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Fusion systems on a Sylow 3-subgroup of the McLaughlin group

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Abstract. We determine all saturated fusion systems $\mathcal{F}$ on a Sylow 3-subgroup of the sporadic McLaughlin group that do not contain any non-trivial normal 3-subgroup and show that they are all realizable.

1 Introduction

Let p be a prime and S a finite p-group. A *fusion system* $\mathcal{F}$ on S is a category whose set $\mathrm{Ob}(\mathcal{F})$ of objects is the set of all subgroups of S, and, for Q and R in $\mathrm{Ob}(\mathcal{F})$, the set $\mathrm{Hom}_{\mathcal{F}}(Q,R)$ of morphisms from Q to R is a set of injective group homomorphisms $Q \to R$ (with composition of morphisms given by the usual composition of maps) such that, for every P, Q and R in $\mathrm{Ob}(\mathcal{F})$,

(FS1) $\mathrm{Hom}_{\mathcal{F}}(S,S)$ contains $\mathrm{Inn}(S)$,

(FS2) if $Q \leq P$ and $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,R)$, then

$$\phi_{|Q} \in \mathrm{Hom}_{\mathcal{F}}(Q,Q^{\phi}) \cap \mathrm{Hom}_{\mathcal{F}}(Q,R),$$

(FS3) if $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ is an isomorphism, then $\phi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(R,Q)$.

The elements of $\mathrm{Hom}_{\mathcal{F}}(R,Q)$ are called $\mathcal{F}$-morphisms. For $x \in S$, denote by c_x the automorphism of S induced by conjugation with x. For $P \in \mathrm{Ob}(\mathcal{F})$, set

$$\mathrm{Aut}_{\mathcal{F}}(P) := \mathrm{Hom}_{\mathcal{F}}(P,P) \quad \text{and} \quad \mathrm{Aut}_S(P) := \{c_{x|P} \mid x \in N_S(P)\},$$

and, for $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$, set

$$N_{\phi} := \{g \in N_S(Q) \mid \text{there exists } h \in N_S(R) \text{ with } q^{c_g\phi} = q^{\phi c_h} \text{ for every } q \in Q\}.$$

A fusion system $\mathcal{F}$ is said to be *saturated* if the following two conditions hold:

(S1) $\mathrm{Aut}_S(S)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$;

(S2) if $P \leq S$ is such that, for every $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P,S)$, $|N_S(P)| \geq |N_S(P^{\alpha})|$, then every $\phi \in \mathrm{Aut}_{\mathcal{F}}(P)$ extends to N_{ϕ}.

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If S is a Sylow p-subgroup of a finite group G, denote by $\mathcal{F}_S(G)$ the category whose objects are all subgroups of S and whose morphisms are the homomorphisms induced by conjugation in G. $\mathcal{F}_S(G)$ is a saturated fusion system on S [7, Theorem 4.12], and a fusion system $\mathcal{F}$ is called *realizable* if $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G (where, by our definition of $\mathcal{F}_S(G)$, S is a Sylow p-subgroup of G).

Let $\mathcal{F}$ be a fusion system on S and H a normal subgroup of S. H is called *normal in* $\mathcal{F}$ if, for every Q and R in $\mathrm{Ob}(\mathcal{F})$ and $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$, ϕ can be extended to a map $\bar{\phi} \in \mathrm{Hom}_{\mathcal{F}}(HQ, HR)$ such that $\bar{\phi}_{|H}$ is an automorphism of H (see [3, Definition I.4.1]). Say $\mathcal{F}$ is *radical free* if S contains no non-trivial subgroup that is normal in $\mathcal{F}$. For $P \in \mathrm{Ob}(\mathcal{F})$, say P is $\mathcal{F}$*-centric* if, for every $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P, S)$, $C_S(P^\alpha) = Z(P^\alpha)$, and say P is *fully* $\mathcal{F}$*-normalized* if, for every $\alpha \in \mathrm{Hom}_{\mathcal{F}}(P, S)$, $|N_S(P)| \geq |N_S(P^\alpha)|$. Say P is $\mathcal{F}$*-essential* if it is proper, $\mathcal{F}$-centric, fully $\mathcal{F}$-normalized and $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup (note that this definition differs from the one in [7], where Craven does not assume an $\mathcal{F}$-essential subgroup to be fully $\mathcal{F}$-normalized). In particular, if P is $\mathcal{F}$-essential, $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$. Denote by $D_{\mathcal{F}}$ the set of $\mathcal{F}$-essential elements of $\mathrm{Ob}(\mathcal{F})$.

Fusion systems over 2-groups of sectional rank at most 4 have been studied in [8, 15]. For p odd, Diaz, Ruiz and Viruel [9, 18] classified saturated fusion systems over p-groups of sectional rank 2, and there is an ongoing project by Parker and Grazian [11–13] to classify all radical free saturated fusion systems over p-groups of sectional rank at most 4. In a different direction, another project [14, 16] aims to obtain a classification of all radical free saturated fusion systems over p-groups with an extraspecial subgroup of index p. In this context, primes strictly greater than 3 usually afford a homogeneous treatment, in contrast 2 and 3 require ad hoc arguments. In this sense, this paper contributes to both the above projects by determining all saturated fusion systems $\mathcal{F}$ on the Sylow 3-subgroups of the McLaughlin sporadic simple group.

By Alperin's theorem for fusion systems [7, Theorem 4.51], $\mathcal{F}$ is completely determined by the automorphism groups of the $\mathcal{F}$-essential subgroups of S. Thus, in Section 2, we determine the possible $\mathcal{F}$-essential subgroups of S (in particular, we get $|D_{\mathcal{F}}| \leq 2$), and, in Section 4, we determine their automorphism groups under the assumption that $|D_{\mathcal{F}}| = 2$.

In Section 5, we prove the following result.

Theorem 1. *Let S be a Sylow 3-subgroup of the McLaughlin sporadic simple group, and let $\mathcal{F}$ be a saturated fusion system on S with $|D_{\mathcal{F}}| > 1$. Then $\mathcal{F}$ is isomorphic to a fusion system $\mathcal{F}_S(G)$ (described in Table 2), where G is one of the following.*

(i) $\tilde{G} \le G \le \mathrm{Aut}(\tilde{G})$, *where* $\tilde{G} \in \{\mathrm{Mc}, U_4(3), \mathrm{Co}_2\}$;

(ii) $G = L_6(q)$, *where* $q \equiv 4, 7 \bmod 9$, *or* $G = U_6(q)$, *where* $q \equiv 2, 5 \bmod 9$;

(iii) $G = L_6(q)\langle\phi\rangle$, *where* $q \equiv 4, 7 \bmod 9$, *or* $G = U_6(q)\langle\phi\rangle$, *where* $q \equiv 2$, $5 \bmod 9$, *and* ϕ *is a field automorphism of order* 2.

Moreover, all groups in (ii) *(respectively in* (iii)*) realize isomorphic fusion systems.*

We refer to [3,7] for fusion systems, to [2] for groups and to the ATLAS [6] for the notation of simple groups and group extensions. In particular, recall that, for $n \ge 4$, S_n has two double covers $2^- S_n$ and $2^+ S_n$ in which transpositions of S_n lift to elements of order 4 or involutions respectively (for $n = 4$, this is elementary; for $n \ge 5$, see [6, p. xxiii]). For $n \ne 6$, these two double covers are not isomorphic. For $n = 6$, the exceptional outer automorphism of S_6 extends to an isomorphism between $2^- S_6$ and $2^+ S_6$, so, up to isomorphism, there is a unique double cover of S_6, which we will simply denote by $2S_6$.

2 $\mathcal{F}$-essential subgroups

Let p, S and $\mathcal{F}$ be as in the previous section. Recall that a *characteristic series* $\mathcal{S}$ of a group P is a series

$$1 = P_0 \le P_1 \le \cdots \le P_n = P,$$

where every P_i is a characteristic subgroup of P. We say that a subgroup H of S *centralizes* the series $\mathcal{S}$ if $[P_i, H] \le P_{i-1}$ for every $i \in \{1, \ldots, n\}$.

Lemma 2. *Let P and H be subgroups of S. If P is $\mathcal{F}$-essential and H centralizes a characteristic series $\mathcal{S}$ in P, then $H \le P$. In particular, if $Z(S)$ is characteristic in P, then $Z_2(S) \le P$.*

Proof. By coprime action, $C_{\mathrm{Aut}(P)}(\mathcal{S})$ is a p-subgroup of $\mathrm{Aut}(P)$, and, since $\mathcal{S}$ is characteristic, $C_{\mathrm{Aut}(P)}(\mathcal{S}) \le O_p(\mathrm{Aut}(P))$. In particular,

$$\begin{aligned}\mathrm{Aut}_H(P) &\le O_p(\mathrm{Aut}(P)) \cap \mathrm{Aut}_{\mathcal{F}}(P)\\ &\le O_p(\mathrm{Aut}_{\mathcal{F}}(P)).\end{aligned}$$

Since P is $\mathcal{F}$-essential, $O_p(\mathrm{Aut}_{\mathcal{F}}(P)) = \mathrm{Inn}(P)$, so $H \le PC_S(P)$. Since P is $\mathcal{F}$-centric, we have $C_S(P) = Z(P)$, whence $PC_S(P) = P$, and the result follows. Clearly, $Z(S) \le Z(P)$; thus, if $Z(S)$ is characteristic in P, then $Z_2(S)$ centralizes the characteristic series $1 \le Z(S) \le P$. □

Lemma 3. *If S is a Sylow 3-subgroup of the McLaughlin group, then S has the presentation*

$$S = \langle x, y, z, a, b, t \mid x^3 = y^3 = z^3 = a^3 = b^3 = t^3 = 1, \\ [x, y] = [a, b] = z, [y, t] = xz, [b, t] = az \text{ and} \\ [c, d] = 1 \text{ for all other } \{c, d\} \subset \{x, y, a, b, t, z\}\rangle. \quad (2.1)$$

Proof. By [6], if $S \in \mathrm{Syl}_3(\mathrm{Mc})$, S is contained in a maximal subgroup of Mc isomorphic to the group $3^4 : M_{10}$. An easy inspection in $3^4 : M_{10}$ shows that S satisfies the presentation in (2.1) (see [4] for details). □

For the remainder of this paper, x, y, a, b, t, z will denote the generators of a 3âĂŇ-group S satisfying the presentation in (2.1).

Denote, as usual, by $J(S)$ the Thompson subgroup of S.

