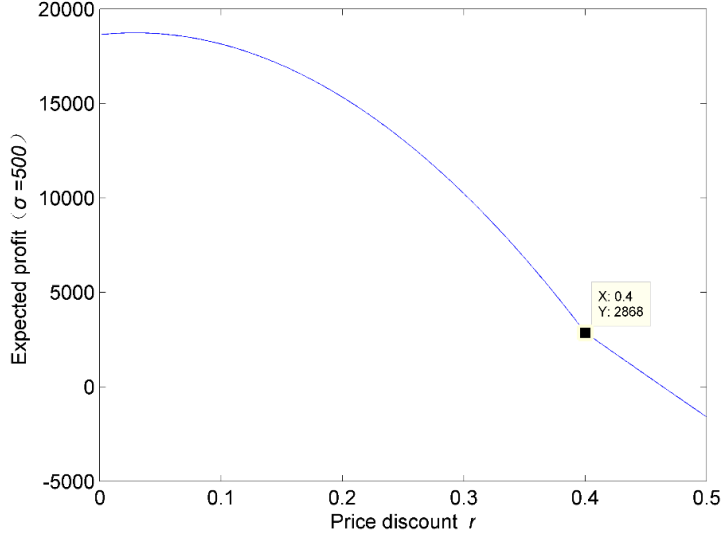
Figure 5.2: Example of profit change with respect to r
$(u_i = u_j = u = 1200, \sigma_i = \sigma_j = \sigma = 500, p = 20, c = 10, q = 4, v = 6, \lambda = 1, d = 0.4)$.

Proposition 5.2. *The optimal price discount r^* is characterized by $D_c = 2pr^*(u_k - L(z_k)\sigma_k)$, where $D_c = [(c-v)z + (p-v+q)L(z)](\alpha\sigma_i + \alpha\sigma_j) - [(c-v)z_k + (p-v+q)L(z_k)]\sigma_k$, $z_i = z_j = z$, and $L(z_i) = L(z_j) = L(z)$.*

Corollary 1. *The condition under which PPS is more profitable than TS is that there exists a positive r^* for which the equation holds.*

Proof. See the Appendix C.

The result is not difficult to understand. D_c means the cost saving of the switched demand due to centralization through PPS and the right hand side of the equation is two times of the expected revenue loss due to the price discount. Note that $u_k - L(z_k)\sigma_k$ is the expected sales of the PPP and supposed to be non-negative, the cost saving D_c should always be non-negative when the retailer adopts PPS. This is consistent with our institution that the retailer should balance the cost saving resulting from risk pooling and the profit loss due to the lower profit margin of the PPP. The best optimal price discount that maximizes profit under PPS can be achieved when the cost saving equals two times of the expected revenue loss. The corollary is intuitive with reference to Proposition 5.2. When $r^* \to 0$, it is not profitable for the retailer to offer the PPP.

Proposition 5.3. *The optimal switch rates and the optimal inventory decisions are characterized by $\alpha^* = \lambda\frac{r^*}{d}, Q_i^{p*} = (1-\alpha^*)u_i + z_i(1-\alpha^*)\sigma_i, Q_j^{p*} = (1-\alpha^*)u_j + z_j(1-\alpha^*)\sigma_j$, and $Q_k^{p*} = \alpha^*(u_i + u_j) + \alpha^* z_k\sqrt{\sigma_i^2 + \sigma_j^2 + 2\rho_{ij}\sigma_i\sigma_j}$.*

Proof. The proof is easy and we omit it.

Note that Q_k^{p*} is a mix of both specific products i and j. Therefore, PPS provides allocation flexibility, which also benefits the retailer when supply shortage occurs. We evaluate the perfor-

mance of PPS in terms of the expected profit, inventory, and customer service level when supply shortage occurs by numerical studies.

Proposition 5.4. $\phi_{pps}^* \geq \phi_{ts}^*$, $Q_{pps}^* \leq Q_{ts}^*$, *i.e., the optimal price discount makes PPS more profitable than TS, yielding a higher profit and lower inventory. Furthermore,* $\frac{\partial(Q_{pps}^* - Q_{Ts}^*)}{\partial r^*} \leq 0$.

Proof. See the Appendix C.

The proposition shows that, because ϕ_{ts}^* is independent of r, ϕ_{pps} is concave in r when $\lambda r/d \leq 1$. Thus, we believe that the improper pricing may make PPS worse than TS. However, PPS can always improve profit through optimal price discount. The total inventory under PPS always decreases with increasing optimal price discount. This is the result of the risk-pooling effect due to demand reshape. This can be explained by Eq.(5.3) that the aggregate demand deviation can be reduced after demand reshape, i.e., $\tilde{\sigma}_i + \tilde{\sigma}_j + \sigma_k \leq \sigma_i + \sigma_j$ since $\rho_{ij} \leq 1$. The decreased demand deviation results in lower safety stock, which can decease the inventory cost. From the perspective of practice, the excess inventory of one product can be used to meet the demand for another product. Similar to the findings on centralized inventory (Snyder and Shen, 2011), the risk-pooling effect is significant especially when the demands for products i and j are negatively correlated. In addition, the chance of this supply-demand match happening under TS is very small, because the customers that intend to buy one product are usually reluctant to accept another product without a price discount.

Proposition 5.5. *The optimal profit and inventory differences have the following properties:* $\frac{\partial(\phi_{pps}^* - \phi_{ts}^*)}{\partial d} < 0$, $\frac{\partial(\phi_{pps}^* - \phi_{ts}^*)}{\partial \lambda} > 0$, $\frac{\partial(Q_{pps}^* - Q_{ts}^*)}{\partial d} > 0$, *and* $\frac{\partial(Q_{pps}^* - Q_{ts}^*)}{\partial \lambda} < 0$, *i.e., the advantage of PPS is decreasing in product differentiation d and increasing in customer price sensitivity (or decreasing in customer product sensitivity).*

Proof. See the Appendix C.

This proposition implies that it is more profitable to adopt PPS to sell products with smaller product differentiation to customers with higher price sensitivity. This is easy to understand as PPS reshapes demand through price discount. A smaller product differentiation or a higher price sensitivity means that with a small price discount, a large proportion of the customers will switch from buying the specific products to buying the PPP, which decreases the aggregate demand uncertainty. So it will cost the retailer less to pool the risk, which increases the profitability of PPS. When d is low enough or λ is large enough to make $\lambda r/d > 1$, the problem reduces to the single-product newsvendor problem and the retailer will only offer PPP.

This can partially explain why the gashapon machines only sell PPPs, and all the Kinder Surprise toys are packaged in opaque chocolate eggs. From the operations management perspective, when the toys are similar and customers' price sensitivity is high, the best strategy for the retailer is to give a very small discount to induce all the customers to buy the PPP, which can efficiently decrease the risk of demand uncertainty. Besides, our conclusion can also partially explain why the retailer of Kinder Joy provides two versions, i.e, the boy's and the girl's version, and there are usually different gashapon machines in a store rather than one big machine.

Because the differentiation between the Kinder Surprise toys for boys and girls are relatively large, the retailer gives a larger price discount to induce all the customers to buy the product, which may be worse than TS due to a loss in profit margin. On the other hand, when the price discount is the same as before, the demand switch is very limited because a girl will not accept the boy's version of the toy.

The conclusion that PPS is more advantageous when customer price sensitivity is higher may partially explain that the sealed "lucky bags" are usually offered at the end of the selling season or during holidays in Japan. Because the customers that postpone their purchases until the holiday time are often highly price sensitive. The retailer can give a small discount to induce demand switch more easily. Therefore, the advantage of offering the PPP at the product release time will not be so apparent.

5.5 Numerical studies

We examine the effects of demand uncertainty, product differentiation, and customer price sensitivity on the performance of PPS to generate some practical insights. Then, we explore the effect of PPS on managing supply shortage risk. For the numerical studies, we assumed that the demands are normally distributed with $\rho_{ij} = -0.5, u_i = u_j = 1200, \sigma_i = [20, 500], \sigma_j = [20, 500], p = 12, c = 10, q = 4, v = 6, \lambda = 1$ and $d = 0.4$. So we required the maximum price discount not to exceed 1/6 to ensure that $(1-r)p > c$. We computed the optimal decisions using the MATLAB software.

5.5.1 Parameter analysis

In this part we explore the optimal decisions and performance of PPS under different demand uncertainty combinations. The observations are as follows:

Observation 5.1: *The optimal price discount increases with demand uncertainty.*

Observation 5.2: *A larger demand uncertainty increases the advantage of PPS over TS in terms of increasing profit while decreasing inventory.*

Fig. 5.3- Fig. 5.5 illustrate the relationship between demand uncertainty and the optimal price discount, total inventory, and expected profit. With increasing demand uncertainty of one product or both products, the retailer should give a greater price discount for the PPP to induce more reshaped demand to pool the risk. The increasing demand uncertainty makes PPS more advantageous than TS in terms of higher profit and lower inventory. Therefore, we can deduce that PPS is more profitable at a higher optimal price discount. This also shows that PS can pool the risk through demand reshape, and its effect on risk pooling will be more effective when demand uncertainty is larger.
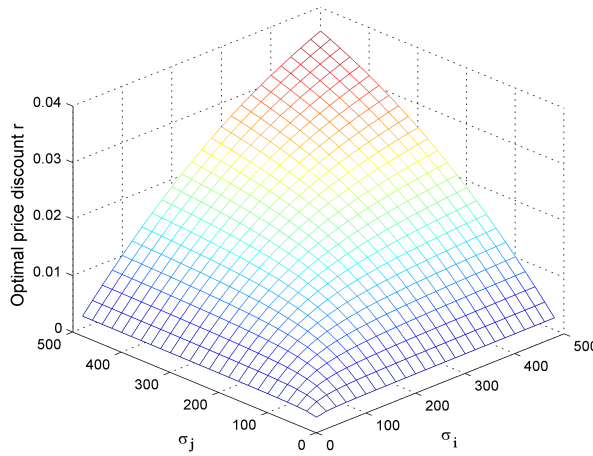
Figure 5.3: Optimal price discount with respect to demand uncertainty
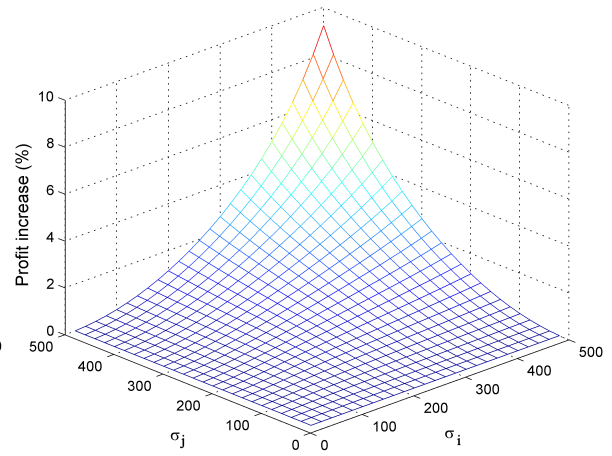
Figure 5.4: Profit change with respect to demand uncertainty
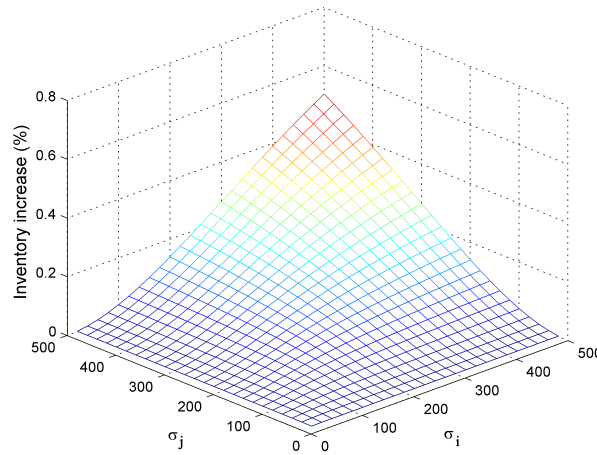
Figure 5.5: Inventory change with respect to demand uncertainty

5.5.2 Effect of supply uncertainty

The analysis above shows the advantage of PPS in pooling the risk of demand uncertainty through demand reshape. It is interesting to further explore the performance of PPS when supply risk occurs. The supply risk in our study is subject to yield uncertainty which means that the delivery quantity of the supplier is random. In reality, it may arise due to shortages of raw and semi-finished materials, failed production schedules, quality problems, etc. We assume that the retailer cannot predict the supply condition and does not have a second chance to order product from other suppliers. Alternatively, the retailer adopts the default passive acceptance strategy mentioned in Tomlin (2006) to react to supply disruption. In this section we consider the case where the supply of one product is in shortage while the other product is unlimited to reflect asymmetrical supply shortage, and the case where the supplies of both products are in shortage

to reflect symmetrical supply shortage. We define the supply shortage rate as the percentage of orders that are not met from supply under TS.

In addition, we define the Type-2 service level [5] as $ESL = \sum_{n=1}^{N} \frac{1}{N} ESL_n = \sum_{n=1}^{N} \frac{1}{N}(1-\sigma_n L(z_n))$, where ESL_n is the service level of product n (Snyder and Shen, 2011). Table 5.1 and Table 5.2 show the optimal inventory, and the resulting profit and service level under both TS and PPS when supply risk occurs. The order of the PPP Q_k^{p*} is a mix of the specific products i and j, and we define the proportion of product j as φ. Therefore, as shown in Table 5.1, the retailer will reduce the proportion of product j allocated to the PPP (denoted by $Q_k^{p*}(\varphi)$) when the supply of product j (denoted by S_j) falls short while the supply of product i is unlimited (denoted by U). When the supplies of both products i and j fall short, we assume that the retailer would ration the limited inventory according to the proportions of the demands for the specific and probabilistic products. We use the same parameters of $p, c, q, v, \rho_{ij}, u, \lambda, d$ as in the above section and $\sigma_i = \sigma_j = \sigma$. We varied the demand standard deviation within $[50, 500]$ and took $\sigma = 500$ as an example to present the results. $\phi\%$ and $ESL\%$ denote the increases in the expected profit and service level under PPS, respectively (compared with those under TS).

Table 5.1: The results for the case of asymmetrical supply shortage

S_j	Q_i^{t*}	Q_j^{t*}	Q_i^{pps}	Q_j^{p*}	$Q_k^{p*}(\varphi)$	$\phi\%$	$ESL\%$
U	1326.7	1326.7	1208.05	1208.05	223.82(0.5)	9.21	2.10
1300	1326.7	1300	1208.05	1208.05	223.83(0.41)	9.53	2.28
1100	1326.7	1100	1208.05	1100	223.83(0)	32.53	4.06
900	1326.7	900	1208.05	900	223.83(0)	150.07	6.67
700	1326.7	700	1208.05	700	223.83(0)	88.36	10.34
500	1326.7	500	1208.05	500	223.83(0)	35.27	15.35
300	1326.7	300	1208.05	300	223.83(0)	21.27	22.16
100	1326.7	100	1208.05	100	223.83(0)	14.87	31.54

Table 5.2: The results for the case of symmetrical supply shortage

S_j	Q_i^{t*}	Q_j^{t*}	Q_i^{pps}	Q_j^{p*}	$Q_k^{p*}(\varphi)$	$\phi\%$	$ESL\%$
U	1326.7	1326.7	1208.05	1208.05	223.82(0.5)	9.21	2.10
1300	1300	1300	1189.78	1189.78	220.44(0.5)	9.50	2.29
1100	1100	1100	1006.74	1006.74	186.52(0.5)	15.50	2.10
900	900	900	823.69	823.69	152.61(0.5)	17.69	1.30
700	700	700	640.65	640.65	118.7 (0.5)	0.53	0.36
500	500	500	457.61	457.61	84.78 (0.5)	-0.19	-0.37
300	300	300	274.56	274.56	50.87(0.5)	-0.16	-0.79
100	100	100	91.52	91.52	17(0.5)	-0.03	-0.68

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[5]The two common definitions of the service level are Type-1 service level and Type-2 service level. The Type-2 service level (also called fill rate) is defined as the percentage of demand that is met from stock. More information can be found in Snyder and Shen (2011).
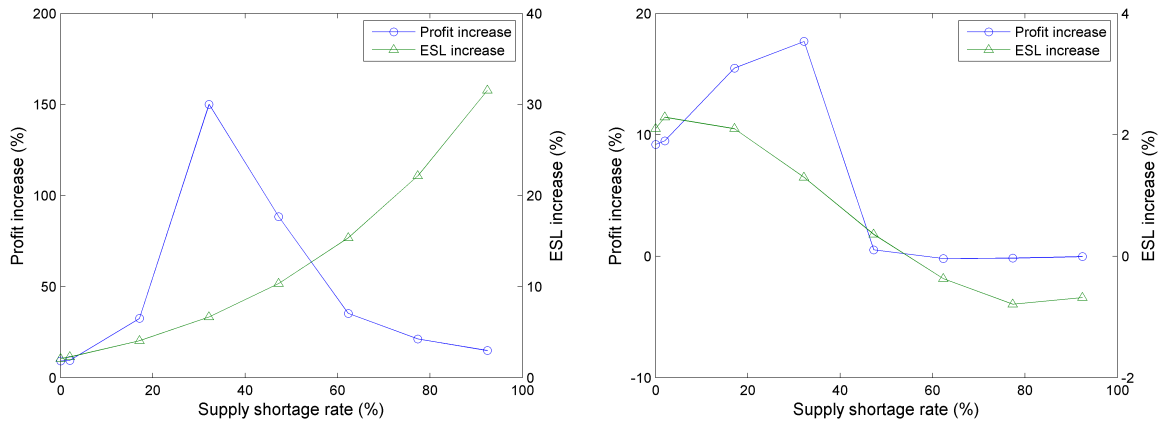
Figure 5.6: Effect of asymmetrical supply shortage   Figure 5.7: Effect of symmetrical supply shortage

Observation 5.3: *PPS will be more advantageous than TS in terms of increasing profit and service level when asymmetrical supply shortage occurs.*

As shown in Fig. 5.6, with increasing supply shortage rate, profit increase will increase first and then decrease afterwards. However, the ESL increase will always increase with the supply shortage rate. Through demand reshape, PPS attracts customers to buy the PPP, which enables the retailer to allocate inventory flexibly. The offer of the PPP diversifies the supply uncertainty. The supply of the PPP comes from either of the specific products, which holds the spirit of dual souring. Therefore, the advantage of PPS in combating supply risk is very impressive when supply shortage is asymmetrical. However, when there is symmetrical supply shortage, the performance of PPS is not impressive, as shown in Fig. 5.7. In fact, at a high symmetrical supply shortage rate, the supply risk may make PPS worse than TS. This may be caused by the fact that when both specific products are subject to shortage risk, the effect of inventory flexibility will vanish. Furthermore, the retailer still has to allocate some inventory for the low-priced PPP even though the demands for the regular-priced products cannot be satisfied.

5.6 Conclusions

To answer the questions of whether and how PS will benefit the retailer when the retailer offers the PPP, we focus on studying the risk-pooling effect of PPS through demand reshape. Modelling demand reshape under PPS, we obtain the optimal price discount for the PPP, the optimal inventory decisions, and the resulting profit. Then we explore the effect of demand variability, customer price (product) sensitivity, and product differentiation on the optimal outcomes. In addition, we examine the performance of PPS in terms of profit and service levels when the retailer faces supply shortage risk. The results show that PPS can alleviate the risks from both the demand and supply sides simultaneously. PPS can improve profit with lower inventory at a

proper discount, which increases with demand uncertainty. PPS is more profitable when product differentiation is smaller and customer price (product) sensitivity is higher (lower). The numerical results show that PPS is a viable strategy to combat asymmetrical supply shortage risk that yields higher profits and service levels.

We prove that offering a PPP can improve the profit of the retailer, even without altering the prices of the original products, postponing product assignment or allocating the inventory dynamically. The implications for the studies on PS are 1) Isolating the demand switch effect in PS, we are surprised to find that the retailer can even benefit from demand cannibalization, i.e., the offering a lower-priced PPP may decrease the demands for the specific products. 2) We consider a new type of PS method and explore its benefits. From a practical point of view, our research findings may help retailers manage demand uncertainty and improve profit through adopting PPS. Furthermore, the offer of the PPP also alleviates the retailer's reliance on product supply. Overall, the retailer can enjoy the benefits of demand centralization while maintaining a decentralized product configuration by offering a low-priced PPP.

Future research may extend our work by considering the package cost under PPS, market competition, and endogenizing the pricing of the specific products. Another direction may extend our work by considering the procurement and production planning for PPP manufacturing when resources are constrained. The PPP may help the manufacturer to maximize the use of the raw materials.

