DECOMPOSITIONS OF MODULES LACKING ZERO SUMS

ZUR IZHAKIAN, MANFRED KNEBUSCH, AND LOUIS ROWEN

Abstract. A module over a semiring **lacks zero sums** (LZS) if it has the property that $v + w = 0$ implies $v = 0$ and $w = 0$. While modules over a ring never lack zero sums, this property always holds for modules over an idempotent semiring and related semirings, so arises for example in tropical mathematics.

A direct sum decomposition theory is developed for direct summands (and complements) of LZS modules: The direct complement is unique, and the decomposition is unique up to refinement. Thus, every finitely generated "strongly projective" module is a finite direct sum of summands of R (assuming the mild assumption that 1 is a finite sum of orthogonal primitive idempotents of R). This leads to an analog of the socle of "upper bound" modules. Some of the results are presented more generally for weak complements and semi-complements. We conclude by examining the obstruction to the "upper bound" property in this context.

1. Introduction

The motivation of this research is to understand direct sum decompositions of submodules of free modules over the max-plus algebra and related structures in tropical algebra (supertropical algebra [4, 9, 11] and symmetrized algebra [1]), as well as in some other settings in algebra. It turns out that direct sum decompositions are unique (not just up to isomorphism), and thus one can develop a theory of direct sum decompositions analogous to the theory of the socle in customary abstract algebra, using the axiom of "**lacking zero sums** (LZS)":

$$v + w = 0 \quad \Rightarrow \quad v = w = 0$$

(termed "zerosumfree" in [8]). This axiom may seem rather peculiar at first glance, but is easily seen to hold in tropical mathematics and also over other semirings of interest, as noted in Examples 1.9, especially in real algebra, such as the positive cone of an ordered field [3, p. 18] or a partially ordered commutative ring [2, p. 32]. More instances are given in Examples 1.9.

After writing the first draft of this paper, we became aware of [14], in which Macpherson already has proved the uniqueness of direct sum decompositions of projective modules in the tropical setting in [14, Theorem 3.9 and Corollary 4.13], working over the Boolean semifield $\mathbb{B}$. (Then he goes on to prove other interesting results about projective modules.) However, our hypotheses are different, based solely on this axiom of "lacking zero sums," which in the language of elementary logic is a quasi-identity (with all of its formal implications) and

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our main tool (Theorem 2.4) is somewhat stronger than the decomposition property for projective modules (to be compared with [14, Aside 3.7]).

Our main results hold for "weak complements," which are more general than complements in direct sums.

Remark 1.1. *We recall that any abelian monoid $(V, +, 0_V)$ can be viewed as a module over the semiring $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, which lacks zero sums, so our results hold in this way for LZS monoids.*

Definition 1.2. *A submodule $T \subset V$ is a **weak complement** (of a submodule $W \subset V$) if*

$$T + W = V \quad \text{and} \quad (w + T) \cap T = \emptyset, \ \forall w \in W \setminus \{0_V\}.$$

In Lemma 2.2 (due to the referee), we show that the weak complement satisfies the following property strengthening the LZS hypothesis.

Definition 1.3. *A subset T of an Abelian monoid $(V, +, 0_V)$ is **summand absorbing** (abbreviated SA) **in** V, if*

$$\forall x, y \in V : \quad x + y \in T \quad \Rightarrow \quad x \in T, y \in T.$$

Our main theorem for an LZS module V:

Theorem 2.4. Suppose V has a submodule T of V which is a weak complement. Then any decomposition of V descends to a decomposition of T, in the sense that if $V = Y + Z$, then $T = (T \cap Y) + (T \cap Z)$.

It follows that the weak complement, if it exists, is unique, and Propositions 2.11 (also due to the referee) gives an explicit description.

Theorem 3.2. Suppose $V = T \oplus W = Y \oplus Z$ are decompositions of modules. Then

$$V = (T \cap Y) \oplus (T \cap Z) \oplus (W \cap Y) \oplus (W \cap Z).$$

Theorem 3.4. Given two module decompositions $V = T \oplus W = Y \oplus Z$ of V, then

$$T + Y = (T \cap Y) \oplus (T \cap Z) \oplus (W \cap Y). \tag{1.1}$$

We say that a module is **strongly projective** if it is a direct summand of a free module.

Theorem 4.8 Any indecomposable strongly projective R-module P is isomorphic to a direct summand of R, and thus has the form Re for some primitive idempotent e of R.

An immediate application is that the analog of the Grothendieck group of a semiring lacking nontrival idempotents (such as the tropical algebra) is just $\mathbb{Z}$. One could also apply this to the Picard group.

Given an LZS module, we start to decompose it, and either we can continue ad infinitum or the process terminates at an indecomposable submodule. The direct sum of all the indecomposable submodules lacking zero sums is called the **decomposition socle**, denoted $\mathrm{dsoc}(V)$, and contains every indecomposable summand of V, in analogy to the socle (the sum of the simple submodules) in classical module theory. In fact, this situation is even tighter than with the classical socle, since $\mathrm{dsoc}(V)$ now is written *uniquely* as a direct sum of indecomposables. Furthermore, under certain conditions, e.g., $R = \mathbb{N}_0$, the set-theoretic complement of $\mathrm{dsoc}(V)$, with $\{0_V\}$ adjoined, also is a submodule of V whose intersection with $\mathrm{dsoc}(V)$ obviously is $\{0_V\}$. (But this is not the direct complement!) Furthermore we can understand $\mathrm{dsoc}(R)$ in terms of the idempotents of R. This approach yields a result analogous to the relation of semisimplicity and the socle in classical ring theory:

Theorem 4.9 $\mathrm{dsoc}(R) = R$ if and only if R has a finite set of primitive orthogonal idempotents whose sum is 1_R, if and only if R is a finite direct sum of indecomposable projective modules.

Section 5 contains some properties of SA modules. We also extend some of our results there to more general decompositions, arising from weak complements and **semi-complements** (Definition 5.6). These would all be the same for modules over rings, but have subtle distinctions in this setting.

Proposition 5.8. If submodules $U := W + S = W \ltimes S$ and $V = U \ltimes T$, then

$$S + T = S \ltimes T,$$
$$W + T = W \ltimes T,$$
$$V = W \ltimes (S \ltimes T) = (W \ltimes S) \ltimes T.$$

(Here the sign $\ltimes$ denotes a semi-decomposition.)