Lemma 4. *The following hold:*

(i) $X(S) := \langle x, y, a, b\rangle$ *is extraspecial of order* 3^5 *and exponent* 3*;*

(ii) $J(S) = C_S(J(S)) = \langle x, a, z, t\rangle$*, and* $J(S)$ *is elementary abelian of order* 3^4*, in particular,* $m_p(S) = 4$*;*

(iii) $Z(S) = Z(X(S)) = X(S)^{(1)} = \langle z\rangle$*;*

(iv) $S^{(1)} = X(S) \cap J(S) = [S, J(S)] = Z_2(S) = \langle x, a, z\rangle$ *and* $|Z_2(S)| = 3^3$*;*

(v) $S^3 = Z(S)$*, and every element of* S *of order* 3 *is contained in* $X(S) \cup J(S)$*;*

Proof. This follows from easy commutator computations (see [4]). □

Lemma 5. *No subgroup of p-rank* 2 *of* $\mathrm{GL}_2(p) \times \mathrm{GL}_2(p)$, $\mathrm{GL}_3(p) \times \mathrm{GL}_1(p)$ *or* $\mathrm{GL}_3(p)$ *contains a strongly p-embedded subgroup.*

Proof. This is immediate for $\mathrm{GL}_2(p) \times \mathrm{GL}_2(p)$; otherwise, it follows by [5, Tables 8.3 and 8.4]. □

Lemma 6. *Let H be a subgroup of* $\mathrm{GL}_4(3)$*, and let U be the natural module for* $\mathrm{GL}_4(3)$*. Suppose that H contains a Sylow* 3*-subgroup H_3 of order* 9 *such that* $|C_U(H_3)| = 3$ *and a strongly* 3*-embedded subgroup. Then H lies in the group of similarities of an orthogonal form on U with Witt index* 1.

Proof. Since H contains a strongly 3-embedded subgroup, $O_3(H) = 1$, and this implies that H cannot stabilize a subspace of U with dimension 1 or 3. Condition $|C_U(H_3)| = 3$ implies that H cannot stabilize a subspace of U with dimension 2, nor normalize a decomposition of U into a direct sum of two subspaces, nor a tensor decomposition, nor an extension field $\mathbb{F}_9$. By Aschbacher's classification of maximal subgroups of finite classical groups [1] and [5, Table 8.9], it follows that either H lies in the group of similarities of an orthogonal form with Witt index 1 or H lies in the group $\mathrm{Sp}_4(3)$. The latter case cannot occur by [5, Tables 8.12 and 8.13]. □

Proposition 7. *Let $\mathcal{F}$ be a saturated fusion system on S. Then the $\mathcal{F}$-essential subgroups of S are in $\{X(S), J(S)\}$.*

Proof. Let P be an $\mathcal{F}$-essential subgroup of S.

Claim 1. $|P| \geq 3^3$ *and P is not properly contained in $J(S)$.*

Since P is $\mathcal{F}$-centric and $J(S)$ is abelian, $|P| \geq 3^2$ and $P \not< J(S)$. Since $Z_2(S) < J(S)$, it follows that $P \not\leq Z_2(S)$. If $|P| = 3^2$, then $P \cap Z_2(S) = Z(S)$ and $|\mathrm{Aut}_{Z_2(S)}(P)| \leq 3$, whence $C_{Z_2(S)}(P) \not\leq P$, a contradiction.

Claim 2. *If* $|N_S(P) : P| \geq 3^2$, *then* $P = J(S)$.

Suppose $|N_S(P) : P| \geq 3^2$. Since $|S| = 3^6$, then $|P| \leq 3^4$, and 3^2 divides $|\mathrm{Out}_{\mathcal{F}}(P)|$. Since $O_3(\mathrm{Out}_{\mathcal{F}}(P)) = 1$, the map

$$\Phi\colon \mathrm{Out}_{\mathcal{F}}(P) \to \mathrm{Aut}(P/Z_2(P)) \times \mathrm{Aut}(Z_2(P)/Z(P)) \times \mathrm{Aut}(Z(P)),$$
$$\phi \mapsto (\phi_{|P/Z_2(P)}, \phi_{|Z_2(P)/Z(P)}, \phi_{|Z(P)})$$

is injective. Since 3^2 divides $|\mathrm{Out}_{\mathcal{F}}(P)|$, 3^2 divides also $|\mathrm{Im}(\Phi)|$, which forces $P = Z_2(P)$. Since $\mathrm{Aut}(P/Z(P)) \times \mathrm{Aut}(Z(P))$ is isomorphic to one of

$$\mathrm{GL}_2(3) \times \mathrm{GL}_2(3), \quad \mathrm{GL}_1(3) \times \mathrm{GL}_3(3), \quad \mathrm{GL}_3(3) \quad \text{or} \quad \mathrm{GL}_4(3)$$

and, by Lemma 5, none of the first three groups contains a subgroup of order divisible by 3^2 with a strongly 3-embedded subgroup, it follows that

$$\mathrm{Aut}(P/Z(P)) \times \mathrm{Aut}(Z(P)) \cong \mathrm{GL}_4(3),$$

which can happen only if P is elementary abelian of order 3^4, that is, $P = J(S)$.

Claim 3. $|P| \neq 3^3$.

Suppose, by means of contradiction, that $|P| = 3^3$. Since $P \not\leq J(S)$ by Claim 1 and $Z_2(S) \leq J(S)$ by Lemma 4 (iv), it follows that $P \not\leq Z_2(S)$. So, by Lemma 2 and Lemma 4 (iv), $Z(S)$ is not characteristic in P. Since

$$|Z(S)| = 3, \quad Z(S) \leq Z(P), \quad P^3 \leq S^3 \leq Z(S)$$

and both $Z(P)$ and P^3 are characteristic in P, it follows that P is elementary abelian. Moreover, since $Z(S) = X(S)^{(1)} \leq P$ and $|X(S)|/|P| = 3^2$, by Claim 2, P cannot be contained in $X(S)$. Similarly, $P \cap Z_2(S) = 3^2$. Therefore, modulo exchanging (x, y) with (a, b), we may assume that there are an integer α and an element $e \in C_{X(S)}(xa^\alpha)$ such that $P = \langle z, xa^\alpha, et\rangle$. Since

$$[et, yb^\alpha] = [e, yb^\alpha]^t [t, yb^\alpha] \in \langle z\rangle xa^\alpha$$

and yb^α normalizes $\langle z, xa^\alpha\rangle$, it follows that $\langle Z_2(S), yb^\alpha\rangle \leq N_S(P)$, whence $|N_S(P) : P| \geq 3^2$, a contradiction to Claim 2.

Claim 4. *If $|P| = 3^4$, then $P = J(S)$.*

Suppose, by means of contradiction, that $|P| = 3^4$ and $P \neq J(S)$. By Claim 2, P is not normal in S, so $Z_2(S) \not\leq P$, whence, by Lemma 2, $Z(S)$ is not characteristic in P. By Lemma 4 (v), P has exponent 3. Since $P \neq J(S)$, P is not abelian, whence $|Z(P)| = 3^2$ and $P^{(1)} \leq Z(P)$, in particular, $|Z(P) : P^{(1)}| \leq 3$. Since $S^{(1)} = Z_2(S)$ is abelian and contains $P^{(1)}$, and $Z(S) \leq Z(P)$ since P is $\mathcal{F}$-essential, it follows that $S^{(1)}$ centralizes the characteristic series

$$1 \leq P^{(1)} \leq Z(P) \leq P,$$

and so, by Lemma 2, $S^{(1)} \leq P$, a contradiction.

Claim 5. *If $|P| = 3^5$, then $P = X(S)$.*

Suppose, by means of contradiction, that P is a maximal subgroup of S and $P \neq X(S)$. Then P is not contained in $X(S) \cup J(S)$, so, by Lemma 4 (v) P has exponent 3^2. As in the previous case, we get $P^3 = S^3 = Z(S)$, $Z_2(S) \leq P$ and $Z_2(S)$ is not characteristic in P. In particular, we have $Z_2(S) < Z_2(P)$, and so $|P/Z_2(P)| \leq 3$, whence $Z_2(P) = P$. Thus, by [19, (3.13)], P is a regular 3-group of exponent 9 with derived subgroup of exponent 3, whence $\Omega_1(P) < P$. Since $X(S)$ is maximal in S and has exponent 3, we get $\Omega_1(P) = P \cap X(S)$, and $X(S)$ centralizes the series $1 < Z(S) < \Omega_1(P) < P$. Lemma 2 now gives the contradiction $X(S) \leq P$. □

Corollary 8. *Let $\mathcal{F}$ be a saturated and radical free fusion system on S. Then its $\mathcal{F}$-essential subgroups are $X(S)$ and $J(S)$.*

Proof. This follows immediately from Proposition 7 and [7, Exercise 9.3]. □

3 The group Aut(S)

In this section, we study the group Aut(S) and, in particular, its relations with Aut($J(S)$) and Aut($X(S)$). Since $J(S)$ and $X(S)$ are characteristic subgroups of S, the restriction maps from Aut(S) to Aut($J(S)$) and Aut($X(S)$) are well defined. For $P \in \{J(S), X(S)\}$, we denote them

$$\mathrm{r}_P\colon \mathrm{Aut}(S) \to \mathrm{Aut}(P).$$

It is straightforward to check that the image of r_P lies in $N_{\mathrm{Aut}(P)}(\mathrm{Aut}_S(P))$.

Lemma 9. *Let* $\phi \in \mathrm{Aut}(S)$. *If* $[J(S), \phi] = 1$ *or* $[X(S), \phi] = 1$, *then* $\phi^3 = \mathrm{id}_S$. *If* $[J(S), \phi] = [X(S), \phi] = 1$, *then* $\phi = \mathrm{id}_S$.

Proof. Let $\phi \in \mathrm{Aut}(S)$, and suppose $[X(S), \phi] = 1$. Then

$$t^\phi = t^m e \quad \text{for some } e \in X(S).$$

From the relation

$$[y, t^\phi] = [y^\phi, t^\phi] = [y, t]^\phi = [y, t],$$

we get $m \equiv 1 \bmod 3$. Hence $[S, \phi] \le X(S)$, and, since $X(S)$ has exponent 3, it follows that $\phi^3 = \mathrm{id}_S$. Suppose now $[J(S), \phi] = 1$. Then we can write

$$y^\phi = y^\alpha b^\beta s \quad \text{with } s \in J(S).$$

From $[y^\phi, a] = [y^\phi, a^\phi] = [y, a]^\phi = 1$, we deduce $\beta \equiv 0 \bmod 3$, and then, from $[y^\phi, x] = [y^\phi, x^\phi] = z^\phi = z$, we get $\alpha \equiv 1 \bmod 3$. Similarly, we get $b^\phi = bs'$ with $s' \in J(S)$. Thus $[S, \phi] \le J(S)$, and, as above, this yields that $\phi^3 = \mathrm{id}_S$. The last claim is clear since S is generated by $J(S)$ and $X(S)$. □

Lemma 10. *For* $P \in \{J(S), X(S)\}$, *the restriction map* r_P *is a surjective homomorphism from* Aut(S) *onto* $N_{\mathrm{Aut}(P)}(\mathrm{Aut}_S(P))$. *Moreover,* $\ker \mathrm{r}_{J(S)}$ *has order* 3^5, *and* $\ker \mathrm{r}_{X(S)}$ *has order* 3.

