EQUILIBRIUM STATES FOR A CLASS OF SKEW PRODUCTS

MARIA CARVALHO AND SEBASTIÁN A. PÉREZ

ABSTRACT. We consider skew products on $M \times \mathbb{T}^2$, where M is the two-sphere or the two-torus, which are partially hyperbolic and semi-conjugate to an Axiom A diffeomorphism. This class of dynamics includes the open sets of Ω-non-stable systems introduced by Abraham, Smale and Shub in [1] and [32]. We present sufficient conditions, both on the skew products and the potentials, for the existence and uniqueness of equilibrium states, and discuss their statistical stability.

1. INTRODUCTION

In a recently disclosed notebook[1], Bowen listed several topics and open problems worthwhile discussing, including the statistical properties of non-Axiom A systems. Some readers of those remarkable pages have emphasized the questions regarding the existence and uniqueness of equilibrium states, having in mind the complete thermodynamic formalism Bowen established in [7] for Axiom A diffeomorphisms and Hölder continuous potentials, besides Bowen's axiomatic description of some systems with unique equilibrium states [6]. The two main ingredients in the latter work, suggested by Smale's Spectral Decomposition Theorem for uniformly hyperbolic systems, are expansiveness and specification. Several weaker forms of these properties have since been introduced in connection with measures of maximal entropy (cf. [30] and references therein). A short time ago, Climenhaga and Thompson [10] developed a new criterium to prove existence and uniqueness of equilibrium states for systems exhibiting non-uniform versions of expansiveness and specification, with respect to Hölder potentials subject to further technical conditions. Although powerful, to apply the main theorem of [10] to particular cases beyond hyperbolicity one has to cope with these additional demands on the dynamics and the potentials. Climenhaga, Fisher and Thompson succeeded in using this machinery to settle a thermodynamic formalism for both Bonatti-Viana diffeomorphisms [3, 11] and the family of partially hyperbolic, robustly transitive, derived from Anosov systems introduced by Mañé [26, 12]. One of the key ideas of these two articles was the reformulation of those supplementary requests on

[1]https://bowen.pims.math.ca/

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the dynamics in terms of the C^0-closeness to an Anosov system and the denseness of each center-stable/center-unstable foliation, both essential to ensure specification at a small scale.

The purpose of this paper is to address similar questions for Abraham-Smale [1] and Shub [32, 23] examples, the ones that paved the way to the proof of the nongenericity of Ω-stable systems. Those examples are skew products obtained through small local C^0-perturbations of a product $\sigma \times L$, where $\sigma \colon M \to M$ is either a Smale's horseshoe map on the two-dimensional sphere $M = \mathbb{S}^2$ (in the case of Abraham-Smale work) or an Anosov diffeomorphism of the two-torus $M = \mathbb{T}^2$ with two fixed points (the choice of Shub), and $L \colon \colon \mathbb{T}^2 \to \mathbb{T}^2$ is a linear hyperbolic toral automorphism. Instead of exploring the approach of [10], we looked for another strategy, benefiting from the point of view introduced in [29].

Abraham-Smale and Shub's examples are homotopic to $\sigma \times L$ and partially hyperbolic, admitting a splitting with a one-dimensional, normally hyperbolic, non-trivial center subbundle. Besides, Abraham-Smale's diffeomorphism is the limit of hyperbolic subshifts of finite type in the sense of [36]; Shub's is robustly transitive [23]. Their non-wandering sets are equal to $\Omega(\sigma) \times \mathbb{T}^2$ (cf. [36, 23]), and no diffeomorphism C^1-close to them is Ω-stable. Exploring the fact that they are extensions of an Axiom A system, and making strong use of the one-dimensionality of the center bundle, Newhouse and L.-S. Young [29] proved that those systems have unique measures of maximal entropy, whose entropy is equal to $h_{\text{top}}(\sigma) + h_{\text{top}}(L)$. So, we decided to endeavor in generalizing their argument to a broader class of Hölder potentials.

Starting with an Axiom A diffeomorphism, we have considered partially hyperbolic skew products that are close and homotopic to that original Axiom A system. A parameterized version of Franks' result [17], due to C. Robinson and included in [29, Lemma 1], then states that there is a semi-conjugacy back to the Axiom A transformation which, under additional requirements, is an almost conjugacy (that is, a topological conjugacy after neglecting a small invariant set). The leading assumption in this reasoning regards the presence of a large invariant region in $M \times \mathbb{T}^2$ within which one detects asymptotic contraction in average along the center foliation. Next, in order to lift the uniqueness of the equilibrium state of the Axiom A system to the skew product, we extended to the pressure Ledrappier-Walters' estimates of the metric entropy of factors [25]. This was enough to deduce uniqueness of the equilibrium states for an open class of Hölder potentials with respect to Abraham-Smale and Shub's examples. Concerning existence of equilibrium states for continuous potentials, we just took into account that Abraham-Smale and Shub's diffeomorphisms are entropy-expansive [13], so their entropy maps are upper semi-continuous [28]. Furthermore, in view of the results in [23] (with noteworthy generalizations in [24] and [2]), these ergodic properties of Abraham-Smale and Shub's examples are also valid for small perturbations thereof, although those C^1-close diffeomorphims may not be skew products.

This line of argument has likewise inspired [8], [9] and [34] in connection with intrinsic ergodicity. Using disintegration of measures along the central foliation, the uniqueness of equilibrium states for partially hyperbolic derived from Anosov systems on the torus $\mathbb{T}^3$, including Mañé family, was also addressed in [14] under the extra assumption that the potential is constant on the intervals that are collapsed by the semi-conjugacy.

The paper is organized as follows. In Section 2 we enumerate the essential properties requested from the skew products we will work with. For the reader's convenience, a few definitions are recalled in Section 3. The statements of the main results may be read in Section 4, where we also give an outline of their proofs. After establishing sufficient conditions for the uniqueness of the equilibrium states in Sections 5, 6 and 7, we analyze in Section 9 the robustness of those conditions in the C^1-topology, and prove in Section 10 the equilibrium states' stability with respect to both the dynamics and the potentials.

The main properties proved in this work are still valid if, instead of $M \times \mathbb{T}^2$, we consider similar dynamical systems on $M \times \mathbb{T}^n$ with $n > 2$. This is due to the fact that the stable and unstable directions of a partially hyperbolic diffeomorphism are integrable (see [23]), and so is the center bundle without restrictions on its dimension for tori diffeomorphisms isotopic to a linear Anosov automorphism along a path of partially hyperbolic diffeomorphisms (cf. [18]).

2. Setting

Let $\sigma \colon M \to M$ be a homeomorphism of a compact metric space M. Take a linear hyperbolic automorphism $A : \mathbb{R}^2 \to \mathbb{R}^2$ such that $A(\mathbb{Z}^2) \subset \mathbb{Z}^2$, $\det A = \pm 1$ and A has a real eigenvalue of multiplicity one whose absolute value is strictly smaller than 1. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus and $L : \mathbb{T}^2 \to \mathbb{T}^2$ the Anosov diffeomorphism induced by A.

Given a compact, connected manifold Z without boundary, denote by $\mathrm{Diff}^r(Z)$ the space of C^r-diffeomorphisms of Z endowed with the norm of the uniform C^r-convergence. Consider a family of C^1-diffeomorphisms $(f_x)_{x \in M}$ acting on $\mathbb{T}^2$, and the skew product of σ and $(f_x)_{x \in M}$ on $M \times \mathbb{T}^2$ defined by

$$
\begin{aligned}
F \colon M \times \mathbb{T}^2 &\to M \times \mathbb{T}^2 \\
(x, y) &\mapsto \big(\sigma(x),\, f_x(y)\big).
\end{aligned}
\tag{2.1}
$$

As in [29], we assume that F satisfies the following conditions:

(1) The map $x \in M \to f_x \in \mathrm{Diff}^1(\mathbb{T}^2)$ is continuous.

(2) F is homotopic to $\sigma \times L$ as a bundle map with trivial fiber $\mathbb{T}^2$, that is, there exists a continuous map $\mathcal{G} \colon M \times \mathbb{T}^2 \times [0, 1] \to \mathbb{T}^2$ satisfying $\mathcal{G}(x, y, 0) = f_x(y)$ and $\mathcal{G}(x, y, 1) = L(y)$, for every $(x, y) \in M \times \mathbb{T}^2$.

(3) There is a one-dimensional lamination $\mathfrak{F}$ of $M \times \mathbb{T}^2$ which is F-invariant and normally expanded. More precisely, for each $(x, y) \in M \times \mathbb{T}^2$, the leaf $\mathfrak{F}_{(x,y)}$ through (x, y) is a smooth immersed line in $\{x\} \times \mathbb{T}^2$ and $F(\mathfrak{F}_{(x,y)}) = \mathfrak{F}_{F(x,y)}$. Besides, there is a splitting $E^{\mathrm{s}}_{(x,y)} \oplus E^{\mathrm{u}}_{(x,y)}$ of the tangent space at (x, y) to the fiber $\{x\} \times \mathbb{T}^2$ which varies continuously with (x, y) and such that

- $D_y f_x(E^{\mathrm{u}}_{(x,y)}) = E^{\mathrm{u}}_{F(x,y)}$ and $D_y f_x(E^{\mathrm{s}}_{(x,y)}) = E^{\mathrm{s}}_{F(x,y)}$;

- $E^{\mathrm{s}}_{(x,y)}$ is the tangent space at (x, y) of $\mathfrak{F}_{(x,y)}$;

- for each $x \in M$, there is a Riemannian metric on $\{x\} \times \mathbb{T}^2$ with induced norm $\| \cdot \|$ such that

$$\inf_{\substack{v \in E_{(x,y)}^{\mathrm{u}} \; : \; \|v\|=1 \\ (x,y) \in M \times \mathbb{T}^2}} \|D_y \, f_x(v)\| \; > \; \max\left\{ 1, \; \sup_{\substack{w \in E_{(x,y)}^{\mathrm{s}} \; : \; \|w\|=1 \\ (x,y) \in M \times \mathbb{T}^2}} \|D_y \, f_x(w)\| \right\}.$$

Under the previous assumptions, Lemma 1 in [29] (see also [17]) ensures the existence of a semi-conjugacy between $\sigma \times L$ and F.