Proposition 5.9. Let W, S, T be submodules of an R-module V, and assume that S is a weak complement of W in $U := W + S$, while T is a semi-complement of U in V. Then $S + T$ is a weak complement of W in V.

Finally, one could recall that the tropical situation often involves the stronger condition (than LZS), called **upper bound** (ub) that $a + b + c = a$ implies $a + b = a$. This leads us in §6 to utilize Green's partial preorder on a semigroup $(V, +)$ by saying that $x \preceq y$ if $x + z = y$ for some z in V. This yields a congruence, the obstruction for a module to be ub, which is studied in terms of a convexity condition, and given in the context of the earlier results of this paper.

1.1. **Background.**

We recall that a **semiring**, denoted in this paper as $(R, +, \cdot\,, 0_R, 1_R)$, is a set R equipped with two binary operations $+$ and $\cdot$, called addition and multiplication, such that:

(i) $(R, +, 0_R)$ is an abelian monoid with identity element 0_R;
(ii) $(R, \cdot\,, 1_R)$ is a monoid with identity element 1_R;
(iii) multiplication distributes over addition.

An element a of a semiring is **additively idempotent** if $a + a = a$. The semiring R is **idempotent** if each element is additively idempotent.

Modules over semirings (often called "semimodules" in the literature, cf. [6]) are defined just as modules over rings, except that now the additive structure is that of a semigroup instead of a group. (Note that subtraction does not enter into the other axioms of a module over a ring.) To wit:

Definition 1.4. *Suppose R is a semiring. A (left) R-**module** V is an abelian monoid $(V, +, 0_V)$ together with scalar multiplication $R \times V \to V$ satisfying the following properties for all $r_i \in R$ and $v, w \in V$:*

(i) $r(v + w) = rv + rw$;
(ii) $(r_1 + r_2)v = r_1v + r_2v$;
(iii) $(r_1r_2)v = r_1(r_2v)$;
(iv) $1_Rv = v$;
(v) $0_Rv = r0_V = 0_V$.

We are concerned with the following property.

Definition 1.5. *An additive monoid* $(V, +, 0_V)$ ***lacks zero sums*** *if* $v_1 + v_2 = 0_V$ *implies* $v_1 = v_2 = 0_V$*, for any* $v_1, v_2 \in V$.

(This condition has other names in the literature – "zerosumfree" in [8]; a semiring lacking zero sums is an "antiring" in the sense of Tan [16] and Dolžna-Oblak [5], also called a **strict semiring** in the literature.) Although this condition never holds when V is a nontrivial group, since we could take $v_2 = -v_1$, it always holds in $V = R^n$ when R is one of the semirings mentioned in Examples 1.9 below. In such situations the LZS condition actually is rather ubiquitous, being a "quasi-identity" in the language of elementary logic.

Examples 1.6.

a) *If* $(V_i \mid i \in I)$ *is a family of* R*-modules which lack zero sums, then* $V := \prod_{i \in I} V_i$ *lacks zero sums.*
b) *Every submodule of an LZS module lacks zero sums. In particular, over any LZS semiring* R*, every submodule of* R^n *lacks zero sums.*
c) *If* V *lacks zero sums, then for any set* S *the module* $\mathrm{Fun}(S, V)$ *of functions from* S *to* V *is LZS.*

Thus, any semiring lacking zero sums supports a wide range of LZS modules. The following observation shows how incompatible LZS is with negations.

Lemma 1.7. *If* $v + s = v$ *and* $v + w = 0$*, then* $s = 0$.

Proof. $s = (v + s) + w = v + w = 0$. □

Proposition 1.8. *Any module over an idempotent semiring lacks zero sums.*

Proof. Take $v = s$ in Lemma 1.7. □

This happens for the max-plus algebra, function semirings, polynomial semirings, and Laurent polynomial semirings over idempotent semirings, and the "boolean semifield" $\mathbb{B} = \{-\infty, 0\}$ (and thus subalgebras of algebras that are free modules over $\mathbb{B}$). This shows that our results pertain to "$\mathbb{F}_1$-geometry," treated in [14].

We have the following other basic examples:

Examples 1.9.

a) *Obviously if* $R \setminus \{0_R\}$ *is closed under addition then* R *lacks zero sums. This includes the supertropical algebra mentioned above, and the more general layered version* [10] *when the "sorting set" is non-negative. Other instances of this phenomenon worth explicit mention:*
   1) *Rewriting the boolean semifield instead as* $\mathbb{B} = \{0, 1\}$ *where* $1 + 1 = 1$*, one can generalize it to* $\{0, 1, \dots, q\}$ $L = [1, q] := \{1, 2, \dots, q\}$ *the "truncated semiring*[†]*" of* [10, Example 2.14]*, where* $a+b$ *is defined to be the minimum of their numerical sum and* q.
   3) *If* F *is a formally real field, i.e.* -1 *is not a sum of squares in* F*, then the subsemiring* $R = \Sigma F^2$*, consisting of all sums of squares in* F*, lacks zero sums. In fact* R *is a semifield; the inverse of a sum of squares*

$$a = x_1^2 + \cdots + x_r^2 \quad \textit{is} \quad a^{-1} = \left(\frac{x_1}{a}\right) + \cdots + \left(\frac{x_r}{a}\right)^2.$$

4) *Let $\mathbb{Z}[t] = \mathbb{Z}[t_1, \dots, t_n]$ denote the polynomial ring in n variables over $\mathbb{Z}$. We choose a non-constant polynomial $f \in \mathbb{Z}[t]$. Then the smallest subsemiring of $\mathbb{Z}[t]$ containing f, namely*

$$\mathbb{N}_0[f] = \mathbb{N}_0 + \mathbb{N}_0 f + \mathbb{N}_0 f^2 + \dots$$

lacks zero sums, since $\mathbb{N}_0[f]$ is a free $\mathbb{N}_0$-module (recalling that $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$).

5) *More generally, the set of positive elements (including 0) of any partially ordered semiring is a sub-semiring lacking zero sums.*

6) *An instance of 5): The set of finite dimensional characters ξ_ρ over a field of characteristic 0 of any group is a semiring lacking zero sums. (Here $\xi_\rho + \xi_\tau = \xi_{\rho\oplus\tau}$ and $\xi_\rho \xi_\tau = \xi_{\rho\otimes\tau}$.)*

Definition 1.10. *We define the* ***direct sum*** *$T \oplus W$ of modules in the usual way (as the Cartesian product, with componentwise operations).*

A submodule $W \subset V$ is a ***direct complement*** *of T if $T \oplus W = V$.*

2. Weak complements and SA-submodules

We assume through the end of §4 that the R-module V lacks zero sums.