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C H A P T E R

6

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PRICING-PRODUCT ALLOCATION POLICY IN PROBABILISTIC SELLING WITH BARTER CHOICE

6.1   Introduction

Many sellers offer probabilistic products based on their existing products, which are called the constituent component products, by concealing some information about the latter from customers to increase revenues (Fay and Xie, 2008; Post, 2010; Post and Spann, 2012), improve inventory efficiency (Fay and Xie, 2014; Zhang et al., 2016), increase market segmentation (Jiang, 2007; Rice et al., 2014) etc. The customer who buys the probabilistic product is allocated one of the component products by the seller and the specific information about the product is unknown to the customer until he pays for it. For example, one can book a hotel room with no specific address of the hotel or buy an air ticket without the airline's name or flight time at a low price from Hotwire.com. One can also book a trip with no specific destination at a large discount from Germanwings.com. However, customers usually have heterogeneous preferences for the component products and may wish to barter their allocated products for the preferred ones. If he is lucky enough, the customer who buys the probabilistic product can increase his satisfaction if he gets his preferred product through bartering. Just like the "Fukubukuro" (also known as the "mystery bag" or "lucky bag"), which is a well-known Japanese New Year's Day custom where merchants make grab bags filled with unknown random products and sell them at substantial discounts. If the customer is not satisfied with the probabilistic product that he is allocated, he can exchange it for a more preferred product with other customers. One innovative e-commerce practice called "Little Black Bag" has extended this Japanese custom from offline to online where

customers can buy and trade fashion accessories[1]. The customer does not know what is exactly in his bag until he places the order. If the customer finds that the product he likes is in another customer's bag, he can make an offer to trade his product with the other customer concerned. Once the customer is happy with all the products in his bag, he can confirm the order and Little Black Bag will ship the products to him. So, in essence, Little Black Bag is engaged in both probabilistic selling and bartering simultaneously.

Our research is motivated by customers' willingness to barter and the emergence of barter platforms associated with probabilistic selling. What benefits can barter bring to the seller? First, barter plays an advertising or marketing role. Buttyán et al. (2010) found that barter can improve message delivery to participants and stimulate them to co-operate. Take Little Black Bag for example, which is not a typical online shopping website, but a combination of an online community and an online shop. Dan Murillo, CEO and co-founder of Little Black Bag, said, "The transactions achieve 2 million per month, and the customers spend about an hour a day using the service. It's like going into the stock exchange" (Daniela, 2012). The information that the customers have revealed in the barter process helps the company understand customers' demand, and make better purchase and product assignment decisions. Besides, the customers can share the barter information through other social networking platforms, which also helps market their products. Therefore, increasing brands join the platform by offering lower wholesale prices and regard it as a marketing platform to promote their products by customers themselves. AliPay is another example that demonstrates the marketing effect of barter. AliPay sponsored the "Collecting five blessing cards" event in 2016, which encouraged people to barter with one another with a view to gathering the full set of cards that would entitle them to monetary rewards. The event attracted a large number of new customers and created a huge customer relationship network, which enabled AliPay to rank first in terms of app attention for eight consecutive days in the 2016 spring festival in China. Second, when offered the chance to barter for the more preferred product in the community before Little Black Bag delivers the order, the customer has a higher valuation for the probabilistic product. This enables the company to charge a higher price for the probabilistic product.

The innovative retailing mode that combines probabilistic selling with barter seems an interesting and promising concept for e-commerce. However, to the best of our knowledge, there is no research on probabilistic selling considering the barter choice. We conduct this research to address the following questions: 1) How does barter affect the seller's optimal decisions in probabilistic selling? 2) Can and when does barter make probabilistic selling more advantageous to the seller?

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[1] Little Black Bag is a social shopping site where shoppers buy a mystery bag of fashion products for their use or for trading with friends. The company was founded on 1 February 2012 and raised US$800 million on 16 August 2012 (source from https://www.crunchbase.com/organization/little-black-bag).

6.2 Literature review

Our research is related to both probabilistic selling and barter. The research on probabilistic selling is scant but growing, and all the related literature assumes that the probabilistic product sales are non-transferable. The literature on barter focuses on exploring the economics, operations, and marketing benefits of barter. However, there is no literature that considers the barter choice in probabilistic selling.

6.2.1 *Probabilistic selling*

Existing studies on probabilistic selling assume that the probabilistic product sales are non-refundable, non-transferable, or non-exchangeable (Fay and Xie, 2008). However, in reality, some customers who buy the probabilistic product may wish to barter for their preferred products with other people through various social channels. There is even a shop that offers its probabilistic product with the barter choice. So it is interesting to explore the motivation of the firm to offer a barter platform that facilitates customers to exchange their probabilistic products. To the best of our knowledge, there is no research exploring the effect of barter on probabilistic selling.

6.2.2 *Barter*

Barter means that businesses or individuals can trade their undesired goods for the goods they need directly without the use of money. With the development of Internet-based technology and rapid globalization, barter services have staged a comeback to become a global form of trade, not only at the individual level but also at the firm level.

Some literature studies the advantages of barter from the economic and operational perspectives. Williams (1996) observed that the barter system provides economic value to its members. Lobo and de Sousa (2014) found that barter can increase value by reducing the depreciation rate. Prendergast and Stole (2001) showed that barter can create liquidity for cash-constrained firms. Chen and Kao (2010) observed that barter is more popular during periods of hyper-inflation. Özer and Özturan (2011) proved that barter can increase the efficiency of allocation and satisfaction of participants in auctions. Plank et al. (1994) studied barter as a tool of moving excess inventory.

Another huge benefit of barter is helping an enterprise to create partnerships and networks with other businesses, which can help refer, promote, and market the former's business. So a few studies consider the marketing benefit of bartering. For example, Ference (2009) observed that barter can help businesses build long-term mutual trust relationships. Guriev and Kvassov (2004) regarded barter as a tool to price-discriminate between customers who pay in cash and those who pay in kind. Oliver and Mpinganjira (2011) observed that barter can increase trade sales volume and facilitate entry into new markets.

No literature has considered barter in probabilistic selling. A barter platform gives customers a chance to improve their satisfaction by bartering their products for preferred ones. Also, when

customers barter with one another on the platform or share their products through other channels
by inviting other people to join the platform, the transaction information delivered by customers
can not only help businesses understand customers' preferences, but also act as a marketing tool
to expand the market.

6.3 The Model

6.3.1 *Assumptions and definitions*

We assume that a seller sells online two functionally similar component products $j = 1, 2$ of
equal costs: $c_1 = c_2 = c \subseteq [0, 1]$. It also offers a probabilistic product that can be one of the two
component products with given probabilities. Besides, the seller offers the customers who buy
the probabilistic product a choice to exchange their products in the seller's barter community.
Specifically, as shown in Fig. 6.1, the product information is revealed to the customer when he
places the order. If the customer gets the product he prefers, he will confirm the order; otherwise,
he chooses to barter his product for another one in the barter community. We assume that the
marketing role of barter is proportional to the quantity of successful barter products. In addition,
we assume that the successful barter probability is $\alpha \subseteq (0, 1)$ in practice. Finally, the customer
will confirm the order no matter whether he gets his preferred product after bartering.
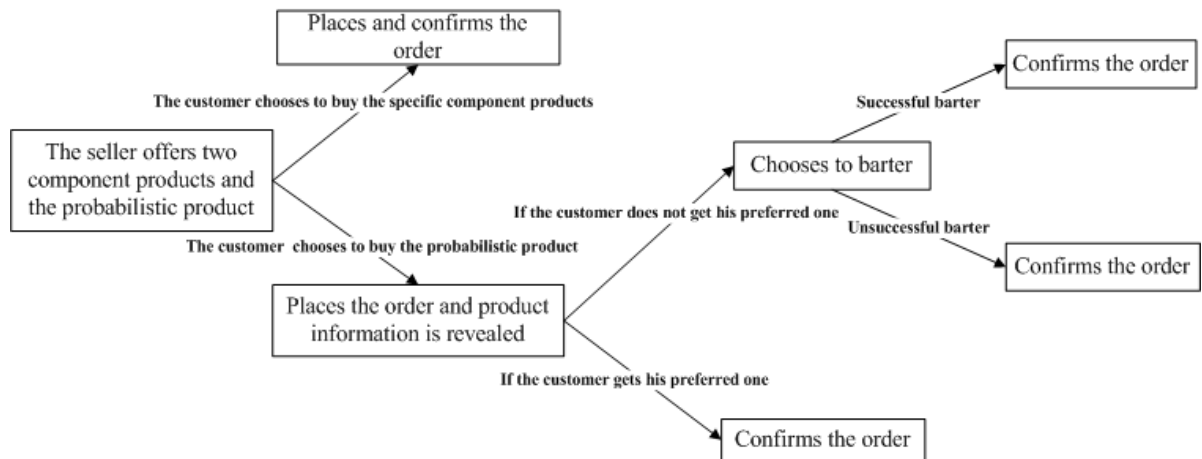
Figure 6.1: Event sequence of probabilistic selling with barter.

Following Fay and Xie (2008), we assume that the customer's valuations for the two component
products follow the Hotelling model. We define v_{ij} as the valuation for product j of customer i,
whose location on the Hotelling line is x_i with a fit-cost-loss coefficient t. So customers' valuations
for the two component products without barter are given in Eq. (6.1) as follows:

$$\begin{cases} v_{1i} & = & 1 - tx_i, \\ v_{2i} & = & 1 - t(1 - x_i). \end{cases}$$

(6.1)

On the other hand, customers' valuations for the two component products with the barter choice are given in Eq. (6.2) as follows:

$$\begin{cases} v_{1i}^* &= v_{1i} + \alpha(v_{2i} - v_{1i})^+, \\ v_{2i}^* &= v_{2i} + \alpha(v_{1i} - v_{2i})^+. \end{cases} \tag{6.2}$$

As shown in Eq. (6.2), if a customer who buys the probabilistic product values product 1 over product 2 (i.e., $v_{1i} > v_{2i}$) while being allocated product 2, he will choose to barter and his valuation for product 2 becomes $v_{2i} + \alpha(v_{1i} - v_{2i})$. If a customer has the same valuations for product 1 and product 2, then he has no motivation to choose barter and his valuations for the products will not change when offered the barter choice. Besides, we assume that product 1 is allocated as the probabilistic product with probability $\varphi \subseteq [0,1]$. Then, customers' valuation for the probabilistic product without the barter choice is $v_{0i} = \varphi v_{1i} + (1-\varphi)v_{2i}$, while with the barter choice, the valuation becomes $v_{0i}^* = \varphi v_{1i}^* + (1-\varphi)v_{2i}^*$. We use the following definitions throughout this chapter.

6.3.2 Probabilistic selling with barter

Without loss of generality, we confine our analysis to the case where $\varphi \geq 1/2$ because we assume that the demand is symmetric, i.e., the results when $\varphi \leqslant 1/2$ (i.e., product 2 is allocated as the probabilistic product with $\varphi \geq 1/2$) is the same. Then according to Eq. (6.2), the valuation for the probabilistic product v_{0i}^* of the customer who prefers product 1 to product 2 (i.e., $x_i < 1/2$) decreases with x_i. The valuation for the probabilistic product v_{0i}^* of the customer who prefers product 2 to product 1 (i.e., $x_i > 1/2$) increases with x_i when $\frac{1}{2} \leq \varphi < \bar{\varphi}$ and decreases with x_i when $\bar{\varphi} \leq \varphi \leq 1$. Therefore, there are two possible outcomes of the Hotelling line of probabilistic selling with barter as shown in Fig. 6.2. It is noted that $\bar{\varphi} = \frac{1}{2(1-\alpha)}$ is the allocation probability that makes customers who prefer product 2(1) to product 1(2) have the same valuation with the customer who is indifferent product 1 and product 2 (i.e., v_{0i}^*=1-t/2). Furthermore, the larger the allocation probability is, the larger the successful barter probability it will need to make the customers have the same valuation. In the first case where $\frac{1}{2} \leq \varphi < \bar{\varphi}$, the Hotelling line is divided into three segments: customers with $x_i \leq \hat{x}_1$ purchase product 1, customers with $x_i \geq \hat{x}_2$ purchase product 2, and customers with $\hat{x}_1 < x_i < \hat{x}_3$ and $\hat{x}_4 < x_i < \hat{x}_2$ purchase the probabilistic product. In the second case where $\bar{\varphi} \leq \varphi \leq 1$, the Hotelling line is divided into three segments: customers with $x_i \leq \hat{x}_1$ purchase product 1, customers with $x_i \geq \hat{x}_2$ purchase product 2, and customers with $\hat{x}_1 < x_i < \hat{x}_3$ purchase the probabilistic product.

The seller that uses probabilistic selling with barter will choose either full coverage or partial coverage of the market, depending on which yields a higher profit. Therefore, in this section we first derive the seller's optimal pricing and optimal allocation probability, and the resulting profits of probabilistic selling with barter in a fully covered market (i.e., $\hat{x}_3=\hat{x}_4=1/2$ in Fig. 6.2(a) and $\hat{x}_3=\hat{x}_2$ in Fig. 6.2(b)) and a partially covered market (i.e., $\hat{x}_3<1/2/\hat{x}_4>1/2$ in Fig. 6.2(a) and $\hat{x}_3<\hat{x}_2$ in Fig. 6.2(b)). We then compare the optimal profits between the two markets to derive the seller's optimal strategy for probabilistic selling with barter.

Variable	Definition
c	The costs of the products $c \subseteq [0,1]$
φ	Product 1 is allocated as the probabilistic product with probability $\varphi \subseteq [0,1]$
q	The marketing benefit brought by per unit of the barter product $q \geq 0$
$\alpha \subseteq (0,1)$	The probability that the customer who buys the probabilistic product can get his preferred one through bartering
$\hat{x}_1$	The maximum value of x_i for which a customer will purchase product 1
$\hat{x}_2$	The minimum value of x_i for which a customer will purchase product 2
$\bar{\varphi}(\bar{\bar{\varphi}})$	The allocation probability that makes customers who prefer product 2(1) to product 1(2) have the same valuation with the customer who is indifferent product 1 and product 2
p_j^{ps}, p_0^{ps}	The price of the component product j and the probabilistic product in probabilistic selling without barter
p_j^{pb}, p_0^{pb}	The price of the component product j and the probabilistic product in probabilistic selling with barter
D_j^{pb}, D_0^{pb}	The sales of the component product j and the probabilistic product in probabilistic selling with barter
G_{Ts}	The profit of traditional selling in which the seller does not offer the probabilistic product
G_b	The marketing benefit for the seller from offering the barter choice
G_{bp}, G_{ps}	The profits of probabilistic selling with and without barter, respectively
G_{Ca}, G_{Ia}	The profits of probabilistic selling with barter in a fully and partially covered market, respectively, when the seller makes positive sales of the component products
G_{Ca}^*, G_{Ia}^*	The profits of probabilistic selling with barter in a fully and partially covered market, respectively, when the seller only sells the probabilistic product

6.3.2.1 Optimal strategy with barter in a fully covered market

As shown in Fig. 6.2(a), the optimal price of the probabilistic product p_0^{pb} should be $1 - t/2$ to make sure that the market is fully covered. Otherwise, as shown in Fig. 6.2(b), the optimal price p_0^{pb} should be the valuation for the probabilistic product v_{0i}^* at location $\hat{x}_2$ (i.e., $p_0^{pb} = \varphi(1 - tx_2 + \alpha t(2x_2 - 1)) + (1-\varphi)(1 - t(1 - x_2)))$. So we compare the optimal profit G_{Ca}^A when $\frac{1}{2} \leq \varphi < \bar{\varphi}$ and optimal profit G_{Ca}^B when $\bar{\varphi} \leq \varphi \leq 1$ to derive the optimal strategy. The profit function is

$$G_{Ca} = \sum_{j=1}^{2} \left(p_j^{pd} - c \right) D_j + \left(p_0^{pd} - c \right) D_0 + G_b. \tag{6.3}$$

With $x_i < 1/2$, customer i prefers product 1 to product 2. When the customer buys the

(a) $\frac{1}{2} \leq \varphi < \bar{\varphi}$

(b) $\bar{\varphi} \leq \varphi \leq 1$
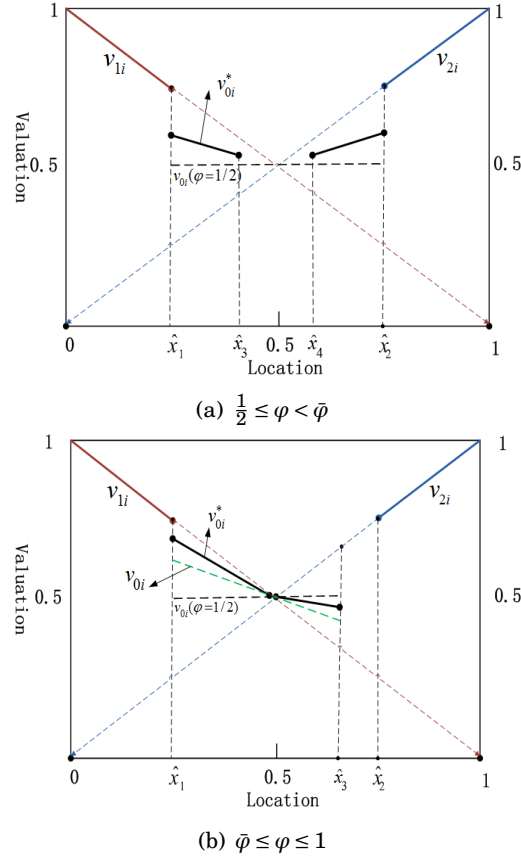
Figure 6.2: Customer valuations

probabilistic product, it turns out to be product 1 with probability φ and product 2 with probability $1 - \varphi$. When the customer gets his preferred product, he confirms the order. Otherwise, he chooses to barter, while offering the barter choice will bring more benefit to the firm. For example in Fig. 6.2(b), when the customer prefers product 1, i.e., $x_i < 1/2$, the benefit (e.g., a lower wholesale price because of the marketing role of bartering, more accurate decisions because of big data about the market etc) brought by barter is expressed as $(1/2 - \hat{x}_1)q\alpha(1 - \varphi)$. When the consumer prefers product 2, i.e., $x_i > 1/2$, the marketing benefit brought by barter in the market where $x_i > 1/2$ is expressed as $(\hat{x}_3 - 1/2)q\alpha\varphi$. Therefore, $G_b = (1/2 - \hat{x}_1)q\alpha(1 - \varphi) + (\hat{x}_3 - 1/2)q\alpha\varphi$ for the case of Fig. 6.2(b).

From Fay and Xie (2008), the optimal profit of traditional selling is

$$G_{Ts} = \begin{cases} 1 - t/2 - c & \text{if } c \leq 1 - t, \\ \frac{(1-c)^2}{2t} & \text{if } c > 1 - t. \end{cases} \tag{6.4}$$

Taking into account that the seller may not offer the probabilistic product, we compare G_{Ca} with G_{Ts} to derive the seller's optimal strategy for probabilistic selling with barter in a fully

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covered market as follows[2]:

With $0 < \alpha \leq \tilde{\alpha}$,

$$G_{Ca} = \begin{cases} -\frac{q^2\alpha^2 - 2t(-1+\alpha)(4+q\alpha) + t^2(-3+2\alpha+\alpha^2)}{8t(-1+\alpha)} - c & \text{if } c \leq \bar{\bar{c}}, \bar{\bar{\varphi}} \leq \varphi^* \leq \bar{\varphi}, \\ \frac{(1-c)^2}{2t} & \text{if } c > \bar{\bar{c}}, \end{cases} \tag{6.5}$$

and with $\tilde{\alpha} < \alpha < 1$,

$$G_{Ca}^* = \begin{cases} 1 - c - \frac{t}{2} + \frac{q\alpha}{2} & \text{if } c \leq \vec{c}, \bar{\bar{\varphi}} \leq \varphi^* \leq \bar{\varphi}, \\ \frac{(1-c)^2}{2t} & \text{if } c > \vec{c}, \end{cases} \tag{6.6}$$

where $\bar{\bar{c}} = 1 - t + t/2\sqrt{\frac{(t(1-\alpha)+q\alpha)^2}{t^2(1-\alpha)}}, \vec{c} = 1 - t + \sqrt{qt\alpha}, \bar{\varphi} = \frac{1}{2(1-\alpha)}, \bar{\bar{\varphi}} = \frac{1-2\alpha}{2(1-\alpha)}$, and $\tilde{\alpha} = \frac{t}{q+t}$.

