Total quantum state in the Einstein–Podolsky–Rosen–Bohm experiment with identical particles

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Abstract

The spin state in the Einstein–Podolsky–Rosen–Bohm *gedankenexperiment* with identical particles is supplemented by the spatial part. This allows one to extract all the information needed in a typical EPR argument, without requiring semi-intuitive steps. Local spin operators are introduced, to describe measurements of spin in given regions of space.

1 Introduction

For a long time, the Einstein–Podolsky–Rosen (EPR) argument [1] has been relegated to the specialised literature on the conceptual foundations of quantum theory. In particular, it is seldom mentioned in textbooks written until the end of the eighties — a remarkable exception being the book by David Bohm [2], which contains a detailed, in-depth analysis of the EPR *gedankenexperiment*. In the last three decades, however, a resurgence of interest in quantum foundations,[1] and the development of the new field of quantum information technology [16],

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[1]For a review of the literature about conceptual issues in quantum theory up to 1992, see [3]. For a sample of more recent papers, some written with an eye towards physics education, see [4–15] and references therein.

have led an increasing number of authors to include a discussion of the EPR argument and of its logical follow-up, Bell's theorem [17], in textbooks at various levels.

The EPR experiment is invariably discussed in the version due to Bohm [2], which considers correlations among spins (the so-called EPRB experiment). Since it requires only a two-dimensional Hilbert space, this is technically simpler than the original presentation in the EPR paper [1], but the conclusions are the same. Sometimes, instead of spins, one considers photon polarisations, which have been used in the majority of real experiments, such as those performed by Aspect's team [18–20].[2]

The typical textbook presentation relies upon the singlet state for a pair of spin-1/2 particles sent in opposite directions to two non-overlapping spatial regions A and B,

$$\frac{1}{\sqrt{2}} \left(|+\rangle|-\rangle - |-\rangle|+\rangle \right), \tag{1.1}$$

where $|+\rangle$ and $|-\rangle$ are eigenstates of the z-component for spin, with eigenvalues $1/2$ and $-1/2$, respectively (in units of $\hbar$). Since this state is rotationally symmetric and exhibits perfect anti-correlations, a measurement of the same component of spin on the two particles always results in opposite values.

This conclusion is usually applied, implicitly, to spin measurements *performed in the two regions*. However, this is not what the state (1.1) actually predicts. Indeed, the state (1.1) does not say anything related to the two regions — the vectors $|+\rangle$ and $|-\rangle$ belong to the Hilbert space associated with the particle spin, not to the space related to the particle location (whose elements are the so-called spatial, or orbital states, described by the Schrödinger wavefunctions, in the position representation). This point becomes even more obvious making the particle labels in the state (1.1) explicit:

$$\frac{1}{\sqrt{2}} \left(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2 \right). \tag{1.2}$$

This implies perfect anti-correlations among spin measurements *performed on particle 1 and on particle 2*, not among those *performed in region A and in region B* — an important point, because in a real experiment it is the spin of the particle in a region that is measured, not the spin of particle 1 or particle 2. In order to conclude that the spin measurements in A and B are also anti-correlated, one needs the essential information that each region contains exactly one particle, which is crucial to establish the correspondence between measurements performed on the particles and measurements performed in the regions. The state (1.2) does not contain, however, such information, that must somehow be added by hand.

All this is not really an issue if one assumes that the two particles can be distinguished and that each particle ends up into a well-defined region. In this case, whatever one can say about a given particle translates into a corresponding statement about a spatial region. However, some presentations of the EPR argument say explicitly that particles 1 and 2 are of the same type (usually, electrons or protons). Then, the lack of any reference to regions A and B in the

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[2]For a comparison between the EPRB arrangement and experiments performed using photons and kaons, see [21].

state (1.2) is indeed a problem, because for indistinguishable particles, a measurement of the spin of particle 1 or 2 is not even a well-defined operation.

It seems, therefore, that the standard textbook description is incomplete, and does not address this specific point adequately. It would be pointless to just blame the presentations which use identical particles for introducing an unnecessary and problematic assumption, because the EPR argument *can* indeed be formulated even if the particles are indistinguishable, as will be shown. The purpose of this note is thus not to correct some mistake in the literature, but rather to fill a gap in the logic, that represents a difficulty for several students. As such, it can be considered an addendum to the expositions found in textbooks, written having in mind students and instructors as main readership.