Suppose that W and T are submodules of V with $W + T = V$ and $W \cap T = \{0_V\}$. In order for T to be a direct complement of W we need the stronger condition that $w_1 + t_1 = w_2 + t_2$ implies $w_1 = w_2$ and $t_1 = t_2$.

The notion of weak complement goes half way. Recall from (Definition 1.2) that $T \subset V$ is a weak complement of W, if and only if $w_1 + t_1 = t_2$ implies $w_1 = 0_V$. We tie this to SA-submodules.

Lemma 2.1. *If W is a submodule of V and T is an SA-submodule of V with $T \cap W = \{0_V\}$, and $a = a' + (w_1 + w_2)$ for $a, a' \in T$ and $w_i \in W$, then $w_1 = w_2 = 0_V$ and $a = a'$.*

Proof. By hypothesis $w_1 + w_2 \in T \cap W = \{0_V\}$, implying $w_1 = w_2 = 0_V$ since W lacks zero sums. □

Lemma 2.2. *Suppose W is a submodule of V. Then T is a weak complement of W, if and only if T is SA with $T \cap V = \{0_V\}$.*

Proof. ($\Rightarrow$): Write $t = x_1 + x_2$ for $x_i \in V$, and write $x_i = t_i + w_i$ for $t_i \in T$, $w_i \in W$. Now $t = (t_1 + t_2) + (w_1 + w_2)$ implies $w_1 + w_2 = 0_V$, and thus $w_1 = w_2 = 0_V$ since V lacks zero sums.

($\Leftarrow$): Let $w \in W \setminus \{0_V\}$ and $t \in T$. Suppose that $w + t \in T$. Then $w \in T$, contradicting $W \cap T = \{0_V\}$. Thus $(w + T) \cap T = \emptyset$. □

We turn to the main computation of this paper.

Lemma 2.3. *Suppose V has an SA-submodule T. Then any decomposition of V descends to a decomposition of T, in the sense that if $V = Y + Z$, then $T = (T \cap Y) + (T \cap Z)$.*

Proof. Namely, for any $a \in T$, write $a = y + z$ for $y \in Y$ and $z \in Z$. By Lemma 2.2, $y, z \in T$. Thus $y \in T \cap Y$ and $z \in T \cap Z$. □

Theorem 2.4. *Suppose V has a submodule T of V which is a weak complement. Then any decomposition of V descends to a decomposition of T, in the sense that if $V = Y + Z$, then $T = (T \cap Y) + (T \cap Z)$.*

Proof. By Lemma 2.2, T is SA, so we are done by Lemma 2.3. □

Corollary 2.5. *Assume that W is a submodule of V. Assume furthermore that T is a weak complement of W in V and U is a submodule of V with $W+U=V$. Then $T \subset U$.*

Proof. Taking $Y=W$ and $Z=U$ in Theorem 2.4, implies $T=T\cap U$ since $T\cap W=\{0_V\}$. □

Corollary 2.6. *Any submodule W of V lacking zero sums has at most one weak complement in V.*

Proof. If T and U are weak complements of W in V, then $T\subset U$ by the corollary. Also $U\subset T$ by symmetry, whence $T=U$. □

This ties in with the following notion, as pointed out by the referee:

Definition 2.7. *For any submodule $W\subset V$, define*

$$W^{\perp}=\{t\in V: \quad t=w+v \text{ for } w\in W, v\in V \text{ implies } w=0\}.$$

$W^{\perp}$ need not be a submodule of V, but we do have:

Lemma 2.8. $W\cap W^{\perp}=0$.

Proof. Take $v=0$ in the definition. □

Here is a way of viewing an SA submodule as a weak complement. For any $W\subset V$ define $W_0^{\mathrm{c}} := (V\setminus W)\cup\{0_V\}$.

Lemma 2.9. $(W_0^{\mathrm{c}})^{\perp}\subset W$.

Proof. $(W_0^{\mathrm{c}})^{\perp}\subset (W_0^{\mathrm{c}})_0^{\mathrm{c}}$ since $W_0^{\mathrm{c}}\cap (W_0^{\mathrm{c}})^{\perp}=\{0_V\}$, and $(W_0^{\mathrm{c}})_0^{\mathrm{c}}=W$. □

Proposition 2.10. *If T is an SA-submodule of V, then T_0^{c} is a submonoid of $(V,+)$, and $T=(T_0^{\mathrm{c}})^{\perp}$.*

Proof. If $x,y\in T_0^{\mathrm{c}}$ we claim that either $x+y=0_V$ or $x+y\notin T$. Indeed, if $x+y\in T$, then $x,y\in T$ so $x,y\in T\cap T_0^{\mathrm{c}}=0_V$.

To see that $T\subset (T_0^{\mathrm{c}})^{\perp}$, suppose that $t=w+v$ for $w\in (T_0^{\mathrm{c}})^{\perp}$ and $v\in V$, then $v,w\in T$ implying $w\in T\cap T_0^{\mathrm{c}}=\{0_V\}$.

$T\supset (T_0^{\mathrm{c}})^{\perp}$, by Lemma 2.9. □

Proposition 2.11.

(i) *If $W^{\perp}$ is a submodule of V with $V=W+W^{\perp}$, then $W^{\perp}$ is a weak complement of W.*

(ii) *If T is a weak complement of W, then $T=W^{\perp}$.*

Proof. (i) By Proposition 2.10.

(ii) ($\subset$): Suppose $t=w+v$ for $w\in W$. Then $w\in T$ since T is SA by Lemma 2.2, so $w=0_V$.

($\supset$): Suppose $t=w+t'\in W^{\perp}$, where $w\in W$ and $t'\in T$. Then $w=0_V$ by definition of $W^{\perp}$, implying $t=t'\in T$. □

3. Direct sum decompositions of LZS modules

We leave further results about weak complements to §5 and turn more specifically to direct complements.

Since direct complements are also weak complements, we may state in consequence of Corollary 2.6 the following.

Corollary 3.1. *Given submodules T,W,Z of V, with $V=W\oplus T=W\oplus Z$, then $T=Z$.*

From Theorem 2.4 we draw the following conclusion.

