# Measuring Satisfaction and Power in Influence Based Decision Systems

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**Abstract**

We introduce collective decision-making models associated with influence spread under the linear threshold model in social networks. We define the *oblivious* and the *non-oblivious influence* models. We also introduce the generalized opinion leader-follower model (gOLF) as an extension of the opinion leader-follower model (OLF) proposed by van den Brink et al. (2011). In our model we allow rules for the final decision different from the simple majority used in OLF. We show that gOLF models are non-oblivious influence models on a two-layered bipartite influence digraph.

Together with OLF models, the *satisfaction* and the *power* measures were introduced and studied. We analyze the computational complexity of those measures for the decision models introduced in the paper. We show that the problem of computing the satisfaction or the power measure is #P-hard in all the introduced models even when the subjacent social network is a bipartite graph. Complementing this result, we provide two subfamilies of decision models in which both measures can be computed in polynomial time.

We show that the collective decision functions are monotone and therefore they define an associated simple game. We relate the satisfaction and the power measures with the Rae index and the Banzhaf value of an associated simple game. This will allow the use of known approximation methods for computing the Banzhaf value, or the Rae index to their practical computation.

*Keywords:* Influence Spread, Game theory, Decision system, Simple game, Power index, Complexity theory

## 1. Introduction

Opinion leadership is a well known and established model for communication in sociology and marketing. It comes from the *two-step flow of communication* theory proposed in the 1940s [24]. This theory recognizes the existence of collective decision-making situations in societies formed by actors called *opinion leaders*, who exert influence over other kind of actors called the *followers*, resulting in a two-step decision process [24, 18]. In the first step, all actors receive information from the environment and they generate their own decisions. In the second step, a flow of influence from some actors over others is able to change the choices of some of them [34]. Following those ideas, van den Brink et al. introduced a collective decision-making model called *opinion leader-follower model* (OLF) [35]. The subjacent influence structure is based on a society with opinion leaders, followers and *independent actors*. This latter kind of actors neither can influence nor can be influenced by other actors of the society. The model includes a procedure to reach an individual final decision, from a given initial decision of the participants. Finally, the global decision is reached after applying the single majority rule to the final decisions of the actors. In the OLF model, the selected decision rule restricts the society to have an odd number of participants. Motivated by the theoretical study of the effects that collective decision-making can have on the participants, *satisfaction* and *power* measures were defined for an OLF model in van den Brink et al. [35]. The satisfaction of an actor is the number of

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initial decisions for which the final collective decision coincides with the initial decision of the actor. The power of an actor is the number of initial decision for which the collective decision changes when the actor changes its initial decision. The measures were intended to provide a measure of the real leadership in the decision systems. Observe that the classification as leader or follower in an OLF model correspond to the topology of the associated node in the network. However, leaders or followers can have different values of satisfaction and power.

From another point of view, *simple games* are the mathematical model used to formulate the situations in which the actors or players have to decide about one alternative. Simple games were firstly introduced in 1944 by von Neumann and Morgenstern [37]. A simple game is determined by its set of *winning coalitions*, i.e., those sets of actors that can force a "yes" decision. Together with the study of simple games, several power indices and values have been proposed to measure the relevance of a player in a simple game. Among them, we highlight the Banzhaf value and the Rae index, because they are relevant to study the satisfaction and power on collective decision-making models. The Banzhaf value measures the proportion of winning coalitions in which a player plays a critical role, i.e., if he/she steps out of the coalition, the remaining players cannot make the issue pass. This value was firstly introduced by Penrose [29], being rediscovered in 1965 by Banzhaf [5] and used again in 1971 by Coleman [7]. That is why the Banzhaf value (actually, the Banzhaf index, which is the normalized version of the Banzhaf value) is also known as the *Penrose index*, the *Penrose-Banzhaf index* or the *Banzhaf-Coleman index*. On the other hand, the Rae index was introduced by Rae [30] for anonymous games and afterwards it was applied by Dubey and Shapley [11] to simple games. For a player *i*, the Rae index counts the proportion of favorable coalitions for the player, i.e, the winning coalitions containing *i* and the coalitions not containing *i* that do not win.

Some relevant subfamilies of simple games are related to voting systems, in particular *weighted voting games* and *influence games*. A weighted voting game on $n$ players is defined by a set of $n + 1$ values, $(q; w_1, \ldots, w_n)$ where $q$ is called *quota* and $w_i > 0$ is the weight of the player $i$ [37]. A coalition of players wins when the sum of the weights of the players in the coalition is equal to or greater than $q$. Weighted voting games are a proper subset of simple games [33] and play a fundamental role in the analysis of electoral systems [32]. Influence games are based on social networks and incorporate a procedure of influence spreading previous to the taking of a vote. A coalition wins depending on whether it can influence enough participants to reach the minimum required. Here the society is modeled by an *influence graph*, where the influence spreads according to the linear threshold model [16, 31, 19]. Influence games are general enough to capture the complete class of simple games, since every simple game can be represented by an influence game [26].

The computational complexity of obtaining the Banzhaf value is a problem of interest in the context of simple games. Computing the Banzhaf value is polynomial time solvable for simple games represented by the set of winning coalitions, but it is #P-complete for simple games represented by the set of minimal[1] winning coalitions [2]. The problem is also known to be #P-hard for weighted voting games [8] and influence games [26]. For the case of influence games, the known #P-hardness result leaves open the question on whether the problem is easy or hard when the influence graph is restricted to be bipartite. An affine-linear relation between the Rae index and the Banzhaf value was given in [23]. Thus, the computational complexity of both measures is the same.

In this paper we consider more general scenarios in which the final decision is obtained from an initial individual vote followed by a process of spread of influence in general social networks. Observe that in general networks we lose the ex-ante classification of the actors as leaders or followers assumed in the OLF model. In our decision models, we consider a set of process initiators that, in analogy with influence games, we call *the players*. It seems natural to consider measures like satisfaction and power as the means to classify the participants (players and non-players) with respect to leadership. The higher the power or satisfaction of a participant seems to indicate higher level of leadership. In this view, it is of interest to analyze the properties of satisfaction and power and its relation to the well established power indices for simple games, in particular the Banzhaf value and the Rae index [14].

Working in this direction we introduce two new simple collective decision-making models. These models are based on the influence spread mechanism of influence games. Our two models differ in the way in which the initial vote of the non-players is taken into consideration. In the *oblivious influence models*, the initial decision of the non-players is not taken into account and it is replaced by a negative initial decision. The final individual decision is taken after running a phase of spread of the 'yes' vote. The final collective decision is set to "yes" when the total number

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[1]A minimal winning coalition is a winning coalition such that by removing any player we obtain a losing coalition.
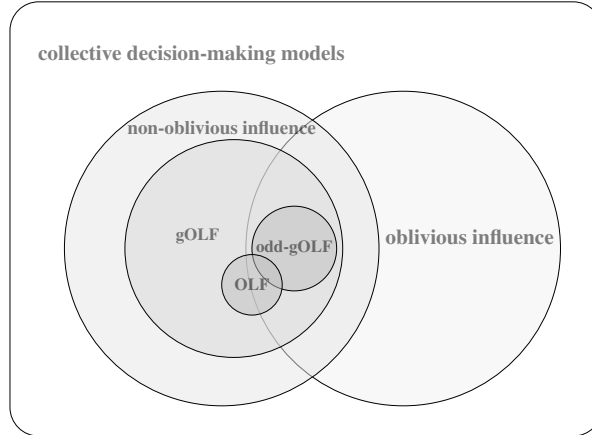
Figure 1: Inclusion relationship between subfamilies of collective decision-making models.

of 'yes' votes reach the required quota (as in weighted voting games). In the *non-oblivious influence models*, after running a phase of spread of the 'yes' vote, non-players determine their final vote in a one step process that takes into consideration intermediate decision of predecessors and its initial inclination. We introduce also an intermediate model that constitute a natural extension of OLF, the *generalized* OLF (gOLF). In a gOLF model, we allow final collective decision mechanisms different from the simple majority rule used in OLF models. We also consider a subfamily of gOLF, the odd-gOLF, in which we require that the nodes in the associated influence graph have an odd in-degree. This subfamily plays an important role in our analysis as it lies in the intersection of oblivious and non-oblivious models.

Once the models are introduced, we are interested first in analyzing their properties and the potential relationships among them and with simple games. We show that the introduced families of models follows the inclusion schema depicted in Figure 1. Interestingly enough, we are able to show that the collective decision-making models introduced in this paper are monotonic. In consequence, they can be reinterpreted, under a simple translation, as simple games. For doing so, we take as winning coalitions those initial decisions of the participants that lead to a "yes" decision. Under this interpretation, we show that the satisfaction measure coincides with the *Rae index* and that the power measure is twice the *Banzhaf value*. Therefore, according to the use of the Banzhaf value in game theory, we get a natural justification of the use of satisfaction or power as measures for leadership [13].

Once the suitability of the two measures to rank leadership is justified, we move into the analysis of the difficulty of computing such measures. We show that computing satisfaction or power are #P-hard problems for odd-gOLF and therefore this hardness result holds in all the decision models containing odd-gOLF. Our hardness proof comes from a reduction from the #$\frac{2}{3}$-VC problem, i.e., the problem of computing the number of vertex covers with exactly $\frac{2}{3}n$ vertices. The later problem is equivalent to the problem of counting independent sets of size $\frac{1}{3}n$ which was shown to be #P-hard, by a reduction from the well known #P-complete #3-SAT problem, the counting version of the 3-satisfiability problem [17]. It is interesting to note that the influence graph associated with an odd-gOLF is formed by a simple two-layered bipartite digraph. On the other hand, as we show that the satisfaction and the power measures are related with the Rae index and the Banzhaf value of an associated simple game, the known approximation methods for computing the Banzhaf value or the Rae index can be applied to these two measures [12, 4].

Given the computational hardness results, we explore the types of influence graphs for which the satisfaction and the power measures can be computed in polynomial time. As our hardness result holds for two-layered bipartite digraphs, there is not much room for finding tractable subfamilies of social networks. We explore the possibility of tractable cases in multilayered bipartite digraphs with additional restrictions. Those structures are still able to represent a "more-than-two-step flow of communication", being more general than the ones considered in the OLF model with respect to the leadership role of the participants. In multilayered bipartite digraphs we still have natural leaders, i.e., nodes with in-degree 0, and followers, i.e., nodes with out-degree 0. Furthermore, we have an additional set of actors that we call the *mediators*. Those actors can be influenced by the opinion leaders and may influence the

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followers or other mediators. Thus, we extend the OLF models by allowing an intermediate set of actors to play the role of mediators between leaders and followers. The first family, the *strong hierarchical influence graphs*, is formed by multilayered bipartite digraphs in which the connection among consecutive layers is restricted to be the disjoint union of complete bipartite graphs. So, influence is exerted in an all-to-all fashion following a hierarchical structure. In the second family, the *star influence graphs*, we have only one mediator, but in this case we allow a two way interaction between the mediator and some opinion leaders. Star influence graphs model a natural mediation schema occurring in society. We show, for these two subfamilies of bipartite influence graphs, that the satisfaction and the power measures can be computed in polynomial time. Our result holds for both the oblivious and the non-oblivious associated decision-making models.

The paper is organized as follows. Section 2 presents the definitions of OLF, the satisfaction and the power measures, simple and influence games. In Section 3, we introduce the *oblivious* and the *non-oblivious* influence models and the gOLF and the odd-gOLF models. We show some of their basic properties and the relationship among the models. We establish here also the connection with simple games. In Section 4, we prove the hardness of the satisfaction problem for odd-gOLF. This result also implies the #P-hardness of computing satisfaction in the other three models that include odd-gOLF, as well as #P-hardness for the power problem in all those models. Sections 5 and 6 are devoted to the definition of strong hierarchical influence graphs and star influence graphs, respectively. We provide algorithms computing satisfaction (and therefore power) in polynomial time, for both the oblivious and the non-oblivious influence models, on the selected influence graph family. Finally, in Section 7 we provide our conclusions and some open problems.

## 2. Preliminary concepts

In this section we introduce the necessary definitions and concepts, such as the opinion leader-follower model, the satisfaction and the power measures, simple games and influence games.

In what follows, we use standard graph notation to represent the basic structure of the collective decision-making models. Let $G = (V, E)$ be a graph, $V(G)$ is its vertex set, and $E(G)$ its edge set. We simply use $V$ and $E$ when there is no risk of confusion. Given a graph $G = (V, E)$ and a subset of vertices $X \subseteq V$, $G[X]$ denotes the subgraph induced by $X$. Let $e = (u, v)$ be an edge, we use sometimes set notation associating $e$ with the set $\{u, v\}$. In this way, for $w \in V$, $w \in e$ represents that $w$ is $u$ or $v$. Further, $n = |V|$ denotes the number of vertices of the graph. All the graphs considered in this paper are directed (i.e., digraphs), unless otherwise stated, without loops and multiple edges.

In order to study the sequential flow in the decision-making processes, we can differentiate between the successors and the predecessors of each vertex. Let $i \in V$ be a vertex, $S_G(i) = \{j \in V \mid (i, j) \in E\}$ denotes the set of *successors* of $i$, and $P_G(i) = \{j \in V \mid (j, i) \in E\}$ the set of *predecessors* of $i$. We extend this notation to vertex subsets $X \subseteq V$, so that $S_G(X) = \{i \in V \mid \exists j \in X, i \in S_G(j)\}$ and $P_G(X) = \{i \in V \mid \exists j \in X, i \in P_G(j)\}$ denote the set of successors and predecessors of all vertices that belong to $X$, respectively. Further, $\delta^-(i) = |P_G(i)|$ and $\delta^+(i) = |S_G(i)|$ denote the in-degree and the out-degree of the vertex $i$, respectively. One of the most basic graph families to distinguish between predecessors and successors is the one of two-layered bipartite digraphs. A *two-layered bipartite digraph* is a bipartite graph $G = (V_1, V_2, E)$ with $V(G) = V_1 \cup V_2$ and $E \subseteq V_1 \times V_2$, i.e., such that for $i \in V_1$, $P_G(i) = \emptyset$ and, for $i \in V_2$, $S_G(i) = \emptyset$. Another graph family that we will use later is that of isolated vertices. We denote by $I_a$ the graph formed by $a = |V(I_a)|$ isolated vertices.

