ON INDEFINITE SUMS WEIGHTED BY PERIODIC SEQUENCES

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Abstract. For any integer $q \geq 2$ we provide a formula to express indefinite sums of a sequence $(f(n))_{n\geq 0}$ weighted by q-periodic sequences in terms of indefinite sums of sequences $(f(qn+p))_{n\geq 0}$, where $p \in \{0,\ldots,q-1\}$. When explicit expressions for the latter sums are available, this formula immediately provides explicit expressions for the former sums. We also illustrate this formula through some examples.

1. Introduction

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of non-negative integers. We assume throughout that $q \geq 2$ is a fixed integer and we set $\omega = \exp(\frac{2\pi i}{q})$. Consider two functions $f\colon \mathbb{N} \to \mathbb{C}$ and $g\colon \mathbb{Z} \to \mathbb{C}$ and suppose that g is q-periodic, that is, $g(n+q) = g(n)$ for every $n \in \mathbb{Z}$. Also, consider the functions $S\colon \mathbb{N} \to \mathbb{C}$ and $T_p\colon \mathbb{N} \to \mathbb{C}$ ($p = 0, \ldots, q-1$) defined by

$$S(n) \ = \ \sum_{k=0}^{n-1} g(k)f(k)$$

and

$$T_p(n) \ = \ \sum_{k=0}^{n-1} f(qk+p),$$

respectively. These functions are indefinite sums (or anti-differences) in the sense that the identities

$$\Delta_n S(n) \ = \ g(n)f(n) \qquad \text{and} \qquad \Delta_n T_p(n) \ = \ f(qn+p)$$

hold on $\mathbb{N}$, where Δ_n is the classical difference operation defined by $\Delta_n f(n) = f(n+1) - f(n)$; see, e.g., [4, §2.6].

In this paper we provide a conversion formula that expresses the sum $S(n)$ in terms of the sums $T_p(n)$ for $p = 0, \ldots, q-1$ (see Proposition 4). Such a formula can sometimes be very helpful for it enables us to find an explicit expression for $S(n)$ whenever explicit expressions for the sums $T_p(n)$ for $p = 0, \ldots, q-1$ are available.

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Example 1. The sum

$$S(n) \;=\; \sum_{k=1}^{n-1}\cos\left(\frac{2k\pi}{3}\right)\log k\,, \qquad n\in\mathbb{N}\setminus\{0\},$$

where the function $g(k)=\cos(\frac{2k\pi}{3})$ is 3-periodic, can be computed from the sums

$$T_p(n) \;=\; \sum_{k=1}^{n-1}\log(3k+p) \;=\; \log\left(3^{n-1}\,\frac{\Gamma(n+p/3)}{\Gamma(1+p/3)}\right), \qquad p\in\{0,1,2\}.$$

Using our conversion formula we arrive, after some algebra, at the following closed-form representation

$$S(n) \;=\; \frac{1}{4}\,\log\left(\frac{4\pi^2}{27}\right)+\sum_{j=0}^{2}\cos\left(\frac{2\pi(n+j)}{3}\right)\,\log\left(3^{j/3}\,\Gamma\left(\frac{n+j}{3}\right)\right).$$

Oppositely, we also provide a very simple conversion formula that expresses each of the sums $T_p(n)$ for $p=0,\ldots,q-1$ in terms of sums of type $S(n)$ (see Proposition 4). As we will see through a couple of examples, this formula can sometimes enable us to find closed-form representations of sums $T_p(n)$ that are seemingly hard to evaluate explicitly.

Example 2. Sums of the form

$$\sum_{k\geq 0}\binom{m}{qk+p}\frac{1}{qk+p+1}$$

where $m\in\mathbb{N}$ and $p\in\{0,\ldots,q-1\}$, may seem complex to be evaluated explicitly. In Example 14 we provide a closed-form representation of this sum by first considering a sum of the form

$$\sum_{k\geq 0}g(k)\binom{m}{k}\frac{1}{k+1},$$

for some q-periodic function $g(k)$. For instance, we obtain

$$\sum_{k\geq 0}\binom{m}{3k+1}\frac{1}{3k+2} \;=\; \frac{1}{6(m+1)}\,\left(2^{m+2}-3\cos\frac{m\pi}{3}-\cos\frac{5m\pi}{3}\right).$$

In this paper we also provide an explicit expression for the (ordinary) generating function of the sequence $(S(n))_{n\geq 0}$ in terms of the generating function of the sequence $(f(n))_{n\geq 0}$ (see Proposition 5). Finally, in Section 3 we illustrate our results through some examples.

