An Operational Standpoint in Electrical Engineering

Frédéric Rotella and Irène Zambettakis

***Abstract*—In electrical engineering education exists a major difficulty for first level students, namely the Laplace transform. The question is: does this ubiquitous tool is needed in an electrical engineering course? Our answer is: Obviously, not. Based on an operational standpoint the paper describes some guidelines and results for a primer on handling signals and linear systems without using the Laplace transform. The main advantage is that the operational standpoint leads to simplified proofs for well-known results.**

***Index Terms*—Transfer operator, operational calculus, Laplace transform, Carson transform, signal generator, Heaviside, Mikusiński.**

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I. Introduction

In signal processing, electrical engineering or automatic control the Laplace transform is used as an ubiquitous tool. First level courses in electrical engineering or basic textbooks on these areas contain lessons on properties and uses of Laplace transforms. For instance, in automatic control the Laplace transform approach leads to define the Laplace transform of a signal or the transfer function of a system. On the one hand it is used to get the corresponding response signal of a system with respect to a given input signal. On the other hand it is important for the analysis and design of control systems [14]. This tool appears thus as a necessary and unavoidable burden for students participating in electrical engineering courses. Nevertheless, the induced mathematical background leads to some problems for teachers from a pure educational standpoint [7], [27], [27] For instance, one of the conclusions of [27] is "*that the Laplace transform is one of the most difficult topics for learning engineering electric circuit theory.*"Roughly speaking, the difficulty arises with the connection between mathematical framework and physical world. However, this transform has some skeletons-in-the-closet [33], [40]. In this article, we discuss an alternative approach to the use of Laplace transform. This approach is based on a pure operational standpoint which has been proposed more than a century ago by Oliver Heaviside. Based on this standpoint an automatic control course has been detailed in [51], [52].

In the following, we describe guidelines for starting a course without using the Laplace transform. The presentation is based on the use of a pure operational point of view that provides an opportunity to link methods developed in electrical engineering with experiments and applications. The paper is organized as follows. In a first part we remind some well-known problems about the use of Laplace transform in operational calculus. Among engineers and mathematicians Heaviside appears, in the historical developments of operational calculus, as the focal point. His ideas on the use of the differential operator and on the definition of the transfer (*resistance*) operator are the basis of guidelines for an engineering course without the Laplace transform. A particular subsection is devoted to Oliver Heaviside. The second part deals with the transfer operator definition. Let us insist here that it must not be confused with the transfer function definition which can be related to the frequency response only. This point is detailed in the third part of the paper. The next part is devoted to some linear models analysis. For instance, we point out that poles and zeros meaning, DC gain calculus or error analysis are obvious within our approach. The final part talks about a recent application of operational calculus, namely algebraic estimation.

The essential proofs based on Laplace transform theorems can be read in standard textbooks on automatic control (*e.g.* [24], [28], [36]). Nevertheless, we will see that the operational standpoint leads to simplified proofs of well-known results.

Concerning the notation, we consider signals as elements belonging to the set C of integrable real valued functions $f = \{f(t)\}$, supposed to be m times continuously differentiable on $[0, \infty)$ except at isolated points where it is assumed that both left limit and right limit exist. As $\{f(t)\}$ denotes the signal f while $f(t)$ stands for its value at time t, we write for two signals a and b in C: for all $t \geq 0$, $a(t) = b(t)$, or $\{a(t)\} = \{b(t)\}$, or $a = b$. However, when no confusion is possible the braces or "for all $t \geq 0$" may be dropped.

II. Operational Calculus

Let us consider a signal $x(t)$ defined for a positive time t and satisfying some appropriate growth conditions. The Laplace transform of $x(t)$ is

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^\infty x(t)e^{-st}\mathrm{d}t, \tag{1}$$

where s is a complex variable. This definition requires advanced mathematical machinery [40], [56] which is very demanding, and usually, beyond the skills of most undergraduate students. This generates difficulties that lead desertion of students from basic electrical engineering classes. Equation (1) assumes that all considerations, diagrams and developments are embedded in a space of transformed signals. Students ask frequently two questions in regards to the usefulness of equation (1). How can we experimentally exhibit or visualize the

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transformed signals for example by mean of an oscilloscope? Are some signals forbidden in controlled systems or linear electronics? For instance, $\exp(t^2)$ has no Laplace transform [43].

Some fundamental theorems linked to Laplace transform have also provided some misunderstandings about the actual meaning of s. For instance, consider the Laplace transform of a derivative function with the initial condition $x(0)$, namely, $\mathcal{L}\{\dot{x}(t)\} = sX(s) - x(0)$. When $x(0)$ vanishes the complex variable s is considered as a time derivative operator. However, it just stands in the space of transformed signals only. Concerning definition (1), the lower limit of integration is often replaced by 0^-, 0^+, or $-\infty$ [14], [36], [46] to overcome discontinuity problems arising in case of particular functions. An attempt to solve this question and to unify the Laplace formalisms is proposed in [38]. For the ∞ case, we are faced with the bilateral Laplace transform, which is questionable as well [40]. The purpose of this article is to show that the mathematical machinery required by the Laplace transform [57] can be avoided. Moreover, the pedagogic difficulty can be cleared in a natural way.

When the transfer function of a linear system has to be defined, Laplace transform is applicable. For a single-input single-output system the transfer function is defined as the quotient of the Laplace transform of the output $y(t)$ to the Laplace transform of the input $u(t)$ with the assumption of zero initial conditions. In other words, the transfer function is defined by

$$F(s) = \frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{u(t)\}} = \frac{Y(s)}{U(s)}.$$

Although the name *transfer function* as a mathematical tool is adequate for $s = j\omega$, where $j^2 = -1$ and ω is the frequency [3], [29], this *ad-hoc* definition generates some interesting questions. The Laplace transform of signals cannot be obtained in practice, and sometimes we wonder how to determine the transfer function of a system? For instance this question is avoided in identification procedures [37] which use ARMAX models involving recurrence relationships instead of transfer functions. In several high quality textbooks on discrete-time systems (*e.g.* [2]), the complex variable z of the Z-transform [32] and the shift-forward operator q are both used. However, the choice between z and q is not always argued. So this subtle differences, which is mysterious for students, is not really useful. According to our personal experience in teaching automatic control, it is very important to be able to give an experimental meaning of the transfer of a system, irrespective of previous formal definitions. In practice students often forget to relate the transfer to the differential operation induced by the system. Indeed, the use of Laplace transforms leads to the diagram depicted in Figure 1 which describes the relationship between the Laplace transforms of input and output signals and the transfer function $F(s)$ of the system. But this formalism hides the time variable and, there is no different notations between signals and systems. So, the essential meaning is lost. The reader can already notice that with the forthcoming developments we will not use anymore the term *transfer function* but just *transfer* for the transfer model of a system.

$$U(s) \longrightarrow \boxed{F(s)} \longrightarrow Y(s)$$

Fig. 1. Basic block diagram.