Our analysis proves that the optimal profit when $\frac{1}{2} \leq \varphi < \bar{\varphi}$ outperforms that when $\bar{\varphi} \leq \varphi \leq 1$, i.e., $G_{Ca}^A \geq G_{Ca}^B$. Thus, the optimal price is $p_0^{pb^*} = 1 - \frac{1}{2}t$. Furthermore, due to symmetry of the model, $\bar{\bar{\varphi}} \leq \varphi^* \leq \frac{1}{2}$ is the optimal allocation probability when $\varphi^* \leq \frac{1}{2}$. Therefore, the optimal allocation probability in a fully covered market is $\bar{\bar{\varphi}} \leq \varphi^* \leq \bar{\varphi}$. We give the optimal pricing for the component products in the Appendix.

Note that $\tilde{\alpha}$ is the dividing point where there are positive demands for the component products. When $0 \leq \alpha < \tilde{\alpha}$, the seller will offer both the probabilistic product and component products in probabilistic selling with barter. Otherwise, when the successful barter probability is high enough, i.e., $\alpha > \tilde{\alpha}$, the seller will only offer the probabilistic product. Eq. (6.5) and Eq. (6.6) indicate that when the product price is relatively low, i.e., $c \leq \bar{\bar{c}}$ when $\alpha \leq \tilde{\alpha}$ and $c \leq \vec{c}$ when $\tilde{\alpha} < \alpha < 1$, the seller can improve its profit through probabilistic selling with barter. When the product price is high, i.e., $c > \bar{\bar{c}}$ when $\alpha \leq \tilde{\alpha}$ and $c > \vec{c}$ when $\tilde{\alpha} < \alpha < 1$, traditional selling is optimal. This finding is consistent with that in Fay and Xie (2008).

6.3.2.2 Optimal strategy with barter in a partially covered market

Similarly, we derive the seller's optimal strategy for probabilistic selling with barter in a partially covered market as follows:

With $0 < \alpha \leq \tilde{\alpha}$,

$$G_{Ia} = \begin{cases} 1 - \frac{t}{2} - c & \text{if } 0 < c \leq 1 - t, \\ \frac{(1-c)^2}{2t} & \text{if } 1 - t < c \leq \tilde{c}, \\ -\frac{(4+4c^2+4q\alpha)(1-\alpha)+t^2(-1+a)^2+q^2\alpha^2+(-1+\alpha)(4t+2\alpha t(-2+q)+8c+4ct(-1+\alpha)+4cq\alpha)}{8t(-1+\alpha)\alpha} & \text{if } \tilde{c} < c < \tilde{\tilde{c}}, \\ \frac{(1-c)^2}{2t} & \text{if } c \geq \tilde{\tilde{c}}, \end{cases} \tag{6.7}$$

and with $\tilde{\alpha} < \alpha < 1$,

$$G_{Ia}^* = \begin{cases} 1 - \frac{t}{2} - c & \text{if } 0 < c \leq 1 - t, \\ \frac{(1-c)^2}{2t} & \text{if } c > 1 - t, \end{cases} \tag{6.8}$$

---

[2]All the proofs in this chapter are given in the Appendix D.

where $\tilde{c} = 1 - \frac{1}{2}t + \frac{q\alpha - \alpha t}{2}$ and $\tilde{\tilde{c}} = 1 - \frac{1}{2}t + \frac{q\alpha}{2(1-\alpha)}$. G_{Ia}^* is the seller's optimal profit when it only offers the probabilistic product. Eq. (6.7) shows that when $0 < \alpha \leq \tilde{\alpha}$ and $\tilde{c} < c < \tilde{\tilde{c}}$, the seller can benefit from probabilistic selling and the optimal allocation probability is $\varphi^* = \frac{1}{2}$. Otherwise, offering no probabilistic product, i.e., traditional selling, is optimal. As shown in Eq. (6.8), when $\tilde{\alpha} < \alpha < 1$, offering merely the probabilistic product will decrease the seller's profit, while offering no probabilistic products is optimal. Therefore, there is no optimal price for the probabilistic product or the optimal allocation probability when $\tilde{\alpha} < \alpha < 1$. We provide the optimal pricing and allocation decisions in a partially covered market in the Appendix.

6.3.2.3 Optimal strategy for probabilistic selling with barter

We compare the profits and the corresponding conditions analyzed above to derive the seller's optimal strategy for probabilistic selling with barter in Proposition 6.1.

Proposition 6.1: *Given the successful barter probability threshold $\tilde{\alpha}$, the seller's optimal strategy for probabilistic selling with barter is as follows:*
With $0 < \alpha \leq \tilde{\alpha}$,

$$G_{bp} = Max[G_{Ca}, G_{Ia}, G_{Ts}] = \begin{cases} G_{Ca}^A & \text{if } c \leq \tilde{c}, \bar{\bar{\varphi}} \leq \varphi^* \leq \bar{\varphi}, \\ G_{Ia}^A & \text{if } \tilde{c} < c < \tilde{\tilde{c}}, \varphi^* = \frac{1}{2}, \\ G_{Ts} & \text{if } c \geq \tilde{\tilde{c}}, \end{cases} \tag{6.9}$$

and with $\tilde{\alpha} < \alpha < 1$,

$$G_{bp}^* = Max\left[G_{Ca}^*, G_{Ia}^*, G_{Ts}\right] = \begin{cases} G_{Ca}^* & \text{if } c \leq \vec{c}, \bar{\bar{\varphi}} \leq \varphi^* \leq \bar{\varphi}, \\ G_{Ts} & \text{if } \vec{c} < c \leq 1. \end{cases} \tag{6.10}$$

From the above results, when $0 < \alpha < \tilde{\alpha}$, we see that when the product cost c is small, i.e., $c \leq \tilde{c}$, it is optimal for the seller to offer the probabilistic product and the market is fully covered. When the cost c is relatively high, i.e., $\tilde{c} < c < \tilde{\tilde{c}}$, it is optimal for the seller to offer the probabilistic product at a higher price and choose to partially cover the market (refer to Table 6.1). When $\tilde{\alpha} < \alpha < 1$, there are no demands for the component products. When $c \leq \vec{c}$, offering merely the probabilistic product with barter and fully covering the market is the optimal strategy for the seller. When the product cost is sufficiently high, i.e., $c > \vec{c}$ when $0 < \alpha < \tilde{\alpha}$ and $c > \tilde{\tilde{c}}$ when $\tilde{\alpha} < \alpha < 1$, the seller should not offer the probabilistic product. We summarize the optimal decisions on price and allocation probability, and the resulting sales in Table 6.1.

Proposition 6.2: *Increasing successful barter probability α and the marketing benefit brought by per barter unit q gradually cannibalize the sales of the component products. The selling decision on offering the component products or not in probabilistic selling depends on the successful barter probability α, marketing benefit per barter unit q, and product differentiation t. In addition, $\frac{\partial \tilde{\alpha}}{\partial q} < 0$ and $\frac{\partial \tilde{\alpha}}{\partial t} > 0$.*

Table 6.1: Optimal decisions of probabilistic selling with barter

	$\alpha \le \tilde{\alpha}$	$\alpha > \tilde{\alpha}$
D_j	$\begin{cases} \frac{-t+q\alpha+t\alpha}{4t(-1+\alpha)} & \text{if } c \le \tilde{c} \\ \frac{-t+q\alpha+t\alpha}{4t(-1+\alpha)} & \text{if } \tilde{c} < c < \tilde{\tilde{c}} \\ \frac{1-c}{2t} & \text{if } c \ge \tilde{\tilde{c}} \end{cases}$	$\begin{cases} 0 & \text{if } c \le \vec{c} \\ \frac{1-c}{2t} & \text{if } c > \vec{c} \end{cases}$
D_0	$\begin{cases} \frac{t\alpha-q\alpha-t}{2t(-1+\alpha)} & \text{if } c \le \tilde{c} \\ \frac{2-2c-t-2\alpha+2c\alpha+q\alpha+t\alpha}{2t\alpha-2t\alpha^2} & \text{if } \tilde{c} < c < \tilde{\tilde{c}} \\ 0 & \text{if } c \ge \tilde{\tilde{c}} \end{cases}$	$\begin{cases} 1 & \text{if } c \le \vec{c} \\ 0 & \text{if } c > \vec{c} \end{cases}$
p_0^{pb}	$\begin{cases} 1-\frac{1}{2}t & \text{if } c \le \tilde{c} \\ \frac{1}{4}(2+2c+t(-1+\alpha)-q\alpha) & \text{if } \tilde{c} < c < \tilde{\tilde{c}} \\ N/A & \text{if } c \ge \tilde{\tilde{c}} \end{cases}$	$\begin{cases} 1-\frac{1}{2}t & \text{if } c \le \vec{c} \\ N/A & \text{if } c > \vec{c} \end{cases}$
p_1^{pb}	$\begin{cases} \frac{1}{2}(2+\alpha(q-t)(1-\varphi)-t\varphi) & \text{if } c \le \tilde{c} \\ 1-\frac{1}{2}t & \text{if } \tilde{c} < c < \tilde{\tilde{c}} \\ \frac{1+c}{2} & \text{if } c \ge \tilde{\tilde{c}} \end{cases}$	$\begin{cases} N/A & \text{if } c \le \vec{c} \\ \frac{1+c}{2} & \text{if } c > \vec{c} \end{cases}$
p_2^{pb}	$\begin{cases} \frac{1}{2}(2+\alpha(q-t)\varphi-t(1-\varphi)) & \text{if } c \le \tilde{c} \\ 1-\frac{1}{2}t & \text{if } \tilde{c} < c < \tilde{\tilde{c}} \\ \frac{1+c}{2} & \text{if } c \ge \tilde{\tilde{c}} \end{cases}$	$\begin{cases} N/A & \text{if } c \le \vec{c} \\ \frac{1+c}{2} & \text{if } c > \vec{c} \end{cases}$
φ^*	$\begin{cases} [\bar{\bar{\varphi}},\bar{\varphi}] & \text{if } c \le \tilde{c} \\ 1/2 & \text{if } \tilde{c} < c < \tilde{\tilde{c}} \\ N/A & \text{if } c \ge \tilde{\tilde{c}} \end{cases}$	$\begin{cases} [\bar{\bar{\varphi}},\bar{\varphi}] & \text{if } c \le \vec{c} \\ N/A & \text{if } c > \vec{c} \end{cases}$

Increasing successful barter probability will increase the valuation for the probabilistic product, which makes the probabilistic product more attractive to customers. Note that $\tilde{\alpha}$ is the threshold that makes all the customers switch to buying the probabilistic product. Therefore, when $\alpha \le \tilde{\alpha}$, the seller will offer both the probabilistic product and component products, while it only offers the probabilistic product when $\alpha > \tilde{\alpha}$. Furthermore, as the marketing benefit per barter unit increases, the threshold decreases, i.e., the seller is more likely to merely offer the probabilistic product. t can be interpreted as the degree of horizontal product differentiation. As product differentiation increases, the threshold increases, i.e., the seller is likely to offer both the component products and the probabilistic product. This means increasing marketing benefit per barter unit encourages the seller to merely offer the probabilistic product, while increasing product differentiation plays the opposite role.

This proposition provides important guidance to the seller as to whether or not to offer the component products in probabilistic selling with barter: If product differentiation is high while customers' successful barter probability is low, the seller may consider offering both the

component and probabilistic products. Offering merely the probabilistic product is advantageous only when customers' successful barter probability is high enough.

6.4 Comparison with probabilistic selling without barter choice

From Fay and Xie (2008), the seller's optimal profit of probabilistic selling without the barter choice is as follows:

$$
G_{ps} = \begin{cases} 1 - c - \frac{3}{8}t & \text{if } c \leq \hat{c}, \\ \frac{(1-c)^2}{2t} & \text{if } c > \hat{c}, \end{cases} \tag{6.11}
$$

where $\hat{c} = 1 - t/2$.

We compare G_{bp} with the profit without barter G_{ps} and define $\Delta_{bp-ps} = G_{bp} - G_{ps}$ (the detailed comparison results are given in the Appendix). Fig. 6.3 summarizes the comparison results between probabilistic selling without barter (PS) and probabilistic selling with barter (PB) when (a) $0 < q < \frac{t}{4}$, (b) $\frac{1}{4}t \leq q \leq \frac{1}{3}t$, (c) $\frac{1}{3}t \leq q < \frac{1}{2}t$, and (d) $\frac{1}{2}t \leq q < t$. In Fig. 6.3, PB denotes the strategy that the seller offers both the component and probabilistic products with barter when $\alpha \leq \tilde{\alpha}$. PB^* denotes the strategy when there are no demands for the component products and the seller only offers the probabilistic product with barter when $\alpha > \tilde{\alpha}$. PS denotes that the seller offers both the component and probabilistic products without the barter choice. TS denotes that there is no demand for the probabilistic product and the seller only offers the component products, i.e., traditional selling without the probabilistic product.

Proposition 6.3: *Barter broadens the application range of probabilistic selling, which increases with the successful barter probability when it is below or above the threshold $\tilde{\alpha}$.*

As shown in Fig. 6.3, the seller cannot benefit from probabilistic selling when the product cost is high enough. This result is consistent with that in Fay and Xie (2008), who proved that probabilistic selling cannot benefit the seller when $c > 1 - t/2$. However, offering the barter choice renders probabilistic selling more profitable than traditional selling when $1 - t/2 < c < \tilde{\tilde{c}}$ if the successful barter probability is below a threshold, i.e., $\alpha \leq \tilde{\alpha}$, and when $1 - t/2 < c < \vec{c}$ if successful barter probability is above a threshold, i.e., $\alpha > \tilde{\alpha}$. It is easy to prove that both $\tilde{\tilde{c}}$ and $\vec{c}$ are greater than $1 - t/2$. Thus, we can deduce that when the product cost is high, i.e., $1 - t/2 < c < \tilde{\tilde{c}}$ when $\alpha \leq \tilde{\alpha}$ and $1 - t/2 < c < \vec{c}$ when $\alpha > \tilde{\alpha}$, the offer of the barter choice makes probabilistic selling beneficial to the seller. In other words, barter broadens the application range of probabilistic selling. However, offering the probabilistic product can never be the strategy to increase profit when the product's value is much more higher, i.e., $c \geq \tilde{\tilde{c}}$ and $c \geq \vec{c}$.

The proposition has significant implications for practice. For the seller, when the product cost is relatively high, it may consider offering the barter choice in probabilistic selling to increase its profit (refer to Proposition 6.4). When the successful barter probability is below or above the
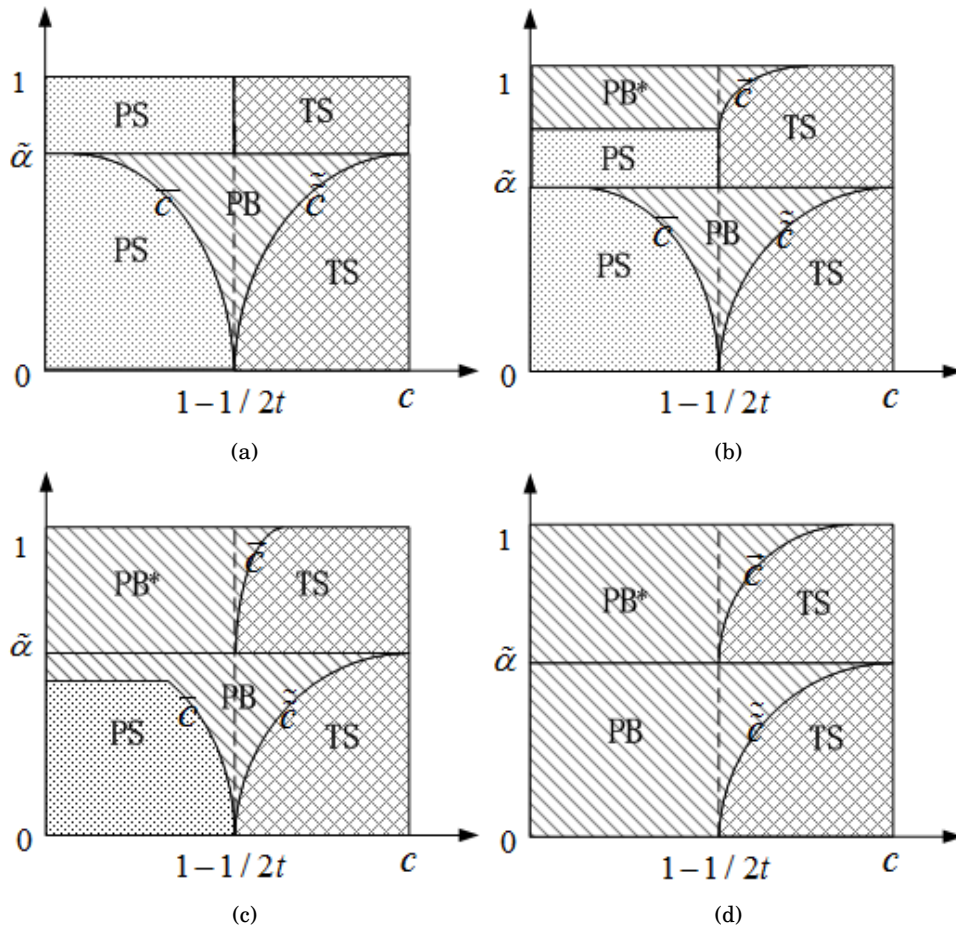
Figure 6.3: Optimal strategies under different conditions.

threshold $\tilde{\alpha}$, the higher the product cost is, the higher the successful barter probability it needs to make probabilistic selling advantageous over traditional selling.

Proposition 6.4: *The decision as to whether or not to offer the barter choice depends on the product cost c, successful barter probability α, and marketing benefit per barter unit q. Specifically,*

1) *When the successful barter probability is below a threshold, i.e., $\alpha \leq \tilde{\alpha}$, the seller considers the three strategies of PS, PB, and TS. If the marketing benefit is small, i.e., $0 < q \leq \frac{t}{3}$, PS is optimal when the cost is low, i.e., $c \leq \bar{c}$, and PB is optimal when the cost is medium, i.e., $\bar{c} < c < \tilde{\bar{c}}$.*

2) *When the successful barter probability is above a threshold, i.e., $\alpha > \tilde{\alpha}$, the seller considers the three strategies of PS, PB^*, and TS. If q is sufficiently low, i.e., $q < 1/4t$, the offer of the barter choice will decrease the profit of probabilistic selling. If the marketing benefit is sufficiently high, i.e., $q \geq 1/3t$, then barter can increase the profit of probabilistic selling.*

3) *When the cost is high enough, i.e., $c \geq \tilde{\bar{c}}$ when $\alpha > \tilde{\alpha}$ and $c \geq \tilde{\bar{c}}$ when $\alpha \leq \tilde{\alpha}$, it is optimal not to offer the probabilistic product.*

4) *When the marketing benefit is sufficiently large, i.e., $q \geq \frac{1}{2}$, barter increases the profit of PS. When the marketing benefit is $q = 0$, barter decreases the profit of PS when $0 < \alpha < 1$.*

The implications of the above findings for practice are that the successful barter probability is not necessarily the higher the better for probabilistic selling with barter. The performance also depends on the marketing benefit brought by bartering. Increasing successful barter probability α increases the valuation for the probabilistic product, which can cannibalize the sales of the high-priced component products. Therefore, barter will weaken the price-discrimination effect of probabilistic selling and make the probabilistic product more attractive. From this perspective, barter will undermine the value of probabilistic selling. However, from another point of view, the more people that buy the probabilistic product, the more marketing benefit the seller can gain from bartering. Thus, when the marketing benefit brought by per unit of the barter product is low while the successful barter probability is high, the profit increase through bartering cannot offset the profit decrease. Just as shown in Fig. 6.3(a), offering the barter choice cannot benefit the seller when the successful barter probability is above a threshold, i.e., $\alpha > \tilde{\alpha}$ while q is sufficiently low, i.e., $0 < q < 1/4t$. In the extreme case where the marketing benefit is $q = 0$, barter will decrease the profit of PS when $0 < \alpha < 1$.

Proposition 6.5: *The cost ranges that make probabilistic selling with barter more profitable than probabilistic selling without barter, i.e., $(\bar{c}, \tilde{\bar{c}})$ and $(0, \bar{c})$ increase with the successful barter probability.*

The implications of the above finding for practice are that when offering probabilistic selling with the barter choice, the seller can choose its products depending on the estimated successful barter probability. Provided that the successful barter probability is above or below a threshold, the higher the successful barter probability is, the wider are the ranges of products that the seller can sell through probabilistic selling with barter.