Lemma 2.1. [29, Lemma 1] *There is a continuous surjective map* $H \colon M \times \mathbb{T}^2 \to M \times \mathbb{T}^2$ *of the form* $H(x,y) = (x, H_x(y))$ *such that* $(\sigma \times L) \circ H = H \circ F$ *and each* $H_x \colon \mathbb{T}^2 \to \mathbb{T}^2$ *is homotopic to the identity.*

3. Preliminaries

In this section we will recall some background material and preliminary useful results.

3.1. **Almost conjugacy.** Given two measures spaces $(X, \mathfrak{A}, \eta)$ and $(Y, \mathfrak{B}, \gamma)$, we say that the measure preserving transformations $\alpha \colon X \to X$ and $\beta \colon Y \to Y$ are *almost conjugate* if there are sets $X_0 \in \mathfrak{A}$ and $Y_0 \in \mathfrak{B}$ such that

(a) $\alpha(X_0) = X_0$, $\eta(X_0) = 0$, $\beta(Y_0) = Y_0$ and $\gamma(Y_0) = 0$;

(b) $\alpha_{|X \setminus X_0}$ is topologically conjugate to $\beta_{|Y \setminus Y_0}$.

3.2. **Equilibrium states.** Let $f \colon X \to X$ be a continuous map on a compact metric space (X, d), $C^0(X)$ the Banach algebra of real-valued continuous functions of X equipped with the supremum norm and $\mathcal{S}(X)$ the collection of all the compact subsets of X. For $\varphi \in C^0(X)$ and $n \in \mathbb{N}$, we denote $\sum_{j=0}^{n-1} \varphi(f^j(x))$ by $S_n(f, \varphi)(x)$. Given a compact subset $Y \subset X$, $\varepsilon > 0$ and a natural number n, a subset E of Y is said to be (n, ε)-*separated* with respect to f if for any $x, y \in E$ with $x \neq y$ there is some $j \in \{0, \cdots, n-1\}$ such that $d\left(f^j(x), f^j(y)\right) > \varepsilon$. The *topological pressure* of f is the operator

$$P_{\mathrm{top}}(f, \cdot, \cdot) \colon \; C^0(X) \times \mathcal{S}(X) \to \mathbb{R} \cup \{+\infty\}$$

which assigns to each $\varphi \in C^0(X)$ and each $Y \subset X$ the value

$$P_{\mathrm{top}}(f, \varphi, Y) = \lim_{\varepsilon \to 0} P(f, \varphi, Y, \varepsilon)$$

where

$$P(f, \varphi, Y, \varepsilon) = \limsup_{n \to +\infty} \frac{1}{n} \log \, P(f, \varphi, Y, \varepsilon, n)$$

$$P(f, \varphi, Y, \varepsilon, n) = \sup \left\{ \sum_{x \in E} e^{S_n(f, \varphi)(x)} \colon \; E \text{ is a } (n, \varepsilon)\text{-separated subset of } Y \right\}.$$

We often write $P_{\mathrm{top}}(f, \varphi)$ in place of $P_{\mathrm{top}}(f, \varphi, X)$. The topological pressure generalizes the notion of topological entropy in the sense that, when $\varphi \equiv 0$, then $P_{\mathrm{top}}(f, 0, Y) = h_{\mathrm{top}}(f, Y)$. We also write $h_{\mathrm{top}}(f)$ instead of $h_{\mathrm{top}}(f, X)$.

Let $\mathscr{P}(X)$ be the set of Borel probability measures on X endowed with the weak*-topology, $\mathscr{P}(X, f)$ be the subset of f-invariant elements of $\mathscr{P}(X)$ and $\mathscr{P}_e(X, f)$ be its subset of ergodic measures. The thermodynamic formalism identifies distinguished elements in $\mathscr{P}(X, f)$, called *equilibrium states*: these are probability measures that maximize the quantity $h_\mu(f) + \int \varphi \, d\mu$, where φ is a fixed potential in $C^0(X)$ and $h_\mu(f)$ denotes the metric entropy of f with respect to μ (definition in [35, §4]). An important motivation for the search of such measures comes from the Variational Principle [35, Theorem 9.10], which states that, for every potential map $\varphi \in C^0(X)$, the supremum of the functional $\mu \in \mathscr{P}(X, f) \to h_\mu(f) + \int \varphi \, d\mu$ coincides with the supremum of its restriction to $\mathscr{P}_e(X, f)$, and is equal to the topological pressure $P_{\text{top}}(f, \varphi)$. In what follows, $\mathscr{P}(f, \varphi)$ stands for the set of equilibrium states of the pair (f, φ).

3.3. **Entropy-expansiveness.** Bowen introduced the notion of entropy-expansive map in [5] referring to systems which seem to be *expansive* with regard to entropy. More precisely, a continuous invertible map $f \colon X \to X$ on a compact metric space (X, d) is expansive if there exists $\varepsilon > 0$ such that, for every $x \in X$,

$$B_\varepsilon^\infty(x) := \left\{ y \in X \colon d(f^k(x), f^k(y)) < \varepsilon \quad \forall\, k \in \mathbb{Z} \right\} = \{x\}.$$

The transformation f is said to be *entropy-expansive* if there exists $\varepsilon > 0$ such that

$$\sup_{x \in X} \; h_{\text{top}}(f, B_\varepsilon^\infty(x)) = 0.$$

It is known that if f is entropy-expansive, then the entropy map

$$\mu \in \mathscr{P}(X, f) \quad \mapsto \quad h_\mu(f)$$

is upper semi-continuous [28], and so $\mathscr{P}(f, \varphi) \neq \emptyset$ for every $\varphi \in C^0(X)$.

3.4. **Partial hyperbolicity.** Let M be a compact Riemannian manifold and a diffeomorphism $f \colon M \to M$. An invariant compact set $K \subset M$ is said to be *partially hyperbolic* by f if the tangent bundle above K admits a Df-invariant splitting $E^s(f) \oplus E^c(f) \oplus E^u(f)$ such that E^s is uniformly contracted and E^u is uniformly expanded, and the possible contraction and expansion of Df in $E^c(f)$ are weaker than those in the complementary bundles. More precisely, there exist constants $N \in \mathbb{N}$ and $\lambda > 1$ such that, for every $x \in K$ and every unit vector $v^* \in E^*(x, f)$, where $* = \mathrm{s}, \mathrm{c}, \mathrm{u}$, we have

(a) $\lambda \|Df_x^N(v^s)\| < \|Df_x^N(v^c)\| < \lambda^{-1} \|Df_x^N(v^u)\|$

(b) $\|Df_x^N(v^s)\| < \lambda^{-1} < \lambda < \|Df_x^N(v^u)\|.$

Generalizing the definition of hyperbolicity, the partially hyperbolic diffeomeorphisms are those systems whose non-wandering sets decompose into finitely many invariant transitive pieces, each of them being partially hyperbolic by f. We will call these subsets *basic pieces*. For comprehensive surveys on this subject, we refer the reader to [4] and [22].

Partial hyperbolicity is a robust property, and a partially hyperbolic diffeomorphism f admits stable and unstable foliations, say $W^s(f)$ and $W^u(f)$, which are f-invariant and tangent to $E^s(f)$ and $E^u(f)$, respectively [4]. By contrast, the center bundle $E^c(f)$ may not have a corresponding tangent foliation (cf. [21]), and the same problem befalls either

$E^{\mathrm{s}}(f) \oplus E^{\mathrm{c}}(f)$ or $E^{\mathrm{c}}(f) \oplus E^{\mathrm{u}}(f)$. When these manifolds exist (as happens with the open set of systems we consider here) we denote them by $W^{\mathrm{c}}(f)$, $W^{\mathrm{cs}}(f)$ and $W^{\mathrm{cu}}(f)$, and refer to them as the center, center-stable and center-unstable foliations, respectively.

Partial hyperbolic diffeomorphisms with a splitting exhibiting a one-dimensional center bundle are entropy-expansive (cf. [13]; see also [15, 16] for generalizations), and so they admit at least one equilibrium state for every continuous potential. However, even if we assume that the system is topologically mixing, there may be, for instance, more than one measure of maximal entropy [20].

4. Main results

Our first result compares the topological pressure operators associated to $\sigma \times L$ and to a skew product as the ones described in Section 2; additionally, it establishes sufficient conditions for the existence of equilibrium states associated to the skew products described in Section 2.

Theorem A. *Let $F \colon M \times \mathbb{T}^2 \to M \times \mathbb{T}^2$ be a skew product as described in Section 2 and H be the semi-conjugacy between F and $\sigma \times L$ given by Lemma 2.1. Then, for every potential $\varphi \in C^0(M \times \mathbb{T}^2)$ the following properties hold:*

(1) *$P_{\mathrm{top}}(F, \varphi \circ H) = P_{\mathrm{top}}(\sigma \times L, \varphi)$.*

(2) *Any equilibrium state of F and $\varphi \circ H$ projects by H_* to an equilibrium state of $\sigma \times L$ and φ.*

(3) *If, in addition, $\sigma \times L$ has an equilibrium state ν_φ for φ satisfying $\nu_\varphi (\mathcal{A}) = 1$ where*

$$\mathcal{A} := \left\{ H(x,y) \in M \times \mathbb{T}^2 \colon H^{-1}(H(x,y)) = \{(x,y)\} \right\}$$

then F has an equilibrium state for the potential $\varphi \circ H$. Moreover, if $\mathscr{P}(\sigma \times L, \varphi)$ contains only one element, then the same is true for $\mathscr{P}(F, \varphi \circ H)$.