Including the spatial part of the state vector in an EPRB experiment provides the missing step required to treat the case of identical particles. As shown in section 2, this allows one to make statements that refer exactly to what is actually measured, without the need to invoke semi-intuitive arguments. In section 3, operators appropriate to describe the spin in a region, rather than the spin of a given particle, are presented. Section 4 contains some final comments.

2 Total quantum state

In this section, which represents the core of the paper, we construct a state appropriate for dealing with the EPRB experimental setup with indistinguishable particles, including explicitly the spatial part. For greater clarity, the discussion is divided into four parts. In section 2.1 we establish the basic notation when there is a single particle with spin 1/2, while in section 2.2 we write the state for two identical particles that can occupy different regions. In section 2.3 the state is further specialised to meet the requirements of the EPR argument. Finally, in section 2.4 we observe that the EPRB state can be factorised into a spatial and a spin part, which provides a partial explanation (but does not justify) why the former is commonly ignored.

2.1 One particle

The Hilbert space for a particle with spin 1/2 is $\mathcal{H} = \mathcal{H}^{(\mathrm{spatial})} \otimes \mathcal{H}^{(\mathrm{spin})}$, where $\mathcal{H}^{(\mathrm{spatial})}$ and $\mathcal{H}^{(\mathrm{spin})}$ are the spaces or spatial and spin states, respectively. For the present discussion, we can assume that $\mathcal{H}^{(\mathrm{spatial})}$ is spanned by only two states $|A\rangle$ and $|B\rangle$, corresponding to the two distinct, non-overlapping spatial regions A and B, so in particular $\langle A|B\rangle = 0$. (For instance, we can take the wavefunctions associated with $|A\rangle$ and $|B\rangle$ to be the characteristic functions of the regions, normalised dividing by the square root of the corresponding volume.) Moreover, let us denote by $|+\rangle$ and $|-\rangle$ the normalised vectors in $\mathcal{H}^{(\mathrm{spin})}$ corresponding to the positive and negative eigenvalues of the spin component along z. That is, if $\hat{S}_z$ is the corresponding operator acting on $\mathcal{H}^{(\mathrm{spin})}$,

$$\hat{S}_z|\pm\rangle = \pm\frac{1}{2}|\pm\rangle. \tag{2.1}$$

Then, the four states $|A,+\rangle$, $|A,-\rangle$, $|B,+\rangle$, $|B,-\rangle$, where $|A,+\rangle := |A\rangle \otimes |+\rangle$, *etc.*, form an orthonormal basis for $\mathcal{H}$. Note that calling, with some abuse of notation, $\hat{S}_z$ the operator in $\mathcal{H}$

that acts on $\mathcal{H}^{(\text{spatial})}$ as the identity and on $\mathcal{H}^{(\text{spin})}$ as in equation (2.1), we have:

$$\hat{S}_z|A,\pm\rangle = \pm\frac{1}{2}|A,\pm\rangle\,; \quad \hat{S}_z|B,\pm\rangle = \pm\frac{1}{2}|B,\pm\rangle\,. \tag{2.2}$$

Similar equations are easily established for the operators $\hat{S}_x$ and $\hat{S}_y$, corresponding to the x- and y-components of spin:

$$\hat{S}_x|A,\pm\rangle = \frac{1}{2}|A,\mp\rangle\,; \quad \hat{S}_x|B,\pm\rangle = \frac{1}{2}|B,\mp\rangle\,; \tag{2.3}$$

$$\hat{S}_y|A,\pm\rangle = \pm\frac{\mathrm{i}}{2}|A,\mp\rangle\,; \quad \hat{S}_y|B,\pm\rangle = \pm\frac{\mathrm{i}}{2}|B,\mp\rangle\,. \tag{2.4}$$