Proof. Let $P \in \{J(S), X(S)\}$. Clearly, the map r_P is a group homomorphism.

With the notation of Section 2, $\mathrm{Aut}_S(J(S)) = \langle c_y, c_b \rangle$, and a direct inspection in the group $\mathrm{Aut}(J(S)) \cong \mathrm{GL}_4(3)$ shows that $N_{\mathrm{Aut}(J(S))}(\mathrm{Aut}_S(J(S)))$ is generated by the three automorphisms $\alpha_1, \alpha_2, \alpha_3$ of $J(S)$ uniquely determined by the conditions

$$\alpha_1\colon \begin{cases} z \mapsto z^{-1}, \\ x \mapsto xa, \\ a \mapsto xa^{-1}z^{-1}, \\ t \mapsto t, \end{cases} \quad \alpha_2\colon \begin{cases} z \mapsto z, \\ x \mapsto x^{-1}z^{-1}, \\ a \mapsto a, \\ t \mapsto t, \end{cases} \quad \alpha_3\colon \begin{cases} z \mapsto z^{-1}, \\ x \mapsto a^{-1}z^{-1}, \\ a \mapsto x^{-1}z, \\ t \mapsto x^{-1}at^{-1}. \end{cases}$$

It is straightforward to check that, for $i \in \{1,2,3\}$, α_i is the restriction to $J(S)$ of the automorphism $\overline{\alpha}_i$ of S uniquely determined by the conditions

$$\overline{\alpha}_1\colon \begin{cases} z \mapsto z^{-1}, \\ x \mapsto xa, \\ a \mapsto xa^{-1}z^{-1}, \\ t \mapsto t, \\ y \mapsto yb, \\ b \mapsto yb^{-1}, \end{cases} \quad \overline{\alpha}_2\colon \begin{cases} z \mapsto z, \\ x \mapsto x^{-1}z^{-1}, \\ a \mapsto a, \\ t \mapsto t, \\ y \mapsto y^{-1}, \\ b \mapsto b, \end{cases} \quad \overline{\alpha}_3\colon \begin{cases} z \mapsto z^{-1}, \\ x \mapsto a^{-1}z^{-1}, \\ a \mapsto x^{-1}z, \\ t \mapsto x^{-1}at^{-1}, \\ y \mapsto b, \\ b \mapsto y. \end{cases}$$

Hence $\mathrm{r}_{J(S)}$ is surjective. For $i \in \{1,2,3\}$, let $\tilde{\alpha}_i$ be the automorphism of $X(S)$ induced by $\overline{\alpha}_i$. Recall that, since $X(S)$ is extraspecial of exponent 3 and order 3^5, $\mathrm{Out}(X(S))$ is isomorphic to the group of similarities of a symplectic space of dimension 4 over the field $\mathbb{F}_3$, which we denote by $\mathrm{GSp}_4(3)$. Then, computing in $\mathrm{GSp}_4(3)$, we get that the image of $\langle \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3 \rangle$ in $\mathrm{Out}(X(S))$ has order 32. Let $\overline{\beta}_1$, $\overline{\beta}_2$, $\overline{\beta}_3$ be the automorphisms of S defined by the positions

$$\overline{\beta}_1\colon \begin{cases} z \mapsto z, \\ x \mapsto x, \\ a \mapsto a, \\ t \mapsto t, \\ y \mapsto y, \\ b \mapsto ab, \end{cases} \quad \overline{\beta}_2\colon \begin{cases} z \mapsto z, \\ x \mapsto x, \\ a \mapsto a, \\ t \mapsto t, \\ y \mapsto xy, \\ b \mapsto b, \end{cases} \quad \overline{\beta}_3\colon \begin{cases} z \mapsto z, \\ x \mapsto x, \\ a \mapsto a, \\ t \mapsto t, \\ y \mapsto ay, \\ b \mapsto xa^{-1}b, \end{cases}$$

and let $\beta_1, \beta_2, \beta_3$ be their restrictions to $X(S)$, respectively. Then it is clear that $\beta_1, \beta_2, \beta_3$ normalize $\mathrm{Aut}_S(X(S))$ and that the image of $\langle \beta_1, \beta_2, \beta_3 \rangle$ in $\mathrm{Out}(X(S))$ is an elementary abelian group of order 27.

Since the group $N_{\mathrm{Out}(X(S))}(\mathrm{Out}_S(X(S)))$ (computed inside $\mathrm{GSp}_4(3)$) has order $2^5 \cdot 3^3$ and $N_{\mathrm{Aut}(X(S))}(\mathrm{Aut}_S(X(S)))/\mathrm{Inn}(X(S)) = N_{\mathrm{Out}(X(S))}(\mathrm{Out}_S(X(S)))$, we get

$$N_{\mathrm{Aut}(X(S))}(\mathrm{Aut}_S(X(S))) = \mathrm{Inn}(X(S))\langle \beta_1, \beta_2, \beta_3, \tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3 \rangle.$$

Hence $\mathrm{r}_{X(S)}$ is surjective.

To prove the claims about kernels, note, first of all, that, by Lemma 9, $\ker \mathrm{r}_{J(S)}$ and $\ker \mathrm{r}_{X(S)}$ are 3-groups and $\ker \mathrm{r}_{J(S)} \cap \ker \mathrm{r}_{X(S)} = 1$. Moreover,

$$\ker \mathrm{r}_{X(S)} \cap \mathrm{Inn}(S) = 1,$$
$$|\ker \mathrm{r}_{J(S)} \cap \mathrm{Aut}_S(X(S))| = |(J(S) \cap X(S))/Z(S)| = 3^2.$$

Therefore, $\ker \mathrm{r}_{X(S)}$ is isomorphic to a subgroup of $N_{\mathrm{Aut}(J(S))}(\mathrm{Aut}_S(J(S)))$ intersecting trivially $\mathrm{Aut}_S(J(S))$. Then we get $|\ker \mathrm{r}_{X(S)}| = 3$ since a Sylow 3-subgroup of $N_{\mathrm{Aut}(J(S))}(\mathrm{Aut}_S(J(S)))$ has order 3^3,

Similarly, $\ker \mathrm{r}_{J(S)}$ is isomorphic to a subgroup of $N_{\mathrm{Aut}(X(S))}(\mathrm{Aut}_S(X(S)))$. Since $\langle \overline{\beta}_1, \overline{\beta}_2, \overline{\beta}_3 \rangle \leq \ker \mathrm{r}_{J(S)}$ and the image of $\langle \beta_1, \beta_2, \beta_3 \rangle$ in $\mathrm{Out}(X(S))$ is elementary abelian of order 27, we get $|\ker \mathrm{r}_{J(S)}| = 3^5$. □

Corollary 11. *The subgroup of* $\mathrm{Aut}(S)$ *that is generated by* $\ker \mathrm{r}_{J(S)}$, $\ker \mathrm{r}_{X(S)}$ *and* $\mathrm{Inn}(S)$ *is a normal Sylow* 3*-subgroup of* $\mathrm{Aut}(S)$ *with order* 3^8 *and index* 2^5.

Proof. By Lemma 10, $\ker \mathrm{r}_{J(S)}$ is a 3-subgroup of order 3^5, and $\mathrm{Aut}(S)/\ker \mathrm{r}_{J(S)}$ is isomorphic to $N_{\mathrm{Aut}(J(S))}(\mathrm{Aut}_S(J(S)))$, which has a normal Sylow 3-subgroup of order 27 and index 32. □

4 Automorphism groups in $\mathcal{F}$

We keep the notation introduced in the previous sections, and we assume that $\mathcal{F}$ is a saturated radical free fusion system on S. In order to obtain the possible fusion systems $\mathcal{F}$, we now need to determine the groups $\mathrm{Aut}_{\mathcal{F}}(J(S))$, $\mathrm{Aut}_{\mathcal{F}}(X(S))$ and $\mathrm{Aut}_{\mathcal{F}}(S)$. We begin with $\mathrm{Aut}_{\mathcal{F}}(J(S))$.

Proposition 12. *Let* $\mathcal{F}$ *be a saturated fusion system on* S*, and assume that* $J(S)$ *is* $\mathcal{F}$*-essential. Then the following holds:*

(i) $\mathrm{Aut}_{\mathcal{F}}(J(S))$ *is contained in a maximal subgroup* M *of* $\mathrm{Aut}(J(S)) \cong \mathrm{GL}_4(3)$ *isomorphic to* $(C_2 \times M_{10}) : C_2$;

(ii) $\mathrm{Aut}_{\mathcal{F}}(J(S))^{(2)} = M^{(2)} \cong A_6$;

(iii) $\mathrm{Aut}_{\mathcal{F}}(J(S))$ *acts irreducibly on* $J(S)$;

(iv) *if* θ *is an element of order* 4 *in* $\mathrm{Aut}_{\mathcal{F}}(J(S))^{(2)}$ *normalizing* $\mathrm{Aut}_S(J(S))$, *then, up to conjugation in* $\mathrm{Aut}_{\mathcal{F}}(J(S))^{(2)}$, $\theta = \zeta_{|J(S)}$, *where* $\zeta \in \mathrm{Aut}_{\mathcal{F}}(S)$ *is such that*

$$x^\zeta = a^{-1}z, \quad y^\zeta = b, \quad a^\zeta = xz^{-1}, \quad b^\zeta = y^{-1},$$
$$t^\zeta = xa^{-1}t^{-1}z, \quad z^\zeta = z^{-1};$$

in particular, $[J(S), \theta] = J(S)$;

(v) *if* $N_{\mathrm{Aut}_{\mathcal{F}}(J(S))}(\mathrm{Aut}_S(J(S)))$ *is contained in two maximal subgroups* M *and* M' *of* $\mathrm{Aut}(J(S))$ *isomorphic to* $(C_2 \times M_{10}) : C_2$*, then* $M' = M^{\xi_{|J(S)}}$*, where* ξ *is an element of order* 3 *in* $\mathrm{Aut}(S)$ *such that* $\xi_{|X(S)} = \mathrm{id}_{X(S)}$ *and* $\xi_{|J(S)}$ *centralizes* $N_{\mathrm{Aut}_{\mathcal{F}}(J(S))}(\mathrm{Aut}_S(J(S)))$.

