Essays on Price and Welfare

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ABSTRACT

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This dissertation is a collection of three essays on price and welfare. The first chapter investigates the optimal price index for central banks to stabilize in a model economy where volatile prices are harmful to welfare through monetary friction. The second chapter estimates the impact of recent technological innovation, namely the internet, on the dynamics of prices and welfare through a variety of real mechanisms. The third chapter analyzes the impact of financial regulation on the prices of financial assets and the welfare of the financial market participants.

There is currently a debate about what price index central banks should target when economies are open and exposed to international price shocks. Chapter 1 derives the optimal price index by solving the Ramsey problem in a New Keynesian small open economy model with an arbitrary number of sectors. This approach improves on existing theoretical benchmarks because (1) it makes an explicit distinction between the consumer price index (CPI) and the producer price index (PPI), and (2) it allows exogenous international price shocks to play a role. Qualitatively, I use the analytical expression of the optimal price index to discuss that popular indices, such as the PPI and the core/headline CPI, are suboptimal because they ignore the heterogeneity in price stickiness and the effect of inflation on the trade surplus. Quantitatively, I calibrate a 35-sector version of the model for 40 countries and show that stabilizing the optimal price index yields significantly higher welfare than alternative indices.

In Chapter 2, which is joint work with Yoon J. Jo and David Weinstein, we estimate the impact of e-commerce on Japanese prices and welfare. First, we consider the possibility that e-commerce may have lowered prices by driving down the average prices of goods available online. Second, we compute the welfare gains due to the ability of e-commerce to enable consumers to purchase goods from other regions. Third, we compute the gains that arise through e-commerce's ability to arbitrage intercity price differences. We find that all three channels produced welfare gains in Japan, but our estimates suggest that the first and second channels are by far the most important, with welfare gains through these channels being eleven to sixteen times larger than through the price arbitrage channel. Overall, we find that increased inter-city arbitrage raised Japanese welfare by 0.12 percent, the gains due to new varieties available through online shopping raised welfare 0.7 percent, and the gains due to overall price reductions for goods available online raised welfare by 1 percent.

In Chapter 3, which is joint work with Sakai Ando, we analyze the impact of dealer regulation on price quality (informativeness and volatility) and its implications for the welfare of market participants. We argue that although price informativeness, volatility, and the dealer's profitability all deteriorate, against conventional wisdom, other market participants are better off due to the dealer's risk-shifting motive. A static model is used to clarify the main intuition, and the robustness of the welfare results, as well as the fragility of the conventional wisdom about price quality, are discussed by incorporating dynamics and endogenizing information acquisition.

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Chapter 1

What Price Index Should Central Banks Target? An Open Economy Analysis

Misaki Matsumura

1.1 Introduction

As many small open economies (SOEs) have shifted their monetary policy from exchange rate pegs to inflation targeting policies, there has been growing interest in which price index they should target. The theory of optimal monetary policy with a multi-sector economy can be used to answer this question, as in Aoki (2001) and Woodford (2010), but such analyses so far have been limited to closed economy setups, leaving open economy questions unanswered, such as the effect of international commodity prices and the role of trade patterns. This lack of the optimal price index theory in an open economy underlies the ongoing debate over the choice between, for instance, the headline consumer price index (CPI) versus the core CPI or the CPI versus the producer price index (PPI).

In this paper, I derive the optimal price index for open economies to stabilize by solving the problem of a central bank attempting to maximize household welfare, i.e., a Ramsey problem. I call the derived index the Ramsey price index (RPI) and present its analytical formula. Due to the openness of my model, the index depends on the export share of output in each sector in addition to the parameters studied in closed economy models such as the consumption share, price stickiness and the elasticity of substitution. By calibrating the model to 40 countries with 35 sectors, I find that (1) RPI stabilization performs better for all countries in terms of welfare than headline CPI, core CPI, or PPI stabilization and (2) the ranking of the indices other than the RPI differs across countries.

To derive the optimal price index, I begin with the multi-sector DSGE model with output price stickiness analyzed in Woodford (2010). The use of a multi-sector model is necessary to answer my research question since different price indices arise due to the difference in weights applied to the prices in different sectors. Output price stickiness is the key monetary friction in my model and the workhorse model in the literature, in keeping with extensive empirical evidence (see Nakamura and Steinsson 2008, for example). Under output price stickiness, volatile inflation causes mispricing by firms, leading to welfare-damaging inefficient production activities.

As the key departure from Woodford (2010), I allow each sector in the economy to export a part of its output. This openness allows for a difference between CPI and PPI because when the economy can trade, what is produced is not necessarily consumed. The choice between the two indices is often the focus of monetary policy discussions especially for commodity exporters and developing countries. For instance, Frankel (2010) numerically analyzes Latin American commodity exporters and concludes that producer price based indices better perform than consumer price based indices in terms of price stability. India changed its target index from PPI to CPI in 2016; see Rajan (2016). The existing theoretical framework is not suitable to answer this type of question since the consumption based weight coincides with the production based weight[1].

Another key feature of my model is the use of an SOE setup rather than a two-country setup. This is to capture the notion of international price movement that is exogenous to the economy. The Bank of Japan, for example, argued that the movement of the international oil price was the most important reason that it failed to achieve its inflation target; see Kawamoto and Nakahama (2017). The SOE framework allows me to answer the question of whether the economy should bear such volatility in inflation that is caused by international

[1]The input-output structure is another reason that the PPI and CPI can differ. I focus on the difference arising from the trade in this paper.

price changes.

In this multi-sector New Keynesian (NK) SOE DSGE environment, I solve the Ramsey problem and obtain the optimal price index that remains constant in the long-run expectation under the Ramsey solution. This means that my proposed optimal price index is based on welfare maximization rather than an arbitrary objective. The welfare maximization problem is subject to optimizing behaviors of the representative household and firms under monetary frictions. The use of the Ramsey framework also means that the monetary policy considered in this paper is not limited to a particular class of monetary policy such as the Taylor rule. Despite the generality of the choice of monetary policy, I show that, in the long-run, a particular price index remains constant. I explore the property of this RPI qualitatively and quantitatively.

The key trade-off between stabilizing one price index versus another can be understood by considering the cost of volatile inflation rates in the sectors with lower weight in each price index. Therefore, the resulting optimal price index takes the form of a weighted sum of the prices in different sectors, where the weight assigned to each sector reflects the cost of inflation in each sector. In other words, in a multi-sector environment, the inflation rates of all the sectors cannot be stabilized simultaneously following a shock that leads to a relative price change. For example, when a change in world demand lowers the efficient relative price of oil, the central bank needs to essentially choose one of two options: (1) a stable oil price and an increase in non-oil price and (2) a stable non-oil price and a decrease in the oil price. Given this trade-off, we should stabilize the price of the sector with the higher cost of inflation.

My first main result is the analytical formula for the RPI. In particular, I highlight three lessons from the formula. The first two lessons come from each of the two components of the formula. The formula is a weighted sum of different log prices in different sectors, where the weight represents the welfare cost of inflation in each sector. I show that the weight consists of two parts, one representing the size of the sector and the other representing the sensitivity of the production wedge to inflation in the sector. I also show that the RPI formula does not directly depend on international prices. The third lesson comes from what is not in the formula.

The first lesson from the first component of the RPI is that the size of the sector in the RPI weight needs to be measured in terms of the production size rather than the consumption size. This is because the cost of inflation in my model is the efficiency loss in production. If there is inefficiency in production, it is welfare damaging either through reduced consumption, more work or a negative effect on the trade balance, which affects the economy through a tighter budget constraint. Therefore, regardless of whether its output is consumed or exported, inefficiency in production is costly in a sector that is large in terms of production. An implication of this is that the central banks should stabilize PPI rather than CPI if everything else is constant. However, there is a caveat in this simple takeaway, as my quantitative analysis shows that the stabilization of PPI does not necessarily perform better than CPI stabilization due to the second component of the RPI weight.

The second component of the RPI weight is a combination of a well-known stickiness parameter and less frequently highlighted but equally important parameter, representing the elasticity of substitution between differentiated goods within a sector. These two parameters govern the sensitivity of inefficiency to inflation in the sector in question. The mechanism comprises two steps. First, volatile inflation causes mispricing by the firms in a sector.

This step depends on the degree of price stickiness. Second, mispricing leads to deviations of demand and production from the efficient level. This step depends on the elasticity of substitution.

The addition of sectoral heterogeneity in the elasticity of substitution provides the second lesson that is important when we discuss core inflation targeting versus headline inflation targeting. Recall that the difference between the two measures is whether they include commodity prices such as food and energy[2]. While the literature to date has focused on one characteristic of commodities, namely price flexibility, the high elasticity of substitution is also an important characteristic[3]. As is standard in the conventional argument, if we base our decision only on the price flexibility of different sectors, we should assign a lower weight to commodity sectors and thus favor the use of core inflation targeting. However, if we focus on the latter characteristic, we should place greater weight on commodity sectors. Given my analytical formula, whether we should place less weight on prices in commodity sectors or not depends on the relative size of price flexibility and elasticity.

The third lesson from the analytical formula is that exogenous international prices do not appear in it. This is despite the fact that I naturally model the effect of exogenous international prices. In my model, the firms respond to the change in the cost of imported material caused by the change in the international price of inputs. The firms also know that a deviation of their export price from those of their international competitors results in a change in export demand. I show that these international prices affect the optimal price

[2] Although the original definition of the core inflation rate involves econometric models that attempt to identify the persistent component of the inflation rate (see, for example, Wynne 2008), the optimal monetary policy literature has practically interpreted the core index as an index excluding food and energy.

[3] See Nakamura and Steinsson (2008) on price flexibility and Broda and Weinstein (2006)on the elasticity of substitution.

index if and only if they affect the output prices of domestic sectors. This is because volatile inflation causes efficiency loss in production regardless of the cause of the volatility, and thus, we do not need to adjust the formula for the price index depending on whether such volatility comes from international prices.

As an implication, although we may tend to think that central banks are not responsible for inflation volatility caused by international price movements, a central bank should be concerned about volatility as long as it affects the RPI. To understand this point, note that although international prices are exogenous, domestic prices can be controlled via changes in the exchange rate. Imagine an economy where all the domestic prices of different sectors are proportional to the international prices in those sectors. The ratio between the vector of international prices and the vector of domestic prices is the exchange rate. If the central bank selects one domestic sector, it is possible to stabilize the domestic price of that sector by adjusting the exchange rate to offset international price movements. Of course, this operation affects all other sectors, so the central bank faces a trade-off between stabilizing one sector and stabilizing another. The RPI indicates how to balance this trade-off.

My second main result is obtained from quantitative analysis, where I compare the welfare under simple stabilization policies for the RPI and three conventional price indices. Here, a simple stabilization policy means a policy in which the inflation rate in terms of the price index in question is zero in both the short and long run. In reality, implementing these policies via either Taylor rules or exchange rate interventions is simpler than implementing the Ramsey solution itself. However, it is not obvious that the simple stabilization of the RPI yields higher welfare than the stabilization of other price indices since the analytical result only states the optimality of long-run stabilization of the RPI, and the Ramsey solution itself, in general, involves short-run deviations from complete stabilization.

Calibrating to 40 countries with 35 sectors, I show that, for all countries in my sample, RPI stabilization performs the best among the stabilization schemes for the four indices considered. The loss from a simple stabilization of the RPI compared with the Ramsey solution turns out to be negligible and less than one-hundredth, on average, of the loss from simple stabilization of the other indices in terms of steady-state consumption. This means that the RPI is suitable not only for long-run stabilization targets but also for short-run targets.

Another important point from the welfare calibration is that there is no simple takeaway other than the RPI. This is because the ranking of other stabilization policies varies across countries depending on the combination of trade patterns and price stickiness. That is, CPI targeting performs better than PPI targeting for some countries while headline CPI performs better than core CPI targeting for other countries, depending on the combination of price stickiness, the elasticity of substitution, and trade patterns. The only result common to all countries in my sample is that RPI stabilization performs better than the stabilization of the other indices.

1.1.1 Related literature

This paper is an open economy extension of the method to derive the optimal price index from the Ramsey problem developed in Woodford (2010). The price index in Woodford (2010) can be obtained as a special case of the RPI proposed in this paper by letting the exports in each sector be zero and requiring the elasticity of substitution to be homogeneous across sectors. However, the other direction, i.e., deriving the RPI from Woodford's index, is not straightforward. This is because the size of each sector in Woodford (2010) can be interpreted either as the size of consumption or the size of production, and one might suggest different open economy extensions of the index depending on the interpretation. My analysis and the resulting formula for the RPI show that the correct interpretation is the size of production.

This paper is the first to theoretically show that the size of sectors in the stabilization objective should be measured by production size rather than consumption size in a multi-sector SOE environment. A similar feature can be seen in the result of Gali and Monacelli (2005), who demonstrate the optimality of output price stabilization in a model with only one production sector. However, having multiple sectors is key to answering the question of which price index to target since this creates the crucial trade-off between stabilizing one sector versus another when the first-best allocation cannot be achieved. In particular, their analysis cannot tell whether the result is coming from the assumption that there is only one sector with sticky prices or the assumption that the economy produces in only one sector. This makes it difficult to generalize their model to various trade patterns commonly observed in the real world such as the commodity importing case. My general formula enables me to separately discuss the effect of production and stickiness and can be applied not only to the special case of Gali and Monacelli (2005) but also to the opposite polar case (commodity exporter) and the intermediate cases.

There is a literature that analyzes the optimal monetary policy in two-country models (see Corsetti et al. 2010 and Engel 2011, for examples) and the models of a monetary union (see Gali and Monacelli (2008) and Kekre (2018), for examples). This paper differs from this literature in two senses. First, although, similarly to Woodford (2010) and this paper, these papers often identify the central bank's trade-off depending on price stickiness, they do not derive the price index that balances the trade-off except for special cases that achieve the first-best allocation. Second, the two-country setups of these papers are essentially closed since the two countries (or the countries in the union) do not trade with the rest of the world. Therefore, their framework cannot answer the question of how to deal with international price movements.

In this paper, I use the term "optimal price index", but the derived price index does not necessarily coincide with the optimal indices in the literature on index theory: see Diewert et al. (2009), for example. This is because the purposes of the index are different. In index theory, Diewert et al. (2009) among others attempt to obtain an accurate measure of the cost of living while my aim is to obtain the index for the central bank's stabilization target. By solving the household's optimization condition in the partial equilibrium sense, we can see that the CPI is the optimal price index in the sense of the cost of living in my model. However, my analysis shows that the optimal price index for the central bank's stabilization target is different from the CPI. It is natural to obtain different optimal price indices for different purposes.

From a technical point of view, the open economy extension in this paper involves two innovations that are also applicable to other SOE problems. The first is the definition of the Ramsey problem, which is consistent with the assumption of the timing of asset markets. Specifically, the Ramsey planner needs to recognize that some of the effects of its policy will be offset by the insurance effect of the asset market. In this way, I can compare the central bank's second-best problem with the planner's first-best problem and offer intuitive discussions comparing the two. The definition of the Ramsey problem is in line with the Ramsey taxation literature, but the previous NK SOE literature has defined the Ramsey problem in a different way, and hence, the first-best allocation cannot serve as a benchmark for the analysis. The definition of the Ramsey problem in this paper can simplify and clarify the analysis by De Paoli (2009), for example, of the case of the inefficient steady state.

The second innovation of this paper is differential tax rates that depend on the place of consumption, which allows me to simplify the analysis under terms of trade externalities without relying on extreme assumptions on parameter values. This is another feature that distinguishes my paper from Gali and Monacelli (2005), who impose a subsidy that partially offsets steady-state inefficiency and eliminate the rest of inefficiency by setting a parameter value such that the value of exports does not respond following any shock. I believe my novel simplification is useful for monetary policy discussions under terms of trade externalities.

The remainder of the paper proceeds as follows. In Section 2, I first explain the SOE NK DSGE model with which I define the Ramsey problem. In Section 3, I explain my analytical results. I first state the key assumptions on tax rates that make the analysis simple before approximating the Ramsey problem. The main theorem states that the RPI is stabilized in the long run, which is the justification for my proposal of RPI stabilization. Section 4 discusses the quantitative welfare comparison. Section 5 concludes the paper.

1.2 Method

I derive the RPI by solving the Ramsey problem of a central bank attempting to maximize the welfare of a representative household given market constraints in an SOE NK DSGE model. This section describes these market constraints and defines the Ramsey problem.

The economy features an arbitrary number of sectors with heterogeneous output price stickiness a la Calvo (1983). There is no domestic input-output structure, but the production requires labor and imported intermediate goods. The output can either be exported or domestically consumed. When exported, the price is sticky in the producer currency. Specifically, I denote the number of sectors by $S \in \mathbb{N}$, within each of which, a continuum of firms produce differentiated goods. The differentiated goods are aggregated within each sector.

The economy is small and open in the sense that international conditions are exogenous. The costs of imported materials are given by the exogenous international price times the endogenous exchange rate. The price of exports is compared with the exogenous prevailing price in the international market, to which the foreign demand for the country's export responds. The economy also takes the asset prices in complete international asset markets as given.

The monetary authority attempts to maximize the welfare of the representative domestic household, which consumes goods from all the sectors and provides labor. The monetary authority takes the optimization behavior of the household and firms under staggered price setting as given. It also takes exogenous international market conditions as given. I assume the timeless perspective following Woodford (2003).

1.2.1 Market conditions

Sectors are heterogeneous in price stickiness and the elasticity of substitution across differentiated goods within a sector. The former is already identified as key to obtaining the optimal price index in the closed economy literature. Although heterogeneity in the elasticity has not been highlighted in the literature, it is quantitatively important and intuitive. That is, a high elasticity of substitution implies that a small mispricing leads to a tremendous swing in demand and is thus costly to welfare.

For the model to be applicable to different countries with different trade patterns, I use a general production technology and a general trade pattern. By adjusting the parameter of the production technology of my model, one can consider a country such as Japan importing commodities, i.e., goods with flexible prices and high elasticities of substitution, and exporting differentiated goods or a country such as Russia doing the opposite.

Compared to the common SOE framework featuring tradable goods and non-tradable goods or that with home goods and foreign goods, the description of the production sector is enriched such that any imported good goes through the domestic sector before being consumed by the household. This allows me to treat different sectors uniformly despite the generality. My model encompasses the common frameworks in the literature as special cases.

1.2.1.1 The representative household

In any period of time $t \in [0, \infty]$, the representative household consumes goods from each of the S sectors denoted by C_{st} for $s \in S$ and supplies labor, denoted by L_{st}, to each of the S sectors. I assume that the amounts of consumption from different sectors are aggregated in a Cobb-Douglas function with the exponential factor ψ_s for sector $s \in S$ summing up to one $\sum_{s\in S} \psi_s = 1$.

$$C_t = \prod_{s\in S} C_{st}^{\psi_s} \tag{1.1}$$

This implies that elasticity of substitution across sectors is one. This is the standard assumption used in multi-sector NK models; see, for example, Aoki (2001) and Eusepi et al. (2011).[4]For the labor supply, I simply assume homogeneous labor that can be summed. This means that the disutility from labor depends only on the aggregate amount of work, not in the distribution of where the household works.

$$L_t = \sum_{s\in S} L_{st} \tag{1.2}$$

An alternative would be to assume increasing disutility from labor supplied to each firm in each sector. This would increase the efficiency cost of price dispersion relative to my case.

Given prices $\{P_{st}\}_{s\in S}, W_t$, profits $\{E_{st}\}_{s\in S}$, a lump sum transfer T_t, all denominated in the local currency, the pricing kernel in the international asset market $\mathcal{M}_t^*$, the exchange rate $\mathcal{E}_t$, and the price Λ of initial debt D_0, where the unit is in the utility in the pre-specified insurance contract over different policies, the household maximizes

$$\max_{D_0, \{C_{st}, L_{st}\}_{s\in S, t\in[0,\infty]}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\phi}}{1+\phi} \right] + \Lambda D_0,$$

[4]This does not mean that the assumption is without loss of generality. Benigno and Benigno (2003), for example, demonstrate that relaxing the assumption of a unitary elasticity between a home good and a foreign good may change the desirability of the flexible price allocation.

subject to

$$E_0 \sum_{t=0}^{\infty} \frac{\mathcal{M}_{0t}^*}{\mathcal{E}_t} \left(\sum_{s \in S} W_t L_{st} + \sum_{s \in S} E_{st} + T_t - \sum_{s \in S} P_{st} C_{st} \right) \geq D_0. \tag{1.3}$$

The first-order conditions are as follows:

$$\begin{aligned} \beta^t \psi_s \frac{C_t^{1-\sigma}}{C_{st}} &= \frac{\mathcal{M}_{0t}^*}{\mathcal{E}_t} \lambda P_{st} \\ \beta^t L_t^{\phi} &= \frac{\mathcal{M}_{0t}^*}{\mathcal{E}_t} \lambda W_t \\ \Lambda &= \lambda. \end{aligned}$$

The first term $\left(\prod_{s \in S} C_{st}^{\psi_s}\right)^{1-\sigma} / (1-\sigma)$ in the objective function represents the instantaneous utility from consumption from each sector $\{C_{st}\}_{s \in S}$ aggregated according to $C_t = \prod_{s \in S} C_{st}^{\psi_s}$. The second term in the objective function represents the disutility from labor supply to each sector $\{L_{st}\}_{s \in S}$. From the expenditure minimization problem, the CPI consistent with this consumption aggregator is

$$P_t = \prod_{s \in S} \left(\frac{P_{st}}{\psi_s} \right)^{\psi_s}. \tag{1.4}$$

Using this, intra-temporal conditions for the household's optimization are expressed as follows:

$$\begin{aligned} \psi_s C_t &= \frac{P_{st}}{P_t} C_{st}, \ \forall s \in S && (1.5) \\ \frac{L_t^{\phi}}{C_t^{-\sigma}} &= \frac{W_t}{P_t}. && (1.6) \end{aligned}$$

I assume that the household trades in the international asset market before the monetary authority chooses its policy. With this timing convention, the marginal utility for the household of having less debt D_0 is fixed at the exogenous level Λ across different possible monetary policies. The constant Λ represents the shadow price of the initial debt in the asset markets. This allows me to subsequently derive an international risk sharing condition that is invariant across policies. The policy-invariant risk sharing condition is standard in the literature, but how to consistently derive the condition in a DSGE setup has not been fully explored. For further discussion, see Senay and Sutherland (2007).

The level of consumption is determined by the tightness of the lifetime budget constraint. Denoting the aggregate consumption of a foreign country and its price by $C_t^*$and P_t^*, we can consider the stochastic discount factor to be equated to the ratio of marginal utilities of the consumer in that foreign country between any two states of the world. In particular, if we let $\mathcal{M}_{0,t}^* = \prod_{\tau=1}^{t} \mathcal{M}_\tau^*$ be the discount factor from period 0, or the planning period, to period t in the future, then, assuming the same utility function for the foreign consumer consuming C_t^* at price P_t^*, we can interpret the stochastic discount factor as

$$\mathcal{M}_{0,t}^* = \beta^t \frac{\left(C_t^*\right)^{-\sigma} / P_t^*}{\left(C_0^*\right)^{-\sigma} / P_0^*} \tag{1.7}$$

under the assumption that the foreign consumer also has access to the same complete asset markets. Gali and Monacelli (2005) also interpret the stochastic discount factor in this way. Combining this with the inter-temporal condition of the household, we have

$$\beta^t \frac{\left(C_t^*\right)^{-\sigma} / P_t^*}{\left(C_0^*\right)^{-\sigma} / P_0^*} \Lambda = \beta^t C_t^{-\sigma} \mathcal{E}_t P_t^{-1}.$$

Thus, we can obtain the international risk sharing condition

$$C_t = \xi C_t^* Q_t^{\frac{1}{\sigma}}, \tag{1.8}$$

where $Q_t = \mathcal{E}_t P_t^* / P_t$ is the real exchange rate and $\xi = (\Lambda P_0^*)^{-\frac{1}{\sigma}} / C_0^*$ is a constant. For this SOE, foreign consumption C_t^* and the foreign consumption price level P_t^* are exogenous, so is the stochastic discount factor $\mathcal{M}_t^*$. Note that if we do not assume the asset markets that insure across different policies, we need to allow Λ to vary across policies and hence the coefficient of the risk sharing condition also varies across policies.

1.2.1.2 The individual firm's technology and aggregation

The production technology for firm i in sector $s \in S$ is given by

$$Y_{sit} + Y_{sit}^X = Z_{s,t} M_{sit}^{\alpha_{sm}} L_{sit}^{\alpha_{sl}}.$$

Y_{sit} and Y_{sit}^X are the output of firm i in sector s at time t shipped for domestic use and exported to foreign, respectively, $Z_{s,t}$ is the stochastic sector-specific productivity, M_{sit} is the imported good, and L_{sit} is labor. Note that the Cobb-Douglas parameters α_{sm} and α_{sl} are allowed to vary across sectors.

I assume that the technology is linear, that is, $\alpha_{sm} + \alpha_{sl} = 1$ for all $s \in S$. When $\alpha_m = 0$, this reduces to the production technology assumed in Gali and Monacelli (2005). The linear technology assumption makes the following calculation simpler by making the marginal cost independent of the amount produced. If one instead assumes decreasing returns to scale, the efficiency cost of price dispersion will be larger. For simplicity, I also assume $\alpha_{sl} > 0$ for all $s \in S$. This means that all sectors use at least some amount of labor. This is empirically true. Some countries, on the other hand, may import nothing in some sectors. Therefore, I do not impose $\alpha_{sm} > 0$.

By setting $\alpha_{sm} \approx 1$ and $\alpha_{sl} \approx 0$, I can consider a country importing in sector s. Alternatively, by setting $\alpha_{sm} \approx 0$ and $\alpha_{sl} \approx 1$, I can consider a country being skilled at producing goods in sector s, and depending on the demand from foreign, it is likely that the country exports in sector s in equilibrium.

There is an aggregation firm in each sector with aggregation technologies

$$Y_{st} = \left(\int Y_{sit}^{\frac{\theta_s - 1}{\theta_s}} di \right)^{\frac{\theta_s}{\theta_s - 1}} \qquad \text{and} \qquad Y_{st}^X = \left(\int \left(Y_{sit}^X \right)^{\frac{\theta_s - 1}{\theta_s}} di \right)^{\frac{\theta_s}{\theta_s - 1}}, \tag{1.9}$$

that operates competitively. The elasticity of substitution parameter θ_s can be heterogeneous across sectors. The cost minimization problem of the aggregator gives the demand schedule

$$Y_{sit} = \left(\frac{P_{sit}}{P_{st}} \right)^{-\theta_s} Y_{st} \qquad \text{and} \qquad Y_{sit}^X = \left(\frac{P_{sit}^X}{P_{st}^X} \right)^{-\theta_s} Y_{st}^X, \tag{1.10}$$

and the price index consistent with the aggregation

$$P_{st} = \left(\int P_{sit}^{1-\theta_s} di \right)^{\frac{1}{1-\theta_s}} \qquad \text{and} \qquad P_{st}^X = \left(\int \left(P_{sit}^X \right)^{1-\theta_s} di \right)^{\frac{1}{1-\theta_s}}. \tag{1.11}$$

Note that the output for domestic use and foreign export are the same goods but labeled and priced differently.

1.2.1.3 The individual firm's pricing decision

Assume that in each sector $s \in S$, a randomly selected fraction $1-\lambda_s$ of the firms can reset the price. The price stickiness parameter λ_s can also vary across sectors. An individual firm in sector s takes wage W_t, import price $\mathcal{E}_t Q^*_{st}$, the demand function in equations (1.10), production function and tax τ_s as given. The unit cost of imported good $\mathcal{E}_t Q^*_{st}$ is given by the product of the endogenous exchange rate $\mathcal{E}_t$ and exogenous and stochastic international price Q^*_{st}. The prices of its output are set by the individual firm to maximize its expected profit.

$$\begin{aligned}\left(P_{sit}(0), P^X_{sit}(0)\right) = \arg \max_{(P,P^X)} \sum_{\tau=0}^{\infty} \lambda_s^\tau E_t \Bigg[\frac{\mathcal{E}_t}{\mathcal{E}_{t+\tau}} \mathcal{M}^*_{t,t+\tau} \\ \times \Bigg\{ \left((1-\tau_s) P - \left(\frac{\mathcal{E}_{t+\tau} Q^*_{s,t+\tau}}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_{t+\tau}}{\alpha_{sl}}\right)^{\alpha_{sl}} Z^{-1}_{s,t+\tau} \right) \left(\frac{P}{P_{s,t+\tau}}\right)^{-\theta_s} Y_{s,t+\tau} \\ + \left(\left(1-\tau^X_s\right) P^X - \left(\frac{\mathcal{E}_{t+\tau} Q^*_{s,t+\tau}}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_{t+\tau}}{\alpha_{sl}}\right)^{\alpha_{sl}} Z^{-1}_{s,t+\tau} \right) \left(\frac{P^X}{P^X_{s,t+\tau}}\right)^{-\theta_X} Y^X_{s,t+\tau} \Bigg\} \Bigg]\end{aligned} \tag{1.12}$$

The realized profit E_{sit} is aggregated within and across sectors $E_{st} = \int E_{sit} di$ and immediately paid out to the household. Note that the firms are taxed differently across sectors and between destinations. The rate for profits earned domestically is τ_s and the rate for profits from foreign is τ^X_s.