III. Historical Standpoint

As a matter of fact, the use of Laplace transforms is one method among many others [35], [39] to justify the Heaviside operational calculus (Figure 2). Note that the operational calculus is used to solve differential equations (in most cases linear) rather than automatic control problems. The history of the Laplace transform has been studied extensively by Deakin [9], [10]. It has been first introduced in the form (1) by Bateman in 1910 to solve the differential equation $\dot{x}(t) = -\lambda x(t)$ where λ is a nonzero real number.

Nevertheless and independently, Oliver Heaviside (1850-1925), electrical engineer at the Great Northern Telegraph Company, has introduced a pure operational calculus. His seminal works have been collected in two books :

- *Electrical Papers*, 1873→1891;
- *Electromagnetic Theory*, 3 volumes, 1891→1893 (*ET1*), 1894→1898 (*ET2*), 1900→1912.

During his life he brought great breakthroughs on different electrical or electromagnetic topics such as :

- Maxwell's field equations. He reworded them in terms of electric and magnetic forces and energy flux. The use of vector analysis to write them as a fourth-order system is due to Heaviside. Consequently, the well-known Maxwell equations should be known as the Maxwell-Heaviside equations;
- atmospheric layers. He predicted the existence of ionized layers by which radio signals are transmitted around the Earth's curvature. The existence of the ionosphere was confirmed in 1923 only. These layers are bearing the name KennellyHeaviside;
- transmission lines. He developed and patented (1880) the coaxial cable;
- fractional derivatives. He introduced a 1/2-order derivative operator to modelize a diffusive process;
- operational calculus. The major part of its ideas about operational calculus are gathered together in *ET2*. The writing is oriented for practice :
  - "*Of course, I do not write for rigourists but for a wider circle of readers who have fewer prejudices,*"(*ET2*);
  - "*There is, however, practicality in theory as well in practice.*"(*ET1*)

  while the proposed developments lead him to name his method "*my operational method*".

The interested reader about life and works of Oliver Heaviside can see the following biographies : G. Lee, *Oliver Heaviside*, 1947; H.J. Josephs, *Oliver Heaviside ; a biography*, 1963; G.F.C. Searle, *Oliver Heaviside, the man*, 1987; P.J. Nahin, *Oliver Heaviside : sage in solitude*, 1988; I. Yavetz, *From obscurity to enigma : the work of Oliver Heaviside*, 1995. Let us briefly, describe the Heaviside operational method.

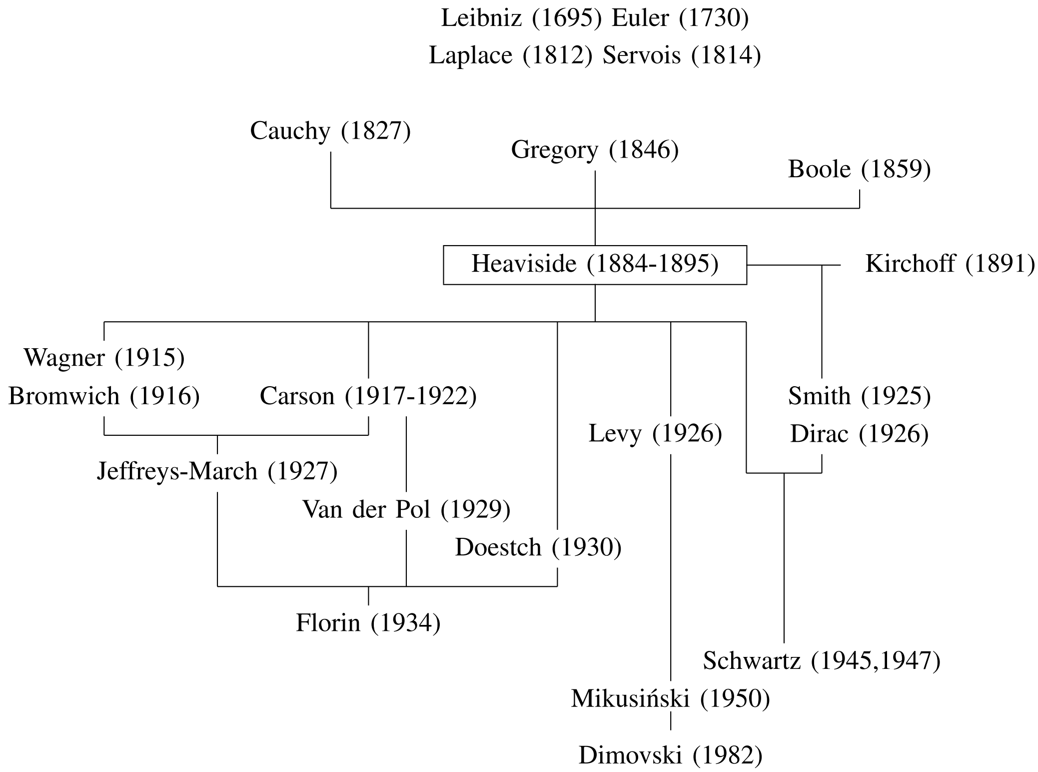

Fig. 2. Genealogy of operational calculus. This diagram is detailed in (Rotella, Zambettakis, 2006). Date indicates publication year of a major contribution in operational calculus.

1) *Coding*. Introducing the derivative operator

$$\frac{\mathrm{d}}{\mathrm{d}t} \triangleq D \text{ (before 1886)} \triangleq p \text{ (after 1886)},$$

he codes a linear differential equation (1884) for an electrical circuit or every system that can be, by analogy, reduced to an electrical circuit. In this framework, he points out the *resistance operator*, namely the transfer operator, $Z(p)$ to defines the *operational solution* $C = \dfrac{E}{Z(p)}$ where C et E are the output and input voltage respectively.