2. The result

The functions $g_p\colon\mathbb{Z}\to\{0,1\}$ $(p=0,\ldots,q-1)$ defined by

$$g_p(n) \;=\; g_0(n-p) \;=\; \begin{cases}1, & \text{if } n=p \pmod q,\\ 0, & \text{otherwise,}\end{cases}$$

form a basis of the linear space of q-periodic functions $g\colon\mathbb{Z}\to\mathbb{C}$. More precisely, for any q-periodic function $g\colon\mathbb{Z}\to\mathbb{C}$, we have

$$g(n) \;=\; \sum_{p=0}^{q-1}g(p)\,g_p(n) \;=\; \sum_{p=0}^{q-1}g(p)\,g_0(n-p)\,, \qquad n\in\mathbb{Z}. \tag{1}$$

To compute the sum $S(n)=\sum_{k=0}^{n-1}g(k)f(k)$, it is then enough to compute the q sums

$$S_p(n) ~=~ \sum_{k=0}^{n-1}g_p(k)f(k) ~=~ \sum_{k=0}^{n-1}g_0(k-p)f(k)\,,\qquad p=0,\ldots,q-1.$$

Indeed, using (1) we then have

$$S(n) ~=~ \sum_{p=0}^{q-1}g(p)\,S_p(n)\,,\qquad n\in\mathbb{N}. \tag{2}$$

It is now our aim to find explicit conversion formulas between $S_p(n)$ and $T_p(n)$, for every $p\in\{0,\ldots,q-1\}$. The result is given in Proposition 4 below. We first consider the following fact that can be immediately derived from the definitions of the sums S_p and T_p.

Fact 3. *For any $n\in\mathbb{N}$ and any $p\in\{0,\ldots,q-1\}$, we have*

$$S_p(n) ~=~ T_p(\lfloor(n-p-1)/q\rfloor+1).$$

In particular, we have $S_p(qn+p-i)=T_p(n)$ for any $i\in\{0,\ldots,q-1\}$ such that $qn+p-i\in\mathbb{N}$.

For any $p\in\{0,\ldots,q-1\}$, let $T_p^+\colon D_p\to\mathbb{C}$ be any extension of T_p to the set $D_p=\{\frac{-p}{q},\frac{1-p}{q},\frac{2-p}{q},\ldots\}$. By definition, we have $\mathbb{N}\subset D_p$ and $T_p^+=T_p$ on $\mathbb{N}$.

Proposition 4. *For any $n\in\mathbb{N}$ and any $p\in\{0,\ldots,q-1\}$, we have $T_p(n)=S_p(qn)$ and*

$$S_p(n) ~=~ \sum_{k=0}^{q-1}g_0(n+k-p)\,T_p^+\left(\frac{n+k-p}{q}\right). \tag{3}$$

Remark 1. We observe that each summand for which $n+k-p<0$ in (3) is zero since in this case we have $n+k\neq p \pmod{q}$.

Proof of Proposition 4. Let $n\in\mathbb{N}$ and $p\in\{0,\ldots,q-1\}$. The identity $T_p(n)=S_p(qn)$ immediately follows from Fact 3. Now, for any $k\in\mathbb{N}$ we have $g_0(n+k-p)=1$ if and only if there exists $M\in\mathbb{Z}$ such that $k=Mq+p-n$. Assuming that $k\in\{0,\ldots,q-1\}$, the latter condition holds if and only if $M=\lfloor(n-p-1)/q\rfloor+1$. Thus, the sum in (3) reduces to $T_p(\lfloor(n-p-1)/q\rfloor+1)$, which is $S_p(n)$ by Fact 3. $\square$

Alternative proof of (3). The identity clearly holds for $n=0$ since we have $S_p(0)=0=T_p(0)$. It is then enough to show that (3) still holds after applying the difference operator Δ_n to each side. Applying Δ_n to the right-hand side, we immediately obtain a telescoping sum that reduces to

$$g_0(n+q-p)\,T_p^+\left(\frac{n+q-p}{q}\right)-g_0(n-p)\,T_p^+\left(\frac{n-p}{q}\right),$$

that is,

$$g_0(n-p)\,(\Delta\,T_p^+)\left(\frac{n-p}{q}\right).$$

If $n\neq p \pmod{q}$, then $g_0(n-p)=0$ and hence the latter expression reduces to zero. Otherwise, it becomes $g_0(n-p)(\Delta\,T_p)(\frac{n-p}{q})$. In both cases, the expression reduces to $g_0(n-p)\,f(n)$, which is nothing other than $\Delta_n S_p(n)$. $\square$

Let $F(z) = \sum_{n\geq 0} f(n)\, z^n$ and $F_p(z) = \sum_{n\geq 0} f(qn+p)\, z^n$ be the generating functions of the sequences $(f(n))_{n\geq 0}$ and $(f(qn+p))_{n\geq 0}$, respectively. The following proposition provides explicit forms of the generating function of the sequence $(S_p(n))_{n\geq 0}$ in terms of $F(z)$ and $F_p(z)$. The proof of this proposition uses a familiar trick for extracting alternate terms of a series; see, e.g., [2, p. 90] and [5, p. 89].

Recall that if $A(z) = \sum_{n\geq 0} f(n)\, z^n$ is the generating function of a sequence $(a(n))_{n\geq 0}$, then $\frac{1}{1-z}A(z)$ is the generating function of the sequence of the partial sums $(\sum_{k=0}^{n} a(k))_{n\geq 0}$; see, e.g., [4, §5.4] and [5, p. 89].