Following the usual procedure, the optimal pricing condition can be aggregated to

$$\frac{P_{s,t}}{P_t} = \frac{P_{s,t-1}}{P_{t-1}} \frac{1}{\Pi_t} \left(\frac{1}{\lambda_s} + \left(1 - \frac{1}{\lambda_s}\right) \left(\frac{\tilde{F}_{s,t}}{\tilde{K}_{s,t}}\right)^{\theta_s - 1} \right)^{\frac{1}{\theta_s - 1}} \tag{1.13}$$

$$\frac{P^X_{s,t}}{P_t} = \frac{P^X_{s,t-1}}{P_{t-1}} \frac{1}{\Pi_t} \left(\frac{1}{\lambda_s} + \left(1 - \frac{1}{\lambda_s}\right) \left(\frac{\tilde{F}^X_{s,t}}{\tilde{K}^X_{s,t}}\right)^{\theta_s - 1} \right)^{\frac{1}{\theta_s - 1}} \tag{1.14}$$

where $\tilde{F}_{s,t}$, $\tilde{K}_{s,t}$, $\tilde{F}^X_{s,t}$, $\tilde{K}^X_{s,t}$ are defined as follows:

$$\tilde{F}_{s,t} = C_t^{-\sigma} \frac{P_{s,t}}{P_t} Y_{s,t} + \lambda_s \beta E_t \left(\Pi_{s,t+1}\right)^{\theta_s - 1} \tilde{F}_{s,t+1} \tag{1.15}$$

$$\tilde{K}_{s,t} = (1-\tau_s)^{-1} \frac{\theta_s}{\theta_s - 1} C_t^{-\sigma} \left(\frac{Q_t Q^*_{s,t}}{\alpha_{sm} P^*_t}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl} P_t}\right)^{\alpha_{sl}} Z_{s,t}^{-1} Y_{s,t} + \lambda_s \beta E_t \left(\Pi_{s,t+1}\right)^{\theta_s} \tilde{K}_{s,t+1} \tag{1.16}$$

$$\tilde{F}^X_{s,t} = C_t^{-\sigma} \frac{P^X_{s,t}}{P_t} Y^X_{s,t} + \lambda_s \beta E_t \left(\Pi^X_{s,t+1}\right)^{\theta_s - 1} \tilde{F}^X_{s,t+1} \tag{1.17}$$

$$\tilde{K}^X_{s,t} = \left(1-\tau^X_s\right)^{-1} \frac{\theta_s}{\theta_s - 1} C_t^{-\sigma} \left(\frac{Q_t Q^*_{s,t}}{\alpha_{sm} P^*_t}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl} P_t}\right)^{\alpha_{sl}} Z_{s,t}^{-1} Y^X_{s,t} + \lambda_s \beta E_t \left(\Pi^X_{s,t+1}\right)^{\theta_s} \tilde{K}^X_{s,t+1} \tag{1.18}$$

Note that the nominal exchange rate is substituted out using the definition of the real exchange rate $Q_t = \mathcal{E}_t P^*_t / P_t \Leftrightarrow \mathcal{E}_t = Q_t P_t / P^*_t$, and I defined CPI inflation rate as $\Pi_t = P_t / P_{t-1}$ and sectoral inflation rates as $\Pi_{s,t} = P_{st}/P_{st-1}, \Pi^X_{s,t} = P^X_{st}/P^X_{st-1}$. For the derivation, see Appendix A.1.1.

Equations (1.13) and (1.14) govern the dynamics of sectoral inflation. Note that the sectoral inflation rate $\Pi_{s,t}$ and the inflation in terms of the CPI Π_t are related through the change in the relative price P_{st}/P_t. Thus, the equations state that sectoral inflation is a function of expected future sectoral inflation $\tilde{F}_{s,t}$ and the expected future marginal cost $\tilde{K}_{s,t}$. The sectoral inflation rate is the weighted sum of one and the ratio $\tilde{F}_{s,t}/\tilde{K}_{s,t}$, where the weight on one becomes larger as the price becomes stickier $\lambda_s \to 1$. When the price is completely sticky $\lambda_s = 1$, then sectoral inflation becomes one, meaning that the nominal sectoral price is fixed at the previous level, and only the relative price may move if the CPI P_t moves. At the other extreme, when the price is fully flexible $\lambda_s \to 0$, these equations hold by having $\tilde{F}_{st} = \tilde{K}_{st}$. In this case, the expectation terms in $\tilde{F}_{st}$ and $\tilde{K}_{st}$ also disappear, restoring the flexible price equilibrium pricing rule

$$\frac{P_{s,t}}{P_t} = (1-\tau_s)^{-1} \frac{\theta_s}{\theta_s - 1} \left(\frac{Q_t Q^*_{s,t}}{\alpha_{sm} P^*_t} \right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl} P_t} \right)^{\alpha_{sl}} Z^{-1}_{s,t}.$$

1.2.1.4 Resource constraints

The market clearing conditions are

$$\sum_{s\in S} \int L_{sit} di = L_t, \int M_{sit} di = M_{st}, C_{st} = Y_{st}, X_{st} = Y^X_{st}.$$

Using the factor demand from individual firms, these reduce to market clearing conditions in aggregate variables

$$C_{st} = Y_{st} \qquad \text{and} \qquad X_{st} = Y^X_{st} \tag{1.19}$$

and the resource constraints in aggregate variables

$$Z_{st} L_{st} = \left(\frac{\alpha_{sl}}{\alpha_{sm}} \frac{Q_t Q^*_{st}/P^*_t}{W_t/P_t} \right)^{\alpha_{sm}} \left(\Delta_{st} C_{st} + \Delta^X_{st} X_{st} \right) \quad \text{and} \quad M_{st} = \frac{\alpha_{sm}}{\alpha_{sl}} \frac{W_t/P_t}{Q_t Q^*_{st}/P^*_t} L_{st}, \tag{1.20}$$

where $\Delta_{st} = \int \left(\frac{P_{sit}}{P_{st}} \right)^{-\theta_s} di \geq 1$ and $\Delta^X_{st} = \int \left(\frac{P^X_{sit}}{P^X_{st}} \right)^{-\theta_s} di \geq 1$ are the production wedges that evolve according to

$$\Delta_{st} = \lambda_s \left(\frac{P_{st}}{P_{st-1}} \right)^{\theta_s} \Delta_{s,t-1} + (1-\lambda_s) \left(f_s \left(\frac{P_{s,t}}{P_t}, \Pi_t; \frac{P_{s,t-1}}{P_{t-1}} \right) \right)^{\theta_s} \tag{1.21}$$

$$\Delta^X_{st} = \lambda_s \left(\frac{P^X_{st}}{P^X_{st-1}} \right)^{\theta_s} \Delta^X_{s,t-1} + (1-\lambda_s) \left(f_s \left(\frac{P^X_{s,t}}{P_t}, \Pi_t; \frac{P^X_{s,t-1}}{P_{t-1}} \right) \right)^{\theta_s}, \tag{1.22}$$

where the function f_s is defined as

$$f_s\left(x, y; z\right) = \left(\frac{1}{1-\lambda_s}\left(1-\lambda_s\left(\frac{xy}{z}\right)^{\theta_s-1}\right)\right)^{\frac{1}{\theta_s-1}}.$$

For the derivation, see Appendix A.1.2.

Equation (1.20) combined with the dynamics (1.21) and (1.22) are the key equations capturing the cost of inflation in sector s.

First, as we can see from the dynamics, sectoral inflation or deflation $\Pi_{st} = P_{st}/P_{st-1}$ causes larger wedges $\Delta_{st}, \Delta_{st}^X$. When sectoral inflation is zero, i.e., $\Pi_{st} = 1$, the wedge decays at the rate λ_s to the steady state of $\Delta_{st} = 1$. When the inflation rate deviates from one, it enlarges the deviation of the wedge from one.[5] The effect of inflation on the wedge is larger when the price is sticky, represented by a larger λ_s, and when the differentiated goods are more substitutable, represented by a larger θ_s. Price stickiness limits the ability of firms to set a uniform price across differentiated goods. A higher elasticity induces a larger response of demand and thus production to the price differential among similar goods within the sector.

Second, the aggregate resource constraint (1.20) states that the wedges $\Delta_{st}, \Delta_{st}^X$ create a gap between the input L_{st} and the outputs C_{st}, X_{st} in effective units, which is the ultimate source of welfare loss in my model. Even if the production function in each firm is not affected by the inflation rate, the distribution of production within the sector is affected by inflation, as explained in the previous paragraph. Since uneven outputs are translated

[5] This happens regardless of inflation or deflation. The first term is increasing in $\Pi_{st} = P_{st}/P_{st-1}$, but the second term is decreasing in $\Pi_{st} = \left(P_{s,t}/P_t\right)\Pi_t/\left(P_{s,t-1}/P_{t-1}\right)$. The overall term behaves like the first term when $\Pi_{st} \gg 1$ and like the second term when $\Pi_{st} \ll 1$.

into a lower effective output under the love of variety assumption represented by the CES aggregator (1.9), sectoral inflation causes the production wedges.

1.2.1.5 Small open economy assumptions

Finally, I assume that foreign demand is price elastic.

$$X_{st} = \left(\frac{P^X_{st}}{\mathcal{E}_t P^*_{st}} \right)^{-\theta^*_s} X^*_{st}, \tag{1.23}$$

where X^*_{st} is the exogenous total international demand for sector s and P^*_{st} is its aggregate price index that is also exogenously given. This assumption can be derived from the cost minimization condition of a foreign buyer who aggregates the composite goods of sector s from different countries with a constant elasticity of substitution θ^*_s aggregator.

1.2.2 The Ramsey problem

The monetary authority's problem is defined as follows.

Definition 1.1. The optimal monetary policy is the solution to the following problem. Given random shocks $\left((Q^*_{st}/P^*_t, P^*_{st}/P^*_t, Z_{st}, X^*_{st})_{s\in S}, C^*_t \right)_{t=0}^{\infty}$, tax $\left(\tau_s, \tau^X_s \right)_{s\in S}$, and initial state variables P_{-1}, $\mathcal{E}_{-1}$, $\left(\Delta_{s,-1}, \Delta^X_{s,-1} \right)_{s\in S}$ the central bank chooses a contingent plan of all the endogenous variables C_t, L_t, $\left(C_{st}, L_{st}, P_{st}/P_t, P^X_{st}/P_t, Y_{st}, Y^X_{st}, X_{st}, M_{st} \right)_{s\in S}$, $\frac{W_t}{P_t}$, Q_t, Π_t, $\left(\Delta_{st}, \Delta^X_{st} \right)_{s\in S}$, $\left(\tilde{K}_{s,t}, \tilde{F}_{s,t}, \tilde{K}^X_{s,t}, \tilde{F}^X_{s,t} \right)_{s\in S}$, D_0 to solve

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\phi}}{1+\phi} \right] + \Lambda D_0$$

subject to equations (1.1), (1.2), (1.4)-(1.6), (1.8), (1.13)-(1.23) and

$$E_0 \sum_{t=0}^{\infty} \left[\mathcal{M}^*_{0,t} P^*_t \sum_{s \in S} \left(X_{st} \frac{P^X_{st}}{Q_t P_t} - M_{st} \frac{Q^*_{st}}{P^*_t} \right) \right] = D_0.$$

The last condition is equivalent to the household's lifetime budget constraint (1.3) under the assumption that all the profit goes to the household as E_t and the balanced government budget. This condition is important for binding the planner with the same trade-off between consumption and labor as that faced by the decentralized economy.

Although the initial level of debt D_0 is mathematically expressed as a choice variable, this does not mean that the central bank can freely choose it. Recall that I assumed in the previous sub-section that the asset markets operate before the monetary authority chooses its policy. Thus, the monetary authority takes into account the change or lack thereof in the initial level of debt D_0 when it chooses its policy. In this sense, the monetary authority indirectly chooses the initial level of debt.

1.3 Analytical Results

In this section, I derive the formula for the RPI and discuss the intuition behind the index. The justification of the index is given in a theorem that states that RPI needs to remain constant in long-run expectation for the economy to achieve the Ramsey optimal allocation. I start by showing two lemmas that help us understand the trade-off faced by the central bank.

The first lemma concerns the steady-state property that makes the analysis tractable.

The second lemma shows how the Ramsey problem can be approximated around the steady state. As studied in Benigno and Woodford (2012), the solution to the approximated problem approximates the solution to the original Ramsey problem under regularity conditions.

Then, I state the theorem on the optimality of stabilizing the RPI. The formula for RPI can be interpreted as a weighted sum of prices in different sectors, where the weight depends on output share of the sector, price stickiness and the elasticity of substitution within the sector. I discuss two points on the formula. First, compared with the CPI, the RPI is closer to PPI since PPI includes prices of exports. However, the PPI is not always better than CPI due to the other two factors: price stickiness and the elasticity of substitution. Second, international prices do not directly appear in the formula. This means that the central bank should be concerned about international prices if and only if they affect output prices that appear in the RPI formula.

1.3.1 Terms of trade externality and the efficiency of the steady state

To focus on the monetary friction in the analysis, it is convenient to assume that the tax rates are set to offset any real distortions that arise under the flexible price equilibrium. There are two types of real distortions in this economy: monopolistic distortions and terms of trade externality. It is widely known what tax rate offsets the former since it also arises in the closed economy setup. Regarding the latter, however, no paper has explicitly defined the distortion and offset it using a tax.

In this subsection, I show that these distortions can be offset by taxes if we assume different tax rates between domestic consumption and exports, as I do in my model. The distortions are defined as wedges between the social planner's allocation and the flexible price equilibrium. The planner's problem is defined as the maximization of the household's welfare subject only to the resource and technology constraint and the conditions in international markets. The flexible price equilibrium is defined as usual. Monopolistic competition leads to monopolistic markups in the price that appear as distortions in the allocation. The terms of trade externality, on the other hand, comes from the inability of the individual firms to exploit monopolistic competition in the international market.

I define the first-best planner's problem as follows.

Definition 1.2. Given $\left(\left(\frac{Q^*_{st}}{P^*_t}, \frac{P^*_{st}}{P^*_t}\right)_{s\in S}, \mathcal{M}^*_{0,t}\right)_{t=0}^{\infty}, \Lambda$, the planner solves

$$\max_{D_0,\left((C_{st},M_{st},X_{st},L_{st})_{s\in S}\right)_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{\left(\prod_{s\in S} C_{st}^{\psi_s}\right)^{1-\sigma}}{1-\sigma} - \frac{\left(\sum_{s\in S} L_{st}\right)^{1+\phi}}{1+\phi}\right] + \Lambda D_0,$$

subject to the technology constraint

$$Z_{s,t} M_{st}^{\alpha_{sm}} L_{st}^{\alpha_{sl}} = C_{st} + X_{st} \ \forall s \in S$$

and the inter-temporal trade balance condition

$$E_0 \sum_{t=0}^{\infty} \left[\mathcal{M}^*_{0,t} P^*_t \sum_{s\in S} \left(X_{st}^{\frac{\theta^*_s - 1}{\theta^*_s}} X_{st}^{*\frac{1}{\theta^*_s}} \frac{P^*_{st}}{P^*_t} - \frac{Q^*_{st}}{P^*_t} M_{st} \right)\right] = D_0.$$

In defining the planner's problem, I use

$$X_{st} = \left(\frac{P_{st}^X}{\mathcal{E}_t P_{st}^*}\right)^{-\theta_s^*} X_{st}^* \Leftrightarrow \frac{P_{st}^X}{\mathcal{E}_t P_{st}^*} = \left(\frac{X_{st}}{X_{st}^*}\right)^{-\frac{1}{\theta_s^*}}$$

to eliminate prices.

The objective function is the same as the welfare of the household in the Ramsey problem in Definition 1.1. The first-best planner is constrained only by the aggregate production technology in each sector and the inter-temporal trade balance condition. In building the aggregate production function, I already imposed uniform production within a sector $Y_{sit} = Y_{st}$ and so forth, as the optimality condition. The inter-temporal trade balance condition does not necessarily require balanced trade in each period, but any trade deficit is financed in the international asset market, and any trade surplus is invested in the international asset market such that the discounted sum of the trade surplus equals the initial level of the external debt D_0.

Appendix A.2.1 shows that the planner's solution is characterized by the following:

$$C_t \frac{\psi_s}{C_{st}} \alpha_{sl} Z_{s,t} \left(\frac{\alpha_{sm}}{\alpha_{sl}} \frac{L_t^\phi}{C_t^{-\sigma}} \frac{P_t^*}{Q_t Q_{st}^*}\right)^{\alpha_{sm}} = \frac{L_t^\phi}{C_t^{-\sigma}} \qquad \forall s \in S \tag{1.24}$$

$$\frac{\theta_s^* - 1}{\theta_s^*} Q_t \frac{P_{st}^*}{P_t^*} \left(X_{st}^*\right)^{\frac{1}{\theta_s^*}} = \left(Z_{s,t} \left(\frac{\alpha_{sm}}{\alpha_{sl}} \frac{L_t^\phi}{C_t^{-\sigma}} \frac{P_t^*}{Q_t Q_{st}^*}\right)^{\alpha_{sm}} L_{st} - C_{st}\right)^{\frac{1}{\theta_s^*}} C_t \frac{\psi_s}{C_{st}} \qquad \forall s \in S \tag{1.25}$$

$$C_t = \xi C_t^* Q_t^{\frac{1}{\sigma}}, \tag{1.26}$$

and

$$D_0 = E_0 \sum_{t=0}^{\infty} \left[\mathcal{M}_{0,t}^* P_t^* \sum_{s \in S} \left(\left(Z_{s,t} \left(\frac{\alpha_{sm}}{\alpha_{sl}} \frac{L_t^\phi}{C_t^{-\sigma}} \frac{P_t^*}{Q_t Q_{st}^*}\right)^{\alpha_{sm}} L_{st} - C_{st}\right)^{\frac{\theta_s^*-1}{\theta_s^*}} {X_{st}^*}^{\frac{1}{\theta_s^*}} \frac{P_{st}^*}{P_t^*} - \frac{\alpha_{sm}}{\alpha_{sl}} \frac{L_t^\phi}{C_t^{-\sigma}} \frac{L_{st}}{Q_t}\right)\right]. \tag{1.27}$$

To compare this with the flexible price allocation, I define the flexible price allocation as the solution to equations (1.1), (1.2), (1.4)-(1.6), (1.8), (1.13), (1.14), (1.15)-(1.18) under $\lambda_s = 0$ for all $s \in S$, and (1.19)-(1.23), and the household's budget constraint. Appendix A.2.2 shows that the equilibrium is characterized by the following:

$$C_t \frac{\psi_s}{C_{st}} \alpha_{sl} Z_{s,t} \left(\frac{\alpha_{sm}}{\alpha_{sl}} \frac{L_t^\phi}{C_t^{-\sigma}} \frac{P_t^*}{Q_t Q_{st}^*} \right)^{\alpha_{sm}} = \chi_s^{-1} \frac{L_t^\phi}{C_t^{-\sigma}} \qquad \forall s \in S \tag{1.28}$$

$$\frac{\theta_s^* - 1}{\theta_s^*} Q_t \frac{P_{st}^*}{P_t^*} (X_{st}^*)^{\frac{1}{\theta_s^*}} = \nu_s^{-1} \left(Z_{s,t} \left(\frac{\alpha_{sm}}{\alpha_{sl}} \frac{L_t^\phi}{C_t^{-\sigma}} \frac{P_t^*}{Q_t Q_{st}^*} \right)^{\alpha_{sm}} L_{st} - C_{st} \right)^{\frac{1}{\theta_s^*}} C_t \frac{\psi_s}{C_{st}} \qquad \forall s \in S \tag{1.29}$$

$$C_t = \xi C_t^* Q_t^{\frac{1}{\sigma}} \tag{1.30}$$

and

$$D_0 = E_0 \sum_{t=0}^{\infty} \left[\mathcal{M}_{0,t}^* P_t^* \sum_{s \in S} \left(\left(Z_{s,t} \left(\frac{\alpha_{sm}}{\alpha_{sl}} \frac{L_t^\phi}{C_t^{-\sigma}} \frac{P_t^*}{Q_t Q_{st}^*} \right)^{\alpha_{sm}} L_{st} - C_{st} \right)^{\frac{\theta_s^*-1}{\theta_s^*}} X_{st}^{*\frac{1}{\theta_s^*}} \frac{P_{st}^*}{P_t^*} - \frac{\alpha_{sm}}{\alpha_{sl}} \frac{L_t^\phi}{C_t^{-\sigma} Q_t} L_{st} \right) \right], \tag{1.31}$$

where the real wedges χ_s, ν_s are defined as

$$\chi_s = (1 - \tau_s) \left(\frac{\theta_s}{\theta_s - 1} \right)^{-1}, \ \nu_s = \frac{1 - \tau_s^X}{1 - \tau_s} \frac{\theta_s^*}{\theta_s^* - 1}.$$

We can see that the characterizations of allocations are equivalent except for the wedges χ_s and ν_s. The wedge χ_s for all s represents distortions coming from domestic monopolistic competition. The wedge ν_s for all s represents distortions coming from the inability of the domestic firms to exert their monopolistic power in the international market, which I call the terms of trade externality.

Thus, the following lemma holds.

Lemma 1.1. *The flexible price allocation is efficient if and only if $\chi_s = \nu_s = 1$ for all $s \in S$.*

That is,

$$1-\tau_s=\frac{\theta_s}{\theta_s-1},\ 1-\tau_s^X=(1-\tau_s)\left(\frac{\theta_s^*}{\theta_s^*-1}\right)^{-1}=\frac{\theta_s}{\theta_s-1}\left(\frac{\theta_s^*}{\theta_s^*-1}\right)^{-1}.$$

There are two types of inefficiency that the tax needs to address. To see this, note that even if the tax in each sector offsets the monopolistic markup in each sector by setting $1-\tau_s=\theta_s/\left(\theta_s-1\right)$, inefficiency remains due to the difference

$$\frac{\theta_s}{\theta_s-1}\left(\frac{\theta_s^*}{\theta_s^*-1}\right)^{-1}$$

between θ_s and θ_s^*. To achieve the efficient allocation, the tax needs to offset both internal distortion due to domestic monopolistic competition and external distortion due to (not utilizing) international monopolistic competition.

The external distortion arises when the elasticity of foreign demand is finite and hence $\theta_s^*/\left(\theta_s^*-1\right)>1$. In this case, the equilibrium consumption of export sector good is too low. The planner can improve welfare by exporting less while simultaneously improving the terms of trade. The market equilibrium cannot achieve this since each export sector takes the total demand for the exports as given, but the planner can strategically increase the sectoral price of exports as a whole to affect the terms of trade and foreign demand. To achieve this allocation in a decentralized manner, the fiscal authority needs to impose different tax rates depending on the destinations of goods.

In the following analysis, I assume such efficient tax rates to focus my analysis on monetary frictions. If I do not assume this efficient level of taxation, the monetary authority will have an incentive to use differential inflation rates across sectors to correct the distorted real allocation. If this force is added to the monetary trade-off that I analyze below, the analysis becomes too complicated. As the first step, I believe this simplification is beneficial in understanding the optimal price index.

1.3.2 Approximation of the Ramsey problem

This subsection derives the approximation to the Ramsey problem around the optimal steady state defined in Appendix A.2.3. I denote the log deviation from the steady state by the lower-case letter of the corresponding symbol of the variable. All domestic nominal variables are expressed relative to domestic CPI P_t. All international nominal variables are expressed in relative terms to foreign CPI P_t^*.

I show that when the steady state is efficient in the sense defined in the previous section, the second-order approximation of the welfare function, i.e., the objective function of the Ramsey problem, becomes purely quadratic without utilizing the second-order approximations of the pricing equations. Therefore, under regularity conditions, we can obtain an accurate first-order approximation to the solution of the non-linear Ramsey problem defined in Definition 1.1 by solving the approximated Ramsey problem that maximizes quadratically approximated welfare subject to linearly approximated constraints.

Note the difference between the optimal steady state and the efficient allocation. As mathematically defined in Appendix A.2.3, the optimal steady state is optimal in the second-best sense, where the monetary authority's problem takes sticky pricing mechanisms and market conditions as given. Therefore, the optimal steady state need not be an efficient allocation in the first-best sense. The appendix also shows that the optimal steady state can be characterized by the equations for flexible price allocation under constant exogenous variables and thus is efficient when the assumption of Lemma 1.1 is satisfied.

Denote the household's welfare by $\mathcal{W}$ and its steady state level by $\bar{\mathcal{W}}$. Define the vector of endogenous real variables as

$$v_t = [\boldsymbol{c}_t', \boldsymbol{x}_t']'$$

where

$$\boldsymbol{c}_t = [c_{1t}, ..., c_{St}]' \quad \text{and} \quad \boldsymbol{x}_t = [x_{1t}, ..., x_{St}]'$$

are the vectors of consumption and exports of all the sectors. Furthermore, define the vector of exogenous variables as

$$\xi_t = [c_t^*, \boldsymbol{x}_t^{*\prime}, \boldsymbol{p}_t^{*\prime}, \boldsymbol{q}_t^{*\prime}, \boldsymbol{z}_t']',$$

where

$$\boldsymbol{x}_t^* = [x_{1t}^*, ..., x_{St}^*]', \quad \boldsymbol{p}_t^* = [p_{1t}^*, ..., p_{St}^*]', \quad \boldsymbol{q}_t^* = [q_{1t}^*, ..., q_{St}^*]', \quad \text{and} \quad \boldsymbol{z}_t = [z_{1t}, ..., z_{St}]'$$

are the vectors of foreign demand for exports, international prices of exports, international prices of imports, and productivity shocks.

Before assuming the efficient tax rate, by using the market conditions except for the pricing equations, I show in Appendix A.2.5 that the approximated welfare can be written as

$$
\begin{aligned}
\frac{\mathcal{W}-\bar{\mathcal{W}}}{L^{1+\phi}} &= \sum_{t=0}^{\infty}\beta^t E_0 \boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_c\right)\left(d\left(\boldsymbol{\chi}\right)^{-1}-I\right)\boldsymbol{c}_t \\
&+\sum_{t=0}^{\infty}\beta^t E_0 \boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right)\left(d\left(\boldsymbol{\chi}\right)^{-1} d\left(\boldsymbol{\nu}\right)^{-1}-I\right)\boldsymbol{x}_t \\
&+\frac{1}{2}\sum_{t=0}^{\infty}\beta^t E_0\left[\left(v_t-N\xi_t\right)'\Gamma_{v2}\left(v_t-N\xi_t\right)+\sum_{s\in S}\frac{\phi_{ls}}{\alpha_{sl}}\left[\phi_{sc}\frac{\theta_s}{\kappa_s}\pi_{s,t}^2+\phi_{sx}\frac{\theta_s}{\kappa_s}\left(\pi_{s,t}^X\right)^2\right]\right]+t.i.p.
\end{aligned}
$$

where L is the steady-state level of aggregate labor supply, $\phi_{sc}=C_s/\left(C_s+X_s\right)$ is the steady-state consumption share of output in sector s, $\phi_{ls}=L_s/L$ is the steady-state labor usage share of sector s, $\phi_{sx}=1-\phi_{sc}$ is the steady-state export share of output and $d\left(\bullet\right)$ is the diagonal matrix of the vector inside the parentheses. The $2S$ by $4S+1$ matrix N defines the natural levels $N\xi_t$ of the endogenous variables defined in the appendix.

The first two lines are linear in the endogenous variables, but when the steady state is efficient $\chi_s=\nu_s=1$ for all $s\in S$, all of the linear terms disappear. Therefore, under the efficient steady state, we can obtain a purely quadratic second-order approximation of welfare.

Appendix A.2.6 shows that under the efficient steady state, the natural levels of the endogenous variables coincide with the flexible price equilibrium denoted by $F\xi_t$ with a $2S$ by $4S+1$ matrix F. In the following, I denote the log deviation from the flexible price equilibrium by $\tilde{v}_t := v_t - F\xi_t$. Furthermore, from the following relationship obtained in Appendix A.2.4

$$
\psi_s=\frac{\chi_s^{-1}\phi_{sc}\frac{\phi_{ls}}{\alpha_{sl}}}{\sum_{s'\in S}\chi_{s'}^{-1}\phi_{sc'}\frac{\phi_{ls'}}{\alpha_{s'l}}},
$$

we can see that the coefficients of the inflation rates can be simplified to

$$\sum_{s\in S}\frac{\phi_{ls}}{\alpha_{sl}}\left[\phi_{sc}\frac{\theta_s}{\kappa_s}\pi_{s,t}^2+\phi_{sx}\frac{\theta_s}{\kappa_s}\left(\pi_{s,t}^X\right)^2\right]=\left(\sum_{s'\in S}\phi_{sc'}\frac{\phi_{ls'}}{\alpha_{s'l}}\right)\sum_{s\in S}\frac{\theta_s}{\kappa_s}\psi_s\left[\pi_{s,t}^2+\frac{\phi_{sx}}{\phi_{sc}}\left(\pi_{s,t}^X\right)^2\right].$$

Therefore, I obtain the following lemma.

Lemma 1.2. *If the steady state is efficient, approximated optimal monetary policy can be obtained by solving the linear-quadratic problem. Given initial conditions* v_{-1} *and pre-commitment, the central bank chooses* $\left\{\tilde{v}_t,\boldsymbol{\pi}_t,\boldsymbol{\pi}_t^X,\pi_t\right\}_{t=0}^{\infty}$ *to minimize*

$$\sum_{t=0}^{\infty}\beta^t E_0\left[\tilde{v}_t'\Gamma_{v2}\tilde{v}_t+\Gamma_\pi\sum_{s\in S}\frac{\theta_s}{\kappa_s}\psi_s\left[\pi_{s,t}^2+\frac{\phi_{sx}}{\phi_{sc}}\left(\pi_{s,t}^X\right)^2\right]\right]$$

subject to (1) the Phillips curves

$$d\left(\boldsymbol{\kappa}\right)^{-1}\left(\boldsymbol{\pi}_t-\beta E_t\left[\boldsymbol{\pi}_{t+1}\right]\right)=\gamma_v^P\tilde{v}_t \quad \text{and} \quad d\left(\boldsymbol{\kappa}\right)^{-1}\left(\boldsymbol{\pi}_t^X-\beta E_t\left[\boldsymbol{\pi}_{t+1}^X\right]\right)=\gamma_{Xv}^P\tilde{v}_t,$$

where $\kappa_s=\left(1-\lambda_s\right)\left(1-\beta\lambda_s\right)/\lambda_s$*, and (2) the identities linking inflation rates and relative prices*

$$\boldsymbol{\pi}_t=\mathbf{1}_{S\times 1}\pi_t+\gamma_v^I\left(\tilde{v}_t-\tilde{v}_{t-1}\right)+\epsilon_t^I-\epsilon_{t-1}^I \text{ and } \boldsymbol{\pi}_t^X=\mathbf{1}_{S\times 1}\pi_t+\gamma_{vX}^I\left(\tilde{v}_t-\tilde{v}_{t-1}\right)+\epsilon_t^{IX}-\epsilon_{t-1}^{IX}.$$

Proof. See Appendix A.2.7. □

The coefficient matrices Γ_{v2}, γ_v^P, γ_{vX}^P, γ_v^I and γ_{vX}^I , the scalar Γ_π and the residuals ϵ_t^I and $\epsilon_t^{IX}$are given in Appendix A.2.7. The choice variables are the vector of consumption of each sector $\boldsymbol{c}_t$ and the vector of exports from each sector $\boldsymbol{x}_t$ contained in the vector of endogenous variables v_t, the vector of inflation rates

$$\boldsymbol{\pi}_t = \left[\pi_{1t}, ..., \pi_{St}\right]', \ \boldsymbol{\pi}_t^X = \left[\pi_{1t}^X, ..., \pi_{St}^X\right]'$$

and CPI inflation π_t. The reason for having CPI inflation here is that nominal variables are normalized by CPI inflation. One can alternatively write the equations with different normalization and still obtain the same result for the optimal price index.