6.5 Extension

We assume in the above study that successful barter probability α is exogenous and independent with the allocation probability φ. In reality, successful barter probability α highly depends on allocation probability: even allocation leads to the highest successful barter probability, and the successful barter probability will be very low when the allocation probability is seriously uneven.

Therefore, we consider successful barter probability α depends on allocation probability in the extension part. The successful barter probability achieves its highest when $\varphi = 1/2$ while becomes zero when $\varphi = 0$ or $\varphi = 1$. This is consistent with the reality that no barter happens when all the customers get the same product. Even when $\varphi = 1/2$, we cannot guarantee the successful barter probability be 1 for reasons like customer patience, customer arrival sequence, and other operational issues. Therefore, we use $k \subseteq [0, 1]$ to capture the effect of operations on the successful barter probability α and define it as barter probability. Making $\alpha \subseteq [0, 1]$, we define the successful barter probability $\alpha = 4k\varphi(1 - \varphi)$.

Keeping other assumptions and definitions the same, we first derive the seller's optimal

Table 6.2: Optimal allocation strategies of probabilistic selling with barter

q	k	φ^*	
$q \le t/2$	$k \le k_1$	$\frac{\partial G_{Ca}^A}{\partial \varphi} > 0, \varphi^* = \bar{\varphi}$	$\frac{\partial G_{Ca}^B}{\partial \varphi} < 0, \varphi^* = \bar{\varphi}$
	$k_1 < k \le \tilde{k}$	$\frac{\partial G_{Ca}^A}{\partial \varphi} < 0, \varphi^* = \frac{1}{2}$	$\frac{\partial G_{Ca}^B}{\partial \varphi} < 0, \varphi^* = \bar{\varphi}$
	$k > \tilde{k}$	$\frac{\partial G_{Ca}^A}{\partial \varphi} < 0, \varphi^* = \frac{1}{2}$	$\frac{\partial G_{Ca}^B}{\partial \varphi} < 0, \varphi^* = \bar{\varphi}$
$q > t/2$	$k \le \tilde{k}$	$\frac{\partial G_{Ca}^A}{\partial \varphi} < 0, \varphi^* = \frac{1}{2}$	$\frac{\partial G_{Ca}^B}{\partial \varphi} < 0, \varphi^* = \bar{\varphi}$
	$k > \tilde{k}$	$\frac{\partial G_{Ca}^A}{\partial \varphi} < 0, \varphi^* = \frac{1}{2}$	$\frac{\partial G_{Ca}^B}{\partial \varphi} < 0, \varphi^* = \bar{\varphi}$

Note: $\bar{\varphi}$ is the solution of the equation $8k\varphi^2(1-\varphi) = 2\varphi - 1$, $k_1 = \frac{-2q+t}{4(q-t)(-\varphi+\varphi^2)}$, and $\tilde{k} = \frac{t}{q+t}$.

allocation strategies of probabilistic selling with barter in Table 6.2. We find that even allocation is optimal except for the case when both the barter probability and the marketing benefit are relatively low, i.e., $k \le k_1$ and $q \le q \le t/2$. In that case, the profit increases with allocation probability (e.g., the optimal profit decreases with the successful barter probability α). That means when the barter probability is very low, the overall marketing benefit cannot offset the loss due to the cannibalized sales of high-priced component products. And then the lower the successful barter probability, the higher the profit. Similarly, comparing the resulting profit results and corresponding conditions analyzed, we then derive the seller's optimal strategy for probabilistic selling with barter in Proposition 6.6.

Proposition 6.6: *Given the barter probability k, the seller's optimal strategy for probabilistic selling with barter is as follows:*
With $k \le k_1$ and $q/2$,

$$G_{bp} = Max[G_{Ca}, G_{Ia}] = \begin{cases} G_{Ca}^A & \text{if } c \le \tilde{c}, \varphi^* = \bar{\varphi}, \\ G_{Ia}^A & \text{if } \tilde{c} < c < \tilde{\tilde{c}}, \varphi^* = \frac{1}{2}, \\ G_{Ts} & \text{if } c \ge \tilde{\tilde{c}}, \end{cases} \quad (6.12)$$

and with $k_1 < k \le \tilde{k}$,

$$G_{bp} = Max[G_{Ca}, G_{Ia}] = \begin{cases} G_{Ca}^A & \text{if } c \le \tilde{c}, \varphi^* = \frac{1}{2}, \\ G_{Ia}^A & \text{if } \tilde{c} < c < \tilde{\tilde{c}}, \varphi^* = \frac{1}{2}, \\ G_{Ts} & \text{if } c \ge \tilde{\tilde{c}}, \end{cases} \quad (6.13)$$

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and with $\tilde{k} < k \leq 1$,

$$G_{bp}^* = Max\left[G_{Ca}^*, G_{Ia}^*\right] = \begin{cases} G_{Ca}^* & \text{if } c \leq \vec{c}, \varphi^* = \frac{1}{2}, \\ G_{Ts} & \text{if } \vec{c} < c \leq 1, \end{cases} \tag{6.14}$$

where $\tilde{c} = 1 - \frac{1}{2}t + \frac{q\alpha - \alpha t}{2}$, $\tilde{\tilde{c}} = 1 - \frac{1}{2}t + \frac{q\alpha}{2(1-\alpha)}$, $\vec{c} = 1 - t + \sqrt{qt\alpha}$, $\tilde{\alpha} = \frac{t}{q+t}$, $k_1 = \frac{-2q+t}{4(q-t)(-\varphi+\varphi^2)}$, $\alpha = k$, $\tilde{\alpha} = \tilde{k} = \frac{t}{q+t}$.

We get the same result that the optimal profit when $\frac{1}{2} \leq \varphi \leq \bar{\varphi}$ outperforms that when $\bar{\varphi} \leq \varphi \leq 1$. And proposition 6.2 still holds. Besides, we also prove that G_{Ca}^A is worse off than G_{Ps} and G_{Ia} when $k \leq k_1$ and $q \leq \frac{1}{2}$. Then the comparison results of PS, PB, and TS considering different conditions of product cost c, barter probability k, successful barter probability α, and the marketing benefit q is the same with Fig. 6.3 except for that the successful barter probability $\alpha = k$ and $\alpha^* = k^* = \frac{t}{q+t}$. Therefore, the propositions 6.3-6.5 still hold. Besides, if the seller chooses PB strategy, even allocation can help to achieve the maximum profit.

6.6 Conclusions

Motivated by existing online shopping platforms that support customers who buy probabilistic products to barter their allocated products for their preferred products, we study the seller's optimal pricing and allocation strategies for probabilistic selling with the barter choice. Considering the effects of product cost, barter probability, and marketing benefit brought by bartering, we explore when barter benefits probabilistic selling and makes it more profitable to the seller. We find a barter probability threshold for offering the component products in probabilistic selling with barter. Above the threshold, the probabilistic product will cannibalize all the component product sales, and then the seller prefers to fully cover the market and offer merely the probabilistic product. When the barter probability is below the threshold, barter can increase the profit of probabilistic selling when the product cost is medium and the cost ranges increase with the barter probability. Besides, barter can broaden the application range of probabilistic selling and the range will increase with the barter probability.

This is the first study on probabilistic selling with the barter choice. Our findings shed light on the practice of probabilistic selling with the barter choice. First, our findings help the seller in making decisions on pricing, allocation probability, and product line design, i.e., the seller merely offers the component products, merely the probabilistic product, or both. Second, our analysis helps the seller understand when and how to offer the barter choice in probabilistic selling to achieve the maximum profit with consideration of product cost, successful barter probability, marketing benefit brought by per barter unit. Specifically, Fay and Xie (2008) proved that probabilistic selling can improve the seller's profit only when the product cost is low, i.e., $c \leq 1/2$. Our research indicates that when the product cost is high, i.e., $c > 1/2$, the seller can offer the barter choice in probabilistic selling to increase its profit under some conditions. When the

barter probability is below or above a threshold, the higher the product cost is, the higher the barter probability is needed to make probabilistic selling advantageous over traditional selling. However, the barter probability is not necessarily the higher the better. When the marketing benefit brought by bartering is sufficiently low and the barter probability is high, the seller will not benefit from offering the barter choice.

In this study we assume that the demand is deterministic and symmetrical. Future studies can extend our model to consider demand uncertainty and asymmetrical demand. In addition, the barter probability is independent of the demand distribution in our model. However, when the demand is asymmetrical, the probability of bartering products A for B and that of bartering products B for A may be different. While we focus on the profit of the seller in our study, future research may consider the effect of barter on consumer welfare in probabilistic selling.

CONCLUSION

Considering different characteristics of the probabilistic product, the buyer, and the seller involved in probabilistic selling, i.e., the probabilistic product form, the buyers' behaviours of demand switch and barter exchange, and the seller's behaviours of product allocation, we establish model and solve the decision problems of pricing, inventory, joint decision of pricing-inventory, and product allocation, etc. Comparing probabilistic selling with traditional selling, probabilistic selling with barter choice with probabilistic selling without barter choice, and probabilistic selling with inventory substitution, we have derived some conditions to optimize probabilistic selling. Based on the analysis of optimal decision and strategy comparison results, we shed some lights on the profitability and the effectiveness of probabilistic selling on managing uncertainty.

The main results show that probabilistic selling can benefit the seller with higher expected profit by reducing demand/supply uncertainty and improving inventory efficiency. The performance of probabilistic selling is closely dependent on customers' price sensitivity, product similarity, and uncertainty level, etc.

The thesis has important implications for the practice. First, the thesis helps the sellers understand the mechanism of probabilistic selling on managing demand uncertainty and supply uncertainty. Second, the thesis helps sellers take full advantage of probabilistic selling by optimizing their inventory, pricing and related implementation issues in more realistic circumstances. Third, the thesis provides some references for the sellers to coordinate marketing and operational decisions in practice to improve their profit.

The thesis has important implications for the existing theory. First, the thesis riches related Operational Management research on inventory management in probabilistic selling. Second, we focus on exploring the mechanism of this innovative marketing strategy as an inventory management tool to combat uncertainty. Then the work also riches the literature on inventory

management. Third, the models and analysis methods in the thesis may also apply to the study of other similar Operations-Marketing interface problems.

The implementation forms of probabilistic selling is diverse and flexible, with many interesting decision-making issues that need to be resolved. The future research directions as follows:

(1) Probability selling strategy considering consumer behavior

Due to the incomplete information of the probabilistic product, customers may behave overconfident, limited rational, optimistic/pessimistic, and brand loyalty, etc. in the purchasing process. These customer behaviours will influence seller's operational decisions in probabilistic selling.

(2) Cooperation mechanism design in probabilistic selling

Sellers using probabilistic selling must determine the product mix, the proposition of one component product in the mix, and if the offer of the probabilistic product is limited, etc. Furthermore, creating probabilistic products may require the cooperation of multiple competing suppliers similar with the opaque intermediary. Therefore, probabilistic selling will lead to selling cooperation and game among channel participants, and furthermore affects other related operations, e.g, advertising cooperation, production cooperation, supply management, etc.

(3)Production operations management in probabilistic selling

Since there is no need to develop additional physical product when the seller extends the product line by creating the probabilistic product. Then expect this selling strategy can reduce manufacturer's cost on product design, procurement, and production.

A.1 Proof of Proposition 3.1

The expected profit function is

$E(Q_i{}^p, Q_j{}^p) =$

$\int_0^{Q_i^p} \int_0^{Q_j^p} \int_0^{Q_i^p + Q_j^p - D_i^p - D_j^p} [pD_i^p + pD_j^p + p_0 D_0^p + s(Q_i^p + Q_j^p - D_i^p - D_j^p - D_0^p)] f^*(x_i, x_j, x_k) dx_i dx_j dx_k$

$+ \int_0^{Q_i^p} \int_0^{Q_j^p} \int_{Q_i^p + Q_j^p - D_i^p - D_j^p}^{\infty} [pD_i^p + pD_j^p + p_0(Q_i^p + Q_j^p - D_i^p - D_j^p)] f^*(x_i, x_j, x_k) dx_i dx_j dx_k$

$+ \int_0^{Q_i^p} \int_{Q_j^p}^{\infty} \int_0^{Q_i^p - D_i^p} [pD_i^p + pQ_j^p + p_0 D_0 + s(Q_i^p - D_i^p - D_0)] f^*(x_i, x_j, x_k) dx_i dx_j dx_k$

$+ \int_0^{Q_i^p} \int_{Q_j^p}^{\infty} \int_{Q_i^p - D_i^p}^{\infty} [pD_i^p + pQ_j^p + p_0(Q_i^p - D_i^p)] f^*(x_i, x_j, x_k) dx_i dx_j dx_k$

$+ \int_{Q_i^p}^{\infty} \int_0^{Q_j^p} \int_0^{Q_j^p - D_j^p} [pQ_i^p + pD_j^p + p_0 D_0 + s(Q_j^p - D_j^p - D_0)] f^*(x_i, x_j, x_k) dx_i dx_j dx_k$

$+ \int_{Q_i^p}^{\infty} \int_0^{Q_j^p} \int_{Q_j^p - D_j^p}^{\infty} [pQ_i^p + pD_j^p + p_0(Q_j^p - D_j^p)] f^*(x_i, x_j, x_k) dx_i dx_j dx_k$

$+ \int_{Q_i^p}^{\infty} \int_{Q_j^p}^{\infty} \int_0^{\infty} [pQ_i^p + pQ_j^p] f^*(x_i, x_j, x_k) dx_i dx_j dx_k$

$- c(Q_i^p + Q_j^p).$

So

$$\frac{\partial E(Q_i^p, Q_j^p)}{\partial Q_i^p} =$$

$$s \int_0^{Q_i^p} \int_0^{Q_j^p} \int_0^{Q_i^p + Q_j^p - D_i^p - D_j^p} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ p_0 \int_0^{Q_i^p} \int_0^{Q_j^p} \int_{Q_i^p + Q_j^p - D_i^p - D_j^p}^{\infty} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ s \int_0^{Q_i^p} \int_{Q_j^p}^{\infty} \int_0^{Q_i^p - D_i^p} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ p_0 \int_0^{Q_i^p} \int_{Q_j^p}^{\infty} \int_{Q_i^p - D_i^p}^{\infty} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ p \int_{Q_i^p}^{\infty} \int_0^{Q_j^p} \int_0^{Q_j^p - D_j^p} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ p \int_{Q_i^p}^{\infty} \int_0^{Q_j^p} \int_{Q_j^p - D_j^p}^{\infty} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ p \int_{Q_i^p}^{\infty} \int_{Q_j^p}^{\infty} \int_0^{\infty} f^*(x_i, x_j, x_k) dx_i dx_j dx_i dx_j dx_k$$

$$- c,$$

$$\frac{\partial E(Q_i^p, Q_j^p)}{\partial Q_j^p} =$$

$$s \int_0^{Q_i^p} \int_0^{Q_j^p} \int_0^{Q_i^p + Q_j^p - D_i^p - D_j^p} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ p_0 \int_0^{Q_i^p} \int_0^{Q_j^p} \int_{Q_i^p + Q_j^p - D_i^p - D_j^p}^{\infty} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ p \int_0^{Q_i^p} \int_{Q_j^p}^{\infty} \int_0^{Q_i^p - D_i^p} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ p \int_0^{Q_i^p} \int_{Q_j^p}^{\infty} \int_{Q_i^p - D_i^p}^{\infty} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ s \int_{Q_i^p}^{\infty} \int_0^{Q_j^p} \int_0^{Q_j^p - D_j^p} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ p_0 \int_{Q_i^p}^{\infty} \int_0^{Q_j^p} \int_{Q_j^p - D_j^p}^{\infty} f^*(x_i, x_j, x_k) dx_i dx_j dx_k$$

$$+ p \int_{Q_i^p}^{\infty} \int_{Q_j^p}^{\infty} \int_0^{\infty} f^*(x_i, x_j, x_k) dx_i dx_j dx_i dx_j dx_k$$

$$- c,$$

$$\frac{\partial E(Q_i^p, Q_j^p)^2}{\partial^2 Q_i^p} =$$

$$(s - p_0) \int_0^{Q_i^p} \int_0^{Q_j^p} f^*(x_i, x_j, Q_i^p + Q_j^p - D_i^p - D_j^p) dx_i dx_j dx_k$$

$$+ (s - p) \int_0^{Q_j^p} \int_0^{Q_j^p - D_j^p} f^*(Q_i^p, x_j, x_k) dx_j dx_k$$

$$+ (s - p_0) \int_0^{Q_i^p} \int_{Q_j^p}^{\infty} f^*(x_i, x_j, Q_i^p - D_i^p) dx_i dx_j$$

$$+ (p_0 - p) \int_{Q_j^p}^{\infty} \int_0^{\infty} f^*(Q_i^p, x_j, x_k) dx_j dx_k$$

$$+ (p_0 - p) \int_0^{Q_2} \int_{Q_2 - D_2}^{\infty} f^*(Q_i^p - D_i^p, x_j, x_k) dx_j dx_k \leq 0,$$

$$\frac{\partial E(Q_i^p, Q_j^p)^2}{\partial^2 Q_j^p} =$$

$$(s - p_0) \int_0^{Q_i^p} \int_0^{Q_j^p} f^*(x_i, x_j, Q_i^p + Q_j^p - D_i^p - D_j^p) dx_i dx_j dx_k$$

$$+ (s - p) \int_0^{Q_i^p} \int_0^{Q_i^p - D_i^p} f^*(x_i, Q_j^p, x_k) dx_i dx_k$$

$$+ (s - p_0) \int_{Q_i^p}^{\infty} \int_0^{Q_j^p} f^*(x_i, x_j, Q_j^p - D_j^p) dx_i dx_j$$

$$+ (p_0 - p) \int_{Q_i^p}^{\infty} \int_0^{\infty} f^*(x_i, Q_j^p, x_k) dx_i dx_k$$

$$+ (p_0 - p) \int_0^{Q_i^p} \int_{Q_i^p - D_i^p}^{\infty} f^*(x_i, Q_j^p, x_k) dx_i dx_k \leq 0.$$

In addition,

$$\frac{\partial E(Q_i^p, Q_j^p)^2}{\partial Q_i^p \partial Q_j^p} = \frac{\partial E(Q_i^p, Q_j^p)^2}{\partial Q_j^p \partial Q_i^p} =$$

$$(s - p_0) \int_0^{Q_i^p} \int_0^{Q_j^p} f^*(x_i, x_j, Q_i^p + Q_j^p - D_i^p - D_j^p) dx_i dx_j dx_k$$

Therefore, the Hessian Matrix

$$\begin{vmatrix} \frac{\partial E(Q_i^p, Q_j^p)^2}{\partial^2 Q_i^p} & \frac{\partial E(Q_i^p, Q_j^p)^2}{\partial Q_i^p \partial Q_j^p} \\ \frac{\partial E(Q_i^p, Q_j^p)^2}{\partial Q_j^p \partial Q_i^p} & \frac{\partial E(Q_i^p, Q_j^p)^2}{\partial^2 Q_j^p} \end{vmatrix} \geq 0.$$

We have the result.

A.2 Proof of Proposition 3.3

The optimal order quantities must satisfy the following equations

$$
\begin{cases}
\frac{\partial E(Q_i^p, Q_j^p)}{\partial Q_i^p} &= p - (p-s)F(Q_i^{p*}) + (p_0 - s)G(Q_i^{p*}, Q_j^{p*}) - c = 0, \\
\frac{\partial E(Q_i^p, Q_j^p)}{\partial Q_j^p} &= p - (p-s)F(Q_j^{p*}) + (p_0 - s)N(Q_i^{p*}, Q_j^{p*}) - c = 0.
\end{cases}
$$

Differentiating the above results with respect to p_0 yields

$$
\begin{cases}
a^* \frac{\partial E(Q_i^{p*})}{\partial p_0} + b^* \frac{\partial E(Q_j^{p*})}{\partial p_0} &= -G(Q_i^{p*}, Q_j^{p*}), \\
c^* \frac{\partial E(Q_i^{p*})}{\partial p_0} + d^* \frac{\partial E(Q_j^{p*})}{\partial p_0} &= -N(Q_i^{p*}, Q_j^{p*}).
\end{cases}
$$

where a^*, b^*, c^*, and d^* denote the values of $\frac{\partial E(Q_i^p, Q_j^p)^2}{\partial^2 Q_i^p}$, $\frac{\partial E(Q_i^p, Q_j^p)^2}{\partial Q_i^p \partial Q_j^p}$, $\frac{\partial E(Q_i^p, Q_j^p)^2}{\partial Q_j^p \partial Q_i^p}$, and $\frac{\partial E(Q_i^p, Q_j^p)^2}{\partial^2 Q_j^p}$ at (Q_i^{p*}, Q_j^{p*}), respectively.