The existence of equilibrium states for F depends not only on the potential map, but also on the dynamics of $\sigma \times L$. And, as L is an Anosov diffeomorphism, the ergodic properties of $\sigma \times L$ strongly relies on the underlying dynamics σ. For instance, if σ has no probability measure of maximal entropy, as the examples described in [35, §8] or [27], then the same happens with F (cf. [29, Theorem 1]). Yet, if σ is expansive, then there exists at least one equilibrium state for F and every continuous potential. Indeed, in this case, the set of points that prevent expansiveness must be contained in a compact subset of the center laminations, inside curves whose lengths remain uniformly bounded after forward iterates of F (more details on [29, page 619]) and so F is entropy-expansive (cf. [13, Proposition 6]). Thus, if σ is expansive, then the skew product F is entropy-expansive.

The uniqueness of equilibrium states is, in general, a harder problem. Yet, besides the existence, Theorem A (3) also provides uniqueness if $\sigma \times L$ has only one equilibrium state associated to φ. So, we are left to find an additional natural assumption which leads to the main hypothesis of item (3) of Theorem A. Our next result establishes one such a sufficient condition: it summons the amount of hyperbolicity available within the skew product F to

prove that the equilibrium state of the pair $(F, \varphi \circ H)$ is unique for some class of potentials φ.

For each $(x, y) \in M \times \mathbb{T}^2$, define the *lower Lyapunov exponent* $\chi(x, y)$ of F^{-1} at (x, y) along the $\mathfrak{F}$-direction by

$$\chi(x, y) := \liminf_{n \to +\infty} \frac{1}{n} \log \|D_{(x,y)} F^{-n}|_{E^s_{(x,y)}}\| \tag{4.1}$$

where $D_{(x,y)} F^{-n}$ denotes the derivative $D_y(\pi_2 \circ F^{-n})$ and $\pi_2 : M \times \mathbb{T}^2 \to \mathbb{T}^2$ is the canonical projection on the second factor. The condition $\chi(x, y) > 0$ roughly states that F^{-n} expands lengths within the leaf $\mathfrak{F}(x, y)$ for large enough $n \in \mathbb{N}$. This happens if, for instance, for a large subset of elements $x \in M$ the diffeomorphism at the fibre $f_x : \{x\} \times \mathbb{T}^2 \to \{\sigma(x)\} \times \mathbb{T}^2$ is Anosov (identifying $\{x\} \times \mathbb{T}^2$ and $\{\sigma(x)\} \times \mathbb{T}^2$ with $\mathbb{T}^2$) with stable foliation equal to $\mathfrak{F}$. Consider the set

$$\mathcal{E} = \{(x, y) \in M \times \mathbb{T}^2 : \quad \chi(x, y) > 0\}.$$

If there exists a σ-invariant set $B \subset M$ such that $B \times \mathbb{T}^2 \subset \mathcal{E}$, then Lemma 4 of [29] shows that the restriction $H : B \times \mathbb{T}^2 \to B \times \mathbb{T}^2$ of the semi-conjugacy H is a conjugacy between $F|_{B \times \mathbb{T}^2}$ and $(\sigma \times L)|_{B \times \mathbb{T}^2}$. Therefore, the set $\mathcal{A}$ in Theorem A is measurable and may be replaced by $B \times \mathbb{T}^2$. This is the way we will relate the equilibrium states of $\sigma \times L$ with the ones of F.

Theorem B. *Let $F : M \times \mathbb{T}^2 \to M \times \mathbb{T}^2$ be a skew product as described in Section 2, and H be the semi-conjugacy between F and $\sigma \times L$ given by Lemma 2.1. Assume that there is a σ-invariant set $B \subset M$ such that $B \times \mathbb{T}^2 \subset \mathcal{E}$. If $\sigma \times L$ has a unique equilibrium state ν_φ associated to the potential $\varphi \in C^0(M \times \mathbb{T}^2)$ and $\nu_\varphi(B \times \mathbb{T}^2) = 1$, then F has a unique equilibrium state $\mu_{\varphi \circ H}$ for the potential $\varphi \circ H \in C^0(B \times \mathbb{T}^2)$. Furthermore, the systems $(\sigma \times L, \nu_\varphi)$ and $(F, \mu_{\varphi \circ H})$ are measure theoretically isomorphic.*

We now apply Theorem A and Theorem B to a non-hyperbolic setting. Let $\sigma : M \to M$ be a C^1-diffeomorphism on a compact connected Riemannian manifold M satisfying Smale's Spectral Decomposition Theorem ([33, Theorem 6.2]). Using the techniques of [23, Section 8], we may find a smooth diffeomorphism $F : M \times \mathbb{T}^2 \to M \times \mathbb{T}^2$ homotopic to $\sigma \times L$ such that its non-wandering set $\Omega(F)$ is $\Omega(\sigma) \times \mathbb{T}^2$ and F satisfies the conditions (1)-(3) of Section 2. Note that, if Λ is a basic set of σ, then so is $\Lambda \times \mathbb{T}^2$ for $\sigma \times L$, and by [7] there exists a unique equilibrium state ν_φ for each Hölder potential φ with respect to the dynamics $(\sigma \times L)|_{\Lambda \times \mathbb{T}^2}$. Besides, F admits a partially hyperbolic splitting of the fiber bundle at $\Lambda \times \mathbb{T}^2$ with a one-dimensional center bundle tangent to $\mathfrak{F}$. Therefore, under the hypotheses of Theorem A and Theorem B, the equilibrium state $\mu_{\varphi \circ H}$ is unique and inherits all ergodic properties of ν_φ.

An important class of non-hyperbolic diffeomorphisms obtained through this process contains the well known examples of Abraham-Smale [1] and Shub [23]. The skew product studied by Abraham and Smale, namely F_{AS}, is a homotopic deformation on $\mathbb{S}^2 \times \mathbb{T}^2$ of the direct product of a Smale's horseshoe $\sigma : \mathbb{S}^2 \to \mathbb{S}^2$ by a linear Anosov diffeomorphism. The one analyzed by Shub, which we denote by F_{Sh}, is obtained deforming on $\mathbb{T}^2 \times \mathbb{T}^2$ the

product of two Anosov diffeomorphisms (not necessarily linear). The global dynamics of these examples satisfy the following properties:

(a) No diffeomorphism C^1-close to F_{AS} (respectively, F_{Sh}) is Ω-stable.

(b) The non-wandering set of F_{AS} (respectively, F_{Sh}) is homeomorphic to the non-wandering set of any diffeomorphism C^1-nearby.

(c) The spectral decomposition of F_{AS} is given by $\Omega(F_{AS}) = \Omega_1 \cup \Omega_2 \cup \Omega_3$, where $\Omega_1 = \{p_0\} \times \mathbb{T}^2$, $\Omega_2 = \Lambda \times \mathbb{T}^2$ and $\Omega_3 = \{p_1\} \times \mathbb{T}^2$, and p_0 is a sink, p_1 is a source and Λ is a horseshoe of the dynamics $\sigma \colon \mathbb{S}^2 \to \mathbb{S}^2$. The transitive set Ω_2 is not hyperbolic due to the presence of two saddles with different stable indices. However, Ω_2 admits a partially hyperbolic splitting.

(d) In Shub's example, the non-wandering set is (robustly) transitive, so $\Omega(F_{Sh}) = \mathbb{T}^2 \times \mathbb{T}^2$. Again, this set is not hyperbolic but has a partially hyperbolic splitting.

In both examples, the corresponding family $x \in M \to f_x \in \mathrm{Diff}^1(\mathbb{T}^2)$ changes continuously from an Anosov diffeomorphism to a derived from Anosov while preserving the foliation $\mathfrak{F}$. Yet, the selection of these families of diffeomorphisms is not unique. As we will see, a suitable choice of $(f_x)_{x \in M}$, complemented with an additional condition on the potential φ, allow us to guarantee, after fixing a basic piece $\Lambda \times \mathbb{T}^2$, the existence of a set $B \subset \Lambda$ as requested in Theorem B.

Let us exemplify this reasoning with F_{AS}. One way to ensure the existence of such a set $B \subset \Lambda$ is to control the sojourns of the orbits by σ inside the small open set $\mathcal{U} \subset M$ where the deformation of $\sigma \times L$ into F_{AS} is performed. For those points $x \in \mathcal{U}$, the diffeomorphism f_x is a derived from Anosov and the foliation $\mathfrak{F}$ in the fiber $\{x\} \times \mathbb{T}^2$ may be expanding by F_{AS}; this is precisely the behavior which, according to Theorem B, we should reduce to a minimum. So, given a basic set Λ of $\Omega(\sigma)$, a Hölder potential φ and the equilibrium state ν_φ with respect to $\sigma \times L$, we will request that F_{AS} and ν_φ satisfy the condition

$$\int_\Lambda \log \zeta(x) \, d(\pi_1)_*(\nu_\varphi)(x) > 0 \qquad (4.2)$$

where

$$\zeta(x) = \inf_{y \,\in\, \mathbb{T}^2} \|D_y f_x^{-1}{}_{|_{E^{\mathrm{s}}_{(x,y)}}}\|$$

and $\pi_1 \colon \Lambda \times \mathbb{T}^2 \to \Lambda$ is the natural projection on the first factor. Notice that $\zeta(x)$ measures the least rate of expansion of $D_y f_x^{-1}$ along the leaf $\mathfrak{F}_{(x,y)}$ at the fiber $\{x\} \times \mathbb{T}^2$; and that condition (4.2) says that, in average, these minimum rates are bigger than 1 (hence expanding). We are left to prove that this assumption is enough to ensure that the set

$$B = \left\{ x \in \Lambda \colon \chi(x,y) > 0 \quad \forall y \in \mathbb{T}^2 \right\}$$

satisfies $B \times \mathbb{T}^2 \subset \mathcal{E}$ and $\nu_\varphi(B \times \mathbb{T}^2) = 1$. This way, by Theorem B, $\mu_{\varphi \circ H}$ is the unique equilibrium state for $(F_{AS}, \varphi \circ H)$. Now, observe that the case $\varphi \equiv 0$ is considerably simpler. Indeed, for this potential, $(\pi_1)_*(\nu_0)$ is the measure of maximal entropy of σ (cf. [29, Theorem 1]), which, besides being uniformly distributed, provides an explicit

connection between the size of $\mathcal{U}$ and its $(\pi_1)_*(\nu_0)$ measure. This enables us to suitably choose, once and for all, the family of diffeomorphisms $(f_x)_{x \in \mathbb{S}^2}$ in order to master the magnitude of the deformation that builds F_{AS} and, at the same time, to ascertain the validity of the assumption (4.2) for $\varphi \equiv 0$ and ν_0. Thus, the set

$$\mathbb{P}(\Lambda \times \mathbb{T}^2) = \left\{ \varphi \in C^0(\Lambda \times \mathbb{T}^2) \colon \varphi \text{ is Hölder and } \int_\Lambda \log \zeta(x) \, d(\pi_1)_*(\nu_\varphi)(x) > 0 \right\}$$

in non-empty (in fact, it contains every constant potential $\varphi \equiv c$, since $P_{\text{top}}(f, c) = h_{\text{top}}(f) + c$ and therefore the equilibrium states ν_c and ν_0 coincide). Moreover, we will show in Section 8 that $\mathbb{P}(\Lambda \times \mathbb{T}^2)$ is an open domain inside the subset of Hölder elements of $C^0(\Lambda \times \mathbb{T}^2)$. The next result states that the class $\mathbb{P}(\Lambda \times \mathbb{T}^2)$ is an adequate choice of potentials for $\sigma \times L$.