We also need basic concepts of set theory. As usual, given a finite set $N$, $\mathcal{P}(N)$ denotes its power set. A family of subsets $\mathcal{W} \subseteq \mathcal{P}(N)$ is said to be *monotonic* if $X \in \mathcal{W}$ and $X \subseteq Z$ implies $Z \in \mathcal{W}$. In some cases we need to establish a relation between vectors $x = (x_1, \ldots, x_n) \in \{0, 1\}^n$ and sets $X \subseteq \{1, \ldots, n\}$. For doing so, we use the notation $X(x) = \{0 \le i \le n \mid x_i = 1\}$, and $x(X) = (x_1, \ldots, x_n)$ with $x_i = 1$ if $i \in X$, and $x_i = 0$ otherwise.

Every collective decision-making model $\mathcal{M}$ considered in this paper can be defined on an unweighted digraph. The actors (vertices) initially choose among two alternatives, 1 or 0. Given a set of $n$ actors, the initial decision of all actors is represented by an *initial decision vector* $x \in \{0, 1\}^n$. These initial decisions may change, depending on the model and the interactions between the actors. All these possible changes lead to a *final decision vector* $c^{\mathcal{M}}(x) \in \{0, 1\}^n$ of the actors. Finally, the *collective decision function* $C_{\mathcal{M}} : \{0, 1\}^n \to \{0, 1\}$ of the model, that depends on $c^{\mathcal{M}}(x)$, assigns a final decision to the system. In general, we drop the explicit reference to $\mathcal{M}$ when the model is clear from the context.

Note that the collective decision function is usually defined through a decision process on a graph. It may include many parameters, and its computational complexity might be high. Nevertheless, for the models considered in this
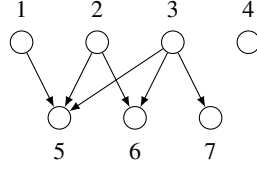
Figure 2: A two-layered bipartite digraph.

paper, the collective decision functions are computable in polynomial time. In Section 2.1 we formally define those vectors and functions for the original model from which all others will be obtained.

### 2.1. The opinion leader-follower model, satisfaction and power

The opinion leader-follower model was introduced by van den Brink et al. [35]. It is based on two-layered bipartite digraphs $G$, whose vertex sets $V(G)$ can be partitioned into three kind of actors:

- the *opinion leaders*, $L(G) = \{i \in V \mid P_G(i) = \emptyset \text{ and } S_G(i) \neq \emptyset\}$,

- the *followers*, $F(G) = \{i \in V \mid S_G(i) = \emptyset \text{ and } P_G(i) \neq \emptyset\}$, and

- the *independent actors*, $I(G) = V \setminus (L(G) \cup F(G)) = \{i \in V \mid S_G(i) = \emptyset \text{ and } P_G(i) = \emptyset\}$.

Note that if $(i, j) \in E(G)$, then $i \in L(G)$ and $j \in F(G)$. When there is no risk of ambiguity, we simply use $S(i)$, $P(i)$, $L$, $F$ or $I$, omitting the corresponding graph $G$. Note also that $S(L) = F$, $P(F) = L$ and $S(I) = P(I) = \emptyset$. For simplicity of notation, we will also sometimes use the set $FI(G) = F(G) \cup I(G)$.

Next, we provide the definition of an opinion leader-follower model [35] in our context of collective decision-making models.

**Definition 1 ([35]).** An *opinion leader-follower model (OLF)* is a tuple $\mathcal{M} = (G, r)$ where $G = (V, E)$ is a two-layered bipartite digraph that represents the actors' relations, and the *fraction value* $r \in [\frac{1}{2}, 1)$ is a rational number. The number of vertices $n = |V|$ is odd. The collective decision function $C_{\mathcal{M}}$ is defined as follows. Let $x \in \{0, 1\}^n$ be an initial decision vector, then the final decision vector $c = c^{\mathcal{M}}(x)$ has the following components, for $1 \leq i \leq n$:

$$
c_i = \begin{cases}
1 & \text{if } |\{j \in P_G(i) \mid x_j = 1\}| > \lfloor r \cdot \delta^{-1}(i) \rfloor \\
0 & \text{if } |\{j \in P_G(i) \mid x_j = 0\}| > \lfloor r \cdot \delta^{-1}(i) \rfloor \\
x_i & \text{otherwise}
\end{cases}
$$

where $\lfloor y \rfloor$ denotes the largest integer not greater than $y$. Finally, the collective decision function $C_{\mathcal{M}} : \{0, 1\}^n \rightarrow \{0, 1\}$ is defined by simple majority voting:

$$
C_{\mathcal{M}}(x) = \begin{cases}
1 & \text{if } |\{i \in V \mid c_i(x) = 1\}| \geq \frac{n+1}{2} \\
0 & \text{otherwise.}
\end{cases}
$$

Note that the values $c_i^{\mathcal{M}}$, for $i \in V$, and $C_{\mathcal{M}}$ are well defined.

Observe that opinion leaders and independent actors always keep their own inclinations in the final decision vector. However, a follower may take a final decision different from its own initial inclination. For a follower, the fraction value $r$ indicates the fraction of predecessors with a common decision required to adopt this common decision. Observe that, if $r = 1/2$, for a follower with an even number of predecessors, it could be the case that half of its predecessors have initial decision 1 and the other half 0. In such a case the follower keeps its initial decision.

**Example 1.** Figure 2 illustrates a two-layered bipartite digraph $G = (V, E)$. Here the opinion leaders are $L(G) = \{1, 2, 3\}$, the only independent actor is $I(G) = \{4\}$, and the followers are $F(G) = \{5, 6, 7\}$. Let us consider the OLF $\mathcal{M} = (G, 1/2)$. Hence, since $\delta^-(5) = 3$, $\delta^-(6) = 2$ and $\lfloor 3/2 \rfloor = \lfloor 2/2 \rfloor = 1$, both followers 5 and 6 need at least two predecessors choosing different from them to change their initial choice. Analogously, since $\delta^-(7) = 1$ and $\lfloor 1/2 \rfloor = 0$,

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| $x$ | $c^{\mathcal{M}}(x)$ | $C_{\mathcal{M}}(x)$ |
|---|---|---|
| 0000000 | 0000000 | 0 |
| ⋮ | ⋮ | ⋮ |
| 0011001 | 0011001 | 0 |
| 0011010 | 0011011 | 1 |
| 0011011 | 0011011 | 1 |
| 0011100 | 0011001 | 0 |
| 0011101 | 0011001 | 0 |
| 0011110 | 0011011 | 1 |
| 0011111 | 0011011 | 1 |
| 0100000 | 0100000 | 0 |
| ⋮ | ⋮ | ⋮ |
| 0101111 | 0101010 | 0 |

| $x$ | $c^{\mathcal{M}}(x)$ | $C_{\mathcal{M}}(x)$ |
|---|---|---|
| 0110000 | 0110111 | 1 |
| ⋮ | ⋮ | ⋮ |
| 0111111 | 0111111 | 1 |
| 1000000 | 1000000 | 0 |
| ⋮ | ⋮ | ⋮ |
| 1001111 | 1001000 | 0 |
| 1010000 | 1010101 | 1 |
| ⋮ | ⋮ | ⋮ |
| 1011111 | 1011111 | 1 |
| 1100000 | 1100100 | 0 |
| 1100001 | 1100100 | 0 |

| $x$ | $c^{\mathcal{M}}(x)$ | $C_{\mathcal{M}}(x)$ |
|---|---|---|
| 1100010 | 1100110 | 1 |
| 1100011 | 1100110 | 1 |
| 1100100 | 1100100 | 0 |
| 1100101 | 1100100 | 0 |
| 1100110 | 1100110 | 1 |
| 1100111 | 1100110 | 1 |
| 1101000 | 1101100 | 1 |
| ⋮ | ⋮ | ⋮ |
| 1101111 | 1101110 | 1 |
| 1110000 | 1110111 | 1 |
| ⋮ | ⋮ | ⋮ |
| 1111111 | 1111111 | 1 |

Table 1: Collective decisions $C_{\mathcal{M}}(x)$, for each of the possible initial decision vectors $x$, in the OLF $\mathcal{M}$ of Example 1.

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Sat($i$) | 88 | 88 | 104 | 72 | 64 | 68 | 64 |
| Pow($i$) | 48 | 48 | 80 | 16 | 0 | 8 | 0 |

Table 2: The satisfaction and the power values for the OLF model given in Example 1.

the follower 7 needs that its only predecessor chooses different from him/her to change its initial choice. Note that the collective decision of the system is 1 if the final decision vector has at least four 1's. Table 1 shows the collective decision, for each of the possible initial decision vectors, starting from $(0, 0, 0, 0, 0, 0, 0)$ and adding $+1$ (in binary system) until reaching the vector $(1, 1, 1, 1, 1, 1, 1)$.

Given an OLF, it is relevant to measure how satisfied each actor is with respect to the collective decision obtained, as well as, the real power exerted by the actor to reach such collective decision. Both the satisfaction and the power measures were defined for OLF models [35]. However, they can be applied to any collective decision-making model.

**Definition 2.** Let $\mathcal{M}$ be a collective decision-making model over a set of $n$ actors. For an initial decision vector $x \in \{0, 1\}^n$, an actor $i$ is *satisfied* when $C_{\mathcal{M}}(x) = x_i$. The *satisfaction* measure of the actor $i$ corresponds to the number of initial decision vectors for which the actor is satisfied, i.e.,

$$\text{Sat}_{\mathcal{M}}(i) = |\{x \in \{0, 1\}^n \mid C_{\mathcal{M}}(x) = x_i\}|.$$

To define the power measure, we need to introduce additional notation. For an initial vector $x \in \{0, 1\}^n$, we denote $\overline{x}_{-i}$ as the decision vector in which actor $i$ changes its decision to $1 - x_i$ while the decisions of the other actors remain unchanged.

**Definition 3.** Let $\mathcal{M}$ be a collective decision-making model over a set of $n$ actors. For an initial decision vector $x \in \{0, 1\}^n$, an actor $i$ can exert *power* when $C_{\mathcal{M}}(\overline{x}_{-i}) = 1 - C_{\mathcal{M}}(x)$. The *power* measure of the actor $i$ corresponds to the number of initial decision vectors for which actor $i$ can exert power, i.e.,

$$\text{Pow}_{\mathcal{M}}(i) = |\{x \in \{0, 1\}^n \mid C_{\mathcal{M}}(\overline{x}_{-i}) = 1 - C_{\mathcal{M}}(x)\}|.$$

In Table 2 we provide the values of the satisfaction and the power measures for the OLF model given in Example 1. Note that the corresponding ordering is the same in both measures. It can also be seen that the participants that are called leaders/followers according to the graph topology do not have always the same value. The level of leadership of 3 is higher than that of 1 or 2. Follower 6 have higher leadership level than 5 or 7.

Associated with the satisfaction and the power measures, we consider the following computational problems.

Instance:       A collective decision-making model $\mathcal{M}$ and an actor $i$.

Output:       $\text{Sat}_{\mathcal{M}}(i)$.

POWER

Instance:       A collective decision-making model $\mathcal{M}$ and an actor $i$.

Output:       $\text{Pow}_{\mathcal{M}}(i)$.

### 2.2. Simple and influence games

In this subsection we define simple and influence games. In the scenario of simple games we follow notation from [33] and, for influence games, notation is taken from [26]. We use $N$ to denote the set of players and $X$ for a subset of $N$.

**Definition 4.** A *simple game* is a tuple $\Gamma = (N, \mathcal{W})$, where $N$ is a finite set of *players* and $\mathcal{W} \subseteq \mathcal{P}(N)$ is a monotonic family of subsets of $N$.

The subsets of the set of players $N$ are called *coalitions*. The set $N$ is the *grand coalition*. A set $X \in \mathcal{W}$ is called a *winning coalition*. The complement of the family of winning coalitions is the family of *losing coalitions* $\mathcal{L}$, i.e., $\mathcal{L} = \mathcal{P}(N) \setminus \mathcal{W}$. Any of those set families determine uniquely the game $\Gamma$ and constitute one of the usual forms of representation for simple games [33]. Observe that the size of such representation is not, in general, polynomial in the number of players [27].

**Example 2.** An example of simple game is the following: the set of players is $N = \{1, 2, 3, 4, 5\}$ and the set of winning coalitions is $\mathcal{W} = \{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 2, 3, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}\}$. Observe that the only requirement is that the set $\mathcal{W}$ must be monotonic.

In the study of simple games, power indices are used to measure the relevance that a player has in the game. We recall here the definition of two classic power indices, the Banzhaf value [29, 5] and the Rae index [30].

**Definition 5.** Let $\Gamma = (N, \mathcal{W})$ be a simple game. The *Banzhaf value* of player $i \in N$ is defined as

$$\text{Bz}_{\Gamma}(i) = |\{X \in \mathcal{W} \mid X \setminus \{i\} \notin \mathcal{W}\}|.$$

The *Rae index* of player $i \in N$ is

$$\text{Rae}_{\Gamma}(i) = |\{X \in \mathcal{W} \mid i \in X\}| + |\{X \notin \mathcal{W} \mid i \notin X\}|.$$

Dubey and Shapley [11] (see also [23]) established an affine-linear relation between the Rae index and the Banzhaf value. Let $\Gamma = (N, \mathcal{W})$ be a simple game and $i \in N$,

$$\text{Rae}_{\Gamma}(i) = 2^{n-1} + \text{Bz}_{\Gamma}(i). \tag{1}$$

Note that, for $i \in N$, as $\text{Bz}_{\Gamma}(i) \geq 0$, then $\text{Rae}_{\Gamma}(i) \geq 2^{n-1}$.

Influence games are simple games based on the linear threshold model for influence spreading [16, 31, 19]. In those games a coalition wins if it can influence enough actors to accept the proposal. Before defining influence games we introduce (unweighted) influence graphs and the activation process by which influence spreads in the network.

**Definition 6.** An *influence graph* is a tuple $(G, f)$, where $G = (V, E)$ is a directed graph and $f : V \to \mathbb{Q}^+$ is a labeling function assigning to any vertex a non-negative rational value.