As is usual in closed economy analysis, we have two parts in the objective function. The first part is the quadratic terms in the gaps in real variables from their respective natural levels. The second part is the nominal part representing the cost of volatile inflation.

The nominal friction is larger when the sector uses more labor, the price is sticky, or the elasticity of substitution is high. This is intuitive because if the inflation rate is volatile in a sector, the price dispersion of the sector increases. This means that to produce a certain effective output in the sector, the sector requires more labor input and imported materials, causing disutility for the household through more labor or a tighter international budget. The overall effect will be larger if the sector uses more labor at the steady state. Inflation volatility leads to higher price dispersion when the price is stickier. Given the same distribution of individual prices within a sector, the degrees of price dispersion $\Delta_{st}, \Delta_{st}^X$ become higher if the elasticity of substitution θ_s is higher.

In the constraints, there are in total $2S$ Phillips curves for domestic prices and export prices in each sector. The last two equations in the constraints are identities linking sectoral inflation rates $\boldsymbol{\pi}_t, \boldsymbol{\pi}_t^X$ and CPI inflation π_t. This means that there is only one degree of freedom left in this problem. Although there are different inflation rates for different sectors, they cannot be freely chosen since relative inflation rate between two sectors determines the evolution of the relative price of the two sectors.

1.3.3 Ramsey price index

This subsection states the main result of this paper. If we define a price index using the coefficients on the inflation rates in the loss function derived in the previous subsection, the price index stays constant in the long-run expectation under the optimal monetary policy. This implies that if the central bank does not stabilize this price index in the long-run, its policy is necessarily sub-optimal. Specifically, Appendix A.2.8 shows the following.

Theorem 1.1. *Define the price index as*

$$\log \mathbb{P}_t = \Phi^{-1} \sum_{s \in S} \psi_s \frac{\theta_s}{\kappa_s} \left(\log P_{st} + \frac{\phi_{sx}}{\phi_{sc}} \log P_{st}^X \right)$$

with

$$\Phi = \sum_{s \in S} \psi_s \frac{\theta_s}{\kappa_s} \left(1 + \frac{\phi_{sx}}{\phi_{sc}} \right).$$

Then, under the solution to the Ramsey problem,

$$\lim_{T \to \infty} E_t \log \mathbb{P}_T = \Phi^{-1} \overline{\log \mathbb{P}}.$$

I call this price index $\mathbb{P}_t$ the RPI since its stabilization is desirable as the solution to the Ramsey problem. The scalar Φ is used to normalize the coefficients to sum to one. This theorem states that the long-run stabilization of the RPI can be obtained as a necessary condition of the solution of the Ramsey problem. The theorem motivates the central bank's policy that stabilizes the inflation rate measured in this index since if this price index is not stabilized in the long run under some policy, the policy must be sub-optimal.

The converse is not necessarily true. That is, complete stabilization of this price index does not necessarily guarantee that the economy follows the optimal path consistent with the first-order conditions. Although it is generally possible to derive the if-and-only-if condition using the method of Giannoni and Woodford (2010), the condition is generally complicated. To keep my discussion simple and intuitive, I propose the use of a simple policy rule that always stabilizes the RPI. The welfare analysis in Section 1.4 shows that the welfare loss from simple RPI stabilization policy is negligible compared to the optimal monetary policy and that it performs better than the stabilization of headline CPI, core CPI, and PPI.

The RPI is a weighted sum of prices in different sectors, where the weights depend on consumption share ψ_s, the elasticity of substitution θ_s, the Phillips curve slope κ_s that contains the information of the price stickiness λ_s and the trade pattern ϕ_{sx}/ϕ_{sc}.

The weight reflects the trade-off that the monetary authority faces. As the derivation indicates, the weight takes the form of the coefficients on inflation rates in the loss function of the Ramsey problem representing the cost of inflation in different sectors. If the volatility of the inflation rate in a sector is relatively more costly to welfare than that in other sectors, the RPI assigns higher weight to the former sector.

Note that this price index will remain constant even if there is a unit-root process in the exogenous variables that may result in a permanent change in the natural levels of endogenous variables. This fact should be noted since if all exogenous variables are stationary, price levels under any price index will eventually coincide after all transitory shocks die out.

1.3.3.1 Comparison with CPI and PPI

To understand the relationship between the RPI and the conventional price indices, let us consider the weight on sector s under $\log P_s = \log P_s^X$. Recalling that $\phi_{sc} + \phi_{sx} = 1$, the weight on the price in sector s becomes

$$\psi_s \frac{\theta_s}{\kappa_s}\left(1 + \frac{\phi_{sx}}{\phi_{sc}}\right) = \frac{\theta_s}{\kappa_s} \underbrace{\overbrace{\psi_s}^{CPI} \frac{1}{\phi_{sc}}}_{PPI}.$$

From this expression, we can see that weighting under the RPI can be seen as that under PPI multiplied by the sensitivity of the wedge to inflation θ_s/κ_s. The PPI weight is relevant because the cost of inflation appears as the wedge in production; see equation (1.20). Therefore, the relevant size of the sector is the production size rather than consumption size.

However, the quantitative result in the next section shows that the sensitivity of the wedge to inflation θ_s/κ_s is important in the sense that PPI targeting sometimes performs worse than CPI targeting. The reason for the inclusion of this additional factor is that a given inflation volatility causes different wedge sizes depending on price stickiness, summarized by κ_s, and the elasticity of substitution, captured by θ_s.

Compared to the CPI weight, ψ_s, the PPI weight is higher for exporting sectors. This is because when some of the output is exported, the consumption weight on the sector is smaller than the optimal weight. In such a case, we can obtain the correct size of the sector by inflating the consumption weight ψ_s by the output-to-consumption ratio $1/\phi_{sc}$.

We can also obtain the price index derived in Woodford (2010) as a special case by assuming no trade $\phi_{sc} = 1$ and a homogeneous elasticity of substitution $\theta_s = \theta$. In this special case, the weight assigned to sector s is[6] ψ_s/κ_s.

The previous literature has argued for core inflation stabilization based on the observation that the non-core sectors have higher degrees of flexibility or higher values of κ_s, resulting in disproportionately smaller weights on those sectors. The RPI adjusts for the elasticity of substitution θ_s and trade $1/\phi_{sc}$. The former has the effect of placing a higher weight on sectors with higher substitutability within the sector. This is important since some non-core sectors do have higher values of the elasticity of substitution. The latter has the effect of placing a higher weight on export sectors. This may shift the optimal weight away from the core weight and closer to the headline weight for commodity exporting countries.

1.3.3.2 Role of international commodity prices

Another lesson that we can learn from the formula for RPI is that international commodity prices P^*_{st}, Q^*_{st} do not appear directly in the index. That is, the formula for RPI in Theorem 1.1 is a weighted sum of prices set by domestic firms P_{st} and P^X_{st}. Even if those prices are influenced by international prices, the formula does not adjust for or offset the influence of external factors.

Note that this is despite the fact that I naturally model the effect of exogenous international prices. As in the pricing equations (1.13)-(1.18), the international price of inputs Q^*_{st} affect the firms' pricing behavior through their marginal costs. As in the export demand equation (1.23), prices of international competitors P^*_{st} affect export demand. The former has a first-order impact on sectoral prices, and the latter has a first-order impact on the

[6] This is not exactly the same as the expression in Woodford (2010) since I am simplifying the analysis in one dimension, namely, heterogeneity in the labor. This will affect the expression for the κ_s reflecting the increasing disutility from uneven labor supply.

trade balance and a second-order impact on sectoral prices.

We can observe from the formula in Theorem 1.1 that these international prices affect the optimal price index if and only if they affect the output prices of domestic sectors. This is because volatile inflation causes efficiency loss in production regardless of the cause of the volatility, and thus, we do not need to adjust the formula for the price index depending on whether such volatility comes from international prices. In other words, output prices in the formula are sufficient statistics in the measure of the most welfare-relevant inflation rate.

As an implication, although we may tend to think that central banks are not responsible for inflation volatility caused by international price movements, a central bank should be concerned about volatility as long as it affects the RPI. To understand this point, note that although international prices are exogenous, domestic prices can be controlled via changes in the exchange rate. Imagine an economy where all the domestic prices of different sectors are proportional to the international prices in those sectors. The ratio between the vector of international prices and that of domestic prices is the exchange rate. If the central bank selects one domestic sector, it is possible to stabilize the domestic price of that sector by adjusting the exchange rate to offset international price movements. Of course, this operation affects all other sectors, so the central bank faces a trade-off between stabilizing one sector and stabilizing another. The RPI indicates how to balance this trade-off.

1.4 Quantitative Results

This section calibrates the model to data on 40 countries with 35 sectors. The purpose of the calibration is twofold: first, to understand the quantitative difference between the optimal price index and conventional price indices and, second, to obtain some insights into the implementation of the optimal monetary policy. That is, as noted above, the long-run stabilization of the optimal price index is insufficient to guarantee that the economy follows the optimal path. Therefore, the performance of the simple policy rule that completely stabilizes the optimal price index would be of interest. I calculate the welfare loss from stabilizing the optimal price index and sub-optimal price indices.

1.4.1 Welfare evaluation

I compare the welfare under the solution to the Ramsey problem, i.e., the optimal policy with those under four simple stabilization policies for the RPI, headline CPI, core CPI, and PPI. The equilibrium dynamics can be obtained by solving for the bounded solution of the set of constraints combined with one of the following monetary policy alternatives.

1. Optimal monetary policy characterized by the first-order conditions (A.4).

2. RPI stabilization[7]

$$\sum_{s\in S}\frac{\theta_s}{\kappa_s}\psi_s\left[\pi_{s,t}+\frac{\phi_{sx}}{\phi_{sc}}\pi^X_{s,t}\right]=0$$

3. Headline CPI stabilization.

$$\pi_t=0$$

4. Core CPI stabilization. Denoting the set of core sectors by $Core\subset S$,

$$\sum_{s\in Core}\psi_s\pi_{st}=0$$

[7]In case $\phi_{sc}=0$, I use the original expression of the weight $\sum_{s\in S}\frac{\phi_{ls}}{\alpha_{sl}}\frac{\theta_s}{\kappa_s}\left[\phi_{sc}\pi_{s,t}+\phi_{sx}\pi^X_{s,t}\right]$.

5. PPI stabilization. Denoting the steady state output by Y_s, Y_s^X for all $s \in S$,

$$\sum_{s\in S}\left(Y_s\pi_{st} + Y_s^X\pi_{st}^X\right) = 0$$

I evaluate the welfare

$$\begin{aligned}\mathcal{W} - \bar{\mathcal{W}} = &\frac{1}{2}L^{1+\phi}\sum_{t=0}^{\infty}\beta^t E_0\left[(v_t - N\xi_t)'\,\Gamma_{v2}\,(v_t - N\xi_t)\right.\\ &\left.+\sum_{s\in S}\frac{\phi_{ls}}{\alpha_{sl}}\left[\phi_{sc}\frac{\theta_s}{\kappa_s}\pi_{s,t}^2 + \phi_{sx}\frac{\theta_s}{\kappa_s}\left(\pi_{s,t}^X\right)^2\right]\right] + t.i.p.\end{aligned}$$

under each of the solutions and report the welfare loss compared to the optimal monetary policy.

1.4.2 Data

To evaluate the welfare loss described in the previous subsection, I need to obtain parameter values, some steady-state variables a description of the exogenous processes. I consider one period to be one month in this section. Parameters common across all countries and sectors, summarized in Table 1.1, are the discount factor $\beta = 0.97^{\frac{1}{12}}$, to match the 3% annual discount rate, the inverse of the elasticity of intertemporal substitution $\sigma = 2$, which is the standard value in the literature, and the inverse of the Frisch elasticity of labor supply $\phi = 0.47$, following De Paoli (2009)De Paoli (2009).

I allow for sectoral heterogeneity in the elasticity of substitution θ_s and price stickiness λ_s . For the stickiness parameters, I use the estimates of Nakamura and Steinsson (2008). For the elasticity of substitution, I use the estimates of Broda and Weinstein (2006). I follow

Table 1.1: Parameters common across countries and sectors

	Parameter	Value	Note
β	Discount rate	$0.97^{\frac{1}{12}}$	3% annual rate
σ	Inverse intertemporal elasticity of substitution	2	e.g. Arellano (2008)
ϕ	Inverse Frisch elasticity of labor supply	0.47	e.g. De Paoli (2009)

the categorization of 35 industrial sectors in the World Input-Output Database (WIOD)[8]. Appendix A.3.2 shows the concordance of the categories across these data sources. The parameter values are summarized in Table 1.2. In the analysis below, these stickiness parameters and elasticity parameters are assumed to be common across countries.

Since the definition of the "core" index varies across countries, I define the set of core sectors $Core \subset S$ as non-commodity sectors for the purposes of cross-country comparison. Table 1.2 also reports whether a sector is the core sector.

I use country-specific values for $\boldsymbol{\psi}, \boldsymbol{\alpha}_m, \boldsymbol{\alpha}_l, \boldsymbol{\phi}_c, \boldsymbol{\phi}_x$ and $\boldsymbol{\phi}_l$. These are constructed for 40 countries in the 2013 release of World Input-Output Database as follows[9]. I use the year 2000 to align with the periods covered in other estimates (Nakamura and Steinsson: 1998-2005, Broda and Weinstein 1990-2001) and the 2013 release for the sake of matching with Rauch's classification.

For a given country, the domestic part of its input-output table is taken from the WIOD and the imports and exports are calculated by summing all the foreign entries for the country. As consumption $\{P_s C_s\}_{s \in S}$, I use the sum of gross fixed capital formation (WIOD column c41) and final consumption by households (c37), non-profit organizations serving households (c38), and government for each sector (c39). The consumption expenditure share $\boldsymbol{\psi}$ is

[8] See Timmer et al. (2015).

[9] The countries included are Australia, Austria, Belgium, Bulgaria, Brazil, Canada, China, Cyprus, Czech Republic, Germany, Denmark, Spain, Estonia, Finland, France, the United Kingdom, Greece, Hungary, India, Indonesia, Ireland, Italy, Japan, Korea, Lithuania, Luxembourg, Latvia, Mexico, Malta, the Netherlands, Poland, Portugal, Romania, Russia, Slovakia, Slovenia, Sweden, Turkey, Taiwan, and the U.S.A.

Table 1.2: Sector-specific parameters common across all countries

Sector	WIOD	θ_s	λ_s	$\frac{\theta_s}{\kappa_s}$	Core
1	Agriculture, Hunting, Forestry and Fishing	9.83	.125	2	1
2	Mining and Quarrying	5.53	.961	3289	1
3	Food, Beverages and Tobacco	6.35	.737	67	0
4	Textiles and Textile Products	3.91	.977	6519	1
5	Leather, Leather and Footwear	3.69	.962	2310	1
6	Wood and Products of Wood and Cork	4.01	.987	19639	1
7	Pulp, Paper, Paper , Printing and Publishing	5.05	.956	2364	1
8	Coke, Refined Petroleum and Nuclear Fuel	5.75	.513	12	0
9	Chemicals and Chemical Products	5.25	.939	1275	1
10	Rubber and Plastics	4.8	.968	4214	1
11	Other Non-Metallic Mineral	3.04	.959	1637	1
12	Basic Metals and Fabricated Metal	7.43	.962	4651	1
13	Machinery, Nec	8.99	.963	5932	1
14	Electrical and Optical Equipment	4.79	.963	3161	1
15	Transport Equipment	13.41	.727	130	1
16	Manufacturing, Nec; Recycling	2.75	.835	83	1
17	Electricity, Gas and Water Supply	2.59	.513	6	0
18	Construction	2.59	.939	629	1
19	Sale, Maintenance and Repair of Motor Vehicles and Motorcycles; Retail Sale of Fuel	2.59	.531	6	0
20	Wholesale Trade and Commission Trade, Except of Motor Vehicles and Motorcycles	2.59	.939	629	1
21	Retail Trade, Except of Motor Vehicles and Motorcycles; Repair of Household Goods	2.59	.939	629	1
22	Hotels and Restaurants	2.59	.939	629	1
23	Inland Transport	2.59	.583	9	1
24	Water Transport	2.59	.583	9	1
25	Air Transport	2.59	.583	9	1
26	Other Supporting and Auxiliary Transport Activities; Activities of Travel Agencies	2.59	.583	9	1
27	Post and Telecommunications	2.59	.939	629	1
28	Financial Intermediation	2.59	.939	629	1
29	Real Estate Activities	2.59	.939	629	1
30	Renting of MandEq and Other Business Activities	2.33	.939	566	1
31	Public Admin and Defence; Compulsory Social Security	2.59	.939	629	1
32	Education	2.59	.939	629	1
33	Health and Social Work	2.59	.939	629	1
34	Other Community, Social and Personal Services	2.85	.939	692	1
35	Private Households with Employed Persons	2.59	.939	629	1

calculated as the share of each sector over aggregate domestic consumption.

As the payment to labor $\{WL_s\}_{s \in S}$, I use value added (WIOD row r64). The labor usage share $\boldsymbol{\phi}_l$ is calculated as the share of each sector over the aggregate value added of all the sectors in the country.

Since I abstract from the input-output linkages in my theoretical analysis, I need to obtain the values of $\boldsymbol{\alpha}_m, \boldsymbol{\alpha}_l, \boldsymbol{\phi}_c, \boldsymbol{\phi}_x$ that correspond to the economy without input-output linkages. To do so, I adjust the raw input shares and usage shares using the input-output matrix. The adjustment described in Appendix A.3.3 counts all indirect usages of labor and imported goods in calculating $\boldsymbol{\alpha}_m, \boldsymbol{\alpha}_l$. In calculating $\boldsymbol{\phi}_c, \boldsymbol{\phi}_x$, all indirect consumption and exports are counted. In this way, I can obtain the property $\boldsymbol{\alpha}_m + \boldsymbol{\alpha}_l = \mathbf{1}_{S \times 1}$ assumed in the analysis and the property $\boldsymbol{\phi}_c + \boldsymbol{\phi}_x = 1$ that needs to hold by definition.

Finally, the dynamics of the exogenous variables are assumed to be described as a vector auto-regressive process with one lag (VAR(1)). I obtain the coefficients and the variance-covariance matrix of the error terms by fitting the following monthly processes to the VAR(1) model. The sample period is from June 2009 to August 2017.

I use the logarithm of US consumption as world consumption c_t^*, US imports as an approximation of world demand $\boldsymbol{x}_t^*$, and US export price indices as an approximation of the prices of international competitors $\boldsymbol{p}_t^*$. The monthly series are accessed through CEIC[10], and the data sources are summarized in Table 1.3 for export demand $\boldsymbol{x}_t^*$ and in Table 1.4 for export prices $\boldsymbol{p}_t^*$. For c_t^*, I use seasonally adjusted series of personal consumption expenditure (PCE) in 2012 prices from Bureau of Economic Analysis. The standard deviation in the sample is 0.94%.

[10]CEIC is a proprietary database, which can be accessed here: https://insights.ceicdata.com.

Table 1.3: Data source

x*	WIOD	Std (%)	Series Name	Source
1	Agriculture, Hunting, Forestry and Fishing	12.7	Imports: 1-Digit: Food and Live Animals	US Census Bureau
2	Mining and Quarrying	9.5	Import Value: SITC: Customs, Aggregate under Metal and Mining Sector	US Census Bureau
3	Food, Beverages and Tobacco	10.2	Imports: 1-Digit: Beverages and Tobacco	US Census Bureau
4	Textiles and Textile Products	10.6	Imports: CIF: 2-Digit: Textile Fibers and Their Wastes	US Census Bureau
5	Leather, Leather and Footwear	7	Imports: 1-Digit: Manufactured Goods Classified Chiefly by Material	US Census Bureau
6	Wood and Products of Wood and Cork	9.9	Imports: 2-Digit: Cork and Wood	US Census Bureau
7	Pulp, Paper, Paper , Printing and Publishing	6.5	Imports: 2-Digit: Paper, Paperboard and Pulp	US Census Bureau
8	Coke, Refined Petroleum and Nuclear Fuel	9.9	Imports: 2-Digit: Petroleum, Petroleum Products	US Census Bureau
9	Chemicals and Chemical Products	6.8	Imports: 1-Digit: Chemicals and Related Products, nes	US Census Bureau
10	Rubber and Plastics	11.2	Imports: 2-Digit: Rubber Manufactures	US Census Bureau
11	Other Non-Metallic Mineral	8.4	Imports: NAICS: Mfg: Non Metallic Mineral	US Census Bureau
12	Basic Metals and Fabricated Metal	13.2	Imports: 2-Digit: Metalliferous Ores and Metal Scrap	US Census Bureau
13	Machinery, Nec	8.1	Imports: 1-Digit: Machinery and Transport Equipment	US Census Bureau
14	Electrical and Optical Equipment	8.5	Imports: 2-Digit: Electrical Machinery, Apparatus and Appliances, nes	US Census Bureau
15	Transport Equipment	10.3	Imports: 2-Digit: Road Vehicles	US Census Bureau
16	Manufacturing, Nec; Recycling	9	Imports: 1-Digit: Miscellaneous Manufactured Articles	US Census Bureau
17	Electricity, Gas and Water Supply	22.4	Imports: 2-Digit: Electric Current	US Census Bureau
18	Construction	1.7	Imports: sa: Service	US Census Bureau
19	Sale, Maintenance and Repair of Motor Vehicles and Motorcycles; Retail Sale of Fuel	10.5	Imports: 2-Digit: Road Vehicles	US Census Bureau
20	Wholesale Trade and Commission Trade, Except of Motor Vehicles and Motorcycles	7.8	Merchant Wholesalers Sales: Total	US Census Bureau
21	Retail Trade, Except of Motor Vehicles and Motorcycles; Repair of Household Goods	6.8	Retail Sales and Food Services: ex Motor Vehicle and Parts	US Census Bureau
22	Hotels and Restaurants	5.4	Retail Sales: FS: ow: Full Service Restaurants	US Census Bureau
23	Inland Transport	5.8	PCE: saar: SE: HCE: TR: PT: Ground Transportation (GT)	Bureau of Economic Analysis
24	Water Transport	2.8	Imports: sa: Service	US Census Bureau
25	Air Transport	12.1	PCE: saar: SE: HCE: TR: PT: Air Transportation	Bureau of Economic Analysis
26	Other Supporting and Auxiliary Transport Activities; Activities of Travel Agencies	5.4	Avg Weekly Earnings: PB: Travel Agency	Bureau of Labor Statistics
27	Post and Telecommunications	4.6	PCE: saar: SE: HCE: Other: CO: Postal and Delivery Services (PDS)	Bureau of Economic Analysis
28	Financial Intermediation	2.7	Avg Weekly Earnings: FA: Credit Intermediation and Rel Activities	Bureau of Labor Statistics
29	Real Estate Activities	2.6	Avg Weekly Earnings: FA: Real Estate	Bureau of Labor Statistics
30	Renting of MandEq and Other Business Activities	6	Avg Weekly Earnings: FA: Machinery and Equip Rental and Leasing	Bureau of Labor Statistics
31	Public Admin and Defence; Compulsory Social Security	2.2	Imports: sa: Service	US Census Bureau
32	Education	1.7	Imports: sa: Service	US Census Bureau
33	Health and Social Work	1.4	Imports: sa: Service	US Census Bureau
34	Other Community, Social and Personal Services	2.4	Avg Weekly Earnings: OS: Personal Care Services	Bureau of Labor Statistics
35	Private Households with Employed Persons	1.5	Imports: sa: Service	US Census Bureau

Table 1.4: Data source

p*	WIOD	Std (%)	Series Name	Source
1	Agriculture, Hunting, Forestry and Fishing	15.2	Export Price Index: Agriculture and Livestock Products (ALP)	Bureau of Labor Statistics
2	Mining and Quarrying	14.2	Export Price Index: Oil, Gas, Meneral and Ores: Mineral and Ores	Bureau of Labor Statistics
3	Food, Beverages and Tobacco	6.3	Export Price Index: Beverages and Tobacco Products	Bureau of Labor Statistics
4	Textiles and Textile Products	7.8	Export Price Index: Textile and Textile Articles (TA)	Bureau of Labor Statistics
5	Leather, Leather and Footwear	9	PPI: Hides, Skins, Leather and Products	Bureau of Labor Statistics
6	Wood and Products of Wood and Cork	3.1	(DC)Export Price Index: Wood Products	Bureau of Labor Statistics
7	Pulp, Paper, Paper , Printing and Publishing	3.2	Export Price Index: Paper	Bureau of Labor Statistics
8	Coke, Refined Petroleum and Nuclear Fuel	26.1	Export Price Index: Petroleum and Coal Products	Bureau of Labor Statistics
9	Chemicals and Chemical Products	5.7	Export Price Index: Chemicals	Bureau of Labor Statistics
10	Rubber and Plastics	3.4	Export Price Index: Plastics and Rubber Products (PRP)	Bureau of Labor Statistics
11	Other Non-Metallic Mineral	.8	Export Price Index: Nonmatalic Mineral Products	Bureau of Labor Statistics
12	Basic Metals and Fabricated Metal	9.8	Export Price Index: Primary Metals (PM)	Bureau of Labor Statistics
13	Machinery, Nec	1.1	Export Price Index: Machinery (MA)	Bureau of Labor Statistics
14	Electrical and Optical Equipment	.6	Export Price Index: Computer and Electronics Products (CEP)	Bureau of Labor Statistics
15	Transport Equipment	.6	Export Price Index: Transportation Equipment	Bureau of Labor Statistics
16	Manufacturing, Nec; Recycling	.5	Export Price Index: Miscellaneous Manufactured Articles (MM)	Bureau of Labor Statistics
17	Electricity, Gas and Water Supply	1.3	CPI U: Services: Utilities and Public Transportation	Bureau of Labor Statistics
18	Construction	1.3	PPI: ME: Construction	Bureau of Labor Statistics
19	Sale, Maintenance and Repair of Motor Vehicles and Motorcycles; Retail Sale of Fuel	.2	CPI U: Transport: Private: MV Maintenance and Repair (MR)	Bureau of Labor Statistics
20	Wholesale Trade and Commission Trade, Except of Motor Vehicles and Motorcycles	1	PPI: Wholesale Trade Services (WTS)	Bureau of Labor Statistics
21	Retail Trade, Except of Motor Vehicles and Motorcycles; Repair of Household Goods	1	CPI U: Housing: HFO: HO: Repair of Household Items	Bureau of Labor Statistics
22	Hotels and Restaurants	3.6	PPI: Accommodation Services: Travel Accommodation	Bureau of Labor Statistics
23	Inland Transport	3.6	PPI: Travel Arrangement Services: Vehicle Rentals and Lodging	Bureau of Labor Statistics
24	Water Transport	2.7	PPI: Travel Arrangement Services: Cruises and Tours	Bureau of Labor Statistics
25	Air Transport	11.1	Export Price Index: Air Passenger Fares	Bureau of Labor Statistics
26	Other Supporting and Auxiliary Transport Activities; Activities of Travel Agencies	2.1	PPI: Travel Arrangement Services: Others	Bureau of Labor Statistics
27	Post and Telecommunications	1.4	PPI: ME: General: Scales and Balances: Retail,Commercial,Hseholdand Mail	Bureau of Labor Statistics
28	Financial Intermediation	2.1	PPI: Credit Intermediation Services (CIS)	Bureau of Labor Statistics
29	Real Estate Activities	2	PPI: Real Estate Services	Bureau of Labor Statistics
30	Renting of MandEq and Other Business Activities	3.6	PPI: Rental and Leasing of Goods	Bureau of Labor Statistics
31	Public Admin and Defence; Compulsory Social Security	1.1	PPI: Selected Security Services	Bureau of Labor Statistics
32	Education	1	PPI: Educational Services	Bureau of Labor Statistics
33	Health and Social Work	.6	CPI U: Medical Care: Services	Bureau of Labor Statistics
34	Other Community, Social and Personal Services	.7	CPI U: GS: PC: Personal Care Services	Bureau of Labor Statistics
35	Private Households with Employed Persons	.2	PCE: PI: sa: Services (SE)	Bureau of Economic Analysis

For import prices $\boldsymbol{q}_t^*$, I combine export price indices using country-specific compositions of imports to sectors. That is, I use the World Input-Output table to calculate how much sector s of a given country imports goods and services from sector s' of all other countries. I denote the share of imports from sector s' over total imports to sector s by $\tilde{\alpha}_{ss'}$. I then use the weighted sum of the log prices of all source sectors s' as the import price index $q_{st}^* = \sum_{s' \in S} \tilde{\alpha}_{s's} p_{s't}^*$. I assume that productivity $\boldsymbol{z}_t$ is constant to focus on observable shocks.

1.4.3 Welfare results

Table 1.5 shows the welfare loss from simple monetary policy rules (i.e., monetary policies 2-5 in Subsection 1.4.1) compared with the optimal monetary policy. The units for these values is 0.01% of steady-state consumption.

As a benchmark, notice that the welfare loss from the stabilization of conventional price indices reported in Table 1.5 is on the order of 0.01% of the steady-state consumption. This is small as a percentage of consumption, but it is typical to obtain such numbers in the standard NK environment. For example, Gali and Monacelli (2005) report 0.0166% for their benchmark case.

The first finding from the welfare calibration is that most of the welfare loss can be eliminated by switching from stabilizing conventional price indices to the RPI. Comparing the second column, labeled Ramsey, with any of the third to the fourth columns in Table 1.5, the welfare loss in terms of consumption decreases to less than one-hundredth of the loss from targeting conventional indices, on average across countries. In other words, mere stabilization of RPI performs as well as the solution to the Ramsey problem.

The second finding shown in Table 1.5 is that, while RPI is always the best, the ranking of the stabilization of other indices varies across countries. This implies that we should not conclude that PPI is superior to CPI just because the analytical expression for the RPI can be interpreted as PPI plus an adjustment. For example, the worst index to target for the U.S., China, and Japan is PPI, core CPI, and headline CPI, respectively. In other words, the adjustment is large enough to make PPI stabilization less desirable than CPI stabilization for some countries, depending on the trade pattern.

1.5 Conclusion

In this paper, I solve a central bank's Ramsey problem and derive the Ramsey price index for small open economies to stabilize. Due to the openness of my model, the index depends on the export share of output in each sector in addition to those parameters that have been studied in closed economy models such as the consumption share, price stickiness and the elasticity of substitution.