Theorem 3.2. *Suppose* $V = T \oplus W = Y \oplus Z$. *Then*

$$V = (T \cap Y) \oplus (T \cap Z) \oplus (W \cap Y) \oplus (W \cap Z).$$

Proof. By Theorem 2.4, $T = (T \cap Y) \oplus (T \cap Z)$, and, symmetrically, $W = (W \cap Y) \oplus (W \cap Z)$. We get the assertion by putting these together. □

We note in passing that Theorem 2.4 also leads to a second proof of Corollary 3.1 as follows:

Second proof of Corollary 3.1. Applying Theorem 2.4 to W instead of T, we have

$$W = (W \cap T) \oplus (W \cap Z) = W \cap Z$$

since $W \cap T = \{0_V\}$. Hence $W \subset Z$, and, by symmetry, $Z \subset W$, yielding $Z = W$. □

Thus any direct summand T of V has a unique direct complement, which we denote as $T^\perp$, in accordance with Definition 2.7. Note this is properly contained in T_0^c whenever $T \neq \{0_V\}$, $T \neq V$, since taking $a \neq 0_V$ in T and $b \neq 0_V$ in T_0^c we have $a + b \in T_0^c \setminus T^\perp$.

Corollary 3.3. *If* $T \subset Y$ *then* $Y^\perp \subset T^\perp$.

Proof. Easily seen by refining the decompositions. □

Theorem 3.4. *Given two decompositions* $V = T \oplus W = Y \oplus Z$ *of* V, *then*

$$T + Y = (T \cap Y) \oplus (T \cap Z) \oplus (W \cap Y). \tag{3.1}$$

Proof. Write $V = (T \cap Y) \oplus (T \cap Z) \oplus (W \cap Y) \oplus (W \cap Z)$, and let π be the projection of V onto $W \cap Z$ sending $(T \cap Y) \oplus (T \cap Z) \oplus (W \cap Y)$ to 0_V. Clearly $\pi^{-1}(0_V) \subset T + Y$. On the other hand, Theorem 2.4 applied to the decomposition of T tells us that $T \subset \pi^{-1}(0_V)$. By symmetry also $Y \subset \pi^{-1}(0_V)$, and thus $T + Y \subset \pi^{-1}(0_V)$. This proves that

$$T + Y = \pi^{-1}(0_V) = (T \cap Y) \oplus (T \cap Z) \oplus (W \cap Y).$$

□

Corollary 3.5. *If* T *and* Y *are direct summands of an* R*-module* V *lacking zero sums, then both* $T \cap Y$ *and* $T + Y$ *are direct summands of* V *and*

$$(T \cap Y)^\perp = (T \cap Y^\perp) \oplus (T^\perp \cap Y) \oplus (T^\perp \cap Y^\perp), \tag{3.2}$$

$$(T + Y)^\perp = T^\perp \cap Y^\perp, \tag{3.3}$$

$$\left(\sum_{i \in I} T_i\right)^\perp = \bigcap_{i \in I} T_i^\perp \text{ for } I \text{ finite }. \tag{3.4}$$

Proof. We get (3.4) by iterating (3.3). □

Proposition 3.6. *Assume that* $(U_i \mid i \in I)$ *is a finite family of direct summands of an* R*-module* V *lacks zero sums, with decompositions* $V = U_i \oplus U_i^\perp$. *For any* $J \subset I$ *define* $U_J := \bigcap_{j \in J} U_j$ *and* $U_J^* := \bigcap_{j \in J} U_j^\perp$. *Then*

$$V = \bigoplus_{J \subset I} \left(U_J \cap U_{I \setminus J}^*\right).$$

Proof. An easy induction on $|I|$ starting from Theorem 3.2, where we add on one U_i at a time. In other words, if we know this up to I and take another module U we apply (3.3) and (3.4). $\square$

We would like to have this also for infinite I, but would need some infinite analog of (3.3).

Definition 3.7. *As usual, we call an R-module V **indecomposable**, if $V \neq \{0_V\}$ and V has no decomposition $V = W_1 \oplus W_2$ with $W_1 \neq \{0_V\}$, $W_2 \neq \{0_V\}$.*

Let us turn to the indecomposable direct summands of V.

Lemma 3.8. *If T and Y are indecomposable direct summands of V, then either $T = Y$ or $T + Y \cong T \oplus Y$.*

Proof. We obtain from $V = Y \oplus Y^\perp$ by Theorem 2.4 that $T = (T \cap Y) \oplus (T \cap Y^\perp)$, and then, since T is indecomposable, that $T \cap Y = T$ or $T \cap Y = \{0_V\}$, i.e., $T \subset Y$ or $T \cap Y = \{0_V\}$.

If $T \subset Y$ we conclude from $V = T \cap T^\perp$ in the same way that $Y = T \oplus (T^\perp \cap Y)$, and then that $Y = T$, since Y is indecomposable. If $T \cap Y = \{0_V\}$ we have from the above that $T = T \cap Y^\perp$, i.e., $T \subset Y^\perp$, and now infer from $V = Y \oplus Y^\perp$ that $Y + T = Y \oplus T$. $\square$

Proposition 3.9. *The indecomposable direct summands of V are independent, in the sense that if T and $\{T_i : i \in I\}$ are distinct indecomposable direct summands of V, then*

$$T \cap \Big(\sum_i T_i\Big) = \{0_V\}.$$

Proof. On the contrary, if $0 \neq a \in T \cap \big(\sum_i T_i\big)$, then a is in some finite sum of the T_i. Thus, we may assume that I is finite, and then we are done by Theorem 3.4 in conjunction with Equation (3.3) (comparing complements) and induction. $\square$

Definition 3.10. *The **decomposition socle** $\operatorname{dsoc}(V)$ is the sum (in V) of the indecomposable direct summands of V.*

Now let $\{T_i : i \in I\}$ denote the set of all indecomposable direct summands of V.

Proposition 3.11. *When I is finite,*

$$\operatorname{dsoc}(V) = \sum_{i \in I} T_i = \bigoplus_{i \in I} T_i,$$

and is a direct summand of V, with direct complement $\bigcap\limits_{i \in I} T_i^\perp$.