Let $(G, f)$ be an influence graph and let $X \subseteq V$ be an initial activation. The *activation process*, at time $t$, $0 \leq t \leq n$, activates a set of vertices $F^t(X)$ defined as follows

$$F^0(X) = X$$
$$F^t(X) = F^{t-1}(X) \cup \{i \in V \mid |P_G(i) \cap F^{t-1}(X)| \geq f(i)\}, \text{ for } 1 \leq t \leq n.$$

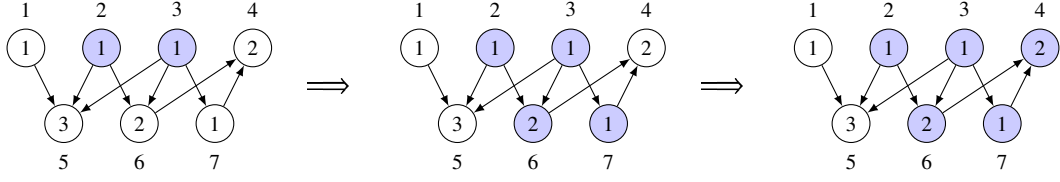The *spread of influence* of $X$ in $(G, f)$ is the set $F(X) = F^n(X)$.

Figure 3: Spread of influence on an influence graph, represented by coloring of nodes from the initial activation $X = \{2, 3\}$.

Observe that, in the activation process, a new vertex is activated whenever the number of vertices activated in the previous step pointing to it is greater or equal to its label. Thus, the labeling function quantifies the number of neighbors that each actor requires to be influenced. By definition, the activation process is monotonic. Note also that, given an initial activation set $X \subseteq V$, $F(X)$ can be computed in polynomial time.

Influence games are simple games defined with the help of an influence graph where a coalition wins if it is able to influence a required number of actors given by a "quota".

**Definition 7.** An *influence game* is a tuple $\Gamma = (G, f, q, N)$, where $(G, f)$ is an influence graph, $q$ is an integer *quota*, and $N \subseteq V(G)$ is the set of players of the game. $X \subseteq N$ is a winning coalition if $|F(X)| \geq q$, otherwise $X$ is a losing coalition. We can assume that $0 \leq q \leq n = |V(G)|$.

Note that, in an influence game, the set of players can be a proper subset of the set of vertices. Observe that the quota $q$ determines the minimum number of final activated nodes required to win. As $F$ is monotonic, influence games are a subfamily of simple games. Moreover, any simple game admits a representation as influence game [26].

**Example 3.** Let $(G, f)$ be an influence graph, where $V(G) = \{1, 2, 3, 4, 5, 6, 7\}$, $E(G) = \{(1, 2), (2, 5), (2, 6), (3, 5), (3, 6), (3, 7), (6, 4), (7, 4)\}$, and the labeling function is defined by $f(1) = f(2) = f(3) = f(7) = 1$, $f(4) = f(6) = 2$ and $f(5) = 3$. Figure 3 shows the spread of influence in $(G, f)$ starting from the initial activation set $X = \{2, 3\} \subseteq V(G)$. After the first step we have $F^1(X) = \{2, 3, 6, 7\}$, and after the second step, the last one, we have $F^2(X) = F(X) = \{2, 3, 4, 6, 7\}$. Note that the actor 5 cannot be influenced unless actors 1, 2, and 3 belong to the initial activation. Now, let $(G, f, q = 4, N = \{1, 2, 3\})$ be an influence game. The only actors that can belong to the initial activation are the players in the set $N = \{1, 2, 3\}$. Furthermore, the quota $q = 4$ ensures, for example, that $X = \{2, 3\}$ is a winning coalition ($|F(X)| = 5 \geq q$), but $X_1 = \{1, 2\}$ or $X_2 = \{1, 3\}$ are losing coalitions ($|F(X_1)| = 2 < q$ and $|F(X_2)| = 3 < q$). Indeed, it is easy to check that this influence game is equivalent to the simple game $(N, \mathcal{W})$, with $\mathcal{W} = \{\{2, 3\}, \{1, 2, 3\}\}$.

## 3. Oblivious, non-oblivious and generalized opinion leader-follower models

From the previous section, it seems natural to consider collective decision-making models which use influence graphs to exert influence before the final decision is taken. This leads us to define two new collective decision-making models. The models differ in the way that, after a process of influence spread, the final decision is taken. After that, we focus on a slight generalization of the OLF, denoted by gOLF, and on a subfamily of the latter, denoted by odd-gOLF, that belongs to the intersection of oblivious and non-oblivious models. We will also establish the inclusion relationships between different families of collective decision-making models.

### 3.1. Oblivious and non-oblivious influence models

In what follows, we focus on models where the actors are the set of vertices of an influence graph. Furthermore, the collective decision function takes into consideration the spread of influence process running on it. In such a setting a player represents either some sort of opinion leader or independent actor, and a non-player a sort of follower. The main difference when considering graphs more general than the two-layered bipartite digraphs, is that some players could be convinced of a change of opinion and take a final decision different from the initial one. This behaviour is not possible in OLF. Based on this idea, we define two collective decision-making models.

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**Definition 8.** An *oblivious influence model* is a collective decision-making model $\mathcal{M} = (G, f, q, N)$, where $(G, f)$ is an influence graph with positive labeling function, $q$ is the quota and $N \subseteq V$ is the set of players. The collective decision function is defined as follows. For $x \in \{0, 1\}^n$, we define the final decision vector $c = c^{\mathcal{M}}(x)$ as

$$c_i = \begin{cases} 1 & \text{if } i \in F(X(x) \cap N) \\ 0 & \text{otherwise} \end{cases}$$

Moreover, the collective decision function, for $x \in \{0, 1\}^n$, is defined as

$$C_{\mathcal{M}}(x) = \begin{cases} 1 & \text{if } |\{i \in V(G) \mid c_i = 1\}| \geq q \\ 0 & \text{otherwise.} \end{cases}$$

Note that condition $|\{i \in V(G) \mid c_i = 1\}| \geq q$ is equivalent to $|F(X(x) \cap N)| \geq q$.

**Definition 9.** A *non-oblivious influence model* is a collective decision-making model $\mathcal{M} = (G, f, q, N)$, where $(G, f)$ is an influence graph with positive labeling function, $q$ is the quota and $N \subseteq V$ is the set of players. The collective decision function is defined as follows. For $x \in \{0, 1\}^n$, $p_i(x) = |F(X(x) \cap N) \cap P(i)|$ and $q_i(x) = |P(i) \setminus F(X(x) \cap N)|$. For $i \in V(G) \setminus N$, we define the final decision vector $c = c^{\mathcal{M}}(x)$ as

$$c_i = \begin{cases} 1 & \text{if } p_i(x) \geq f(i) \text{ and } q_i(x) < f(i) \\ 0 & \text{if } q_i(x) \geq f(i) \text{ and } p_i(x) < f(i) \\ x_i & \text{otherwise} \end{cases}$$

and, for $i \in N$,

$$c_i = \begin{cases} 1 & \text{if } i \in F(X(x) \cap N) \\ 0 & \text{otherwise.} \end{cases}$$

Finally, the collective decision function, for $x \in \{0, 1\}^n$, is defined as

$$C_{\mathcal{M}}(x) = \begin{cases} 1 & \text{if } |\{i \in V(G) \mid c_i = 1\}| \geq q \\ 0 & \text{otherwise.} \end{cases}$$

In order to simplify notation, we often make an abuse of notation and use $\Gamma = (G, f, q, N)$ as denoting an influence game. In this context, in order to avoid confusion, we use $\mathcal{M}^o(\Gamma)$ to denote the associated oblivious influence model and $\mathcal{M}^n(\Gamma)$ to denote the associated non-oblivious influence model.

When modeling collective decision models, it is natural to require that $f(i) > 0$, for $i \in V$. In other words, only some positive level of influence might change an opinion. Nevertheless, for technical reasons, we keep open the possibility of having $f(i) = 0$ for some actor $i$ in an influence graph.

Observe that, when $N = V$, for $\mathcal{M}^n(G, f, q, V)$, we have $|\{i \in V \mid c_i = 1\}| = |\{i \in V \mid i \in F(X(x))\}|$, therefore $C_{\mathcal{M}^o} = C_{\mathcal{M}^n}$. In oblivious influence models, as in influence games, the initial decision of the actors in $V \setminus N$ is not taken into account and a negative initial decision is assumed. In non-oblivious influence models, the initial decision of actors in $V \setminus N$ sometimes is taken into account. This happens when, at the end of the influence phase, the number of supporters of each alternative are both above or both below the threshold, so that $c_i = x_i$.

**Example 4.** Let $(G, f)$ be an influence graph, where $V(G) = \{1, 2, 3, 4, 5, 6, 7\}$, $E(G) = \{(1, 2), (2, 5), (2, 6), (3, 5), (3, 6), (3, 7), (6, 4), (7, 4)\}$, and the labeling function is defined by $f(1) = f(2) = f(3) = f(6) = f(7) = 1$, $f(4) = 2$ and $f(5) = 3$. See Figure 4. Let $\Gamma = (G, f, q = 5, N = \{1, 2, 3\})$ be an influence game. The models $\mathcal{M}^o(\Gamma)$ and $\mathcal{M}^n(\Gamma)$ do not coincide. In effect, consider the initial decision vector $x = (1, 0, 1, 0, 0, 0, 0)$.

For the oblivious case, we have $C_{\mathcal{M}^o(\Gamma)}(x) = 1$, because $|F(\{1, 3\} \cap \{1, 2, 3\})| = |F(\{1, 3\})| = |\{1, 3, 4, 6, 7\}| = 5 \geq q$.

For the non-oblivious case, we have that $F(\{1, 3\}) = \{1, 3, 4, 6, 7\}$. So, $c_1 = c_3 = 1$ and $c_2 = 0$. Moreover,

- $p_4(x) = |\{1, 3, 4, 6, 7\} \cap \{6, 7\}| = 2 \geq f(4) = 2$, and $q_4(x) = |\{6, 7\} \setminus \{1, 3, 4, 6, 7\}| = 0 < f(4) = 2$ implies $c_4 = 1$.
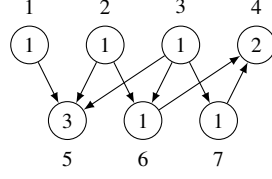
Figure 4: Influence graph such that, for the initial decision vector $x = (1, 0, 1, 0, 0, 0, 0)$, we have $C_{\mathcal{M}^o(\Gamma)}(x) = 1$ and $C_{\mathcal{M}^n(\Gamma)}(x) = 0$.

- $p_5(x) = |\{1, 3, 4, 6, 7\} \cap \{1, 2, 3\}| = 2 < f(5) = 3$, and $q_5(x) = |\{1, 2, 3\} \setminus \{1, 3, 4, 6, 7\}| = 1 < f(5) = 3$ implies $c_5 = 0$.

- $p_6(x) = |\{1, 3, 4, 6, 7\} \cap \{2, 3\}| = 1 \geq f(6) = 1$, and $q_6(x) = |\{2, 3\} \setminus \{1, 3, 4, 6, 7\}| = 1 \geq f(6) = 1$ implies $c_6 = 0$.

- $p_7(x) = |\{1, 3, 4, 6, 7\} \cap \{3\}| = 1 \geq f(7) = 1$, and $q_7(x) = |\{3\} \setminus \{1, 3, 4, 6, 7\}| = 0 < f(7) = 1$ implies $c_7 = 1$.

Thus, $C_{\mathcal{M}^n(\Gamma)}(x) = 0$, because $c(x) = (1, 0, 1, 1, 0, 0, 1)$.

Now we analyze some basic properties of both oblivious and non-oblivious influence models. We start relating some particular decision-making models with simple games. We say that a decision-making model $\mathcal{M}$ is *monotonic*, on a set of actors $V$, if its collective decision function is monotonic with respect to inclusion, on $\mathcal{P}(V)$. For a monotonic model, we define an associated simple game $\Gamma_{\mathcal{M}}$ in the natural way, i.e., a coalition $X$ is winning in $\Gamma_{\mathcal{M}}$ if and only if $C_{\mathcal{M}}(x(X)) = 1$. Our next results relate the satisfaction and the power measures to the Rae index and the Banzhaf value.

**Lemma 1.** *Let $\mathcal{M}$ be a monotonic decision-making model on a set of actors $V$. For $i \in V$, $\text{Sat}_{\mathcal{M}}(i) = \text{Rae}_{\Gamma_{\mathcal{M}}}(i)$.*

*Proof.* Recall that, in $\Gamma_{\mathcal{M}}$, we have $\mathcal{W} = \{X \subseteq V \mid C_{\mathcal{M}}(x(X)) = 1\}$. Furthermore, for $i \in V$, $\text{Rae}(i) = |\{X \in \mathcal{W} \mid i \in X\}| + |\{X \notin \mathcal{W} \mid i \notin X\}|$. For $i \in V$ and $X \subseteq V$, we consider four cases.

- If $X \in \mathcal{W}$ and $i \in \mathcal{W}$, then $1 = C_{\mathcal{M}}(x(X)) = x_i$.

- If $X \in \mathcal{W}$ and $i \notin X$, then $1 = C_{\mathcal{M}}(x(X)) \neq x_i = 0$.

- If $X \notin \mathcal{W}$ and $i \in X$, then $0 = C_{\mathcal{M}}(x(X)) \neq x_i = 1$.

- If $X \notin \mathcal{W}$ and $i \notin X$, then $0 = C_{\mathcal{M}}(x(X)) = x_i$.

The first and fourth cases are counted in $\text{Rae}$ and in $\text{Sat}$. The second and third cases are not counted neither in $\text{Rae}$ nor in $\text{Sat}$. Therefore, the claim follows. $\square$

**Lemma 2.** *Let $\mathcal{M}$ be a monotonic decision-making model on a set of actors $V$. For $i \in V$, $\text{Pow}_{\mathcal{M}}(i) = 2\text{Bz}_{\Gamma_{\mathcal{M}}}(i)$.*

*Proof.* Recall that, in $\Gamma_{\mathcal{M}}$, we have $\mathcal{W} = \{X \subseteq V \mid C_{\mathcal{M}}(x(X)) = 1\}$. Furthermore, for $i \in V$, $\text{Bz}(i) = |\{X \in \mathcal{W} \mid X \setminus \{i\} \notin \mathcal{W}\}|$ and $\text{Pow}(i) = |\{x \in \{0, 1\}^n \mid C(\overline{x}_{-i}) = 1 - C(x)\}|$. For $i \in V$ observe that, for $x \in \{0, 1\}^n$, we have again four cases.

- If $x_i = 1$ and $C(x) = 0$ by monotonicity $C(\overline{x}_{-i}) = C(x)$.