2) *Algebrization*. When E is fixed at the outset he proposes different methods to get the solution of the differential equation. For example, let E be the Heaviside function or step signal $H(t)$ defined by $H(t) = 1$ for $t \geq 0$ and 0 elsewhere. Using the *expansion theorem* (1886) he gets

$$C = \frac{E}{Z(0)} + E \sum_k \frac{e^{\lambda_k t}}{\lambda_k Z'(\lambda_k)},$$

where $Z(\lambda_k) = 0$. Later on, using the series expansions with respect to p (1888)

$$C = \sum_{n \geq 0} \alpha_n p^n H(t),$$

and supposing $p^n H(t) = 0$ he gets $C = \alpha_0$. Then, using the series expansions with respect to p^{-1} (1892)

$$C = \sum_{n \geq 0} a_n p^{-n} H(t),$$

and because $p^{-n} H(t) = \dfrac{t^n}{n!}$ he gets

$$C = \sum_{n \geq 0} a_n \frac{t^n}{n!};$$

In 1895, he deals with the sinusoidal input $E = \sin(nt) = \Im(\exp(nt))$. Using of the Euler shifting theorem

$$\varphi(p) e^{at} f(t) = e^{at} \varphi(p + a) f(t),$$

he proposes to solve the differential equation by successively replacing p with ni and, then i with $\dfrac{\mathrm{d}}{\mathrm{d}(nt)}$.

Let us consider for example The RL circuit with $Z(p) = R + Lp$. For the input $E = H(t)$, the expansion theorem yields

$$Z(0) = R,\ k = 1,\ \lambda_k = -\frac{R}{L},\ Z'(\lambda_k) = L,$$

thus

$$C = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right).$$

The series expansion with respect to p^{-1} leads to

$$\begin{aligned} C &= \frac{E}{Lp} \frac{1}{1 + \frac{R}{Lp}}, \\ &= \frac{E}{Lp}\left(1 - \frac{R}{Lp} + \frac{R^2}{L^2 p^2} - \cdots\right), \\ &= \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right). \end{aligned}$$

For the input $E = \sin(nt)$, the successive replacements method induces

$$\begin{aligned} C &= \frac{1}{R+Lp}\sin(nt) = \frac{R-Lni}{R^2+n^2L^2}\sin(nt), \\ &= \frac{R}{R^2+n^2L^2}\sin(nt) - \frac{Ln}{R^2+n^2L^2}\frac{\mathrm{d}}{\mathrm{d}(nt)}\sin(nt). \end{aligned}$$

Despite the fact that Oliver Heaviside was disapproved by mathematicians, at the begining of the twentieth century the operational calculus became a high challenge. For instance, in 1928 E.T. Whittaker quotes

"*We should now place the operational calculus with Poincaré's discovery of automorphic functions and Ricci's discovery of the tensor calculus as the three most important mathematical advances of the last quarter of the nineteenth century. Applications, extensions and justifications of it constitute a considerable part of the mathematical activity of today.*"

Justifications of the operational calculus (see [39], [47] and references therein) can be gathered together in two different kinds of methods : on the one hand the integral transforms such as the Laplace transform and, on the other hand the algebraic methods based on the Paul Lévy standpoint linked to the integral operator. These last ones, which to our point of view are the only ones to preserve the practical meaning have lead to the Mikusiński operational calculus. In the sequel we carry on with the way paved by Heaviside approach, keeping in mind that the mathematical background must not cover up the practical meaning.

IV. Transfer Operator

We begin by considering a linearized system around an equilibrium point. We suppose this system can be described by the linear differential equation

$$\begin{aligned} &y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + a_{n-2}y^{(n-2)}(t) + \cdots \\ &+a_1y^{(1)}(t) + a_0y(t) = \\ &b_mu^{(m)}(t) + b_{m-1}u^{(m-1)}(t) + \cdots \\ &+b_1u^{(1)}(t) + b_0u(t), \end{aligned} \tag{2}$$

where $y(t)$ and $u(t)$ stand for the differences of output and input signals with their setpoint values respectively and n and m are two integers. In (2) the coefficients a_i and b_j are constant parameters.

A. Coding

The aim is to provide a tool making easy the manipulation of linear time-invariant differential equations and, which allows to separate input and output variables from the system. Following Heaviside [30], [47] or Carson [5], we introduce the *derivative operator*

$$p \triangleq \frac{\mathrm{d}}{\mathrm{d}t},$$

which applied on $x(t)$ in C gives the codings

$$\dot{x}(t) = px(t),\ \ddot{x}(t) = p^2x(t), \ldots,\ x^{(n)}(t) = p^nx(t), \ldots \tag{3}$$

In view of these codings and using the distributivity property we get, for every real numbers α and β,

$$[\alpha p^n + \beta p^m]x(t) = \alpha x^{(n)}(t) + \beta x^{(m)}(t).$$

So, equation (2) becomes

$$\begin{aligned} &\left[p^n + a_{n-1}p^{n-1} + a_{n-2}p^{n-2} + \cdots + a_1p + a_0\right]y(t) = \\ &\qquad \left[b_mp^m + b_{m-1}p^{m-1} + \cdots + b_1p + b_0\right]u(t). \end{aligned} \tag{4}$$

To separate input and output signals from the system we divide equation (4) by the polynomial factor $p^n + a_{n-1}p^{n-1} + \cdots + a_0$, which yields to code the input-output relationship as $y(t) = F(p)u(t)$ with

$$F(p) = \frac{b_mp^m + b_{m-1}p^{m-1} + \cdots + b_1p + b_0}{p^n + a_{n-1}p^{n-1} + a_{n-2}p^{n-2} + \cdots + a_1p + a_0}. \tag{5}$$

We must insist here that $y(t) = F(p)u(t)$ cannot be considered as the solution of the differential equation (2). Indeed, the initial conditions are not known. $y(t)$ is determined with this writing as with the writing (2). In equation (5), $F(p)$ stands for the transfer operator. In essence, it is the transfer, which represents the operation induced by the system to transform the input signal into the output signal. The operational approach provides an opportunity to relate the transfer operator (5) to the differential equation (2). The diagram, depicted in figure 3, corresponds to an experimental situation. Notice that, in this figure, t denotes the time variable and p the derivative operator. The difference between signals and system is due to the use of these notations. The action performed by a system on an input signal is retained. The essential meaning of the transfer $F(p)$ is the linear differential equation that links the output signal to the input signal.

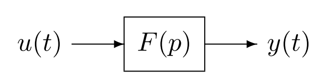

Fig. 3. Operational block diagram. The output signal $y(t)$ is obtained by $F(p)u(t)$ where $u(t)$ is the input signal.