Proposition 5. *If $F(z)$ converges in some disk $|z| < R$, then*

$$\sum_{n\geq 0} S_p(n)\, z^n \;=\; \frac{z}{q(1-z)}\sum_{k=0}^{q-1}\omega^{-kp}\, F(\omega^k z) \;=\; \frac{z^{p+1}}{1-z}\, F_p(z^q).$$

Proof. We first observe that the identity $\frac{1}{q}\sum_{k=0}^{q-1}\omega^{kn} = g_0(n)$ holds for any $n \in \mathbb{Z}$. For any $z \in \mathbb{C}$ such that $|z| < R$, we then have

$$\begin{aligned}\frac{z}{q(1-z)}\sum_{k=0}^{q-1}\omega^{-kp}\, F(\omega^k z) \;&=\; \frac{z}{1-z}\sum_{n\geq 0} g_p(n) f(n)\, z^n\\ &=\; \sum_{n\geq 0} S_p(n+1)\, z^{n+1} \;=\; \sum_{n\geq 0} S_p(n)\, z^n,\end{aligned}$$

which proves the first formula. For the second formula, we simply observe that

$$\sum_{n\geq 0} g_p(n) f(n)\, z^n \;=\; \sum_{n\geq 0} f(qn+p)\, z^{qn+p} \;=\; z^p\, F_p(z^q).$$

This completes the proof. □

We end this section by providing explicit forms of the function $g_0(n)$. For instance, it is easy to verify that

$$g_0(n) \;=\; \left\lfloor\frac{n}{q}\right\rfloor - \left\lfloor\frac{n-1}{q}\right\rfloor \;=\; \Delta_n \left\lfloor\frac{n-1}{q}\right\rfloor.$$

As already observed in the proof of Proposition 5, we also have

$$g_0(n) \;=\; \frac{1}{q}\sum_{j=0}^{q-1}\omega^{jn} \;=\; \frac{1}{q}\sum_{j=0}^{q-1}\cos\left(j\,\frac{2n\pi}{q}\right). \tag{4}$$

Alternatively, we also have the following expression (see also [3, p. 41])

$$g_0(n) \;=\; \begin{cases}\frac{1}{q} + \frac{2}{q}\sum_{j=1}^{(q-1)/2}\cos(j\,\frac{2n\pi}{q}), & \text{if } q \text{ is odd},\\ \frac{1}{q} + \frac{1}{q}(-1)^n + \frac{2}{q}\sum_{j=1}^{(q/2)-1}\cos(j\,\frac{2n\pi}{q}), & \text{if } q \text{ is even},\end{cases}$$

or equivalently,

$$g_0(n) \;=\; \frac{1}{q} + \frac{(-1)^n + (-1)^{n+q}}{2q} + \frac{2}{q}\sum_{j=1}^{\lfloor (q-1)/2\rfloor}\cos\left(j\,\frac{2n\pi}{q}\right).$$

3. Some applications

In this section we consider some examples to illustrate and demonstrate the use of Propositions 4 and 5. A few of these examples make use of the *harmonic number* with a complex argument, which is defined by the series

$$H_z \ = \ \sum_{n\geq 1}\left(\frac{1}{n}-\frac{1}{n+z}\right), \qquad z\in\mathbb{C}\setminus\{-1,-2,\ldots\},$$

(see, e.g., [4, p. 311, Ex. 6.22] and [5, p. 95, Ex. 19]).

Remark 2. Formula (3) is clearly helpful to obtain an explicit expression for the sum $S_p(n)$ whenever closed-form representations of the associated sums $T_p(n)$ for $p=0,\ldots,q-1$ are available. Otherwise, the formula might be of little interest. For instance, the formula will not be very useful to obtain an explicit expression for the *number of derangements* (see, e.g., [4, p. 195])

$$d(n) \ = \ n!\sum_{k=0}^{n}(-1)^k\,\frac{1}{k!}\,.$$

Indeed, the associated sums $\sum_{k=0}^{n}\frac{1}{(2k)!}$ and $\sum_{k=0}^{n}\frac{1}{(2k+1)!}$ have no known closed-form representations.

3.1. **Sums weighted by a 4-periodic sequence.** Suppose we wish to provide a closed-form representation of the sum

$$S(n) \ = \ \sum_{k=0}^{n-1}\sin\left(k\frac{\pi}{2}\right)f(k)\,, \qquad n\in\mathbb{N},$$

where the function $g(k)=\sin(k\frac{\pi}{2})$ is 4-periodic. By (2) we then have

$$S(n) \ = \ \sum_{p=0}^{3}\sin\left(p\frac{\pi}{2}\right)S_p(n) \ = \ S_1(n)-S_3(n)\,, \qquad n\in\mathbb{N},$$

where

$$S_p(n) \ = \ \sum_{k=0}^{n-1}g_0(k-p)f(k)\,, \qquad p\in\{1,3\},$$

and

$$g_0(n) \ = \ \frac{1}{4}+\frac{1}{4}(-1)^n+\frac{1}{2}\cos\left(n\frac{\pi}{2}\right), \qquad n\in\mathbb{Z}.$$

Now, if an explicit expression for the sum $T_p(n)=\sum_{k=0}^{n-1}f(4k+p)$ for any $p\in\{1,3\}$ is available, then a closed-form expression for $S_p(n)$ can be immediately obtained by (3).

Also, if $F(z)$ denotes the generating function of the sequence $(f(n))_{n\geq 0}$, then by Proposition 5 the generating function of the sequence $(S(n))_{n\geq 0}$ is simply given by

$$\frac{iz}{2(1-z)}\big(F(-iz)-F(iz)\big).$$

To illustrate, let us consider a few examples.