Proof. We may assume that $I = \{1, \dots, n\}$. Let $V_r = \sum_{i=1}^r T_i$, for $r \leq n$. By an easy induction, we obtain from Corollary 3.5 that every V_r is a direct summand of V, written

$$V = V_r \oplus W_r. \tag{3.5}$$

Furthermore, from Proposition 3.9

$$V_r \cap T_{r+1} = \{0_V\}. \tag{3.6}$$

By Theorem 2.4 we conclude that

$$T_{r+1} = (V_r \cap T_{r+1}) + (W_r \cap T_{r+1}) = W_r \cap T_{r+1},$$

i.e.,

$$T_{r+1} \subset W_r. \tag{3.7}$$

Given elements $u, u' \in V_r$, $t, t' \in W_r$ with $u + t = u' + t'$ it follows from (3.5) and (3.7) that $u = u'$ and $t = t'$. Thus

$$V_{r+1} = V_r \oplus T_{r+1}$$

for every $r < n$. The proposition now follows, up to the last assertion, which can be obtained from (3.3) in Corollary 3.5 by another easy induction. □

For I infinite, it is seen in the same way that $\mathrm{dsoc}(V)$ is the direct sum of the T_i, but now it need not be a direct summand of V. Furthermore, we must cope with the possibility that

$$\mathrm{dsoc}(V)_0^{\mathrm{c}} := \mathrm{dsoc}(V)^{\mathrm{c}} \cup \{0_V\}$$

is not an R-submodule of V, but just a submonoid.

To rectify the situation, we view V as an $\mathbb{N}_0$-module, and let

$$\mathrm{Indc}(V) := \{W_i : i \in I'\}$$

denote the set of all indecomposable $\mathbb{N}_0$-direct summands of V, and

$$W := \sum_{i \in I'} W_i = \bigoplus_{i \in I'} W_i.$$

Then $W_0^{\mathrm{c}} := (V \setminus W) \cup \{0_V\}$ does not contain any $\mathbb{N}_0$-indecomposable direct summands of V, and in particular none of the T_i.

Let $R^{\times}$ denote the group of units of R. Every $\lambda \in R^{\times}$ yields an automorphism $v \mapsto \lambda v$ of $(V, +, 0_V)$, and so the group $R^{\times}$ operates on $\mathrm{Indc}(V)$. When the semiring R is additively generated by $R^{\times}$, the T_i are precisely the sums $\sum_{i \in J} W_i = \bigoplus_{i \in J} W_i$ where $\{W_i : i \in J\}$ is an orbit of $R^{\times}$ on $\mathrm{Indc}(V)$. The following result follows immediately from these observations.

Theorem 3.12. *Assume that the semiring R is additively generated by $R^{\times}$. Then* $\mathrm{dsoc}(V)$ *is the direct sum of all indecomposable direct summands of V, and the additive monoid*

$$\mathrm{dsoc}(V)^{\mathrm{c}} := (V \setminus \mathrm{dsoc}(V)) \cup \{0_V\}$$

is an R-submodule of V. (But if there are infinitely many such indecomposable direct summands, $\mathrm{dsoc}(V)$ *need not be a direct summand of V.)*

Examples where the theorem applies are:

(a) $R = \mathbb{N}_0$;
(b) R is a semifield;
(c) R is a so-called **supersemifield**, i.e., a supertropcial semiring (cf. [11], [12]) where both $R \setminus (eR)$ and $(eR) \setminus \{0_R\}$ are groups, with $e = 1_R + 1_R$;
(d) R is replaced by the semiring $R[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ of Laurent polynomials in n variables over any of the previous semirings R.

4. Projective R-modules

We are ready to apply these results to the case that $V = R^n$. Assume throughout this section that R lacks zero sums.

Projective modules are treated in [15] and [8], also cf. [7]. We use the equivalent form of [8, Proposition 17.16]:

Definition 4.1. *A module P is* ***free*** *if it is isomorphic to $R^{(I)}$ for some index set I. The module P is* ***projective*** *if there is a split epic from $R^{(I)}$ to P for some I (which can be taken to have order m if P is finitely generated by m elements).*

Definition 4.2. *An element $e \in R$ is* ***multiplicatively idempotent*** *if $e^2 = e$. From now on, "idempotent" means multiplicatively idempotent. Two idempotents e, f are* ***orthogonal*** *if $ef = fe = 0_R$.*

We write $\mathrm{End}_R(M)$ for the semiring of R-module homomorphisms from M to M.

Lemma 4.3. *Any projective module P has the form $g(R^{(I)})$ for some idempotent $g \in \mathrm{End}_R(M)$.*

Proof. Taking the split epic $g : R^{(I)} \to P$, we can view $P \subset R^{(I)}$, with g restricting to the identity on P, i.e., $gf = 1_P$, where $f : P \to R^{(I)}$ is the natural inclusion. But then $g^2 = gfg = 1_P g = g$. □

Lemma 4.4. *Any cyclic projective module P has the form Re for e an idempotent of R, which is primitive if and only if P is indecomposable.*

Proof. As in ring theory, we need to display the isomorphism $\mathrm{End}_R(R) \to R$, given by $f \mapsto f(1_R)$. But for any $a \in P$ we have $a = f(a) = f(a1_R) = af(1_R)$; the reverse map is given by $r \mapsto f_r$ where $f_r(a) = ra$. □

Lemma 4.5. [8, Proposition 17.19] *A direct sum $\bigoplus_{i\in I} P_i$ of R-modules is projective, if and only if each P_i is projective.*

Thus, every direct summand of a free R-module is projective. So far this looks like the classical theory but in contrast to the classical situation we also have the following example.

Example 4.6. *Let R be the truncated max-plus algebra $\{-\infty, 1, \dots, n\}$ where we treat n as the infinite element insofar as we stipulate $ab = n$ if under usual addition the sum of a and b is at least n. For example, for $n = 8$ we have $4 \cdot 5 = 8$. Let $P = Rn = \{-\infty, n\}$. Any epic $R \to P$ must send n to n, and thus 1 to n, and this defines an epic, so P is projective. On the other hand, if U is a submodule of R with $R = U + P$, then U contains an element $a \neq -\infty$ and n, which implies $na = n \in U \cap P$, whence $U \cap P \neq \{-\infty\}$. Thus, P is certainly not a direct summand of the R-module R.*

This necessitates the following definition.