2.2 Two particles

Consider, now, a system of two particles 1 and 2 with spin $1/2$. The total Hilbert space is now $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, and an orthonormal basis for it can be obtained considering the sixteen vectors formed by products like, *e.g.*, $|A,+\rangle_1 \otimes |A,-\rangle_2$, where $|A,+\rangle_1 \in \mathcal{H}_1$ and $|A,-\rangle_2 \in \mathcal{H}_2$. For notational convenience, we shall henceforth drop the tensor product symbol and the subscripts "1" and "2", writing for instance $|A,+\rangle_1 \otimes |A,-\rangle_2 =: |A,+\rangle|A,-\rangle$. This simplification does not entail a risk of ambiguity, because the first and the second vectors always belong to the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Such basis vectors can be organised, for convenience, as in the following 4×4 table:

	1	2	3	4
1	$|A,+\rangle|A,+\rangle$	$|A,+\rangle|A,-\rangle$	$|A,+\rangle|B,+\rangle$	$|A,+\rangle|B,-\rangle$
2	$|A,-\rangle|A,+\rangle$	$|A,-\rangle|A,-\rangle$	$|A,-\rangle|B,+\rangle$	$|A,-\rangle|B,-\rangle$
3	$|B,+\rangle|A,+\rangle$	$|B,+\rangle|A,-\rangle$	$|B,+\rangle|B,+\rangle$	$|B,+\rangle|B,-\rangle$
4	$|B,-\rangle|A,+\rangle$	$|B,-\rangle|A,-\rangle$	$|B,-\rangle|B,+\rangle$	$|B,-\rangle|B,-\rangle$

$$\tag{2.5}$$

We can then write, for the arbitrary vector $|\Psi\rangle \in \mathcal{H}$:

$$|\Psi\rangle = c_{11}\,|A,+\rangle|A,+\rangle + c_{12}\,|A,+\rangle|A,-\rangle + \ldots + c_{44}\,|B,-\rangle|B,-\rangle, \tag{2.6}$$

where c_{ij} are complex coefficients such that

$$\sum_{i,j=1}^{4} |c_{ij}|^2 = 1\,. \tag{2.7}$$

If the two particles are identical, antisymmetry requires that $\hat{P}|\Psi\rangle = -|\Psi\rangle$, where $\hat{P}$ is the exchange operator [22]. Since

$$\hat{P}|A,+\rangle|A,+\rangle = |A,+\rangle|A,+\rangle,$$
$$\hat{P}|A,+\rangle|A,-\rangle = |A,-\rangle|A,+\rangle, \tag{2.8}$$
$$\hat{P}|A,+\rangle|B,+\rangle = |B,+\rangle|A,+\rangle,$$

etc., this implies that the matrix with elements c_{ij} must be antisymmetric; that is, $c_{ij} = -c_{ji}$. In particular, $c_{11} = c_{22} = c_{33} = c_{44} = 0$. The vectors

$$|A,+;A,-\rangle := \frac{1}{\sqrt{2}} \left(|A,+\rangle|A,-\rangle - |A,-\rangle|A,+\rangle \right), \tag{2.9}$$

$$|A,+;B,+\rangle := \frac{1}{\sqrt{2}} \left(|A,+\rangle|B,+\rangle - |B,+\rangle|A,+\rangle \right), \tag{2.10}$$

$$|A,+;B,-\rangle := \frac{1}{\sqrt{2}} \left(|A,+\rangle|B,-\rangle - |B,-\rangle|A,+\rangle \right), \tag{2.11}$$

$$|A,-;B,+\rangle := \frac{1}{\sqrt{2}} \left(|A,-\rangle|B,+\rangle - |B,+\rangle|A,-\rangle \right), \tag{2.12}$$

$$|A,-;B,-\rangle := \frac{1}{\sqrt{2}} \left(|A,-\rangle|B,-\rangle - |B,-\rangle|A,-\rangle \right), \tag{2.13}$$

$$|B,+;B,-\rangle := \frac{1}{\sqrt{2}} \left(|B,+\rangle|B,-\rangle - |B,-\rangle|B,+\rangle \right), \tag{2.14}$$

form an orthonormal basis in the Hilbert space of antisymmetric states for the system, and the state can be written as

$$|\Psi\rangle = \alpha_{12}|A,+;A,-\rangle + \alpha_{13}|A,+;B,+\rangle + \alpha_{14}|A,+;B,-\rangle$$
$$+ \alpha_{23}|A,-;B,+\rangle + \alpha_{24}|A,-;B,-\rangle + \alpha_{34}|B,+;B,-\rangle. \tag{2.15}$$

where $\alpha_{ij} := \sqrt{2}\, c_{ij}$.