Corollary C. *Let $F_{AS} : \mathbb{S}^2 \times \mathbb{T}^2 \to \mathbb{S}^2 \times \mathbb{T}^2$ be Abraham-Smale's example, H the semi-conjugacy between F_{AS} and $\sigma \times L$ given by Lemma 2.1 and Λ be a basic set of σ. Take a potential $\varphi \in \mathbb{P}(\Lambda \times \mathbb{T}^2)$ and the unique equilibrium state ν_φ associated to the $(\sigma \times L)_{|_{\Lambda \times \mathbb{T}^2}}$ and the potential φ. Then:*

(1) *There exists a σ-invariant set $B \subset \Lambda$ such that $B \times \mathbb{T}^2 \subset \mathcal{E}$ and $\nu_\varphi(B \times \mathbb{T}^2) = 1$.*

(2) *F_{AS} has a unique equilibrium state $\mu_{\varphi \circ H}$ for the potential $\varphi \circ H$.*

(3) *$(F_{AS}, \mu_{\varphi \circ H})$ and $(\sigma \times L, \nu_\varphi)$ are measure theoretically isomorphic. In particular, $\mu_{\varphi \circ H}$ is Bernoulli.*

An analogous statement is true for Shub's example.

We now proceed studying C^1-diffeomorphisms $G \colon \mathbb{S}^2 \times \mathbb{T}^2 \to \mathbb{S}^2 \times \mathbb{T}^2$ in a small open neighborhood $\mathcal{V}$ of F_{AS} in $\text{Diff}^1(\mathbb{S}^2 \times \mathbb{T}^2)$. A similar discussion may be pursued with F_{Sh}. We may select $\mathcal{V}$ so that, for each $G \in \mathcal{V}$, the following properties remain valid:

(a) There exists a partially hyperbolic splitting on $\Omega(G)$ with integrable subbundles $E^*(G)$, where $* = \text{s, c, u, cs}$; besides, $\dim E^c(G) = 1$ and the center foliation is normally hyperbolic (cf. [23]).

(b) G may not be a skew product, but there exists a homeomorphism $\tau_G \colon \Omega(G) \to \Omega(F_{AS})$ such that $\tau_G \circ G \circ \tau_G^{-1}$ is a bundle map covering σ and C^0-close to G and F_{AS} (cf. [23, §8]). Moreover, the skew product $\tau_G \circ G \circ \tau_G^{-1}$ satisfies the conditions (1)-(3) of Section 2. Denote by H_G the corresponding semi-conjugacy provided by Lemma 2.1.

(c) $\Omega(G) = \Omega_1(G) \cup \Omega_2(G) \cup \Omega_3(G)$, where $\Omega_i(G) = \tau_G^{-1}(\Omega_i)$, for $i = 1, 2, 3$, and $\Omega_i(G)$ is a transitive, locally maximal and partially hyperbolic set.

(d) The restriction of G to each $\Omega_i(G)$ is entropy-expansive.

Therefore, the existence of equilibrium states for each restriction $G_{|_{\Omega_i(G)}}$ and every continuous potential is guaranteed. The next result provides uniqueness of these equilibrium states for a certain class of potentials.

Theorem D. *Consider $G \in \mathcal{V}$, a basic set Λ of $\Omega(\sigma)$ and a potential $\varphi \in \mathbb{P}(\Lambda \times \mathbb{T}^2)$. Then there is a unique equilibrium state $\mu_{\varphi \circ H_G \circ \tau_G}$ for the restriction of G to the piece $\tau_G^{-1}(\Lambda \times \mathbb{T}^2)$ of $\Omega(G)$ and the continuous potential $\varphi \circ H_G \circ \tau_G$. An analogous statement applies to Shub's example.*

Having established the existence and uniqueness of equilibrium states for Hölder potentials (up to the semi-conjugacies with $\sigma \times L$) for every diffeomorphism in a C^1-neighborhood of F_{AS} (respectively, F_{Sh}), a natural question is how this equilibrium state varies with the underlying dynamics and the potential. Given a basic set Λ of $\Omega(\sigma)$ and a potential $\varphi \in \mathbb{P}(\Lambda \times \mathbb{T}^2)$, we say that the equilibrium state $\mu_{\varphi \circ H}$ of F_{AS} is *statistically stable* if, given a sequence $(G_n)_{n \in \mathbb{N}}$ of diffeomorphisms in $\mathcal{V}$ converging in the C^1-topology to F_{AS}, a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of maps in $\mathbb{P}(\Lambda \times \mathbb{T}^2)$ converging in the C^0-norm to φ and the sequence of equilibrium states μ_n for the restriction of G_n to $\tau_{G_n}^{-1}(\Lambda \times \mathbb{T}^2)$ and $\varphi_n \circ h_{G_n} \circ \tau_{G_n}$, then any accumulation point of the sequence $(\mu_n)_{n \in \mathbb{N}}$ in the weak*-topology is the equilibrium state $\mu_{\varphi \circ H}$ of $(F_{AS}, \varphi \circ H)$. We will prove the statistical stability of the equilibrium states of F_{AS} obtained in Corollary C.

Corollary E. *Given a basic set Λ of $\Omega(\sigma)$ and $\varphi \in \mathbb{P}(\Lambda \times \mathbb{T}^2)$, then the map*

$$(G, \, \psi) \, \in \, \mathcal{V} \times \mathbb{P}(\Lambda \times \mathbb{T}^2) \quad \mapsto \quad \mu_{\psi \circ H_G \circ \tau_G} \, \in \, \mathscr{P}(\mathbb{S}^2 \times \mathbb{T}^2)$$

is continuous at $(F_{AS}, \, \varphi)$. A similar result holds for F_{Sh}.

5. Proof of Theorem A

Let H be the semi-conjugacy given by Lemma 2.1. The inequality

$$P_{\text{top}}(F, \, \varphi \circ H) \, \geq \, P_{\text{top}}(\sigma \times L, \, \varphi), \qquad \forall \, \varphi \in C^0(M \times \mathbb{T}^2)$$

follows directly from [35, Theorem 9.8]. Regarding the reverse inequality, we start showing another connection between the pressure of two semi-conjugate dynamics.

Proposition 5.1. *Let $T : X \to X$ and $S : Y \to Y$ be two continuous transformations on compact metric spaces X and Y. Suppose there exists a surjective continuous map $\pi : X \to Y$ satisfying $\pi \circ T = S \circ \pi$. Then, for every potential $\varphi \in C^0(Y)$ we have*

$$P_{\text{top}}(T, \, \varphi \circ \pi) \leq P_{\text{top}}(S, \, \varphi) + \sup_{\nu \in \mathscr{P}(Y, S)} \int h_{\text{top}}\big(T, \pi^{-1}(y)\big) \, d\nu(y).$$

Proof. For each $\nu \in \mathscr{P}(Y, S)$, consider the set $\mathscr{P}_\nu(X, T) \subset \mathscr{P}(X, T)$ defined by

$$\mathscr{P}_\nu(X, T) = \Big\{ \mu \in \mathscr{P}(X, T) \colon \, \pi_* \mu = \nu \Big\}.$$

Observe that

$$\mathscr{P}(X, T) = \bigcup_{\nu \in \mathscr{P}(Y, S)} \mathscr{P}_\nu(X, T). \tag{5.1}$$

Moreover, Ledrappier-Walters' formula ([25]) establishes that, for every $\nu \in \mathscr{P}(Y, S)$,

$$\sup_{\mu \in \mathscr{P}_\nu(X, T)} h_\mu(T) = h_\nu(S) + \int h_{\text{top}}\big(T, \, \pi^{-1}(y)\big) \, d\nu(y). \tag{5.2}$$

As the operator

$$\mu \in \mathscr{P}_\nu(X, T) \quad \mapsto \quad \int \varphi \circ \pi \, d\mu$$

is constant and equal to $\int \varphi \, d\nu$, adding $\int \varphi \, d\nu$ to both sides of the equality (5.2) we obtain

$$\sup_{\mu \in \mathscr{P}_\nu(X, T)} \left\{ h_\mu(T) + \int \varphi \circ \pi \, d\mu \right\} = h_\nu(S) + \int \varphi \, d\nu + \int h_{\text{top}}\big(T, \pi^{-1}(y)\big) \, d\nu(y). \quad (5.3)$$

Taking in (5.3) the supremum over all the measures $\nu \in \mathscr{P}(Y, S)$ and applying the Variational Principle to $S : Y \to Y$ and $\varphi : Y \to \mathbb{R}$, we get

$$\sup_{\nu \in \mathscr{P}(Y,S)} \left\{ \sup_{\mu \in \mathscr{P}_\nu(X,T)} \left\{ h_\mu(T) + \int \varphi \circ \pi \, d\mu \right\} \right\}$$

$$\leq P_{\text{top}}(S, \varphi) + \sup_{\nu \in \mathscr{P}(Y,S)} \int h_{\text{top}}\big(T, \pi^{-1}(y)\big) \, d\nu(y).$$