Proof. Since $J(S)$ is elementary abelian of rank 4, $\mathrm{Aut}(J(S)) \cong \mathrm{GL}_4(3)$. Clearly, $C_{J(S)}(\mathrm{Aut}_S(J(S)) = C_{J(S)}(S) = Z(S)$ has order 3 by Lemma 4. Since $\mathcal{F}$ is saturated, $\mathrm{Aut}_S(J(S))$ is a Sylow 3-subgroup of $\mathrm{Aut}_{\mathcal{F}}(J(S))$ of order 9, and, since $J(S)$ is $\mathcal{F}$-essential and abelian, $\mathrm{Aut}_{\mathcal{F}}(J(S))$ has a strongly 3-embedded subgroup. Thus, by Lemma 6, $\mathrm{Aut}_{\mathcal{F}}(J(S))$ is contained in the group of similarities of an orthogonal form with Witt index 1, that is, a maximal subgroup $M \cong (C_2 \times M_{10}) : C_2$. Then we have $M^{(2)} \cong A_6$ and $M/M^{(2)} \cong D_8$. Let T be a Sylow 3-subgroup of $\mathrm{Aut}_{\mathcal{F}}(J(S))$. Then $T \leq M^{(2)} \cap \mathrm{Aut}_{\mathcal{F}}(J(S))$, and, since $O_3(\mathrm{Aut}_{\mathcal{F}}(J(S))) = 1$, we get $O_3(M^{(2)} \cap \mathrm{Aut}_{\mathcal{F}}(J(S))) = 1$. Since $O_3(H) \neq 1$ for every proper subgroup H of A_6 of order divisible by 3^2, $M^{(2)} \leq \mathrm{Aut}_{\mathcal{F}}(J(S))$. Since $M^{(2)}$ acts irreducibly on $J(S)$, claim (iii) follows. To prove claim (iv), note that, since $J(S)$ is normal in S and θ normalizes $\mathrm{Aut}_S(J(S))$, we have $N_\theta = S$, and axiom (S2) yields that there exists $\zeta \in \mathrm{Aut}_{\mathcal{F}}(S)$ such that $\theta = \zeta_{|J(S)}$. Since there is a unique semidirect product of $J(S)$ by A_6 via a non-trivial action (see also [17, Lemma 3.4 (iv)]), it follows that, up to conjugation in $\mathrm{Aut}_{\mathcal{F}}(J(S))^{(2)} \cong A_6$,

$$x^\zeta = a^{-1}z, \quad a^\zeta = xz^{-1}, \quad t^\zeta = xa^{-1}t^{-1}z, \quad z^\zeta = z^{-1}.$$

Set

$$y^\zeta = x^r y^s a^l b^m z^k, \quad b^\zeta = x^\alpha y^\beta a^\gamma b^\delta z^\varepsilon$$

for some $r, s, l, m, k, \alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{F}_3$ (note that $y^\zeta, b^\zeta \in X(S)$ since $X(S)$ is characteristic in S). From the identity $[a^\zeta, b^\zeta] = z^\zeta$, we get

$$z^{-1} = z^\zeta = [a^\zeta, b^\zeta] = [xz^{-1}, x^\alpha y^\beta a^\gamma b^\delta z^\varepsilon] = [x, y^\beta] = z^\beta,$$

whence $\beta = -1$, and similarly, from $[x^\zeta, y^\zeta] = z^\zeta$, we get $m = 1$. From

$$[x^\zeta, b^\zeta] = [a^\zeta, y^\zeta] = 1,$$

we get $\delta = s = 0$. Further, up to replacing ζ by its product with some element of $\mathrm{Inn}(S)$ (namely, powers of c_x, c_a, c_t), we may also assume $l = \alpha = 0$ and $k = \varepsilon = 0$. Then, the identity $[y^\zeta, b^\zeta] = 1$ gives

$$1 = [y^\zeta, b^\zeta] = [x^r b, y^{-1}a^\gamma] = [x^r, y^{-1}][b, a^\gamma] = z^{-r-\gamma},$$

whence $\gamma = -r$ and $y^\zeta = x^r b$, $b^\zeta = y^{-1}a^{-r}$. Finally, by Lemma 10, we may assume that ζ has order 4, and this last condition yields $r = 0$, as claimed.

To prove (v), suppose that $\mathrm{Aut}_{\mathcal{F}}(J(S))$ is contained in two maximal subgroups M and M' isomorphic to $(C_2 \times M_{10}) : C_2$. Then M and M' are conjugate in

$\mathrm{Aut}(J(S)) \cong \mathrm{GL}_4(3)$, and clearly they contain $\mathrm{Aut}_S(J(S))$. Comparing the number of conjugates of $\mathrm{Aut}_S(J(S))$ in M and in $\mathrm{Aut}(J(S))$ and the number of conjugates of M in $\mathrm{Aut}(J(S))$, we get that $\mathrm{Aut}_S(J(S))$ is contained in exactly 3 conjugates of M. Let $\xi \in \mathrm{Aut}(S)$ be defined by

$$x^\xi = x, \quad y^\xi = y, \quad a^\xi = a, \quad b^\xi = b, \quad t^\xi = tz, \quad z^\xi = z,$$

and set $\bar{\xi} := \xi_{|J(S)}$ and $N := N_{\mathrm{Aut}(J(S))}(\mathrm{Aut}_S(J(S)))$. It is clear from the definition that $\xi_{|X(S)} = \mathrm{id}_{X(S)}$ and ξ has order 3. Moreover, $\bar{\xi} \in N$, but $\bar{\xi} \notin M$ since $\mathrm{Aut}_S(J(S))$ is a Sylow 3-subgroup of M. Hence, up to replacing ξ by ξ^{-1}, we have $M' = M^{\xi_{|J(S)}}$. Now set

$$N_M := N_M(\mathrm{Aut}_S(J(S))) \quad \text{and} \quad C_M := C_N(\bar{\xi}) \cap M.$$

Then N_M has index 3 in N. If $\alpha_1, \alpha_2, \alpha_3$ are the automorphisms of $J(S)$ defined in the proof of Lemma 9, then α_1 does not centralize $\bar{\xi}$, and $\langle \alpha_1^2, \alpha_2, \alpha_3 \rangle \leq C_N(\bar{\xi})$. Hence $C_N(\bar{\xi})$ has index 2 in N, and C_M has index 2 in N_M. Moreover, we have $(N_M)^{\bar{\xi}} \neq N_M$, and, since $N = \langle N_M, N_M^{\bar{\xi}} \rangle$ is not contained in M, it follows that $(N_M)^{\bar{\xi}}$ is not contained in M. On the other hand, C_M is contained in $M \cap M'$ (since $M' = M^{\bar{\xi}}$), whence $C_M = M \cap M' \cap N$. Hence

$$N_{\mathrm{Aut}_{\mathcal{F}}(J(S))}(\mathrm{Aut}_S(J(S))) \leq C_M,$$

and the claim is proved. □

We turn now to $\mathrm{Aut}_{\mathcal{F}}(X(S))$. Note that $\mathrm{Aut}_{\mathcal{F}}(X(S))$ is completely determined once we determine $\mathrm{Out}_{\mathcal{F}}(X(S))$ up to conjugacy in $\mathrm{Out}(X(S))$ since $\mathrm{Aut}_{\mathcal{F}}(X(S))$ contains the group $\mathrm{Inn}(X(S))$. Now we have $\mathrm{Aut}_S(X(S)) = \langle c_t \rangle$, so, by Proposition 12 (iv), $\zeta_{|X(S)} \in N_{\mathrm{Aut}_{\mathcal{F}}(X(S))}(\langle c_t \rangle)$. Since $X(S)$ is extraspecial of exponent 3, $X(S)/Z(X(S))$ has, as usual, a natural structure of a symplectic space over $\mathbb{F}_3$, the form being defined by the commutator and identifying $Z(X(S))$ with the defining field. Denote by V this space, and let

$$v_1 := xZ(X(S)), \quad v_2 := aZ(X(S)), \quad u_1 := bZ(X(S)), \quad u_2 := yZ(X(S))$$

so that $\mathrm{Out}(X(S)) \cong \mathrm{GSp}(V)$ and $\mathcal{B} := (v_1, v_2, u_1, u_2)$ is a hyperbolic basis of V with mutually orthogonal hyperbolic subspaces $\langle v_1, u_2 \rangle$ and $\langle u_1, v_2 \rangle$. Further, denote by I the image in $\mathrm{Out}(X(S))$ of $C_{\mathrm{Aut}(X(S))}(Z(X(S)))$ so that $I \cong \mathrm{Sp(V)}$. Set $\tilde{\zeta} := \mathrm{Inn}(X(S))\zeta_{|X(S)}$ and $\tilde{t} := \mathrm{Inn}(X(S))c_t$ so that, with respect to the basis $\mathcal{B}$ of V, the matrices associated to $\tilde{t}$ and $\tilde{\zeta}$ are

$$\tilde{t}\colon \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{\zeta}\colon \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $G := \mathrm{Out}(X(S))$, $H := \mathrm{Out}_{\mathcal{F}}(X(S))$ and $T := \langle \tilde{t} \rangle = \mathrm{Out}_S(X(S))$. The following lemma summarizes the properties of H that are needed in the sequel.

Lemma 13. *H contains $\tilde{t}$ and $\tilde{\zeta}$. Further, $T \in \mathrm{Syl}_3(H)$, and T is not normal in H.*

Proof. H contains $\tilde{t}$ by (FS1) and (FS2) and contains $\tilde{\zeta}$ by Proposition 12 (iv). Since $\mathcal{F}$ is saturated, $T \in \mathrm{Syl}_3(H)$. Since $X(S)$ is $\mathcal{F}$-essential, H has a strongly 3-embedded subgroup, so T is not normal in H. □

Lemma 14. *With the above notation,*

(i) $C_V(\tilde{t}) = \langle v_1, v_2 \rangle$;

(ii) *for every non-zero vector $\bar{v}_1 \in C_V(\tilde{t})$, there exist $\bar{v}_2 \in C_V(\tilde{t})$, $\bar{u}_1, \bar{u}_2 \in V$ such that*

$$f(\bar{v}_i, \bar{u}_j) = \delta_{ij} \quad \text{for } i, j \in \{1,2\}, \quad f(\bar{u}_1, \bar{u}_2) = 0,$$
$$\bar{v}_i^{\tilde{t}} = \bar{v}_i, \quad \bar{u}_i^{\tilde{t}} = \bar{v}_{3-i} + \bar{u}_i$$

and either $\tilde{\zeta}$ or $\tilde{\zeta}^{-1}$ maps $\bar{v}_i$ to $(-1)^i \bar{v}_{3-i}$ and $\bar{u}_i$ to $(-1)^i \bar{u}_{3-i}$ for $i \in \{1,2\}$;

(iii) *$C_V(\tilde{t})$ is the unique maximal isotropic subspace of V normalized by $\tilde{t}$.*