By calibrating the formula to 40 countries, I find that RPI stabilization eliminates almost all of the welfare loss obtained under stabilization policies for headline CPI, core CPI, or PPI. In other words, the loss coming from a simple stabilization of RPI compared with the Ramsey optimal solution is negligible.

Regarding the ranking of stabilization policies for other indices, there is no common tendency applicable to all countries. Therefore, one should not ignore the price stickiness and elasticity components of RPI and prefer CPI or PPI.

Steady-state efficiency represents the key assumption that substantially simplifies the

Table 1.5: Welfare loss from simple policy rules

	Welfare Loss (0.01%)				Ranking			
Country	Ramsey	Headline	Core CPI	PPI	Best	2nd	3rd	Worst
AUS	.001	.007	.016	.004	Ramsey	PPI	Headline	Core CPI
AUT	.002	.099	.099	.074	Ramsey	PPI	Core CPI	Headline
BEL	.009	.177	.261	.51	Ramsey	Headline	Core CPI	PPI
BGR	.016	.073	.168	.134	Ramsey	Headline	PPI	Core CPI
BRA	0	.005	.004	.005	Ramsey	Core CPI	Headline	PPI
CAN	.002	.044	.06	.025	Ramsey	PPI	Headline	Core CPI
CHN	0	.006	.006	.004	Ramsey	PPI	Headline	Core CPI
CYP	.008	.052	.047	.104	Ramsey	Core CPI	Headline	PPI
CZE	.005	.215	.253	.147	Ramsey	PPI	Headline	Core CPI
DEU	.001	.043	.038	.033	Ramsey	PPI	Core CPI	Headline
DNK	.002	.095	.046	.083	Ramsey	Core CPI	PPI	Headline
ESP	.002	.073	.03	.086	Ramsey	Core CPI	Headline	PPI
EST	.006	.363	.507	.324	Ramsey	PPI	Headline	Core CPI
FIN	.002	.064	.103	.091	Ramsey	Headline	PPI	Core CPI
FRA	.001	.056	.022	.058	Ramsey	Core CPI	Headline	PPI
GBR	0	.012	.01	.012	Ramsey	Core CPI	Headline	PPI
GRC	.002	.029	.015	.035	Ramsey	Core CPI	Headline	PPI
HUN	.003	.255	.167	.159	Ramsey	PPI	Core CPI	Headline
IDN	.001	.033	.032	.036	Ramsey	Core CPI	Headline	PPI
IND	.001	.012	.007	.016	Ramsey	Core CPI	Headline	PPI
IRL	.005	.332	.323	.241	Ramsey	PPI	Core CPI	Headline
ITA	.001	.071	.031	.075	Ramsey	Core CPI	Headline	PPI
JPN	0	.008	.003	.007	Ramsey	Core CPI	PPI	Headline
KOR	.002	.18	.086	.358	Ramsey	Core CPI	Headline	PPI
LTU	.011	.053	.099	.177	Ramsey	Headline	Core CPI	PPI
LUX	.03	1.343	1.466	.932	Ramsey	PPI	Headline	Core CPI
LVA	.007	.102	.128	.101	Ramsey	PPI	Headline	Core CPI
MEX	.001	.011	.015	.008	Ramsey	PPI	Headline	Core CPI
MLT	.008	1.669	.769	1.379	Ramsey	Core CPI	PPI	Headline
NLD	.007	.215	.085	.451	Ramsey	Core CPI	Headline	PPI
POL	.001	.023	.027	.022	Ramsey	PPI	Headline	Core CPI
PRT	.001	.122	.034	.148	Ramsey	Core CPI	Headline	PPI
ROU	.003	.018	.058	.015	Ramsey	PPI	Headline	Core CPI
RUS	0	.011	.013	.007	Ramsey	PPI	Headline	Core CPI
SVK	.01	.193	.266	.27	Ramsey	Headline	Core CPI	PPI
SVN	.008	.216	.279	.15	Ramsey	PPI	Headline	Core CPI
SWE	.001	.078	.089	.159	Ramsey	Headline	Core CPI	PPI
TUR	.001	.036	.01	.035	Ramsey	Core CPI	PPI	Headline
TWN	.003	.198	.121	.164	Ramsey	Core CPI	PPI	Headline
USA	0	.015	.003	.015	Ramsey	Core CPI	Headline	PPI

analysis. Relaxing this assumption would give the central bank an additional incentive to stabilize one sector rather than another to influence their equilibrium relative price. Extending the analysis in this direction represents a fruitful area of future research.

I abstract from input-output networks across different sectors in the economy. Adding this feature would result in a different formula for the RPI.

Chapter 2

The Impact of E-Commerce on Urban Prices and Welfare

Yoon J. Jo, Misaki Matsumura, and David E. Weinstein

2.1 Introduction

How has e-commerce affected prices and welfare? Economists have suggested a number of approaches to answering this question. In early work, Brynjolfsson et al. (2003) suggested that the internet raises welfare by giving consumers better access to new varieties. More recent work, has considered impacts through impacts on average prices (Goolsbee and Klenow (2018)) or through the ability of information technology to generate price convergence (c.f. Brown and Goolsbee (2002), Jensen (2007), and Steinwender (2018)).One of the limitations of prior work has been that it has focused on small samples of goods (e.g., books), had no data on price trends prior to the creation of the internet (which makes assigning the impact of e-commerce difficult to measure), or focused on different information technologies (cell phones or telegraphs).

This paper makes use of a unique Japanese data set covering hundreds of products over close to three decades to examine the impact of the internet on Japanese prices and welfare using a number of popular modeling techniques. First, we consider the possibility that e-commerce may have lowered prices by driving down the average prices of goods available online. Second, following Brynjolfsson et al. (2003) and Arkolakis et al. (2012), we compute the welfare gains due to the ability of e-commerce to enable consumers to purchase goods from other regions. Third, following Jensen (2007), we compute the gains that arise through e-commerce's ability to arbitrage intercity price differences. We find that all three channels produced welfare gains in Japan, but our estimates suggest that the first and second channels are by far the most important, with welfare gains through these channels being eleven to sixteen times larger than through the price arbitrage channel. Overall, we find that increased inter-city arbitrage raised Japanese welfare by 0.06 percent, the gains due to new varieties available through online shopping raised welfare 0.12 percent, and the gains due to overall price reductions for goods available online raised welfare by 1 percent.

We make use of two unique datasets. The first, a quinquennial government survey of households that reports the purchase locations of Japanese consumers, and the second covering all sales transacted by Japan's largest e-retailer, Rakuten. We merge these data with the price data underlying the Japanese consumer price index (CPI), covering 30,000 price quotes per year in physical stores across Japan over the period 1991 to 2016. We also use these estimates to compute the welfare gains arising from the ability of e-retail to arbitrage away regional price differences.

In order to identify the impact of e-commerce, we exploit the fact that the growth of e-retail happened rapidly in Japan, so it is relatively easy to divide recent history into clear pre and post e-retail periods. Rakuten was one of the earliest entrants to the e-retail business, starting in February of 1997. Unlike Amazon, Rakuten's business model was not to sell goods directly to consumers, but rather to serve as a platform through which consumers could easily find stores willing to sell products online. This difference enabled them to expand without developing warehouses and extensive logistics networks. Thus, while only 40 percent of Amazon's transaction value comes from third-party sellers, virtually all of Rakuten's transactions come from this source. Since its entry into Japan, Rakuten experienced explosive growth. By April of 2000, when it announced its initial public offering and a year before the entry of Amazon into Japan, Rakuten had grown to be a platform in which consumers had access to goods available from 2,300 merchants.[1] Already at this point,

[1]Phred Dvorak, "Japan's Highly Popular Rakuten Plans IPO Despite Shaky Market," *Wall Street Journal*,

the Rakuten website was getting 95 million hits per month—almost one hit for every man, woman, and child in Japan. Thus, within five years, Japanese consumers in any city went from only being able to buy locally or from catalogs to being able to purchase goods from thousands of merchants located across Japan. By 2010, Rakuten had a 30 percent market share in the Japanese e-commerce market, eclipsing global giants like Amazon and Yahoo.[2] Rakuten transaction values continued to skyrocket reaching 12.7 trillion yen by 2017: equal to about 2.3 percent of Japanese GDP.[3]

Since we can observe both which goods tend to be sold intensively online and can observe how pricing behavior for goods changed before and after the entry of Rakuten, we can adopt differences-in-differences strategy to identify how pricing behavior changed for goods suitable for e-retail relative to goods not suitable for e-retail. We first characterize the set of goods suitable for e-retail by calculating the difference between the share of consumer expenditure for a good purchased through the internet with that in the Japanese Family Income and Expenditure Survey (FIES), which is used to compute the expenditure weights for the Japanese consumer price index. While we find that virtually all goods are sold online, there is enormous variation in e-retail intensity. Not surprisingly, highly perishable, non-standardized goods are not sold intensively online. Neither are highly time-sensitive goods and goods requiring consumer identification (e.g., medicines) or goods with high weight-to-value ratios. To a first approximation, goods sold online closely resemble the set of goods that historically dominated the catalog business: books, clothing, footwear, hardware, gardening supplies, and (later) electronics. [4] Moreover, we find that regional variation in e-commerce

April 18, 2000.

[2]Rakuten Annual Report (2010).

[3]Rakuten Annual Report (2017).

[4]see https://dingley.com/five-pivotal-moments-in-catalog-history/

sales intensity is entirely driven by the share of college-educated people, with no role for urban-rural or young-old divides once one controls for education.

We use this feature of the data to first document evidence that the emergence of e-commerce is associated with price drops in physical retailers selling goods that are available online. This result is complementary to that of Goolsbee and Klenow (2018) who found that goods available online in the U.S. have a lower rate of inflation than goods not available online. Our result differs in two important dimensions. First, we show that this price impact is apparent in the pricing decisions of physical retailers. Second, we show that the association between goods that can be sold online and their relative rates of price changes did not exist prior to the entry of large e-commerce firms. Thus, we provide evidence that advent of e-commerce changed relative pricing behavior by lowering the relative prices of goods suitable for e-retail by reducing relative retail markups, improving logistics, or generating some other efficiency gain. If we assume that the monetary policy is such that e-commerce did not affect prices in sectors untouched by e-retail, this appears to have lowered the Japanese price level by about one percentage point between 1997 and 2016.

We next consider the impact of e-commerce on welfare through enhancing consumer access to new varieties. Computing these gains for the class of models introduced by Krugman (1980), Melitz (2003), and Eaton and Kortum (2002) can be done quite easily with our data by simply computing the share of online purchases. This share rose rapidly to levels quite similar to those observed in the U.S. Based on these sales shares, we obtain an estimate of the welfare gain from e-commerce of 0.7 percent in 2016.

Finally, we examine how the difference in convergence rates across cities between goods available online and those not available online changed after the entry of Rakuten into the market. We estimate that intercity convergence rates for Japan pre-Rakuten are higher than those obtained in Parsley and Wei (1996) and Cecchetti et al. (2002). These studies found no convergence or half-lives of price differentials of nine years. In contrast, we find that half-lives for price differentials across Japanese cities are only 4.8 years for both goods that are suitable and unsuitable for online sales prior to Rakuten's entrance into the market. The difference probably arises from the fact that Japanese CPI data is based on the sampling of identical or extremely similar goods across cities, which makes it superior for this type of analysis. Rakuten's entry is associated with a 25 percent drop in this half-life for goods traded intensively online, which we assume is due to the difficulty of maintaining intercity price differences in a world with e-retail.

With these estimates in hand, we next compute the impact of e-retail on Japanese welfare through improved price arbitrage using the model developed in Jensen (2007). An interesting feature of this model is that not only is it the case that consumers in low-priced cities lose while consumers in high-priced cities gain, the price movements imply that on net consumers lose. This result doesn't mean that arbitrage isn't beneficial. The net welfare gain occurs because producers in the low-priced cities can now sell in the high-priced cities. While this does entail a loss for producers in the high-priced cities, the increase in surplus for producers in the low-priced cities exceeds the losses for producers in the high-priced cities and the net loss of consumer surplus, resulting in higher real income in the economy. Thus, the internet acts very much like a reduction in trade costs in that there are lots of winners and losers. Relative producer surplus falls and consumer surplus rises in the sectors with comparative disadvantage (locations with relatively high-priced goods), but relative producer surplus rises and consumer surplus falls in the sectors with comparative advantage (locations with relatively low-priced goods). However, just like a reduction in trade costs, the net gains are positive.

In order to estimate these effects, we compute counterfactual price paths for the Japanese economy. The first is based on our estimates of how intercity price differentials should dissipate in the presence of e-retail, and the second is based on how intercity price differences would dissipate in the absence of e-retail.The faster rate of price convergence in the presence of e-retail implies that welfare gains will be higher after Rakuten entered the market. When we quantify this for Japan, we estimate that the entry of Rakuten raised the welfare of Japanese consumers by about 0.06 percent per year.

2.1.1 Related Literature

Our results are related to a number of papers related to how information technology has affected pricing and welfare. A large number of papers have demonstrated that information technology serves to reduce price dispersion and promote trade. Freund and Weinhold (2004) show that countries with more web hosts export more to each other. Jensen (2007), Aker (2010), and Allen (2014) examined the impact of the introduction of mobile phones on fish or agricultural markets in India, Niger, and the Philippines, and Steinwender (2018) examined the impact of the transatlantic telegraph cables on 19th century textile prices and exports. Our work is complementary to these papers in that we also show that e-retail serves to reduce price dispersion. However, our work differs in focus and scope—our study examines the role played by e-commerce in an advanced, modern economy on the prices of hundreds of goods in physical retailers.

Our paper is also related to studies of the impact of e-commerce on welfare. Many of these studies have focused on the gains from variety that arise as consumers can purchase products that are not available in local stores. For example, Brynjolfsson et al. (2003) compute the variety gains from internet book sales; Fan et al. (2018) examine the relative variety gains in large and small Chinese cities associated with internet usage; and Einav et al. (2017) estimate the gains from e-retail due to shopping convenience and and new varieties. An important difference between these studies and ours is that we make use of government surveys of e-commerce sales to compute e-commerce sales shares (an important input into welfare calculations) rather than using private Alibaba or credit-card data to assess e-commerce expenditure shares. These private measures of e-commerce intensity deviate substantially from official measures. For example, the U.S. Census Bureau reports that in 2014, e-commerce firms accounted for about 6 percent of total retail sales, and the Japanese Ministry of Economy Trade and Industry reports that in the same year, Japanese e-commerce firms had a 4.4 percent market share.[5] By contrast, the e-commerce sales intensity in credit card or Ali Research data is around 20 percent for the U.S. and 10.6 percent for China. Part of the discrepancy comes from the fact that credit cards are more likely to be used for e-commerce transactions and balance-sheet sales numbers for e-commerce firms often include revenues from other sources—finance, telecommunications, etc.—that do not necessarily involve e-commerce. For example, of Rakuten's net sales, 42 percent comes from e-commerce, 7 percent from travel, and of the remainder most comes from credit cards, banking, securities, and telecommunications.

Our paper also relates to studies of how the internet affects local markets. Goldmanis

[5] https://census.gov/retail/mrts/www/data/pdf/ec_current.pdf

et al. (2010) examine regional patterns in online purchase behavior change the market structure in bookstores, travel agencies and car dealers. Goyal (2010) finds that the introduction of internet kiosks raised soy prices in rural India. Couture et al. (2018) conduct a randomized control trial in eight rural Chinese counties and find little effect of the introduction of e-commerce on the local economy. Brown and Goolsbee (2002) show that the creation of online insurance sales systems reduced the variance of insurance pricing. Our work differs from these studies in terms of scope (the large number of different sectors considered), the link to physical retail prices across an entire economy, and identification strategy (the ability to examine differential rates of price convergence before and after the advent of e-commerce).

The paper also relates to the literature on internet pricing. In particular, Cavallo (2017) shows that online prices and prices in physical stores are quite similar. This fact helps motivate our assumption that local retailers with high prices should face stiff competition from online retailers. Goolsbee and Klenow (2018) show that price trends of goods sold online exhibited lower inflation rates in the US between 2014 and 2017 than in the CPI. While we also show that goods available online exhibit lower inflation rates than goods not available online, there are two important differences between our study and theirs. First, we can show that this pattern was not present prior to the introduction of e-commerce. This fact makes it clear that the differential patterns in inflation rates is not a characteristic of the goods themselves but rather it was an effect that appeared for goods sold online only after the introduction of the internet. Second, we show that goods sold intensively online exhibit lower rates of price increase in physical stores. This is consistent with the idea that online price trends are mirrored in the prices of physical retailers.

Finally, our paper is also related to the large literature on PPP convergence regressions.

Parsley and Wei (1996) were the first to document that differences in convergence coefficients across cities was linked to trade costs, an insight that we build upon in this paper. Bergin et al. (2017) employ a similar triple difference strategy to show that rates of price convergence across European countries increased after joining the euro area. Our contribution lies in adapting some of the techniques developed in these earlier papers for understanding how e-commerce affects regional price convergence.

The remainder of the paper is organized as follows. Section 2.2 introduces the the estimation strategy and provides the theory for the welfare calculation. Section 2.3 presents the data and provides some stylized facts about e-retail suitability. We present our main estimates for the impact of Rakuten on price convergence and welfare in Section 2.4, and Section 2.5 concludes.

2.2 Theory

Estimating the impact of e-commerce on average prices and in New Trade Theory is very straightforward following Arkolakis et al. (2012), so we will skip the theoretical discussion of how to do this and just deal with the estimation issues in Sections 2.4.1 and 2.4.2. In Section 2.2.1, we model the impact that e-commerce has had on interregional price differential, and we show how the decline in these differentials raises welfare in Section 2.2.2.

2.2.1 Estimating the Impact of the E-Retail on Price Arbitrage

We begin by defining some notation. Let $p_{ict} \equiv \ln P_{ict}$ be the log price of item i in city c in time t. Define the Δ^k operator as $\Delta^k p_{ict} \equiv p_{ict} - p_{ic,t-k}$; thus, if we set $k = 1$, we can examine annual changes, but we can also examine longer differences by setting k equal to a whole number larger than one. Let $x_i \in [0, 1]$ be the "e-commerce suitability" of the good, where zero indicates it is not suitable for e-commerce and one indicates that it is the most suitable good for e-retail. Let D_t be an indicator variable that is one if e-commerce are positive in period t and zero otherwise. We assume that the change in the price of any item in a city c can be written as a standard purchasing price parity specification in which we introduce a modification that allows the rate of price converge for goods available online to change, i.e.,

$$\Delta^k p_{ict} = \alpha_{it} + \beta_{ct} + (\gamma + \delta_1 x_i + \delta_2 D_t x_i)\, p_{ic,t-k} + \epsilon_{ict}, \tag{2.1}$$

where α_{it} is a item-time fixed effect; β_{ct} is a city-time fixed effect; γ is a parameter that captures the rate of intercity price convergence for goods not available online; δ_1 is a parameter that captures the rate of price convergence for goods available online prior to the entry of e-commerce firms; δ_2 captures the increase in rate of price convergence for online goods after the entry of e-commerce firms; and ϵ_{ict} is an iid error term. We think of this error as price shocks arising from period t local supply-and-demand conditions for an item in a city that are not shared by all items in the city and are uncorrelated with past prices.

In this specification, a critical parameter is the rate of convergence given by $(\gamma + \delta_1 x_i + \delta_2 D_t x_i)$, which we expect to be between -1 and 0. A value of -1 means that equation (2.1) collapses to $p_{ict} = \alpha_{it} + \beta_{ct} + \epsilon_{ict}$, and therefore the price of any item can be decomposed into its national price (α_{it}), a common local market premium (β_{ct}), and an iid error term that is not persistent. In this case, any idiosyncratic price shock to a good in a city (ϵ_{ict}) has no impact on prices in the next period. Hence, price convergence occurs in one period, and prices always equal their conditional mean of $(\alpha_{it} + \beta_{ct})$ plus a random iid shock. At the other extreme, we have the case of where $(\gamma + \delta_1 x_i + \delta_2 D_t x_i) = 0$, which implies that the price of that good i in city c follows a random walk with a drift term given by $(\alpha_{it} + \beta_{ct})$. In intermediate cases where $(\gamma + \delta_1 x_i + \delta_2 D_t x_i) \in (-1, 0)$, price differences across cities can persist for more than k years.

In our setup, we can write the approximate half-life[6] of any price deviation from the steady-state price (measured in intervals of length k) as[7]

$$H_t \equiv \frac{\ln(0.5)}{\ln\left(1 + \hat{\gamma} + \hat{\delta}_1 x_i + \hat{\delta}_2 D_t x_i\right)}. \tag{2.2}$$

As one can see from this formula, the change in the rate of convergence depends on all of the estimated convergence parameters, therefore there is not a simple mapping from changes in δ_t into rates of convergence. Thus, the impact of e-retail on the rate of convergence for any good i can be written as:

$$\Delta H_t \equiv \frac{\ln(0.5)}{\ln\left(1 + \hat{\gamma} + \hat{\delta}_1 x_i + \hat{\delta}_2 x_i\right)} - \frac{\ln(0.5)}{\ln\left(1 + \hat{\gamma} + \hat{\delta}_1 x_i\right)}. \tag{2.3}$$

2.2.2 Welfare

We can use these estimates to inform us about the welfare gains from e-commerce by using the framework developed in Jensen (2007). Jensen considered a technological change that enabled arbitrage between a high-priced region (H) and a low-priced region (L). This

[6] As Goldberg and Verboven (2005) note, this formula is only correct for AR1 processes.

[7] The steady state price is given by the price at which setting $\Delta p_{ict} = 0$. This price equals $p_{ic} = -(\alpha_{it} + \beta_{ct}) / (\gamma + \delta_1 x_i + \delta_2 D_t x_i)$.

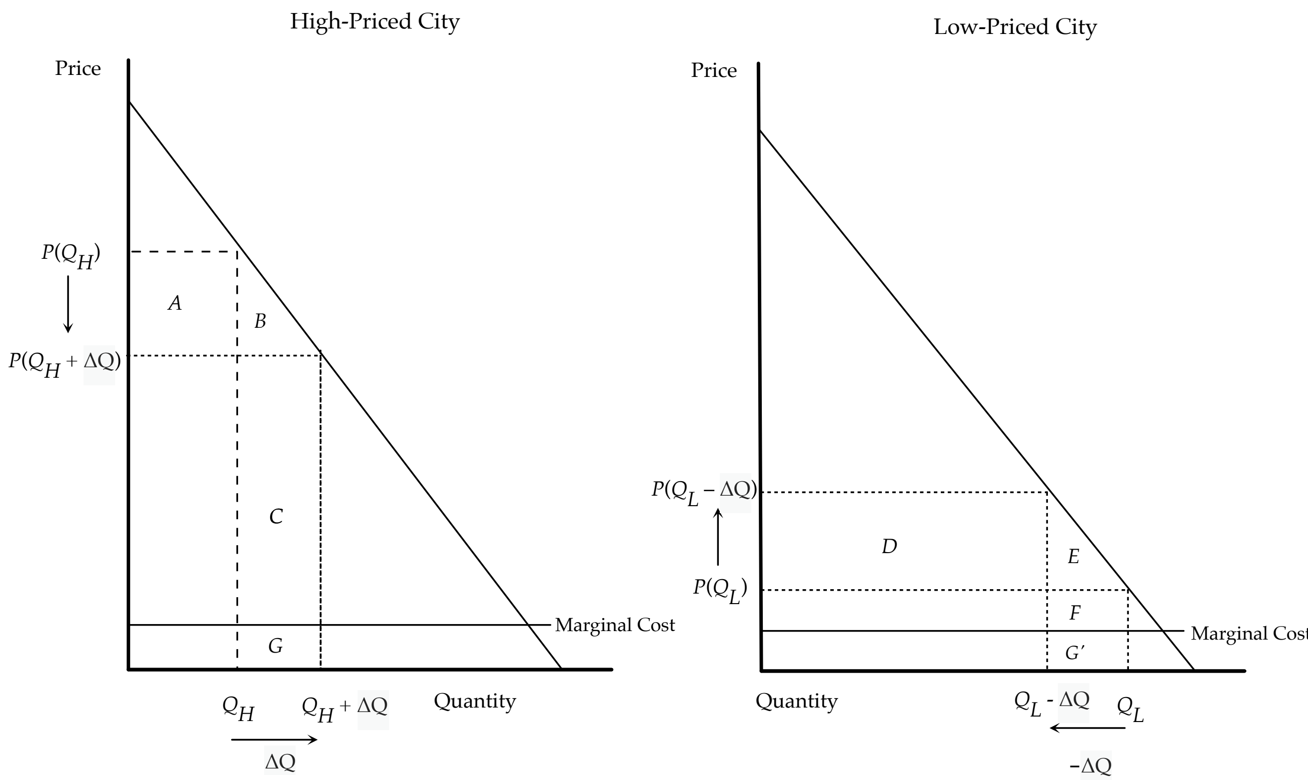
framework can easily be applied to the e-commerce context since e-commerce firms provide a platform that enables consumers in any city to purchase goods from a large number of retailers spread across Japan. If e-commerce enables local retailers in the low-priced region to make ΔQ units of sales to the high-priced region, we should expect the price in region H to fall and the price in L to rise as shown in Figure 2.1. Consumers in H will gain $(A + B)$, and sellers will gain $(C - A)$, yielding a net gain of $(B + C)$. Similarly, in region L, consumers will *lose* $(D + E)$ and sellers will gain $(D - F)$, yielding a net loss of $(E + F)$. Overall, the welfare gain is $(B + C) - (E + F)$, which will necessarily be positive in the case of linear demands with equal slopes as long as the price in H is at least as large as the price in the region L after arbitrage (i.e., $P(Q_H + \Delta Q) \geq P\left(Q_L - \Delta Q\right)$). One can also see this condition holds in the figure because both trapezoids $(B + C)$ and $(E + F)$ have identical bases and differ only in the heights of their parallel sides.

Jensen (2007) considered a case in which the marginal cost of supplying a market is zero, which enabled him to compute the lengths of the parallel sides of the quasi-trapezoids by just using the prices. When thinking about production more generally, however, marginal costs are likely to be positive, so technically we should subtract marginal costs from prices when computing the lengths of the parallel sides of the quasi-trapezoids. However, as one can see from Figure2.1, if we assume constant and equal marginal costs of production, then $G = G'$, and we can still compute the welfare gain as $(B + C + G) - \left(E + F + G'\right) = (B + C) - (E + F)$.[8]

In order to compute the welfare gain, we need to compute the price change associated

[8]The assumption of equal marginal costs is probably not extreme for Japan given the small physical size of the country (most major cities are within a few hours drive of Tokyo), which means that transport costs are unlikely to produce large price differences across cities.

Figure 2.1: Welfare Gains from Arbitrage in the Jensen Model



with the arbitrage opportunity associated with e-commerce. We begin by writing the change in welfare due to the price change of good i in city c over k periods as

$$\Delta W_{ict} = \frac{1}{2}\left(2P_{ic,t-1} + \Delta P_{ict}\right)\Delta Q_{ic,t} - m\Delta Q_{ic,t}, \tag{2.4}$$

where m is the marginal cost of producing the good. Without loss of generality we can decompose prices and quantities into two components: a national component that captures national movements in the price of good i (ΔP_{it}), a city-specific component that captures relative movements in prices in that city (ΔP^R_{ict}):

$$\Delta P_{ict} = \Delta P_{it} + \Delta P^R_{ict}. \tag{2.5}$$

Let the total quantity demanded be denoted by $Q_{it} \equiv \sum_c Q_{ict}$. We can now decompose quantity movements (ΔQ_{ict}) into a national component ($\Delta Q^N_{ict} \equiv (Q_{ic,t}/Q_{i,t})\,\Delta Q_{it}$), which tells us how consumption in the city would have moved if it followed the national trend, and a city-specific ($\Delta Q^R_{ict} \equiv \Delta Q_{ic,t} - \Delta Q^N_{ic,t}$) component. This lets us rewrite equation (2.4) as

$$\Delta W_{ict} = \frac{1}{2}\left(2P_{ic,t-1} + \Delta P_{it} + \Delta P^R_{ict}\right)\left(\Delta Q^N_{ict} + \Delta Q^R_{ict}\right) - m\Delta Q_{ic,t-1} \tag{2.6}$$

Rearranging terms produces

$$\begin{aligned}
\Delta W_{ict} &= \frac{\left(2P_{ic,t-1} + \Delta P_{it} + \Delta P^R_{ict}\right)\Delta Q^N_{ict}}{2} + \frac{\left(2P_{ic,t-1} + \Delta P_{it} + \Delta P^R_{ict}\right)\Delta Q^R_{ict}}{2} - m\Delta Q_{ict} \\
&= \frac{\left(2P_{ic,t-1} + \Delta P_{it}\right)\Delta Q^N_{ict}}{2} + \frac{\left(2P_{ic,t-1} + \Delta P^R_{ict}\right)\Delta Q^R_{ict}}{2} \\
&\quad + \frac{\Delta P^R_{ict}\Delta Q^N_{ict}}{2} + \frac{\Delta P_{it}\Delta Q^R_{ict}}{2} - m\Delta Q_{ict} \\
&= \left(\Delta W^N_{ict} - m\Delta Q^N_{ict}\right) + \left(\Delta W^R_{ict} - m\Delta Q^R_{ict}\right) + \frac{\Delta P^R_{ict}\Delta Q^N_{ict}}{2} + \frac{\Delta P_{it}\Delta Q^R_{ict}}{2}
\end{aligned} \tag{2.7}$$

In other words, the change in welfare in a city can be decomposed into a term that depends on how the national change in prices and quantities affected welfare, a second term that depends on how price movements in that city relative to the national average affected welfare, and two second-order terms that capture the fact that a relative price decline matters more for welfare if it occurs for a good that is on average growing in consumption and one that captures the fact that a national drop in prices raises welfare more if local demand is also rising. Our focus will be on the aggregate gains arising from the second term ($\sum_c \Delta W^R_{ict} {=} \sum_c \Delta W^R_{ict} - m\sum_c \Delta Q^R_{ict}$), which captures the first-order impact of relative price movements on welfare.