Example 6. Suppose that $f(n)=\log(n+1)$ for all $n\in\mathbb{N}$. It is not difficult to see that

$$T_p(n) \;=\; \log\left(4^n\,\frac{\Gamma\left(n+\frac{p+1}{4}\right)}{\Gamma\left(\frac{p+1}{4}\right)}\right), \qquad p\in\{1,3\}.$$

Defining T_p^+ on $D_p=\{\frac{-p}{4},\frac{1-p}{4},\frac{2-p}{4},\ldots\}$ by

$$T_p(x) \;=\; \log\left(4^x\,\frac{\Gamma\left(x+\frac{p+1}{4}\right)}{\Gamma\left(\frac{p+1}{4}\right)}\right), \qquad p\in\{1,3\},$$

and then using (3) we obtain

$$S_p(n) \;=\; \sum_{k=0}^{3} g_0(n+k-p)\,\log\left(4^{\frac{n+k-p}{4}}\,\frac{\Gamma\left(\frac{n+k+1}{4}\right)}{\Gamma\left(\frac{p+1}{4}\right)}\right), \qquad p\in\{1,3\}.$$

Since $S(n)=S_1(n)-S_3(n)$, after some algebra we finally obtain

$$S(n) \;=\; \log\left(\frac{2}{\sqrt{\pi}}\right)+\cos\left(n\frac{\pi}{2}\right)\,\log\left(\frac{\Gamma\left(\frac{n+2}{4}\right)}{2\,\Gamma\left(\frac{n+4}{4}\right)}\right)+\sin\left(n\frac{\pi}{2}\right)\,\log\left(\frac{\Gamma\left(\frac{n+1}{4}\right)}{2\,\Gamma\left(\frac{n+3}{4}\right)}\right).$$

Example 7. Suppose that $f(n)=\frac{1}{n+1}$ for all $n\in\mathbb{N}$. Here, one can show that

$$T_p(n) \;=\; \frac{1}{4}\left(H_{n+\frac{p+1}{4}-1}-H_{\frac{p+1}{4}-1}\right), \qquad p\in\{1,3\}.$$

Defining T_p^+ on $D_p=\{\frac{-p}{4},\frac{1-p}{4},\frac{2-p}{4},\ldots\}$ by

$$T_p(x) \;=\; \frac{1}{4}\left(H_{x+\frac{p+1}{4}-1}-H_{\frac{p+1}{4}-1}\right), \qquad p\in\{1,3\},$$

and then using (3) we obtain

$$S_p(n) \;=\; \sum_{k=0}^{3} g_0(n+k-p)\,\frac{1}{4}\left(H_{\frac{n+k+1}{4}-1}-H_{\frac{p+1}{4}-1}\right), \qquad p\in\{1,3\}.$$

After simplifying the resulting expressions, we finally obtain

$$S(n) \;=\; \frac{1}{4}\,\log 4+\frac{1}{4}\,\cos\left(n\frac{\pi}{2}\right)\left(H_{\frac{n-2}{4}}-H_{\frac{n}{4}}\right)+\frac{1}{4}\,\sin\left(n\frac{\pi}{2}\right)\left(H_{\frac{n-3}{4}}-H_{\frac{n-1}{4}}\right).$$

Since $F(z)=-\frac{1}{z}\log(1-z)$, the generating function of the sequence $(S(n))_{n\geq 0}$ is given by

$$\frac{1}{2(1-z)}\big(\log(1-iz)+\log(1+iz)\big).$$

Example 8. Suppose that $f(n)=H_n$ for all $n\in\mathbb{N}$. That is, we are to evaluate the sum

$$S(n) \;=\; \sum_{k=0}^{n-1}\sin\left(k\frac{\pi}{2}\right)H_k\,, \qquad n\in\mathbb{N}.$$

Using summation by parts we obtain

$$\begin{aligned} S(n) \;&=\; -\frac{1}{\sqrt{2}}\sum_{k=0}^{n-1}\left(\Delta_k\sin\left(k\frac{\pi}{2}+\frac{\pi}{4}\right)\right)H_k \\ &=\; -\frac{1}{\sqrt{2}}\,\sin\left(n\frac{\pi}{2}+\frac{\pi}{4}\right)H_n+\frac{1}{\sqrt{2}}\sum_{k=0}^{n-1}\sin\left(k\frac{\pi}{2}+\frac{3\pi}{4}\right)\frac{1}{k+1}\,, \end{aligned}$$

where the latter sum can be evaluated as in Example 7. After simplification we obtain

$$\begin{aligned} S(n) &= \frac{\pi - 2\ln 2}{8} + \frac{1}{8}\cos\left(n\frac{\pi}{2}\right)\left(H_{\frac{n}{4}} - H_{\frac{n-1}{4}} - H_{\frac{n-2}{4}} + H_{\frac{n-3}{4}} - 4H_n\right) \\ &\quad + \frac{1}{8}\sin\left(n\frac{\pi}{2}\right)\left(H_{\frac{n}{4}} + H_{\frac{n-1}{4}} - H_{\frac{n-2}{4}} - H_{\frac{n-3}{4}} - 4H_n\right). \end{aligned}$$

Using the classical multiplication formula (see, e.g., [1, §6.4.8])

$$4H_x \;=\; H_{\frac{x}{4}} + H_{\frac{x-1}{4}} + H_{\frac{x-2}{4}} + H_{\frac{x-3}{4}} + 4\ln 4$$

we finally obtain

$$\begin{aligned} S(n) &= \frac{\pi - 2\ln 2}{8} - \frac{1}{4}\cos\left(n\frac{\pi}{2}\right)\left(H_{\frac{n-1}{4}} + H_{\frac{n-2}{4}} + 4\ln 2\right) \\ &\quad - \frac{1}{4}\sin\left(n\frac{\pi}{2}\right)\left(H_{\frac{n-2}{4}} + H_{\frac{n-3}{4}} + 4\ln 2\right). \end{aligned}$$