On the other hand, it follows from (5.1) that

$$\sup_{\mu \in \mathscr{P}(X,T)} \left\{ h_\mu(T) + \int \varphi \circ \pi \, d\mu \right\} = \sup_{\nu \in \mathscr{P}(Y,S)} \left\{ \sup_{\mu \in \mathscr{P}_\nu(X,T)} \left\{ h_\mu(T) + \int \varphi \circ \pi \, d\mu \right\} \right\}$$

and so, again by the Variational Principle, we conclude that

$$P_{\text{top}}(T, \varphi \circ \pi) \leq P_{\text{top}}(S, \varphi) + \sup_{\nu \in \mathscr{P}(Y,S)} \int h_{\text{top}}\big(T, \pi^{-1}(y)\big) \, d\nu(y)$$

thus completing the proof of the proposition. $\qquad\square$

Now, for the skew products introduced in Section 2, as the leaves of the lamination $\mathfrak{F}$ are one-dimensional, each fiber of the semi-conjugacy carries no entropy (this is not necessarily true if the dimension of those leaves is larger than one; we refer the reader to [29, Remark at page 628]). More precisely:

Lemma 5.2. [29, Lemma 3] *For every* $(x, y) \in M \times \mathbb{T}^2$*, one has* $h_{\text{top}}\big(F, H^{-1}(x, y)\big) = 0$.

Consequently, in this setting, Proposition 5.1 may be rewritten as

$$P_{\text{top}}(F, \varphi \circ H) \leq P_{\text{top}}(\sigma \times L, \varphi), \qquad \forall \, \varphi \in C^0(M \times \mathbb{T}^2).$$

This ends the proof of part (1) of Theorem A.

Concerning statement (2) of Theorem A, let $\mu_{\varphi \circ H}$ be an equilibrium state for $(F, \varphi \circ H)$. We need to show that $\nu_\varphi = H_*(\mu_{\varphi \circ H})$, which belongs to $\mathscr{P}(M \times \mathbb{T}^2, \sigma \times L)$, is an equilibrium state for $(\sigma \times L, \varphi)$. By Lemma 5.2, equation (5.3) and part (1) of Theorem A, we deduce

that

$$h_{\nu_\varphi}(\sigma \times L) + \int \varphi \, d\nu_\varphi = \sup_{\mu \in \mathscr{P}_{\nu_\varphi}(M \times \mathbb{T}^2, F)} \left\{ h_\mu(F) + \int \varphi \circ H \, d\mu \right\}$$

$$\geq h_{\mu_{\varphi \circ H}}(F) + \int \varphi \circ H \, d\mu_{\varphi \circ H}$$

$$= P_{\text{top}}(F, \varphi \circ H)$$

$$= P_{\text{top}}(\sigma \times L, \, \varphi).$$

Thus, ν_φ is an equilibrium state of $\sigma \times L$ and φ. This ends the proof of Theorem A (2).

Let us now show item (3) of Theorem A. Assume that ν_φ is an equilibrium state of $\sigma \times L$ and φ, and that the set $\mathcal{A}$ of points in $M \times \mathbb{T}^2$ where H is injective has full ν_φ measure.

Lemma 5.3. $\mathcal{A}$ *is* $(\sigma \times L)$*-invariant.*

Proof. As F and $\sigma \times L$ are bijections and $H \circ F = (\sigma \times L) \circ H$, we have $H^{-1} \circ (\sigma \times L)^{-1}(\cdot) \supseteq F^{-1} \circ H^{-1}(\cdot)$. Therefore, if $z \in \mathcal{A}$, then

$$H^{-1}(z) = H^{-1} \circ (\sigma \times L)^{-1}((\sigma \times L)(z)) \supseteq F^{-1} \circ H^{-1}((\sigma \times L)(z))$$

and so, as $H^{-1}(z)$ is a singular set and F is injective, $H^{-1}((\sigma \times L)(z))$ must be singular as well. Thus, $(\sigma \times L)(\mathcal{A}) \subset \mathcal{A}$. A similar argument proves the reverse inclusion. $\square$

Thus, $H^{-1}(\mathcal{A})$ is F-invariant, and, since $H_{|H^{-1}(\mathcal{A})}$ is injective, we conclude that:

Lemma 5.4. $H_{|H^{-1}(\mathcal{A})}$ *is a topological conjugacy between* $F_{|_{H^{-1}(\mathcal{A})}}$ *and* $(\sigma \times L)_{|\mathcal{A}}$.

Proof. The main difficulty in proving this statement is the fact that, having in mind possible applications (as happens with the subset $B \times \mathbb{T}^2$ in Theorem B), we are not assuming that the set $\mathcal{A}$ is compact. Thus, it is not immediate that the injectivity of H at $H^{-1}(\mathcal{A})$ yields the continuity of H^{-1} in $\mathcal{A}$. Consider a sequence $\left((x_n, y_n) \right)_{n \in \mathbb{N}}$ of points in $H^{-1}(\mathcal{A})$ such that the sequence $\left(H(x_n, y_n) \right)_{n \in \mathbb{N}}$ converges to $H(x_0, y_0)$ for some $(x_0, y_0) \in \mathcal{A}$. We need to show that $\lim_{n \to +\infty} (x_n, y_n) = (x_0, y_0)$. As $M \times \mathbb{T}^2$ is compact, the sequence $\left((x_n, y_n) \right)_{n \in \mathbb{N}}$ has accumulation points in $M \times \mathbb{T}^2$. Suppose that β_1 and β_2 are accumulated by the sequence $\left((x_n, y_n) \right)_{n \in \mathbb{N}}$. Then, by the continuity of H, we get

$$\lim_{n \to +\infty} H(x_n, y_n) = H(x_0, y_0) = H(\beta_1) = H(\beta_2).$$

Therefore, as H is injective at (x_0, y_0), we infer that $\beta_1 = \beta_2 = (x_0, y_0)$. Hence, $\left((x_n, y_n) \right)_{n \in \mathbb{N}}$ converges to (x_0, y_0), and so H^{-1} is continuous in $\mathcal{A}$. $\square$

Consequently, the map $\left((H^{-1})_{|\mathcal{A}} \right)_*$ is an isomorphism between $\mathscr{P}(\mathcal{A}, (\sigma \times L)_{|\mathcal{A}})$ and $\mathscr{P}(H^{-1}(\mathcal{A}), F_{|_{H^{-1}(\mathcal{A})}})$. Thus, $(H^{-1})_*(\nu_\varphi)$ is an equilibrium state for the pair $(F, \varphi \circ H)$.

6. Proof of Theorem B

The key idea to prove Theorem B is Lemma 4 of [29] which ascertains that, under the assumption that F^{-m} expands lengths in the central leaves $\mathfrak{F}$ for large enough values of $m \in \mathbb{N}$ (a property that is conveyed by the set $B \times \mathbb{T}^2$), then the restriction of the semi-conjugacy $H_{|_{B \times \mathbb{T}^2}}$ becomes an almost conjugacy.

Lemma 6.1. [29, Lemma 4] *Suppose there is a σ-invariant set $B \subset \Lambda$ such that $\chi(x, y) > 0$ for every $(x, y) \in B \times \mathbb{T}^2$. Then the restriction of H to the set $B \times \mathbb{T}^2$ is a homeomorphism.*

Notice that this lemma also indicates that $B \times \mathbb{T}^2 \subset \mathcal{A}$. So, as $\nu_\varphi(B \times \mathbb{T}^2) = 1$, we get $\nu_\varphi(\mathcal{A}) = 1$ and Theorem A (3) applies.

Assume now that $\sigma \times L$ has a unique equilibrium state ν_φ with respect to $\sigma \times L$ and a Hölder potential φ. We wish to show that F also has only one equilibrium state for $\varphi \circ H$. Suppose, otherwise, that F has two equilibrium states μ_1 and μ_2 for $\varphi \circ H$. Then Theorem A (2) indicates that $H_*(\mu_1) = H_*(\mu_2) = \nu_\varphi$. Moreover, by assumption, there exists a σ-invariant set B such that

$$B \times \mathbb{T}^2 \subset \mathcal{E} = \left\{ (x, y) \in M \times \mathbb{T}^2 \colon \chi(x, y) > 0 \right\}$$

and $\nu_\varphi(B \times \mathbb{T}^2) = 1$. Thus, as H is surjective and is the identity on the first coordinate,

$$\mu_1(B \times \mathbb{T}^2) = \mu_1(H^{-1}(B \times \mathbb{T}^2)) = \nu_\varphi(B \times \mathbb{T}^2) = 1 \tag{6.1}$$

and similarly $\mu_2(B \times \mathbb{T}^2) = 1$. Besides, since $H_{|B \times \mathbb{T}^2}$ is a topological conjugacy between $F_{|_{B \times \mathbb{T}^2}}$ and $(\sigma \times L)_{|_{B \times \mathbb{T}^2}}$, the map H_* is an isomorphism between $\mathscr{P}(B \times \mathbb{T}^2, F_{|_{B \times \mathbb{T}^2}})$ and $\mathscr{P}(B \times \mathbb{T}^2, (\sigma \times L)_{|_{B \times \mathbb{T}^2}})$. Thus, $\mu_1 = (H^{-1})_*(\nu_\varphi) = \mu_2$.

6.1. A generalization.

Theorems A and B together admit a more general statement whose proof is a straightforward adaptation of the arguments presented in this and the previous section.

Let $f \colon X \to X$ be an expansive homeomorphism of a compact metric space X with the specification property. Consider a continuous extension F of f through a continuous surjective map H, and take a Hölder potential φ and the unique (ergodic) equilibrium state ν_φ of f and φ. Suppose that the two following conditions are fulfilled:

(1) $h_{\text{top}}(F, H^{-1}(x)) = 0$ for every $x \in X$.