B. Operational Calculus As Polynomial Calculus

Operational calculus is understood as algebraic methods for solving differential or recurrence equations, specifically in a linear time-invariant framework. In our point of view solving a differential equation for a given input is a mathematical exercise only [40]. In electrical engineering or, more generally, in automatic control operational calculus means rules for transfer connections or decompositions through polynomial calculus. Thus the coding (4) is useless when we are not allowed to associate transfer operators. From the operational standpoint the connection rules can be demonstrated through the following steps. The transfers of the connected system provide differential equations. The connections and the elimination of intermediate signals lead to a differential equation between the output and input signals. The encoding of this differential equation with p ensures the results. Although we can use this procedure in every case, it is sufficient to exemplify it with respect to series or parallel connections for two first-order systems.

Let us consider such two linear systems described by $y_1(t) = F_1(p)u_1(t)$ and $y_2(t) = F_2(p)u_2(t)$ where $u_1(t)$ and

$u_2(t)$ are the input signals, $F_1(p)$ and $F_2(p)$ the transfers of the systems, and $y_1(t)$ and $y_2(t)$ the corresponding output signals. In this regard, we have

$$F_1(p) = \frac{b_1 p + b_0}{a_1 p + a_0} \text{ and } F_2(p) = \frac{\beta_1 p + \beta_0}{\alpha_1 p + \alpha_0},$$

where a_0, a_1, b_0, b_1, α_0, α_1, β_0, and β_1 are constant parameters. The series connection is defined by $u_2(t) = y_1(t)$, $u(t) = u_1(t)$, and $y(t) = y_2(t)$. The application of the procedure yields

$$y(t) = \frac{\beta_1 b_1 p^2 + (\beta_0 b_1 + \beta_1 b_0)p + \beta_0 b_0}{\alpha_1 a_1 p^2 + (\alpha_0 a_1 + \alpha_1 a_0)p + \alpha_0 a_0} u(t),$$

where we recognize the product $F_1(p)F_2(p)$. The parallel connection is defined by $u_2(t) = u_1(t) = u(t)$ and $y(t) = y_1(t) + y_2(t)$, which yields

$$y(t) = \frac{\begin{array}{r}((a_1\beta_1 + \alpha_1 b_1)p^2 + (a_1\beta_0 + b_1\alpha_0 + a_0\beta_1 \\ + b_0\alpha_1)p + (a_0\beta_0 + b_0\alpha_0))\end{array}}{\alpha_1 a_1 p^2 + (\alpha_0 a_1 + \alpha_1 a_0)p + \alpha_0 a_0} u(t),$$

where we recognize the sum $F_1(p) + F_2(p)$. These results can be extended to any order by induction, thus the transfer of the series connection of two systems is the product of their transfer and the transfer of the parallel connection of two systems is their sum.

Parallel and series rules give a meaning to the decompositions and the handling of transfer operators with polynomial calculus. These operations on transfer operators are the basis of operational calculus in automatic control. We can apply usual techniques as Mason's rule associated to the signal-flow graphs [41]. This operational calculus can be applied also for multiple-input multiple-output systems with the difference that commutativity does not occur anymore. Let us note that the polynomial formalism is used in several textbooks on multivariable systems [33], [34] with no need of the Laplace transform.

C. The Delay Operator

A pure time delay of T between input and output signals induces $y(t) = u(t - T)$. This particular linear system cannot be associated to a differential equation as (2). A special treatment must be used for delay equations. Following an idea of Euler [15], the Taylor expansion of $u(t - T)$ yields

$$u(t-T) = u(t) - \dot{u}(t)T + \ddot{u}(t)\frac{T^2}{2} - \cdots + u^{(n)}(t)\frac{(-T)^n}{n!} + \cdots,$$

which is encoded to give

$$\begin{aligned} y(t) &= u(t) - pTu(t) + p^2\frac{T^2}{2}u(t) - \cdots + p^n\frac{(-T)^n}{n!}u(t) + \cdots, \\ &= \left(\sum_{n\geq 0}\frac{(-pT)^n}{n!}\right)u(t) = (\exp(-pT))\,u(t). \end{aligned}$$

We obtained the transfer operator for the time delay T as $F(p) = e^{-pT}$.

V. System Responses

System analysis is often the study of some particular responses of the system and, specially, the step and frequency responses.

A. Step Response

The step response of a system is the solution of the differential equation of the system to a step input signal with zero initial conditions. With operational calculus, we can expand transfer operator as a linear combination of simple transfers

$$\frac{a^n}{(p+a)^n} \text{ or } \frac{1}{p^n},$$

where a stands for a nonzero complex number and n for an integer. The step response of a multiple integrator with the transfer $\frac{1}{p^n}$ is $\frac{t^n}{n!}$. Let us consider the step response $s_n(t)$ of the Strejć system [48] with the transfer $\frac{a^n}{(p+a)^n}$. For $n = 1$ the associated differential equation to the transfer $\frac{a}{p+a}$ is $ay(t) + \dot{y}(t) = au(t)$ and we obtain by usual methods [61] the corresponding response to a given input $u(t)$ with the initial condition $y(0)$

$$y(t) = e^{-at}\left(y(0) + a\int_0^t e^{a\nu}u(\nu)\mathrm{d}\nu\right).$$

For $u(t) = 1$ and zero initial conditions the step response becomes

$$s_1(t) = 1 - e^{-at}.$$

For $n = 2$ we have $s_2(t) = \frac{a}{p+a}s_1(t)$ from which it follows

$$s_2(t) = ae^{-at}\int_0^t \left(e^{a\nu} - 1\right)\mathrm{d}\nu = 1 - e^{-at}(1 + at). \quad (6)$$

For the general case $n \geq 1$, let us suppose

$$s_n(t) = 1 - e^{-at}\left(\sum_{i=0}^{n-1} c_{i,n}t^i\right),$$

and

$$s_{n+1}(t) = 1 - e^{-at}\left(\sum_{i=0}^{n} c_{i,n+1}t^i\right),$$

where the coefficients $c_{i,n}$ and $c_{i,n+1}$ are constant parameters. These signals are linked by the differential equation

$$\dot{s}_{n+1}(t) + as_{n+1}(t) = as_n(t),$$

which leads to the the relationships $c_{0,n+1} = 1$ and, for $i = 1$ to n,

$$c_{i,n+1} = \frac{a}{i}c_{i-1,n} = \frac{a^i}{i!}.$$

We deduce the well known result that, for the Strejć model, $\frac{a^n}{(p+a)^n}$, the step response is

$$s_n(t) = 1 - e^{-at}\left(\sum_{i=0}^{n-1}\frac{a^i t^i}{i!}\right).$$

So, the step response of a given system defined by a transfer operator can be calculated using polynomial calculus.