Example 9. Let us consider the following series

$$S \;=\; \sum_{k=1}^{\infty}\sin\left(k\frac{\pi}{2}\right)\frac{1}{k^2}\,.$$

In this case the function $f\colon \mathbb{N} \to \mathbb{R}$ is defined by $f(0) = 0$ and $f(k) = 1/k^2$ for all $k \geq 1$. By using the identity $S_p(4n) = T_p(n)$ (see Proposition 4) we immediately obtain

$$S \;=\; S_1(\infty) - S_3(\infty) \;=\; T_1(\infty) - T_3(\infty) \;=\; \sum_{k=0}^{\infty}\left(\frac{1}{(4k+1)^2} - \frac{1}{(4k+3)^2}\right).$$

Actually, this expression is nothing other than the Catalan constant

$$G \;=\; \sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)^2} \;=\; 0.915965594\ldots$$

3.2. **Alternating sums.** Consider the alternating sum

$$S(n) \;=\; \sum_{k=0}^{n-1}(-1)^k f(k), \qquad n \in \mathbb{N}.$$

Here we clearly have $q = 2$. Using (2) and (3) we immediately obtain

$$\begin{aligned} S(n) \;=\; S_0(n) - S_1(n) &= g_0(n)\,T_0^+\left(\frac{n}{2}\right) + g_0(n+1)\,T_0^+\left(\frac{n+1}{2}\right) \\ &\quad - g_0(n-1)\,T_1^+\left(\frac{n-1}{2}\right) - g_0(n)\,T_1^+\left(\frac{n}{2}\right), \end{aligned}$$

which requires the explicit computation of the sums T_0 and T_1. Alternatively, since $(-1)^k = 2\,g_0(k) - 1$, setting $S_f(n) = \sum_{k=0}^{n-1} f(k)$ we also have

$$S(n) \;=\; 2\,S_0(n) - S_f(n) \;=\; 2\,g_0(n)\,T_0^+\left(\frac{n}{2}\right) + 2\,g_0(n+1)\,T_0^+\left(\frac{n+1}{2}\right) - S_f(n),$$

that is,

(5)
$$S(n) \;=\; \left(T_0^+\left(\frac{n}{2}\right) + T_0^+\left(\frac{n+1}{2}\right) - S_f(n)\right) + (-1)^n\left(T_0^+\left(\frac{n}{2}\right) - T_0^+\left(\frac{n+1}{2}\right)\right),$$

which requires the explicit computation of the sums T_0 and S_f. It is then easy to compute the sum T_1 since by Proposition 4 we have

$$T_1(n) \;=\; S_1(2n) \;=\; \frac{1}{2}(S_f(2n)-S(2n)).$$

The following proposition shows that the expression $T_0^+\left(\frac{n}{2}\right)+T_0^+\left(\frac{n+1}{2}\right)-S_f(n)$ in (5) can be made independent of n by choosing an appropriate extension T_0^+. In this case, that expression is simply given by the constant $T_0^+(\frac{1}{2})$ and hence no longer requires the computation of $S_f(n)$.

Proposition 10. *The following conditions are equivalent.*

(i) $T_0^+(\frac{n}{2})+T_0^+(\frac{n+1}{2})-S_f(n)$ *is constant on* $\mathbb{N}$.
(ii) *We have* $T_0^+(\frac{n}{2}+1)-T_0^+(\frac{n}{2}) \;=\; f(n)$ *for all odd* $n\in\mathbb{N}$.
(iii) $T_0^+(n+\frac{1}{2})-T_1(n)$ *is constant on* $\mathbb{N}$.

Proof. Assertion (i) holds if and only if

$$\Delta_n\left(T_0^+\left(\frac{n}{2}\right)+T_0^+\left(\frac{n+1}{2}\right)\right) \;=\; f(n), \qquad n\in\mathbb{N},$$

or equivalently,

$$T_0^+\left(\frac{n}{2}+1\right)-T_0^+\left(\frac{n}{2}\right) \;=\; f(n), \qquad n\in\mathbb{N},$$

where the latter identity holds whenever n is even. This proves the equivalence between assertions (i) and (ii).

Replacing n by $2n+1$ in the latter identity, we see that assertion (ii) is equivalent to

$$T_0^+\left(n+\frac{3}{2}\right)-T_0^+\left(n+\frac{1}{2}\right) \;=\; f(2n+1), \qquad n\in\mathbb{N},$$

that is,

$$\Delta_n T_0^+\left(n+\frac{1}{2}\right) \;=\; \Delta_n T_1(n), \qquad n\in\mathbb{N}.$$

This proves the equivalence between assertions (ii) and (iii). □

Interestingly, we also have

$$\sum_{k=0}^{2n}(-1)^k f(k) \;=\; \sum_{k=0}^{n} f(2k)-\sum_{k=0}^{n-1} f(2k+1) \;=\; T_0(n+1)-T_1(n).$$

Also, if $F(z)$ denotes the generating function of the sequence $(f(n))_{n\geq 0}$, then by Proposition 5 the generating function of the sequence $(S(n))_{n\geq 0}$ is simply given by $\frac{z}{1-z}\,F(-z)$.