Definition 4.7. *A module P is* ***strongly projective*** *if it is a direct summand of a free R-module. An idempotent is* ***primitive*** *if it cannot be written as the sum of two nonzero orthogonal idempotents.*

Theorem 4.8. *Any indecomposable strongly projective module has the form Re for some primitive idempotent e of R.*

Proof. Write the free module $F = P \oplus P^{\perp} = \bigoplus_{i\in I} R\varepsilon_i$ with base $\{\varepsilon_i : i \in I\}$. For each $i \in I$, $P = (R\varepsilon_i \cap P) \oplus \sum_{j\neq i}((R\varepsilon_j) \cap P)$.

Thus we may assume that there is $i \in I$ with $(R\varepsilon_i) \cap P \neq \{0_F\}$. Now $(R\varepsilon_i) \cap P = P$, since P is indecomposable, whence P is a direct summand of $R\varepsilon_i$, and we may assume that $F = R$.

Thus $P = Re$ where e is idempotent. If e were not primitive then writing $e = e_1 + e_2$ for orthogonal idempotents e_1, e_2 would yield $Re = Re_1 \oplus Re_2$. (The standard proof for modules over rings also holds here.) This would contradict the indecomposability of P. □

The same argument as in [14] yields the analogous conclusion.

Theorem 4.9. $\operatorname{dsoc}(R) = R$*, if and only if* R *has a finite set of orthogonal primitive idempotents whose sum is* 1_R*, if and only if* R *is a finite direct sum of indecomposable strongly projective modules.*

Proof. In view of Theorem 4.8, the only thing remaining to check here is the finiteness. But this is another standard argument taken from ring theory. If $R = \bigoplus_i P_i$, then the unit element 1_R is in this sum, and thus is some finite sum of elements $\sum_i r_i e_i$, implying R is the sum of these P_i. □

Corollary 4.10. *If* R *has a finite set of orthogonal primitive idempotents* $\{e_1, \dots, e_m\}$ *whose sum is* 1_R*, then every finitely generated strongly projective* R*-module* P *is a finite direct sum of indecomposable strongly projective modules, i.e.,* $P \cong \bigoplus_{i=1}^m (Re_i)^{n_i}$ *for suitable* n_i*, and this direct sum decomposition is unique.*

5. Submodules satisfying the summand absorbing property

Throughout, R is a semiring and V an R-module. Perhaps surprisingly at first glance, uniqueness of decompositions can be proved in settings where we drop the requirement that V lacks zero sums (but strengthen the requirement on W). So we drop this hypothesis and focus instead on its submodule W, first in conjunction with SA

$$(\forall x, y \in V: \quad x + y \in T \quad \Rightarrow \quad x \in T, y \in T),$$

and then in terms of "semi-complements."

Proposition 5.1. *Assume that* W *is a submodule of* V *and that* T *is a weak complement of* W *in* V*. Assume also that* $V \setminus T$ *is closed under addition. Then* W *lacks zero sums (and so* T *is the unique weak complement of* W *in* V*).*

Proof. Let $w_1, w_2 \in W \setminus \{0_V\}$. Then $w_i \notin T$ for $i = 1, 2$, implying $w_1 + w_2 \notin T$. Thus certainly $w_1 + w_2 \neq 0_V$. □

Lemma 5.2. *The following conditions are equivalent for a submodule* $W \subset V$*:*

(i) W *is SA in* V*;*

(ii) W_0^{c} *is an additive (monoid) ideal, in the sense that* $w + v \in W_0^{\mathrm{c}}$ *for all* $w \in W_0^{\mathrm{c}}$*,* $v \in V$*;*

(iii) *If* $\sum_{i=1}^{m} a_i \in W$*, then each* $a_i \in W$*.*

Proof. *(i)* ⇔ *(ii)* is clear, and *(i)* ⇒ *(iii)* by induction on m. *(iii)* ⇒ *(i)* is immediate. □

We pass to the case of R-modules (which is not much of a transition, since an additive monoid is an $\mathbb{N}_0$-module).

Lemma 5.3. *Assume that* $\varphi : V_1 \to V_2$ *is a homomorphism of* R*-modules over an arbitrary semiring* R*. If* T *is a SA-submodule of* V_2*, then* $\varphi^{-1}(T)$ *is a SA-submodule of* V_1*.*

Proof. Let $x, y \in V$, and assume $x + y \in \varphi^{-1}(T)$. Then $\varphi(x) + \varphi(y) = \varphi(x + y) \in T$, and so $\varphi(x), \varphi(y) \in T$, whence $x, y \in \varphi^{-1}(T)$. □

Remark 5.4.

a) *If* $(U_i \mid i \in I)$ *is a family of SA submodules of an* R*-module* V*, then the intersection* $\bigcap_{i \in I} U_i$ *clearly also is SA in* V*.*

b) *If* $(U_i \mid i \in I)$ *is an upward directed family of SA submodules of* V*, i.e. for any* $i, j \in I$ *there exists* $k \in I$ *with* $U_i \subset U_k$*,* $U_j \subset U_k$*, then the union* $\bigcup_{i \in I} U_i$ *is a SA submodule of* V*.*

Note in Definition 1.10 that for $R = \mathbb{N}_0$, Lemma 5.2(ii) implies that any SA submodule T is a weak complement of T_0^{c}.

Theorem 5.5. *Assume that* $V = W + T$*, where* T *is the unique weak complement of* W*. If in addition* U *is a submodule of* V *with* $W + U = V$ *and also* U *is SA in* V*, then* $T \subset U$*, and* T *is the unique weak complement of* $W \cap U$ *in* U*.*

Proof. $V \setminus T$ is closed under addition. By Proposition 5.1 we know that W lacks zero sums and thus $T \subset U$. Since U is SA in V we conclude from $V = W + T$ that $U = (W \cap U) + T$. By part (i), T is the unique weak complement of $W \cap U$ in U, since T is SA in U. □

Here is one nice kind of weak complement.

Definition 5.6. *Let* W *and* T *be* R*-submodules of* V*.* T *is a* ***semi-complement of*** W ***in*** V *if* $W + T = V$ *and*

$$\forall w_1, w_2 \in W : \quad w_1 \neq w_2 \quad \Rightarrow \quad (w_1 + T) \cap (w_2 + T) = \emptyset. \tag{5.1}$$

In this case, we also write $V = W \ltimes T$*.*

Condition (5.1) can be recast as follows: For any $w_1, w_2 \in W$, $t_1, t_2 \in T$,

$$w_1 + t_1 = w_2 + t_2 \quad \Rightarrow \quad w_1 = w_2.$$

This means that there exists an R-linear projection $p : V \to V$ given by $p(w + t) = w$, with image $p(V) = W$ and kernel $p^{-1}(0_V) = T$. We sometimes write

$$p = \pi_{W,T}. \tag{5.2}$$

In summary, we have the following hierarchy of conditions on modules T, W satisfying $W + T = V$, each implying the next, which are all equivalent in classical module theory over a ring:

(i) T is a direct complement of W;
(ii) T is a semi-complement of W;
(iii) T is a weak complement of W;
(iv) $W \cap T = \{0_V\}$.