It is worth commenting on the meaning of the states (2.9)–(2.14). All the states of the basis (2.5) admit an interpretation where particle 1 is in a definite region with a definite z-component for spin, and particle 2 is also in a definite region with a definite z-component for spin (for instance, $|A,+\rangle|B,-\rangle$ corresponds to particle 1 in A with positive spin component, and particle 2 in B with negative spin component). The states (2.9)–(2.14) are entirely different, because they carry information only about the regions occupied by the particles, and the correlations between the region and the spin component, but do not refer in any way to the individual particles — a feature that is only expected, after all, for indistinguishable particles. For example, in the state $|A,+;B,-\rangle$ we just know that there is a particle with positive spin component in region A, and a particle with negative spin component in region B. Technically, this happens because all the vectors (2.9)–(2.14) represent entangled states for the two particles.

2.3 EPRB state

Let us now come to the requirements that are specific of the typical EPR state. We do not want both particles to be in the same region, so the coefficients in front of basis vectors containing A or B twice must vanish; that is, $\alpha_{12} = \alpha_{34} = 0$. Moreover, requiring that the total spin along the z-direction be equal to zero, we must have $\hat{S}_z |\Psi\rangle = 0$, where $\hat{S}_z = \hat{S}_z^{(1)} \otimes \hat{1}^{(2)} + \hat{1}^{(1)} \otimes \hat{S}_z^{(2)}$, with obvious notation. Acting on the various states on the right-hand side of equation (2.15) with $\hat{S}_z^{(1)}$ and $\hat{S}_z^{(2)}$, and using equations (2.2), we find $\alpha_{13} = \alpha_{24} = 0$, so we remain with

$$|\Psi\rangle = a\,|A, +; B, -\rangle + b\,|A, -; B, +\rangle, \tag{2.16}$$

where $a := \alpha_{14}$ and $b := \alpha_{23}$ satisfy the condition $|a|^2 + |b|^2 = 1$.

The state (2.16) predicts perfect anti-correlations among measurements of the z-component of spin *performed in A and B*. This is satisfactory, because a real measurement of spin takes place in the region A or B, not on particle 1 or particle 2 (as we already said in section 1, for indistinguishable particles this would not even make sense). The state (2.16) is, however, more general than the one appropriate to deal with the EPRB experimental setup. In order to obtain the EPRB state, let us require that also the spin along another direction (say, x) vanishes. That is, $\hat{S}_x |\Psi\rangle = 0$, where $\hat{S}_x = \hat{S}_x^{(1)} \otimes \hat{1}^{(2)} + \hat{1}^{(1)} \otimes \hat{S}_x^{(2)}$ is the x-component of total spin. Applying equations (2.3), this implies that $b = -a$, so the state is

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \big(|A, +; B, -\rangle - |A, -; B, +\rangle \big), \tag{2.17}$$

up to an irrelevant global phase. It is easy to check, using equations (2.4), that also $\hat{S}_y |\Psi\rangle = 0$, where $\hat{S}_y = \hat{S}_y^{(1)} \otimes \hat{1}^{(2)} + \hat{1}^{(1)} \otimes \hat{S}_y^{(2)}$.

2.4 Factorisation

The EPRB state vector (2.17) is such that a measurement of the spin component along an arbitrary direction in both regions, always gives opposite results — a fundamental ingredient of the EPR argument.