Stability analysis does not use the Laplace transform. However, we may insist again on the link between the transfer operator, the differential equation and the transient behavior. Let us consider the transfer operator $\frac{a^n}{(p+a)^n}$ where a and n have the same meaning as before. From the previous paragraph we can see that the step response is composed of a constant term and a time-dependent term. The first term is the forced response and the second term is the transient behavior. The transient behavior tends asymptotically to zero if and only if the real part of a is strictly negative. More generally, consider the n-th order transfer

$$F(p) = \frac{b_m p^m + \cdots + b_1 p + b_0}{(p-p_1)^{\rho_1}(p-p_2)^{\rho_2}\cdots(p-p_r)^{\rho_r}},$$

where $p_1,\, p_2, \ldots,\, p_r$ are the r complex poles of $F(p)$ and ρ_i are the respective multiplicities with $n=\sum_{i=1}^{r}\rho_i$. The poles generate terms associated with the signals $e^{p_1 t},\, e^{p_2 t}, \ldots,\, e^{p_r t}$ weighted by time polynomials of order $\rho_i - 1$ respectively. So, when all the poles have strictly negative real part, the transient behavior tends asymptotically to zero. Namely, the system is asymptotically stable.

B. Frequency Response

For asymptotically stable systems the frequency response is deduced from the steady-state output response to a given sinusoidal input signal $u(t)=e^{j\omega t}$ where ω is the frequency and $j^2=-1$. Consider a system defined by the transfer operator $F(p)=\frac{B(p)}{A(p)}$ where $A(p)$ and $B(p)$ are two polynomials. The operational approach leads to the encoded input-output differential equation as $A(p)y(t)=B(p)u(t)$. For the sinusoidal input $u(t)=e^{j\omega t}$, we obtain

$$B(p)u(t) = |B(j\omega)|\, e^{j(\omega t + \arg(B(j\omega)))},$$

where $|B(j\omega)|$ and $\arg(B(j\omega))$ denote the module and the argument of the complex number $B(j\omega)$ respectively. The output $y(t)$ is the sum of a particular solution of the differential equation and the general solution of the differential equation without second member. The general solution characterizes the transient response that vanishes in case of asymptotically stable systems. For a particular solution, we look for the steady-state behavior as the form $y(t)=Ye^{j(\omega t+\varphi)}$ where Y and φ are constant parameters. Replacing this expression for $y(t)$ in the differential equation yields

$$\begin{aligned} Y &= \frac{|B(j\omega)|}{|A(j\omega)|} = |F(j\omega)|\,, \\ \varphi &= \arg(B(j\omega)) - \arg(A(j\omega)) = \arg(F(j\omega)). \end{aligned}$$

The frequency response is defined by the evolution of $(|F(j\omega)|\,, \arg(F(j\omega)))$ as the frequency ω varies from 0 to $+\infty$. We can notice that $F(j\omega)$ is the transfer function of the system such as Harris defined it [29]. In our standpoint, this transfer function must not be confused with the transfer operator $F(p)$. Nevertheless, $F(j\omega)$ such as a function of the frequency is the only actual transfer function. Graphic representations such as Bode, Black-Nichols, or Nyquist loci may be used to analyze the frequency response [36]. For unstable systems the loci are valid as calculated representations only. While for stable systems experiments can allow to get the frequency response as well.

VI. Analysis

A. Poles and Zeros

The names of poles and zeros come from the interpretation of a transfer operator $F(p)$ as a function of a complex variable p. This interpretation is a consequence of the formulation of Laplace transform and it misunderstands the physical meaning of these notions. The consideration of a transfer operator as a coding of a differential equation provides an immediate physical interpretation. Namely, let us consider the transfer operator

$$F(p) = \frac{p+a}{p+b}, \tag{7}$$

where a and b are constant parameters. In the operational standpoint the transfer (7) corresponds to the input-output differential equation $\dot{y}(t)+by(t)=\dot{u}(t)+au(t)$ where $y(t)$ and $u(t)$ are the output and input signals. First, consider $u(t)=0$ for $t>0$ and a nonzero initial condition $y(0)$ we then get $y(t)=y(0)e^{-bt}$ for $t>0$. Second, consider a zero initial condition for the output and the input signal $u(t)=e^{-at}$ for $t>0$ we obtain $\dot{u}(t)+au(t)=0$ so $y(t)=0$ for $t>0$.

More generally, poles correspond to signals generated by the system with zero input. Zeros correspond to signals absorbed or blocked by the system. Let us write the transfer operator (5) as

$$F(p) = k\frac{(p-z_1)^{\nu_1}(p-z_2)^{\nu_2}\cdots(p-z_d)^{\nu_d}}{(p-p_1)^{\rho_1}(p-p_2)^{\rho_2}\cdots(p-p_r)^{\rho_r}},$$

where $k=b_m,\ z_i,\ i=1,\ldots,d$ and $p_i,\ i=1,\ldots,r$ are complex numbers, and $\nu_i,\ i=1,\ldots,d$ and $\rho_i,\ i=1,\ldots,r$ are integers. For $i=1,\ldots,r,\ e^{p_i t}$ is solution of the coded differential equation

$$(p-p_1)^{\rho_1}(p-p_2)^{\rho_2}\cdots(p-p_r)^{\rho_r}y(t) = 0,$$

and for $i=1,\ldots,d,\ e^{z_i t}$ is solution of the coded differential equation

$$(p-z_1)^{\nu_1}(p-z_2)^{\nu_2}\cdots(p-z_d)^{\nu_d}u(t) = 0.$$

On the one hand we can use the correspondence between $e^{p_i t}$ and the transfer denominator roots p_i that characterizes the transient rate in the linear constant parameter framework only. The same remark can be said about the correspondence between $e^{z_i t}$ and the transfer numerator roots z_i. On the other hand this signal approach for the pole and zeros meaning can be extended to time-varying or nonlinear multivariable systems with an algebraic standpoint [16], [17].

In order to underline and to exemplify the important problem of pole/zero cancellation let us consider the series connection with the systems :

$$y(t) = \frac{1}{p-1}u(t),$$
$$z(t) = \frac{p-1}{p+1}y(t).$$

The pole 1 induces, in the transient behavior or in the initial conditions effect for the first system, an e^t signal. This signal is blocked by the second system, which has 1 as zero. As $\lim_{t\to\infty} e^t = \infty$, this fact forbids such a connection. Indeed, while the input and output signals are zero, there exists in the system a non observed and non controlled unbounded signal. The conclusion is different if we consider the series connection with the systems :

$$y(t) = \frac{1}{p+1}u(t),$$
$$z(t) = \frac{p+1}{p-1}y(t).$$

Due to the pole/zero cancellation at -1, $y(t)$ has an e^{-t} component that vanishes at ∞. Except during the transient behavior, the pole/zero cancellation is acceptable for asymptotically stable canceled zeros.