- If $x_i = 0$ and $C(x) = 1$ by monotonicity $C(\overline{x}_{-i}) = C(x)$.

- If $x_i = 1$ and $C(x) = 1$, $i \in X(x)$ and $C(\overline{x}_{-i}) = 1 - C(x)$ is equivalent to $X(x) \in \mathcal{W}$ and $X(\overline{x}_{-i}) = X(x) \setminus \{i\} \notin \mathcal{W}$.

- If $x_i = 0$ and $C(x) = 0$, $i \in X(\overline{x}_{-i})$ and $C(\overline{x}_{-i}) = 1 - C(x)$ is equivalent to $X(\overline{x}_{-i}) \in \mathcal{W}$ and $X(x) = X(\overline{x}_{-i}) \setminus \{i\} \notin \mathcal{W}$.

The last two cases are counted once in $\text{Bz}$ and twice in $\text{Pow}$, and the claim follows. $\square$

Next we prove that both the oblivious and non-oblivious models associated with an influence game are monotonic.

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**Lemma 3.** *Let* $\Gamma = (G, f, q, N)$ *be an influence game.* $\mathcal{M}^o(\Gamma)$ *and* $\mathcal{M}^n(\Gamma)$ *are monotonic.*

*Proof. Oblivious model.* Let $\mathcal{M} = \mathcal{M}^o(\Gamma)$. For $X \subseteq X' \subseteq V$, $(X \cap N) \subseteq (X' \cap N) \subseteq N$. Thus, by the monotonicity of the spread of influence process, we know that $F(X) \subseteq F(X')$. Thus, we have $C_{\mathcal{M}}(X) \leq C_{\mathcal{M}}(X')$.

    *Non-oblivious model.* Let $\mathcal{M} = \mathcal{M}^n(\Gamma)$. For $X \subseteq V$ and $i \notin X$, we consider two cases.

- If $i \in N$, then $C_{\mathcal{M}}(X) \leq C_{\mathcal{M}}(X \cup \{i\})$ because of the monotonicity of $F$.

- If $i \notin N$, then $(F(X \cap N) \cap P(i)) \subseteq (F((X \cup \{i\}) \cap N) \cap P(i))$. Thus, by the definition of the collective decision function, we have $C_{\mathcal{M}}(X) \leq C_{\mathcal{M}}(X \cup \{i\})$.

$\square$

### 3.2. Generalized opinion leader-follower models

Now we define a generalization of the OLF models presented in Definition 1. As we will see, these models are a subfamily of non-oblivious influence models. Despite of its simplicity, this model will allow us to prove in Section 4 that Satisfaction and Power problems are computationally hard even for simple models. Analogously to OLF, the new model is also based on two-layered bipartite digraphs, but allows any integer number $n$ of actors, not just odd. Further, the fraction value $r$ extends the possible values from $1/2 < r \leq 1$ to $0 \leq r \leq 1$, and there is a quota $q$ that extends the collective decision function, beyond the simple majority.

**Definition 10.** A *generalized opinion leader-follower model (gOLF)* is a triple $\mathcal{M} = (G, r, q)$ where $G = (V, E)$ is a two-layered bipartite digraph that represents the actors' relations, the fraction value $r \in [0, 1]$ is a rational number, and the *quota* $q$ is a natural number, $0 < q \leq n$. The collective decision function $C_{\mathcal{M}}$ is defined as follows. Let $x \in \{0, 1\}^n$ be an initial decision vector, then the final decision vector $c = c^{\mathcal{M}}(x)$ has the following components, for $1 \leq i \leq n$:

$$c_i = \begin{cases} 1 & \text{if } |\{j \in P_G(i) \mid x_j = 1\}| \geq \lceil r \cdot \delta^{-1}(i) \rceil \text{ and } |\{j \in P_G(i) \mid x_j = 0\}| < \lceil r \cdot \delta^{-1}(i) \rceil \\ 0 & \text{if } |\{j \in P_G(i) \mid x_j = 0\}| \geq \lceil r \cdot \delta^{-1}(i) \rceil \text{ and } |\{j \in P_G(i) \mid x_j = 1\}| < \lceil r \cdot \delta^{-1}(i) \rceil \\ x_i & \text{otherwise} \end{cases}$$

where $\lceil y \rceil$ is the least integer greater than or equal to $y$. The collective decision function $C_{\mathcal{M}} : \{0, 1\}^n \to \{0, 1\}$ is defined as

$$C_{\mathcal{M}}(x) = \begin{cases} 1 & \text{if } |\{i \in V \mid c_i(x) = 1\}| \geq q \\ 0 & \text{otherwise.} \end{cases}$$

The values $c_i^{\mathcal{M}}$, for $i \in V$, and $C_{\mathcal{M}}$ are again well defined. Moreover, the restriction $0 \leq r \leq 1$ could be replaced by $0 < r \leq 1$, without changing the model, because a follower having $r = 0$ can be replaced by an independent actor.

Remarkably, we can associate a gOLF model $\mathcal{M} = (G, r, q)$ with an influence game $\Gamma(\mathcal{M}) = (G, f, q, N)$, taking $N = \mathtt{L}(G) \cup \mathtt{I}(G)$ and defining the labeling function $f$ as

$$f(i) = \begin{cases} \lceil r \cdot \delta^-(i) \rceil & \text{if } i \in \mathtt{F}(G) \\ 1 & \text{if } i \in \mathtt{L}(G) \cup \mathtt{I}(G). \end{cases}$$

Note that $N$ does not include the set of followers, because the followers never can enforce their personal conviction and their final decision depends exclusively on whether the opinion leaders can influence them or not. We denote the influence graph $(G, f)$ of the associated influence game $\Gamma(\mathcal{M})$ as $G(\mathcal{M})$, and the corresponding set of players as $N(\mathcal{M})$.

The following result proves that gOLF models are non-oblivious influence models.

**Lemma 4.** *Let* $\mathcal{M}$ *be a gOLF model and let* $\mathcal{M}' = \mathcal{M}^n(\Gamma(\mathcal{M}))$. *Then the collective decision functions of* $\mathcal{M}$ *and* $\mathcal{M}'$ *coincide.*

*Proof.* Let $\mathcal{M}' = \mathcal{M}^n(\Gamma(\mathcal{M}))$ be the non-oblivious influence model associated with $\mathcal{M} = (G, r, q)$. Let $X \subseteq V$ be a coalition so that $x(X)$ is the initial decision of the actors. Set $c = c^{\mathcal{M}}(X)$ and $c' = c^{\mathcal{M}'}(X)$ to be the corresponding final decision vectors.

Observe that, for $i \in \text{L} \cup \text{I}$, actor $i$ can not be influenced by any other actor in $V$; therefore, $i \in F(X \cap N)$ if and only if $i \in X$ and $c_i = c_i'$. For $i \in \text{F}$, $\{j \in P(i) \mid x_j = 1\} = F(X \cap N) \cap P(i)$ and $f(i) = \lceil r\delta^-(i) \rceil$.

Thus, according to Definitions 10 and 9, $c_i = c_i'$. As in both models we get the same final decision vector, we conclude that $C_{\mathcal{M}} = C_{\mathcal{M}'}$. $\qquad\square$

As a consequence of the previous result we have a way to map gOLF models to a subfamily of the non-oblivious influence models. However, in general a gOLF model cannot be cast as an oblivious influence model because the final decision vectors are different. Those facts justify the inclusions depicted in Figure 1. Nevertheless, we can consider a subfamily in which the final decision vector under the two models coincide.

**Definition 11.** An *odd-gOLF* model is a gOLF model $\mathcal{M} = (G, r, q)$ in which $r = 1/2$ and for all $i \in \text{F}$, $\delta^-(i)$ is odd.

**Lemma 5.** *Let $\mathcal{M}$ be an odd-gOLF model and let $\mathcal{M}' = \mathcal{M}^o(\Gamma(\mathcal{M}))$. Then the collective decision functions of $\mathcal{M}$ and $\mathcal{M}'$ coincide.*

*Proof.* Let $\mathcal{M} = (G, 1/2, q)$ be an odd-gOLF, let $\Gamma(\mathcal{M})$ be the influence game associated with $\mathcal{M}$, and let $X \subseteq V$ be a coalition so that $x(X)$ is the initial decision of the actors. Let $\mathcal{M}' = \mathcal{M}^o(\Gamma(\mathcal{M}))$.

For $i \in \text{L} \cup \text{I}$, Lemma 4 shows that $c_i = c_i'$. For $i \in \text{F}$, $\{j \in P(i) \mid x_j = 1\} = F(X \cap N) \cap P(i)$.

Since $r = 1/2$ and $\delta^-(i)$ is odd, $|F(X \cap N) \cap P(i)| \neq |P(i)| - |F(X \cap N) \cap P(i)|$. Therefore, there are no ties arising in the predecessors' decision and the final decision vector in both models coincides.

So, the oblivious influence model verifies $C_{\mathcal{M}'} = C_{\mathcal{M}}$. $\qquad\square$

Thus odd-gOLF models lie in the intersection of oblivious and non-oblivious influence models (as shown in Figure 1). As we will show later odd-gOLF is the smallest subfamily in which we have been able to show that computing satisfaction or power is computationally hard. Note that Lemma 5 is not true for a gOLF model where some follower has even in-degree, as we can see in the following example.

**Example 5.** Let $\mathcal{M} = (G, 1/2, 4)$ be a gOLF, where $G$ is the two-layered bipartite digraph depicted in Figure 2, and the associated influence graph $(G, f)$ takes a labeling function defined by $f(5) = 2$ and $f(i) = 1$, for all $i \in V(G) \setminus \{5\}$. Let $x = (1, 1, 0, 0, 0, 0, 0)$ be an initial decision vector. For $\mathcal{M}$, the final decision vector is $c^{\mathcal{M}}(x) = (1, 1, 0, 0, 1, 0, 0)$, and thus $C_{\mathcal{M}}(x) = 0$. For $\mathcal{M}' = \mathcal{M}^o(\Gamma(\mathcal{M}))$, the final decision vector is $c^{\mathcal{M}'}(x) = (1, 1, 0, 0, 1, 1, 0)$, and thus $C_{\mathcal{M}'}(x) = 1$.

## 4. The hardness of computing the satisfaction and the power measures

In this section we show that the SATISFACTION problem is #P-hard for odd-gOLF. The relationship among the measures implies the same hardness result for the POWER problem. In order to do so we introduce some notation and define an auxiliary computational problem.

Let $(G, f)$ be an influence graph. For $i \in V(G)$, $\text{F}_i(G, f)$ denotes the set $\{j \in S_G(i) \mid |P_G(j)| = 1 \text{ and } f(j) = 1\}$. For $N \subseteq V(G)$ and $1 \leq k \leq n$, $F_k(N, G, f)$ denotes the set $\{X \subseteq V(G) \mid |F(X \cap N)| = k\}$. When there is no risk of ambiguity, we simply say $\text{F}_k$ or $F_k(N)$. Note that $F_k(V) = \{X \subseteq V \mid |F(X)| = k\}$. Now we are able to define the auxiliary problem.

EXPANSION

Instance: An influence graph $(G, f)$, a set of vertices $N \subseteq V(G)$ and an integer $k$.

Output: $|F_k(N, G, f)|$.

We sometimes consider that an instance of the EXPANSION problem is an odd-gOLF $\mathcal{M} = (G, r, q)$ by taking $G = G(\mathcal{M})$, $N = N(\mathcal{M})$ and some adequate value for $k$.

The following results show the relationship among the SATISFACTION and the EXPANSION problems for some oblivious influence models.

**Lemma 6.** *Let* $\Gamma = (G, f, q, N)$ *be an influence game and let* $\mathcal{M}^o(\Gamma)$. *For* $i \in V(G) \setminus N$ *or* $i \in N$ *with* $f(i) = 0$, $\text{S}_{\text{AT}}_{\mathcal{M}^o}(i) = 2^{n-1}$.

*Proof.* Let $\mathcal{M} = \mathcal{M}^o(\Gamma) = (G, f, q, N)$, let $Z = \{i \in N \mid f(i) = 0\}$. Let $x$ be an initial decision vector and set $X = X(x) \cap (N \setminus Z)$. Observe that $F(X) = F(X \cup Z)$. Therefore, the final decision is independent on the initial decision of those players in the set $Z$.

For $i \notin N$ or $i \in Z$, we provide the necessary and sufficient condition for actor $i$ being satisfied. We consider two cases:

1. If $X \in \mathcal{W}$, $C_\mathcal{M}(x) = 1$. Player $i$ is satisfied only when $x_i = 1$.
2. If $X \notin \mathcal{W}$, $C_\mathcal{M}(x) = 0$. Player $i$ is satisfied only when $x_i = 0$.

Therefore, for an initial decision vector of the players in $N \setminus Z$, there is only one way, for player $i$, to complete it in such a way that the collective decision coincides with player $i$'s decision. Thus we obtain $\text{S}_{\text{AT}}(i) = 2^{n-1}$. $\square$

In the following result we make use of a particular game construction. Let $(G, f, q, N)$ be an influence game. Define $R(G, f, i)$ as the influence graph $(G', f')$, where $G' = G[V(G) \setminus (\text{F}_i \cup \{i\})]$, for $j \notin S_G(i)$, $f'(j) = f(j)$, and, for $j \in S_G(i)$, $f'(j) = \max\{f(j) - 1, 0\}$.