Proof. We have

$$C_V(\tilde{t}) = [V, \tilde{t}]^{\perp} = \langle v_1, v_2 \rangle^{\perp} = \langle v_1, v_2 \rangle,$$

and (i) follows. Claim (ii) follows by Witt's lemma (see, e.g., [2, p. 81]) and elementary computations. In order to prove (iii), suppose that $\tilde{t}$ normalizes a maximal isotropic subspace U of V. Since $\tilde{t}$ is an isometry of V of order 3, it has a fixed point u on U, and hence $u \in U \cap C_V(\tilde{t})$. It follows that $U \leq \langle u \rangle^{\perp}$. By (ii), we may assume $u = v_1$, and a direct check shows that $U = C_V(\tilde{t})$. □

Lemma 15. *$H \cap I$ normalizes no non-trivial isotropic subspace of V.*

Proof. Let W be a non-trivial isotropic subspace of maximal dimension among those normalized by $H \cap I$. By Lemma 14 (iii), $W \leq C_V(\tilde{t}) = \langle v_1, v_2 \rangle$. Since $\tilde{\zeta}$ normalizes $H \cap I$, $H \cap I$ normalizes $W^{\tilde{\zeta}}$ too, and, again by Lemma 14 (iii), we have $W^{\tilde{\zeta}} \leq C_V(\tilde{t}) = \langle v_1, v_2 \rangle$. Since $\langle \tilde{\zeta} \rangle$ is irreducible on $\langle v_1, v_2 \rangle$, it follows that $W = \langle v_1, v_2 \rangle$. Since $\tilde{t}$ centralizes the series

$$\{0\} < \langle v_1, v_2 \rangle = \langle v_1, v_2 \rangle^{\perp} < V,$$

we get $T = O_3(N_I(W)) \cap H$, a contradiction, since T is not normal in H. □

Corollary 16. *One of the following holds:*

(a) *H stabilizes a decomposition of V into the direct orthogonal sum of two hyperbolic lines;*

(b) *H normalizes a cyclic subgroup of order 4 of G not contained in I;*

(c) *H is contained in the normalizer in G of a group $Q \cong 2_-^{1+4}$.*

Proof. This follows from Lemma 15 and Aschbacher's classification of maximal subgroups of classical groups (see [1] and also [5, Table 8.12]). □

We investigate now cases (a), (b) and (c) of Corollary 16. We start with case (a).

Lemma 17. *$C_G(\langle \tilde{t}, \tilde{\xi} \rangle)$ acts transitively on the set of decompositions of V into an orthogonal sum of two hyperbolic lines $U_1 \perp U_2$ stabilized by $\tilde{t}$.*

Proof. Suppose that $\tilde{t}$ stabilizes a decomposition of V into an orthogonal sum of two hyperbolic lines $U_1 \perp U_2$. We show that there exists $\gamma \in C_G(\langle \tilde{t}, \tilde{\xi} \rangle)$ such that $U_1 = \langle v_1, u_2 \rangle^\gamma$ and $U_2 = \langle v_2, u_1 \rangle^\gamma$. We may of course assume that $U_1 \neq \langle v_1, u_2 \rangle$. Since $\tilde{t}$ has a fixed point in U_1, by Lemma 14 (ii), we may assume $v_1 \in U_1$. Since $\tilde{t}$ centralizes both $\langle v_1, u_2 \rangle / \langle v_1 \rangle$ and $U_1 / \langle v_1 \rangle$, it follows that $\tilde{t}$ centralizes the quotient space $(\langle v_1, u_2 \rangle + U_1)/\langle v_1 \rangle$. On the other hand,

$$C_{V/\langle v_1 \rangle}(\tilde{t}) = \langle v_1, v_2, u_2 \rangle / \langle v_1 \rangle,$$

so $\langle v_1, u_2 \rangle + U_1 = \langle v_1, v_2, u_2 \rangle$ and $U_1 = \langle v_1, v_2 + \beta u_2 \rangle$ for some $\beta \in \{\pm 1\}$. Then

$$U_2 = U_1^\perp = \{\lambda v_1 + \mu v_2 + \nu u_1 \mid \lambda, \mu, \nu \in \mathbb{F}_3 \text{ and } \nu = \lambda\beta\} = \langle v_2, v_1 + \beta u_1 \rangle,$$

and the linear map $\gamma \colon V \to V$, defined by $v_i^\gamma := \beta v_i$ and $u_i^\gamma := v_i + \beta u_i$ for $i \in \{1, 2\}$, has the required properties. □

Fix the basis $\mathcal{B}_1 := (v_1, u_2, v_2, u_1)$ of V, and identify every element of G with its associated matrix with respect to $\mathcal{B}_1$ so that

$$\tilde{t} = \begin{pmatrix} \tau & 0 \\ 0 & \tau \end{pmatrix}, \qquad \text{where } \tau := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

$$\tilde{\xi} = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix}, \quad \text{where } \mu := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let

$$D = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \;\middle|\; \alpha \in \mathrm{Sp}_2(3) \right\}.$$

Lemma 18. *Assume H is contained in the stabilizer M in G of a decomposition of V into an orthogonal sum of two hyperbolic lines $U_1 \perp U_2$. Then one of the following holds:*

(i) *H is conjugate in G to $D\langle\tilde{\zeta}\rangle$, $H \cong 2^- S_4$, and there is an element of order 4 in $G \setminus I$ that centralizes H;*

(ii) *H is conjugate in G to $D\langle\tilde{\zeta}, \tilde{\eta}\rangle$, $H \cong (2 \times \mathrm{SL}_2(3)) : 2$, and H normalizes a cyclic subgroup of order 4 in $G \setminus I$, where $\tilde{\eta}$ is the linear map that swaps v_1 with v_2 and u_1 with u_2;*

(iii) *H is conjugate in G to $Z(K)D\langle\tilde{\zeta}, \tilde{\eta}\rangle$, where $K = N_M(U_1) \cap N_M(U_2)$, and $H \cong (2 \times \mathrm{GL}_2(3)) : 2$;*

(iv) *$H = O_2(M)\langle\tilde{\iota}, \tilde{\zeta}\rangle$;*

(v) *H is conjugate in G to $O_2(M)\langle\tilde{\iota}, \tilde{\zeta}, \tilde{\eta}\rangle$.*

Proof. By Lemma 17, we may assume $U_1 = \langle v_1, u_2\rangle$ and $U_2 = \langle v_2, u_1\rangle$. For $i \in \{1, 2\}$, denote by S_i the subgroup of M normalizing U_i and acting trivially on U_{3-i}. Set $K := S_1 S_2$, and let R be the unique Sylow 3-subgroup of K containing T. Then

$$S_i \cong \mathrm{SL}_2(3) \quad \text{for } i \in \{1, 2\}, \quad S_1^{\tilde{\zeta}} = S_2 \quad \text{and} \quad [S_1, S_2] = 1; \tag{4.1}$$

$$M = K\langle\tilde{\zeta}, \tilde{\eta}\rangle, \quad \langle\tilde{\zeta}, \tilde{\eta}\rangle \cong D_8;$$

$$N_M(T) = RZ(K)\langle\tilde{\zeta}, \tilde{\eta}\rangle, \quad \text{so} \quad T\langle\tilde{\zeta}\rangle \le N_H(T) \le Z(K)T\langle\tilde{\zeta}, \tilde{\eta}^\rho\rangle \tag{4.2}$$

for a suitable $\rho \in R$.

Since $T \le H \cap K \trianglelefteq H$, by the Frattini argument, $H = (H \cap K)N_H(T)$, so, by Lemma 13, T is not normal in $H \cap K$, and 12 divides $|H \cap K|$. Moreover, by (4.2), either $H = (H \cap K)\langle\tilde{\zeta}\rangle$ or $H = (H \cap K)\langle\tilde{\zeta}, \tilde{\eta}^\rho\rangle$. If $|H \cap K| = 12$, then $H \cap K \cong A_4$ (the unique group of order 12 with no normal Sylow 3-subgroups) and $(H \cap K) \cap Z(K) = 1$. It follows that $m_2(K) \ge 4$, a contradiction as $m_2(\mathrm{GSp}_4(3)) = 3$ (see [10, Theorem 4.10.5]). Hence we get $|H \cap K| \ge 24$. Let $\{i, j\} = \{1, 2\}$, and assume $H \cap S_i = 1$ for some $i \in \{1, 2\}$. Since $\tilde{\zeta} \in H$, by (4.1), $H \cap S_{3-i} = 1$. Since $|H \cap K| \ge 24$, it follows that there is $\gamma \in \mathrm{GL}_2(3)$ such that

$$H \cap K = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^\gamma \end{pmatrix} \text{ with } \alpha \in \mathrm{SL}_2(3) \right\}.$$

Since $\tilde{\iota} \in H \cap K$, γ has to centralize τ, so $\gamma \in \langle\tau, Z(\mathrm{GL}_2(3))\rangle$. Let

$$\epsilon := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \text{and} \quad \sigma(\gamma) := \begin{pmatrix} I & 0 \\ 0 & \gamma \end{pmatrix}.$$

Then, for every $\alpha \in \mathrm{SL}_2(3)$,

$$\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}^{\sigma(\gamma)} = \begin{pmatrix}\alpha & 0\\ 0 & \alpha^{\gamma}\end{pmatrix}, \quad \epsilon^{\sigma(\gamma)} = \begin{pmatrix}0 & \gamma\\ -\gamma^{-1} & 0\end{pmatrix}, \quad \eta^{\sigma(\gamma)} = \begin{pmatrix}0 & \gamma\\ \gamma^{-1} & 0\end{pmatrix}$$

and

$$\begin{pmatrix}0 & \mu\\ -\mu & 0\end{pmatrix}^{\sigma(\gamma)} \begin{pmatrix}\gamma & 0\\ 0 & \gamma\end{pmatrix}^{\sigma(\gamma)} = \begin{pmatrix}0 & \mu\\ -\mu & 0\end{pmatrix}.$$

Thus, if $H = (H \cap K)\langle\tilde{\zeta}\rangle$, then H is conjugate via $(\sigma(\gamma))^{-1}$ to $D\langle\tilde{\zeta}\rangle$, which is isomorphic to $2^{-}S_4$, and centralizes the element ϵ (of order 4). If

$$H = (H \cap K)\langle\tilde{\zeta}, \tilde{\eta}^{\rho}\rangle \quad \text{for some } \rho \in R,$$

then $\tilde{\eta}^{\rho}$ normalizes $H \cap K$, whence

$$\tilde{\eta}^{\rho} = \begin{pmatrix}0 & \gamma\\ \gamma^{-1} & 0\end{pmatrix}.$$

It follows that H is conjugate via $(\sigma(\gamma))^{-1}$ to $D\langle\tilde{\zeta}, \tilde{\eta}\rangle$, which normalizes $\langle\epsilon\rangle$ and is isomorphic to $(2 \times \mathrm{SL}_2(3)) : 2$.