B. DC Gain

Let us keep in mind that the transfer operator $F(p)$ in equation (5) is just a coding of the differential equation (2). In the case of an asymptotically stable system, with a constant value U as input, the step response analysis indicates that the output tends, as t goes to $+\infty$, to a constant value Y given by the relationship $a_0Y = b_0U$. The ratio $\frac{Y}{U}$ defines the DC gain of the system G_{DC} . The stability condition implies $a_0 \neq 0$, and from (5) we obtain $G_{DC} = F(0)$.

C. Steady-State Error Analysis

In all this section systems are supposed to be asymptotically stable, namely the transient behavior vanishes and only the permanent behavior remains. For the reference inputs $r_i(t)$ defined by, for $t > 0$, $r_i(t) = \frac{t^i}{i!}$, and for $t < 0$, $r_i(t) = 0$, the corresponding outputs are $y_i(t) = F(p)r_i(t)$. The input-output error $\varepsilon_i(t) = r_i(t) - y_i(t)$ is called the system error of order i. Two notions can be pointed out here. First a norm of the instantaneous system error $\varepsilon_i(t)$ characterizes the system performance. Secondly, the value $\varepsilon_i(\infty) = \lim_{t\to\infty} \varepsilon_i(t)$ during the permanent behavior characterizes the steady-state error. In a basic lecture of automatic control this last notion is usually considered. We detail it according to our formulation, namely without the use of the final value theorem.

A steady-state error of order N is ensured if $\varepsilon_i(\infty) = 0$, for $i = 0$ to N, and $\varepsilon_{N+1}(\infty) \neq 0$. Consider the transfer operator $F(p)$ of an asymptotically stable system defined in (5). The corresponding permanent step response value is given by the DC gain $\frac{b_0}{a_0}$, so $\varepsilon_0(\infty) = 0$ if and only if $b_0 = a_0$. Thus we conclude that a steady-state error of zero order is fulfilled whether the DC gain is equal to 1. In other words since the input-error transfer is $1 - F(p)$, we obtain a steady-state error of zero order when the input-error DC gain is zero. This is a fundamental remark for the following.

Let us notice that $r_1(t)$ is the integral of $r_0(t)$. Namely, $r_1(t) = \frac{1}{p}r_0(t)$, thus we have

$$\begin{aligned}\varepsilon_1(t) &= r_1(t) - y_1(t),\\ &= \frac{1}{p}r_0(t) - F(p)\frac{1}{p}r_0(t),\\ &= \frac{1-F(p)}{p}r_0(t).\end{aligned}$$

Clearly, from the previous result for $\varepsilon_0(\infty)$, $\varepsilon_1(\infty)$ vanishes if and only if the DC gain of the transfer operator $\frac{1-F(p)}{p}$ is equal to zero. Since

$$\frac{1-F(p)}{p} = \frac{(a_0-b_0)+(a_1-b_1)p+(a_2-b_2)p^2+\cdots}{p(a_0+a_1p+a_2p^2+\cdots+a_np^n)}$$

we obtain $\varepsilon_1(\infty) = 0$ if and only if $a_0 = b_0$ and $a_1 = b_1$. It can be seen that :

- when $a_0 \neq b_0$, we have $\varepsilon_0(\infty) \neq 0$ and $\varepsilon_1(\infty) = \lim_{p\to 0} \frac{a_0-b_0}{pa_0} = \pm\infty$;
- when $a_0 = b_0$, we obtain $\varepsilon_0(\infty) = 0$ and $\varepsilon_1(\infty) = \frac{a_1-b_1}{a_0}$. Moreover, $\varepsilon_1(\infty) = 0$ when $a_1 = b_1$.

In the same way we can show by induction that the system has a steady-state error of order N if and only if its transfer $F(p)$ in equation (5) is such that, for $i = 0$ to N, $a_i = b_i$. The steady-state error of order $N+1$ is then

$$\varepsilon_{N+1}(\infty) = \frac{a_{N+1} - b_{N+1}}{a_0},$$

and the next ones have an infinite module. Thus, the degree of the steady-state error can be obtained just by a visual inspection of the transfer operator of the system.

VII. Algebraic Estimation

Recently, M. Fliess and his co-workers [21], [22], [42] have proposed new methods to estimate the parameters of a system or the first derivatives of a measured signal. For instance, the last point is implemented in a model-free control of a system [19], [20]. These methods are based on operational calculus. According to the presented operational framework of our paper, let us describe the used skills for the particular case of derivative estimation.

Firstly, the signal $s(t)$ is approximated on a short-time window $[-T, 0]$ with the polynomial :

$$s(t) = s_0 + s_1t + s_2t^2, \tag{8}$$

where s_1 stands for the estimation of the first derivative, $s_1 = \widehat{\dot{s}(0)}$.

Secondly, the time functions 1, t and t^2 are associated to operational forms, namely, their signal generators.

A. Signal Generators

The main reason of using the Laplace transform is the tables we have at our disposal. Firstly, they contain information to determine the response of a system with respect to a given input signal. Secondly, they allow to get the discrete transfer operator of a computer controlled system with a formula given in [52]. Although transforms are not used in our presentation, we show that these tables can be used without any change. To do that, let us introduce the notion of generator of a continuous signal, which consists in writing the time expression of this signal by means of the operator p.

The previous parts show that the transfer operator allows to link input $u(t)$ and output $y(t)$ signals of a linear system by a differential equation coded as $y(t) = F(p)u(t)$. Until now, we get the step response by solving this differential equation when initial conditions are all zero. In case of no input and non zero initial conditions, such a transfer operator produces an output signal solution of the associated homogeneous differential equation. The coding of this differential equation with the p operator defines then the generator of this signal. Two ways can be considered to take into account initial conditions in this coding. Namely, on the one hand the Mikusiński operational calculus and on the other hand the integral form of a differential equation.