Then we get

$$
\begin{cases}
\frac{\partial E(Q_i^{p*})}{\partial p_0} &= \frac{b^* N(Q_i^{p*}, Q_j^{p*}) - d^* G(Q_i^{p*}, Q_j^{p*})}{a^* d^* - b^* c^*}, \\
\frac{\partial E(Q_j^{p*})}{\partial p_0} &= \frac{c^* G(Q_i^{p*}, Q_j^{p*}) - a^* N(Q_i^{p*}, Q_j^{p*})}{a^* d^* - b^* c^*}.
\end{cases}
$$

There are four cases to consider as follows:

Case 1 If $\{c^*G > a^*N$ & $b^*N < d^*G\}$, then $\{G < \frac{a^*}{c^*}N$ & $G < \frac{b^*}{d^*}N$ & $\frac{a^*}{c^*} > 1$ & $\frac{b^*}{d^*} < 1\}$. Therefore, we can deduce that $\frac{\partial E(Q_i^{p*})}{\partial p_0} < 0$ and $\frac{\partial E(Q_j^{p*})}{\partial p_0} > 0$ with $G < \frac{b^*}{d^*}N$.

Case 2 If $\{c^*G < a^*N$ & $b^*N > d^*G\}$, then $\{G > \frac{a^*}{c^*}N$ & $G > \frac{b^*}{d^*}N$ & $\frac{a^*}{c^*} > 1$ & $\frac{b^*}{d^*} < 1\}$. Therefore, we can deduce that $\frac{\partial E(Q_i^{p*})}{\partial p_0} > 0$ and $\frac{\partial E(Q_j^{p*})}{\partial p_0} < 0$ with $G > \frac{a^*}{c^*}N$.

Case 3 If $\{c^*G > a^*N$ & $b^*N > d^*G\}$, then $\{G < \frac{a^*}{c^*}N$ & $G > \frac{b^*}{d^*}N$ & $\frac{a^*}{c^*} > 1$ & $\frac{b^*}{d^*} < 1\}$. Therefore, we can deduce that $\frac{\partial E(Q_i^{p*})}{\partial p_0} > 0$ and $\frac{\partial E(Q_j^{p*})}{\partial p_0} > 0$ with $\frac{b^*}{d^*}N < G < \frac{a^*}{c^*}N$.

Case 4 If $\{c^*G < a^*N$ & $b^*N < d^*G\}$, then $\{G > \frac{a^*}{c^*}N$ & $G < \frac{b^*}{d^*}N$ & $\frac{a^*}{c^*} > 1$ & $\frac{b^*}{d^*} < 1\}$. There is no intersection set. Therefore, we can deduce that $\frac{\partial E(Q_i^{p*})}{\partial p_0} < 0$ and $\frac{\partial E(Q_j^{p*})}{\partial p_0} < 0$ can not coexist.

Assuming that $E\left(Q_i^p, Q_j^p\right)$ is continuations and differentiable, we take the first and second partial derivatives of $E\left(Q_i^p, Q_j^p\right)$ with respect to Q_i^p and Q_j^p, respectively under the four cases:

(1) Case I: $rQ_i^p + rQ_j^p \leq Q_i^p \, and \, rQ_i^p + rQ_j^p \leq Q_j^p$

$$\frac{\partial E\left(Q_i^p, Q_j^p\right)}{\partial Q_i^p} = \int_0^{Q_i^p/(1-\alpha)} \int_0^\infty ((1-r)p + q)f(x,y)dxdy + \int_{Q_i^p/(1-\alpha)}^\infty \int_0^\infty (p+q)f(x,y)dxdy$$
$$+ \int_{Q_i^p - Q_j^p \alpha/(1-\alpha)}^{Q_i^p/(1-\alpha)} \int_0^{Q_i^p + Q_j^p - x} (s - (1-r)p - q)f(x,y)dxdy$$
$$+ \int_0^{Q_i^p - Q_j^p \alpha/(1-\alpha)} \int_0^{(Q_i^p - x)/\alpha} (s - (1-r)p - q)f(x,y)dxdy - c$$

$$\frac{\partial E\left(Q_i^p, Q_j^p\right)}{\partial Q_j^p} = \int_0^\infty \int_0^{Q_j^p/(1-\alpha)} ((1-r)p + q)f(x,y)dxdy + \int_0^\infty \int_{Q_j^p/(1-\alpha)}^\infty (p+q)f(x,y)dxdy$$
$$+ \int_{Q_i^p - Q_j^p \alpha/(1-\alpha)}^{Q_i^p/(1-\alpha)} \int_0^{Q_i^p + Q_j^p - x} (s - (1-r)p - q)f(x,y)dxdy$$
$$+ \int_{Q_i^p/(1-\alpha)}^{Q_j^p/\alpha} \int_0^{Q_j^p - \alpha x} (s - (1-r)p - q)f(x,y)dxdy - c$$

$$\frac{\partial E\left(Q_i^p, Q_j^p\right)^2}{\partial^2 Q_i^p} = \int_0^\infty \frac{1}{1-\alpha}(-rp)f(\frac{Q_i^p}{1-\alpha}, y)dy + \int_0^{Q_i^p + Q_j^p - Q_i^p/(1-\alpha)} \frac{1}{1-\alpha}(s - (1-r)p - q)f(\frac{Q_i^p}{1-\alpha}, y)dy$$
$$+ \int_{Q_i^p - Q_j^p \alpha/(1-\alpha)}^{Q_i^p/(1-\alpha)} (s - (1-r)p - q)f(x, Q_i^p + Q_j^p - x)dx$$
$$+ \int_0^{Q_i^p - Q_j^p \alpha/(1-\alpha)} \frac{1}{\alpha}(s - (1-r)p - q)f(x, \frac{Q_i^p - x}{\alpha})dx < 0$$

$$\frac{\partial E\left(Q_i^p, Q_j^p\right)^2}{\partial^2 Q_j^p} = \int_0^\infty \frac{1}{1-\alpha}(-rp)f(x, \frac{Q_j^p}{1-\alpha})dx + \int_{Q_i^p/(1-\alpha)}^{Q_j^p/\alpha} \frac{1}{1-\alpha}(s-(1-r)p-q)f(x, Q_j^p - \alpha x)dx$$

$$+ \int_{Q_i^p - Q_j^p \alpha/(1-\alpha)}^{Q_i^p/(1-\alpha)} (s-(1-r)p-q)f(x, Q_i^p + Q_j^p - x)dx$$

$$+ \int_0^{Q_i^p - Q_j^p \alpha/(1-\alpha)} \frac{1}{1-\alpha}(s-(1-r)p-q)f(x, \frac{Q_j^p}{1-\alpha})dx < 0$$

$$\frac{\partial E\left(Q_i^p, Q_j^p\right)^2}{\partial Q_i^p \partial Q_j^p} = \frac{\partial E\left(Q_i^p, Q_j^p\right)^2}{\partial Q_j^p \partial Q_i^p} = \int_{Q_i^p - Q_j^p \alpha/(1-\alpha)}^{Q_i^p/(1-\alpha)} (s-(1-r)p-q)f(x, Q_i^p + Q_j^p - x)dx < 0$$

(2) Case II : $rQ_i^p + rQ_j^p \le Q_i^p \, and \, rQ_i^p + rQ_j^p \ge Q_j^p$

$$\frac{\partial E\left(Q_i^p, Q_j^p\right)}{\partial Q_i^p} = \int_{Q_i^p/(1-\alpha)}^\infty \int_0^\infty (p+q)f(x,y)dxdy + \int_0^{Q_i^p/(1-\alpha)} \int_0^\infty ((1-r)p+q)f(x,y)dxdy$$

$$\int_0^{Q_i^p - Q_j^p \alpha/(1-\alpha)} \int_0^{(Q_i^p - x)/\alpha} (s-(1-r)p-q)f(x,y)dxdy$$

$$+ \int_{Q_i^p - Q_j^p \alpha/(1-\alpha)}^{Q_i^p + Q_j^p} \int_0^{Q_i^p + Q_j^p - x} (s-(1-r)p-q)f(x,y)dxdy - c$$

$$\frac{\partial E\left(Q_i^p, Q_j^p\right)}{\partial Q_j^p} = \int_0^\infty \int_0^{Q_j^p/(1-\alpha)} ((1-r)p+q)f(x,y)dxdy + \int_0^\infty \int_{Q_j^p/(1-\alpha)}^\infty (p+q)f(x,y)dxdy$$

$$\int_{Q_i^p - Q_j^p \alpha/(1-\alpha)}^{Q_i^p + Q_j^p} \int_0^{Q_i^p + Q_j^p - x} (s-(1-r)p-q)f(x,y)dxdy$$

$$+ \int_0^{Q_i^p - Q_j^p \alpha/(1-\alpha)} \int_0^{Q_j^p/(1-\alpha)} (s-(1-r)p-q)f(x,y)dxdy - c$$

$$\frac{\partial E\left(Q_i^p, Q_j^p\right)^2}{\partial^2 Q_i^p} = \int_0^\infty \frac{1}{1-\alpha}(-rp)f(\frac{Q_i^p}{1-\alpha}, y)dy$$

$$+ \int_{Q_i^p - Q_j^p \alpha/(1-\alpha)}^{Q_i^p + Q_j^p} (s-(1-r)p-q)f(x, Q_i^p + Q_j^p - x)dx$$

$$+ \int_0^{Q_i^p - Q_j^p \alpha/(1-\alpha)} \frac{1}{\alpha}(s-(1-r)p-q)f(x, \frac{Q_i^p - x}{\alpha})dx < 0$$

$$\frac{\partial E\left(Q_i^p, Q_j^p\right)^2}{\partial^2 Q_j^p} = \int_0^\infty \frac{1}{1-\alpha}(-rp)f(x, \frac{Q_j^p}{1-\alpha})dx$$

$$+ \int_{Q_i^p - Q_j^p \alpha/(1-\alpha)}^{Q_i^p + Q_j^p} (s-(1-r)p-q)f(x, Q_i^p + Q_j^p - x)dx$$

$$+ \int_0^{Q_i^p - Q_j^p \alpha/(1-\alpha)} \frac{1}{1-\alpha}(s-(1-r)p-q)f(x, \frac{Q_j^p}{1-\alpha})dx < 0$$

$$\frac{\partial E\left(Q_i^p, Q_j^p\right)^2}{\partial Q_i^p \partial Q_j^p} = \frac{\partial E\left(Q_i^p, Q_j^p\right)^2}{\partial Q_j^p \partial Q_i^p}$$

$$= \int_{Q_i^p - Q_j^p \alpha/(1-\alpha)}^{Q_i^p + Q_j^p} (s - (1-r)p - q) f(x, Q_i^p + Q_j^p - x) dx$$

(3) Case III: $rQ_i^p + rQ_j^p \geq Q_i^p \, and \, rQ_i^p + rQ_j^p \leq Q_j^p$

$$\frac{\partial E\left(Q_i^p, Q_j^p\right)}{\partial Q_i^p} = \int_0^{Q_i^p/(1-\alpha)} \int_0^\infty ((1-r)p+q) f(x,y) dx dy + \int_{Q_i^p/(1-\alpha)}^\infty \int_0^\infty (p+q) f(x,y) dx dy$$

$$+ \int_0^{Q_i^p/(1-\alpha)} \int_0^{Q_i^p + Q_j^p - x} (s - (1-r)p - q) f(x,y) dx dy - c$$

$$\frac{\partial E\left(Q_i^p, Q_j^p\right)}{\partial Q_j^p} = \int_0^\infty \int_0^{Q_j^p/(1-\alpha)} ((1-r)p+q) f(x,y) dx dy + \int_0^\infty \int_0^{Q_j^p/(1-\alpha)} (p+q) f(x,y) dx dy$$

$$+ \int_0^{Q_i^p/(1-\alpha)} \int_0^{Q_i^p + Q_j^p - x} (s - (1-r)p - q) f(x,y) dx dy$$

$$+ \int_{Q_i^p/(1-\alpha)}^{Q_j^p/\alpha} \int_0^{Q_j^p - \alpha x} (s - (1-r)p - q) f(x,y) dx dy - c$$

$$\frac{\partial E\left(Q_i^p, Q_j^p\right)^2}{\partial^2 Q_i^p} = \int_0^{Q_i^p/(1-\alpha)} (s - (1-r)p + q) f(x, Q_i^p + Q_j^p - x) dx$$

$$+ \int_0^{Q_i^p + Q_j^p - Q_i^p/(1-\alpha)} \frac{1}{1-\alpha} (s - (1-r)p + q) f(\frac{Q_i^p}{1-\alpha}, y) dy$$

$$+ \int_0^\infty \frac{1}{1-\alpha} (-rp) f(\frac{Q_i^p}{1-\alpha}, y) dy$$

$$\frac{\partial E\left(Q_i^p, Q_j^p\right)^2}{\partial^2 Q_j^p} = \int_0^{Q_i^p/(1-\alpha)} (s - (1-r)p + q) f(x, Q_i^p + Q_j^p - x) dx$$

$$+ \int_{Q_i^p/(1-\alpha)}^{Q_j^p/\alpha} (s - (1-r)p + q) f(x, Q_i^p - \alpha x) dy$$

$$+ \int_0^\infty \frac{1}{1-\alpha} (-rp) f(x, \frac{Q_j^p}{1-\alpha}) dx$$

$$\frac{\partial E\left(Q_i^p, Q_j^p\right)^2}{\partial Q_i^p \partial Q_j^p} = \frac{\partial E\left(Q_i^p, Q_j^p\right)^2}{\partial Q_j^p \partial Q_i^p} = \int_0^{Q_i^p/(1-\alpha)} (s - (1-r)p + q) f(x, Q_i^p + Q_j^p - x) dx$$

(4) Case IV: $rQ_i^p + rQ_j^p \geq Q_i^p \, and \, rQ_i^p + rQ_j^p \geq Q_j^p$

$$\frac{\partial E\left(Q_i^p, Q_j^p\right)}{\partial Q_i^p} = \int_0^{Q_i^p + Q_j^p} \int_0^{Q_i^p + Q_j^p - x} (s - (1-r)p - q) f(x,y) dx dy + \int_0^{Q_i^p/(1-\alpha)} \int_0^{\infty} ((1-r)p - q) f(x,y) dx dy$$

$$+ \int_{Q_i^p/(1-\alpha)}^{\infty} \int_0^{\infty} (p+q) f(x,y) dx dy - c$$

$$\frac{\partial E\left(Q_i^p, Q_j^p\right)}{\partial Q_j^p} = \int_0^{Q_i^p + Q_j^p} \int_0^{Q_i^p + Q_j^p - x} (s - (1-r)p - q) f(x,y) dx dy + \int_0^{\infty} \int_0^{Q_j^p/(1-\alpha)} ((1-r)p - q) f(x,y) dx dy$$

$$+ \int_0^{\infty} \int_{Q_j^p/(1-\alpha)}^{\infty} (p+q) f(x,y) dx dy - c$$

$$\frac{\partial E\left(Q_i^p, Q_j^p\right)^2}{\partial^2 Q_i^p} = \int_0^{\infty} \frac{-rp}{1-\alpha} f(\frac{Q_i^p}{1-\alpha}, y) dy + \int_0^{Q_i^p + Q_j^p} (s - (1-r)p - q) f(x, Q_i^p + Q_j^p - x) dx < 0$$

$$\frac{\partial E\left(Q_i^p, Q_j^p\right)^2}{\partial^2 Q_j^p} = \int_0^{\infty} \frac{-rp}{1-\alpha} f(x, \frac{Q_j^p}{1-\alpha}) dx + \int_0^{Q_i^p + Q_j^p} (s - (1-r)p - q) f(x, Q_i^p + Q_j^p - x) dx < 0$$

$$\frac{\partial E\left(Q_i^p, Q_j^p\right)^2}{\partial Q_i^p \partial Q_j^p} = \frac{\partial E\left(Q_i^p, Q_j^p\right)^2}{\partial Q_j^p \partial Q_i^p} = \int_0^{Q_i^p + Q_j^p} (s - (1-r)p - q) f(x, Q_i^p + Q_j^p - x) dx < 0$$

Then, we get the the Hessian Matrix under the above four cases as follows:

$$\begin{vmatrix} \frac{\partial E(Q_i^p, Q_j^p)^2}{\partial^2 Q_i^p} & \frac{\partial E(Q_i^p, Q_j^p)^2}{\partial Q_i^p \partial Q_j^p} \\ \frac{\partial E(Q_i^p, Q_j^p)^2}{\partial Q_j^p \partial Q_i^p} & \frac{\partial E(Q_i^p, Q_j^p)^2}{\partial^2 Q_j^p} \end{vmatrix} > 0.$$

Therefore, the expected profit functions have a maximum value. We have the result.

C.1   Proof of Proposition 5.1

Noting that z_i and z_j are independent of r, while z_k depends on r, we first derive the derivatives of z_k and $L(z_k)$ with respect to r as follows: $\frac{\partial z_k}{\partial r} = \frac{pv - pc}{[(1-r)p - v + q]^2 f_s(z_k)}$, $\frac{\partial L(z_k)}{\partial z_k} = \frac{v - c}{(1-r)p - v + q}$, $\frac{\partial L(z_k)}{\partial r} = \frac{p(v-c)^2}{[(1-r)p - v + q]^3 f_s(z_k)}$.

Then

1) When $\lambda \frac{r}{d} \leqslant 1$, the expected profit function is shown in Eq.(**??**). Then the first derivative is as follows:

$$
\begin{aligned}
\frac{d\phi_{pps}}{dr} = {} & [(c - v)z_i + (p - v + q)L(z_i)]\frac{\lambda\sigma_i}{d} + [(c - v)z_j + (p - v + q)L(z_j)]\frac{\lambda\sigma_j}{d} \\
& - [(c - v)z_k + (p - v + q)L(z_k)]\frac{\lambda\sqrt{\sigma_i^2 + \sigma_j^2 + 2\rho_{ij}\sigma_i\sigma_j}}{d} \\
& + 2\frac{pr}{d}\lambda L(z_k)\sqrt{\sigma_i^2 + \sigma_j^2 + 2\rho_{ij}\sigma_i\sigma_j} - 2\frac{pr}{d}(\lambda u_i + \lambda u_j),
\end{aligned}
\tag{C.1}
$$

and

$$
\frac{d^2\phi_{pps}}{dr^2} = 2\frac{p}{d}\lambda\left[L(z_k)\sqrt{\sigma_i^2 + \sigma_j^2 + 2\rho_{ij}\sigma_i\sigma_j} - (u_i + u_j)\right] - \frac{p^2 r(v - c)^2}{[(1-r)p - v + q]^3 f_s(z_k)}.
$$

Multiplying the first term of $\frac{d^2\phi_{pps}}{dr^2}$ by $r > 0$, we obtain

$$
2p\left[\alpha L(z_k)\sqrt{\sigma_i^2 + \sigma_j^2 + 2\rho_{ij}\sigma_i\sigma_j} - \alpha(u_i + u_j)\right].
\tag{C.2}
$$

Because $r > 0$, Eq.(C.2) does not change sign. The first term $\alpha L(z_k)\sqrt{\sigma_i^2 + \sigma_j^2 + 2\rho_{ij}\sigma_i\sigma_j}$ is the loss function, which defines the expected demand excessing the order quantity (see, e.g., Silver et

al., 1998). We subtract the expected loss sales from the mean u_k to get the expected sales of the probabilistic product as $\alpha(u_i + u_j) - \alpha L(z_k)\sqrt{\sigma_i^2 + \sigma_j^2 + 2\rho_{ij}\sigma_i\sigma_j}$, which is supposed to be positive. So we have $2p\alpha\left[L(z_k)\sqrt{\sigma_i^2 + \sigma_j^2 + 2\rho_{ij}\sigma_i\sigma_j} - (u_i + u_j)\right] < 0$. As $\frac{p^2 r(v-c)^2}{[(1-r)p-v+q]^3 f_s(z_k)} > 0$, $\frac{d^2\phi_{pps}}{dr^2} < 0$.