This property is most easily seen noticing that the state (2.17) can be factorised into a spatial part and a spin part,[3]

$$|\Psi\rangle = |\text{spatial}\rangle|\text{spin}\rangle, \tag{2.18}$$

where

$$|\text{spatial}\rangle = \frac{1}{\sqrt{2}} \big(|A\rangle_1 |B\rangle_2 + |B\rangle_1 |A\rangle_2 \big) \tag{2.19}$$

and

$$|\text{spin}\rangle = \frac{1}{\sqrt{2}} \big(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2 \big) \tag{2.20}$$

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[3]We are now writing the state as an element of $\mathcal{H}_1^{(\text{spatial})} \otimes \mathcal{H}_2^{(\text{spatial})} \otimes \mathcal{H}_1^{(\text{spin})} \otimes \mathcal{H}_2^{(\text{spin})}$, rather than of $\mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{H}_1^{(\text{spatial})} \otimes \mathcal{H}_1^{(\text{spin})} \otimes \mathcal{H}_2^{(\text{spatial})} \otimes \mathcal{H}_2^{(\text{spin})}$. This reshuffling is harmless, because the two representations are equivalent, as it is obvious if one chooses to represent single-particle spatial states by wavefunctions, and spin states by two-component columns.

(particle labels have been reintroduced for greater clarity). The perfect anti-correlation between spin measurements along an arbitrary direction then follows from the rotational symmetry of the spin state (2.20).

The state (2.20) coincides with the "textbook state" (1.2), but now it is only a part of the total state $|\Psi\rangle$, and gives the physically relevant information only when it is supplemented by the spatial state (2.19). For instance, we can write, for the probability $P(A, -; B, +)$ to get the outcomes $-1/2$ at A and $1/2$ at B:

$$P(A, -; B, +) = P(1, -; 2, +) P(1, A; 2, B) + P(1, +; 2, -) P(1, B; 2, A), \qquad (2.21)$$

where $P(1, -; 2, +)$ is the probability that particle 1 has spin $-1/2$ and particle 2 has spin $1/2$, $P(1, A; 2, B)$ is the probability that particle 1 is in A and particle 2 is in B, and the other probabilities are defined in a similar way. These probabilities can all be found using either equation (2.19) or equation (2.20). For instance:

$$P(1-; 2, +) = |\langle \text{spin}|-\rangle_1 |+\rangle_2|^2 = \frac{1}{2} ; \qquad (2.22)$$

$$P(1, A; 2, B) = |\langle \text{spatial}|A\rangle_1 |B\rangle_2|^2 = \frac{1}{2} . \qquad (2.23)$$

Substituting all the factors in equation (2.21), one finds $P(A, -; B, +) = 1/2$. Of course, this is most easily computed working with the total state $|\Psi\rangle$:

$$P(A, -; B, +) = |\langle \Psi|A, -; B, +\rangle|^2 = \frac{1}{2} . \qquad (2.24)$$

The important point is, however, that one cannot obtain this probability from the state (1.2) alone, which yields only probabilities like the one in equation (2.22).

3   Local spin operators

We have said several times that what is measured in an EPRB setting is the spin at the regions A and B. Although it is intuitively clear what this means, it is appropriate to provide the corresponding formal counterpart — Hermitian operators $\hat{S}_z^{(A)}$ and $\hat{S}_z^{(B)}$ for which (2.9)–(2.14) are eigenvectors with appropriate eigenvalues. These can be called *local spin operators*,[4] as opposed to $\hat{S}_z^{(1)} \otimes \hat{1}^{(2)}$ and $\hat{1}^{(1)} \otimes \hat{S}_z^{(2)}$, which correspond to the z-component of spin for particles 1 and 2, respectively, whose spatial location is ill-defined in a state of the type (2.16).

The local spin operators are

$$\hat{S}_z^{(A)} := |A\rangle\langle A| \otimes \hat{S}_z^{(1)} \otimes \hat{1}^{(2)} + \hat{1}^{(1)} \otimes |A\rangle\langle A| \otimes \hat{S}_z^{(2)} \qquad (3.1)$$

and

$$\hat{S}_z^{(B)} := |B\rangle\langle B| \otimes \hat{S}_z^{(1)} \otimes \hat{1}^{(2)} + \hat{1}^{(1)} \otimes |B\rangle\langle B| \otimes \hat{S}_z^{(2)}. \qquad (3.2)$$

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[4]The concept of local spin is useful when dealing with the spatial distribution of the electrons in a molecule [23].