Indeed, all the previous developments can be rigorously proved by means of operational calculus of Mikusiński [44], which is based on convolution algebra of operators. Let us briefly describe this operational calculus whereas keeping in mind that the considerations below are not needed in a first level course. Convolution product is a fundamental tool in dynamic systems field [54], [55], specifically in case of linear systems [11], [25]. This tool is defined by

$$(f, g) \mapsto gf = \int_0^t f(\tau)g(t-\tau)d\tau,$$

while the Heaviside function $H = \{1\}$ is of great importance due to the fact that we have for every f in the set of integrable function C

$$Hf = \left\{ \int_0^t f(x)\mathrm{d}x \right\}.$$

Consequently, H appears as the integration operator. The successive powers of H with respect to the convolution product are, for all n in $\mathbb{N}$, $n \geq 1$,

$$H^n = \left\{ \frac{t^{n-1}}{(n-1)!} \right\}.$$

To distinguish a constant signal $\{\alpha\}$ with the operator defined by the constant gain α we denote it $[\alpha]$. For all f in C

$$\{\alpha\} f = \left\{ \alpha \int_0^t f(\tau)d\tau \right\} \text{ and } [\alpha] f = \{\alpha f(t)\}.$$

The unit element for the convolution is [1] and we can give a meaning to H^0 as $H^0 = [1]$. We define then the derivative operator as the solution of the convolution equation $pH = [1]$, and we write $p = H^{-1}$. With the understanding $p^0 = H^{-0} = [1]$, we have $p^n = H^{-n}$ for n in $\mathbb{N}$.

Mikusiński [44] has proved the two results below, which are essential to our purpose.

Theorem 1 *For every continuous function f in C, $\left\{f^{(1)}(t)\right\} = pf - [f(0)]$. More generally, for every integer k*

$$\left\{ f^{(k)}(t) \right\} = p^k f - \sum_{i=0}^{k-1} \left[f^{(i)}(0) \right] p^{k-i-1}. \tag{9}$$

Theorem 2 *For every f in C such that $\int_0^\infty e^{-tp} f(t)dt$ exists*

$$f = \int_0^\infty e^{-tp} f(t)dt.$$

Theorem 1 allows to write the generator of a signal $\{f(t)\}$ when the differential equation whose this signal is solution is known. Indeed, let us suppose that this differential equation is

$$\sum_{i=0}^{n} \alpha_i f^{(i)}(t) = 0, \tag{10}$$

with initial conditions $f(0) = f_0$, $\dot{f}(0) = f_1, \ldots, f^{(n-1)}(0) = f_{n-1}$, where n is an integer and the α_i are real numbers. With (9) the coding of (10) leads to

$$\left[\sum_{i=0}^{n} \alpha_i p^i \right] f(t) - P_{\text{IC}}(p, f_0, \ldots,\ f_{n-1}) = 0,$$

where $P_{\text{IC}}(p, f_0, \ldots,\ f_{n-1})$ is a polynomial in p that depends on the initial conditions and the coefficients α_i. We obtain then the generator of $\{f(t)\}$

$$\{f(t)\} = \frac{P_{\text{IC}}(p, f_0, \ldots,\ f_{n-1})}{\left[\sum_{i=0}^{n} \alpha_i p^i\right]}. \tag{11}$$

Theorem 2 indicates that when the one-sided Laplace transform of a signal exists, its expression is identical to the generator of the signal, the complex variable s of Laplace transform being changed into the derivative operator p (to avoid any confusion). A major consequence is that the tables [13], [58], can be used. Since $H = p^{-1}$ we remark that the generator can be written indifferently with the operators H or p.

For example, when we look for the generator of $\sin(\omega t)$, we have just to observe that $\sin(\omega t)$ is the solution of the differential equation

$$\ddot{x}(t) + \omega^2 x(t) = 0,\ x(0) = 0,\ \dot{x}(0) = 1. \tag{12}$$

Using (9) the coded form is then

$$p^2 x(t) - 1 + \omega^2 x(t) = 0,$$

which leads to the generator of $\sin(\omega t)$

$$\{\sin(\omega t)\} \underset{\text{M}}{=} \frac{1}{p^2 + \omega^2},$$

where the symbol "M" denotes "in the Mikusiński sense".

Indeed, this definition for the generator of a signal is not unique because it depends on the used integral transformation. For instance, the reader can find in [52] another way to handle a differential equation which leads to the signal generator of $f(t)$ in the Carson sense. It is based on the result below [53]

$$\{\dot{x}(t) = f(t),\ x(0) = x_0\} \text{ if and only if} \\ x(t) = x_0 + \int_0^t f(\tau)\mathrm{d}\tau, \tag{13}$$

but for shortness sake we don't develop this point here. The Carson transform was introduced in 1926 [6] and it differs from the Laplace transform by a factor p. The Carson tables can be used in this framework as well.

Let us mention that the Mikusiński operational calculus has been extended recently by the convolutional calculus [12].

For a system defined by the transfer operator $F(p)$ we can calculate the response $y(t)$ to an input $u(t)$ by using Carson or Laplace transform tables. Indeed when $U(p)$ is a generator of $u(t)$ the generator of the output $y(t)$ is $F(p)E(p)$. We must remark here that the generators can be obtained in any sense as defined (Mikusiński or Carson). However, we must keep the consistency in using tables. For example, following the Heaviside series expansion, when we want to know the beginning of the response we can write the power series with respect to p^{-1} of the generator of $y(t)$. However, different functions may be associated to p^{-k} according to the adopted generator sense.

Moreover, in order to see the importance of the generator for operational calculus, let us consider the following example where two signals y_1 and y_2 are defined by the differential equations

$$(p-1)y_1(t) = u(t), \tag{14}$$

$$(p-1)y_2(t) = u(t), \tag{15}$$

and the initial conditions y_1^0 and y_2^0 respectively. Let us consider the parallel connection $y(t) = y_1(t) - y_2(t)$. It yields

$$y(t) = \frac{1}{p-1}u(t) - \frac{1}{p-1}u(t) = 0.$$

This conclusion is obviously wrong. Indeed, our setting indicates that we consider formal differential equations, namely, without initial conditions. When we write

$$y(t) = \frac{1}{p-1}u(t),$$

it is just a coding of differential equations (14) and (15). The initial conditions can be taken into account by means of the generator notion. We can write (14) and (15) as, respectively,

$$y_1(t) \underset{M}{=} \frac{1}{p-1}u(t) + \frac{y_1^0}{p-1},$$

$$y_2(t) \underset{M}{=} \frac{1}{p-1}u(t) + \frac{y_2^0}{p-1},$$

in the Mikusiński's generator sense. It yields for $y(t) = y_1(t) - y_2(t)$ the generator

$$y(t) \underset{M}{=} \frac{y_1^0 - y_2^0}{p-1}.$$

This result indicates that $y(t)$ is solution of the differential equation

$$(p-1)y(t) = 0,\ y(0) = y_1^0 - y_2^0,$$

or, equivalently, $y(t) = (y_1^0 - y_2^0)e^t$.