2) When $\lambda\frac{r}{d} > 1, \alpha = 1$. Then all the customers would switch to buying the probabilistic product and the expected profit function can be re-written as follow:

$$\phi_{pps} = ((1-r)p - c)(u_i + u_j) - [(c-v)z_k + ((1-r)p - v + q)L(z_k)]\sqrt{\sigma_i^2 + \sigma_j^2 + 2\rho_{ij}\sigma_i\sigma_j}. \qquad \text{(C.3)}$$

Then first derivative $\frac{d\phi_{pps}}{dr} = -p\left((u_i + u_j) - L(z_k)\sqrt{\sigma_i^2 + \sigma_j^2 + 2\rho_{ij}\sigma_i\sigma_2}\right)$. Similarly, we can recognize that $(u_i + u_j) - L(z_k)\sqrt{\sigma_i^2 + \sigma_j^2 + 2\rho_{ij}\sigma_i\sigma_j}$ is the expected sales of the probabilistic product which is supposed to be positive. Therefore, $\frac{d\phi_{pps}}{dr} < 0$.

According to Eq.(6.1), given λ and d, we can deduce that ϕ_{pps} is also concave in α. Thus the proposition is proved.

C.2 Proof of Proposition 5.2

From Proposition 5.1, we can find that the optimal price discount r^* is located within the range $(0, min(d/\lambda, 1)]$. Therefore, we can just consider the expected profit when $\lambda\frac{r}{d} \leq 1$ and set Eq.(C.1) to zero to determine the optimal price discount.

Let the first derivative $\frac{d\phi_{pps}}{dr} = 0$,

$$r = \Big([(c-v)z_i + (p-v+q)L(z_i)]\alpha\sigma_i + [(c-v)z_j + (p-v+q)L(z_j)]\alpha\sigma_j$$

$$-[(c-v)z_k + (p-v+q)L(z_k)]\alpha\sqrt{\sigma_i^2 + \sigma_j^2 + 2\rho_{ij}\sigma_i\sigma_j}\Big)/(2p(u_k - L(z_k)\sigma_k)).$$

Because the specific products have the same price/cost structure and $\int_{-\infty}^{z_i} f_s(x)dx = \frac{p-c+q}{p-v+q}$, $\int_{-\infty}^{z_j} f_s(x)dx = \frac{p-c+q}{p-v+q}$, and $\int_{-\infty}^{z_k} f_s(x)dx = \frac{(1-r^*)p-c+q}{(1-r^*)p-v+q}$, where $f_s(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. Then $z_i = z_j$ and $L(z_i) = L(z_j)$. However, z_k and $L(z_k)$ are dependent on the optimal discount r^*. Therefore, the optimal discount of r^* meets the equation as follows:

$$D_c = 2pr(u_k - L(z_k)\sigma_k), \qquad \text{(C.4)}$$

where $D_c = [(c-v)z + (p-v+q)L(z)](\alpha\sigma_i + \alpha\sigma_j) - [(c-v)z_k + (p-v+q)L(z_k)]\sigma_k$, $z_i = z_j = z$, and $L(z_i) = L(z_j) = L(z)$. Thus the proposition is proved.

C.3 Proof of Proposition 5.4

1) The optimal profit ϕ_{ts}^* and total order quantity Q_{ts}^* in the traditional selling (TS) is as follows:

$$\phi_{ts}^* = (p-c)u_i - [(c-v)z_i + (p-v+q)L(z_i)]\sigma_i$$
$$+ (p-c)u_j - [(c-v)z_j + (p-v+q)L(z_j)]\sigma_j. \qquad \text{(C.5)}$$

$$Q_{ts}^* = u_i + u_j + z_i\sigma_i + z_j\sigma_j.$$

Therefore,

$$\phi_{pps}^* - \phi_{ts}^* = [(c-v)z + (p-v+q)L(z)](\alpha\sigma_i + \alpha\sigma_j) - [(c-v)z_k + (p-v+q)L(z_k)]\sigma_k$$
$$- pr^*(u_k - L(z_k)\sigma_k).$$

Substituting Eq.(C.4) into Eq.(C.3), we get $\phi_{pps}^* - \phi_{ts}^* = pr^*(u_k - L(z_k)\sigma_k) \geq 0$. Thus, the proposition that $\phi_{pps}^* \geq \phi_{ts}^*$ has been proved.

2) Next we compare the optimal order quantity Q_{pps}^* in probabilistic selling with Q_{ts}^* as $Q_{pps}^* - Q_{ts}^* = \alpha^* Q_d^*$, where $Q_d^* = z_k\sqrt{\sigma_i^2 + \sigma_j^2 + 2\rho_{ij}\sigma_i\sigma_j} - z_i\sigma_i - z_j\sigma_j$. Taking the first derivative of Q_d^* with respect to r^*, we get $\frac{\partial Q_d^*}{\partial r^*} = \frac{p(v-c)\sqrt{\sigma_i^2+\sigma_j^2+2\rho_{ij}\sigma_i\sigma_j}}{[(1-r)p-v+q]^2 f_s(z_k)} < 0$. Then Q_d^* can achieves the maximum when $r^* \to 0$. When $r^* \to 0$, we find that $Q_d^* \leq 0$ and $\lim_{r^* \to 0}(Q_{pps}^* - Q_{ts}^*) = 0$. Thus, the proposition that $Q_{pps}^* \leq Q_{ts}^*$ has been proved.

3) With $Q_d^* \leq 0$ and $\frac{\partial Q_d^*}{\partial r^*} < 0$, we can get that $\frac{\partial(Q_{pps}^* - Q_{ts}^*)}{\partial r^*} = \frac{\lambda}{d}Q_d^* + \alpha^*\frac{\partial Q_d^*}{\partial r^*} \leq 0$.

C.4 Proof of Proposition 5.5

1) Taking the first derivative of $\phi_{pps}^* - \phi_{ts}^*$ with respect to λ, we obtain

$$\frac{\partial(\phi_{pps}^* - \phi_{ts}^*)}{\partial\lambda} = pr^*\left(u_i\frac{r^*}{d} + u_j\frac{r^*}{d} - L(z_k)\frac{r^*}{d}\sqrt{\sigma_i^2 + \sigma_j^2 + 2\sigma_i\sigma_j}\right). \tag{C.6}$$

Multiplying Eq.(C.6) by $\lambda(\lambda > 0)$ does not change the sign. Then we obtain

$$\lambda\frac{\partial(\phi_{pps}^* - \phi_{ts}^*)}{\partial\lambda} = pr^*(u_k - L(z_k)\sigma_k) > 0. \tag{C.7}$$

Similarly, we can prove that

$$\frac{\partial(\phi_{pps}^* - \phi_{ts}^*)}{\partial d} = -\frac{1}{d}[pr^*(u_k - L(z_k\sigma_k))] < 0. \tag{C.8}$$

2) Taking the first derivative of $Q_{pps}^* - Q_{ts}^*$ with respect to λ, we obtain

$$\frac{\partial(Q_{pps}^* - Q_{ts}^*)}{\partial\lambda} = \frac{r^*}{d}Q_d^* < 0. \tag{C.9}$$

Similarly, we can prove that

$$\frac{\partial(Q_{pps}^* - Q_{ts}^*)}{\partial d} = -\frac{r^*}{d^2}Q_d^* > 0. \tag{C.10}$$

Appendix. Proposition 6.1

D.1 Probabilistic selling with barter in a fully covered market

(1) When $\frac{1}{2} \leq \varphi < \bar{\varphi}$, for given $\hat{x}_1$ and $\hat{x}_2$, the optimal prices that can extract the maximum consumer surplus are expressed as follows (we let $\hat{x}_3 = \hat{x}_4 = 1/2$ in Figure 6.2(a) to make sure that the market is fully covered):

$$
\begin{aligned}
p_0^{pb} &= \varphi(1 - t/2) + (1 - \varphi)(1 - t(1 - 1/2)), \\
p_1^{pb} &= \varphi t \hat{x}_1 + (1 - \varphi)t(1 - \hat{x}_1) - t\hat{x}_1 - (1 - \varphi)\alpha t(1 - 2\hat{x}_1) + p_0^{pb}, \\
p_2^{pb} &= \varphi t \hat{x}_2 + (1 - \varphi)t(1 - \hat{x}_2) - \varphi \alpha t(2\hat{x}_2 - 1) - t(1 - \hat{x}_2) + p_0^{pb}.
\end{aligned}
\tag{D.1}
$$

Taking the first derivative of the profit with respect to $\hat{x}_1$, we get $\frac{\partial G_c}{\partial \hat{x}_1} = (q\alpha + t(1 - 4x_1)(-1 + \alpha))(-1 + \varphi)$. Solving $\frac{\partial G_c}{\partial \hat{x}_1} = 0$, we get $\hat{x}_1 = \frac{-t + q\alpha + t\alpha}{4t(-1+\alpha)}$. Solving $\frac{\partial G_c}{\partial \hat{x}_2} = (t(-3 + 4\hat{x}_2)(-1 + \alpha) + q\alpha)\varphi = 0$, we get $\hat{x}_2 = \frac{-3t - q\alpha + 3t\alpha}{4t(-1+\alpha)}$, $\forall \varphi \geq 1/2$. With optimal $\hat{x}_1$ and $\hat{x}_2$, the prices of the component products are $p_1^{pb} = \frac{1}{2}(2 + \alpha(q - t)(1 - \varphi) - t\varphi)$, $p_2^{pb} = \frac{1}{2}(2 + q\alpha\varphi - t + \varphi t - \alpha t\varphi)$, and $p_0^{pb} = 1 - t/2$. The profit of probabilistic selling with barter in a fully covered market is

$$
G_{ca}^A = -\frac{q^2\alpha^2 - 2t(-1 + \alpha)(4 + q\alpha) + t^2(-3 + 2\alpha + \alpha^2)}{8t(-1 + \alpha)} - c.
\tag{D.2}
$$

Obviously, G_{ca}^A is independent of the allocation probability and the optimal allocation probability is $\frac{1}{2} \leq \varphi^* < \bar{\varphi}$.

   (2) When $\bar{\varphi} \leq \varphi < 1$, for given $\hat{x}_1$ and $\hat{x}_2$, the optimal price that can extract the maximum consumer surplus are expressed as follows (we let $\hat{x}_3 = \hat{x}_2$ in Figure 6.2(b) to make sure that the

market is fully covered):

$$p_0^{pb} = \varphi(1 - t\hat{x}_2) + (1 - \varphi)(1 - t(1 - \hat{x}_2) + \alpha(2\hat{x}_2 - 1)),$$
$$p_1^{pb} = \varphi t\hat{x}_1 + (1 - \varphi)t(1 - \hat{x}_1) - t\hat{x}_1 - (1 - \varphi)\alpha t(1 - 2\hat{x}_1) + p_0^{pb}, \qquad \text{(D.3)}$$
$$p_2^{pb} = \varphi t\hat{x}_2 + (1 - \varphi)t(1 - \hat{x}_2) - \varphi\alpha t(2\hat{x}_2 - 1) - t(1 - \hat{x}_2) + p_0^{pb}.$$

The profit function is the same as Eq. (6.3). Similarly, we get $\hat{x}_1 = \frac{-t + q\alpha + t\alpha}{4t(-1 + \alpha)}$ and $\hat{x}_2 = \frac{-t - t\varphi - q\alpha\varphi + t\alpha\varphi}{4t(-1 + \alpha)\varphi}$, and a resulting optimal profit is

$$G_{ca}^B = -\frac{q^2\alpha^2\varphi + t^2\left(1 + (-5 + 4\alpha + \alpha^2)\varphi\right) - 2t\varphi(-4 - 4c(-1 + \alpha) + 2\alpha(2 + q(-1 + \varphi)) + \alpha^2(q - 2q\varphi))}{8t(-1 + \alpha)\varphi}.$$

$$\text{(D.4)}$$

Taking the first derivative, we can find that $\frac{\partial G_{ca}^B}{\partial \varphi} = \frac{t - 4q(-1 + \alpha)\alpha\varphi^2}{8(-1 + \alpha)\varphi^2} < 0$. Therefore, the optimal $\varphi^* = \frac{1}{2(1-\alpha)}$, which makes $G_{ca}^B = G_{ca}^A$. Therefore, the allocation probability that extracts the maximum profit when $\varphi \geq \frac{1}{2}$ is $\frac{1}{2} \leq \varphi^* \leq \bar{\varphi}$. Similarly, when $\varphi \leq \frac{1}{2}$, we can extract the allocation probability $\frac{1-2\alpha}{2(1-\alpha)} \leq \varphi^* \leq \frac{1}{2}$. Thus, the optimal allocation probability for probabilistic selling with barter is $\bar{\bar{\varphi}} \leq \varphi^* \leq \bar{\varphi}$

Furthermore, we set $\hat{x}_1 = \frac{-t + q\alpha + t\alpha}{4t(-1 + \alpha)} \geq 0$, i.e., $\alpha \leq \frac{t}{q+t} = \tilde{\alpha}$, to make sure that there are non-negative demands for the component products. When $\alpha > \tilde{\alpha}$, i.e., $\hat{x}_1 = 0$ and $\hat{x}_2 = 1$, the seller only sells the probabilistic product. Thus, the resulting profit is

$$G_{Ca}^* = 1 - c - \frac{t}{2} + \frac{q\alpha}{2}. \qquad \text{(D.5)}$$

(3) Referring to the optimal decisions and resulting profit of traditional selling in Fay and Xie (2008) (see Eq. (6.4)), we compare G_{Ca} and G_{Ca}^* with G_{Ts} to get the optimal strategy for probabilistic selling with barter.

a) G_{Ca} vs G_{Ts}

When $c < 1 - t$, $G_{Ca} - G_{Ts} = -\frac{(t(-1+\alpha) - q\alpha)^2}{8t(-1+\alpha)} > 0$. When $c \geq 1 - t$, $G_{Ca} - G_{Ts} = \Delta_1 = -\frac{q^2\alpha^2 - 2t(-1+\alpha)(4 + q\alpha) + t^2(-3 + 2\alpha + \alpha^2)}{8t(-1+\alpha)} - c - \frac{(1-c)^2}{2t}$. Notice $\frac{\partial \Delta_1}{\partial c} = \frac{1-t-c}{t} < 0$ and the comparison result is given by

$$G_{Ca} = \begin{cases} G_{Ca}^A & \text{if } c \leq \bar{\bar{c}}, \\ \frac{(1-c)^2}{2t} & \text{if } c > \bar{\bar{c}}, \end{cases} \qquad \text{(D.6)}$$

where $\bar{\bar{c}} = 1 - t + t/2\sqrt{\frac{(t(1-\alpha) + q\alpha)^2}{t^2(1-\alpha)}}$.

b) G_{Ca}^* vs G_{Ts}

When $c < 1 - t$, $G_{Ca}^* - G_{Ts} = \frac{q\alpha}{2} > 0$. When $c \geq 1 - t$, $G_{Ca}^* - G_{Ts} = \Delta_2 = 1 - c - \frac{t}{2} + \frac{q\alpha}{2} - \frac{(1-c)^2}{2t}$. Notice $\frac{\partial \Delta_2}{\partial c} = \frac{1-t-c}{t} < 0$ and the comparison result is given by

$$G_{Ca}^* = \begin{cases} 1 - c - \frac{t}{2} + \frac{q\alpha}{2} & \text{if } c \leq \vec{c}. \\ \frac{(1-c)^2}{2t} & \text{if } c > \vec{c}, \end{cases} \qquad \text{(D.7)}$$

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where $\vec{c} = 1 - t + \sqrt{qt\alpha}$.

Thus, we have derived the optimal strategy for probabilistic selling with barter in a fully covered market. We summarize the optimal decisions on price and allocation probability, and the resulting sales and profit in Table D.1.

Table D.1: Optimal decisions for probabilistic selling with barter when market is fully covered

	$\alpha \leq \tilde{\alpha}$		$\alpha > \tilde{\alpha}$	
D_j	$\begin{cases} \frac{-t+q\alpha+t\alpha}{4t(-1+\alpha)} & \text{if } c \leq \bar{\bar{c}} \\ \frac{1-c}{2t} & \text{if } c > \bar{\bar{c}} \end{cases}$		$\begin{cases} 0 & \text{if } c \leq \vec{c} \\ \frac{1-c}{2t} & \text{if } c > \vec{c} \end{cases}$	
D_0	$\begin{cases} \frac{t\alpha-q\alpha-t}{2t(-1+\alpha)} & \text{if } c \leq \bar{\bar{c}} \\ 0 & \text{if } c > \bar{\bar{c}} \end{cases}$		$\begin{cases} 1 & \text{if } c \leq \vec{c} \\ 0 & \text{if } c > \vec{c} \end{cases}$	
p_0^{pb}	$\begin{cases} 1 - \frac{1}{2}t & \text{if } c \leq \bar{\bar{c}} \\ N/A & \text{if } c > \bar{\bar{c}} \end{cases}$		$\begin{cases} 1 - \frac{1}{2}t & \text{if } c \leq \vec{c} \\ N/A & \text{if } c > \vec{c} \end{cases}$	
p_1^{pb}	$\begin{cases} \frac{1}{2}(2+\alpha(q-t)(1-\varphi)-t\varphi) & \text{if } c \leq \bar{\bar{c}} \\ \frac{1+c}{2} & \text{if } c > \bar{\bar{c}} \end{cases}$		$\begin{cases} N/A & \text{if } c \leq \vec{c} \\ \frac{1+c}{2} & \text{if } c > \vec{c} \end{cases}$	
p_2^{pb}	$\begin{cases} \frac{1}{2}(2+\alpha(q-t)\varphi-t(1-\varphi)) & \text{if } c \leq \bar{\bar{c}} \\ \frac{1+c}{2} & \text{if } c > \bar{\bar{c}} \end{cases}$		$\begin{cases} N/A & \text{if } c \leq \vec{c} \\ \frac{1+c}{2} & \text{if } c > \vec{c} \end{cases}$	
G_{pb}	$\begin{cases} G_{Ca}^A & \text{if } c \leq \bar{\bar{c}} \\ \frac{(1-c)^2}{2t} & \text{if } c > \bar{\bar{c}} \end{cases}$		$\begin{cases} G_{Ca}^* & \text{if } c \leq \vec{c} \\ \frac{(1-c)^2}{2t} & \text{if } c > \vec{c} \end{cases}$	
φ^*	$\begin{cases} \bar{\bar{\varphi}} \leq \varphi \leq \bar{\varphi} & \text{if } c \leq \bar{\bar{c}} \\ N/A & \text{if } c > \bar{\bar{c}} \end{cases}$		$\begin{cases} \bar{\bar{\varphi}} \leq \varphi \leq \bar{\varphi} & \text{if } c \leq \vec{c} \\ N/A & \text{if } c > \vec{c} \end{cases}$	

Note: $\vec{c} = 1 - t + \sqrt{qt\alpha}$ and $\bar{\bar{c}} = 1 - t + t/2\sqrt{\frac{(t(1-\alpha)+q\alpha)^2}{t^2(1-\alpha)}}$.