Here the reverse implications may fail. We now address "transitivity" of these various complements.

Question 5.7. *Assume that* W, S, T *are submodules of an* R*-module* V *such that* S *is a complement of* W *in* $U := W + S$ *of a certain type (direct, semi-, weak) and* T *is a complement of* U *in* V *of the respective type. Then is* $S + T$ *a complement of* W *in* V*, of this respective type?*

This is obviously true for direct complements. It also holds for semi-complements. More explicitly, we have the following facts.

Proposition 5.8. *If* $U := W + S = W \ltimes S$ *and* $V = U \ltimes T$*, then*

$$S + T = S \ltimes T, \tag{5.3}$$

$$W + T = W \ltimes T, \tag{5.4}$$

$$V = W \ltimes (S \ltimes T) = (W \ltimes S) \ltimes T. \tag{5.5}$$

We give two proofs of these facts, having different flavors.

First proof. Here we use the definition of semi-complements given in (5.1). Let $w_1, w_2 \in W$ and $w_1 \neq w_2$. Then $(w_1 + S) \cap (w_2 + S) = \emptyset$ and so $w_1 + s_1 \neq w_2 + s_2$ for any $s_1, s_2 \in S$. Since T is a semi-complement of $W + S$ in V we have in turn

$$(w_1 + s_1 + T) \cap (w_2 + s_2 + T) = \emptyset.$$

This proves that

$$(w_1 + S + T) \cap (w_2 + S + T) = \emptyset, \tag{5.6}$$

and it follows that

$$(w_1 + T) \cap (w_2 + T) = \emptyset.$$

If $s_1 \neq s_2$ in S then, since s_1 and s_2 are different elements of $W + S$, we also conclude from (5.6) that $(s_1 + T) \cap (s_2 + T) = \emptyset$. □

Second proof. We employ the projections associated to semi-decompositions, cf. (5.2), identifying any projection $p : X \to X$ onto an R-module X with the induced surjection $X \twoheadrightarrow p(X)$. We have projections $p := \pi_{U,T} : V \twoheadrightarrow U$ and $q := \pi_{W,S} : U \twoheadrightarrow W$ with respective kernels T and S. Then $r := q \circ p : V \twoheadrightarrow W$ is a projection with kernel $S+T$, yielding $V := W \ltimes (S+T)$. The projection $r : V \twoheadrightarrow W$ restricts to maps $r|(S + T) \twoheadrightarrow S$ and $r|(W + T) \twoheadrightarrow T$, which both are projections with kernel T. Thus $S + T = S \ltimes T$ and $W + T = W \ltimes T$. □

For weak complements we cannot expect a transitivity statement such as (4.5) above. But a "mixed transitivity" holds for weak and semi-complements.

Proposition 5.9. *Let W, S, T be submodules of an R-module V, and assume that S is a weak complement of W in $U := W + S$, while T is a semi-complement of U in V. Then $S + T$ is a weak complement of W in V.*

Proof. Let $w \in W \setminus \{0_V\}$ and $s_1, s_2 \in S$. Then $(w + S) \cap S = \emptyset$. Thus $w + s_1$ and s_2 are different elements of $W + S = U$, which implies that $(w + s_1 + T) \cap (s_2 + T) = \emptyset$. This proves that $(w + S + T) \cap (S + T) = \emptyset$, as desired. □

We finally mention a result of independent interest, which can be obtained by a slight amplification of the proof of Theorem 5.5(ii).

Proposition 5.10. *Assume that W, T, U are submodules of an R-module V with $W + T \subset W + U$ and $W \cap T \subset W \cap U$. Assume furthermore that T is SA in V, then $T \subset U$.*

Proof. Let $t \in T$ be given. We write $t = w + u$ with $w \in W$, $u \in U$. Since T is SA in V, this implies that $w \in T$, whence $w \in W \cap T \subset W \cap U$. We conclude that $t = w + u \in U$. □

6. The obstruction to the "upper bound" condition

Recall from [12] that an additive monoid $(V, +, 0_V)$ is **upper bound** if $x + y + z = x$ implies $x + y = x$. This property instantly implies LZS. The object of this section is to study the obstruction to this condition.

Definition 6.1. *Define **Green's partial preorder** on a monoid $(V, +, 0_V)$ by saying*

$$x \preceq y \quad \text{if} \quad x + z = y \text{ for some } z \text{ in } V.$$

We write $x \equiv y$ if $x \preceq y$ and $y \preceq x$.

(Green's partial preorder is generated by the relations $0_V \preceq z$, since $x + z = y$ implies $x = x + 0_V \le x + z = y$.) Clearly $\preceq$ is reflexive and transitive, implying that $\equiv$ is an equivalence relation; in fact, $\equiv$ is a congruence, since if $x \preceq y$ then $x + a \preceq y + a$ for any $a \in V$. (Indeed, if $x + z = y$, then $x + a + z = y + a$.) Accordingly, $\overline{V} := V/ \equiv$ also is a monoid, with the induced operation $\bar{x} + \bar{y} = \overline{x + y}$, where $\bar{x}$ denotes the equivalence class of x. $\preceq$ induces a partial order $\le$ on $\overline{V}$, given by

$$\bar{x} \le \bar{y} \quad \text{if} \quad x \preceq y. \tag{6.1}$$

Proposition 6.2.

(i) *The monoid $\overline{V}$ is upper bound.*

(ii) *V is upper bound if and only if the congruence $\equiv$ is trivial.*

Proof. (i) Suppose that $\bar{x} + \bar{y} + \bar{z} = \bar{x}$. Then $\overline{x + y + z} = \bar{x}$, so $x + y + z + z' = x$ for some z', implying $x + y \preceq x$. But obviously $x \preceq x + y$, so $x + y \equiv x$, and $\bar{x} + \bar{y} = \bar{x}$.