This standpoint can also be explained in a more algebraic framework as the Fliess' module-theoretic approach [18]. Nevertheless, let us quote a sentence of a recent paper [22] where this point of view is used for the design of an algebraic identification procedure : "*Let us add we tried to write the examples in such a way that they might be grasped without the necessity of reading the sections on the algebraic background. Our standpoint on parametric identification should therefore be accessible to most engineers.*"

B. Derivative Estimation

Let us associate to the signal $\frac{t^n}{n!}$, $n \geq 0$, its signal generator in the Mikusiński sense $\frac{1}{p^{n+1}}$. The polynomial approximation (8) of a signal $s(t)$ can then be written

$$s(t) = s_0\frac{1}{p} + s_1\frac{1}{p^2} + s_2\frac{1}{p^3}.$$

To obtain s_1 let us follow the steps :

1) Multiply with p^3

$$p^3 s(t) = p^2 s_0 + p s_1 + s_2.$$

2) Derivate with respect to p

$$3p^2 s(t) + p^3 s'(t) = 2ps_0 + s_1,$$

where $s'(t)$ stands for $\frac{\mathrm{d}s(t)}{\mathrm{d}p}$.

3) Divide with p

$$3ps(t) + p^2 s'(t) = 2s_0 + \frac{s_1}{p}.$$

4) Derivate with respect to p

$$3s(t) + 5ps'(t) + p^2 s''(t) = -\frac{s_1}{p^2}.$$

As p^2 stands for a double time-derivative it cannot be implemented. Thus, the following step consists in a division with p^3. So, we obtain the operational estimation

$$3p^{-3}s(t) + 5p^{-1}s'(t) + p^{-1}s''(t) = -p^{-5}s_1.$$

Notice that the use of a sufficient number of time-integrals induces noise filtering.

In order to implement the derivative estimator we have then to come back in the time domain. There is no difficulty to translate the time-integrals on $[-T, 0]$

$$-p^{-5}s_1 = -\frac{T^4}{24}s_1,$$

$$\frac{1}{p^3}s(t) = \int\int\int s(\tau)\mathrm{d}\tau = \int \frac{(t-\tau)^2}{2}s(\tau)\mathrm{d}\tau.$$

For the other terms we must prove in a operational way the following well known result

$$\frac{\mathrm{d}^k}{\mathrm{d}p^k}s(t) = (-1)^k t^k s(t). \tag{16}$$

For simplicity sake, let us consider the signal

$$x(t) = \sum_{\nu=0}^{\infty} x^{\nu}(0)\frac{t^{\nu}}{\nu!}.$$

With the signal generators $\frac{t^{\nu}}{\nu!} = \frac{1}{p^{\nu+1}}$, we get

$$x(t) = \sum_{\nu=0}^{\infty} x^{\nu}(0)\frac{1}{p^{\nu+1}},$$

which leads to

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p}x(t) &= -\sum_{\nu=0}^{\infty}(\nu+1)x^{\nu}(0)\frac{1}{p^{\nu+2}},\\
&= -\sum_{\nu=0}^{\infty}(\nu+1)x^{\nu}(0)\frac{t^{\nu+1}}{(\nu+1)!},\\
&= -\sum_{\nu=0}^{\infty}x^{\nu}(0)\frac{t^{\nu+1}}{\nu!},\\
&= -t\sum_{\nu=0}^{\infty}x^{\nu}(0)\frac{t^{\nu}}{\nu!}.
\end{aligned}$$

Thus $\frac{\mathrm{d}}{\mathrm{d}p}x(t) = -tx(t)$. It is easy to state the announced result (16) by induction.

We can then state the final form of the derivative estimation as

$$\widehat{s^{(1)}(0)} = -\frac{24}{T^4}\int_{-T}^{0}\left[\frac{3}{4}T^2 - \frac{13}{2}Tt + \frac{27}{4}t^2\right]s(t)\mathrm{d}t.$$

An example of a real-time application is given in figure 4 where we show the signal corrupted with noise and the real-time estimation of its derivative.

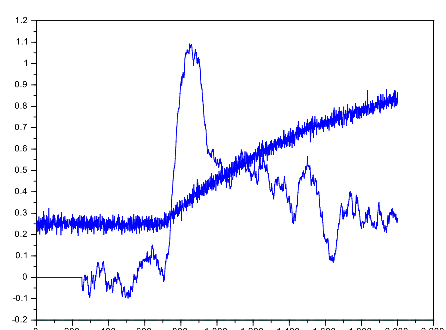

Fig. 4. Algebraic derivative estimation.

VIII. Conclusion

We show in this short survey that from the use of the differential operator we obtain all the usual results derived by means of the Laplace formulations. The teaching of a basic lecture in electrical engineering and in automatic control using the operational method offers some advantages. The integral or derivative operators allow to link every notion to its physical meaning. We keep in mind that a transfer operator is always related to a differential equation. Mathematical background is minimized, however, when a rigorous justification is needed the Mikusiński operational calculus may be used. This operational calculus is based on the convolution operator, which is a natural tool for linear equations.

Moreover, we meet here, through a pedagogical step, the operational standpoints adopted directly in some advanced textbooks to modelize the input-output relationship induced by a linear system. For instance, the discrete-time autoregressive moving-average model $A(q^{-1})y_k = B(q^{-1})u_k + C(q^{-1})\epsilon_k$ where $A(q^{-1})$, $B(q^{-1})$, and $C(q^{-1})$ are polynomials in the delay operator, $\{\epsilon_k\}$ a noise signal, is used in [1], [8] for identification purposes to describe the difference equation between the sampled input u_k and the sampled output signals y_k of a given system. For multivariable continuous-time linear systems the following model is introduced in [49], [59], [60]

$$\begin{aligned}
P(p)\xi(t) &= Q(p)u(t),\\
y(t) &= R(p)\xi(t) + W(p)u(t),
\end{aligned}$$

where $P(p)$, $Q(p)$, $R(p)$, and $W(p)$ are matrix polynomials in the differential operator and $\xi(t)$ is a vector-valued function of time named the partial state. More recently, the generator of a multivariable system is defined in [4] as the polynomial matrix $M(p)$ in the derivative operator, which allows to write the relationship between input and output signals as $M(p)\begin{bmatrix} y(t) \\ u(t) \end{bmatrix} = 0$. The generator in the sense defined in [4] must not be confused with signal generators. The interested reader can see the quoted literature.

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