**Lemma 7.** *Let* $\Gamma = (G, f, q, N)$ *be an influence game. For* $i \in N$ *with* $f(i) > 0$ *and* $P_G(i) = \emptyset$, *we have that*

$$\text{S}_{\text{AT}}_{\mathcal{M}^o}(i) = 2^{n-1} + 2^{n-|N|} \sum_{j=1}^{1+|\text{F}_i|} |F_{q-j}(N \setminus \{i\}, R(G, f, i))|.$$

*Proof.* Let $\mathcal{M} = \mathcal{M}^o(\Gamma) = (G, f, q, N)$, let $x$ be an initial decision vector and set $X = X(x) \cap N$. For $i \in N$ with $f(i) > 0$ and $P_G(i) = \emptyset$, we provide the conditions for $i$ to be satisfied. We consider three cases.
1. If $X \setminus \{i\} \in \mathcal{W}$, then $C_\mathcal{M}(x) = 1$, so it must be $x_i = 1$.
2. If $X \setminus \{i\} \notin \mathcal{W}$ and $X \notin \mathcal{W}$, then $C_\mathcal{M}(x) = 0$, so it must be $x_i = 0$.
3. If $X \setminus \{i\} \notin \mathcal{W}$ and $X \in \mathcal{W}$, then $C_\mathcal{M}(x) = 1$, so it must be $x_i = 1$.
The first two cases provide a total of $2^{n-1}$ initial decision vectors for which the collective decision coincides with the initial decision of player $i$. To count the initial decision vectors for the third case, we consider the influence graph $R(G, f, i)$. We have to separate those vertices in the set $\text{F}_i$ that can be influenced directly and only by $i$. Observe that all the vertices in $S_G(i) \setminus \text{F}_i$ have in-degree at least 2. Now, for a coalition $Y$, it holds that $Y \in \mathcal{L}$ and $Y \cup \{i\} \in \mathcal{W}$ if and only if $Y \in F_{q-j}(N \setminus \{i\}, R(G, f, i))$, for some $1 \le j \le 1 + |\text{F}_i|$. Therefore, since the influence model is oblivious, there are $2^{n-|N|} \sum_{j=1}^{1+|\text{F}_i|} |F_{q-j}(N \setminus \{i\}, R(G, f, i))|$ additional initial decision vectors $z$ with $z_i = C_\mathcal{M}(z) = 1$. $\square$

Note that, for $i \in N$ with $f(i) > 0$, $P_G(i) = S_G(i) = \emptyset$ and $\text{F}_i = \emptyset$,

$$\text{S}_{\text{AT}}(i) = 2^{n-1} + 2^{n-|N|}|F_{q-1}(N \setminus \{i\}, R(G, f, i))|.$$

Note also that, in the odd-gOLF models, $N = \text{L} \cup \text{I}$, $V \setminus N = \text{F}$ and, for $i \in N$, $P(i) = \emptyset$. Therefore, Lemmas 6 and 7 provide the formulas for the satisfaction measure in those models. These results also show that, as expected, the opinion leaders have always a satisfaction greater than or equal to that of the independent actors, and that both have always a satisfaction greater or equal than the followers.

The previous lemma does not provide a formula for the case in which the vertex $i \in N$ has $f(i) > 0$ and $P_G(i) \neq \emptyset$. Although this case never occurs in gOLF, it can be handled in other models by considering another graph construction. Let $R_2(G, f, i)$ be the influence graph $(G'', f'')$, where $V(G'') = (V(G) \setminus \text{F}_i) \cup Z$, with $Z = \{z_1, \ldots z_{2n}\}$ a set of new vertices, and $E(G'')$ is formed by the edges in $G[V(G) \setminus \text{F}_i]$ and the set $\{(i, z_j) \mid 1 \le j \le 2n\}$. The labeling function is given by $f''(j) = f(j)$, for $j \in V(G) \setminus S_G(i)$, by $f''(j) = \max\{f(j) - 1, 0\}$, for $j \in S_G(i)$, and by $f''(z_j) = 1$, for $1 \le j \le 2n$. The following result can be proved in the same way than Lemma 7, by replacing $R(G, f, i)$ by $R_2(G, f, i)$.

**Lemma 8.** *Let* $(G, f, q, N)$ *be an oblivious influence model, for* $i \in N$ *with* $f(i) > 0$ *and* $P_G(i) \neq \emptyset$,

$$\text{S}_{\text{AT}}(i) = 2^{n-1} + 2^{n-|N|} \sum_{j=1}^{1+|\text{F}_i|} |F_{q-j}(N \setminus \{i\}, R_2(G, f, i))|$$

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For our hardness result we consider a variation of the *counting vertex cover problem* [15]:

$\#\frac{2}{3}$-VC

Instance: An undirected graph $G = (V, E)$.

Output: Number of vertex covers of $G$ with size $\frac{2}{3}|V|$, i.e.,

$$\left| \{X \subseteq V \mid \forall \, \{i, j\} \in E, \ \ \{i, j\} \cap X \neq \emptyset \text{ and } |X| = \tfrac{2}{3}|V|\} \right|.$$

It is known that the problem of computing in a graph the number of independent sets with size exactly $\frac{1}{3}|V|$ is #P-hard [17]. Hence, as the complement of an independent set is a vertex cover, the same result shows that $\#\frac{2}{3}$-VC is #P-hard.

**Theorem 1.** *The* Expansion *problem for odd-gOLF is #P-hard.*

*Proof.* We provide a reduction from the $\#\frac{2}{3}$-VC problem.

Let $G = (V, E)$ be an instance of $\#\frac{2}{3}$-VC. Without loss of generality we assume that $G$ is connected, $n = |V|$ is a multiple of 3, and $n \geq 6$. Let $m = |E(G)|$ and $E = \{e_1, \ldots, e_m\}$.

We construct a two-layered bipartite digraph $G_1$ associated with $G$ as follows. The set of vertices is $V(G_1) = V \cup E_1 \cup E_2 \cup \cdots \cup E_{n+2} \cup \{z\}$, where $z$ is a new vertex and $E_j = \{e_1^j, \ldots, e_m^j\}$, $1 \leq j \leq n + 2$, is formed by vertices associated with the edges of $E$. Observe that $n_1 = |V(G_1)| = n + (n+2)m + 1$. The set of edges is the following:

$$E(G_1) = \{(u, e_k^j) \mid u \in V, 1 \leq j \leq n + 2, 1 \leq k \leq m \text{ and } u \in e_k\}$$
$$\cup \{(z, a) \mid a \in E_j, 1 \leq j \leq n + 2\}.$$

Note that in $G_1$ all the vertices have in-degree either 0 or 3.

Consider the labeling function $f_1$ associated with the influence graph of the game $\Gamma(G_1, 1/2, n_1)$. We have $f_1(u) = 1$, for $u \in V$, $f_1(z) = 1$ and $f_1(u) = 2$, for $u \notin (V \cup \{z\})$.

Now we define the reduction from $\#\frac{2}{3}$-VC to Expansion, which associates with $G$ the following instance $h(G)$ for the Expansion problem:

$$h(G) = \left( (G_1, f_1), V \cup \{z\}, \frac{2}{3}n + (n+2)m + 1 \right),$$

where the *instance* has the influence graph $(G_1, f_1)$, the set of vertices $V \cup \{z\}$ and the integer $\frac{2}{3}n + (n+2)m + 1$.

Let $X \subseteq V$ and let $\alpha = |X|$. We analyze the expansion of the sets $X \cup \{z\}$ and $X$ in the influence graph $(G_1, f_1)$.

Figure 5 provides an example showing how to construct the influence graph $(G_1, f_1)$ from a graph $G$. Note that, to clarify such construction, this example just considers $n = 3$ vertices.

When the initial activation set is $X \cup \{z\}$, we have two cases, either $X$ is a vertex cover or not. When $X$ is a vertex cover, all vertices corresponding to edges get activated, so $|F(X \cup \{z\})| = \alpha + (n+2)m + 1$. This last quantity is equal to the required size only when $\alpha = \frac{2}{3}n$. When $X$ is not a vertex cover, then $\alpha \leq n - 2$ and at least one edge $e \in E$ is not covered. In consequence, $F(X \cup \{z\})$ can not influence all the vertices in $E_1 \cup \ldots \cup E_{n+2}$. Therefore, $|F(X \cup \{z\})| \leq \alpha + (n+2)(m-1) + 2 \leq n - 2 + (n+2)m - (n+2) + 2 = (n+2)m - 2$ which is strictly smaller than the required size.

Now consider the case when the initial activation set is $X$. Recall that $G$ is connected. If for $\{u, v\} \in E$ it holds that $\{u, v\} \subseteq X$, then we have $|F(X \cup \{z\})| = n + (n+2)m$, which is greater than the required size. Otherwise, $F(X)$ can not influence all the vertices in $E_1 \cup \ldots \cup E_{n+2}$, we have $\alpha \leq n - 1$, and hence $|F(X)| \leq \alpha + (n+2)(m-1) \leq n - 1 + (n+2)m - (n+2) = (n+2)m - 3$ which is strictly smaller than the required size.

From the previous case analysis, we have that the elements in $F_k(V \cup \{z\})$, for $(G_1, f_1)$, are in a one-to-one correspondence with the vertex covers of size $\frac{2}{3}n$ in $G$. As the reduction can be computed trivially in polynomial time, the claim holds. $\qquad \square$

The hardness of the Expansion problem does not rule out the possibility of having some particular cases for which the Expansion problem is computationally easy. For example, when $f$ is strictly positive and the parameter $k$ of the problem is smaller than the minimum label over the actors not in $N$, i.e., $k < \min\{f(i) \mid i \in V \setminus N\}$, it is easy to see that, for an oblivious influence model, $|F_k(N)| = \binom{n-|N|}{k}$.

Now we can combine the previous results of this section to provide a hardness proof for the Satisfaction problem.
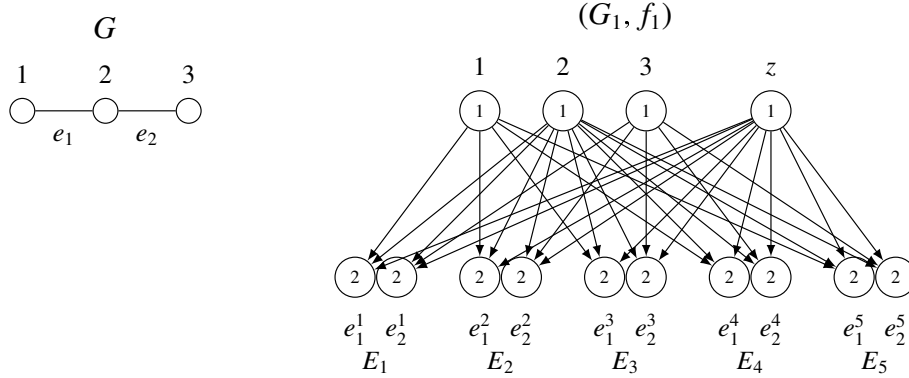
Figure 5: Influence graph $(G_1, f_1)$ obtained from an undirected graph $G$ with just 3 vertices.

**Theorem 2.** *The* SATISFACTION *problem for the odd-gOLF models is #P-hard.*

*Proof.* We prove hardness by showing a polynomial time reduction from the EXPANSION problem on bipartite influence graphs to the SATISFACTION problem. Consider an instance of EXPANSION given by a bipartite influence graph $(G, f)$, a set $N \subseteq V(G)$ and an integer $k$. Let $(G', f')$ be the influence graph obtained from $(G, f)$ by adding an isolated vertex $z$ with label 1. Finally, we consider the influence game $\Gamma(G, 2/3, k+1) = (G', f', N \cup \{z\}, k+1)$ and the instance $(\Gamma(G, f'), z)$ of the SATISFACTION problem.

In order to compute SAT$(z)$ in $\Gamma(G, f)$, according to Lemma 7, we have to consider the reduced influence graph $R(G', f', z)$. Recall that $q = k + 1$. By construction $R(G', f', z) = (G, f)$ and thus we have

$$\text{SAT}(z) = 2^n + 2^{n+1-|N|}|F_k(N, (G, f))|.$$

Therefore, if we could solve the SATISFACTION problem in polynomial time, we would also be able to solve EXPANSION in polynomial time. □

As a consequence of the previous result and Lemma 5, taking into account the relationship among models depicted in Figure 1, we have the following results.

**Corollary 1.** *The* SATISFACTION *problem for the gOLF models and the oblivious and non-oblivious influence models is #P-hard.*

**Corollary 2.** *The* POWER *problem for the gOLF models and the oblivious and non-oblivious influence models is #P-hard.*

As a consequence of the previous result and Lemmas 1 and 2, the problems of computing the Rae index and the Banzhaf value are #P-hard for the families of simple games associated with oblivious and non-oblivious influence models, odd-gOLF and gOLF models.

## 5. Strong hierarchical influence models

In this section we focus our attention on one particular topology of the influence graphs where we can show that the SATISFACTION and POWER problems are polynomial time solvable. In the previous section we have shown #P-hardness when the influence spreads in a two-layered bipartite digraph. Thus, to try to find tractable cases we must consider particular classes of bipartite digraphs. We consider subfamilies of multilayered bipartite digraphs having strong restrictions on the connections among the layers. In a multilayered bipartite digraph, there are some intermediate actors that we call *mediators*, who act as intermediate layers of influence expansion between opinion leaders, having in-degree 0, and followers, having out-degree 0.
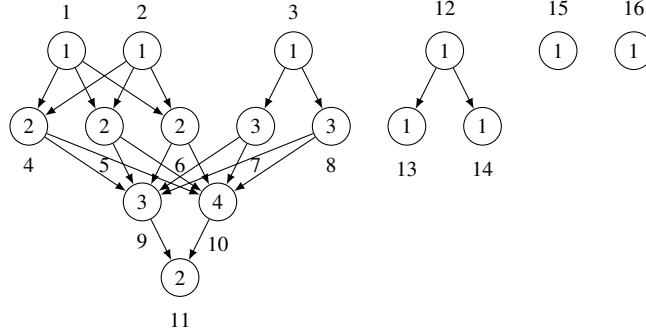
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Figure 6: A strong hierarchical influence graph.

The family of graphs is defined recursively using two graph operations. Given two graphs $H_1$ and $H_2$ with $V(H_1) \cap V(H_2) = \emptyset$ their *disjoint union* is the graph $H_1 + H_2 = (V(H_1) \cup V(H_2), E(H_1) \cup E(H_2))$. Given a graph $H$, the *one layer extension to a set $V' \neq \emptyset$ of new vertices* $(V(H) \cap V' = \emptyset)$ is the graph $H \otimes V'$ constructed as follows

$$V(H \otimes V') = V(H) \cup V' \text{ and}$$
$$E(H \otimes V') = E(H) \cup \{(u, v) \mid u \in \mathtt{FI}(H), v \in V'\}.$$

Observe that we have $\mathtt{L}(H_1 + H_2) = \mathtt{L}(H_1) \cup \mathtt{L}(H_2)$, $\mathtt{I}(H_1 + H_2) = \mathtt{I}(H_1) \cup \mathtt{I}(H_2)$, $\mathtt{F}(H_1 + H_2) = \mathtt{F}(H_1) \cup \mathtt{F}(H_2)$ and $\mathtt{FI}(H_1 + H_2) = \mathtt{FI}(H_1) \cup \mathtt{FI}(H_2)$. Furthermore, $\mathtt{L}(H \otimes V') = \mathtt{L}(H) \cup \mathtt{I}(H)$, $\mathtt{I}(H \otimes V') = \emptyset$, and $\mathtt{F}(H \otimes V') = \mathtt{FI}(H \otimes V') = V'$.