Indeed, the states (2.9)–(2.14) are all eigenvectors of these operators corresponding, respectively, to the eigenvalues: 0, 0; 1/2, 1/2; 1/2, −1/2; −1/2, 1/2; −1/2, −1/2; 0, 0. Note that

$$\left[\hat{S}_z^{(A)}, \hat{S}_z^{(B)}\right] = \hat{0} \,, \tag{3.3}$$

as expected, given that these operators have common eigenvectors that form a basis.

Of course, one could also construct operators $\hat{S}_i^{(A)}$ and $\hat{S}_i^{(B)}$ for the generic i-th component of spin in the two regions. Remarkably,

$$\left[\hat{S}_i^{(A)}, \hat{S}_j^{(B)}\right] = \hat{0} \,, \qquad \forall\, i, j \,, \tag{3.4}$$

and one can find common eigenstates for every pair of operators $\hat{S}_i^{(A)}$ and $\hat{S}_j^{(B)}$ associated with different regions. Within each single region, spin behaves in the usual way: For arbitrary components i and j,

$$\left[\hat{S}_i^{(A)}, \hat{S}_j^{(A)}\right] = i\, \varepsilon_{ijk}\, \hat{S}_k^{(A)}, \quad \left[\hat{S}_i^{(B)}, \hat{S}_j^{(B)}\right] = i\, \varepsilon_{ijk}\, \hat{S}_k^{(B)}, \tag{3.5}$$

where ε_{ijk} is the Levi-Civita alternating symbol, and a sum over the repeated index k is understood.

4 Final comments

We have seen that including the spatial part of the state vector is mandatory if one wants to give a precise formulation of the EPR argument, which involves measurements performed in different regions, that holds for indistinguishable particles. Specifying only the spin part of the state as in equation (1.2), is not enough from a formal point of view, because such a state is compatible with other total states describing a system of two identical particles with spin 1/2. For instance,

$$|B, +; B, -\rangle = |B\rangle_1 |B\rangle_2 \frac{1}{\sqrt{2}} \left(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2\right) \tag{4.1}$$

shares the same spin state as (2.17), but of course cannot be used for the EPR argument, or to find a contradiction with locality from the violation of Bell's inequality, because it describes two particles that are both in the same region. The information about the spatial location must be added by hand if one uses the state vector (1.2), whereas the state (2.17) allows one to extract all the relevant predictions applying automatically the quantum-mechanical rules.

Perhaps, the reasons why the spatial part is usually omitted in the textbook presentations, are that (2.17) is the *only* state (up to a global phase) that satisfies all the requirements needed to run the EPR argument, and that such a state can in fact be factorised as in equation (2.18). Indeed, it is only for $a = \pm b$ that a state like (2.16) can be factorised into the product of a spatial and a spin state. It is because the EPRB state vector (2.17) is of this very peculiar type, that one tends to ignore the spatial part and represent the state simply as (1.1). The state corresponding to the choice $b = a$ is

$$\frac{1}{\sqrt{2}}\left(|A, +; B, -\rangle + |A, -; B, +\rangle\right), \tag{4.2}$$

for which

$$|\text{spatial}\rangle = \frac{1}{\sqrt{2}} \left(|A\rangle_1 |B\rangle_2 - |B\rangle_1 |A\rangle_2 \right) \tag{4.3}$$

and

$$|\text{spin}\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle_1 |-\rangle_2 + |-\rangle_1 |+\rangle_2 \right). \tag{4.4}$$

Although the state (4.2) is also entangled, it is not an eigenstate of $\hat{S}_x$ and $\hat{S}_y$, so one cannot use it for the purposes of the EPR argument. Other states, like

$$|A, +; B, +\rangle = \frac{1}{\sqrt{2}} \left(|A\rangle_1 |B\rangle_2 - |B\rangle_1 |A\rangle_2 \right) |+\rangle_1 |+\rangle_2 \,, \tag{4.5}$$

are entangled only in their spatial part. Again, this is not enough to formulate the EPR argument and to apply Bell's theorem.

Summarising, the authors of textbooks and papers that use identical particles when discussing the EPR argument do not draw incorrect conclusions, because they consider measurements at a region rather than measurements on a given particle. This is possible, however, only because of the reasons discussed in this article, which are normally not included in the presentation. Remarkably, a similar analysis is required even for distinguishable particles, if their spatial state happens to be entangled.

Acknowledgements

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