Defining the terms this way lets us write $\Delta Q^R_{ict}/Q_{ict} = \Delta Q_{ict}/Q_{ict} - \Delta Q_{it}/Q_{it}$, which is more convenient to write as a log approximation given by $\Delta q^R_{ict} \equiv \Delta q_{ict} - \Delta q_{it}$. If we assume that there is no regional variation in demand elasticities ($\eta_c = \eta \ \forall c$), we then have

$$\Delta q_{ict}^R \equiv \Delta q_{ict} - \Delta q_{it} = -\eta \left(\Delta p_{ict} - \Delta p_{it} \right). \tag{2.8}$$

If we multiply this expression by $Q_{ic,t-5}$ and sum across all cities we obtain an expression for the aggregate change in quantity due to relative price movements across the cities:

$$\sum_c \Delta Q_{ict}^R \equiv \sum_c Q_{ic,t-1} \Delta q_{ict}^R = -\eta \left(\sum_c Q_{ic,t-1} \Delta p_{ict} - \sum_c Q_{ic,t-1} \Delta p_{it} \right) = 0, \tag{2.9}$$

where the last equality follows from our assumption (which is the same as that of Jensen (2007)) that relative price changes arising from new arbitrage opportunities do not affect aggregate demand for the good. This expression lets us solve for the expression for national prices:

$$\Delta p_{it} = \frac{\sum_c Q_{ic,t-1} \Delta p_{ict}}{\sum_c Q_{ic,t-1}} \tag{2.10}$$

In other words, the log-change in national price index is just a quantity-weighted average of the log price change in each city. We now can conduct our counterfactual welfare analysis. Based on equation (2.1), we can write the best estimate of price change:

$$\widehat{\Delta p}_{ict} = \hat{\alpha}_{it} + \hat{\beta}_{ct} + \left(\hat{\gamma} + \hat{\delta}_1 x_i + \hat{\delta}_2 D_t x_i \right) p_{ic,t-1}. \tag{2.11}$$

Therefore, the aggregate price change for any good is

$$\widehat{\Delta p}_{it} = \hat{\alpha}_{it} + \sum_c \frac{Q_{ic,t-1}\hat{\beta}_{ct}}{\sum_c Q_{ic,t-1}} + \left(\hat{\gamma} + \hat{\delta}_1 x_i + \hat{\delta}_2 D_t x_i\right) \sum_c \frac{Q_{ic,t-1} p_{ic,t-1}}{\sum_c Q_{ic,t-1}}, \tag{2.12}$$

The evolution of prices in each city relative to the national price increase can be expressed as the difference between equations (2.11) and (2.12):

$$\widehat{\Delta p}^R_{ict}(D_t) = \left[\hat{\beta}_{ct} - \sum_c \frac{Q_{ic,t-1}\hat{\beta}_{ct}}{\sum_c Q_{ic,t-1}}\right] + \left(\hat{\gamma} + \hat{\delta}_1 x_i + \hat{\delta}_2 D_t x_i\right) \left[p_{ic,t-k} - \sum_c \frac{Q_{ic,t-1} p_{ic,t-1}}{\sum_c Q_{ic,t-1}}\right]. \tag{2.13}$$

We now can write the impact of a price change on welfare in a city as

$$\Delta W^R_{ict}(D_t) = \frac{-\eta}{2}\left[2P_{ic,t-k} + P_{ic,t-k}\widehat{\Delta p}^R_{ict}(D_t)\right] Q_{ic,t-k}\widehat{\Delta p}^R_{ict}(D_t) \tag{2.14}$$

The national gain in welfare can then be written as:

$$\widehat{\Delta W}^R_t(D_t) \equiv \sum_c \sum_i \widehat{\Delta W}^R_{ict}(D_t). \tag{2.15}$$

The real gain due to enhanced arbitrage from e-retail can then be written as:

$$\widehat{\Delta W}^E_t \equiv \left[\widehat{\Delta W}^R_t(D_t = 1) - \widehat{\Delta W}^R_t(D_t = 0)\right]/P_t, \tag{2.16}$$

where P_t is the price level in year t. Note that this gain only applies to any particular period. If we want to compute the aggregate gain over more than one period, we can calculate the cumulative value of these gains between t and T:

$$\widehat{\Delta W}_T^E = \sum_{s=t}^{T} \widehat{\Delta W}_s^R. \tag{2.17}$$

2.3 Data

A major challenge in prior research is the lack of a product-wise measure of e-retail sales. Thus, prior work has not been able to observe the value of e-retail sales and instead uses online expenditure share of each city's total retail sale or . Fortunately, we make use of two datasets providing a product-wise measure of e-retail sales in Japan: the National Survey of Family Income and Expenditures (NSFIE) and Rakuten sales data.

The NSFIE is a representative survey of households that report the purchase channels by product of Japanese households:small retail store, supermarket, convenience store, department store, co-op purchasing, discount store, catalog, Internet, and others. Starting in 2004, the NSFIE also began a quinquennial recording the expenditure share of each product from online merchants.

Rakuten, the largest e-retail company in Japan, provided us with 2010 internet sales data (aggregated across buyers and merchants) for each of their narrowest product categories covering the universe of transactions on their platform. In that year, Rakuten had a 30 percent market share of all Japanese e-commerce, so we believe that its transactions are likely to be representative of the overall internet sales intensity of goods.[9] When uploading the product information on to the Rakuten platform, sellers designate whether their product falls into one of approximately 40,000 product categories or "genres". We then matched these

[9]Rakuten, Inc. (2010) *Annual Report.*

genres to the expenditure categories in the 2010 Japanese Family Income and Expenditure Survey (FIES) and the categories that appear in the Japanese consumer price index. This generated a matched sample in which we have 312 tradable goods in a typical year, which we use in our main specifications, and of 364 expenditure items that we use to characterize all Japanese expenditures.

We construct e-commerce intensity of an expenditure category by comparing average household's total expenditure on that category with average household's online expenditure on it. We measure total expenditure e_i on category i by using national average expenditures per household in 2009 taken from FIES. We denote online expenditure in category i from NSFIE by s_i.[10] We then define e-commerce intensity x_i of category i by taking the ratio of the online to total expenditures, normalized by the maximum value of this ratio:

$$x_i = \frac{s_i}{e_i} / \max_j \left(\frac{s_j}{e_j} \right).$$

In order to see how e-commerce intensity varies across products, we aggregated the FIES codes into some broader categories in Table 2.1 so that we could display the data in a compact form. As most of services are not available online, we will focus on e-commerce's impact on goods prices for all of our main results. The rows are ordered by a category's share of Japanese expenditures on goods. The first column of Table 2.1 reports the percentage of expenditures in category ℓ among goods in 2009 as reported in the FIES ($E_\ell \equiv \sum_{i \in \Omega^\ell} e_i / \sum_j e_j \times 100$), where Ω^ℓ is the set of items in some more aggregated category ℓ. In the second column, we report the percentage of online expenditure in 2009 that corre-

[10]We compute s_i from Rakuten by summing gross merchant sales across all genres contained in the set of genres in category i (Ω^i), i.e., $s_i \equiv \sum_{j \in \Omega^i} s_j$.

sponds to that category (($S_\ell \equiv \sum_{i \in \Omega^\ell} s_i / \sum_j s_j \times 100$), where s_i is online expenditure from NSFIE). The fourth column reports the "e-commerce intensity" in 2009, which we define to be the ratio of the two previous columns divided it by the maximum value of S_ℓ / E_ℓ (i.e., $x_i \equiv S_i/E_i/\left[\max_j \{S_j/E_j\}\right]$). Thus, our measure of e-commerce intensity takes on a value of zero if there are no transactions involving an expenditure category and a value of 1 if the online expenditure relative to those in the economy is the highest among all categories of goods. Expressing e-commerce intensity this way makes our e-commerce intensity (x_i) invariant to the size of sector i.

Table 2.1: Internet intensity of consumer expenditure on goods

Category	Share of Total Expenditure 2009	Share of Internet Expenditure 2009	E-commerce Intensity 2004	E-commerce Intensity 2009	E-commerce Intensity 2014	E-commerce Intensity Rakuten 2010
Fruits and vegetables	10.24	1.76	0.03	0.01	0.05	0.03
Household consumables	10.19	18.00	0.36	0.15	0.28	0.58
Clothing	9.61	13.45	0.28	0.11	0.22	0.42
Store-bought cooked food	7.62	1.10	0.03	0.03	0.04	0.03
Cereal	6.21	1.54	0.05	0.02	0.05	0.07
Fish and shellfish	6.13	1.40	0.05	0.02	0.05	0.04
Cakes and candies	5.72	1.62	0.06	0.03	0.04	0.08
Meat	5.55	0.73	0.03	0.01	0.04	0.03
Recreational goods	4.65	12.71	0.55	0.30	0.47	0.93
Household applicances	4.05	6.32	0.31	0.21	0.36	0.35
Electronics	3.88	19.32	1.00	1.00	1.00	0.53
Alcoholic beverages	3.36	1.32	0.08	0.05	0.10	0.26
Medicine and nutritional supplements	3.35	4.85	0.29	0.23	0.31	0.23
Non-alcoholic beverages	3.17	2.20	0.14	0.09	0.15	0.16
Oils, fats and seasonings	3.11	0.73	0.05	0.02	0.07	0.05
Newspapers and magazines	2.96	0.00	0.00	0.00	0.00	0.00
Dairy products and eggs	2.81	0.29	0.02	0.01	0.04	0.01
Transportation equiment	2.14	3.01	0.28	0.23	0.18	0.58
Domestic utensils	2.06	4.04	0.39	0.14	0.49	0.53
Furniture and furnishings	1.78	3.45	0.39	0.33	0.51	1.00
Footwear	1.40	2.13	0.30	0.14	0.28	0.92
Total/Mean	100.00	100.00	0.22	0.15	0.23	0.32

Notes: Shares are expressed as percentages. Internet intensity is calculated as Data s ource: FIES, NSFIE, Rakuten, and authors' calculation. The first column is from FIES. Column 2- 5 are from NSFIE and the last column is from Rakuten.

Table 2.1 makes clear some basic stylized facts of our data. First, within goods categories we see that there are no zeros in the table indicating that at this level of aggregation all categories of goods were available online in in 2009. Second, there is enormous variation in the e-retail intensity. Some of this reflects the fact that highly perishable, non-standardized items (e.g. fresh foods), restricted/time-sensitive items (e.g., medicine and physical newspapers), and high weight-to-value items (non-perishable groceries) are not sold much online. At the other end of the spectrum, we see that more standardized goods—e.g., electronics, books, clothing, footwear, and furniture and furnishings—are sold very intensively online. Interestingly, we see that domestic utensils, household consumables (which includes non-durable household supplies like paper products and cleaning agents), and recreational goods (which includes items like sports equipment and gardening supplies) are sold very intensively online as well. In other words, goods sold intensively online tend have characteristics that are similar to those goods historically available in catalogs—i.e., goods that are non-perishable, low weight-to-value, standardized, and storable.Third, e-commerce intensity from NSFIE in 2009 is highly correlated with that from Rakuten in 2010.

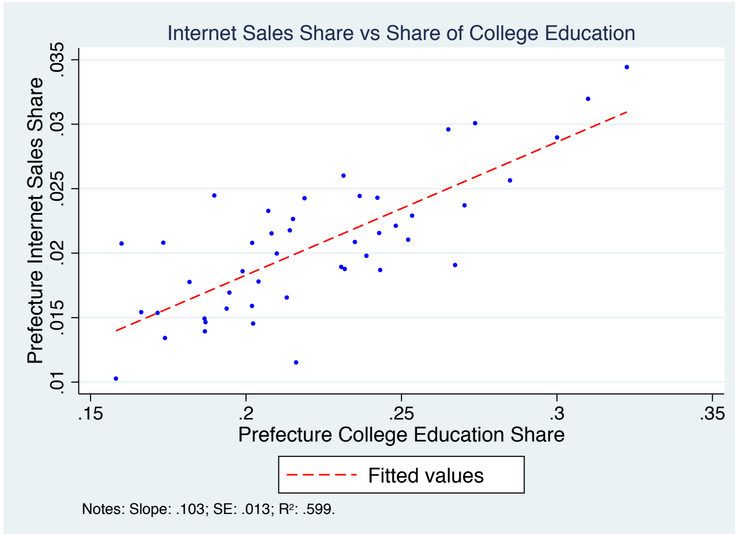
There is an important difference between *aggregate* e-commerce sales shares point-of-sale data reported by Japan's Ministry of Economy, Trade and Industry (METI) and the numbers reported in the NSFIE. The difference is likely to due to the way in which the data are collected. METI bases its e-commerce sales shares by conducting a survey of the major e-commerce firms and dividing e-commerce sales by the sales of of all Japanese manufacturers. The NSFIE, however, is based on household surveys and suffers from the problem that a growing share of households have been not answering questions related to where they made their purchases. In particular, in 1999 only 7 percent of expenditures were from undisclosed locations, but by 2014, this number had risen to 13 percent. By contrast, the share of e-commerce sales in the NSFIE is rising over time, but the the level in 2014 was only 2.4 percent of aggregate expenditures, which is about half the METI number and implausibly small given the magnitude of Rakuten sales. After discussing the discrepancies with people knowledgeable about Japanese statistics, we came to the conclusion that an important part of the discrepancy is due to households that buy goods online not reporting the sales channel and the purchase channel being left blank.[11] We therefore use the METI data to identify aggregate internet sales shares and the NSFIE and Rakuten data for understanding the *distribution* of internet sales shares.

The NSFIE data also enables us to examine e-commerce intensity by location for the 47 prefectures of Japan. As one can see from Figure 2.2, there is a very strong association between the share of prefectures online expenditures and the share of a prefecture's population over the age of 15 that went to college. This variable alone explains 60 percent of the variance across prefectures. Interestingly, when we tried to control for other prefectural covariates that we thought might be associated with e-commerce expenditure shares—per capita income, population (urbanization), average age, or other education levels—none of these were even close to statistically or economically significant in any specification in which we also included the share of the population with a college degree. This suggests that e-commerce is largely a technology that serves highly educated people and urban-rural or old-young digital divides are secondary.[12]

[11] We thank Takashi Unayama of Hitotsubashi University for insights into this issue.

[12] Interestingly, Fan et al. (2018) find no link between education and internet sales intensity. One possibility for the difference in the results is that Chinese education levels in their data set are much lower than in our data. The average number of years of education in Fan et al. (2018) is only 8.8 years whereas the average in our sample of Japanese cities is 11.9 years .

Figure 2.2: Education and e-commerce Purchases



We also make use of the fact that the Japan Statistical Bureau (JSB), which produces the Japanese CPI provides detailed information on representative prices of the products in the FIES categories. These prices are sampled in all cities that are either a prefectural government or have population of 150,000 or more with an aim of not only tracking product prices across time but also across space. This information typically identifies the brand of an item or a detailed description (e.g., "Big-eyed tuna, sliced (for sashimi), lean, 100g"). While the data is not sufficiently detailed to always pin down the exact barcode, the data leaves limited scope for unobserved quality differences to affect intercity price differentials. For example, Imai et al. (2012) find that it is sufficiently detailed to rule out approximately 85 percent of all bar codes in a CPI product category. Moreover, since the objective of the JSB sampling is to make meaningful intercity price comparisons, there is a tendency to select the same products by, for example always picking the largest selling item within a sampling frame if available. Thus, while US CPI data typically is based on different baskets of goods in different cities, the JSB's "purposive" sampling generate samples in which the same good or very similar goods are sampled in different cities. Therefore, it is reasonable to believe that intercity prices are informative about true price differences across locations.[13] One problem in the data is that we have periodic product substitutions that arise as goods are added to or dropped from the CPI sample. Fortunately, we have official quality-adjusted price quotes for Tokyo computed by the JSB[14], which we use to adjust the prices in other cities. This procedure is equivalent to assuming that the quality change associated with a product substitution in the CPI is identical across cities.

[13]In order to further clean the data, we drop all observations in which the item only appears in one city. We also trimmed 3 smallest and 3 largest price quotes within an item-year observation. Finally, we dropped the bottom and top 1% of log price changes.

[14]http://www.e-stat.go.jp/SG1/estat/List.do?bid=000001033703&cycode=0, accessed on April 5th, 2017.

Table 2.2: Summary Statistics for the Sample of Goods

	Mean	Standard Deviation	Min	p10	p50	p90	Max
Period: 1992 to 1996							
$\Delta^1 p_{ict}$	-0.004	0.099	-0.858	-0.102	0.000	0.089	0.953
$x_{i(t=2009)}$	0.057	0.073	0.000	0.004	0.022	0.157	0.456
$x_{i(t=2009)} \times p_{ic(t-1)}$	0.455	0.679	0.000	0.018	0.136	1.337	4.107
Observations	74,992						
Period: 1997 to 2001							
$\Delta^1 p_{ict}$	-0.008	0.106	-1.124	-0.117	-0.001	0.084	1.165
$x_{i(t=2009)}$	0.053	0.070	0.000	0.004	0.022	0.151	0.456
$x_{i(t=2009)} \times p_{ic(t-1)}$	0.415	0.645	0.000	0.018	0.123	1.212	4.075
Observations	109,853						
Period: 2002 to 2006							
$\Delta^1 p_{ict}$	-0.010	0.115	-1.798	-0.129	-0.003	0.104	1.679
$x_{i(t=2009)}$	0.052	0.074	0.000	0.004	0.022	0.146	1.000
$x_{i(t=2009)} \times p_{ic(t-1)}$	0.399	0.689	0.000	0.020	0.113	1.165	11.690
Observations	164,011						
Period: 2007 to 2011							
$\Delta^1 p_{ict}$	-0.001	0.125	-1.695	-0.129	0.000	0.126	1.556
$x_{i(t=2009)}$	0.053	0.081	0.000	0.004	0.020	0.146	1.000
$x_{i(t=2009)} \times p_{ic(t-1)}$	0.410	0.750	0.000	0.019	0.112	1.195	10.684
Observations	164,531						
Period: 2012 to 2016							
$\Delta^1 p_{ict}$	0.014	0.100	-1.276	-0.085	0.010	0.121	1.092
$x_{i(t=2009)}$	0.052	0.080	0.000	0.004	0.020	0.141	1.000
$x_{i(t=2009)} \times p_{ic(t-1)}$	0.398	0.724	0.000	0.019	0.109	1.173	10.300
Observations	168,592						

Notes: Prices are in natural log. Δp_{ict} is the one-year log difference in prices; $x_i = \frac{s_i/e_i}{\max_j(s_j/e_j)}$.

Table 2.2 reports the sample statistics for our data. As one can see from the table, we have more than 100,000 price quotes in any of our five-year periods since e-retail has become available in Japan in 1997. The first line of the table shows the average annual rate of inflation across the sample period. As one can see, on average goods prices fell before 2011, which reflects the deflation that can be observed in Japan over this time period. The second line reports information on the e-commerce intensity of the goods in our sample (x_i). The values of x_i across goods tell us about the relative importance of online sales. Here we see that goods in the the upper 90^{th} percentile of the distribution have an e-retail intensity of 0.146 over the full sample period, which is more than seven times higher than a good with the median intensity. Moreover, at the upper tail of the distribution, we observe goods with an e-commerce intensity that is more than 50 times higher than that of the median good. These summary statistics reflect the skewness in the distribution of e-retail sales intensity that we saw in Table 2.1. Some goods are sold very intensively online, but most goods are purchased predominantly in physical stores.

One concern about our data is that it is not at the barcode level. This might create problems if the JSB is sampling goods of very different quality across cities. Hottman et al. (2016) show that the correlation between price and quality in bar-code data is 0.9, so we should expect sampling problems to produce high levels of price dispersion in our sample. We can check whether we have excessive price dispersion (caused by including goods of very different quality within categories) by computing the the price of each good in each city less the average price of that good across all cities and then taking the standard deviation of this difference. When we do this in the third row of each panel, we find that intercity price differences for the same good in Japan is about 15 percent. By contrast, Broda and Weinstein (2008) find the standard deviation in intercity prices of bar-coded goods is 22 percent in the US and 19 percent for Canadian provinces. The fact that intercity price dispersion of goods in the Japanese CPI is lower than that for bar-coded goods in the US and Canada suggests that the JSB is probably not including goods that differ substantially in quality when sampling goods in different cities and therefore that quality variation across cities for the same product is unlikely to be a major problem in our data.

2.4 Estimation

In Section 2.4.3, we first do some data exploration to first show that price convergence is a central tendency in the data and that there is visual evidence that the internet changed the rate of convergence. We also show that there was no important change in the overall rates of price change for goods available online. This provides some prima facie evidence that our focus on relative intercity price movements of goods sold by e-retailers as opposed to absolute price declines of online goods is in line with the data. We next estimate the impact of e-retail on the rate of price convergence in Section 2.4.3.1. Finally, in Section 2.4.3.2, we present our estimates of the welfare gain from e-retail.

2.4.1 E-commerce and National Prices

Goolsbee and Klenow (2018) have explored whether we see any differences in rates of inflation for goods available online. Here, we build on this work to show that these differential rates of inflation only appeared after the entry of e-commerce firms. In order to examine this in the data, we regress annual log price changes of goods (Δp_{ict}) on good (i) and city (c) fixed effects along with an indicator variable, D_t, that is one starting in 1997 (the year Rakuten opened) and zero before as well as the internet sales intensity of the good interacted with this dummy ($x_i D_t$):

$$\Delta p_{ict} = \alpha_i + \beta_c + \phi D_t + \theta x_i D_t + \epsilon_{ict}, \tag{2.18}$$

The coefficient on D_t (ϕ) tells us whether there was any differential trend in price inflation for goods available online after the entrance of e-commerce firms and θ, the coefficient on the e-commerce intensity interaction term ($x_i D_t$) tells us about the differential rate of price change for goods traded online after the entry of e-commerce firms. We do this for two time periods (1992-2001) and (1992-2016) to see if there is any difference in the results we obtain by looking at the period immediately after the entry of e-commerce firms versus the full time period.

Table 2.3: Relative Price Changes

	(1) $\Delta^1 p_{ict}$	(2) $\Delta^1 p_{ict}$	(3) $\Delta^1 p_{ict}$	(4) $\Delta^1 p_{ict}$	(5) $\Delta^1 p_{ict}$	(6) $\Delta^1 p_{ict}$	(7) $\Delta^1 p_{ict}$
D_t	-0.0024***	-0.0003	-0.0003	0.0069***	0.0090***	0.0090***	0.0093***
	(0.0006)	(0.0006)	(0.0006)	(0.0005)	(0.0006)	(0.0006)	(0.0005)
E-commerce Intensity	0.0229***	-0.0060	-0.0058	-0.0583***	-0.0860***	-0.0859***	-0.0882***
$\times D_t$	(0.0048)	(0.0068)	(0.0068)	(0.0026)	(0.0059)	(0.0059)	(0.0059)
Constant	-0.0040***	-0.0119***	-0.0094**	-0.0041***	-0.0175***	-0.0172***	-0.0175***
	(0.0004)	(0.0040)	(0.0045)	(0.0004)	(0.0027)	(0.0030)	(0.0030)
Sample	Goods	Goods	Goods	Goods	Goods	Goods	Goods and Service
Fixed Effects		Product	Product and City		Product	Product and City	Product and City
t	1992-2001	1992-2001	1992-2001	1992-2016	1992-2016	1992-2016	1992-2016
Observations	152,958	152,958	152,958	394,663	394,663	394,663	397,670
R^2	0.000	0.040	0.040	0.001	0.038	0.039	0.038
E-commerce intensity year	2009	2009	2009	2009	2009	2009	2009

Standard errors in parentheses
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Data source: RPS, NSFIE, and authors' calculation. First three columns use periods from 1992 - 20001 immediately after the entry of e-commerce firms and column (4) - (7) use the full-time period.

We present these results in Table 2.3. Overall, the constant term is negative, which reflects the fact that Japan experienced deflation over much of this time period. In all the regressions with full time period, we see a significant negative coefficient on the post-e-commerce e-commerce intensity interaction ($x_i D_t$). Even more interesting is the fact that this effect becomes statistically significant and larger once we extend the sample to 2016. In terms of economic magnitudes, the results in column 6 imply that a good at the 90^{th} percentile of internet sales intensity had an inflation rates that fell by 1 percentage points per year faster after the entry of e-commerce firms. This is consistent with results in Goolsbee and Klenow (2018) who found that goods traded online have lower inflation rates than goods not available online. However, our results differ in that we show that this difference is observable in the prices in physical stores and that these differential price trends did not exist prior to the entry of e-commerce firms. These results suggest that while there may be some evidence that better logistics, reduced markups, and other factors associated with e-retail might have affected price increases of goods available online.

We can also use these estimates to compute the implied gain from the differential impact on prices estimated in Table 2.3. If we assume that the entry of e-commerce did not affect the price level in sectors with no e-commerce sales, i.e., we ignore general equilibrium effects, we can obtain an estimate of e-commerce on the price level by computing

$$\Delta p_t^N \equiv \hat{\theta} \sum_i x_i \frac{e_{i,t-1}}{\sum_i e_{i,t-1}} \tag{2.19}$$

where $\hat{\theta}$ is the estimated coefficient on the internet intensity interaction term in equation (2.18). Based on the specification in column 7, we estimate that e-commerce lowered the Japanese price level by 0.05 percent per year or about 1 percentage point between 1997 and 2016 due to reductions in markups and efficiency gains. Converting this number to welfare depends on whether one thinks the drop in the Japanese price level was just due to a reduction in retail markups (in which case most of the gain was a transfer from producers to consumers) or a rise in efficiency in which case it was a pure welfare gain.

2.4.2 Gains in "New Trade Models"

Although Brynjolfsson et al. (2003) computed the variety gains from internet book sales, we know of no study that has done this generally. Fortunately, this is trivial to do with our data. Arkolakis et al. (2012) show that in the Krugman (1980), Melitz (2003), and Eaton and Kortum (2002) models, the log change in welfare following a trade liberalization equals $\frac{1}{\epsilon}\ln\lambda$, where $\lambda \in [0,1)$ is the share of consumer of expenditures on sales from retailers other than e-commerce firms, and ϵ equals the "trade elasticity." The trade elasticity has a different interpretation in different models. For example, in the Krugman (1980) model it equals one minus the elasticity of demand whereas in the Eaton and Kortum (2002) model, it equals the negative of the Fréchet shape parameter. However, in order to make progress, we will use a standard estimate for the trade elasticity of -5.

The computation of the share of household purchases that was made from sources other than e-retailers and be computed straightforwardly. Based on the NSFIE data, we know that the share of household expenditures purchased from all retailers (χ) was 0.62 in 2014, with the remaining expenditures covering utilities, education, and other expenditure items that we will assume do not represent e-commerce. If we couple this with the share of expenditures in 2014 from all e-retailers, $s = 0.0437$, we we have $\lambda = (1-s)\,\chi + 1 - \chi = 0.97$. This gives us an estimate of the welfare gain from e-commerce in Japan in 2014 of 0.6 percent. We can also easily do some robustness checks for different values of the trade elasticity. Allowing the trade elasticity to range -3 to -7 results in welfare gains that range from 0.4 percent to 0.9

percent. Similarly, if we had computed the Japanese gains achieved in 2017, we would obtain a number of 0.7 percent. Thus, our point estimate for the welfare gains these e-commerce gains is 0.6, with reasonable estimates ranging from 0.4 percent to 0.9 percent.

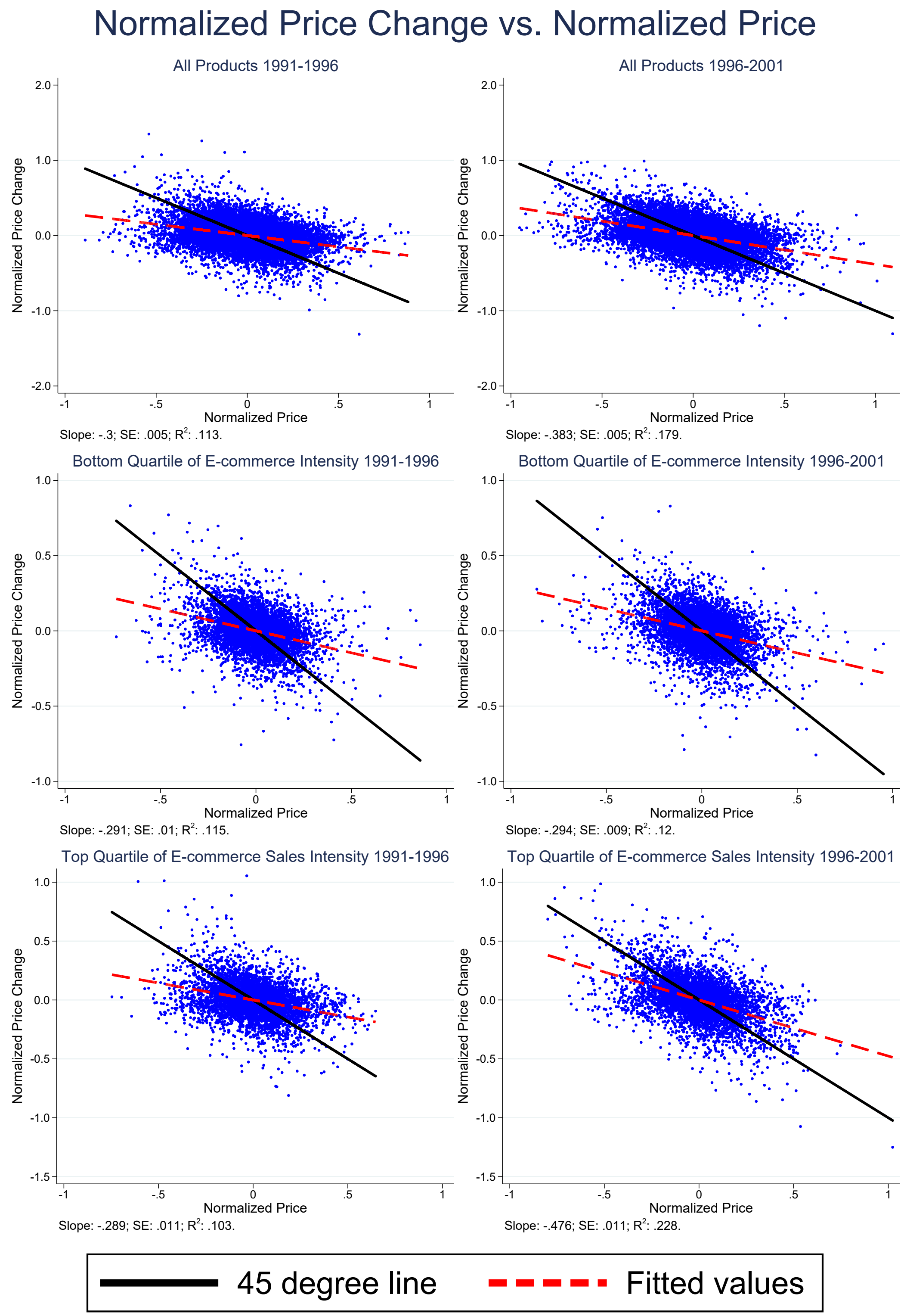
2.4.3 Gains Due to Price Arbitrage

In order to visualize what is going to drive our results, we first consider two five-year periods. The first five-year period (1991-1996), pre-dates the formation of e-commerce by at least a year, so we can call this period the "pre-e-commerce period." We start the second period in 1996 because we assume that in 1996, the distribution of prices was reflective of a world without e-commerce but by 2001, Rakuten was already a prominent, listed company, with tens of millions of hits and thousands of stores selling on its platform.