Assume now that $H \cap S_i \neq 1$. As above, $H \cap S_{3-i} \neq 1$. Since T is not conjugate in G to a Sylow 3-subgroup of S_i (their generators having different Jordan normal forms), $H \cap S_i$ is a 2-group. Since $T \leq K \leq N_K(S_i)$, $H \cap S_i$ is normalized by T, so either $H \cap S_i = Z(S_i)$ or $H \cap S_i = O_2(S_i) \cong Q_8$. In all cases, $Z(K) \leq H$. Moreover, by (4.1), $|H \cap S_1| = |H \cap S_2|$.

If $|H \cap S_i| = 2$, then $|(H \cap K)/Z(K)| \leq 12$ and, as T is not normal in H, $(H \cap K)/Z(K) \cong A_4$. As above, it follows that there is $\gamma \in \mathrm{GL}_2(3)$ such that $H \cap K$ is the product of the groups

$$\left\{\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{\gamma}\end{pmatrix} \;\middle|\; \alpha \in \mathrm{Sp}_2(3)\right\} \text{ and } Z(K),$$

whence $H \cap K = (DZ(K))^{\sigma(\gamma)}$, which is isomorphic to $2 \times \mathrm{SL}_2(3)$.

Thus, if $H = (H \cap K)\langle\tilde{\zeta}\rangle$, then H is conjugate in G to $DZ(K)\langle\tilde{\zeta}\rangle$, which is, in turn, conjugate to $D\langle\tilde{\zeta}, \tilde{\eta}\rangle$, and (ii) follows. If $H = (H \cap K)\langle\tilde{\zeta}, \tilde{\eta}^{\rho}\rangle$, then H is conjugate in G to $Z(K)D\langle\tilde{\zeta}, \tilde{\eta}\rangle$, and (iii) follows.

Finally, if $H \cap S_i = O_2(S_i)$ for $i = 1, 2$, then $H \cap K = O_2(M)T$, and we get (iv) and (v). □

Note that groups described in Lemma 18 satisfy both case (a) and (b) of Corollary 16. In order to deal with case (b), we need the following elementary result.

Lemma 19. *Let L be a group, E and F subgroups of L with $|F| = 2$ and such that L is the direct product of E and F. Let L_1 and L_2 be subgroups of L. Then L_1 and L_2 are conjugate in L if and only if there is an element $e \in E$ such that $(L_1F)^e = L_2F$ and $(L_1 \cap E)^e = (L_2 \cap E)$.*

Proof. Assume there is an element $e \in E$ such that

$$(L_1F)^e = L_2F \quad \text{and} \quad (L_1 \cap E)^e = (L_2 \cap E).$$

Let $i \in \{1, 2\}$. If either $F \le L_i$ or $L_i \le E$, the result follows immediately. Otherwise, $L_2F/(L_2 \cap E)$ is elementary abelian of order 4. So, if f is the generator of F, there is an element $d \in E$ such that the three maximal subgroups of L_2F containing $L_2 \cap E$ are $(L_2 \cap E)F$, $(L_2 \cap E)\langle df\rangle$ and $(L_2 \cap E)\langle d\rangle$. So the only possibility is $L_1^e = L_2 = (L_2 \cap E)\langle df\rangle$. The converse is obvious. □

Assume now that H normalizes a cyclic subgroup of order 4 in $G \setminus I$ (case (b) of Corollary 16). In this case, H is contained in a maximal subgroup M of G such that $M = \langle\gamma\rangle A$ with γ of order 4, $\gamma^2 \in Z(A)$, $[A^{(1)}, \gamma] = 1$ and $[A, \langle\gamma\rangle] = Z(A)$, and there is an isomorphism $\varphi\colon A \to 2S_6$ such that $\tilde{t}Z(A)$ is mapped to the product of two 3-cycles in S_6 (see [6, p. 26]). Since $\tilde{\zeta}$ has order 4, inverts $\tilde{t}$ and supplements $A^{(1)}\langle\gamma\rangle$ in M, it follows that $\tilde{\zeta} = \alpha\gamma^m$ for suitable $\alpha \in A \setminus A^{(1)}$ of order 4 and $m \in \mathbb{N}$. By the choice of φ, $\alpha Z(A)$ must map to the product of three disjoint transpositions.

Lemma 20. *With the above notation, H is one of the groups listed in the fifth column of Table 1 and H is uniquely determined, up to conjugation in M, by its isomorphism type.*

Proof. Set $K := A^{(1)}\langle\gamma\rangle$. Then $H = (H \cap K)\langle\tilde{\zeta}\rangle$. Note that $Z(M) = \langle\tilde{\zeta}^2\rangle \le H$, and hence H is completely determined by its image $H/Z(M)$ in the quotient group $M/Z(M)$. For an element ψ, or a subgroup L, of M, denote by $\overline{\psi}$, respectively $\overline{L}$, its image in $M/Z(M)$. Thus, in particular,

$$\overline{M} = \overline{A} \times \langle\overline{\gamma}\rangle.$$

Since T has order 3 and $T \le A^{(1)} \trianglelefteq M$, we have $T^H \le A^{(1)}$, and, since T is a non-normal Sylow 3-subgroup of H, T^H is isomorphic either to $2A_4$ or to $2A_5$. Moreover,

$$\overline{T}^{\overline{H}} \le \overline{(H \cap K)} \le N_{\overline{K}}(\overline{T}^{\overline{H}}) = N_{\overline{A}^{(1)}}(\overline{T}^{\overline{H}}) \times \langle\overline{\gamma}\rangle.$$

If $T^H \cong 2A_5$, then $\overline{T}^{\overline{H}}$ is a maximal subgroup of $\overline{A}^{(1)}$, so $\overline{T}^{\overline{H}} = N_{\overline{A}^{(1)}}(\overline{T}^{\overline{H}})$; if $T^H \cong 2A_4$, then $N_{\overline{A}^{(1)}}(\overline{T}^{\overline{H}}) \cong S_4$. In both cases, we get one of the configurations listed in Table 1. Finally, let L be a subgroup of M isomorphic to H and

T^H	$\overline{H \cap K}$	$\overline{H}$	$\overline{H}\langle\overline{\gamma}\rangle$	H	Structure
$2A_5$	$\overline{T}^{\overline{H}}$	S_5	$S_5 \times 2$	$T^H\langle\tilde{\zeta}\rangle$	$2^- S_5$
	$\overline{T}^{\overline{H}} \times \langle\overline{\gamma}\rangle$	$S_5 \times 2$	$\overline{H}$	$(T^H \circ \langle\gamma\rangle)\langle\tilde{\zeta}\rangle$	$2^- S_5 \circ 4$
$2A_4$	$\overline{T}^{\overline{H}}$	S_4	$S_4 \times 2$	$T^H\langle\tilde{\zeta}\rangle$	$2^- S_4$
	$\overline{T}^{\overline{H}} \times \langle\overline{\gamma}\rangle$	$S_4 \times 2$	$\overline{H}$	$(T^H \circ \langle\gamma\rangle)\langle\tilde{\zeta}\rangle$	$(\mathrm{SL}_2(3) \times 2) : 2$
	$\overline{T}^{\overline{H}}\langle\overline{\gamma\sigma}\rangle$ (with $\sigma \in N_{A^{(1)}}(T^H) \setminus T^H$)	S_4	$S_4 \times 2$	$T^H\langle\gamma\sigma, \tilde{\zeta}\rangle$	$\mathrm{GL}_2(3) : 2$
	$N_{\overline{A}^{(1)}}(\overline{T}^{\overline{H}})$	$S_4 \times 2$	$S_4 \times 2 \times 2$	$N_{A^{(1)}}(T^H)\langle\tilde{\zeta}\rangle$	$2^- S_4 : 2$
	$N_{\overline{A}^{(1)}}(\overline{T}^{\overline{H}}) \times \langle\overline{\gamma}\rangle$	$S_4 \times 2 \times 2$	$\overline{H}$	$(N_{A^{(1)}}(T^H) \circ \langle\gamma\rangle)\langle\tilde{\zeta}\rangle$	$(2^- S_4 : 2) : 2$

Table 1. Possibilities for H in case (b) of Corollary 16.

containing T, $Z(M)$ and $\tilde{\zeta}$. A direct check inside A_6 shows that there is an element $g \in A$ such that

$$(\overline{H}\langle\overline{\gamma}\rangle)^{\overline{g}} = \overline{L}\langle\overline{\gamma}\rangle \quad \text{and} \quad (\overline{H \cap A})^{\overline{g}} = \overline{L \cap A}.$$

Thus, by Lemma 19, $H^g = L$. □

We turn finally to case (c) of Corollary 16. Here we use the isomorphism $PSp_4(3) \cong GO_6^-(2)$ (see [6, p. 26]) and identify $G/Z(G)$ with the latter group so that the natural action of $GO_6^-(2)$ on an orthogonal space Y of dimension 6 over the field of order 2 with Witt defect 1 extends to a representation ν of G on Y. Then, by [6, p. 26], H is contained in the stabilizer M of a singular vector v_0 in Y, and ν induces a representation $\overline{\nu}$ of M onto the full permutation group on the set of the five singular non-zero vectors of $v_0^{\perp}/\langle v_0\rangle$ such that $\ker(\overline{\nu})$ is the unipotent radical U of M, which is isomorphic to 2_-^{1+4}.

Lemma 21. *With the above notation, one of the following holds:*

(i) *the order of H is divisible by 5, in which case either $H \cong 2^- S_5$ or $H = M$;*

(ii) *H stabilizes a totally singular line in Y;*

(iii) *H centralizes a non-singular vector of Y;*

(iv) *$H \cap U$ is equal either to U or to $[U, T]Q$, where Q is the cyclic subgroup of order 4 of $C_U(T)$ and $H/(H \cap U)) \cong S_3 \times 2$.*

Proof. Since Y has Witt defect -1, there is a basis $(e_1, e_2, e, f, f_2, f_1)$ of Y such that

- $e_1 = v_0$ (so H fixes e_1),
- for $i \in \{1, 2\}$, (e_i, f_i) is a hyperbolic pair,

- the subspace $\langle e, f\rangle$ does not contain any singular non-zero vector,
- f is not orthogonal to e.