(2) $\nu_\varphi \left(\left\{ H(x) \in X \colon H^{-1}(H(x)) = \{x\} \right\} \right) = 1$.

Then the measure $(H^{-1})_*(\nu_\varphi)$ is F-invariant, ergodic, and the unique equilibrium state of F and $\varphi \circ H$. This result may be compared with [8, Theorem 1.5].

7. Proof of Corollary C

We start recalling the construction of Abraham-Smale and Shub's examples. Consider a diffeomorphism $\sigma \colon M \to M$ satisfying the Spectral Decomposition Theorem. The tangent

space of M admits a hyperbolic splitting $TM = E^{ss} \oplus E^{uu}$ such that, for some uniform constant $0 < \lambda < 1$,

$$\max\left\{\|D\sigma|_{E^{ss}}\|, \ \|D\sigma^{-1}|_{E^{uu}}\|\right\} < \lambda.$$

Assume that σ has two fixed points p and q. Note that they are homoclinically related (that is, both transversal intersections $W^s(p) \pitchfork W^u(q)$ and $W^u(p) \pitchfork W^s(q)$ are non-empty). Afterwards, take a smooth family of torus diffeomorphisms $f_x \colon \mathbb{T}^2 \to \mathbb{T}^2$ indexed by $x \in M$ and satisfying the following properties:

(a) $T\,\mathbb{T}^2 = E^c(f_x) \oplus E^u(f_x)$, a splitting invariant under Df_x and for which there exist constants $0 < \gamma_1 < \gamma_2 < 1$ such that

$$\|Df_x^{-1}|_{E^u(f_x)}\| < \gamma_1 \qquad \text{and} \qquad \gamma_2 < \|Df_x|_{E^c(f_x)}\| \leq \gamma_1^{-1}.$$

We may assume, taking a power of σ if necessary, that $\lambda < \gamma_1$.

(b) For every $x \in M$, the diffeomorphism f_x preserves cone fields $\mathcal{C}^{cs}$ and $\mathcal{C}^u$.

(c) The map f_p is Anosov, while f_q is a derived from Anosov.

(d) There is $\theta \in \mathbb{T}^2$ such that $f_x(\theta) = \theta$ for every x, and θ is a saddle of f_p and a source for f_q.

Now, consider the skew product

$$F \colon M \times \mathbb{T}^2 \to M \times \mathbb{T}^2, \qquad F(x,y) = (\sigma(x), f_x(y)).$$

In [23] it was shown that F can be obtained as a deformation of $\sigma \times L$, where L is a linear Anosov diffeomorphism of the two-torus, satisfying the conditions (1)-(3) made explicit in Section 2. Moreover, the resulting F is partially hyperbolic, with central foliation $\mathfrak{F}$ whose leaves are tangent to the E^c-direction (cf. [31]).

The example of Abraham-Smale F_{AS} is obtained by the previous reasoning considering M to be equal to the two-sphere $\mathbb{S}^2$ with the dynamics σ of Smale's horseshoe. The family $(f_x)_{x \in M}$ is the result of a homotopic deformation of the linear Anosov diffeomorphism L induced by the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ at p until the deformation reaches a derived from Anosov diffeomorphism when x attains q. In the case of Shub's example, the diffeomphism F_{Sh} is obtained taking $M = \mathbb{T}^2$ and σ an Anosov diffeomorphism having two fixed points. Both skew products F_{AS} and F_{Sh} are C^0-perturbations of $\sigma \times L$ supported on a small ball with radius $\rho > 0$ centered at the fixed point q of σ in M. Outside that ball, the maps coincide with $\sigma \times L$, hence we may carry out the above construction so that the C^0-distance between F_{AS} (respectively, F_{Sh}) and $\sigma \times L$ is of the order of ρ.

Let us be more accurate, exemplifying with the case of F_{AS}. The matrix A is diagonalizable, with eigenvalues $0 < \lambda_s < 1$ and $\lambda_u > 1$ such that $\lambda_s \lambda_u = 1$. Denote by $B_r(x)$ the r-ball centered at $x \in \mathbb{S}^2$. Then (see Figure 1):

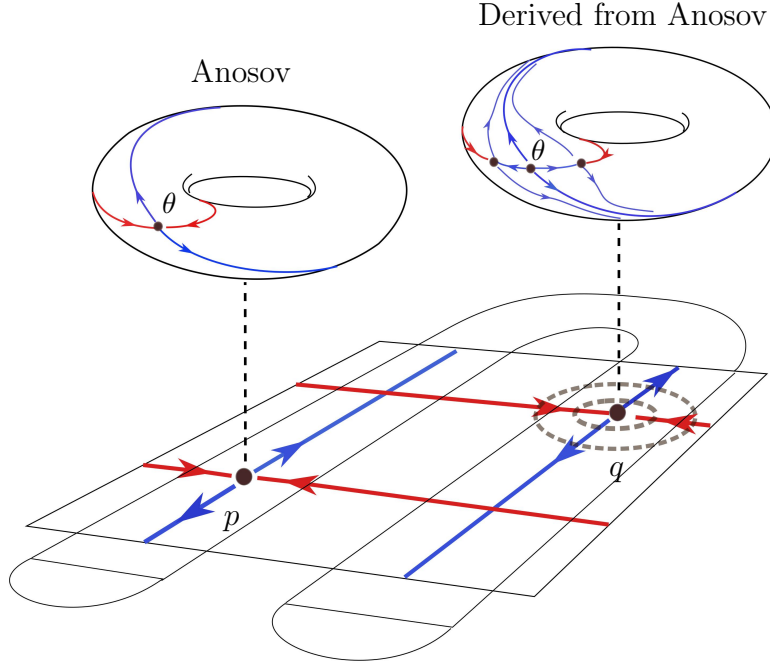(a) For $x \in \mathbb{S}^2 \setminus B_\rho(q)$, the diffeomorphism f_x is equal to L.

FIGURE 1. Homotopic deformation from an Anosov diffeomorphism to a derived from Anosov.

(b) For $x \in B_{\rho/2}(q)$, the diffeomorphism f_x is equal to a derived from the Anosov L, whose (lower triangular) linear part is $\begin{pmatrix} \lambda_u & 0 \\ * & \lambda_c \end{pmatrix}$ with $1 < \lambda_c < \lambda_u$.

(c) For $x \in B_{\rho}(q) \setminus B_{\rho/2}(q)$, the diffeomorphism f_x is a transition between L and the previous derived from Anosov, with linear part $\begin{pmatrix} \lambda_u & 0 \\ * & \gamma(x) \end{pmatrix}$ where $\gamma(x) > 0$.

Observe that we have control on the size ρ of the neighborhood of q where the perturbation occurs, on the value of λ_c and on the map $x \in B_{\rho}(q) \setminus B_{\rho/2}(q) \mapsto \gamma(x)$. Selecting them appropriately, we are able to bind to this construction the condition (4.2) for $\varphi \equiv 0$ and the measure of maximal entropy ν_0 of $\sigma \times L$. Indeed, the map ζ is given by

$$\zeta(x) = \begin{cases} \lambda_s^{-1} & \text{if } x \in \mathbb{S}^2 \setminus B_{\rho}(q) \\ \lambda_c^{-1} & \text{if } x \in B_{\rho/2}(q) \\ \gamma(x)^{-1} & \text{if } x \in B_{\rho}(q) \setminus B_{\rho/2}(q). \end{cases}$$

So, we construct F_{AS} requesting that

(H1) ρ is small enough so that $(\pi_1)_*(\nu_0)(B_{\rho}(q)) < \frac{\log \lambda_u}{\log \lambda_u + \log \lambda_c}$;

(H2) $\gamma(x) \le \lambda_c$ for every $x \in B_{\rho}(q) \setminus B_{\rho/2}(q)$.

This way, we obtain

$$\int_\Lambda \log \zeta(x) \, d(\pi_1)_*(\nu_0)(x) \geq \log(\lambda_u) - [\log \lambda_u + \log \lambda_c](\pi_1)_*(\nu_0)(B_\rho(q)) > 0.$$

Having finished the construction of F_{AS}, take a basic set Λ of $\Omega(\sigma)$, a potential $\varphi \in \mathbb{P}(\Lambda \times \mathbb{T}^2)$ and the equilibrium state ν_φ with respect to $\sigma \times L$ and φ. We will show that the set

$$B = \left\{ x \in \Lambda : \ \chi(x, y) > 0 \quad \forall \, y \in \mathbb{T}^2 \right\}$$

satisfies $B \times \mathbb{T}^2 \subset \mathcal{E}$ and $\nu_\varphi(B \times \mathbb{T}^2) = 1$. Accordingly, by Theorem B, $\mu_\varphi = (H^{-1}_{|B \times \mathbb{T}^2})_*(\nu_\varphi)$ is the unique equilibrium state for F_{AS} and $\varphi \circ H$.