D.2 Probabilistic selling with barter in a partially covered market

(1) When $\frac{1}{2} \leq \varphi < \bar{\varphi}$ (as shown in Figure 2(a)), the profit function is given by

$$G_{Ia}^A = (p_{01}^{pd}-c)(\hat{x}_3-\hat{x}_1)+(p_{02}^{pd}-c)(\hat{x}_2-\hat{x}_4)+(p_1-c)\hat{x}_1+(p_2-c)(1-\hat{x}_2)+(\hat{x}_3-\hat{x}_1)q\alpha(1-\varphi)+(\hat{x}_2-\hat{x}_4)q\alpha\varphi,$$

(D.8)

where p_{01}^{pd} and p_{02}^{pd} are the price for the probabilistic product. For given $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$, and $\hat{x}_4$, the optimal prices that can extract the maximum consumer surplus are expressed as

$$
\begin{aligned}
p_{01}^{pd} &= \varphi(1 - t\hat{x}_3) + (1 - \varphi)(1 - t(1 - \hat{x}_3) + t\alpha(1 - 2\hat{x}_3)), \\
p_{02}^{pd} &= \varphi(1 - t\hat{x}_4 + \alpha t(2\hat{x}_4 - 1)) + (1 - \varphi)(1 - t(1 - \hat{x}_4)), \\
p_1^{pd} &= \varphi t\hat{x}_1 + (1 - \varphi)t(1 - \hat{x}_1) - t\hat{x}_1 - (1 - \varphi)\alpha t(1 - 2\hat{x}_1) + p_{01}^{pd}, \\
p_2^{pd} &= \varphi t\hat{x}_2 + (1 - \varphi)t(1 - \hat{x}_2) - \varphi\alpha t(2\hat{x}_2 - 1) - t(1 - \hat{x}_2) + p_{02}^{pd}.
\end{aligned}
\tag{D.9}
$$

Taking the first derivatives of the profit with respect to $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$, and $\hat{x}_4$, and setting them to zero, we get $\hat{x}_1 = \frac{-t + q\alpha + t\alpha}{4t(-1 + \alpha)}$, $\hat{x}_2 = \frac{-3t - q\alpha + 3t\alpha}{4t(-1 + \alpha)}$, $\hat{x}_3 = \frac{-1 + c + t - q\alpha - t\alpha - t\varphi + q\alpha\varphi + t\alpha\varphi}{2t(1 - 2\alpha - 2\varphi + 2\alpha\varphi)}$, and $\hat{x}_4 = \frac{-1 + c + 2t - 3t\varphi - q\alpha\varphi + 3t\alpha\varphi}{2t(1 - 2\varphi + 2\alpha\varphi)}$. Given that the price of the probabilistic product is $p_0^{pd} = p_{01}^{pd} = p_{02}^{pd}$, the optimal $\varphi^* = 1/2$. Substituting φ^* into $\hat{x}_3$ and $\hat{x}_4$, we get the optimal $\hat{x}_3 = \frac{2 - 2c - t + q\alpha + \alpha t}{4\alpha t}$ and $\hat{x}_4 = \frac{-2 + 2c + t - q\alpha + 3\alpha t}{4\alpha t}$. Notice that the seller chooses to make positive sales of the probabilistic product when $\hat{x}_3 > \hat{x}_1$ or $\hat{x}_4 < \hat{x}_2$, and the market is partially covered if $\hat{x}_3 < 1/2$ or $\hat{x}_4 > 1/2$. Thus we can deduce the condition on the product cost as $\tilde{c} < c < \tilde{\tilde{c}}$, where $\tilde{c} = 1 - \frac{1}{2}t + \frac{q\alpha - \alpha t}{2}$ and $\tilde{\tilde{c}} = 1 - \frac{1}{2}t + \frac{q\alpha}{2(1-\alpha)}$. The profit is expressed as follows:

$$
G_{Ia}^A = \frac{t(1 - \alpha)}{8\alpha} + \frac{q^2\alpha^2}{8t(-1 + \alpha)\alpha} + \frac{4t + 2(-2 + q)\alpha t + 8c + 4ct(-1 + \alpha) + 4cq\alpha + 4 + 4c^2 + 4q\alpha}{8t\alpha}. \tag{D.10}
$$

(2) When $1 > \varphi \geq \bar{\varphi}$, we get the optimal $\hat{x}_1 = \frac{-t + q\alpha + t\alpha}{4t(-1 + \alpha)}$, $\hat{x}_2 = \frac{-1 + c + 2t}{2t}$, and $\hat{x}_3 = \frac{-1 + c + t - t\varphi - q\alpha\varphi + t\alpha\varphi}{2t(1 - 2\varphi + 2\alpha\varphi)}$. And we can get the optimal profit as follows:

$$
\begin{aligned}
G_{Ia}^B = -\frac{1}{8t(-1 + \alpha)(1 + 2(-1 + \alpha)\varphi)}&[-4(-1 + c)^2(-1 + \alpha)^2\varphi - 4(-1 + c)q(-1 + \alpha)\alpha\varphi \\
&+ t^2(-1 + \alpha^2(1 - 3\varphi) + \varphi + 2\alpha^3\varphi) + q^2\alpha^2(1 + (-3 + 2\alpha)\varphi) + 2t(-1 + \alpha) \\
&(-2 + 2\varphi - 2\alpha\varphi + 2c(1 + (-1 + \alpha)\varphi) + q\alpha(-1 + (3 - 2\alpha)\varphi + 4(-1 + \alpha)\varphi^2))].
\end{aligned}
\tag{D.11}
$$

Because $\frac{\partial G_{Ia}^B}{\partial \varphi} < 0$, the optimal $\varphi = \frac{1}{2(1-\alpha)}$. Notice that the denominator of $\hat{x}_3$ is zero when $\varphi = \frac{1}{2(1-\alpha)}$. Thus, $\hat{x}_3$ approaches to 1 and the market reduces to a fully covered market, i.e., $\hat{x}_3 = \hat{x}_2 = 1$. Therefore, the optimal profit in the partially covered market is $G_{Ia} = G_{Ia}^A$ with $\tilde{c} < c < \tilde{\tilde{c}}$. Also when $\alpha > \tilde{\alpha}$, i.e., $\hat{x}_1 = 0$ and $\hat{x}_2 = 1$), the seller only sells the probabilistic product. Thus, the resulting profit is

$$
G_{Ia}^* = \frac{4c^2 + (2 + q\alpha)^2 - 4c(2 + t(-1 + \alpha) + q\alpha) + t^2(1 - 6\alpha + 5\alpha^2) - 2t(2 + (-2 + q)\alpha + q\alpha^2)}{8t\alpha}. \tag{D.12}
$$

(3) We then compare G_{Ia} and G_{Ia}^* with G_{Ts} to derive the optimal strategy for probabilistic selling with barter in a partially covered market.

a) G_{Ia} vs G_{Ts}

Because $\tilde{c} > 1 - t$, we can only consider the case where $c > 1 - t$. Notice that $\alpha \leq \tilde{\alpha}$ and $\tilde{\tilde{c}} \leq 1$, so $G_{Ia} - G_{Ts} = \frac{(2 + 2c(-1 + \alpha) + t(-1 + \alpha) - 2\alpha + q\alpha)^2}{8t(1 - \alpha)\alpha} > 0$. Therefore, the strategy for probabilistic selling when

$\alpha \le \frac{t}{q+t}$ in a partially covered market is given by

$$G_{Ia} = \begin{cases} 1 - t/2 - c & \text{if } c \le 1 - t, \\ \frac{(1-c)^2}{2t} & \text{if } 1 - t < c \le \tilde{c}, \\ G_{Ia}^A & \text{if } \tilde{c} < c < \tilde{\tilde{c}}, \\ \frac{(1-c)^2}{2t} & \text{if } c \ge \tilde{\tilde{c}}. \end{cases} \tag{D.13}$$

b) G_{Ia}^* vs G_{Ts}

Because $\tilde{c} > 1 - t$, we can only consider the case where $c > 1 - t$.

$G_{Ia}^* - \frac{(1-c)^2}{2t} = \Delta_3 = \frac{4 - 4c^2(-1+\alpha) - 4\alpha + 4q\alpha + q^2\alpha^2 - 4c(2 + t(-1+\alpha) + (-2+q)\alpha) + t^2(1 - 6\alpha + 5\alpha^2) - 2t(2 + (-2+q)\alpha + q\alpha^2)}{8t\alpha}$.

$\frac{\partial^2 \Delta_3}{\partial c^2} = \frac{1-\alpha}{t\alpha} > 0$. And we get c_1 and c_2 to make $\Delta_3 = 0$. Furthermore, we prove that $c_1 > \tilde{c}, c_1 > 1$, and $c_2 < \tilde{c}$. Therefore, $\Delta_3 < 0$ and the strategy for probabilistic selling when $\alpha > \tilde{\alpha}$ in a partially covered market is given by

$$G_{Ia}^* = \begin{cases} 1 - t/2 - c & \text{if } c \le 1 - t, \\ \frac{(1-c)^2}{2t} & \text{if } c > 1 - t. \end{cases} \tag{D.14}$$

Thus, we have derived the optimal strategy for probabilistic selling with barter in a partially covered market. Also referring to the optimal decision results of traditional selling in Fay and Xie (2008), we summarize the optimal decisions on the price and allocation probability, and the resulting sales and profit when market is partially covered in Table D.2.

D.3 Proof of Proposition 6.1

(1) When $0 < \alpha \le \tilde{\alpha}$,

Notice that $\bar{\bar{c}} - \tilde{c} = 1 - t + \frac{t+q\alpha-t\alpha}{2\sqrt{1-\alpha}} - (1 - t + \frac{t+q\alpha-t\alpha}{2}) > 0$, then we compare G_{Ca} with G_{Ia} when $\tilde{c} < c \le \bar{\bar{c}}$. We get $G_{Ia} - G_{Ca} = \frac{(-2+2c+t-q\alpha+t\alpha)^2}{8t\alpha} > 0$. Therefore, the optimal strategy for probabilistic selling with barter can be expressed as follows:

$$G_{bp} = Max[G_{Ca}, G_{Ia}, G_{Ts}] = \begin{cases} G_{Ca}^A & \text{if } c \le \tilde{c}, \bar{\bar{\varphi}} \le \varphi^* \le \bar{\varphi}, \\ G_{Ia}^A & \text{if } \tilde{c} < c < \tilde{\tilde{c}}, \varphi^* = \frac{1}{2}, \\ G_{Ts} & \text{if } c \ge \tilde{\tilde{c}}. \end{cases} \tag{D.15}$$

(2) When $\tilde{\alpha} < \alpha < 1$,

It is easy to obtain the optimal optimal strategy for probabilistic selling with barter as follows:

$$G_{bp}^* = Max[G_{Ca}^*, G_{Ia}^*, G_{Ts}] = \begin{cases} G_{Ca}^* & \text{if } c \le \vec{c}, \bar{\bar{\varphi}} \le \varphi^* \le \bar{\varphi}, \\ G_{Ts} & \text{if } \vec{c} < c \le 1. \end{cases} \tag{D.16}$$

Table D.2: Optimal decisions of probabilistic selling with barter when market is partially covered

	$\alpha \le \tilde{\alpha}$	$\alpha > \tilde{\alpha}$
D_j	$\begin{cases} \frac{1}{2} & \text{if } c \le 1-t \\ \frac{1-c}{2t} & \text{if } 1-t < c \le \tilde{c} \\ \frac{-t+q\alpha+t\alpha}{4t(-1+\alpha)} & \text{if if } \tilde{c} < c < \tilde{\tilde{c}} \\ \frac{1-c}{2t} & \text{if } c \ge \tilde{\tilde{c}} \end{cases}$	$\begin{cases} \frac{1}{2} & \text{if } c \le 1-t \\ \frac{1-c}{2t} & \text{if } c > 1-t \end{cases}$
D_0	$\begin{cases} \frac{2-2c-t-2\alpha+2c\alpha+q\alpha+t\alpha}{2t\alpha-2t\alpha^2} & \text{if } \tilde{c} < c < \tilde{\tilde{c}} \\ 0 & \text{if } others \end{cases}$	N/A
p_0^{pb}	$\begin{cases} \frac{1}{4}(2+2c+t(-1+\alpha)-q\alpha) & \text{if } \tilde{c} < c < \tilde{\tilde{c}} \\ N/A & \text{if } others \end{cases}$	N/A
p_j^{pb}	$\begin{cases} \frac{2-t}{2} & \text{if } c \le 1-t \\ \frac{1+c}{2} & \text{if } c > 1-t \end{cases}$	$\begin{cases} \frac{2-t}{2} & \text{if } c \le 1-t \\ \frac{1+c}{2} & \text{if } c > 1-t \end{cases}$
G_{pb}	$\begin{cases} 1-\frac{t}{2}-c & \text{if } c \le 1-t \\ \frac{(1-c)^2}{2t} & \text{if } 1-t < c \le \tilde{c} \\ G_{Ia}^{A} & \text{if if } \tilde{c} < c < \tilde{\tilde{c}} \\ \frac{(1-c)^2}{2t} & \text{if } c \ge \tilde{\tilde{c}} \end{cases}$	$\begin{cases} 1-\frac{t}{2}-c & \text{if } c \le 1-t \\ \frac{(1-c)^2}{2t} & \text{if } c > 1-t \end{cases}$
φ	$\begin{cases} \frac{1}{2} & \text{if } \tilde{c} < c < \tilde{\tilde{c}} \\ N/A & \text{if } others \end{cases}$	N/A

Note: $\tilde{c} = 1 - \frac{1}{2}t + \frac{q\alpha - \alpha t}{2}$ and $\tilde{\tilde{c}} = 1 - \frac{1}{2}t + \frac{q\alpha}{2(1-\alpha)}$.

D.4   Proof of Proposition 6.2

(1) Full coverage
$\hat{x}_2 - \hat{x}_1 = \frac{1}{2} + \frac{q\alpha}{2t(1-\alpha)}$, $\frac{\partial(\hat{x}_2-\hat{x}_1)}{\partial\alpha} = \frac{q}{2t(-1+\alpha)^2} > 0$, and $\frac{\partial(\hat{x}_2-\hat{x}_1)}{\partial q} = \frac{\alpha}{2t(1-\alpha)} > 0$.

(2) Partial coverage
$\frac{\partial(\hat{x}_3-\hat{x}_1)}{\partial q} = \frac{\alpha}{4t\alpha-4t\alpha^2} > 0$, and $\frac{\partial(\hat{x}_2-\hat{x}_4)}{\partial q} = \frac{\alpha}{4t\alpha-4t\alpha^2} > 0$.

D.5   Proof of Proposition 6.5

From proposition 5.5, offering barter is more advantageous when $\bar{c} < c < \tilde{\tilde{c}}$ if $\alpha \le \tilde{\alpha}$, and $c < \vec{c}$ if $\alpha > \tilde{\alpha}$ & $q \ge 1/3t$. Taking the first derivatives of $\bar{c}$, $\tilde{\tilde{c}}$, and $\vec{c}$ with respective to α and simplifying, we have $\frac{\partial\bar{c}}{\partial\alpha} = \frac{q^2\alpha^2(3-2\alpha)+2(q-t)(-1+\alpha)(1+t(-1+\alpha)\alpha)+2qt\alpha(-1+\alpha)^2}{2(-1+\alpha)\sqrt{(-1+\alpha)\alpha^2(-2qt(-1+\alpha)+t^2(-1+\alpha)+q^2\alpha)}} < 0$, $\frac{\partial\tilde{\tilde{c}}}{\partial\alpha} = \frac{q}{2(-1+\alpha)^2} > 0$, and $\frac{\partial\vec{c}}{\partial\alpha} = \frac{qt}{2\sqrt{qt\alpha}} > 0$. Therefore, the result holds.

D.6 Proof of the comparison results in Figure 6.3

(1) G_{Ca}^{A} vs G_{ps}

$G_{Ca}^{A} - G_{ps} = \Delta_4 = \frac{\alpha(-2qt(-1+\alpha)+t^2(-1+\alpha)+q^2\alpha)}{8t(1-\alpha)}$, and $\frac{\partial^2 \Delta_4}{\partial \alpha^2} = -\frac{q^2}{4t(-1+\alpha)^3} > 0$. Therefore, we solve $\Delta_4 = 0$ and get $\alpha_1 = 0$ and $\alpha_2 = \frac{-2qt+t^2}{(q-t)^2}$. Notice that $\alpha_2 \leq 0$ when $q \geq \frac{t}{2}$, and $\alpha_2 \geq \tilde{\alpha}$ when $q \leq \frac{t}{3}$. Thus, the comparison results are as follows:

When $q \leq \frac{t}{3}$,

$$\alpha \leq \tilde{\alpha}, \quad G_{Ca}^{A} - G_{ps} < 0 \quad \text{if } c \leq \tilde{c},$$

and when $\frac{t}{3} < q < \frac{t}{2}$,

$$\alpha \leq \alpha_2, \quad G_{Ca}^{A} - G_{ps} \leq 0 \quad \text{if } c \leq \tilde{c},$$
$$\alpha_2 \leq \tilde{\alpha}, \quad G_{Ca}^{A} - G_{ps} > 0 \quad \text{if } c \leq \tilde{c},$$

and when $\frac{t}{2} \leq q < t$,

$$\alpha \leq \tilde{\alpha}, \quad G_{Ca}^{A} - G_{ps} > 0 \quad \text{if } c \leq \tilde{c},$$

and when $q \geq t$,

$$\alpha \leq \tilde{\alpha}, \quad G_{Ca}^{A} - G_{ps} > 0 \quad \text{if } c \leq \hat{c}.$$

(2) G_{Ia}^{A} vs G_{ps}

$G_{Ia}^{A} - G_{ps} = \Delta_5$, and $\frac{\partial^2 \Delta_5}{\partial \alpha^2} = \frac{1}{t\alpha} > 0$. Therefore, the solutions of $\Delta_5 = 0$ are c_3 and c_4 as follows:

$$c_3 = \frac{-2+t+2\alpha-q\alpha+q\alpha^2-t\alpha^2-\sqrt{(-1+\alpha)\alpha^2((t^2-2qt)(-1+\alpha)+q^2\alpha)}}{2(-1+\alpha)},$$
$$c_4 = \frac{-2+t+2\alpha-q\alpha+q\alpha^2-t\alpha^2+\sqrt{(-1+\alpha)\alpha^2((t^2-2qt)(-1+\alpha)+q^2\alpha)}}{2(-1+\alpha)}.$$

Notice that $\alpha \leq \alpha_2$ makes $(-1+\alpha)\alpha^2(-2qt(-1+\alpha)+t^2(-1+\alpha)+q^2\alpha) \geq 0$, and then c_3 and c_4 exist. We derive that when $q < \frac{t}{2}$, then $\alpha_2 > 0$ and when $q \geq \frac{t}{3}$, then $\alpha_2 \leq \tilde{\alpha}$. Furthermore, $\tilde{c} - c_3 < 0$ and $\tilde{c} - c_4 > 0$. Let $\bar{c}$ denote c_3, the comparison results are as follows:

When $q \leq \frac{t}{3}$,

$$\alpha \leq \tilde{\alpha}, \quad \begin{cases} G_{Ia}^{A} - G_{ps} \leq 0 & \text{if } \tilde{c} < c \leq \bar{c}, \\ G_{Ia}^{A} - G_{ps} > 0 & \text{if } \bar{c} < c \leq \hat{c}, \end{cases}$$

and when $\frac{t}{3} < q < \frac{t}{2}$,

$$\alpha \leq \alpha_2, \quad \begin{cases} G_{Ia}^{A} - G_{ps} \leq 0 & \text{if } \tilde{c} < c \leq \bar{c}, \\ G_{Ia}^{A} - G_{ps} > 0 & \text{if } \bar{c} < c \leq \hat{c}, \end{cases}$$
$$\alpha_2 < \alpha \leq \tilde{\alpha}, \quad G_{Ia}^{A} - G_{ps} > 0 \quad \text{if } \tilde{c} < c \leq \hat{c},$$

and when $\frac{t}{2} \le q \le t$,

$$\alpha \le \tilde{\alpha}, \qquad G^A_{Ia} - G_{ps} > 0 \quad \text{if } \tilde{c} < c \le \hat{c}.$$

(3) G^*_{Ca} vs G_{ps}

$G^*_{Ca} - G_{ps} = \Delta_6 = q\alpha - \frac{t}{4}$. Thus, $\Delta_6 > 0$ when $\alpha > \frac{t}{4q}$, and $\Delta_6 \le 0$ when $\alpha \le \frac{t}{4q}$. Therefore, the comparison results are as follows:

When $0 < q \le \frac{t}{4}$,

$$\tilde{\alpha} < \alpha < 1, \quad G^*_{Ca} - G_{ps} < 0 \text{ if } c < \vec{c},$$

and when $\frac{t}{4} < q \le \frac{t}{3}$,

$$\tilde{\alpha} < \alpha \le \frac{t}{4q}, \quad G^*_{Ca} - G_{ps} < 0 \text{ if } c < \vec{c},$$

$$\frac{t}{4q} < \alpha < 1, \quad G^*_{Ca} - G_{ps} > 0 \text{ if } c < \hat{c},$$

and when $q > \frac{t}{3}$,

$$\tilde{\alpha} < \alpha < 1, \quad G^*_{Ca} - G_{ps} > 0 \text{ if } c < \hat{c}.$$