It is difficult to compare price changes across goods and cities in their raw form because different goods exhibit different average price changes in different years. We therefore normalized the data by regressing Δp_{ict} and p_{ict} on product and city fixed effects and construct normalized price changes ($\Delta^5 p_{ict} - \hat{\alpha}_{it} - \hat{\beta}_{ct}$) and normalized price levels ($p_{ic,t-5} - \hat{\alpha}'_{it} - \hat{\beta}'_{ct}$), where $\hat{\alpha}_{it}$ ($\hat{\alpha}'_{it}$) and $\hat{\beta}_{ct}$ ($\hat{\beta}'_{ct}$) are the estimated fixed effects from the regression of Δp_{ict} (p_{ict}) on product and city fixed effects. Thus, these normalized prices remove the effect of any common price movements at the product or city level. Figure 2.3 presents plots of normalized five-year change in prices ($\Delta^5 p_{ict} - \hat{\alpha}_{it} - \hat{\beta}_{ct}$) against the normalized five-year lag of prices in each city ($p_{ic,t-5} - \hat{\alpha}'_{it} - \hat{\beta}'_{ct}$).

The first panel shows how normalized price changes vary with normalized prices before and after the entry of e-commerce. There is a clear negative relationship between initial

Figure 2.3: Normalized Price Change vs. Normalized Price



Data source: RPS, NSFIE, and authors' calculation. Notes: This graph plots normalized price changes against normalized price levels. The left panel shows normalized price changes before the entry of e-commerce and the right panel shows them after the entry of e-commerce. The first panel plots for all goods, the second panel plots for goods with e-commerce intensity lower than the bottom quartlie, and the third panel shows for gods with e-commerce intensity higher than the top quartile.

price deviations and future price growth, which indicates that goods that had high prices in cities tended to have lower rates of inflation than goods with low prices. This mean reversion in the data is the convergence that is likely to be product of price arbitrage. As one can see from these two plots, there is the average rate of price unconditional rate price convergence is about 30 percent and rose to 39 percent in the four-years after the entry of e-commerce. These plots also speak to the relatively high quality of the Japanese data. For example, studies using U.S. data (c.f., Parsley and Wei (1996)) find no evidence of price convergence once one controlled for city fixed effects. One plausible reason for the weaker evidence of price convergence in the U.S. is that that the American Chamber of Commerce Researchers Association data used in this study is not based on purposive sampling, so price changes in cities are based on a changing mix goods of different qualities across locations (as shown in Handbury and Weinstein (2015)).

The next two pictures show what was driving this increase in the intercity rate of price convergence. Here, we divide the sample into the set of goods with an internet sales intensity in the lowest first quartile of the distribution in 2009 ($x_i < 0.076$) and the set of goods in the highest quartile of the distribution ($x_i > 0.013$). As one can see from the second panel in Figure 2.3, there was almost no change in the rate of convergence for goods not sold on the e-commerce. The slope of the line for goods not available online in the early period is -0.29, which is almost identical to the slope in the pre-e-commerce period (-0.30) and the slope in the post-e-commerce period for goods not traded much online (-0.29). In other words, the entry of e-commerce firms did not seem to have had much of an effect on goods not traded on e-commerce. However, we see a very different pattern for goods with an internet intensity in the upper quartile of the distribution. The slope steepens by 66 percent, rising in magnitude from -0.29 to -0.48. In other words, Figure 2.3 provides the visual version of the result we will explore econometrically in Section 2.2.1—following the entry of e-commerce firms, the rate of price convergence for goods traded intensively rose substantially.

2.4.3.1 Estimating Convergence Rates

We now turn to obtaining estimates of the impact of Rakuten on the rate of intercity price convergence. Table 2.4 presents the results of estimating equation (2.1) for five- and one-year intervals. In the first column, we present separate regressions for a specification analogous to what we showed in the plots. In the first two columns, we present separate regressions for 1996 and 2001 where we let the convergence rates vary across the two time periods. Comparing the first rows of columns 1 and 2 reveals the convergence rates for goods not suitable for e-commerce (i.e., those where $x_i = 0$) were almost identical before and after the entry of e-commerce. The coefficient on e-commerce intensity interacted with lagged prices ($x_i p_{ic,t-5}$) in column 1 indicates that the rate of convergence for goods suitable for e-commerce sales was not significantly different than the convergence rate of other goods prior to to the entry of e-commerce. However, the negative and significant coefficient on the interaction term ($D_t x_i p_{ic,t-5}$) in the post-e-commerce sample (where we dropped the $x_i p_{ic,t-5}$ term from the specification because we do not have any pre-e-commerce observations) indicates a significantly faster rate for goods available online after the entry of e-commerce firms. This specification confirms the basic insight about the increase in the rates of convergence of goods sold intensively online after the entry of e-commerce that we saw in Figure 2.3.

Table 2.4: Estimates Over Period 1991-2001

Dependent variable	(1) $\Delta^5 p_{ict}$	(2) $\Delta^5 p_{ict}$	(3) $\Delta^5 p_{ict}$	(4) $\Delta^1 p_{ict}$
Lagged price	-0.291***	-0.326***	-0.312***	-0.137***
	(0.007)	(0.007)	(0.005)	(0.002)
E-commerce intensity	-0.181**		-0.028	0.050**
(t=2009) × Lagged price	(0.091)		(0.084)	(0.025)
E-commerce intensity		-1.115***	-1.222***	-0.753***
× Lagged price × Post e-commerce		(0.084)	(0.097)	(0.048)
t	{1996}	{2001}	{1996,2001}	Annual 1992-2001
Observations	25,923	27,509	51,185	152,958
R^2	0.52	0.53	0.52	0.51
E-commerce intensity Year	2009	2009	2009	2009

Standard errors in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Data source: RPS, NSFIE, and authors' calculation.

In column 3, we estimate our baseline differences-in-differences specification of equation (2.1) using a five-year difference by letting t take on two values: 1996 and 2001. The most important result for our purposes is the estimate of the coefficient on the interaction term on the internet intensity coefficient. As one can see from the table, the coefficient is negative and precisely measured. Not surprisingly, the estimated coefficient on $p_{ic,t-k}$, $\hat{\gamma}$, doesn't change much, and we continue to get a negative and significant coefficient on the e-commerce intensity interaction term ($\hat{\delta}_2 = -1.222$). Interestingly, the estimate of δ_1, the differential rate of price convergence for e-commerce intensive goods flips sign and becomes small and positive, which implies that if we pool the data over the two time periods, goods suitable for e-commerce had, if anything, a slightly slower rate of price convergence than goods not available online.

One way to assess the economic significance our results is in terms of price-convergence half-lives: i.e., the amount of time it takes for half of any price differential to disappear. Although five-year differences lend themselves easily to the standard differences-in-differences specification, it is not standard to discuss half-lives in terms of five-year periods. The conventional way to estimate half-lives is to use higher frequency data. In column 4 of Table 2.4, we redo our estimation at an annual frequency. The coefficients fall in magnitude because a given percentage movement in prices over a five-year period implies a smaller movement over any single year. However, we observe the same pattern in the data. The entry of e-commerce is associated with a significant increase in inter-regional rates of price convergence.

We next compute the half-lives for three variants of our specification: one for goods not traded online ($x_i = 0$); one for goods in the 90$^{\text{th}}$ percentile of e-commerce intensity in the pre-e-commerce period ($x_i = \text{p90}, D_t = 0$); and finally one for one for goods in the 90$^{\text{th}}$ percentile of e-commerce intensity in the post-e-commerce period ($x_i = \text{p90}, D_t = 1$). For goods not sold online ($x_i = 0$), the coefficient of -0.137 on lagged prices corresponds to a half-life for price differences of 4.7 years. Similarly, we also see that prior to the entry of e-commerce firms (i.e., goods for which $x_i = \text{p90}, D_t = 0$) had a similar half-life of 5 years. Thus, prior to the entry to e-commerce firms, we see only a marginal difference in convergence rates for goods available online and off. By contrast, when Cecchetti et al. (2002) estimated this half-life using U.S. data, they found it to be nine years: almost twice as long. Similarly, Parsley and Wei (1996) found a unit root, i.e., an infinite half-life, when running a similar regression with city fixed effects. The differences in these results probably highlights the intercity comparability of the price data in Japan. However, there is a much larger impact of e-commerce on the convergence rate for Japanese goods sold online ($x_i = \text{p90}, D_t = 1$).

For these goods, the half-life fell from 5 years to 2.5 years after Rakuten's entry: a 50percent increase in the convergence rate.

Table 2.5: Estimates Over Alternative Periods

Dependent variable	(1) $\Delta^5 p_{ict}$	(2) $\Delta^5 p_{ict}$	(3) $\Delta^5 p_{ict}$	(4) $\Delta^5 p_{ict}$	(5) $\Delta^1 p_{ict}$
Lagged price	-0.391***	-0.460***	-0.389***	-0.391***	-0.161***
	(0.005)	(0.006)	(0.005)	(0.004)	(0.001)
E-commerce intensity (t=2009) × Lagged Price	0.638***	1.077***	0.477***	0.488***	0.017
	(0.089)	(0.094)	(0.088)	(0.086)	(0.017)
E-commerce intensity × Lagged price × Post e-commerce	-1.319***	-2.459***	-1.380***	-1.297***	-0.345***
	(0.102)	(0.105)	(0.097)	(0.089)	(0.015)
t	{1996,2006}	{1996,2011}	{1996,2016}	{1996,2001, 2006,2016}	Annual 1992-2016
Observations	52,017	43,388	42,683	87,818	394,663
R^2	0.55	0.60	0.64	0.61	0.46
E-commerce intensity year	2009	2009	2009	2009	2009

Standard errors in parentheses

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

Data source: RPS, NSFIE, and authors' calculation.

There are a number of potential problems with the evidence that we have just presented. The first is that we have mismeasurement in e-retail suitability (x_i). If this is correct, one should expect that measurement error in e-commerce suitability would cause us to underestimate the effects of this variable on price convergence. A second concern is that perhaps there is something special about the first five years after e-commerce entered the Japanese market. In Table 2.5, we also examine alternative time periods following the entry of e-commerce firms. In the first three columns, we do a differences in differences based comparing the five years prior to the entry of e-commerce firms (1991-1996) with three alternative non-overlapping periods: 2001-2006, 2006-2011, and 2011-2016. These results confirm that the choice of end period does not matter qualitatively for our basic finding that the rate of price convergence for goods available online rose after the entry of e-commerce firms. Finally, in the last column of Table 2.5 we repeat the estimation over the full period (1992-2016) at the annual frequency. The results are similar to the estimates over the shorter time period (1992-2001) reported in Table 2.4.

The results we have presented so far have only included a sample of goods because prices are easier to measure for goods than services and many services are hard to provide at a distance. However, e-commerce also conducts online sales activities in the services sector, so a reasonable question might be whether the results change significantly if we include all goods and services in the analysis. In Table 2.6, we repeat our analysis using the universe of all goods and services in the Japanese consumer expenditure survey. The results are quite similar to the set of results we obtained only looking at a sample of goods. The coefficient on the post-e-commerce, e-commerceintensity interaction term does not move significantly, indicating that our results are robust to the choice of sample.

Table 2.6: Robustness Check Using All Goods and Services

	(1)	(2)	(3)
Dependent variable	$\Delta^5 p_{ict}$	$\Delta^5 p_{ict}$	$\Delta^1 p_{ict}$
Lagged price	-0.312***	-0.390***	-0.161***
	(0.005)	(0.004)	(0.001)
E-commerce intensity	-0.028	0.500***	0.017
(t=2009) × Lagged price	(0.083)	(0.085)	(0.017)
E-commerce intensity	-1.220***	-1.305***	-0.345***
× Lagged price × Post e-commerce	(0.096)	(0.088)	(0.015)
t	{1996,2001}	{1996,2001 ,2006,2016}	Annual 1992-2016
Observations	51,685	88,930	397,670
R^2	0.52	0.62	0.46
E-commerce intensity year	2009	2009	2009

2.4.3.2 Welfare Gain

Aggregate consumer gains due to faster price convergence can be calculated from the equation (2.16) and (2.17). One of the interesting features of these equations is that the welfare gain is proportional to the choice of demand elasticity. Since this elasticity has been estimated in other papers, we calibrate the welfare gain using an elasticity of six, which corresponds to a trade elasticity of -5. Note that halving this elasticity would halve the welfare gain and doubling it would double the gain. In all cases, we base our estimates of the impact of e-retail on the rate of convergence on Table 2.5 column 5. All of the data has been converted into 2016 yen. The first columns shows the estimated welfare gains due to price convergence in that year and the second column gives the counterfactual welfare gain that would have occurred if price convergence for goods available online had remained at the pre-e-commerce rate (i.e., $(D_t = 0)$) as given by equation (2.15). We see that price convergence across regions during this time period led to an average welfare gain of 227.8 billion yen per year (about 2 billion dollars) for residents of our sampled cities. In the second column, we compute the counterfactual gain that would have occurred if the speed of convergence had remained at the pre-e-commerce rate. This gain is somewhat lower: only 180.2 billion yen per year. Thus, the difference between these two columns—47.7 billion yen— (approximately 433 million dollars) constitutes the average annual welfare gain for consumers in our sample of cities.

This number is difficult to interpret because it is only computed for residents in our sampled cities. To obtain a sense of how much this matters for welfare, we should deflate the number by the total amount of expenditures of our sampled households, which is reported in the fifth column of Table 2.7. We obtain an estimate of the welfare gain which equals 0.12

Table 2.7: Counterfactual Welfare Gain

Year	$\Delta\hat{W}_t^R(D_t=1)$	$\Delta\hat{W}_t^R(D_t=0)$	$\Delta\hat{W}_t^E$	Total Expenditure	Expenditure on Goods
1997	198.7	156.3	42.4	32,287	20,690
1998	190.4	149.8	40.6	32,076	20,557
1999	205.0	162.1	42.8	35,561	22,231
2000	240.5	191.0	49.4	39,809	24,781
2001	247.8	200.3	47.6	38,222	23,716
2002	216.3	172.1	44.2	38,635	23,928
2003	224.9	176.3	48.6	38,564	24,365
2004	236.4	187.4	49.1	37,897	23,995
2005	227.5	178.8	48.6	42,102	25,538
2006	240.0	189.8	50.3	41,178	25,114
2007	274.3	219.0	55.3	41,483	25,137
2008	257.7	203.4	54.2	40,850	25,073
2009	253.8	199.5	54.3	40,982	25,410
2010	251.6	197.6	54.0	43,400	27,975
2011	242.8	190.5	52.2	42,339	27,429
2012	234.0	185.5	48.5	42,849	27,548
2013	216.6	171.6	45.0	43,501	27,505
2014	203.5	160.4	43.2	42,550	27,174
2015	203.5	160.7	42.8	40,698	26,964
2016	191.6	151.6	40.0	40,210	26,809
Average	227.8	180.2	47.7	39,760	25,097

Note: Unit is in billions of yen.

percent of consumer expenditures. [15]

A concern that one might have about this counterfactual is that the level of price dispersion itself might be a function of the existence of e-retail. One simple way of seeing how important shifts in the distribution of prices are for understanding the welfare gains is to conduct the counterfactual using the pre-e-commerce period (1991-1996). The shocks to prices during this period were independent of e-retail and hence the counterfactual answers the question, "what would the gain to consumers have been given the pre-e-commerce dis-

[15] To get some sense of how large this is, we can compare the gain to Brynjolfsson et al. (2003) estimate of the gains due to Amazon's entry into U.S. book market. That paper estimated a gain of less than 1 billion dollars in 2000—only 0.015 percent of U.S. personal consumption expenditures in that year. In other words, our estimate is about eight times as large.

Table 2.8: Counterfactual Welfare Gain pre-Rakuten Period

Year	$\Delta\hat{W}_t^R(D_t=1)$	$\Delta\hat{W}_t^R(D_t=0)$	$\Delta\hat{W}_t^E$	Total Expenditure	Expenditure on Goods
1992	176.4	137.2	39.2	33,020	21,001
1993	180.6	143.7	36.9	32,613	20,745
1994	166.0	131.6	34.4	32,368	20,582
1995	170.0	133.9	36.1	32,844	21,049
1996	196.3	156.6	39.7	32,848	21,055
Average	177.8	140.6	37.2	32,738	20,886

Note: Unit is in billions of yen.

tribution of prices if e-commerce had entered in those years." We report the results from this exercise in Table 2.8. Our estimates imply that had e-commerce entered the Japanese market five years earlier, the welfare gain would have been about 37.2 billion yen per year for consumers in our sample of cities. This estimated gain is comparable to what we observed in the post-e-commerce period (47.7 billion), indicating that it doesn't make a substantial difference whether one computes the welfare gain based on price differential that existed before or after e-commerce entered the market.

2.5 Conclusion

This paper makes use of a unique Japanese data set covering hundreds of products over close to three decades to examine the impact of the internet on Japanese prices and welfare using a number of popular modeling techniques. First, we consider the possibility that e-commerce may have lowered prices by driving down the average prices of goods available online. Second, following Brynjolfsson et al. (2003) and Arkolakis et al. (2012) we compute the welfare gains due to the ability of e-commerce to enable consumers to purchase goods from other regions. Third, following Jensen (2007), we compute the gains that arise through e-commerce's ability to arbitrage intercity price differences. We find that all three channels produced welfare gains in Japan, but our estimates suggest that the first and second channels are by far the most important, with welfare gains through these channels being eleven to sixteen times larger than through the price arbitrage channel. Overall, we find that increased inter-city arbitrage raised Japanese welfare by 0.12 percent, the gains due to new varieties available through online shopping raised welfare 0.7 percent, and the gains due to overall price reductions for goods available online raised welfare by at most 1 percent.

Chapter 3

Intensive Margin of the Volcker Rule: Price Quality and Welfare

Sakai Ando and Misaki Matsumura

3.1 Introduction

The Volcker rule, named after Paul Volcker (Volcker (2010)), is one of the most important regulations after the Great Recession designed to prevent future financial crises. It limits the ability of banking entities as dealers to take risks on their own books by banning proprietary tradings and other activities described in the Final Rule.[1] Although the necessity of restricting the excessive risk taking by banking entities is widely agreed, as discussed in Duffie (2012) and summarized in the Final Rule, the specific regulation targeted at the risk taking of dealers has invoked various concerns about its economic implications. Despite the importance of the policy and active public debates, there has been no theoretical analysis on the validity of these concerns. In particular, one of the immediate concerns is about the intensive margin (short-term impacts);[2] the dealer regulation that ties the hands of price making institutions deteriorates the quality of price (informativeness and volatility) and therefore has a negative welfare impact. (Duffie (2012) Section 1, the Final Rule Section 3.b.2.b.)

This paper studies the robustness of this conventional wisdom about the intensive margin. Our main result is that the dealer regulation may lower price quality and dealer's profitability as in the conventional wisdom, but at the same time, if we take into account the fact that somebody has to hold the traded financial assets eventually, the dealer regulation may *raise* the welfare of other market participants against the conventional wisdom. In two extensions

[1] Prohibitions and Restrictions on Proprietary Trading and Certain Interests in, and Relationships with, Hedge Funds and Private Equity Funds, Office of the Federal Register, Vol.79, No. 21, National Archives and Records Administration.

[2] The extensive margin, on the other hand, is mainly about the unknown consequences of the migration of dealers to unregulated areas. Given that the extensive margin has not materialized as Kelleher et al. (2016) argues, this paper focuses on the intensive margin.

of the baseline model, we also argue that this novel insight on the welfare is robust, while the conventional wisdom about the deteriorating price quality is fragile to the introduction of dynamics and endogenous information acquisition.

Our welfare analysis deepens the debate on the dealer regulation by identifying the distribution of cost bearing. The typical discussion of dealer regulation focuses on the trade-off between the benefit from lower systemic risk and the cost due to efficiency loss caused by constraints imposed on dealers. This paper elaborates on the discussion of the cost by identifying dealers as the only group of market participants who bear the cost even though all the market participants face a less informative and more volatile price.

To demonstrate the seemingly counter-intuitive results rigorously, the baseline model formally describes the force that improves the welfare of non-dealers despite the price quality deterioration. The price quality deterioration that the baseline model captures is based on the same mechanism as the one widely discussed in public comments summarized in the Final Rule; if the dealer is not allowed to buffer temporary supply and demand imbalances, the price she quotes has to reflect those imbalances rather than economic fundamentals, and therefore becomes less informative and more volatile. Accordingly, one can show her expected profit decreases. The intuition behind the seemingly counter-intuitive welfare results is that, if the total asset is fixed, somebody has to hold risky assets. If a dealer regulation restricts the dealer's risk-taking ability, the risky assets held previously by the dealer have to be held by other market participants. The only way for a dealer to induce other market participants to hold risky assets is to quote an attractive price for them, resulting in a welfare redistribution from the dealer to other market participants. At the same time, for other market participants to accept more risks, this welfare redistribution has to be large enough to make them better off.

To critically evaluate the robustness of our mechanism, we relax the two simplifying assumptions in the baseline model in the two extensions. These exercises not only validate the robustness of our welfare results but also point out the fragility of the conventional wisdom about price quality. In the first extension, we extend the assumption of exogenous initial endowments in the baseline model by constructing a dynamic model and obtaining the endogenous steady state inventories of risky assets. The analysis suggests that the welfare implications in the baseline model survive in the steady state, but the price volatility decreases against the conventional wisdom. In the second extension, we endogenize the information acquisition activities. The analysis suggests that the welfare results of the baseline model are again robust, but the price informativeness might remain the same after regulation against conventional wisdom.

The Volcker rule and how we map it to a stylized model. In analyzing the effect of the Volcker rule, the obvious challenge is how to map the complicated actual regulation into an economic model. In particular, a large portion of the debates on the Volcker rule focuses on the difficulty of telling the proprietary trading from pure dealing. We argue that even if the regulators cannot cleanly detect proprietary tradings, the Volcker rule works as a deterrence device so that the impact of the Volcker rule, at least qualitatively, can be analyzed by considering the "effective" risk aversion of the dealer. In the following, we illustrate the institutional details of the Volcker rule and the justification for using the "effective" risk aversion of the dealer as the modeling device. For the complete description, see the Final Rule.

The Volcker rule is a section of the Dodd-Frank Wall Street Reform and Consumer

Protection Act. It bars banks from engaging in proprietary trading and having relationships with covered funds, with several exemptions including market making. To achieve this goal, the rule mandates that each bank under regulation runs a compliance program and requires big banks to report to regulators seven quantitative measures of trading activities, which are calculated every day for each individual trading desk.[3] Enforcement tools include criminal and civil penalties,[4] which are presumably used as a threat to incentivize banks to police themselves as part of their compliance program. Although the Volcker rule has not been applied against any cases yet, some argue that a good predictor of actual punishment procedure is the so-called London Whale trading mess, which eventually cost J.P. Morgan Chase $920 million in fines (Henning (2013)).

We map the implementation of the Volcker rule to the increase in the dealer's effective risk aversion parameter. In section 3.2.3, we micro-found our comparative statics by showing how a severer threat of possible regulatory intervention can be mapped to a higher effective risk aversion parameter of the dealer.

One can also simply interpret that our choice of modeling describes an idealistic regulation; since the absolute goal of the Volcker rule is to reduce dealers' excessive risk taking activities, the comparative statics in the dealer's effective risk aversion is a direct thought experiment on its effect. Therefore, our analysis is robust to possible future changes in the actual implementation of the Volcker rule, and is applicable to potential similar dealer regulations outside of the U.S. that share the same goal.

In reality, however, due to the difficulty of detecting proprietary trading, the Volcker

[3]See the Final Rule, III.D and III.E for overviews of metrics reporting and compliance program requirement.

[4]See the Final Rule, IV.C.4. for other possible enforcement tools.

rule might not be able to perfectly achieve what it intends. In this case, one may be more interested in the effect of the Volcker rule when it can only achieve its goal to some extent, i.e., when the Volcker rule can eliminate some but not all risk takings on dealers' own books. Our comparative statics in a continuum of the effective risk aversion levels can provide insights on all such intermediate cases. We provide further discussions in section 3.2.3.

Literature. To the best of our knowledge, this paper is the first to formally analyze the Volcker rule based on an internally consistent equilibrium model, although the economic implications of the dealer regulation themselves have attracted interest after the financial crisis of 2007. The Final Rule summarizes public comments that reflect a wide range of opinions from both academia and industries. Duffie (2012) is one of the critical assessments of the Volcker rule based on various empirical and theoretical research. Trebbi and Xiao (2015) report that the Volcker rule has not produced structural deterioration in market liquidity. Kelleher et al. (2016) argues that the incumbent big dealers are still in their positions since they find legal loopholes and discourage new entrants. We complement these contributions by (1) formalizing the mechanisms behind some of the concerns about the intensive margin and (2) pointing out a novel insight on welfare implications of the dealer regulation.

From the modeling point of view, our baseline model is an extension of Grossman and Stiglitz (1980) with a monopolistic and risk averse dealer who quotes the price, and our infinite horizon model is a descendant of Wang (1994). Another paper that also extends Grossman and Stiglitz (1980) with a risk averse dealer is Liu and Wang (2016), which studies the bid-ask spreads under information asymmetry in a static model. As a modeling contribution, we show that a price making dealer can be cleanly embedded in both the static and dynamic frameworks, enabling us to isolate the effect of the dealer regulation from initial inventories of risky assets. See Vives (2010) for a coherent summary of the extensive literature on the dealership in general.

The negative correlation between informativeness and welfare of market participants in our model may remind some readers of the Hirshleifer effect, which describes that better information can make market participants worse off by destroying insurance opportunity. (Hirshleifer (1971)) However, as opposed to Hirshleifer effect, a change in informativeness is not the cause of a change in welfare in our model. Both price quality and welfare redistribution are the consequences of the dealer's risk shifting. In this sense, our model points out a mechanism that is different from that of the Hirshleifer effect.

In this paper, we focus on the intensive margin, although the extensive margin is also a big portion of debates on the Volcker rule. For instance, Duffie (2012) discusses unpredictable consequences of dealer's migration to unregulated sectors. A direct analysis of entry and exit in OTC markets can be found in Atkeson et al. (2015). An interesting result of the three-type entry model of Atkeson et al. (2015) is that regulating dealer banks improves welfare because in their model there is an excessive intermediation relative to a socially optimal level. Although our analysis demonstrates that all other market participants than the dealer are better off, our mechanism is different from Atkeson et al. (2015); we focus on welfare redistribution through price level, while this channel is muted in Atkeson et al. (2015), in which price is determined by Nash bargaining.

3.2 Baseline model

In this section, we describe the main intuition in a static model. We show as a result of dealer regulation, price quality deteriorates and dealer's expected profit declines, but the welfare of other market participants improves.

3.2.1 Environment and definition

The baseline model contains two ingredients necessary to analyze the effects of dealer regulation on price quality and welfare: a price quoting dealer and a signal extraction problem. First, to formally think about the dealer's pricing channel, it is necessary to introduce a dealer who quotes price optimally. As a consequence of introducing optimal pricing, the standard pricing mechanism based on the market clearing of demand and supply is extended to an optimal inventory management problem, in which the dealer's optimal price pins down the trade-off between higher expected profit and riskier inventory, instead of clearing the excess demand.[5] Second, to discuss price informativeness, we introduce a signal extraction problem so that agents endogenously learn valuable information from equilibrium objects. In particular, we introduce heterogeneous traders such that some agents have private information about fundamentals. The dealer learns such private information through order flows and quotes price optimally based on it. Since the price quoted is based on the extracted private information, the price itself is also informative about the fundamentals. Thus, the

[5] This connection between a dealer's inventory and her pricing has been emphasized in the literature (Amihud and Mendelson (1980), Ho and Stoll (1983), Treynor (1987), Grossman and Miller (1988), Hansch et al. (1998), and Liu and Wang (2016)). The connection became salient in the financial crisis of 2007 - 2009 where "the reduced dealer capacity resulted in dramatic downward distortions in corporate bond prices." (Duffie (2012))

informativeness of equilibrium objects is determined endogenously.[6]

Formally, there is one asset in the economy traded by three types of agents, the informed agents I with mass $\lambda \in (0,1)$, the uninformed agents U with mass $1-\lambda$, and a dealer D. We call the informed and uninformed agents, combined, traders. The dealer is monopolistic for the purpose of parsimonious modeling, but as will be clear, a more sophisticated setup such as monopolistic competition among dealers do not alter the insights as long as dealer's pricing decision takes into account average payoff and inventory risks.

Each player has a constant absolute risk aversion utility function (CARA), with parameters $\theta > 0$ for the traders and $\theta_D > 0$ for the dealer. The economy has three independent sources of uncertainty, which follow a common prior distribution

$$X = \begin{bmatrix} d \\ s \\ \epsilon \end{bmatrix} \sim N\left(\begin{bmatrix} \bar{d} \\ \bar{s} \\ 0 \end{bmatrix}, \begin{bmatrix} \kappa_d^{-1} & 0 & 0 \\ 0 & \kappa_s^{-1} & 0 \\ 0 & 0 & \kappa_\epsilon^{-1} \end{bmatrix} \right). \tag{3.1}$$

d denotes the return of the risky asset in the economy, which nobody can observe. $z := d + \epsilon$ is the signal on d that only the informed agents can observe. s is the dealer's inventory of the asset, which only the dealer can observe. Risky inventory s can be positive or negative, reflecting that the dealer can take either a long or short position. In the baseline model, we do not model the traders' inventory risks to illustrate the intuition parsimoniously. We relax this assumption in section 3.3. The CARA normality assumption is for tractability. It is known that the equity returns are not normal, and the approximation is worse for bonds

[6] The asymmetric information is not only a modeling device but also has empirical relevance. For instance, Lu et al. (2010) and Han and Zhou (2013) study asymmetric information in the corporate bond market.

than it is for equities. Although we conjecture that the intuition applies to a broader class of fundamentals, the extension is beyond the scope of this paper. For an analysis of non-normal noisy rational expectations models, see Breon-Drish (2015).