(ii) ($\Rightarrow$) Suppose that $x \equiv y$, i.e., $x + z_1 = y$ and $y + z_2 = x$ for $z_i \in V$. Then $x + (z_1 + z_2) = x$. Since V is presumed to be upper bound, we have $x = x + z_1 = y$.

($\Leftarrow$) Suppose that $x + y + z = x$. Then $x + y \preceq x$ and clearly $x \preceq x + y$, implying $x \equiv x + y$, and thus $x = x + y$ by assumption. □

This construction respects other topological notions.

Definition 6.3. *A subset $S \subset V$ is* ***convex*** *if, for any s_i in S and v in V, $s_1 \preceq v \preceq s_2$ implies $v \in S$.*

Lemma 6.4. *The convex hull of a point $s \in S$ is its equivalence class in $\overline{V}$.*

Proof. ($\subset$): If $s + y + z = s$, then $s \preceq s + y \preceq s$, implying $s \equiv s + y$.

($\supset$): If $s \equiv s + y$, then $s + y + z = s$ for some z, implying $s \preceq s + y \preceq s$. □

Proposition 6.5. *S is convex in V if and only if $\overline{S}$ is convex in $\overline{V}$ and S is a union of equivalence classes.*

Proof. Take the convex hull and apply Lemma 6.4. □

This also ties in with the SA property.

Lemma 6.6. *A subset S containing 0_V is convex in V, if and only if S is SA.*

Proof. ($\Rightarrow$): If $a + b \in S$ then $0_V \preceq a \preceq a + b$ implies $a \in S$, and likewise $b \in S$.

($\Leftarrow$): If $s_1 \preceq a \preceq s_2$, then writing $s_2 = a + z_2$, we have $a \in S$. □

Proposition 6.7. *A submodule $S \subset V$ is SA in V, if and only if S is a union of equivalence classes and $\overline{S}$ is SA in $\overline{V}$.*

Proof. This follows from Proposition 6.5 and Lemma 6.6, applied to S and $\overline{S}$. □

Some concluding observations:

Remark 6.8.

(i) *If R is a semiring, then (taking $V = R$) the equivalence $\equiv$ also respects multiplication, so $R/\equiv$ is a ub semiring.*

(ii) *If V is an R-module, then $\overline{V}$ is an $\overline{R}$-module, where scalar multiplication is given by $\bar{a}\bar{v} = \overline{av}$.*

(iii) *Any decomposition $V = W_1 \oplus W_2$ induces a decomposition $\overline{V} = \overline{W}_1 \oplus \overline{W}_2$.*

(iv) *Any decomposition* $V = W_1 \oplus W_2$ *induces a decomposition* $\overline{V} = \overline{W}_1 \oplus \overline{W}_2$.

(v) *If* $W \subset V$ *is a submodule, then* $W^\perp$ *(Definition 2.7) is a convex subset of* V *under* $\preceq$, *consisting of full equivalence classes under* $\equiv$. *Its image* $\overline{W^\perp}$ *in* $\overline{V}$ *coincides with* $\overline{W}^\perp$, *where* $\overline{W}$ *is the image of* W *in* $\overline{V}$, *and* $\overline{W^\perp}$ *is the largest convex subset of* $\overline{V}$ *having zero intersection with* $\overline{W}$.

References

[1] M. Akian, S. Gaubert, and A. Guterman. Linear independence over tropical semirings and beyond. In *Tropical and Idempotent Mathematics*, G.L. Litvinov and S.N. Sergeev, (eds.), *Contemp. Math.* 495:1–38, 2009.

[2] G.W. Brumfiel. *Partially ordered rings and semialgebraic geometry*, London Math. Soc. Lecture Notes 37, Cambridge Univ. Press, 1979.

[3] J. Bochnak, M. Coste, and M.-F. Roy. *Real algebraic geometry*, Ergebnisse Math. Grenzgeb. 36, Springer, 1998.

[4] A. Connes and C. Consani. Characteristic 1, entropy, and the absolute point. *Noncommutative Geometry, Arithmetic, and Related Topics*, Proc. of the 21st JAMI Conference, Johns Hopkins Univ. Press, 75–139, 2011.

[5] D. Dolžna and P. Oblak. Invertible and nilpotent matrices over antirings. *Lin. Alg. and Appl.*, 430(1):271–278, 2009.

[6] M. Dubey. *Some results on semimodules analogous to module theory*, Doctoral Dissertation, University of Delhi, 2008.

[7] N. Durov. A new approach to Arakelov geometry, arXiv 0704.2030, 2007.

[8] J. Golan. *Semirings and their Applications*, Springer-Science + Business, Dordrecht, 1999. (Originally published by Kluwer Acad. Publ., 1999.)

[9] Z. Izhakian, M. Knebusch, and L. Rowen. Supertropical linear algebra. *Pacific J. of Math.* 266(1):43–75, 2013.

[10] Z. Izhakian, M. Knebusch, and L. Rowen. Layered tropical mathematics, *J. Alg.* 416:200–273, 2014.

[11] Z. Izhakian and L. Rowen. Supertropical algebra. *Adv. in Math.* 225(4):2222–2286, 2010.

[12] Z. Izhakian, M. Knebusch, and L. Rowen. Supertropical semirings and supervaluations. *J. Pure and Appl. Alg.* 215(10):2431–2463, 2011.

[13] M. Knebusch and D. Zhang. Convexity, valuations, and Prüfer extensions in real algebra, *Doc. Math.* 10:1–109, 2005.

[14] A.W. Macpherson. Projective modules over polyhedral semirings, arXiv:1507.07213v1, 26 Jul 2015.

[15] M. Takahashi. On the bordism categories III, *Math. Seminar Notes Kobe University*, 10:211–236, 1982.

[16] Y. Tan. On invertible matrices over antirings. *Lin. Alg. and Appl.* 423(2):428–444, 2007.

[17] S. Yoshimoto. Generators of modules in tropical geometry, preprint at arXiv:1001.0448v2, 2010.

Institute of Mathematics, University of Aberdeen, AB24 3UE, Aberdeen, UK.
E-mail address: `zzur@abdn.ac.uk; zzur@math.biu.ac.il`

Department of Mathematics, NWF-I Mathematik, Universität Regensburg 93040 Regensburg, Germany
E-mail address: `Manfred.Knebusch@mathematik.uni-regensburg.de`

Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel
E-mail address: `rowen@math.biu.ac.il`