Example 11. Let us provide an explicit expression for the sum

$$\sum_{k=1}^{n-1}(-1)^k\,\frac{1}{k}\,, \qquad n\in\mathbb{N}\setminus\{0\}.$$

Let $f\colon\mathbb{N}\to\mathbb{R}$ be defined by $f(0)=0$ and $f(k)=\frac{1}{k}$ for all $k\geq 1$. Then $S_f\colon\mathbb{N}\to\mathbb{R}$ is defined by $S_f(0)=0$ and $S_f(n)=\sum_{k=1}^{n-1}\frac{1}{k}=H_{n-1}$ for all $n\geq 1$. Also, $T_0\colon\mathbb{N}\to\mathbb{R}$ is defined by $T_0(0)=0$ and $T_0(n)=\sum_{k=1}^{n-1}\frac{1}{2k}=\frac{1}{2}\,H_{n-1}$ for all $n\geq 1$.

Finally, define the function T_0^+ on $D_0 = \frac{1}{2}\mathbb{N} = \{\frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \ldots\}$ by $T_0^+(0) = 0$ and $T_0^+(x) = \frac{1}{2}\, H_{x-1}$ if $x > 0$. It is then easy to see that

$$T_0^+\left(\frac{n}{2}+1\right) - T_0^+\left(\frac{n}{2}\right) ~=~ f(n), \qquad n\in\mathbb{N}.$$

Using (5) and Proposition 10, we finally obtain

$$\sum_{k=1}^{n-1}(-1)^k\,\frac{1}{k} ~=~ -\ln 2+\frac{1}{2}(-1)^n\left(H_{\frac{n-2}{2}}-H_{\frac{n-1}{2}}\right), \qquad n\in\mathbb{N}\setminus\{0\}.$$

In particular, using the classical duplication formula $2H_x = H_{\frac{x}{2}} + H_{\frac{x-1}{2}} + 2\ln 2$ (see, e.g., [1, §6.3.8]), we obtain

$$\sum_{k=1}^{2n}(-1)^{k+1}\,\frac{1}{k} ~=~ H_{2n}-H_n ~=~ \sum_{k=1}^{n}\frac{1}{n+k}\,, \qquad n\in\mathbb{N}.$$

Also, since $F(z) = -\log(1-z)$, the generating function of the sequence $(S(n))_{n\geq 0}$ is given by $-\frac{z}{1-z}\,\log(1+z)$.

3.3. **Sums weighted by a 3-periodic sequence.** Suppose we wish to evaluate the sum

$$S(n) ~=~ \sum_{k=0}^{n-1}\cos\left(\frac{2k\pi}{3}\right)f(k), \qquad n\in\mathbb{N},$$

where the function $g(k) = \cos(\frac{2k\pi}{3})$ is 3-periodic. By (2), we have

$$S(n) ~=~ S_0(n)-\frac{1}{2}S_1(n)-\frac{1}{2}S_2(n),$$

which, using (3), requires the explicit computation of the sums T_0, T_1, and T_2. Alternatively, since $\cos(\frac{2k\pi}{3}) = \frac{3}{2}g_0(k) - \frac{1}{2}$, setting $S_f(n) = \sum_{k=0}^{n-1} f(k)$ we also have

$$S(n) ~=~ \frac{3}{2}S_0(n)-\frac{1}{2}S_f(n),$$

that is, using (3),

$$\begin{aligned}
S(n) ~~=~~ & \frac{1}{2}\left(T_0^+\left(\frac{n}{3}\right)+T_0^+\left(\frac{n+1}{3}\right)+T_0^+\left(\frac{n+2}{3}\right)-S_f(n)\right)\\
&+\cos\left(\frac{2n\pi}{3}\right)\left(T_0^+\left(\frac{n}{3}\right)-\frac{1}{2}\,T_0^+\left(\frac{n+1}{3}\right)-\frac{1}{2}\,T_0^+\left(\frac{n+2}{3}\right)\right)\\
&+\sin\left(\frac{2n\pi}{3}\right)\left(-\frac{\sqrt{3}}{2}\,T_0^+\left(\frac{n+1}{3}\right)+\frac{\sqrt{3}}{2}\,T_0^+\left(\frac{n+2}{3}\right)\right),
\end{aligned}$$

which requires the explicit computation of the sums T_0 and S_f only.

We then have the following proposition, which can be established in the same way as Proposition 10. We thus omit the proof.

Proposition 12. *The following conditions are equivalent.*

(i) $T_0^+(\frac{n}{3})+T_0^+(\frac{n+1}{3})+T_0^+(\frac{n+2}{3})-S_f(n)$ *is constant on* $\mathbb{N}$.
(ii) *For any* $p\in\{1,2\}$*, we have* $(\Delta\, T_0^+)(n+\frac{p}{3}) = f(3n+p)$.
(iii) *For any* $p\in\{1,2\}$*,* $T_0^+(n+\frac{p}{3})-T_p(n)$ *is constant on* $\mathbb{N}$.

Also, if $F(z)$ denotes the generating function of the sequence $(f(n))_{n\geq 0}$, then by Proposition 5 the generating function of the sequence $(S(n))_{n\geq 0}$ is simply given by $\frac{z}{2(1-z)}\,(F(\omega z)+F(\omega^{-1}z))$.

