Computing the Green Function for a Dirichlet problem on Spider Networks

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Abstract

Product networks are the weigthed version of the cartesian product of graphs. Our research group used separation of variable techniques in order to work with this family of networks and express their Green operators on a subset of vertices in terms of the eigenvalues and eigenfuctions of one of the factor and the Green operator of the other factor; see [1]. Some authors have dealt with similar problems. For example, in [2], Chung and Yau obtained the classical Green function with respect to the normalized laplacian of the cartesian product of two regular graphs.

In this work we used the separation of variables technique to get the expression of the Green function on a subset of vertices of *spider networks*; see Figure 1. The difficulty to obtain this function by means of direct methods is high. However, any spider network can be seen as the modification of a certain product network. A

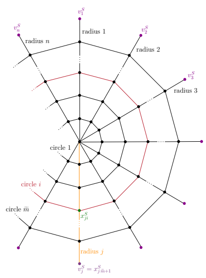

Figure 1: Structure of a spider network.

spider network (Γ_S, c_S) is a circular planar network with n boundary nodes and the following structure: n *radius* and $\widetilde{m}$ *circles* distributed as in Figure 1, where the vertices lay on the intersections and the edges are given by these radius and circle lines. The vertex x_{ji}^S is defined as the intersection between the radius j and the circle i for all $i = 1, \dots, \widetilde{m}+1$, $j = 1, \dots, n$, whereas the vertex x_{00}^S is the intersection of all the radius, that is, the vertex on the center. Note that $x_{j0}^S = x_{00}^S$ for all $j = 1, \dots, n$.

The boundary circle does not give any edge as it is not a proper circle of the network –it is an imaginary one such that the vertices on it are the n boundary nodes. For all $j = 1, \dots, n$ we call $v_j = x_{j\,\widetilde{m}+1}^S$ the vertices on the boundary circle. Let F_S be the set of interior vertices of the spider network, where

$F_S = \{x_{11}^S, \dots, x_{n1}^S, \dots, x_{1\widetilde{m}}^S, \dots, x_{n\widetilde{m}}^S, x_{00}^S\}$, and let the set of boundary vertices be $\delta(F_S) = \{v_1^S, \dots, v_n^S\}$.

The spider network can be obtained from the product network $\Gamma = C_n \times P_m$ of a cycle and a path in the following way. Taking $m = \widetilde{m}+2$, we consider the product network of an n–cycle and an m–path. Observe that the removal of all the edges between boundary vertices and the identification of all the vertices of the first level (H_1) into one only vertex leads to the obtention of a spider network (see Figure 2). For all $j = 1, \dots, n$ and $i = 1, \dots, \widetilde{m}$ there exists a correspondence between

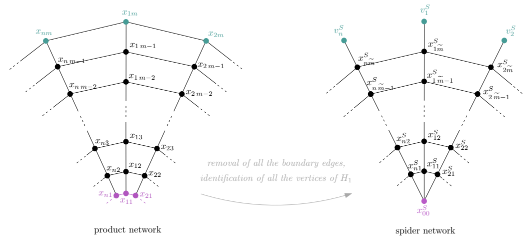

Figure 2: How to obtain a spider from a product network.

the vertex x_{ji}^S on the spider network and the vertex $x_{j\,i+1}$ on this modification of the product network, between v_j^S and x_{jm} and between x_{j0}^S and x_{j0}. As a consequence, we can arbitrarily get the conductances of a spider network from a product network: $c_S(x_{ji}^S, x_{lk}^S) = c(x_{j\,i+1}, x_{l\,k+1})$, $c_S(x_{ji}^S, x_{00}^S) = c(x_{j\,i+1}, x_{j1})$, $c_S(x_{ji}^S, v_l^S) = c(x_{j\,i+1}, x_{lm})$ and $c_S(v_j^S, v_l^S) = 0$ for all $j, l = 1, \dots, n$ and $i, k = 1, \dots, \widetilde{m}$.

Then, we can obtain the Green function of a Spider network which will be an appropriate modification of the Green function of the product $C_n \times P_m$.

Acknowledgements: This work has been partly supported by the Spanish Research Council (Comisión Interministerial de Ciencia y Tecnología) and I+D+i State Program (Ministerio de Economía y Competitividad) under projects MTM2014-60450-R and MTM2017-85996-R. This work has not deserved financial support at all by the Generalitat de Catalunya under its research program SGR2017.

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