As base case we use graphs with only isolated vertices. The family is completed by taking the closure under the two graph operations defined above.

**Definition 12.** The family of *strong hierarchical graphs* is defined recursively as follows.

- The graph $I_a$, for $a > 0$, is a strong hierarchical graph.

- If $H_1$ and $H_2$ are disjoint strong hierarchical graphs, the graph $H_1 + H_2$ is a strong hierarchical graph.

- If $H$ is a strong hierarchical graph and $V' \neq \emptyset$ is a set of vertices with $V(H) \cap V' = \emptyset$, the graph $H \otimes V'$ is a strong hierarchical graph.

A *strong hierarchical influence graph* is an influence graph $(G, f)$ where $G$ is a strong hierarchical graph. A *strong hierarchical influence game* is an influence game $(G, f, q, N)$ where $G$ is a strong hierarchical graph and $N = \mathtt{L}(G) \cup \mathtt{I}(G)$.

In Figure 6 we provide an example of a strong hierarchical influence graph. Observe that the vertices of a strong hierarchical graph can be partitioned into layers so that edges occur only between vertices of consecutive layers. Furthermore, by removing the vertices with out-degree zero in a connected strong hierarchical graph, we obtain a decomposition formed by connected strong hierarchical graphs and possibly a set of independent vertices. By repeatedly applying this process we can obtain a decomposition allowing to reconstruct the graph according to the allowed operations. The graph given in Figure 6 can be obtained as

$$[([(H_1 \otimes \{4, 5, 6\}) + (H_2 \otimes \{7, 8\})] \otimes \{9, 10\}) \otimes \{11\}] + [(H_3 \otimes \{13, 14\}) + H_4]$$

where $H_1 = (\{1, 2\}, \emptyset)$, $H_2 = (\{3\}, \emptyset)$, $H_3 = (\{12\}, \emptyset)$ and $H_4 = (\{15, 16\}, \emptyset)$.

We start devising an algorithm to solve the Expansion problem for strong hierarchical influence games. Our dynamic programming algorithm uses the recursive construction of the graph to guide the computation of the adequate tabulated values.

**Lemma 9.** *Let $(G, f, q, N)$ be a strong hierarchical influence game. For $1 \leq k \leq n$, the values $|F_k(N, G, f)|$ can be computed in polynomial time.*

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*Proof.* Let $n = |V(G)|$, for $0 \le b \le a \le n$ and $0 \le b \le |\text{FI}(G)|$, consider the following values $T(a,b)$:

$$T(a,b) = |\{X \subseteq N \mid |F(X)| = a \text{ and } |F(X) \cap \text{FI}(G)| = b\}|.$$

Observe that, if we can compute in polynomial time an array holding all the $T(a,b)$ values, then we can obtain $|F_k(N)|$ in polynomial time as

$$|F_k(N)| = 2^{n-|N|} \sum_{0 \le b \le |\text{FI}(G)|} T(k,b).$$

Let us show, by induction on the structure of the graph $G$, how an array storing the desired values of $T$ can be obtained from the corresponding $T$ arrays for the adequate subgraphs. According to Definition 12, the base case are sets of isolated vertices.

**Base case:** $H = I_\alpha$.

In this case all the actors are independent, $|\text{FI}(H)| = \alpha$. Furthermore, for $X \subseteq V$, $F(X) = X \cup \{i \in V \mid f(i) = 0\}$. Therefore, for $0 \le b \le a \le \alpha$, we have

$$T(a,b) = \begin{cases} \binom{\alpha-\gamma}{b-\gamma} 2^\gamma & \text{if } a = b > \gamma \\ 0 & \text{otherwise} \end{cases}$$

where $\gamma = |\{i \in V \mid f(i) = 0\}|$. Recall that $\binom{n}{k}$ is the number of $k$-element subsets of an $n$-element set, and $2^n$ is the number of subsets of an $n$-element set. Observe that all those values can be computed in polynomial time.

The inductive step of the proof is divided into two cases.

**Case 1:** $H = H_1 + H_2$, for some strong hierarchical graphs $H_1$, $H_2$, recall that we also have that $V(H_1) \cap V(H_2) = \emptyset$.

For $1 \le i \le 2$, let $T_i$ be the tables corresponding to the graphs $H_i$, respectively, and let $n_i = |V(H_i)|$. As the graphs are disjoint, for $0 \le b \le a \le n$ and $0 \le b \le |\text{FI}(H)|$, we have:

$$T(a,b) = \sum_{\substack{a_1 + a_2 = a \\ b_1 + b_2 = b \\ 0 \le b_i \le a_i \le n_i \text{ and } 0 \le b_i \le |\text{FI}(H_i)|, 1 \le i \le 2}} T_1(a_1, b_1) T_2(a_2, b_2).$$

Those values can be computed in polynomial time using a multiple scanning as follows:

> for $a$ from 0 to $n$, $b$ from 0 to $a$, $b$ from 0 to $|\text{FI}(H)|$ do
>     Initialize $T(a,b)$ to 0
>     for $a_1$ from 0 to $n_1$, $b_1$ from 0 to $a_1$, $b_1$ from 0 to $|\text{FI}(H_1)|$,
>         $a_2$ from 0 to $n_2$, $b_2$ from 0 to $a_2$, $b_2$ from 0 to $|\text{FI}(H_2)|$ do
>         $T(a_1 + a_2, b_1 + b_2) = T(a_1 + a_2, b_1 + b_2) + T_1(a_1, b_1) T_2(a_2, b_2)$.

**Case 2:** $H = H' \otimes V'$ for some strong hierarchical graph $H'$ and a set $V' \ne \emptyset$ with $V(H') \cap V' = \emptyset$.

In such a case $F(H) = V'$ and the graph $H'$ is a strong hierarchical graph with one layer less. Let $T'$ be the tables corresponding to the graph $H'$, and let $n' = |V(H')|$. Recall that, in the construction of $H$, the edges added to $H'$ connect in an all-to-all fashion the vertices in $\text{FI}(H')$ with the vertices in $F(H)$.

To compute the values of $T$, we use an auxiliary table $R(c)$, $0 \le c \le n$, defined as

$$R(c) = |\{v \in F(H) \mid f(v) \le c\}|.$$

A vector storing the values of $R$ can be computed in polynomial time by sorting the set $F(H)$ in increasing order of labels and performing a scanning of the sorted table.

We get the following expression, for $0 \le b \le a \le n$ and $0 \le b \le \beta = |\text{FI}(H)|$:

$$T(a,b) = \sum_{\substack{a' + R(b') = a \\ R(b') = b \\ 0 \le b' \le a' \le n', 0 \le b' \le |\text{FI}(H')|}} T'(a', b').$$

Those values can be computed in polynomial time using a double scanning as follows:

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for $a$ from 0 to $n$, $b$ from 0 to $a$, $b$ from 0 to $|\mathrm{FI}(H)|$ do
    Initialize $T(a, b)$ to 0
    for $a'$ from 0 to $n'$, $b'$ from 0 to $a'$, $b'$ from 0 to $|\mathrm{FI}(H')|$ do
        $T(a' + R(b'), R(b')) = T(a' + R(b'), R(b')) + T'(a', b')$.

Note that given a graph, it is possible to know whether it is a strong hierarchical graph in polynomial time. Moreover, given a strong hierarchical graph it is possible to find a decomposition, according to the definition, in polynomial time. So, the claim follows. □

It is easy to see that, for a strong hierarchical influence graph $(G, f)$ and an actor $i \in \mathrm{L}(G) \cup \mathrm{F}(G)$, the graph $R(G, f, i)$, constructed as in Lemma 7, is a strong hierarchical influence graph. Therefore, we can use Lemma 7 together with the previous algorithm to compute $\mathrm{Sat}(i)$ in polynomial time. Using the relationship among problems, we have the following result.

**Theorem 3.** *The* Satisfaction *and the* Power *problems, for oblivious influence models associated with strong hierarchical influence games, are polynomial time solvable.*

Finally, we extend the previous computation to the non-oblivious models. In this case we provide a recursive algorithm that allow us to compute directly the satisfaction measure.

**Theorem 4.** *The* Satisfaction *and the* Power *problems, for non-oblivious models associated with strong hierarchical influence games, are polynomial time solvable.*

*Proof.* Let $\mathcal{M} = (G, f, q, N)$ be an oblivious influence model. Assume that $G$ is a strong hierarchical graph. We first compute a decomposition of $G$ according to the recursive definition. Recall that such a decomposition can be obtained in polynomial time.

Given a vertex $u$, our algorithm to compute $\mathrm{Sat}(u)$ fills first some tables with partial results, one for each of the subgraphs in the decomposition of $G$. Finally, the algorithm combines the tables corresponding to $G$ to get $\mathrm{Sat}(u)$. Observe, the vertex $u$ is present only in a subset of the graphs appearing in the decomposition of $G$. For a subgraph $H$, we have to compute information when $u \in V(H)$ different from the one needed when $u \notin H$. When $u \in V(H)$, we have to keep track of $u$'s initial decision.

Let $n = |V(H)|$ and $\mathcal{M} = \mathcal{M}^n(H)$. When $u \notin V(H)$, we consider the values $S(a, b)$, for $0 \leq a \leq n$ and $0 \leq b \leq |\mathrm{FI}(H)|$, defined as follows:

$$S(a, b) = |\{x \in \{0, 1\}^n \mid |\{i \mid c_i^{\mathcal{M}}(x) = 1\}| = a \text{ and } |F(X(x) \cap N) \cap \mathrm{FI}(H)| = b\}|.$$

When $u \in V(H)$, we consider two sets of values $S_0(a, b)$ and $S_1(a, b)$, for $0 \leq a \leq n$ and $0 \leq b \leq |\mathrm{FI}(H)|$, defined as

$$S_0(a, b) = |\{x \in \{0, 1\}^n \mid x_u = 0 \text{ and } |\{i \mid c_i^{\mathcal{M}}(x) = 1\}| = a \text{ and } |F(X(x) \cap N) \cap \mathrm{FI}(G)| = b\}|,$$
$$S_1(a, b) = |\{x \in \{0, 1\}^n \mid x_u = 1 \text{ and } |\{i \mid c_i^{\mathcal{M}}(x) = 1\}| = a \text{ and } |F(X(x) \cap N) \cap \mathrm{FI}(G)| = b\}|.$$

Observe that, if we can compute in polynomial time arrays holding all the $S_0(a, b)$ and $S_1(a, b)$ values, for the graph $G$ (which indeed contains $u$), we can express $\mathrm{Sat}(u)$ as

$$\mathrm{Sat}(u) = \sum_{0 \leq a < q} \sum_{0 \leq b \leq \mathrm{FI}(G)} S_0(a, b) + \sum_{q \leq a \leq n} \sum_{0 \leq b \leq \mathrm{FI}(G)} S_1(a, b).$$

Thus $\mathrm{Sat}(u)$ could be computed in polynomial time.

Let us show, by induction on the structure of the graph $G$, how the array $S$, storing the desired values, can be obtained from the corresponding $S$ arrays for adequate subgraphs. Let us recall that, according to the definition, the labeling function is strictly positive. From Definition 12, the base case is a set of isolated vertices.

**Base case:** $H = I_\alpha$, for some $\alpha > 0$.

As all the actors are independent, $|\mathrm{FI}(H)| = \alpha$ and, for $X \subseteq V$, $F(X) = X$. Furthermore, $c_i^{\mathcal{M}} = 1$ if and only if $i \in F(X(x))$. For $0 \leq a \leq \alpha$ and $0 \leq b \leq \alpha$, we have two cases.

First, when $u \notin V(H)$, we have the expression

$$S(a,b) = \begin{cases} \binom{\alpha}{b} & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

Second, when $u \in V(H)$, we have to derive expressions for the two cases. If $x_u = 0$, $u$ does not form part of the initial $X$, so we can select vertices from $V \setminus \{u\}$. If $x_u = 1$, $u$ must be part of any $X$, so we have to select one vertex less. Therefore, we have

$$S_0(a,b) = \begin{cases} \binom{\alpha-1}{b} & \text{if } a = b, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad S_1(a,b) = \begin{cases} \binom{\alpha-1}{b-1} & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

Note that all those values can be computed in polynomial time.

**Case 1:** $H = H_1 + H_2$, for some disjoint strong hierarchical influence graphs, therefore $V(H_1) \cap V(H_2) = \emptyset$.

For $1 \leq i \leq 2$, let $S^i$, $S_0^i$ and $S_1^i$ be the tables corresponding to the graph $H_i$, repectively, and $n_i = |V(H_i)|$. Set $I_{H_1+H_2}(a,b) = \{(a_1, a_2, b_1, b_2) \mid a_1 + a_2 = a, \; b_1 + b_2 = b, \; 0 \leq a_1 \leq n_1, \; 0 \leq b_1 \leq |\mathrm{FI}(H_1)|, \; 0 \leq a_2 \leq n_2, \; 0 \leq b_2 \leq |\mathrm{FI}(H_2)|\}$.

As the graphs are disjoint, for $0 \leq b \leq a \leq n$ and $0 \leq b \leq |\mathrm{FI}(H_1)| + |\mathrm{FI}(H_2)|$, we have two different cases.