Example 13. Let us compute the sum

$$S(n) ~=~ \sum_{k=1}^{n-1}\cos\left(\frac{2k\pi}{3}\right)\log(k),\qquad n\in\mathbb{N}\setminus\{0\}.$$

Let $f\colon\mathbb{N}\to\mathbb{R}$ be defined by $f(0)=0$ and $f(k)=\log(k)$ for all $k\geq 1$. As observed in Example 1, $T_0\colon\mathbb{N}\to\mathbb{R}$ is defined by $T_0(0)=0$ and $T_0(n)=\log(3^{n-1}\Gamma(n))$ for all $n\geq 1$. Also, define the function T_0^+ on $D_0=\frac{1}{3}\mathbb{N}$ by $T_0^+(0)=0$ and $T_0^+(x)=\log(3^{x-1}\Gamma(x))$ if $x>0$. We then see that condition (ii) of Proposition 12 holds.

Finally, after some algebra we obtain

$$\begin{aligned} S(n) ~&=~ \log\left(\frac{\sqrt{2\pi}}{3^{3/4}}\right)+\frac{1}{2}\cos\left(\frac{2n\pi}{3}\right)\log\left(\frac{3^{n-3/2}\,\Gamma(\frac{n}{3})^3}{2\pi\,\Gamma(n)}\right)\\ &\quad +\frac{\sqrt{3}}{2}\sin\left(\frac{2n\pi}{3}\right)\log\left(\frac{3^{1/3}\,\Gamma(\frac{n+2}{3})}{\Gamma(\frac{n+1}{3})}\right), \end{aligned}$$

which, put in another form, is the function obtained in Example 1.

3.4. **Computing $T_p(n)$ from $S_p(n)$.** We now present two examples where the computation of the sum $T_p(n)$ can be made easier by first computing the sum $S_p(n)$. According to Proposition 4, the corresponding conversion formula is simply given by $T_p(n)=S_p(qn)$ for all $n\in\mathbb{N}$.

Example 14. Let $m\in\mathbb{N}\setminus\{0\}$ and $p\in\{0,\ldots,q-1\}$. Suppose we wish to provide a closed-form representation of the sum

$$\sum_{k\geq 0}\binom{m}{qk+p}h(qk+p)\,.$$

for some function $h\colon\mathbb{N}\to\mathbb{C}$. Considering the function $f(k)=\binom{m}{k}h(k)$, the sum above is nothing other than $T_p(n)$ for any $n\geq\lfloor\frac{m-p}{q}\rfloor+1$. Actually, we can even write

$$\sum_{k\geq 0}\binom{m}{qk+p}h(qk+p) ~=~ T_p(\infty) ~=~ S_p(\infty) ~=~ \sum_{k\geq 0}\binom{m}{k}g_0(k-p)h(k)\,.$$

Using (4), the latter expression then becomes

$$\frac{1}{q}\sum_{j=0}^{q-1}\sum_{k\geq 0}\binom{m}{k}\omega^{j(k-p)}h(k) ~=~ \frac{1}{q}\sum_{j=0}^{q-1}\omega^{-jp}\sum_{k\geq 0}\binom{m}{k}\omega^{jk}\,h(k),$$

where the inner sum can sometimes be evaluated explicitly.

For instance, if $h(k)=1$ (see, e.g., [5, p. 71, Ex. 38]), then using the classical identity $e^{i\theta}+1=2\,e^{i\frac{\theta}{2}}\cos\frac{\theta}{2}$ the latter expression reduces to

$$\frac{1}{q}\sum_{j=0}^{q-1}\omega^{-jp}\,(\omega^j+1)^m ~=~ \frac{2^m}{q}\sum_{j=0}^{q-1}e^{i\frac{j(m-2p)\pi}{q}}\cos^m\frac{j\pi}{q}\,.$$

Since this quantity is known to be real, we may take the real part and finally obtain

$$\sum_{k\geq 0}\binom{m}{qk+p} ~=~ \frac{2^m}{q}\sum_{j=0}^{q-1}\cos\frac{j(m-2p)\pi}{q}\,\cos^m\frac{j\pi}{q}\,.$$

To give a second example, if $h(k)=\frac{1}{k+1}=\int_0^1 x^k\,dx$, then we proceed similarly and obtain

$$\sum_{k\geq 0}\binom{m}{qk+p}\frac{1}{qk+p+1} ~=~ \frac{1}{q(m+1)}\sum_{j=0}^{q-1}\left(2^{m+1}\cos\frac{j(m-2p-1)\pi}{q}\,\cos^{m+1}\frac{j\pi}{q}-\cos\frac{2j(p+1)\pi}{q}\right).$$

The case where m is not an integer is more delicate. Fixing $z\in\mathbb{C}$ and letting $f(k)=\binom{z}{k}$, we have for instance

$$\sum_{k=0}^{n-1}\binom{z}{2k+1} ~=~ T_1(n) ~=~ S_1(2n) ~=~ S_1(2n+1) ~=~ \sum_{k=0}^{2n}\frac{1-(-1)^k}{2}\binom{z}{k}.$$

Using the identity $\sum_{k=0}^n(-1)^k\binom{z}{k}=(-1)^n\binom{z-1}{n}$ (see, e.g., [4, p. 165]), we obtain

$$\sum_{k=0}^{n-1}\binom{z}{2k+1} ~=~ \frac{1}{2}\sum_{k=0}^{2n}\binom{z}{k}-\frac{1}{2}\binom{z-1}{2n},$$

which is not a closed-form expression.