(4) The comparison results

When $0 \le q < \frac{1}{4}t$,

$$0 < \alpha \le \tilde{\alpha}, \qquad \Delta_{bp-ps} = \begin{cases} G^A_{Ca} - G_{ps} < 0 & \text{if } c \le \tilde{c}, \\ G^A_{Ia} - G_{ps} \le 0 & \text{if } \tilde{c} < c \le \bar{c}, \\ G^A_{Ia} - G_{ps} > 0 & \text{if } \bar{c} < c < \hat{c}, \\ G^A_{Ia} - G_{Ts} > 0 & \text{if } \hat{c} \le c < \tilde{\tilde{c}}, \\ G_{Ts} - G_{Ts} = 0 & \text{if } c \ge \tilde{\tilde{c}}, \end{cases}$$

$$\tilde{\alpha} < \alpha < 1, \qquad \Delta_{bp-ps} = \begin{cases} G^*_{Ca} - G_{ps} < 0 & \text{if } c \le \vec{c}, \\ G_{Ts} - G_{ps} < 0 & \vec{c} < c \le \hat{c}, \\ G_{Ts} - G_{Ts} \le 0 & \text{if } \hat{c} < c \le 1, \end{cases}$$

and when $\frac{1}{4}t < q \le \frac{1}{3}t$,

$$0 < \alpha \le \tilde{\alpha}, \qquad \Delta_{bp-ps} = \begin{cases} G^A_{Ca} - G_{ps} < 0 & \text{if } c \le \tilde{c}, \\ G^A_{Ia} - G_{ps} \le 0 & \text{if } \tilde{c} < c \le \bar{c}, \\ G^A_{Ia} - G_{ps} > 0 & \text{if } \bar{c} < c < \hat{c}, \\ G^A_{Ia} - G_{Ts} > 0 & \text{if } \hat{c} \le c < \tilde{\tilde{c}}, \\ G_{Ts} - G_{Ts} = 0 & \text{if } c \ge \tilde{\tilde{c}}, \end{cases}$$

$$\tilde{\alpha} < \alpha \le \frac{t}{4q}, \qquad \Delta_{bp-ps} = \begin{cases} G^*_{Ca} - G_{ps} < 0 & \text{if } c \le \vec{c}, \\ G_{Ts} - G_{ps} < 0 & \vec{c} < c \le \hat{c}, \\ G_{Ts} - G_{Ts} \le 0 & \text{if } \hat{c} < c \le 1, \end{cases}$$

$$\frac{t}{4q} < \alpha < 1, \qquad \Delta_{bp-ps} = \begin{cases} G^*_{Ca} - G_{ps} > 0 & \text{if } c \le \hat{c}, \\ G^*_{Ca} - G_{Ts} > 0 & \text{if } \hat{c} < c \le \vec{c}, \\ G_{Ts} - G_{Ts} = 0 & \text{if } \vec{c} < c \le 1, \end{cases}$$

and when $\frac{1}{3}t < q \le \frac{1}{2}t$,

$$0 < \alpha \le \frac{t^2 - 2qt}{(q-t)^2}, \qquad \Delta_{bp-ps} = \begin{cases} G^A_{Ca} - G_{ps} < 0 & \text{if } c \le \tilde{c}, \\ G^A_{Ia} - G_{ps} \le 0 & \text{if } \tilde{c} < c \le \bar{c}, \\ G^A_{Ia} - G_{ps} > 0 & \text{if } \bar{c} < c < \hat{c}, \\ G^A_{Ia} - G_{Ts} > 0 & \text{if } \hat{c} \le c < \tilde{\tilde{c}}, \\ G_{Ts} - G_{Ts} = 0 & \text{if } c \ge \tilde{\tilde{c}}, \end{cases}$$

$$\frac{t^2 - 2qt}{(q-t)^2} < \alpha \le \tilde{\alpha}, \qquad \Delta_{bp-ps} = \begin{cases} G^A_{Ca} - G_{ps} > 0 & \text{if } c \le \tilde{c}, \\ G^A_{Ia} - G_{ps} > 0 & \text{if } \tilde{c} < c \le \hat{c}, \\ G^A_{Ia} - G_{Ts} > 0 & \text{if } \hat{c} < c < \tilde{\tilde{c}}, \\ G_{Ts} - G_{Ts} = 0 & \text{if } \tilde{\tilde{c}} \le c < 1, \end{cases}$$

$$\tilde{\alpha} < \alpha < 1, \qquad \Delta_{bp-ps} = \begin{cases} G^*_{Ca} - G_{ps} > 0 & \text{if } c \le \hat{c}, \\ G^*_{Ca} - G_{Ts} > 0 & \text{if } \hat{c} < c \le \vec{c}, \\ G_{Ts} - G_{Ts} = 0 & \text{if } \vec{c} < c \le 1, \end{cases}$$

and when $\frac{1}{2}t < q < t$,

$$0 < \alpha \le \tilde{\alpha}, \qquad \Delta_{bp-ps} = \begin{cases} G^A_{Ca} - G_{ps} > 0 & \text{if } c \le \tilde{c}, \\ G^A_{Ia} - G_{ps} > 0 & \text{if } \tilde{c} < c \le \hat{c}, \\ G^A_{Ia} - G_{Ts} > 0 & \text{if } \hat{c} < c < \tilde{\tilde{c}}, \\ G_{Ts} - G_{Ts} = 0 & \text{if } \tilde{\tilde{c}} \le c \le 1, \end{cases}$$

$$\tilde{\alpha} < \alpha \le 1, \qquad \Delta_{bp-ps} = \begin{cases} G^*_{Ca} - G_{ps} > 0 & \text{if } c \le \hat{c}, \\ G^*_{Ca} - G_{Ts} > 0 & \text{if } \hat{c} < c < \vec{c}, \\ G_{Ts} - G_{Ts} = 0 & \text{if } \vec{c} \le c \le 1, \end{cases}$$

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and when $q \geq t$,

$$0 < \alpha \leq \tilde{\alpha}, \qquad \Delta_{bp-ps} = \begin{cases} G^A_{Ca} - G_{ps} > 0 & \text{if } c \leq \hat{c}, \\ G^A_{Ca} - G_{Ts} > 0 & \text{if } \hat{c} < c \leq \tilde{c}, \\ G^A_{Ia} - G_{Ts} > 0 & \text{if } \tilde{c} < c < \tilde{\tilde{c}}, \\ G_{Ts} - G_{Ts} = 0 & \text{if } \tilde{\tilde{c}} \leq c \leq 1, \end{cases}$$

$$\tilde{\alpha} < \alpha \leq 1, \qquad \Delta_{bp-ps} = \begin{cases} G^*_{Ca} - G_{ps} > 0 & \text{if } c \leq \hat{c}, \\ G^*_{Ca} - G_{Ts} > 0 & \text{if } \hat{c} < c \leq \vec{c}, \\ G_{Ts} - G_{Ts} = 0 & \text{if } \vec{c} < c \leq 1, \end{cases}$$

where $\bar{c} = 1 - \frac{t}{2} + \frac{q\alpha - t\alpha}{2} + \frac{\sqrt{(-1+\alpha)\alpha^2(-2qt(-1+\alpha)+t^2(-1+\alpha)+q^2\alpha)}}{2(1-\alpha)}, \hat{c} = 1 - \frac{1}{2}t, \tilde{c} = 1 - \frac{1}{2}t + \frac{q\alpha - \alpha t}{2}$, and $\tilde{\tilde{c}} = 1 - \frac{1}{2}t + \frac{q\alpha}{2(1-\alpha)}$.

D.7   Proof of Table. 6.2

We replace the successful barter probability α with $4k\varphi(1-\varphi)$ in the pricing, i.e., Eq. (D.1), and corresponding profit functions, i.e., Eq. (6.3), in the fully covered market.

(1) When $\frac{1}{2} \leq \varphi \leq \bar{\varphi}$,

Taking the first derivatives of the profit with respective to $\hat{x}_1$ and $\hat{x}_2$, we get the optimal $\hat{x}_1 = \frac{t+4k(q+t)(-1+\varphi)\varphi}{4t(1+4k(-1+\varphi)\varphi)}$, $\hat{x}_2 = \frac{3t-4k\varphi(q-3t)(-1+\varphi)}{4t(1+4k\varphi(-1+\varphi))}$, and a resulting profit is

$$G^A_{Ca} = \frac{16k^2(-1+\varphi)^2\varphi^2(q^2+1)+t^2(-3-8k(-1+\varphi)\varphi)}{8t(1+4k(-1+\varphi)\varphi)} + 1 - kq(-1+\varphi)\varphi - c. \tag{D.17}$$

Taking the first derivative of G^A_{Ca}, we get $\frac{\partial G^A_{Ca}}{\partial \varphi} = \frac{k(-1+2\varphi)(8kq^2(-1+\varphi)\varphi(1+2k(-1+\varphi)\varphi))}{2t(1+4k(-1+\varphi)\varphi)^2} - q + \frac{t}{2}$. Taking the first and second derivative of $\frac{\partial G^A_{Ca}}{\partial \varphi}$ with respect to k, we find $\frac{\partial G^A_{Ca}}{\partial \varphi}$ is concave with respect to k with the second derivative $-\frac{8q^2(-1+\varphi)\varphi(-1+2\varphi)(-1+2k(-1+\varphi)\varphi)}{t(1+4k(-1+\varphi)\varphi)^4} < 0$. Thus, we set $\frac{\partial G^A_{Ca}}{\partial \varphi}$ to zero and get solutions $k_0 = 0, k_1 = \frac{-2q+t}{4(q-t)(-\varphi+\varphi^2)}, k_2 = -\frac{t}{4(-q+t)(-\varphi+\varphi^2)}$. Furthermore, $\hat{x}_1 \leq 0$ and $\hat{x}_2 \leq 1$ (i.e. $\tilde{k} \leq \frac{t}{4(q+t)(1-\varphi)\varphi}$) to make sure that there are non-negative demands for the component products. Therefore, we deduce that $\frac{\partial G^A_{Ca}}{\partial \varphi} < 0$ except for the case when $q < \frac{t}{2}$ and $k \leq k_1$.

When $k > \tilde{k}$, the seller only sellers the probabilistic product with $\hat{x}_1 = 0$ and $\hat{x}_2 = 1$. Thus, the resulting profit is

$$G^*_{Ca} = 1 - c - \frac{t}{2} + 2qk\varphi(1-\varphi). \tag{D.18}$$

It is easy to find that $\frac{\partial G^*_{Ca}}{\partial \varphi} < 0$ and the optimal $\varphi^* = \frac{1}{2}$. Therefore, the result holds.

(2) When $\varphi \geq \bar{\varphi}$, we get the optimal $\hat{x}_1 = \frac{t+4k(q+t)(-1+\varphi)\varphi}{4t(1+4k(-1+\varphi)\varphi)}$ and $\hat{x}_2 = \frac{-4k\varphi q(-1+\varphi)\varphi+t}{4t(1+4k\varphi(-1+\varphi))\varphi} + \frac{1}{4}$, and a resulting profit as follows:

$G_{Ca}^B = \frac{1}{8t(1+4k\varphi(-1+\varphi))\varphi}(16k\varphi^2q^2(-1+\varphi)^2\varphi + t^2(1+(-5+16k\varphi+16k\varphi^2)\varphi - 16k\varphi(1+2k\varphi)\varphi^2 + 16k\varphi^5) + 8t\varphi(1+c(-1-4k\varphi(-1+\varphi)) + 2k\varphi(2+q(-1+\varphi))(-1+\varphi) + 4k\varphi^2q(-1+\varphi)^2(-1+2\varphi)))$.

The first derivative is $\frac{\partial G_{Ca}^B}{\partial \varphi} = \frac{1}{8t(1+4k\varphi(-1+\varphi))^2\varphi^2}(32k\varphi^2q^2(1+2k\varphi(-1+\varphi))(-1+\varphi)\varphi^2 + t^2(-1+ 4k\varphi(-1+\varphi)^2 + 32k\varphi^2(-1+\varphi)\varphi^2 + 64k\varphi^3(-1+\varphi)^2\varphi^2) + 32k\varphi q t(-1+\varphi)\varphi^2(1+k\varphi(-6+8\varphi) + 4k\varphi^2(3-7\varphi+4\varphi^2)))$. And it is easy to prove that the first two terms are negative, and we have to prove that if $\mathbb{I} = 1+k\varphi(-6+8\varphi) + 4k\varphi^2(3-7\varphi+4\varphi^2)$ is positive. Taking the first, second, and third derivative of $\mathbb{I}$ with respect to φ, we get $\frac{\partial \mathbb{I}}{\partial \varphi} = 8k\varphi + 4k\varphi^2(-7+8\varphi) + k(-6+8\varphi) + 8k\varphi(3-7\varphi+4\varphi^2)$, $\frac{\partial^2 \mathbb{I}}{\partial \varphi^2} = 2k(20-84\varphi+96\varphi^2)$, and $\frac{\partial^3 \mathbb{I}}{\partial \varphi^3} = 24k(-7+16\varphi) > 0$. Thus, $\frac{\partial \mathbb{I}}{\partial \varphi}$ is convex. Setting $\frac{\partial \mathbb{I}}{\partial \varphi}$ to zero, we get the solution $\varphi = \frac{1}{16}(7 + \frac{(-225+8\sqrt{894})^{1/3}}{3^{2/3}} - \frac{13}{(3(-225+8\sqrt{894}))^{1/3}}$ which is less than $1/2$. Therefore, $\frac{\partial \mathbb{I}}{\partial \varphi} > 0$. The minimum $\mathbb{I}_{\varphi=1/2} = 1 - \frac{k}{2}$ is positive. And then $\frac{\partial G_{Ca}^B}{\partial \varphi} < 0$, and the optimal is $\varphi^* = \bar{\varphi}$ which makes $p_0^{pd} = 1 - t/2$ and $G_{Ca}^B = G_{Ca}^A$. Therefore, the optimal $G_{Ca} = G_{Ca}^A \geq G_{Ca}^B$.

D.8 Proof of Proposition 6.6

The seller that uses probabilistic selling with barter will choose either full coverage, i.e., G_{Ca}^A, or partial coverage, i.e., G_{Ia}, depending on which yields a higher profit. The comparison results when $k > k_1$ or $q > t/2$ will be the same with previous proof for $\varphi^* = \frac{1}{2}$. The difference is $\alpha = k$ and $\tilde{\alpha} = \tilde{k} = \frac{t}{q+t}$. We just need to give the sketch proof when $k \leq k_1$ and $q/2$.

(1) G_{Ca}^A vs G_{Ts} when $k \leq k_1$ and $q/2$

When $c < 1-t$, $G_{Ca}^A - G_{Ts} = \frac{(t+q(-1+2\varphi))^2}{16t\varphi} > 0$. When $c > 1-t$, $\Delta_{G_{Ca}^A - G_{Ts}} = (t^2(1-8\varphi) - 8(c-1)^2\varphi + (2qt+q^2(2\varphi-1))(2\varphi-1))/(16t\varphi) + 1 - c$. $\frac{\partial \Delta}{\partial c} = \frac{-16(-1+c)\varphi - 16t\varphi}{16t\varphi}$ and $\frac{\partial^2 \Delta}{\partial c^2} = -\frac{1}{t} < 0$, the comparison result is given by:

$$G_{Ca} \begin{cases} G_{Ca}^A & \text{if } c \leq c', \\ \frac{(1-c)^2}{2t} & \text{if } c > c', \end{cases}$$

where $c' = 1 - t + \frac{(2q\bar{\varphi}-q+t)\sqrt{2\bar{\varphi}}}{4\bar{\varphi}}$.

(2) G_{Ca}^A vs G_{Ps}

When $k \leq k_1$ and $q/2$, the optimal allocation strategy is $\varphi^* = \bar{\varphi}$. And $\bar{\varphi}$ is the solution of the equation $8k\varphi^2(1-\varphi) = 2\varphi - 1$. Substituting $k = (2\varphi-1)/(8\varphi^2(1-\varphi))$ into Eq. (D.17), we get $\frac{q^2(1-2\varphi)^2 + 2qt(-1+2\varphi) + t(t+16\varphi-8t\varphi)}{16t\varphi} - c$.

Then $\Delta_{G_{Ca} - G_{Ps}} = -\frac{(-2qt+t^2+q^2(1-2\varphi))(-1+2\varphi)}{16t\varphi}$. Solving $\Delta_{G_{Ca} - G_{Ps}}$ to zero, we get solutions $\varphi_1 = \frac{1}{2}$ and $\varphi_2 = \frac{(q-t)^2}{2q^2}$. Taking the derivatives of $\Delta_{G_{Ca} - G_{Ps}}$ with respect to φ, we get $\frac{\partial \Delta}{\partial \varphi} = -\frac{q^2 - 2qt + t^2 - 4q^2\varphi^2}{16t\varphi^2}$, $\frac{\partial^2 \Delta}{\partial \varphi^2} = \frac{(q-t)^2}{8t\varphi^3} > 0$. Because the maximum $\bar{\varphi}$ can be achieved with the maximum k_1. Therefore, we substitute k_1 into $8k\varphi^2(1-\varphi) = 2\varphi - 1$ and get maximum $\bar{\varphi} = \frac{t-q}{2q}$. And it's easy to prove that $\bar{\varphi} < \varphi_2$ and $\Delta_{\varphi=\bar{\varphi}} = -\frac{(-2q+t)^2}{8t} < 0$. Then $\Delta_{G_{Ca} - G_{Ps}} < 0$. The result is proved.

(3) G_{Ca}^A vs G_{Ia}

Due to the complexity of the equations of G_{Ia}, we conduct the simulation in Mathematics to prove $G_{Ia} > G_{Ca}$. Then we prove $\tilde{c} \leq c' \leq \tilde{\tilde{c}}$ to derive the condition for the proposition 6.6.

Substituting $k = \frac{2\bar{\varphi}-1}{8(1-\bar{\varphi})\bar{\varphi}^2}$ into $\tilde{\tilde{c}}$, we get $\tilde{\tilde{c}} = \frac{q-2q\bar{\varphi}-(-2+t)(-1+2\bar{\varphi}-8\bar{\varphi}^2+8\bar{\varphi}^3)}{2(-1+2\bar{\varphi}-8\bar{\varphi}^2+8\bar{\varphi}^3)}$. Then $\Delta_{c'-\tilde{\tilde{c}}} = \frac{1}{4}(4-4t+$
$\frac{\sqrt{2}(t+q(-1+2\bar{\varphi}))}{\sqrt{\bar{\varphi}}} + \frac{2(q(-1+2\bar{\varphi})+(-2+t)(-1+2\bar{\varphi}-8\bar{\varphi}^2+8\bar{\varphi}^3))}{-1+2\bar{\varphi}-8\bar{\varphi}^2+8\bar{\varphi}^3})$ and $\frac{\partial \Delta}{\partial q} = (-1+2\bar{\varphi})\frac{\sqrt{2}\left[\sqrt{2\bar{\varphi}}-1+2(1-2\bar{\varphi})^2\bar{\varphi}\right]}{(-1+2(1-2\bar{\varphi})^2\bar{\varphi})\sqrt{\bar{\varphi}}}$. Notice that $8k\bar{\varphi}^2(1-\bar{\varphi}) = 2\bar{\varphi}-1$, the maximum $\bar{\varphi}$ can be achieved when $k = 1$ and $-1+2(1-2\bar{\varphi})^2\bar{\varphi} = 0$. Therefore, $-1+2(1-2\bar{\varphi})^2\bar{\varphi} \leq 0$, and $\sqrt{2\bar{\varphi}}-1+2(1-2\bar{\varphi})^2\bar{\varphi} \geqslant 0$. Finally, we prove that $\frac{\partial \Delta}{\partial q} < 0$. Then $\Delta_{c'-\tilde{\tilde{c}}}$ is highest when $q = 0$ and $\Delta_{c'-\tilde{\tilde{c}}} = \frac{1}{4}t(-2+\frac{\sqrt{2}}{\sqrt{\varphi}}) \leq 0$. Therefore, $c' \leq \tilde{\tilde{c}}$.

Similarly, we prove that $c' \geq \tilde{c}$. Substituting $k = \frac{2\bar{\varphi}-1}{8(1-\bar{\varphi})\bar{\varphi}^2}$ into $\tilde{c}$, we get $\tilde{c} = 1-\frac{t}{2}-\frac{(q-t)(-1+2\bar{\varphi})}{16(-1+\bar{\varphi})\bar{\varphi}^2}$. $\Delta_{c'-\tilde{c}} = \frac{1}{4}\left(-2t+\frac{2q(-1+2\bar{\varphi})}{-1+2\bar{\varphi}-8\bar{\varphi}^2+8\bar{\varphi}^3}+\frac{\sqrt{2}(t+q(-1+2\bar{\varphi}))}{\sqrt{\bar{\varphi}}}\right)$. Taking the first derivative of $\Delta_{c'-\tilde{c}}$, we get $\frac{\partial \Delta_{c'-\tilde{c}}}{\partial q} = \frac{1}{4}\left(\frac{\sqrt{2}(-1+2\bar{\varphi})}{\sqrt{\bar{\varphi}}}+\frac{2(-1+2\bar{\varphi})}{-1+2\bar{\varphi}-8\bar{\varphi}^2+8\bar{\varphi}^3}\right) < 0$. Therefore, the lowest $\Delta_{c'-\tilde{c}}$ can be achieved when $q = t/2$. Furthermore, the maximum $\bar{\varphi} = \frac{t-q}{2q} = \frac{1}{2}$. We then substitute $\bar{\varphi}$ into $\Delta_{c'-\tilde{c}}$ and find that $\Delta_{c'-\tilde{c}} \geq 0$. The result is proved.

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