Since all elements of order 3 of M are conjugate in M, we may choose ν in such a way that $\nu(\tilde{t})$ acts trivially on $\langle e_1, e_2, f_2, f_1\rangle$ and maps e to f and f to $e+f$. Since $\tilde{\zeta}$ acts trivially on $\langle e_1, e_2, f_2, f_1\rangle$, it maps e to f and f to e, and $\tilde{\zeta} \notin U$. Since $\overline{\nu}(\tilde{\zeta}) \in \overline{\nu}(H) \setminus A_5$ and $\{1\} \neq \overline{\nu}(T) \leq \overline{\nu}(H)$, $\overline{\nu}(H)$ is a subgroup of S_5 not contained in A_5 and divisible by 6. If 5 divides $|H|$, then $\overline{\nu}(H) = S_5$, and (i) follows since $\langle \tilde{\zeta}^2\rangle = Z(U)$ and M is irreducible on $U/Z(U)$. Assume 5 does not divide $|H|$. Then $\overline{\nu}(H)$ is contained in a subgroup of S_5 isomorphic either to S_4 or to $2 \times S_3$. In the former case, $\overline{\nu}(H)$ fixes a singular non-zero vector in $\langle e_1\rangle^{\perp}/\langle e_1\rangle$, and hence H stabilizes a totally singular line in Y as in (ii). In the latter case, $\overline{\nu}(H)$ fixes a non-singular vector in $\langle e_1\rangle^{\perp}/\langle e_1\rangle$ (see [6, p. 2]), which has to be $e_2 + f_2 + \langle e_1\rangle$ since this is the unique non-singular vector in $e_1^{\perp}/\langle e_1\rangle$ which is fixed by $\langle \tilde{t}, \tilde{\zeta}\rangle$. Thus H acts on the subspace $\langle e_1, e_2 + f_2\rangle$ of Y. If this action is trivial, then (iii) holds. Otherwise, H contains an element α that swaps the two non-singular vectors $e_2 + f_2$ and $e_2 + f_2 + e_1$ of $\langle e_1, e_2 + f_2\rangle$. Since $\overline{\nu}(T)$ is normal in $\overline{\nu}(H)$, possibly substituting α with $\alpha\tilde{\zeta}$, we may assume $[\tilde{t}, \alpha] \in H \cap U$. Thus α maps the basis (e_1, e_2, e, f, f_2) of $\langle e_1\rangle^{\perp}$ to

$$(e_1, f_2 + he_1, e + le_1, f + me_1, e_2 + ne_1) \quad \text{for some } h, l, m, n \in \mathbb{F}_2.$$

Since α swaps $e_2 + f_2$ and $e_2 + f_2 + e_1$, we have $n = h + 1$. It follows that

$$1 \neq \alpha^2 \in H \cap C_U(\tilde{t}).$$

Since $e_1^{\perp}/\langle e_1\rangle$ is canonically isometric to the factor $U/Z(U)$ of the extraspecial 2-group endowed with the usual quadratic form induced by the squaring [2, (23.10)] and $Z(U)\alpha^2$ is non-singular, $Q := \langle \alpha^2\rangle$ is a subgroup of order 4 in $C_U(\tilde{t})$. Assume, by means of contradiction, that $[H \cap U, T] = 1$. Since

$$H/(H \cap U) \cong S_3 \times 2,$$

it follows that $T \trianglelefteq T(H \cap U) \trianglelefteq H$. Since T is a Sylow 3-subgroup of H, this implies that T is normal in H, against the hypothesis. So $[H \cap U, T] \neq 1$. Since T acts irreducibly on $[U, T]/Z(U)$, it follows that $[U, T] \leq H$, and (iv) holds. □

5 Fusion systems

Lemma 22. *For $P \in \{J(S), X(S)\}$, the restriction map*

$$\mathrm{r}_P^{\mathcal{F}} \colon \mathrm{Aut}_{\mathcal{F}}(S) \to N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P))$$

is a surjective homomorphism such that $\ker \mathrm{r}_P^{\mathcal{F}} \leq \mathrm{Inn}(S)$.

Proof. Let $P \in \{J(S), X(S)\}$. By the surjectivity property [7, p. 190], the restriction to P induces a surjective homomorphism

$$\mathrm{r}_P^{\mathcal{F}} \colon \mathrm{Aut}_{\mathcal{F}}(S) \to N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P)).$$

Since P is $\mathcal{F}$-essential, $C_S(P) \le P$. By Thompson's $A \times B$ lemma [19, (1.15)′], this implies that $\ker \mathrm{r}_P^{\mathcal{F}}$ is a 3-group, whence, since $\mathcal{F}$ is saturated, $\ker \mathrm{r}_P^{\mathcal{F}}$ is contained in $\mathrm{Inn}(S)$. □

Theorem 23. *Let $\mathcal{F}$ and $\mathcal{E}$ be saturated fusion systems on a Sylow 3-subgroup S of the McLaughlin group* Mc *with $|D_{\mathcal{F}}| = 2$. If $\mathrm{Aut}_{\mathcal{F}}(X(S))$ is conjugate to $\mathrm{Aut}_{\mathcal{E}}(X(S))$ in $\mathrm{Aut}(X(S))$, then $\mathcal{F}$ and $\mathcal{E}$ are isomorphic fusion systems.*

Proof. Suppose $\mathrm{Aut}_{\mathcal{F}}(X(S))$ is conjugate to $\mathrm{Aut}_{\mathcal{E}}(X(S))$ in $\mathrm{Aut}(X(S))$. Since $\mathrm{Aut}_{\mathcal{F}}(X(S))$ and $\mathrm{Aut}_{\mathcal{E}}(X(S))$ contain (by definition of saturated) $\mathrm{Aut}_S(X(S))$ as a Sylow 3-subgroup, there exists $\overline{\delta} \in N_{\mathrm{Aut}(X(S))}(\mathrm{Aut}_S(X(S)))$ such that

$$\mathrm{Aut}_{\mathcal{F}}(X(S))^{\overline{\delta}} = \mathrm{Aut}_{\mathcal{E}}(X(S)).$$

By Lemma 10, there exists $\delta \in \mathrm{Aut}(S)$ such that $\delta_{|X(S)} = \overline{\delta}$. Since the fusion system

$$\mathcal{F}^{\delta} = \langle \mathrm{Aut}_{\mathcal{F}}(S)^{\delta}, \mathrm{Aut}_{\mathcal{F}}(J(S))^{\delta}, \mathrm{Aut}_{\mathcal{F}}(X(S))^{\delta} \rangle$$

is isomorphic to $\mathcal{F}$, it is enough to show that $\mathcal{F}^{\delta}$ is isomorphic to $\mathcal{E}$. Hence

(a) we may assume $\mathrm{Aut}_{\mathcal{F}}(X(S)) = \mathrm{Aut}_{\mathcal{E}}(X(S))$ and, in particular,

$$N_{\mathrm{Aut}_{\mathcal{F}}(X(S))}(\mathrm{Aut}_S(X(S))) = N_{\mathrm{Aut}_{\mathcal{E}}(X(S))}(\mathrm{Aut}_S(X(S))).$$

Let Q be the preimage of $N_{\mathrm{Aut}_{\mathcal{F}}(X(S))}(\mathrm{Aut}_S(X(S)))$ via the map $r_{X(S)}$ defined in Section 3. Then, by Lemma 10 and Corollary 11, $\mathrm{Aut}_{\mathcal{F}}(S)/\mathrm{Inn}(S)$ and $\mathrm{Aut}_{\mathcal{E}}(S)/\mathrm{Inn}(S)$ are Sylow 2-subgroups of

$$Q/\mathrm{Inn}(S) = \mathrm{Aut}_{\mathcal{F}}(S) \ker \mathrm{r}_{X(S)}/\mathrm{Inn}(S),$$

and so there exists $\mu \in \ker \mathrm{r}_{X(S)}$ such that $\mathrm{Aut}_{\mathcal{F}}(X(S))^{\mu} = \mathrm{Aut}_{\mathcal{F}}(X(S))$. Up to replacing $\mathcal{F}$ by $\mathcal{F}^{\mu}$,

(b) we may assume

$$\mathrm{Aut}_{\mathcal{F}}(X(S)) = \mathrm{Aut}_{\mathcal{E}}(X(S)) \quad \text{and} \quad \mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Aut}_{\mathcal{E}}(S).$$

Then, by Lemma 22,

$$N_{\mathrm{Aut}_{\mathcal{F}}(J(S))}(\mathrm{Aut}_S(J(S))) = N_{\mathrm{Aut}_{\mathcal{E}}(J(S))}(\mathrm{Aut}_S(J(S))).$$

By Proposition 12 (v), there exists an automorphism $\xi \in \ker \mathrm{r}_{X(S)}$ such that $\xi_{|J(S)}$ centralizes $N_{\mathrm{Aut}_{\mathcal{F}}(J(S))}(\mathrm{Aut}_S(J(S)))$ and $\mathrm{Aut}_{\mathcal{F}}(J(S))^{\xi}$ and $\mathrm{Aut}_{\mathcal{E}}(J(S))$ are contained in the same maximal subgroup $M \cong (C_2 \times M_{10}) : C_2$ of $\mathrm{Aut}(J(S))$. Since, by Lemma 22,

$$[\mathrm{Aut}_{\mathcal{F}}(S), \xi]^{r_{J(S)}} = [N_{\mathrm{Aut}_{\mathcal{F}}(J(S))}(\mathrm{Aut}_S(J(S))), \xi_{|J(S)}] = 1,$$

by Lemma 9, we have

$$[\mathrm{Aut}_{\mathcal{F}}(S), \xi] \le \ker r_{J(S)} \cap \ker r_{X(S)} = 1.$$

Thus $\mathrm{Aut}_{\mathcal{F}}(S)^{\xi} = \mathrm{Aut}_{\mathcal{F}}(S)$. By Proposition 12 and the Frattini argument, since $\mathrm{Aut}_S(J(S))$ is a Sylow 3-subgroup of $\mathrm{Aut}_{\mathcal{E}}(J(S))$, $\mathrm{Aut}_{\mathcal{F}}(J(S))^{\xi}$ and of $M^{(2)}$, we have

$$\begin{aligned}\mathrm{Aut}_{\mathcal{F}}(J(S))^{\xi} &= M^{(2)} N_{\mathrm{Aut}_{\mathcal{F}}(J(S))}(\mathrm{Aut}_S(J(S))) \\ &= M^{(2)} N_{\mathrm{Aut}_{\mathcal{E}}(J(S))}(\mathrm{Aut}_S(J(S))) = \mathrm{Aut}_{\mathcal{E}}(J(S)).\end{aligned}$$

Thus $\mathcal{F}^{\xi} = \mathcal{E}$, and we have the claim. □

Theorem 24. *Let $\mathcal{F}$ be a fusion system on a Sylow 3-subgroup S of the McLaughlin group* Mc *with $|D_{\mathcal{F}}| = 2$. Then $\mathcal{F}$ is isomorphic to one of the fusion systems listed in Table* 2.[1] *In particular, Theorem* 1 *holds.*

Proof. By Alperin's theorem for fusion systems [7, Theorem 4.51] and Proposition 7, we have $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(J(S)), \mathrm{Aut}_{\mathcal{F}}(X(S))\rangle$. By Theorem 23, it is enough to find the triples $(\mathrm{Aut}_{\mathcal{F}}(J(S)), \mathrm{Aut}_{\mathcal{F}}(S), \mathrm{Aut}_{\mathcal{F}}(X(S)))$ up to conjugation of $\mathrm{Aut}_{\mathcal{F}}(X(S))$ in $\mathrm{Aut}(X(S))$.