Example 15. Let us illustrate the use of Proposition 4 by proving the following Gauss formula, which provides an explicit representation of the harmonic number for fractional arguments (see, e.g., [5, p. 95] and [6, p. 30]). For any integer $p\in\{1,\ldots,q-1\}$, we have

$$H_{\frac{p}{q}} ~=~ \frac{q}{p}-\ln(2q)-\frac{\pi}{2}\cot\frac{p\pi}{q}+2\sum_{j=1}^{\lfloor (q-1)/2\rfloor}\cos\left(\frac{2jp\pi}{q}\right)\ln\left(\sin\frac{j\pi}{q}\right).$$

To establish this formula, define $f\colon\mathbb{N}\to\mathbb{R}$ by $f(0)=0$ and $f(k)=\frac{1}{k}$ for $k\geq 1$. We then have

$$\begin{aligned}
\frac{1}{q}\,H_{\frac{p}{q}} &= \sum_{n\geq 1}\left(\frac{1}{qn}-\frac{1}{qn+p}\right) ~=~ \frac{1}{p}+\lim_{N\to\infty}\left(\sum_{n=1}^{N-1}\frac{1}{qn}-\sum_{n=0}^{N-1}\frac{1}{qn+p}\right)\\
&= \frac{1}{p}+\lim_{N\to\infty}\left(T_0(N)-T_p(N)\right)\\
&= \frac{1}{p}+\lim_{N\to\infty}\left(S_0(qN)-S_p(qN+p)\right)\\
&= \frac{1}{p}+\lim_{N\to\infty}\left(\sum_{n=1}^{qN-1}\frac{g_0(n)-g_0(n-p)}{n}-\sum_{n=qN}^{qN+p-1}\frac{g_0(n-p)}{p}\right)\\
&= \frac{1}{p}+\sum_{n\geq 1}\frac{1}{n}\left(g_0(n)-g_0(n-p)\right),
\end{aligned}$$

that is, using (4),

$$H_{\frac{p}{q}} \;=\; \frac{q}{p}+\sum_{j=1}^{q-1}(1-\omega^{-jp})\sum_{n\geq 1}\frac{\omega^{jn}}{n}\,.$$

Since $\omega^j\neq 1$ for $j=1,\ldots,q-1$, the inner series converges to $-\log(1-\omega^j)$, where the complex logarithm satisfies $\log 1=0$.

Using the identity $1-e^{i\theta}=-2\,i\,e^{i\frac{\theta}{2}}\sin\frac{\theta}{2}$, we obtain

$$\log(1-\omega^j) \;=\; \ln\left(2\sin\frac{j\pi}{q}\right)+i\left(\frac{j\pi}{q}-\frac{\pi}{2}\right).$$

It follows that

$$\begin{aligned}
H_{\frac{p}{q}} &= \frac{q}{p}-\sum_{j=1}^{q-1}\mathrm{Re}\left(\left(1-e^{-i\frac{2jp\pi}{q}}\right)\left(\ln\left(2\sin\frac{j\pi}{q}\right)+i\left(\frac{j\pi}{q}-\frac{\pi}{2}\right)\right)\right)\\
&= \frac{q}{p}-\sum_{j=1}^{q-1}\left(\left(1-\cos\frac{2jp\pi}{q}\right)\ln\left(2\sin\frac{j\pi}{q}\right)-\left(\frac{j\pi}{q}-\frac{\pi}{2}\right)\sin\frac{2jp\pi}{q}\right).
\end{aligned}$$

Now, it is not difficult to show that

$$\sum_{j=1}^{q-1}\sin\frac{2jp\pi}{q} \;=\; 0,\qquad \sum_{j=1}^{q-1}\cos\frac{2jp\pi}{q} \;=\; -1,$$

and

$$\sum_{j=1}^{q-1}j\,\sin\frac{2jp\pi}{q} \;=\; -\frac{q}{2}\,\cot\frac{p\pi}{q}\,.$$

Thus, we obtain

$$H_{\frac{p}{q}} \;=\; \frac{q}{p}-q\ln 2-\frac{\pi}{2}\cot\frac{p\pi}{q}-\ln\left(\prod_{j=1}^{q-1}\sin\frac{j\pi}{q}\right)+\sum_{j=1}^{q-1}\cos\frac{2jp\pi}{q}\,\ln\left(\sin\frac{j\pi}{q}\right),$$

where the product of sines, which can be evaluated by means of Euler's reflection formula and then the multiplication theorem for the gamma function, is exactly $q\,2^{1-q}$. Finally, the expression for $H_{\frac{p}{q}}$ above reduces to

$$H_{\frac{p}{q}} \;=\; \frac{q}{p}-\ln(2q)-\frac{\pi}{2}\cot\frac{p\pi}{q}+\sum_{j=1}^{q-1}\cos\frac{2jp\pi}{q}\,\ln\left(\sin\frac{j\pi}{q}\right).$$

To conclude the proof, we simply observe that both $\cos\frac{2jp\pi}{q}$ and $\sin\frac{j\pi}{q}$ remain invariant when j is replaced with $q-j$.

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