We work on the following equilibrium. Fix exogenous parameters $\left\{\theta, \theta_D, \bar{d}, \bar{s}, \kappa_d, \kappa_s, \kappa_\epsilon, \lambda\right\}$. Let E_i be the conditional expectation operator conditional on the σ-algebra generated by the information set $\mathcal{F}_i$ of agent $i = I, U, D$.

Definition 3.1. A set of price and demand functions $\left\{p(z,s), x_I^B(z,p), x_U^B(p), x^B(z,p)\right\}$ constitutes an equilibrium if

1. (Traders) Demand functions $x_I^B(z,p)$ and $x_U^B(p)$ are best responses to the price quoted by the dealer $p(z,s)$.

$$x_I^B(z,p) = \arg\max_x E_I\left[-e^{-\theta(d-p)x}\right], \ \mathcal{F}_I = \{p(z,s), z\}, \tag{3.2}$$

$$x_U^B(p) = \arg\max_x E_U\left[-e^{-\theta(d-p)x}\right], \ \mathcal{F}_U = \{p(z,s)\}. \tag{3.3}$$

   Aggregate demand satisfies $x^B(z,p) = \lambda x_I^B(z,p) + (1-\lambda)x_U^B(p)$.

2. (Dealer) Price quoted by the dealer $p(z,s)$ is a best response to the aggregate demand $x^B(z,p)$.

$$p(z,s) = \arg\max_p E_D\left[-e^{-\theta_D\left\{sd+(p-d)x^B(z,p)\right\}}\right], \ \mathcal{F}_D = \left\{s, x^B(z,p)\right\}. \tag{3.4}$$

Note that it is straightforward to introduce a risk-free interest rate $1+r$ and the initial wealth of the traders w_0. Without loss of generality due to CARA normal framework, r and

w_0 can be set to be 0.

The interpretation of each agent's problem is as follows. The traders are price takers and extract information about the return d from the price $p(z,s)$. Given the information, the traders optimally choose the demand schedule $p \mapsto x_i^B$, which sums up to the aggregate demand $x^B(z,p)$. The superscript B represents best response. The dealer is a demand taker and extracts information about the return d from $x^B(z,p)$. Based on the information, she quotes the price optimally by controlling the sum of the risk-less cash flow $px^B(z,p)$ and the risky inventory $\left\{s - x^B(z,p)\right\} d$. The equilibrium requires the quoted price to coincide with the price $p(z,s)$ taken as given by the traders. Thus, the pair of price and demand $\left\{p(z,s), x^B(z,p)\right\}$ that closes this loop constitutes a fixed point. In equilibrium, the dealer can infer the signal z from the demand schedule due to the affine structure stated in theorem 3.1, so that the the uninformed agents are the only group of agents who do not observe the signal z.

The equilibrium objects can be used to define equilibrium trading volumes and welfare. The trading volumes of individual agents and the total trading volume in equilibrium can be obtained by substituting p in demand functions with $p(z,s)$, i.e., $x_I(z,s) := x_I^B(z, p(z,s))$, $x_U(z,s) := x_U^B(p(z,s))$, and $x(z,s) := x^B(z, p(z,s))$. With these equilibrium objects, we can define the welfare of each agent. The welfare of the traders is measured by the ex-ante utility

$$u_i := E\left[-e^{-\theta\{d-p(z,s)\}x_i(z,s)}\right], \ i = I, U. \tag{3.5}$$

For the dealer, since we conduct a comparative statics with respect to θ_D as explained in section 3.2.3, we adopt the expected cash flow as her welfare criterion to avoid mechanical welfare changes.

$$u_D := E\pi\left(d,z,s\right) = E\left[sd + \left\{p\left(z,s\right) - d\right\} x\left(z,s\right)\right]. \tag{3.6}$$

The expected cash flow can be considered a form of profit, which is in itself interesting since some of the public comments in the Final Rule show the concern that U.S. banks lose international competitiveness due to decreasing profitability.

3.2.2 Characterization of equilibrium

This section characterizes the affine structure of the equilibrium and defines the price informativeness.

The next theorem states that the equilibrium price and quantities are affine.

Theorem 3.1. *There is a unique equilibrium such that the price function $p\left(z,s\right)$ is affine. In this equilibrium, the demand and the trading volume functions are also affine. The unique set of coefficients $\{\alpha_I, \beta_I, \gamma_I, \alpha_U, \beta_U, \alpha, \beta, \gamma, A, B, C\}$ of the equilibrium*

$$p\left(z,s\right) = A + B\left(z + Cs\right), \; x^B\left(z,p\right) = \alpha + \beta p + \gamma z, \tag{3.7}$$

$$x_I^B\left(z,p\right) = \alpha_I + \beta_I p + \gamma_I z, \; x_U^B\left(p\right) = \alpha_U + \beta_U p \tag{3.8}$$

satisfies $BC\beta_I\gamma_I\beta_U\beta\gamma \neq 0$, $C < 0 < B$, and

$$-\frac{\kappa_d + \kappa_\epsilon}{\theta} = \beta_I < \beta = \lambda\beta_I + \left(1-\lambda\right)\beta_U < \beta_U < 0 < \gamma = \lambda\gamma_I < \gamma_I. \tag{3.9}$$

Proof. See Appendix C.1.[7] □

The signs of $(\beta_I, \beta_U, \beta)$ and (γ_I, γ) reflect respectively the law of demand and the fact that a higher signal z on the return d pushes up demand of the asset. The difference in the price elasticity $|\beta_U| < |\beta_I|$ can be understood by noting that for the uninformed agents, two forces are at work in the opposite directions; an increase in price not only reduces their demand by lowering the total return $d - p$, but also increases their demand by signaling the dealer's information about the higher return d. Since the informed agents observe z, they are only subject to the former force. As a result, the aggregate demand function $x^B(z, p)$, which is a convex combination of the traders' demand functions, is more responsive to price than the uninformed agents $x_U^B(p)$ but less than the informed agents $x_I^B(z, p)$. The positive sign of B reflects the incentive for the dealer to raise her price when she faces a higher demand as a result of a better signal z. The sign of C is negative since with a higher volume of inventory the risk averse dealer wants to cut the price to dispose of her inventory.

Eq. (3.7) in theorem 3.1 suggests a natural way to define price informativeness. For the uninformed agents, the term Cs is a noise that prevents them from inferring the signal z out of price $p(z, s)$. When the variance of the noise term $V(Cs)$ is large relative to that of the signal $V(z)$, the variation of price function is dominated mainly by noise Cs, so that the price is not informative about the valuable signal z. In contrast, if $C = 0$, the price fully reveals z. Since the sign of C does not matter, we define the price informativeness as follows.

Definition 3.2. Consider the equilibrium characterized in theorem 3.1. The price informa-

[7] Whether the affine price function $p(z, s)$ is unique in a larger set of functions, say, $\mathcal{C}^1$ or continuous functions, remains to be open. Papers about the uniqueness in models of Grossman and Stiglitz (1980) and Kyle (1985) include Boulatov et al. (2012), Palvolgyi and Venter (2015), and Breon-Drish (2015).

tiveness Q is defined by

$$Q := \frac{1}{|C|}. \tag{3.10}$$

Now that we have equilibrium objects, we are ready to discuss the dealer regulation.

3.2.3 Mapping the dealer regulation to the model

This section explains how we map the Volcker rule into the stylized model and provides a formal micro-foundation as well as a verbal justification.

The way we map dealer regulation into the baseline model is to raise the dealer's risk aversion parameter $\theta_D \in (0, \infty)$. Such modeling choice reflects the interpretation that θ_D represents the *effective* risk aversion rather than the deep preference parameter; due to the dealer regulation that bans the dealer from taking risks, the dealer behaves as if she becomes more risk averse. In particular, we analyze the comparative statics of (1) price informativeness Q, (2) price volatility $V(p(z,s))$, and (3) welfare of agents u_i for $i = I, U, D$, with respect to the dealer's effective risk aversion $\theta_D \in (0, \infty)$.

As a micro-foundation, we justify our modeling choice by showing that controlling the risk aversion of the dealer is observationally equivalent to the dealer regulation with imperfect monitoring and pecuniary punishment. Suppose the dealer's true risk aversion is $\theta > 0$ and when she makes her pricing decision she knows that she has to submit a report about her inventory to regulators, such as quantitative measures and annual certification by CEOs as specified in the Volcker rule. We assume that regulators can only get noisy information m about the risky inventory

$$m(p) = s - x^B(z,p) + \xi \tag{3.11}$$

where the noise $\xi \sim N\left(0, \sigma_\xi^2\right)$ is independent of the uncertainty $X' = [d\, s\, \epsilon]$ and its accuracy $\sigma_\xi^2 > 0$ can be chosen by regulators for some costs. Based on m, regulators impose a pecuniary punishment $F(m)$. Suppose regulators are reluctant to impose one when $|m|$ is close to 0 out of concern about false accusation, but are willing to impose a larger one when $|m| \to \infty$. A simple parametrization of such punishment is $F(m) = \frac{\alpha}{2}m^2$ where $\alpha > 0$ represents how strictly regulators punish the dealer. In such setting, given the dealer regulation $\left(\sigma_\xi^2, \alpha\right)$, the dealer solves

$$\max_p -E_D e^{-\theta\left\{\left(s-x^B(z,p)\right)d+px^B(z,p)-\frac{\alpha}{2}\left(s-x^B(z,p)+\xi\right)^2\right\}}. \tag{3.12}$$

Recall that the dealer's problem (3.4) in the baseline model is

$$\max_p -E_D e^{-\theta_D\left\{\left(s-x^B(z,p)\right)d+px^B(z,p)\right\}}. \tag{3.13}$$

The following proposition states that the two problems are equivalent, and therefore justifies our parsimonious policy variable θ_D.

Proposition 3.1. *For each regulation* $\left(\sigma_\xi^2, \alpha\right) \in \mathbb{R}_{++}^2$*, there is an effective risk aversion* $\theta_D \in (\theta, \infty)$ *such that the prices quoted in problem (3.12) and (3.13) are identical. The converse is also true. For each effective risk aversion* $\theta_D \in (\theta, \infty)$*, there is a regulation* $\left(\sigma_\xi^2, \alpha\right) \in \mathbb{R}_{++}^2$ *such that the prices quoted in the two problems are identical.*

Proof. See Appendix C.2. □

An implication of proposition 3.1 is that we can always order the severity of regulations linearly. Such simplification not only makes the interpretations of the policy analyses simpler but also makes the results robust to the details of the implementation in the stylized model.

A verbal justification is to interpret that our analysis describes the ideal scenario of the Volcker rule, i.e., a scenario in which it can directly control the dealer's risk-taking on her own books. In other words, we investigate the dealer regulation's mechanics which are invariant to all possible implementations as long as they make the dealer effectively more risk averse. Thus, our analysis provides a benchmark result that is robust to possible future changes in the exact implementation of the Volcker rule, and is applicable to potential dealer regulations outside of the U.S. that share the same goal.

For such justification to be valid, it is desirable to show that the policy variable θ_D can fully span the situations of interest. Indeed, as the following proposition shows, by controlling θ_D, we can capture both pure market making and pure proprietary trading.

Proposition 3.2. *θ_D is a parameter that connects pure market making and pure proprietary trading. That is, for any (z, s), $x(z, s) \to s$ as $\theta_D \to \infty$, and $x(z, s) \to 0$ as $\theta_D \to 0$.*

Proof. See Appendix C.3. □

When the dealer becomes extremely risk averse $\theta_D \to \infty$, all assets are held by the traders and therefore the dealer just makes the market. This equilibrium is exactly the same as Grossman and Stiglitz (1980). When $\theta_D \to 0$, the risk neutral dealer takes all the risks on her own book. The intermediate cases $0 < \theta_D < \infty$ correspond to the realistic situation where the Volcker rule deters proprietary trading only to some extent. Thus, by observing the entire range of θ_D, we can obtain policy-relevant insights even if the regulation is not as effective as is intended to be.

3.2.4 Results

The following theorem presents the analytical results.

Theorem 3.2. *Suppose the dealer becomes more risk averse. Price informativeness decreases. Price volatility eventually increases with sufficiently large inventory shocks. The welfare of the traders improves. The welfare of the dealer deteriorates. Formally,*

1. *The price informativeness Q is decreasing in θ_D with $\lim_{\theta_D \to \infty} Q = \frac{\lambda \kappa_\epsilon}{\theta}$ and $\lim_{\theta_D \to 0} Q = \infty$.*

2. *Price volatility eventually increases, $\lim_{\theta_D \to 0} V\left(p\left(z, s\right)\right) < \lim_{\theta_D \to \infty} V\left(p\left(z, s\right)\right)$, if κ_s is sufficiently small.*

3. *The welfare of the traders $\{u_I, u_U\}$ defined in (3.5) is higher when $\theta_D > 0$ than when $\theta_D \to 0$.*

4. *The welfare of the dealer u_D defined in (3.6) is higher when $\theta_D \to 0$ than when $\theta_D \to \infty$.*

Proof. See Appendix C.4. $\square$

We also provide numerical results in Fig.3.1 to help illustrate the global behaviors of equilibrium objects of interest and their intuitions. The parameter values are chosen so that excess return of the asset is 10% with standard deviation 30% and the dealer's inventory is positive for 95% of the time so the dealer is almost always in the sell-side and the traders are in the buy-side. The choice is for a descriptive purpose and the qualitative behaviors remain the same for other parameter values.

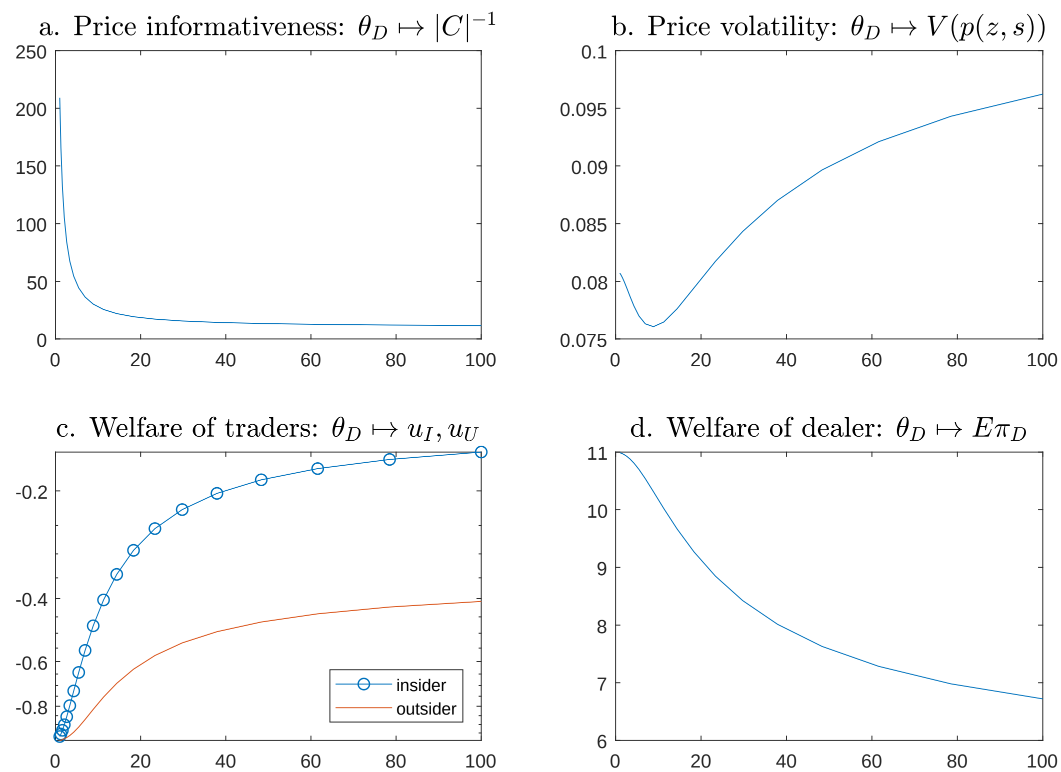


Figure 3.1: The four figures describe the impact of the increase in θ_D on equilibrium objects of interest. All the x axes are θ_D. Parameter values are set to be $\theta = 1$, $\bar{d} = 1.1$, $\bar{s} = 10$, $\kappa_d = 1/.09$, $\kappa_s = 1/25$, $\kappa_\epsilon = 1/.01$ and $\lambda = 0.1$. Panel a shows that price informativeness deteriorates. Panel b shows that the price volatility eventually increases. Panel c shows that the welfare of traders improves, and the welfare difference widens. Panel d shows that the welfare of the dealer declines.

The driving force behind theorem 3.2 and Fig. 3.1 is the risk-shifting motive of the dealer. Note that the dealer's objective function is equivalent to

$$\left(s - x^{B}(z,p)\right) E[d|z] + p x^{B}(z,p) - \frac{\theta_D}{2}\left(s - x^{B}(z,p)\right)^{2} V[d|z]. \tag{3.14}$$

The first two terms are the expected return of the inventory and the income from selling $x^{B}(z,p)$ units of the asset for the price p, and the third term governs the dis-utility from holding risky assets. As she becomes more risk averse $\theta_D \to \infty$, the third term dominates the dealer's incentive so that she puts more weight on disposing her risky inventory, trying to shift risky assets to other market participants.

With this in mind, the effect of the dealer regulation on price quality can be understood in the same way as the conventional wisdom. When the dealer becomes more risk averse, she is more interested in clearing her inventory, so that the information of z, which is useful in raising expected profit but not in reducing the risk of her inventory, is reflected less in her pricing decision. Therefore, the price she quotes becomes less informative about the return d, and more about the inventory risk s, leading to lower price informativeness. Accordingly, since the price has to move together with the inventory shock s to reduce the inventory risks, if the volatility of inventory $Vs = \kappa_s^{-1}$ is high enough, price volatility eventually increases. One force that could confound this intuition is the rise in the traders' price sensitivity $|\beta|$, which contributes to the non-monotonicity of panel b. of Fig 3.1. Recall $|\beta_U| < |\beta_I|$ since the uninformed agents infer higher signal z from higher price p. As the dealer's risk aversion θ_D increases, price informativeness decreases so that the uninformed agents no longer infer signal z from the price function $p(z,s)$. As a result, the uninformed agents' price sensitivity

$|\beta_U| = \left|\partial x_U^B(z,p)/\partial p\right|$ increases, which then raises the price sensitivity of aggregate demand $|\beta| = \left|\partial x^B(z,p)/\partial p\right|$. This higher responsiveness $|\beta| = \left|\partial\left(s - x^B(z,p)\right)/\partial p\right|$ makes it easier for the dealer to adjust her inventory, so that the dealer may not have to change price as much to control inventory. Although this force creates the non-monotonicity, as the figure suggests, it is of second-order importance in the limit.[8]

What is new relative to the conventional wisdom is the result on the traders' welfare. This welfare result can also be explained by the shift of the dealer's incentive. Since the dealer cannot absorb shocks using her own inventory due to additional risk aversion, some of the risks have to be shifted to the traders. The only instrument the dealer has is the price. Therefore, in order for her to shift risks to the traders, she has to compensate them for their riskier positions by quoting a more attractive price than ever before. For the traders to accept riskier positions, this benefit has to be large enough to make them better off. As a result, welfare is redistributed from the dealer to the traders to the extent that the traders are better off. Accordingly, the dealer's profitability deteriorates in exchange for less risky inventory.

One can also see in Fig.3.1.c that the traders' utility difference $u_I - u_U$ increases. This is a corollary of the deteriorating price informativeness. Since price is less informative as $\theta_D \to \infty$, the informational advantage of the informed agents over the uninformed agents increases, which is then reflected in the welfare difference. In other words, $u_I - u_U$ measures

[8] Another way to understand the result is to note the volatility identity

$$Vd = E\left[V\left(d|p(z,s)\right)\right] + V\left(E\left[d|p(z,s)\right]\right).$$

When price informativeness declines, $V(d|p(z,s))$ increases, so $V(E[d|p(z,s)])$ decreases. Since $E[d|p(z,s)] = c_0 + c_1 p(z,s)$ with a positive slope $c_1 > 0$, $Vp(z,s)$ should decrease if c_1 is a constant over θ_D. This argument is intuitive but is not complete since c_1 is also a function of θ_D.

the value of information, and it increases as the dealer becomes more risk averse.

3.2.5 Discussion

We discuss two potential concerns about the assumptions embedded in the baseline model: (1) the exogenous initial endowments and (2) the exogenous information acquisition. These assumptions are relaxed in the following extensions so that the robustness or the fragility of the results in the baseline model are examined.

First, in the baseline model, the traders are not endowed with any risky asset for parsimony. However, such simplification implies that the immediacy of trading comes from the dealer. If the traders are endowed with a large number of risky assets, the immediacy of trading comes from the traders, in which case the dealer regulation might make tradings harder and deteriorate the welfare of the traders. To address such concern, in section 3.3, we endogenize the risky asset holdings before transactions by deriving the steady state in a dynamic model. In particular, we extend Wang (1994) to build a dynamic model with steady state risky asset holdings on both the dealer's and the traders' sides, and conduct an analogous policy analysis of the dealer regulation.

Second, we like to address the exogenous information acquisition by the informed agents. In the baseline model, the number of the informed agents λ is exogenous. However, when the risk attitude of the dealer changes, the change in price informativeness might affect the resulting welfare difference between the informed and uninformed agents, and therefore, might incentivize more traders to invest in information acquisition activities. To see if the results in the baseline model are robust to the introduction of the endogenous information acquisition, in section 3.4, we allow λ to depend on θ_D, and conduct the same policy analysis as in the baseline model.

3.3 Dynamic inventory management

This section argues that the welfare results of the baseline model survive even when the exogenous initial endowments are endogenized as the steady state of a dynamic model. We also show that the increase in price volatility, which is a part of the conventional wisdom derived in the baseline model, does not survive.

As discussed in section 3.2.5, the exogenous endowments in the baseline model have to be endogenized to deal with the concerns about the immediacy of tradings. For this purpose, we extend the baseline model along Wang (1994) by adding a price-making dealer.

3.3.1 Environment and definition

There are one price-making dealer D and a continuum of identical price-taking informed traders $I = [0, 1]$.[9] Their preference parameters are the rate of absolute risk aversion θ_i, $i = D, I$ and the common discount factor $\beta \in (0, 1)$. They trade a single risky asset x_t with risky dividend d_{t+1} and a risk-free bond y_t with gross interest rate R. The total supply of the risky asset is $\bar{x}$, and the supply of the risk-free asset is infinitely elastic so that R is exogenously given. Each agent $i = D, I$ has a private investment opportunity s_t^i, generating a risky dividend d_{t+1}^i. At each period t, each agent divides revenue $p_t x_{t-1}^i + d_t x_{t-1}^i + R y_{t-1}^i + s_{t-1}^i d_t^i$ into consumption π_t^i, investment in the risk-free bond y_t and investment in the risky asset

[9] Since both the dealer and the traders have private shocks, we do not need to introduce uninformed traders to prevent full information revelation.

$p_t x_t^i$. Hence, the flow budget for each agent $i = D, I$ can be written as

$$\pi_t^i + y_t^i + p_t x_t^i = p_t x_{t-1}^i + m_t^i,\ m_t^i := d_t x_{t-1}^i + R y_{t-1}^i + s_{t-1}^i d_t^i \tag{3.15}$$

where m_t^i is the amount of money at the beginning of period t that is independent of the period t price.

To define the equilibrium, we need to specify the information set of each agent. As in the baseline model, the informed traders receive a noisy signal about the dividend $z_t = d_{t+1} + \epsilon_{t+1}$. The information set for the informed traders $\mathcal{F}_t^I = \left\{p_{\tau-1}, x_{\tau-1}^I, d_\tau, d_\tau^I, s_\tau^I, z_\tau\right\}_{\tau \le t}$ contains all the past prices, asset holdings, dividends, private investment opportunities and signals. Given the prices (p_t, R) and the information $\mathcal{F}_t^I$, the informed traders submit the demand schedule $x_t^I = x_t^I\left(z_t, s_t^I, p_t\right)$ where the notation emphasizes that the demand schedule is a noisy signal about z_t. The information set for the dealer $\mathcal{F}_t^D = \left\{p_{\tau-1}, x_{\tau-1}^D, d_\tau, d_\tau^D, s_\tau^D, x_\tau^I\left(z_\tau, s_\tau^I, p_\tau\right)\right\}_{\tau \le t}$ contains the demand schedule $x_t^I\left(z_t, s_t^I, p_t\right)$ so that the dealer extracts information about the signal z_t. The dealer quotes price optimally knowing that her inventory is determined by $x_t^D(p_t) := \bar{x} - x_t^I\left(z_t, s_t^I, p_t\right)$.

We assume the exogenous uncertainty $X_t = \left[d_t, d_t^I, d_t^D, s_t^I, s_t^D, \epsilon_t\right]'$ is *i.i.d.* over time t and is joint normal

$$X_t \sim N\left(\begin{bmatrix} \bar{d} \\ \bar{d}^I \\ \bar{d}^D \\ 0_{3\times 1} \end{bmatrix}, \begin{bmatrix} \Sigma_{3\times 3} & 0 & 0 & 0 \\ 0 & \sigma_{sI}^2 & 0 & 0 \\ 0 & 0 & \sigma_{sD}^2 & 0 \\ 0 & 0 & 0 & \sigma_\epsilon^2 \end{bmatrix}\right). \tag{3.16}$$

The correlation among $\left(d_t, d_t^D, d_t^I\right)$, denoted by the 3×3 matrix $\Sigma_{3\times 3}$, generates reasons to trade, and the $i.i.d.$ assumption suffices to yield AR(1) equilibrium asset holdings as stated in proposition 3.3.

The equilibrium is defined as follows. Let E_t^i be the expectation operator conditional on the σ-algebra generated by $\mathcal{F}_t^i$, $i = D, I$.

Definition 3.3. The sequence of asset holdings and prices $\left(x_t^D, p_t\right)_t$ is an equilibrium if the following conditions are satisfied.

1. At each t, given the information $\mathcal{F}_t^I$ and prices (p_t, R), the traders' asset holding $x_t^I\left(z_t, s_t^I, p_t\right)$ solves

$$J_t^I = \max_{\pi,x,y} -E_t^I \sum_{u=0}^{\infty} \beta^u e^{-\theta_I \pi_{t+u}} \ s.t. \begin{cases} m_t^I = d_t x_{t-1}^I + R y_{t-1} + s_{t-1}^I d_t^I \\ \\ \pi_t + y_t + p_t x_t = p_t x_{t-1}^I + m_t^I \end{cases} . \tag{3.17}$$

2. At each t, given the information $\mathcal{F}_t^D$ and the price R, the dealer's price p_t solves

$$J_t^D = \max_{\pi,p,y} -E_t^D \sum_{u=0}^{\infty} \beta^u e^{-\theta_D \pi_{t+u}} \ s.t. \begin{cases} m_t^D = d_t x_{t-1}^D + R y_{t-1} + s_{t-1}^D d_t^D \\ \\ \pi_t + y_t + p x_t^D = p x_{t-1}^D + m_t^D \\ \\ x_t^D = \bar{x} - x_t^I\left(z_t, s_t^I, p\right) \end{cases} . \tag{3.18}$$

Note that this equilibrium shares the same spirit as the baseline model; the dealer is a demand taker and the traders are price takers. In this sense, the price and the demand are best responses to each other.

The main difference is that since the problem is dynamic, the risky asset purchase of the last period is directly tied to the current period risky inventory. Therefore, both agents can control their inventory shocks as opposed to the baseline model.

3.3.2 Characterization of equilibrium

This section characterizes the affine equilibrium and shows that the dynamic model has a well-defined steady state distribution of the risky asset holdings.

Fix the following sixteen exogenous parameters

$$\left\{\beta, R, \theta_I, \theta_D, \bar{x}, \bar{d}, \bar{d}^I, \bar{d}^D, \sigma_d^2, \sigma_\epsilon^2, \sigma_{ddI}, \sigma_{ddD}, \sigma_{dI}^2, \sigma_{dD}^2, \sigma_{sI}^2, \sigma_{sD}^2\right\} \tag{3.19}$$

where $\left(\bar{d}, \bar{d}^i\right)$ are the means of (d_t, d_t^i), $(\sigma_d^2, \sigma_\epsilon^2, \sigma_{dI}^2, \sigma_{dD}^2, \sigma_{sI}^2, \sigma_{sD}^2)$ are the variances of the random variables $\left(d_t, \epsilon_t, d_t^I, d_t^D, s_t^I, s_t^D\right)$,and $(\sigma_{ddI}, \sigma_{ddD})$ are the covariances of $\left(d_t, d_t^I\right)$ and $\left(d_t, d_t^D\right)$.[10]

Proposition 3.3. *There is an affine equilibrium $\left(x_t^D, p_t\right)$, i.e., for some constants of price $A_0, A_x, B, C = [C_I, C_D]'$ and constants of risky asset holdings (ρ_0, ρ_1), the equilibrium has the form of*

$$p_t = A_0 + A_x x_{t-1}^D + B\left(z_t + C's_t\right), \ \ s_t = \left[s_t^I, s_t^D\right]', \tag{3.20}$$

$$x_t^D = \rho_0 + \rho_1 x_{t-1}^D + \epsilon_t^D, \ \ \epsilon_t^D \sim N\left(0, V\epsilon_t^D\right), \tag{3.21}$$

where ϵ_t^D is a function of current shocks (z_t, s_t).

Proof. See Appendix C.5. □

This proposition satisfies our motivation to introduce dynamics; if $|\rho_1| < 1$, we can obtain

[10]We do not have to specify the covariance of $\left(d_t^I, d_t^D\right)$ since it does not affect individual optimization, and therefore equilibrium objects.

the average asset holding

$$Ex^D := \frac{\rho_0}{1-\rho_1}. \tag{3.22}$$

Another observation is that the equilibrium has the state-space representation where Eq.(3.21) is the state equation and Eq.(3.20) is the observation. The reason for which the equilibrium objects are persistent despite the *i.i.d.* shock assumption is the inventory management. The transaction of the last period on the risky asset affects the amount of risks to begin with in the current period, which then affects the transaction in the current period.

With the affine equilibrium structure, we can conduct an analogous policy analysis as the baseline model.

3.3.3 Policy analysis

As in the baseline model, we focus on the comparative statics of equilibrium objects with respect to the effective risk aversion $\theta_D \in (0, \infty)$. The interpretation of the thought experiment is as follows. Suppose the economy is in the steady state. If suddenly the dealer regulation is introduced, what will happen to the price quality and welfare? In particular, we note that our welfare analysis takes into account the transitional dynamics, not just the comparison of the steady states before and after the policy intervention.