First, when $u \notin V(H)$ we have the following expression

$$S(a,b) = \sum_{I_{H_1+H_2}(a,b)} S^1(a_1, b_1) \, S^2(a_2, b_2).$$

Second, when $u \in V(H)$ we assume w.l.o.g. that $u \in V(G_1)$ and we have

$$S_0(a,b) = \sum_{I_{H_1+H_2}(a,b)} S_0^1(a_1, b_1) \, S^2(a_2, b_2) \qquad \text{and} \qquad S_1(a,b) = \sum_{I_{H_1+H_2}(a,b)} S_1^1(a_1, b_1) \, S^2(a_2, b_2).$$

Those values can be computed in polynomial time using a multiple scanning similar to the one used in the proof of Lemma 9:

> for $a$ from 0 to $n$, $b$ from 0 to $a$, $b$ from 0 to $|\mathrm{FI}(H_1)| + |\mathrm{FI}(H_2)|$ do
>     Initialize $S(a,b)$ to 0
>     for $a_1$ from 0 to $n_1$, $b_1$ from 0 to $|\mathrm{FI}(H_1)|$,
>         $a_2$ from 0 to $n_2$, $b_2$ from 0 to $|\mathrm{FI}(H_2)|$ do
>         $S(a_1 + a_2, b_1 + b_2) = S(a_1 + a_2, b_1 + b_2) + S^1(a_1, b_1) \, S^2(a_2, b_2)$

and, for $0 \leq j \leq 2$,

> for $a$ from 0 to $n$, $b$ from 0 to $a$, $b$ from 0 to $|\mathrm{FI}(H_1)| + |\mathrm{FI}(H_2)|$ do
>     Initialize $S_j(a,b)$ to 0
>     for $a_1$ from 0 to $n_1$, $b_1$ from 0 to $|\mathrm{FI}(H_1)|$,
>         $a_2$ from 0 to $n_2$, $b_2$ from 0 to $|\mathrm{FI}(H_2)|$ do
>         $S_j(a_1 + a_2, b_1 + b_2) = S_j(a_1 + a_2, b_1 + b_2) + S_j^1(a_1, b_1) \, S^2(a_2, b_2)$.

**Case 2:** $H = H' \otimes V'$, for some strong hierarchical graph $H$ and a set $V' \neq \emptyset$ and $V' \cap V(H') = \emptyset$.

In such a case $\mathrm{F}(G) = V'$. Let $S'$, $S_0'$ and $S_1'$ be the tables corresponding to $H'$, and let $n' = |V(H')|$. Recall that, in the construction of $G$, the edges added to $G'$ connect in an all-to-all fashion the vertices in $\mathrm{FI}(H')$ with the vertices in $\mathrm{F}(H)$.

To express the values of $S$, $S_0$ and $S_1$ we use, as before, an auxiliary table $R(c)$, $0 \leq c \leq \alpha$, defined as

$$R(c) = |\{v \in \mathrm{F}(G) \mid f(v) \leq c\}|$$

which can be computed in polynomial time. Note that $R(c)$ accounts for the number of actors in the added layer when $c$ followers of $G'$ are influenced.

We need also information for other relevant sets. For $0 \le c \le |\mathtt{FI}(H')|$, define

$$A_1(c) = \{v \in \mathtt{F}(H) \mid f(v) \le c \text{ and } |\mathtt{FI}(H')| - c < f(v)\},$$
$$A_2(c) = \{v \in \mathtt{F}(H) \mid f(v) \le |\mathtt{FI}(H')| - c \text{ and } c < f(v)\},$$
$$A_3(c) = \mathtt{F}(H) - A_1(c) - A_2(c).$$

Finally set $R_1(c) = |A_1(c)|$, $R_2(c) = |A_2(c)|$ and $R_3(c) = |A_3(c)|$. All those sets and values can be precomputed, for any possible value of $c$, in polynomial time.

As the connection to the final layer is complete and $f$ is positive, for a set $X \subseteq \mathtt{FI}(H')$, we have $F(X) = X \cup \{u \in \mathtt{F}(H) \mid |X| \ge f(u)\}$. Using this information, we know that a subset of opinion leaders $X \subseteq \mathtt{L}(H)$ with $\mu = |F(X) \cap \mathtt{FI}(H')|$ will expand its influence to all the followers $i$ for which $f(i) \le \mu$.

For an initial decision vector $x \in \{0, 1\}^n$, let $\mu(x) = |F(X(x) \cap \mathtt{L}(H)) \cap \mathtt{FI}(H')|$. Observe that the associated final decision vector, for $i \in \mathtt{F}(G)$, can be expressed as

$$c_i^{\mathcal{M}}(x) = \begin{cases} 1 & \text{if } \mu(x) \ge f(i) \text{ and } \alpha - \mu(x) < f(i) \\ 0 & \text{if } \alpha - \mu(x) \ge f(i) \text{ and } \mu(x) < f(i) \\ x_i & \text{otherwise.} \end{cases} = \begin{cases} 1 & \text{if } i \in A_1(\mu(x)), \\ 0 & \text{if } i \in A_2(\mu(x)), \\ x_i & \text{if } i \in A_3(\mu(x)). \end{cases}$$

Taking into account the last expression, we can count those initial decision vectors giving raise to the prescribed number of 1's in the final decision vector. Set $I_{H' \otimes V'}(a, b) = \{(a', b') \mid a' + R_1(b') + \delta = a, R(b') = b, 0 \le a' \le n', 0 \le b' \le |\mathtt{FI}(H')|, 0 \le \delta \le R_3(b')\}$. For $0 \le a \le n$ and $0 \le b \le \beta$, we get the following expressions.

First, when $u \notin V(H)$, we have that

$$S(a, b) = \sum_{I_{H' \otimes V'}(a,b)} S'(a', b') \binom{R_3(b')}{\delta} 2^{R_1(b') + R_2(b')}.$$

Second, when $u \in V(H)$, we have two cases.
If $u \in V(H')$,

$$S_0(a, b) = \sum_{I_{H' \otimes V'}(a,b)} S_0'(a', b') \binom{R_3(b')}{\delta} 2^{R_1(b') + R_2(b')} \quad \text{and} \quad S_1(a, b) = \sum_{I_{H' \otimes V'}(a,b)} S_1'(a', b') \binom{R_3(b')}{\delta} 2^{R_1(b') + R_2(b')}.$$

If $u \in V' = \mathtt{F}(H)$,

$$S_0(a, b) = \sum_{\substack{(a', b') \in I_{H' \otimes V'}(a,b) \\ u \notin A_3(b')}} S'(a', b') \binom{R_3(b')}{\delta} 2^{R_1(b') + R_2(b') - 1} \; + \sum_{\substack{a' + R_1(b') + \delta = a \\ R(b') = b \\ 0 \le a' \le n', \, 0 \le b' \le \alpha \\ 0 \le \delta < R_3(b') \\ u \in A_3(b')}} S'(a', b') \binom{R_3(b') - 1}{\delta} 2^{R_1(b') + R_2(b')}$$

and

$$S_1(a, b) = \sum_{\substack{(a', b') \in I_{H' \otimes V'}(a,b) \\ u \notin A_3(b')}} S'(a', b') \binom{R_3(b')}{\delta} 2^{R_1(b') + R_2(b') - 1} \; + \sum_{\substack{a' + R_1(b') + \delta + 1 = a \\ R(b') = b \\ 0 \le a' \le n', \, 0 \le b' \le \alpha \\ 0 \le \delta < R_3(b') \\ u \in A_3(b')}} S'(a', b') \binom{R_3(b') - 1}{\delta} 2^{R_1(b') + R_2(b')}.$$

Note that, for the last situation $u \in V' = \mathtt{F}(H)$, when $u \in A_3(b')$, for $S_0(a, b)$ we have $x_u = 0$, but for $S_1(a, b)$ we have $x_u = 1$.

All the required values can be computed in polynomial time using a double scanning similar to the one used in **Case 1**.

Thus, taking into account the relationship among the problems, the claim holds. $\qquad \square$

## 6. Star influence models

Now we consider a family of influence graphs with a star-topology which was previously studied in the context of influence games [25]. In such a graph the two-layered bipartite topology is restricted to be a star graph and extended by allowing some bidirectional connections to the center of the star. In a star influence graph, in addition to the sets $L$, $I$ and $F$, we have the central node $c$ wich acts as mediator and the set $R$ of *reciprocal* actors.

**Definition 13.** A *star influence graph* is an influence graph $(G, f)$, where $V(G) = L \cup I \cup R \cup \{c\} \cup F$ and $E(G) = \{(u, c) \mid u \in L \cup R\} \cup \{(c, v) \mid v \in R \cup F\}$. A *star influence game* is a game $\Gamma = (G, f, q, N)$, where $N = L \cup R \cup I$ and $(G, f)$ is a star influence graph.

Without loss of generality we can assume that the labeling function of a star influence game satisfies $f(i) \in \{0, 1\}$, for $i \in V(G) \setminus \{c\}$. Note that a reciprocal actor with a label greater than 1 could never be influenced by the mediator, so we remove the edges from the center to the actor which becomes an additional opinion leader in the new graph. We can also assume that $f(c) \in \{0, 1, \ldots, |L| + |R| + 1\}$.

**Lemma 10.** *Let $(G, f, q, N)$ be a star influence game. For $1 \leq k \leq n$, the values $|F_k(N, G, f)|$ can be computed in polynomial time.*

*Proof.* We consider three cases and provide either a closed formula or a recursion allowing to compute $|F_k(N, G, f)|$ in polynomial time. Let $RF = R \cup F$.

**Case 1:** $f(i) > 0$, for $i \in V(G)$ .

When $k < f(c)$ and $X \in F_k(N)$, we know that $c \notin F(X \cap N)$. Thus, $F_k(N)$ only contains those sets $X$ with $|X \cap N| = k$. When $k \geq f(c)$, $F_k(N)$ can be divided into two subsets: those with $c \notin F(X \cap N)$ and those with $c \in F(X \cap N)$. If $c \notin F(X \cap N)$, we know that $F(X) = X$ and that $|X \cap (L \cup R)| < f(c)$. To fulfill this condition it is enough to take $0 \leq i \leq f(c) - 1$ vertices from $L \cup R$ and the remaining $k - i$ vertices from $I$ . If $c \in F(X \cap N)$, we know that $|X \cap (L \cup R)| \geq f(c)$ and, therefore, $R \cup F \cup \{c\} \subseteq F(X \cap N)$. To attain those restrictions, together with $F(X \cap N) = k$, we have to take $i$ vertices from $L$, $j$ vertices from $R$, for some values of $i, j$ verifying $i + j \geq f(c)$ and $k - i - (|RF| + 1) \geq 0$, and complete the set with $k - i - (|RF| + 1)$ elements from $I$. Putting all together, we have the following expressions.

When $k < f(c)$,

$$\frac{|F_k(N)|}{2^{|F|+1}} = \binom{|L| + |R| + |I|}{k}$$

When $k \geq f(c)$,

$$\frac{|F_k(N)|}{2^{|F|+1}} = \sum_{i=0}^{f(c)-1} \binom{|L| + |R|}{i}\binom{|I|}{k - i} + \sum_{\substack{i+j \geq f(c) \\ 0 \leq i \leq |L|\ , 0 \leq j \leq |R| \\ k-i-(|RF|+1) \geq 0}} \binom{|L|}{i}\binom{|R|}{j}\binom{|I|}{k - i - (|RF| + 1)}$$

**Case 2:** $f(c) = 0$.

Observe that in this case, for $X \subseteq V$, $F(X \cap N) = (X \cap (L \cup I)) \cup R \cup F \cup \{c\}$. Thus either $k < |RF| + 1$ or $k \geq |RF| + 1$. In the first case, there are no sets expanding to $k$ vertices. In the second, we have to select the adequate number of vertices from $L \cup I$ and any number from $R$. This leads to the following expression:

$$\frac{|F_k(N)|}{2^{|F|+1}} = \begin{cases} 0 & \text{if } k < |RF| + 1 \\ \binom{|L| + |I|}{k - (|RF| + 1)} 2^{|R|} & \text{if } k \geq |RF| + 1. \end{cases}$$

**Case 3:** $f(c) > 0$ and, for some $i \in V(G) \setminus \{c\}$, $f(i) = 0$.

Let $Z_1 = \{i \in I \cup F \mid f(i) = 0\}$, $Z_2 = \{i \in L \cup R \mid f(i) = 0\}$, $z_1 = |Z_1|$ and $z_2 = |Z_2|$.

Let $X \subseteq V$. Observe that, for $i \in Z_1 \cup Z_2$, $i \in F(X \cap N)$. Therefore, we can remove those vertices from the influence graph taking care of reducing the label of $c$ whenever a vertex from $Z_2$ is removed and reducing the size of the required
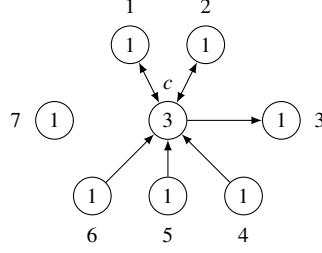
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Figure 7: A star influence graph.

expansion. So, we construct the influence graph $(G', f')$ where $G' = G[V(G) \setminus (Z_1 \cup Z_2)]$ and $f'(u) = f(u)$, for $u \neq c$, and $f'(c) = \max\{f(c) - z_2, 0\}$. We have

$$|F_k(N, G, f)| = |F_{k-z_1-z_2}(N \setminus (Z_1 \cup Z_2), G', f')|2^{z_1+z_2}.$$

As, only $f'(c)$ can be zero, the later expression can be computed using the formulas provided in the previous cases and the claim follows. $\qquad\square$

It is known that the problem of counting the number of winning or losing coalitions of a given influence game is #P-complete [26]. The expressions provided in Lemma 10 allow us to count the number of winning and losing coalitions, for star influence games, in polynomial time, by computing $\sum_{i=q}^{n} |F_i(N)|$ and $\sum_{i=0}^{q-1} |F_i(N)|$, respectively.

**Example 6.** Consider the star influence graph $(G, f)$ of Figure 7 and the star influence model $(G, f, q = 4, N = \{1, 2, 4, 5, 6, 7\})$. Here $\mathtt{I} = \{7\}$, $\mathtt{F} = \{3\}$, $\mathtt{R} = \{1, 2\}$ and $\mathtt{L} = \{4, 5, 6\}$. Hence, $|\mathcal{W}| = \sum_{i=4}^{8} |F_i(N)| = 4(0+3+12+13+4) = 128$, and $|\mathcal{L}| = \sum_{i=0}^{3} |F_i(N)| = 4(1 + 6 + 15 + 10) = 128$. Note that it holds that $|\mathcal{W}| + |\mathcal{L}| = 256 = 2^8$, as expected.

In order to solve the Satisfaction problem for star influence models we first show that the Expansion problem can be solved in polynomial time in a slightly extended family of influence graphs.

**Definition 14.** An *extended star influence graph* is an influence graph that is obtained from a star influence graph $(G', f')$, by selecting one vertex $u \in \mathrm{R}(G')$ and adding a set of vertices $F_u$ with label 1 and the set of edges $\{(u, v) \mid v \in F_u\}$. An *extended star influence model* is an influence model $(G, f, q, N)$, where $(G, f)$ is an extended star influence graph and $N = \mathtt{L} \cup \mathtt{R} \cup \mathtt{I}$.