By Proposition 12, up to conjugation in $\mathrm{Aut}(J(S))$, $\mathrm{Aut}_{\mathcal{F}}(J(S))$ is isomorphic to a subgroup of $(2 \times M_{10}) : 2$ containing a copy A of A_6. Since $(2 \times M_{10}) : 2/A$ is isomorphic to D_8, up to conjugation in $(2 \times M_{10}) : 2$, there are exactly 8 such subgroups, and these are listed in the first column of Table 2. This and Lemma 22 give the isomorphism classes for $\mathrm{Out}_{\mathcal{F}}(S)$ listed in the third column of Table 2.

We turn now to $\mathrm{Aut}_{\mathcal{F}}(X(S))$. Note that $\mathrm{Aut}_{\mathcal{F}}(X(S))$ is completely determined up to conjugation in $\mathrm{Aut}(X(S))$ once we determine $\mathrm{Out}_{\mathcal{F}}(X(S))$ up to conjugation in $\mathrm{Out}(X(S))$ since $\mathrm{Aut}_{\mathcal{F}}(X(S))$ contains the group $\mathrm{Inn}(X(S))$. As remarked after Proposition 12 and with the same notation, we may identify $\mathrm{Out}(X(S))$ with the group $\mathrm{GSp}_4(3)$, and $\mathrm{Out}_{\mathcal{F}}(X(S))$ is then a subgroup H of $\mathrm{GSp}_4(3)$ containing $\tilde{t}$ and $\tilde{\zeta}$ such that $T := \langle \tilde{t} \rangle$ is a Sylow 3-subgroup of H. Moreover, T is not normal in H since, by definition of $\mathcal{F}$-essential subgroups, H has a strongly 3-embedded subgroup. Then T is not normal in $H \cap I$, and H falls into one of the three cases of Corollary 16.

If H is as in case (a), (i)–(v) of Lemma 18 imply that H is isomorphic to one of the groups listed in rows 1, 3, 7, 4 and 8 of the second column of Table 2,

[1] Note that, by Lemma 18, Lemma 20, and Lemma 21, the structure of the groups in the first three columns of Table 2 determines their isomorphism class.

$\mathrm{Aut}_{\mathcal{F}}(J(S))$	$\mathrm{Out}_{\mathcal{F}}(X(S))$	$\mathrm{Out}_{\mathcal{F}}(S)$	Groups
A_6	$2^{-}S_4$	4	$U_4(3)$
$2\times A_6$	$\mathrm{GL}_3(2):2$	4×2	$U_4(3).2_1$
S_6	$(2\times \mathrm{SL}_2(3)):2$ $(Q_8\times Q_8).S_3$	D_8 D_8	$U_4(3).2_2$ $L_6(q)$, $q\equiv 4,7 \bmod 9$ $U_6(q)$, $q\equiv 2,5 \bmod 9$
M_{10}	$(2^{-}S_4):2$ $2^{-}S_5$	Q_8 Q_8	$U_4(3).2_3$ Mc
$2\times S_6$	$(2\times \mathrm{GL}_2(3)):2$ $(Q_8\times Q_8).(3:D_8)$	$2\times D_8$ $2\times D_8$	$U_4(3).2^2_{122}$ $L_6(q)\langle\phi\rangle$, $q\equiv 4,7 \bmod 9$ $U_6(q)\langle\phi\rangle$, $q\equiv 2,5 \bmod 9$ ϕ field automorphism of order 2
$2\times M_{10}$	$(2^{-}S_4:2):2$ $2^{-}S_5:2$	$2\times Q_8$ $2\times Q_8$	$U_4(3).2^2_{133}$ Aut(Mc)
$A_6:4$	$(Q_8\circ 4).(S_3\times 2)$	2×8	$U_4(3).4$
$(2\times M_{10}):2$	$2^{1+4}_{-}.(S_3\times 2)$ $2^{1+4}_{-}.S_5$	$2\times QD_{16}$ $2\times QD_{16}$	$\mathrm{Aut}(U_4(3))$ Co_2

Table 2. Radical free fusion systems on S.

respectively. If H is as in case (b), then rows 1–7 of the fifth column of Table 1 imply that H is isomorphic to one of the groups listed in rows 6, 10, 1, 3, 2, 5 and 9, respectively. Note that, by Lemma 18, the groups obtained in cases (a) and (b) are isomorphic if and only if they are conjugate.

Assume now that H satisfies case (c) and does not satisfy cases (a) and (b). Then H falls into cases (i) or (iv) of Lemma 21. In case (i), either H is a maximal subgroup of G isomorphic to the normalizer in G of a group of type 2^{1+4}_{-}, which gives the last row of Table 2, or $H\cong 2^{-}S_5$. The latter case cannot occur, for H would satisfy case (b) since the normalizer of a cyclic subgroup of order 4 of G, not contained in I, contains subgroups isomorphic to $2^{-}S_5$, and these are contained in a single G-conjugacy class. In case (iv), we get rows 11 and 12 of the second column of Table 2.

Finally, a direct check in [6] shows that the fusion systems corresponding to the rows of Table 2, except, possibly, for those in rows 4 and 8, are realized by the related groups listed in the last column. Routine computation shows that the same holds for fusion systems in rows 4 and 8. For example, consider the group $L_6(q)$ with $q \equiv 4, 7 \bmod 9$. Then $q-1$ is divisible by 3 but not by 9. Let P be a Sylow 3-subgroup of $L_6(q)$, and let $\mathbb{F}_q$ be the field of order q. By [10, Theorem 4.10.2], P is contained in the normalizer of a frame $\mathcal{D}$ of $\mathbb{F}_q^6$ in $L_6(q)$ and $P = AP_W$, where A is the Sylow 3-subgroup of $C_{L_6(q)}(\mathcal{D})$ and P_W faithfully permutes the elements of $\mathcal{D}$ as a Sylow 3-subgroup of the alternating group over $\mathcal{D}$. Since $|\mathcal{D}| = 6$ and A_6 has a unique, up to equivalence, irreducible representation of degree 4 on the field of order 3, it follows that P is isomorphic to S. By [10, Remark 4.10.4], $N_{L_6(q)}(A)/C_{L_6(q)}(A)$ is isomorphic to a section of S_6. Since A is characteristic in D, we get $N_{L_6(q)}(A)/C_{L_6(q)}(A) \cong S_6$. This means that $\mathcal{F}_S(L_6(q))$ corresponds either to line 3 or to line 4 of Table 2. Moreover, $L_6(q)$ has a subgroup isomorphic to $\mathrm{SL}_3(q) \circ \mathrm{SL}_3(q)$, and $\mathrm{SL}_3(q)$ contains a maximal subgroup isomorphic to $3_+^{1+3} : Q_8$. It follows that, in $L_6(q)$, there is a subgroup isomorphic to 3_+^{1+4}, whose normalizer contains a copy of $Q_8 \times Q_8$, which implies that $\mathcal{F}_S(L_6(q))$ is the fusion system corresponding to line 4. □

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Bibliography

[1] M. Aschbacher, On the maximal subgroups of the finite classical groups, *Invent. Math.* **76** (1984), no. 3, 469–514.

[2] M. Aschbacher, *Finite Group Theory*, 2nd ed., Cambridge Stud. Adv. Math. 10, Cambridge University Press, Cambridge, 2000.

[3] M. Aschbacher, R. Kessar and B. Oliver, *Fusion Systems in Algebra and Topology*, London Math. Soc. Lecture Note Ser. 391, Cambridge University Press, Cambridge, 2011.

[4] E. Baccanelli, *Reduced fusion systems on Sylow* 3*-subgroups of the Mclaughlin group*, PhD Thesis, Università di Milano Bicocca, 2017.

[5] J. N. Bray, D. F. Holt and C. M. Roney-Dougal, *The Maximal Subgroups of the Low-Dimensional Finite Classical Groups*, London Math. Soc. Lecture Note Ser. 407, Cambridge University Press, Cambridge, 2013.

[6] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups. Maximal Subgroups and Ordinary Characters for Simple Groups*, Oxford University Press, Oxford, 1985.

[7] D. A. Craven, *The Theory of Fusion Systems. An Algebraic Approach*, Cambridge Stud. Adv. Math. 131, Cambridge University Press, Cambridge, 2011.

[8] D. A. Craven and A. Glesser, Fusion systems on small p-groups, *Trans. Amer. Math. Soc.* **364** (2012), no. 11, 5945–5967.

[9] A. Diaz, A. Ruiz and A. Viruel, All p-local finite groups of rank 2 for odd prime p, *Trans. Amer. Math. Soc.* **359** (2007), no. 4, 1725–1764.

[10] D. Gorenstein, R. Lyons and R. Solomon, *The Classification of the Finite Simple Groups. Number 3. Part I. Chapter A. Almost Simple K-groups*, Math. Surveys Monogr. 40.3, American Mathematical Society, Providence, 1998.

[11] V. Grazian, *Fusion systems on p-groups of sectional rank* 3, PhD Thesis, University of Birmingham, 2017.

[12] V. Grazian, Fusion systems containing pearls, *J. Algebra* **510** (2018), 98–140.

[13] V. Grazian, Fusion systems on p-groups of sectional rank 3, preprint (2018), `https://arxiv.org/abs/1804.00883v1`.

[14] R. Moragues Moncho, *Fusion systems on p-groups with an extraspecial subgroup of index p*, PhD Thesis, University of Birmingham, 2018, expected.

[15] B. Oliver, Reduced fusion systems over 2-groups of sectional rank at most 4, *Mem. Amer. Math. Soc.* **1131** (2016), 1–100.

[16] C. Parker and J. Semeraro, Fusion systems over a Sylow p-subgroup of $G_2(p)$, *Z. Math.* **289** (2018), 629–662.

[17] C. Parker and G. Stroth, An improved 3-local characterization of McL and its automorphism group, *J. Algebra* **406** (2014), 69–90.

[18] A. Ruiz and A. Viruel, The classification of p-local finite groups over the extraspecial group of order p^3 and exponent p, *Math. Z.* **248** (2004), 45–65.

[19] M. Suzuki, *Group Theory II*, Springer, New York, 1986.

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