We begin noticing that any ergodic $(\sigma \times L)$-invariant Borel probability measure projects via π_1 to an ergodic σ-invariant probability measure. Besides, as $\sigma \times L$ is a C^1 Axiom A diffeomorphism and φ is Hölder, there is a unique equilibrium state of φ with respect to $\sigma \times L$, which is a Bernoulli probability measure [7] since σ is topologically mixing for both $\Lambda = $ Horseshoe and $\Lambda = \mathbb{T}^2$. By Theorem A, such a unique equilibrium state must be ν_φ. Recalling the definition of the lower Lyapunov exponent of F^{-1} along the central foliation $\mathfrak{F}$ (cf. 4.1) and taking into account that the leaves of $\mathfrak{F}$ are one-dimensional, one gets

$$\chi(x, y) = \liminf_{n \to +\infty} \frac{1}{n} \log \| D_{(x,y)} F^{-n}_{\ |E^s_{(x,y)}} \|$$

$$= \liminf_{n \to +\infty} \frac{1}{n} \log \| D_y (\pi_2 \circ F^{-n})_{\ |E^s_{(x,y)}} \|$$

$$= \liminf_{n \to +\infty} \frac{1}{n} \log \| D_y (f_x^{-1} \circ f_{\sigma(x)}^{-1} \circ \cdots \circ f_{\sigma^{n-1}(x)}^{-1})_{\ |E^s_{(x,y)}} \|$$

$$= \liminf_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \psi \big(F^{-i}(x, y) \big)$$

where $\psi(x, y) = \| D_y f_x^{-1}|_{E^s_{(x,y)}} \|$. On the other hand, applying the Ergodic Theorem of Birkhoff to the map

$$x \in \Lambda \quad \mapsto \quad \log \zeta(x) := \log \Big(\inf_{y \in \mathbb{T}^2} \psi(x, y) \Big)$$

we guarantee the existence of a full $(\pi_1)_*(\nu_\varphi)$-measure set $B \subset \Lambda$ such that for every $x \in B$ one has

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \zeta(\sigma^{-i}(x)) = \int_\Lambda \log \zeta(t) \, d(\pi_1)_*(\nu_\varphi)(t).$$

Selecting the potential φ in the set $\mathbb{P}(\Lambda \times \mathbb{T}^2)$, we are sure that $\int_\Lambda \log \zeta(t) \, d(\pi_1)_*(\nu_\varphi)(t) > 0$ and thereby

$$\chi(x, y) \geq \int_\Lambda \log \zeta(t) \, d(\pi_1)_*(\nu_\varphi)(t) > 0, \qquad \forall \, (x, y) \in B \times \mathbb{T}^2.$$

This means that $\nu_\varphi(B \times \mathbb{T}^2) = \nu_\varphi(\pi_1^{-1}(B)) = (\pi_1)_*(\nu_\varphi)(B) = 1$, which is what we needed to finish the proof of Corollary C.

8. Selection of the potentials

In what follows $\mathcal{H}(\Lambda \times \mathbb{T}^2) \subset C^0(\Lambda \times \mathbb{T}^2)$ stands for the subset of Hölder real-valued continuous maps with domain $\Lambda \times \mathbb{T}^2$. We will explain why the set $\mathbb{P}(\Lambda \times \mathbb{T}^2)$ of Hölder potentials which comply with the condition (4.2), with respect to either F_{AS} or F_{Sh}, is an open subset of $\mathcal{H}(\Lambda \times \mathbb{T}^2)$.

Proposition 8.1. *Given $\varphi \in \mathcal{H}(\Lambda \times \mathbb{T}^2)$, consider a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of maps in $\mathcal{H}(\Lambda \times \mathbb{T}^2)$ which converges in the C^0-norm to φ. Take the sequence of unique equilibrium states ν_n for $\sigma \times L$ and φ_n, and assume that the sequence $(\nu_n)_{n \in \mathbb{N}}$ converges in the weak*-topology to a Borel probability measure η in $\Lambda \times \mathbb{T}^2$. Then η is the equilibrium state ν_φ of $\sigma \times L$ and φ.*

Proof. It is immediate that the probability measure η is $(\sigma \times L)$-invariant. So, we are left to show that

$$P_{\text{top}}(\sigma \times L, \varphi) = h_\eta(\sigma \times L) + \int \varphi \, d\eta.$$

As $\sigma \times L$ is expansive, its entropy map is upper semi-continuous, so

$$\limsup_{n \to +\infty} h_{\nu_n}(\sigma \times L) \leq h_\eta(\sigma \times L).$$

On the other hand,

$$\lim_{n \to +\infty} \int \varphi_n \, d\nu_n = \int \varphi \, d\eta$$

since

$$\left| \int \varphi_n \, d\nu_n - \int \varphi \, d\eta \right| \leq \left| \int \varphi_n \, d\nu_n - \int \varphi \, d\nu_n \right| + \left| \int \varphi \, d\nu_n - \int \varphi \, d\eta \right|$$

$$\leq \|\varphi_n - \varphi\|_{C^0} + \left| \int \varphi \, d\nu_n - \int \varphi \, d\eta \right| \quad \overset{n \to +\infty}{\longrightarrow} \quad 0.$$

From [35, Theorem 9.7] and the fact that $h_{\text{top}}(\sigma \times L) < +\infty$, we conclude that the pressure operator of $\sigma \times L$ is Lipschitz with respect to the potential. Thus,

$$\lim_{n \to +\infty} P_{\text{top}}(\sigma \times L, \varphi_n) = P_{\text{top}}(\sigma \times L, \varphi)$$

and so

$$
\begin{aligned}
P_{\text{top}}(\sigma \times L, \varphi) &= \lim_{n \to +\infty} P_{\text{top}}(\sigma \times L, \varphi_n) \\
&= \limsup_{n \to +\infty} \left[h_{\nu_n}(\sigma \times L) + \int \varphi_n \, d\nu_n \right] \\
&= \limsup_{n \to +\infty} h_{\nu_n}(\sigma \times L) + \lim_{n \to +\infty} \int \varphi_n \, d\nu_n \\
&\leq h_\eta(\sigma \times L) + \int \varphi \, d\eta.
\end{aligned}
$$

Therefore, $P_{\text{top}}(\sigma \times L, \varphi) = h_\eta(\sigma \times L) + \int \varphi \, d\eta$ and η is an equilibrium state of $\sigma \times L$ and φ. By uniqueness of this equilibrium state, $\eta = \nu_\varphi$ and the proof of the proposition is complete. □

Consider now the skew product F_{AS} (the argument is similar if we use F_{Sh} instead) and the transformation

$$
\begin{aligned}
\mathcal{L} \colon \mathcal{H}(\Lambda \times \mathbb{T}^2) &\to \mathbb{R} \\
\varphi &\mapsto \int_\Lambda \log \zeta(x) \, d(\pi_1)_*(\nu_\varphi)(x).
\end{aligned}
$$

Observe that, if $\mathcal{L}$ is continuous, then the set $\mathbb{P}(\Lambda \times \mathbb{T}^2) = \mathcal{L}^{-1}(]0, +\infty[)$ is open, as claimed.

Lemma 8.2. *The map $\mathcal{L}$ is continuous.*

Proof. Take a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of maps in $\mathcal{H}(\Lambda \times \mathbb{T}^2)$ converging in the C^0-topology to $\varphi \in \mathcal{H}(\Lambda \times \mathbb{T}^2)$. Let ν_{φ_n} and ν_φ the unique equilibrium states of $\sigma \times L$ associated to φ_n and φ, respectively. By Proposition 8.1, the map

$$
\varphi \in \mathcal{H}(\Lambda \times \mathbb{T}^2) \quad \mapsto \quad \nu_\varphi \in \mathscr{P}(\Lambda \times \mathbb{T}^2, \sigma \times L)
$$

varies continuously. Moreover, as F_{AS} is C^1 and partially hyperbolic, the function

$$
x \in \Lambda \quad \mapsto \quad \zeta(x) = \inf_{y \,\in\, \mathbb{T}^2} \|D_y f_x^{-1}{}_{|E^s_{(x,y)}}\|
$$

belongs to $C^0(\Lambda)$ (see [19]), and so the map $\log(\zeta \circ \pi_1)$ belongs to $C^0(\Lambda \times \mathbb{T}^2)$. Finally, we notice that, as the sequence $(\nu_{\varphi_n})_{n \in \mathbb{N}}$ converges in the weak*-topology to ν_φ, we get

$$
\mathcal{L}(\varphi_n) = \int_\Lambda \log \zeta \, d(\pi_1)_*(\nu_{\varphi_n}) = \int_\Lambda \log(\zeta \circ \pi_1) \, d\nu_{\varphi_n}
$$

and

$$
\lim_{n \to +\infty} \int_\Lambda \log(\zeta \circ \pi_1) \, d\nu_{\varphi_n} = \int_\Lambda \log(\zeta \circ \pi_1) \, d\nu_\varphi = \mathcal{L}(\varphi).
$$

This ends the proof of the lemma. □

9. Proof of Theorem D

Recall that the neighborhood $\mathcal{V}$ of F_{AS} has been chosen so that there exists a homeomorphism $\tau_G \colon \Omega(G) \to \Omega(F_{AS})$ such that $\tau_G \circ G \circ \tau_G^{-1}$ is a skew product C^0-close to G and F_{AS}, and defined by

$$
\begin{aligned}
\tau_G \circ G \circ \tau_G^{-1} \colon \quad \Omega(\sigma) \times \mathbb{T}^2 &\to \Omega(\sigma) \times \mathbb{T}^2 \\
(x, y) &\mapsto (\sigma(x), g_x(y)).
\end{aligned}
$$

Moreover, $\tau_G \circ G \circ \tau_G^{-1}$ is normally hyperbolic to the foliation $\left\{\{x\} \times \mathbb{T}^2\right\}_{x \in \Omega(\sigma)}$ and plaque expansive with respect to the center foliation. From the results of [23], the skew product $\tau_G \circ G \circ \tau_G^{-1}$ satisfies the properties (1)-(3) of Section 2. So, from Lemma 2.1, we may find

a continuous surjective map $H_G \colon \Omega(F_{AS}) \to \Omega(\sigma \times L)$ such that $H_G \circ (\tau_G \circ G \circ \tau_G^{-1}) = (\sigma \times L) \circ H_G$. Hence, Theorem A may be applied to $\tau_G \circ G \circ \tau_G^{-1}$, and so

$$P_{\mathrm{top}}(G, \, \varphi \circ H_G \circ \tau_G) = P_{\mathrm{top}}(\tau_G \circ G \circ \tau_G^{-1}, \, \varphi \circ H_G) = P_{\mathrm{top}}(\sigma \times L, \, \varphi). \qquad (9.1)$$