Observe that all the additional vertices attached to a reciprocal actor are followers. By taking $F_u = \emptyset$ we obtain a star influence graph.

**Lemma 11.** *Let $(G, f)$ be an extended star influence graph and let $N = \mathrm{L}(G) \cup \mathrm{I}(G) \cup \mathrm{R}(G)$. For $1 \leq k \leq N$, $|F_k(N, G, f)|$ can be computed in polynomial time.*

*Proof.* Let $(G, f)$ be an extended star influence graph. Let $u$ be the selected vertex in $\mathrm{R}$. Assume that, for $i \in V(G)$, $f(i) > 0$. Set $\mathtt{LR} = \mathtt{L} \cup \mathtt{R}$ and $\mathtt{RF} = \mathtt{R} \cup \mathtt{F}$.

If $k < f(c)$ and $X \in F_k(N)$, we know that $c \notin F(X \cap N)$. Therefore, $F_k(N)$ only contains those sets $X$ with $|X \cap N| = k$. If $u \notin X$, then $F(X \cap N) = X \cap N$; but if $u \in X$, then $F(X \cap N) = (X \cap N) \cup F_u$.

On the other hand, if $k \geq f(c)$, the set $F_k(N)$ can be divided into two subsets, those with $c \notin F(X \cap N)$ and those with $c \in F(X \cap N)$. As in the proof of Lemma 10, when $c \notin F(X \cap N)$, we need $|X \cap (\mathtt{L} \cup \mathtt{R})| < f(c)$; but when $c \in F(X \cap N)$, $\mathtt{R} \cup \mathtt{F} \cup F_u \subseteq F(X \cap N)$. Therefore, we have the following expressions. When $k < f(c)$,

$$\frac{|F_k(N)|}{2^{|\mathtt{F}|+|F_u|+1}} = \binom{|\mathtt{LR}| - 1 + |\mathtt{I}|}{k} + \binom{|\mathtt{LR}| - 1 + |\mathtt{I}|}{k - |F_u| - 1}.$$

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When $k \geq f(c)$,

$$\frac{|F_k(N)|}{2^{|F|+|F_u|+1}} = \sum_{i=0}^{f(c)-1} \left[ \binom{|LR|-1}{i}\binom{|I|}{k-i} + \binom{|LR|-1}{i}\binom{|I|}{k-i-|F_u|} \right]$$
$$+ \sum_{\substack{i+j \geq f(c) \\ 0 \leq i \leq |L|, \, 0 \leq j \leq |R| \\ k-i-(|RF|+1) \geq 0}} \binom{|L|}{i}\binom{|R|}{j}\binom{|I|}{k-i-(|R|+|F|+|F_u|+1)}.$$

Using this expression, $|F_k(N)|$ can be computed in polynomial time. An argument similar to the one in the proof of Lemma 10 allows us to devise a polynomial time algorithm to compute $|F_k(N)|$ when some of the labels are zero. $\quad\square$

We transfer the previous results to an algorithm for solving the SATISFACTION problem (and therefore the POWER problem) for the corresponding oblivious models. Note that, for a given star influence model $(G, f, q, N)$, the graphs $R(G, f, i)$ and $R_2(G, f, i)$, as required in Lemmas 7 and 8, are extended star influence graphs. As a consequence of those results we have the following.

**Theorem 5.** *The* SATISFACTION *and* POWER *problems, for oblivious models associated with star influence games, are polynomial time solvable.*

Finally, we show how to solve the SATISFACTION and POWER problems in non-oblivious models.

**Theorem 6.** *The* SATISFACTION *and* POWER *problems, for non-oblivious models associated with star influence games, are polynomial time solvable.*

*Proof.* Let $\Gamma = (G, f, q, N)$ be a star influence game. We analyze the differences among actors in the oblivious $\mathcal{M}^o$ and non-oblivious $\mathcal{M}^n$ associated models. Recall that, for $x \in \{0, 1\}^n$, $p_i(x) = |F(X(x) \cap N) \cap P(i)|$ and $q_i(x) = |P(i) \setminus F(X(x) \cap N)|$ and that the final decision vectors in both models are defined as follows.

In the oblivious model, for $i \in V(G)$,

$$c_i^{\mathcal{M}^o} = \begin{cases} 1 & \text{if } i \in F(X(x)), \\ 0 & \text{otherwise.} \end{cases}$$

In the non-oblivious model we have two cases, for $i \in V(G) \setminus N$,

$$c_i^{\mathcal{M}^n} = \begin{cases} 1 & \text{if } p_i(x) \geq f(i) \text{ and } q_i(x) < f(i), \\ 0 & \text{if } q_i(x) \geq f(i) \text{ and } p_i(x) < f(i), \\ x_i & \text{otherwise,} \end{cases}$$

and, for $i \in N$,

$$c_i^{\mathcal{M}^n} = \begin{cases} 1 & \text{if } i \in F(X(x)), \\ 0 & \text{otherwise.} \end{cases}$$

Let $x \in \{0, 1\}^n$ be an initial decision vector. Note that, for $i \in V \setminus \{c\}$, $f(i) \in \{0, 1\}$ and the in-degree of $i$ is either 0 or 1. Thus, according to the above expressions, for $i \in V \setminus \{c\}$, $c_i^{\mathcal{M}^o}(x) = c_i^{\mathcal{M}^n}(x)$. This implies that the final decision vectors in $\mathcal{M}^o$ and $\mathcal{M}^n$ can only differ in the final decision of $c$.

When $p_c(x), q_c(x) \geq f(i)$, $c_c^{\mathcal{M}^o}(x) = 1$ but $c_c^{\mathcal{M}^n}(x) = x_c$. When $p_c(x), q_c(x) < f(i)$, $c_c^{\mathcal{M}^o}(x) = 0$ but $c_c^{\mathcal{M}^n}(x) = x_c$. In all the remaining cases we have $c_c^{\mathcal{M}^o}(x) = c_c^{\mathcal{M}^n}(x)$.

The different final decision of $c$ has relevance only if it implies a change in the final collective decision. Therefore we have to examine only two cases in which a difference can arise.

**Case 1** $p_c(x), q_c(x) \geq f(i)$ and $|F(X(x))| = q$.
In this case $c_{\mathcal{M}^o}(x) = 1$, thus in the oblivious model $c$ is satisfied only when $x_c = 1$. But, $c_{\mathcal{M}^n}(x) = x_c$, thus in the non-oblivious model $c$ is satisfied independently of its initial choice.

**Case 2** $p_c(x), q_c(x) < f(i)$ and $|F(X(x))| = q - 1$.

In this case $c_{\mathcal{M}^o}(x) = 0$, thus in the oblivious model $c$ is satisfied only when $x_c = 0$. But, $c_{\mathcal{M}^n}(x) = x_c$, thus in the non-oblivious model $c$ is satisfied independently of its initial choice. Using the above properties, we can obtain an expression for $\text{S}\textsc{at}_{\mathcal{M}^n}(i)$.

When $i \neq c$, the final collective decision in both $\mathcal{M}^o$ and $\mathcal{M}^n$ is independent of the value of $x_i$. Therefore, $\text{S}\textsc{at}_{\mathcal{M}^n}(i) = \text{S}\textsc{at}_{\mathcal{M}^o}(i) = 2^{n-1}$.

For the central vertex $c$ we have,

$$\text{S}\textsc{at}_{\mathcal{M}^n}(c) = \text{S}\textsc{at}_{\mathcal{M}^o}(c) + |\{x \in \{0, 1\}^n \mid x_c = 0, \ p_c(x), q_c(x) \geq f(c) \text{ and } |F(X(x))| = q\}|$$

$$+ |\{x \in \{0, 1\}^n \mid x_c = 1, \ p_c(x), q_c(x) < f(c) \text{ and } |F(X(x))| = q - 1\}|.$$

It is easy to derive closed formulas for the sizes of the sets appearing in the above expressions. Therefore, from Theorem 5 the claim follows. □

## 7. Conclusions and open problems

Firstly, we have introduced collective decision-making models that extend the opinion leader-follower decision model to general social networks. We allow a phase of influence spread under the linear threshold model before taking the final individual decision. This leads us to the definition of oblivious and non-oblivious influence decision models reflecting the different use of influence spread in OLF models and influence games. Our models capture the gOLF model, which is an extension of the OLF model proposed by van den Brink et al. [35]. In our models the final decision is taken by computing the sum of the 'yes' votes when individual final decisions are set. Our formalism allows a flexibility in the required quota to win. As in weighted voting games, the quota allows to consider different decision rules including traditional ones like simple majority or consensus.

We have established a connection between influence decision models and simple games. This connection allows us to relate the satisfaction and power measures proposed by [35] with well known power indices for simple games. In particular we show that both measures are polynomially related to the Banzhaf value in the associated simple game. This justifies further the initial idea that satisfaction and power are good measures to rank the leadership of the participants.

Secondly, we have studied the computational complexity of the satisfaction and the power measures. We have shown that computing satisfaction or power is computationally hard even in decision systems in which the influence graph is restricted to be a two-layered bipartite digraph with constant in-degree. Interestingly enough, we established a close relation to the well known Banzhaf value and Rae index on the associated simple games. This allows the use of known approximation algorithms for computing Banzhaf value or Rae index [12, 4] to compute the satisfaction and the power measures. A future line of research is to analyze the quality of the solutions provided by such algorithms on oblivious and non-oblivious influence decision models.

Finally, in order to complement the computationally hardness results, we have explored the possibility of having models in which the S\textsc{atisfaction} and the P\textsc{ower} problems are tractable. We defined two subfamilies of oblivious influence models, the strong hierarchical influence models and the star influence models, in which satisfaction can be computed in polynomial time. The first family considers multilayered bipartite influence graphs having a hierarchy of mediators moderating the spread of influence from opinion leaders to followers. In the star influence models we also allow a restricted form of mutual influence. For strong hierarchical influence models and star influence models, besides computing in polynomial time the satisfaction and the power measures, we have show efficiently algorithms to compute additional information. In particular, we can compute in polynomial time the number of initial decision vectors that expand the 'yes' vote to $k$ actors being $b$ of them followers.

The result for star influence models can easily be extended to a more general version of the star influence graphs, in which the central vertex is replicated in a set of independent vertices. Since these vertices are a complete replica of the central vertex, then they keep the existing connections from the central vertex to the remaining vertices. The algorithm follows the same lines of the dynamic programming algorithm for star influence models. We can consider a *mixed oblivious influence model* whose structure is obtained from an influence graph $(G, f)$ and a strong hierarchical or a star influence graph $(G', f')$ adding a complete bipartite graph between $V(G')$ and a set $U \subseteq V(G)$. It is easy to show that the set $F_k(\text{L}(G') \cup \text{I}(G') \cup \text{R}(G'))$ in a mixed oblivious influence model can be computed in polynomial time.

As the connections are among sets of vertices and forming complete bipartite graphs, only a polynomial number of subsets of $U$ can be activated. Thus strong hierarchical or star influence graph can be used as mechanism to control the spread of influence in a generic social network.

Among several problems that remain open, we want to point out two that relate directly to the hardness proofs provided in this paper. The first one is the complexity of the Satisfaction problem for OLF models under the simple majority rule as our reduction does not construct an odd-gOLF with this restriction. Although we conjecture that the problem is hard we have been unable to straighten the reduction. The second one relates to other aspects of the complexity of counting independent sets of size $\frac{1}{3}n$. In [17] it was shown a stronger result for this problem. The reduction from #3-SAT, given a formula with $m$ clauses, constructs a graph with $O(m)$ vertices. Furthermore, the number of independent sets of size $\frac{1}{3}|V|$ of the constructed graph is equal to the number of satisfying assignments of the original formula. Such property implies that the problem of computing the number of independent sets with size exactly $\frac{2}{3}|V|$ in a graph is harder, in the sense that it cannot be solved by a sub-exponential time algorithm, unless the #3-Sat problem can be solved in sub-exponential time [17]. The graph constructed in our reduction does not have linear size with respect to the input graph. Hence, we can only deduce that the problem is #P-hard. It remains open to show if sub-exponential time algorithms can also be ruled out.

One restrictive decision model that has received some attention is the one in which the unanimity of the predecessors is required to adopt a decision. This is equivalent to set $r = 1$ in the OLF models or set $f(u) = \delta^-(u)$ in influence models. There exists an axiomatization of the satisfaction measure for OLFs with $r = 1$ [36]. The influence games on undirected influence graphs with $f(u) = \delta^-(u)$ are called *maximum influence* games [26]. Interestingly enough, several problems that are hard for influence games became tractable for maximum influence games. None of the reductions in this paper produces a model in which unanimity of predecessors is required. Therefore, the complexity of computing the satisfaction measure in such subfamilies of decision models remains as an interesting open problem.

Another line for further work is to study the computational complexity of other measures for collective decision-making models. The satisfaction measure is closely related to the *Chow parameters* for simple games [11], but considering also losing coalitions. Chow parameters were initially defined in the 1960s in the context of threshold Boolean functions [6], and it is known that their computation, for simple games given by their minimal winning coalitions, is #P-complete [2]. In a similar way, the power measure is related with another power index called the *Holler index* [3]. It will be of interest to see whether the results on this paper can be extended to the computation of those and other measures. In this line, due to the relationship that we have established between these decision models and simple games, each of the power indices introduced in the context of simple games can be extended to a corresponding measure for influence models. It is of interest to analyze which of those power indices provide interesting measures for decision systems.

The decision models considered in this paper are simplified *social networks group decision* models defined over some opinion dynamics process (see, for example, [9, 38, 10] and references herein). It would be interesting to consider simple collective decision-making models based on opinion dynamics processes and analyze the suitability of the satisfaction and the power measures to rank leadership. In particular, a closer model is the two competing opinion *bounded confidence model* [38]. Here the society is divided in three classes: leaders supporting the issue, leaders against the issue, and followers. Here again the agent's role is directly related to the topology of the network. It will be of interest to analyze whether this kind of opinion dynamics can lead to the definition of a collective decision model and see if the satisfaction or the power measures still provide good measures of leadership. Under this approach, it is also interesting to devise decision models, and the adequate measures for leadership, based on well established group decision dynamics, discrete or continuous, having multiple opinion dynamics or/and multicriteria decision making, like the ones considered in [22, 21, 20, 39, 1, 28].

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