In particular, we deduce that the topological entropy is constant in a neighborhood of F_{AS}. We will now check condition (4.2) for $G \in \mathcal{V}$. Given a basic set Λ of $\Omega(\sigma)$, the following diagrams commute

$$
\begin{array}{ccc}
\tau_G^{-1}(\Lambda \times \mathbb{T}^2) & \xrightarrow{\;G\;} & \tau_G^{-1}(\Lambda \times \mathbb{T}^2) \\
\downarrow{\scriptstyle \tau_G} & & \downarrow{\scriptstyle \tau_G} \\
\Lambda \times \mathbb{T}^2 & \xrightarrow{\;\tau_G \circ G \circ \tau_G^{-1}\;} & \Lambda \times \mathbb{T}^2 \\
\downarrow{\scriptstyle H_G} & & \downarrow{\scriptstyle H_G} \\
\Lambda \times \mathbb{T}^2 & \xrightarrow{\;\sigma \times L\;} & \Lambda \times \mathbb{T}^2
\end{array}
$$

and the condition (4.2), which is C^0-open with respect to the dynamics, is also valid for $\tau_G \circ G \circ \tau_G^{-1}$. Indeed, as G is C^1-near F_{AS}, the homeomorphism τ_G is C^0-near the inclusion of $\Omega(G)$ in $\mathbb{S}^2 \times \mathbb{T}^2$ and the center foliations for $\tau_G \circ G \circ \tau_G^{-1}$ and F_{AS} are also C^0-close. Therefore, the map ζ for $\tau_G \circ G \circ \tau_G^{-1}$ still satisfies condition (4.2) of Section 7 with respect to the reference measure ν_φ (which is intrinsic to the dynamics $\sigma \times L$ and the potential φ, and does not depend on G nor on τ_G). Thus, Corollary C is also valid for $\tau_G \circ G \circ \tau_G^{-1}$. Consequently, if we fix a basic set Λ of σ, the uniqueness of equilibrium states for the restriction of $\tau_G \circ G \circ \tau_G^{-1}$ to $\Lambda \times \mathbb{T}^2$ may be obtained from Theorem B, and conveyed to G through the conjugacy τ_G.

10. Proof of Corollary E

Here we will analyze the statistical stability of the equilibrium states of F_{AS}. The argument may easily be reformulated for F_{Sh}, whose main difference to F_{AS} is the fact that $\Omega(F_{Sh})$ has a unique basic piece, which is a manifold. To simplify the presentation, firstly we will vary the potential, then the dynamics, and only afterwards will we blend these two perturbations.

Lemma 10.1. *Let $G_n \in \mathcal{V}$ be a sequence of diffeomorphisms converging to F_{AS} in the C^1-topology and let $(\eta_n)_{n \in \mathbb{N}}$ be a sequence of G_n-invariant probability measures which converges to a probability measure η in the weak*-topology. Then η is F_{AS}-invariant.*

Proof. We need to show that, given a continuous map $\psi : \Lambda \times \mathbb{T}^2 \to \mathbb{R}$, then

$$\int (\psi \circ F_{AS}) \, d\eta = \int \psi \, d\eta.$$

As $(\eta_n)_{n \in \mathbb{N}}$ converges in the weak*-topology to η, then

$$\lim_{n \to +\infty} \int \psi \, d\eta_n = \int \psi \, d\eta.$$

By the G_n-invariance of η_n, we also have

$$\int (\psi \circ G_n) \, d\eta_n = \int \psi \, d\eta_n, \quad \forall \, n \in \mathbb{N}.$$

Besides,

$$\lim_{n \to +\infty} \int (\psi \circ G_n) \, d\eta_n = \int (\psi \circ F_{AS}) \, d\eta$$

since

$$\left| \int (\psi \circ G_n) \, d\eta_n - \int (\psi \circ F_{AS}) \, d\eta \right|$$

$$\leq \quad \left| \int (\psi \circ G_n) \, d\eta_n - \int (\psi \circ F_{AS}) \, d\eta_n \right| + \left| \int (\psi \circ F_{AS}) \, d\eta_n - \int (\psi \circ F_{AS}) \, d\eta \right|$$

$$\leq \quad \|(\psi \circ G_n) - (\psi \circ F_{AS})\|_{C^0} + \left| \int (\psi \circ F_{AS}) \, d\eta_n - \int (\psi \circ F_{AS}) \, d\eta \right|$$

and the first term of the previous estimate goes to 0 due to the uniform continuity of ψ and the C^0-convergence of $(G_n)_{n \in \mathbb{N}}$ to F, while the second goes to 0 because $(\mu_n)_{n \in \mathbb{N}}$ converges in the weak*-topology to μ_0.   □

Having fixed a basic set Λ in $\Omega(\sigma)$, let us now perturb the potential in $\mathbb{P}(\Lambda \times \mathbb{T}^2)$.

Lemma 10.2. *Given a basic set Λ of σ and $\varphi \in \mathbb{P}(\Lambda \times \mathbb{T}^2)$, consider a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of maps in $\mathbb{P}(\Lambda \times \mathbb{T}^2)$ which converges in the C^0-norm to φ. Take the sequence of unique equilibrium states μ_n for F_{AS} and $\varphi_n \circ H$, and assume that $(\mu_n)_{n \in \mathbb{N}}$ converges in the weak*-topology to a probability measure η in $\Lambda \times \mathbb{T}^2$. Then η is the equilibrium state $\mu_{\varphi \circ H}$ of F_{AS} and $\varphi \circ H$.*

Proof. We already know (cf. Lemma 10.1) that η is F_{AS}-invariant, so we are left to show that

$$P_{\text{top}}(F_{AS}, \varphi \circ H) = h_\eta(F_{AS}) + \int (\varphi \circ H) \, d\eta.$$

As F_{AS} is entropy-expansive in $\Lambda \times \mathbb{T}^2$, the entropy map of F_{AS} is upper semi-continuous, so

$$\limsup_{n \to +\infty} h_{\mu_n}(F_{AS}) \leq h_\eta(F_{AS}).$$

Besides, as $\lim_{n \to +\infty} \mu_n = \eta$ and $\lim_{n \to +\infty} \varphi_n = \varphi$,

$$\lim_{n \to +\infty} \int (\varphi_n \circ H) \, d\mu_n = \int (\varphi \circ H) \, d\eta.$$

So, by [35, Theorem 9.7], $\lim_{n\to+\infty} P_{\text{top}}(F_{AS}, \varphi_n \circ H) = P_{\text{top}}(F_{AS}, \varphi \circ H)$. Hence

$$
\begin{aligned}
P_{\text{top}}(F_{AS}, \varphi \circ H) &= \lim_{n\to+\infty} P_{\text{top}}(F_{AS}, \varphi_n \circ H) \\
&= \limsup_{n\to+\infty} \left[h_{\mu_n}(F_{AS}) + \int (\varphi_n \circ H)\, d\mu_n \right] \\
&= \limsup_{n\to+\infty} h_{\mu_n}(F_{AS}) + \lim_{n\to+\infty} \int (\varphi_n \circ H)\, d\mu_n \\
&\leq h_\eta(F_{AS}) + \int (\varphi \circ H)\, d\eta.
\end{aligned}
$$

Therefore, η is an equilibrium state of F_{AS} and $\varphi \circ H$. And, by Corollary C, this is unique. $\qquad\square$

Let us now vary simultaneously the diffeomorphism and the potential.

Proposition 10.3. *Given a basic set Λ of σ and a potential $\varphi \in \mathbb{P}(\Lambda \times \mathbb{T}^2)$, consider a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of maps in $\mathbb{P}(\Lambda \times \mathbb{T}^2)$ converging in the C^0-norm to φ. Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of diffeomorphisms in $\mathcal{V}$ converging in the C^1-topology to F_{AS}. Take the unique equilibrium state η_n for G_n and the potential $\varphi_n \circ H_{G_n} \circ \tau_{G_n}$, and suppose that the sequence $(\eta_n)_{n\in\mathbb{N}}$ converges in the weak*-topology to a probability measure η in $\Lambda \times \mathbb{T}^2$. Then η is the equilibrium state $\mu_{\varphi\circ H}$ of F_{AS} and $\varphi \circ H$.*

Proof. We already know that

$$
P_{\text{top}}(G_n, \varphi_n \circ H_{G_n} \circ \tau_{G_n}) = P_{\text{top}}(\sigma \times L, \varphi_n) = P_{\text{top}}(F_{AS}, \varphi_n \circ H)
$$

and

$$
\lim_{n\to+\infty} P_{\text{top}}(F_{AS}, \varphi_n \circ H) = \lim_{n\to+\infty} P_{\text{top}}(F_{AS}, \varphi \circ H)
$$

where the first two equalities come from (9.1) and the last one is a direct consequence of [35, Theorem 9.7]. Moreover, $\nu_{\varphi_n} = (H_{G_n} \circ \tau_{G_n})_*(\eta_n)$ is the unique equilibrium state of $\sigma \times L$ and φ_n (cf. Theorem A); and the sequence $(\nu_{\varphi_n})_{n\in\mathbb{N}}$ converges to $H_*(\eta)$ since

Lemma 10.4. [29, Lemma 1] *and* [23, §8]

    (a) $\lim_{n\to+\infty} \| \tau_{G_n} - identity \|_{C^0} = 0$.
    (b) $\lim_{n\to+\infty} \| H_{G_n} - H \|_{C^0} = 0$.

Thus, by Proposition 8.1, $H_*(\eta) = \nu_\varphi$, the unique equilibrium state of $\sigma \times L$ and φ. Besides, by Corollary C, $\mu_\varphi = (H^{-1}_{|_{B\times\mathbb{T}^2}})_*(\nu_\varphi)$ is the unique equilibrium state of F_{AS} and $\varphi \circ H$. Finally,

$$
\mu_\varphi = \left(H^{-1}_{|_{B\times\mathbb{T}^2}} \right)_* (\nu_\varphi) = \left(H^{-1}_{|_{B\times\mathbb{T}^2}} \right)_* \left(H_{|_{B\times\mathbb{T}^2}} \right)_* (\eta) = \eta.
$$

$\qquad\square$

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Maria Carvalho, Centro de Matemática da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal

E-mail address: mpcarval@fc.up.pt

Sebastián Pérez Opazo, Centro de Matemática da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal

E-mail address: sebastian.opazo@fc.up.pt