Let us formalize the equilibrium objects of interest. The price informativeness is defined as $|C_D|^{-1}$ which reflects how much the information that the dealer extracts from the order flow goes into the equilibrium price. We report the conditional and unconditional variances $V\left(p_t|x_{t-1}^D\right)$ and $V\left(p_t\right)$ as price volatility. The welfare of the traders is defined as $E\left[J_0^I|x_{-1}^D, m_0^I\right]$ where the asset holding is the steady state value derived in Eq. (3.22) when

$\theta_D = \theta_I$, denoted by $x^D_{-1} = Ex^D\left(\theta_I = \theta_D\right)$, and the initial money is set to be $m^I_0 = 0$ without loss of generality. For the welfare of the dealer, we see the path of expected consumption $t \mapsto E\left[\pi^D_t | x^D_{-1}, m^D_0\right]$, again with $x^D_{-1} = Ex^D\left(\theta_I = \theta_D\right)$ and $m^D_0 = 0$. Fig. 3.2 shows a numerical example where the parameter values are specified in the caption.

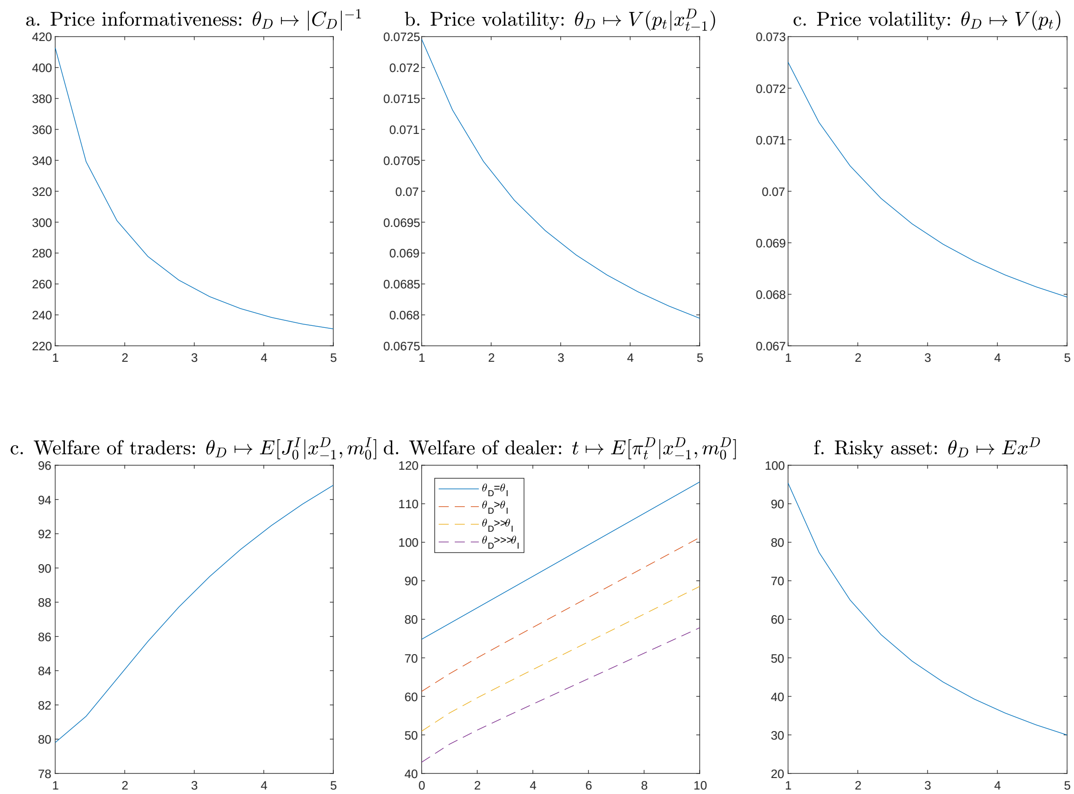


Figure 3.2: The parameters are $\beta = R^{-1} = .9$, $\theta_I = 1$, $\bar{x} = 200$, $\bar{d} = 1.3$, $\bar{d}^I = \bar{d}^D = 1$, $\sigma^2_d = .09$, $\sigma^2_\epsilon = .01$, $\sigma_{ddI} = \sigma_{ddD} = .5$, $\sigma^2_{dI} = \sigma^2_{dD} = \sigma^2_{sI} = \sigma^2_{sD} = .1$. All the x axes except for panel d are θ_D. The x axis for panel d is time t. Panel a shows that the price informativeness deteriorates as a result of dealer regulation from the steady state. Panel b and c show that price volatility decreases as opposed to Fig. 3.1. Panel c shows that the traders' welfare improves. Panel d shows that the expected consumption of the dealer declines at each time horizon. Panel f shows that as the dealer becomes more risk averse, the steady state asset holding for the dealer decreases.

What is different from the baseline model is the declining price volatility, which increases according to the conventional wisdom. The logic of decreasing price volatility can be understood from observing panel f. As the dealer becomes more risk averse, the dealer reduces the risky asset holding. Since the absolute amount of risk in her inventory decreases, she does not have to fluctuate price as much as in the baseline model where the inventory risks from the endowments are exogenously fixed. In other words, since in the dynamic model the inventory risks can be endogenously chosen to be small by reducing the risky asset holding, there is less need to make price fluctuate.

All other results in the baseline model including the welfare implications survive in this dynamic setting. Price informativeness deteriorates, the welfare of the traders improve and the welfare of the dealer decreases. In particular, the expected consumption for the dealer decreases not just in the sense of the discounted sum, but also at each future period.

In summary, we observe that the welfare results in the baseline model are robust, but the increase in price volatility is flipped by introducing dynamics, suggesting the fragility of the conventional wisdom.

3.4 Endogenous information acquisition

The section extends the baseline model by endogenizing information acquisition. We show that the welfare results of the baseline model survive, but the decreasing price informativeness, derived in the baseline model as a description of the conventional wisdom, does not.

3.4.1 Definition of equilibrium

To endogenize the information acquisition, we follow Grossman and Stiglitz (1980) and construct a simple two-period model, in which the traders decide whether to invest in information acquisition in the first period, and all players play the baseline model in the second period. Accordingly, we use the same notation as the baseline model and modify the definition by incorporating the free entry condition.

Let $c > 0$ denote the cost that traders have to pay when they decide to invest in information acquisition.

Definition 3.4. Fix exogenous parameters $\left\{\theta, \theta_D, \bar{d}, \bar{s}, \kappa_d, \kappa_s, c\right\}$. A set of the number of the informed agents, price and demand functions $\left\{\lambda, p\left(z, s\right), x_I^B\left(z, p\right), x_U^B\left(p\right), x^B\left(z, p\right)\right\}$ is an equilibrium if

1. Given λ, price and demand functions $\left\{p\left(z, s\right), x_I^B\left(z, p\right), x_U^B\left(p\right), x^B\left(z, p\right)\right\}$ constitute an equilibrium in the baseline model with the informed agents problem (3.2) replaced by

$$x_I^B\left(z, p\right) = \arg\max_x E_I\left[-e^{-\theta\{(d-p)x-c\}}\right],\ \mathcal{F}_I = \{p\left(z, s\right), z\}, \tag{3.23}$$

2. Free entry condition is satisfied, *i.e.*, one of the three cases holds

   (a) $u_I = u_U,\ 0 \leq \lambda \leq 1,$

   (b) $u_I > u_U,\ \lambda = 1,$

   (c) $u_I < u_U,\ \lambda = 0,$

where (u_I, u_U) are the ex-ante welfare of the traders

$$u_I := E\left[-e^{-\theta\{(d-p(z,s))x_I(z,s)-c\}}\right],\ x_I(z,s) := x_I^B(z, p(z,s)) \tag{3.24}$$

$$u_U := E\left[-e^{-\theta(d-p(z,s))x_U(z,w)}\right],\ x_U(z,s) := x_U^B(p(z,s)). \tag{3.25}$$

Compared to the baseline model, there is one more endogenous variable λ, which is pinned down by the free entry condition, and one more exogenous parameter c. The cost to invest in information acquisition c shows up only in the calculation of the free entry and does not alter the equilibrium conditions in Appendix C.1. In terms of the free entry condition, we choose c such that the first case holds to make the analysis interesting.

3.4.2 Results and discussion

The same four equilibrium objects as the baseline model are of interest, i.e., price informativeness Q, price volatility $V(p(z,s))$, the welfare of the traders $\{u_I, u_U\}$, and the welfare of the dealer u_D. In addition, we are also interested in the behavior of the number of the informed agents λ, which is a new endogenous variable in the extension.

Thanks to the CARA normal framework, we can analytically derive the comparative statics in terms of the price informativeness Q and λ.

Theorem 3.3. *Suppose the cost of information acquisition c is such that the number of the informed agents is in the interior point $0 < \lambda < 1$.*

1. *Then, the price informativeness is constant over the strictness of regulation*

$$\frac{\partial Q}{\partial \theta_D} = 0$$

2. *For a large θ_D, the information acquisition activities increase as the regulation becomes more stringent*

$$\frac{\partial \lambda}{\partial \theta_D} > 0.$$

The constant price informativeness is a knife-edge result specific to CARA normal framework, but it cleanly delivers an insight on the endogenous information acquisition. As can be seen from Fig.3.1, if the number of the informed agents λ is exogenously fixed, a stricter regulation enlarges the welfare difference between the informed and uninformed agents as a result of decreasing price informativeness. Therefore, as in the second claim, traders have incentive to pay the cost c to acquire information until the welfare levels equalize. As more traders become informed, the price informativeness increases, acting as the opposing force to the original downward pressure due to the dealer's risk aversion. In CARA normal framework, these two forces exactly cancel out, so the price informativeness Q remains constant over regulation θ_D as in the first claim.

The result for all the objects are summarized in Fig.3.3. The parameter values are identical to the baseline model for those that overlap.

The most salient difference from the baseline model is the constant price informativeness. It implies that the conventional wisdom about the decreasing price informativeness is not robust once we take into account endogenous information acquisition.

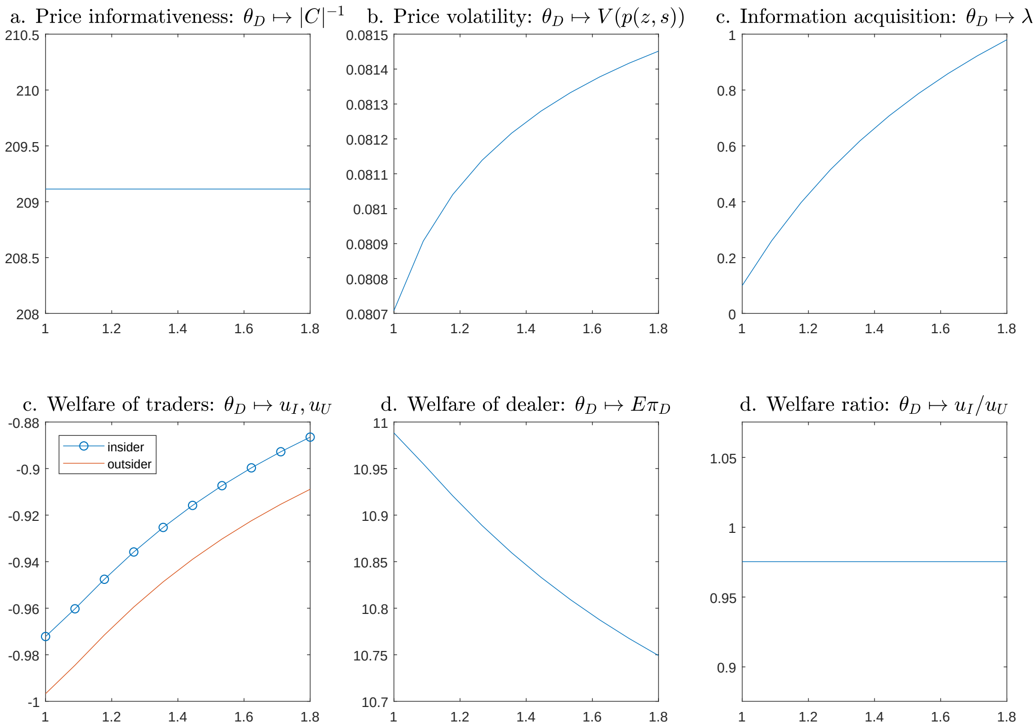


Figure 3.3: Endogenous information acquisition. All the x axes are θ_D. Parameter values are set to be $\theta = 1$, $\bar{d} = 1.1$, $\bar{s} = 10$, $\kappa_d = 1/.09$, $\kappa_s = 1/25$, $\kappa_\epsilon = 1/.01$, and c is chosen so that the equilibrium $\lambda = .1$ when $\theta_D = \theta$. For the purpose of comparison, we plot the same objects in the baseline model when $\kappa_\epsilon = 1$. All but the price informativeness show the same comparative statics results as the baseline model.

Other panels follow the same patterns as in the baseline model except for the last panel highlighting that the welfare ratio between the informed and uninformed agents is constant due to the free entry condition. In terms of the level, one can see the same welfare redistribution from the dealer to the traders as the baseline model. This observation confirms the robustness of the welfare results in the baseline model.

3.5 Final Remarks

We have analyzed the effects of dealer regulation on the properties of price and the resulting welfare consequences. The baseline model shows that the price quality deterioration can coexist with the welfare improvement of other market participants than the dealer. The two extensions then demonstrate the robustness of the novel welfare implications as well as the fragility of the conventional wisdom on the price quality deterioration. We are going to conclude the paper by describing other important aspects of dealer regulation that this paper does not address.

In this paper, we have limited our scope to the intensive margin of dealer regulation. However, considering the extensive margin is also imperative for a comprehensive assessment of the Volcker rule. Although Kelleher et al. (2016) reports that the migration has not happened due to the efforts by the incumbent dealers to discourage entrance, as emphasized in Duffie (2012), "a potential migration of market making to the outside of the regulated bank sector might have unpredictable and potentially important adverse consequences for financial stability." See Whitehead (2011) for more discussion.

Another interesting topic that our paper does not address is the impact of the Volcker rule on the real economy. For instance, a stricter regulation in financial markets might increase capital cost and dampen real investment. Although the Volcker rule tries to mitigate such impact by, for example, not restricting underwriting, the actual impact needs to be investigated empirically.

These are the points worth more investigation as well as the important caveats in understanding the results of this paper.

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Appendices

Appendix A

Appendix to Chapter 1

A.1 Proofs and Derivations for Section 1.2

A.1.1 Derivation of equations (1.13) and (1.14)

I derive equations (1.13) and (1.14) replicated here

$$\frac{P_{s,t}}{P_t} = \frac{P_{s,t-1}}{P_{t-1}}\frac{1}{\Pi_t}\lambda_s\left(1-(1-\lambda_s)\left(\frac{\tilde{K}_{s,t}}{\tilde{F}_{s,t}}\right)^{1-\theta_s}\right)^{\frac{1}{\theta_s-1}}$$
$$\frac{P^X_{s,t}}{P_t} = \frac{P^X_{s,t-1}}{P_{t-1}}\frac{1}{\Pi_t}\lambda_s\left(1-(1-\lambda_s)\left(\frac{\tilde{K}^X_{s,t}}{\tilde{F}^X_{s,t}}\right)^{1-\theta_s}\right)^{\frac{1}{\theta_s-1}}$$

from the following conditions.

1. Optimal pricing problem of individual firms in equation (1.12) replicated here

$$
\begin{aligned}
\left(P_{sit}(0), P_{sit}^{X}(0)\right) = \arg\max_{(P,P^X)} \sum_{\tau=0}^{\infty} \lambda_s^{\tau} E_t \Bigg[\frac{\mathcal{E}_t}{\mathcal{E}_{t+\tau}} \mathcal{M}_{t,t+\tau}^{*} \\
\times \Bigg\{ \left((1-\tau_s) P - \left(\frac{\mathcal{E}_{t+\tau} Q_{s,t+\tau}^{*}}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_{t+\tau}}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{s,t+\tau}^{-1} \right) \left(\frac{P}{P_{s,t+\tau}}\right)^{-\theta_s} Y_{s,t+\tau} \\
+ \left(\left(1-\tau_s^{X}\right) P^{X} - \left(\frac{\mathcal{E}_{t+\tau} Q_{s,t+\tau}^{*}}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_{t+\tau}}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{s,t+\tau}^{-1} \right) \left(\frac{P^{X}}{P_{s,t+\tau}^{X}}\right)^{-\theta_X} Y_{s,t+\tau}^{X} \Bigg\} \Bigg]
\end{aligned}
$$

2. The household's condition

$$\mathcal{M}_{0t}^{*} = \beta^{t} \frac{\mathcal{E}_t}{P_t} \frac{C_t^{-\sigma}}{\Lambda}$$

derived from the definition of C_t^* and P_t^* given in equation (1.7) and the risk sharing condition (1.8)

3. Aggregate price dynamics connecting the sectoral price to the price in the previous period and the newly set price $P_{sit}(0), P_{sit}^{X}(0)$

$$P_{s,t} = \left(\lambda_s \left(P_{s,t-1}\right)^{1-\theta_s} + (1-\lambda_s) P_{s,t}(0)^{1-\theta_s}\right)^{\frac{1}{1-\theta_s}}$$

$$P_{s,t}^{X} = \left(\lambda_s \left(P_{s,t-1}^{X}\right)^{1-\theta_s} + (1-\lambda_s) P_{s,t}^{X}(0)^{1-\theta_s}\right)^{\frac{1}{1-\theta_s}},$$

which follows from the aggregation (1.11) and the i.i.d. likelihood of resetting prices.

The derivation closely follows that in Benigno and Woodford (2005).

First, take the first-order conditions of the pricing problem. The first-order conditions are

$$P\sum_{\tau=0}^{\infty}\lambda_s^{\tau}E_t\left[\frac{\mathcal{E}_t}{\mathcal{E}_{t+\tau}}\mathcal{M}^*_{t,t+\tau}P^{\theta_s}_{s,t+\tau}Y_{s,t+\tau}\right]$$
$$=(1-\tau_s)^{-1}\,\frac{\theta_s}{\theta_s-1}\sum_{\tau=0}^{\infty}\lambda_s^{\tau}E_t\left[\frac{\mathcal{E}_t}{\mathcal{E}_{t+\tau}}\mathcal{M}^*_{t,t+\tau}\left(\frac{\mathcal{E}_{t+\tau}Q^*_{s,t+\tau}}{\alpha_{sm}}\right)^{\alpha_{sm}}\left(\frac{W_{t+\tau}}{\alpha_{sl}}\right)^{\alpha_{sl}}Z^{-1}_{s,t+\tau}P^{\theta_s}_{s,t+\tau}Y_{s,t+\tau}\right]$$

and

$$P^X\sum_{\tau=0}^{\infty}\lambda_s^{\tau}E_t\left[\frac{\mathcal{E}_t}{\mathcal{E}_{t+\tau}}\mathcal{M}^*_{t,t+\tau}\left(P^X_{s,t+\tau}\right)^{\theta_s}Y^X_{s,t+\tau}\right]$$
$$=\left(1-\tau_s^X\right)^{-1}\frac{\theta_s}{\theta_s-1}\sum_{\tau=0}^{\infty}\lambda_s^{\tau}E_t\left[\frac{\mathcal{E}_t}{\mathcal{E}_{t+\tau}}\mathcal{M}^*_{t,t+\tau}\left(\frac{\mathcal{E}_{t+\tau}Q^*_{s,t+\tau}}{\alpha_{sm}}\right)^{\alpha_{sm}}\left(\frac{W_{t+\tau}}{\alpha_{sl}}\right)^{\alpha_{sl}}Z^{-1}_{s,t+\tau}\left(P^X_{s,t+\tau}\right)^{\theta_s}Y^X_{s,t+\tau}\right].$$

Using the household's condition $\mathcal{M}^*_{0t}=\beta^t\frac{\mathcal{E}_t}{P_t}\frac{C_t^{-\sigma}}{\Lambda}$,

$$P\sum_{\tau=0}^{\infty}\lambda_s^{\tau}E_t\left[\frac{C_{t+\tau}^{-\sigma}}{P_{t+\tau}}P^{\theta_s}_{s,t+\tau}Y_{s,t+\tau}\right]$$
$$=(1-\tau_s)^{-1}\,\frac{\theta_s}{\theta_s-1}\sum_{\tau=0}^{\infty}\lambda_s^{\tau}E_t\left[\frac{C_{t+\tau}^{-\sigma}}{P_{t+\tau}}\left(\frac{\mathcal{E}_{t+\tau}Q^*_{s,t+\tau}}{\alpha_{sm}}\right)^{\alpha_{sm}}\left(\frac{W_{t+\tau}}{\alpha_{sl}}\right)^{\alpha_{sl}}Z^{-1}_{s,t+\tau}P^{\theta_s}_{s,t+\tau}Y_{s,t+\tau}\right].$$

$$P^X\sum_{\tau=0}^{\infty}\lambda_s^{\tau}E_t\left[\frac{C_{t+\tau}^{-\sigma}}{P_{t+\tau}}\left(P^X_{X,t+\tau}\right)^{\theta_M}Y^X_{X,t+\tau}\right]$$
$$=\left(1-\tau_s^X\right)^{-1}\frac{\theta_s}{\theta_s-1}\sum_{\tau=0}^{\infty}\lambda_s^{\tau}E_t\left[\frac{C_{t+\tau}^{-\sigma}}{P_{t+\tau}}\left(\frac{\mathcal{E}_{t+\tau}Q^*_{s,t+\tau}}{\alpha_{sm}}\right)^{\alpha_{sm}}\left(\frac{W_{t+\tau}}{\alpha_{sl}}\right)^{\alpha_{sl}}Z^{-1}_{s,t+\tau}\left(P^X_{s,t+\tau}\right)^{\theta_s}Y^X_{s,t+\tau}\right].$$

Thus, for each sector $s \in S$

$$\frac{P_{s,t}(0)}{P_{s,t}} = \left(\underbrace{\sum_{\tau=0}^{\infty} \lambda_s^{\tau} \beta^{\tau} E_t \left[\overbrace{C_{t+\tau}^{-\sigma} \frac{P_{s,t+\tau}}{P_{t+\tau}} \left(\frac{P_{s,t+\tau}}{P_{s,t}} \right)^{\theta_s - 1} Y_{s,t+\tau}}^{=:F_{s,t,t+\tau}} \right]}_{=:\tilde{F}_{s,t}} \right)^{-1}$$

$$\times \underbrace{\sum_{\tau=0}^{\infty} \lambda_s^{\tau} \beta^{\tau} E_t \left[\overbrace{(1-\tau_s)^{-1} \frac{\theta_s}{\theta_s - 1} C_{t+\tau}^{-\sigma} \left(\frac{Q_{t+\tau} Q_{s,t+\tau}^{*}}{\alpha_{sm} P_{t+\tau}^{*}} \right)^{\alpha_{sm}} \left(\frac{W_{t+\tau}}{\alpha_{sl} P_{t+\tau}} \right)^{\alpha_{sl}} Z_{s,t+\tau}^{-1} \left(\frac{P_{s,t+\tau}}{P_{s,t}} \right)^{\theta_s} Y_{s,t+\tau}}^{=:K_{s,t,t+\tau}} \right]}_{=:\tilde{K}_{s,t}}$$

and for exports

$$\frac{P_{s,t}^{X}(0)}{P_{s,t}} = \left(\underbrace{\sum_{\tau=0}^{\infty} \lambda_s^{\tau} \beta^{\tau} E_t \left[\overbrace{C_{t+\tau}^{-\sigma} \frac{P_{s,t+\tau}^{X}}{P_{t+\tau}} \left(\frac{P_{s,t+\tau}^{X}}{P_{s,t}^{X}} \right)^{\theta_s - 1} Y_{s,t+\tau}^{X}}^{=:F_{s,t,t+\tau}^{X}} \right]}_{=:\tilde{F}_{s,t}^{X}} \right)^{-1} .$$

$$\times \underbrace{\sum_{\tau=0}^{\infty} \lambda_s^{\tau} \beta^{\tau} E_t \left[\overbrace{\left(1-\tau_s^{X}\right)^{-1} \frac{\theta_s}{\theta_s - 1} C_{t+\tau}^{-\sigma} \left(\frac{Q_{t+\tau} Q_{s,t+\tau}^{*}}{\alpha_{sm} P_{t+\tau}^{*}} \right)^{\alpha_{sm}} \left(\frac{W_{t+\tau}}{\alpha_{sl} P_{t+\tau}} \right)^{\alpha_{sl}} Z_{s,t+\tau}^{-1} \left(\frac{P_{s,t+\tau}^{X}}{P_{s,t}^{X}} \right)^{\theta_s} Y_{s,t+\tau}^{X}}^{=:K_{s,t,t+\tau}^{X}} \right]}_{=:\tilde{K}_{s,t}^{X}} .$$

Next, rewrite the dynamics and insert the above conditions:

$$\begin{aligned} 1 &= \left(\lambda_s \left(\frac{P_{s,t-1}}{P_{s,t}} \right)^{1-\theta_s} + (1-\lambda_s) \left(\frac{P_{s,t}(0)}{P_t} \frac{P_t}{P_{s,t}} \right)^{1-\theta_s} \right)^{\frac{1}{1-\theta_s}} \\ &= \left(\lambda_s \left(\frac{P_{s,t-1}}{P_{s,t}} \right)^{1-\theta_s} + (1-\lambda_s) \left(\frac{\tilde{K}_{s,t}}{\tilde{F}_{s,t}} \right)^{1-\theta_s} \right)^{\frac{1}{1-\theta_s}} . \end{aligned}$$

$$
\begin{aligned}
1 &= \left(\lambda_s \left(\frac{P^X_{s,t-1}}{P^X_{s,t}}\right)^{1-\theta_s} + (1-\lambda_s)\left(\frac{P^X_{s,t}(0)}{P_t}\frac{P_t}{P^X_{s,t}}\right)^{1-\theta_s}\right)^{\frac{1}{1-\theta_s}} \\
&= \left(\lambda_s \left(\frac{P^X_{s,t-1}}{P^X_{s,t}}\right)^{1-\theta_s} + (1-\lambda_s)\left(\frac{\tilde{K}^X_{s,t}}{\tilde{F}^X_{s,t}}\right)^{1-\theta_s}\right)^{\frac{1}{1-\theta_s}}.
\end{aligned}
$$

By rearranging this, we can obtain equation (1.13) and (1.14).

Finally, note that under the assumption of a bounded solution,

$$
\tilde{F}_{s,t} = \sum_{\tau=0}^{\infty} \lambda_s^\tau \beta^\tau E_t F_{s,t,t+\tau}
$$

can equivalently be written as

$$
\begin{aligned}
\tilde{F}_{s,t} &= F_{s,t,t+1} + \lambda_s \beta \sum_{\tau=0}^{\infty} \lambda_s^\tau \beta^\tau E_t F_{s,t,t+1+\tau} \\
&= F_{s,t,t} + \lambda_s \beta \left(\Pi_{s,t+1}\right)^{\theta_s - 1} E_t \tilde{F}_{s,t+1}.
\end{aligned}
$$

Similarly, for $\tilde{K}_{s,t}$, $\tilde{F}^X_{s,t}$, $\tilde{K}^X_{s,t}$. Thus we obtain the equivalent definitions given in equation (1.15)-(1.18).

A.1.2 Derivation of equation (1.20)

I derive the aggregate resource constraint (1.20) replicated here

$$
Z_{st} L_{st} = \left(\frac{\alpha_{sl}}{\alpha_{sm}} \frac{Q_t Q^*_{st}/P^*_t}{W_t/P_t}\right)^{\alpha_{sm}} \left(\Delta_{st} C_{st} + \Delta^X_{st} X_{st}\right), \; M_{st} = \frac{\alpha_{sm}}{\alpha_{sl}} \frac{W_t/P_t}{Q_t Q^*_{st}/P^*_t} L_{st}
$$

together with the evolution of the price dispersion (1.21) and (1.22) from the following conditions.

1. Market clearing conditions

$$\sum_{s\in S}\int L_{sit}di = L_t, \int M_{sit}di = M_{st}$$

2. Factor demand from firm's optimization conditions

$$M_{it} = \left(\frac{\alpha_{sm}}{\alpha_{sl}}\frac{W_t}{\mathcal{E}_t Q^*_{st}}\right)^{\alpha_{sl}}\left(\left(\frac{P_{sit}}{P_{st}}\right)^{-\theta_s}\frac{Y_{st}}{Z_{st}} + \left(\frac{P^X_{sit}}{P^X_{st}}\right)^{-\theta_s}\frac{Y^X_{st}}{Z_{st}}\right)$$

$$L_{it} = \left(\frac{\alpha_{sl}}{\alpha_{sm}}\frac{\mathcal{E}_t Q^*_{st}}{W_t}\right)^{\alpha_{sm}}\left(\left(\frac{P_{sit}}{P_{st}}\right)^{-\theta_s}\frac{Y_{st}}{Z_{st}} + \left(\frac{P^X_{sit}}{P^X_{st}}\right)^{-\theta_s}\frac{Y^X_{st}}{Z_{st}}\right)$$

3. Optimal pricing equation obtained in Appendix A.1.1.

$$\frac{P_{s,t}(0)}{P_t}\frac{P_t}{P_{s,t}} = \frac{\tilde{K}_{s,t}}{\tilde{F}_{s,t}} = \left(\frac{1-\lambda_s\left(\frac{P_{s,t-1}}{P_{s,t}}\right)^{1-\theta_s}}{1-\lambda_s}\right)^{\frac{1}{1-\theta_s}}.$$

The derivation here closely follows that in Benigno and Woodford (2005).

To obtain the aggregate resource constraint (1.20) , combine these conditions 1 and 2. Then,

$$L_{st} = \int L_{sit}di = \left(\frac{\alpha_{sl}}{\alpha_{sm}}\frac{\mathcal{E}_t Q^*_{st}}{W_t}\right)^{\alpha_{sm}}\left(\underbrace{\int\left(\frac{P_{sit}}{P_{st}}\right)^{-\theta_s}di}_{:=\Delta_{st}}\frac{Y_{st}}{Z_{st}} + \underbrace{\int\left(\frac{P^X_{sit}}{P^X_{st}}\right)^{-\theta_s}di}_{:=\Delta^X_{st}}\frac{Y^X_{st}}{Z_{st}}\right),$$

$$M_{st} = \frac{\alpha_{sm}}{\alpha_{sl}} \frac{W_t/P_t}{Q_t Q_{st}^*/P_t^*} L_{st}.$$

Note that the second condition also uses the definition of the real exchange rate $Q_t \equiv \frac{\mathcal{E}_t P_t*}{P_t}$.

To obtain the dynamics of price dispersion, rewrite the definition of the dispersion using the optimal pricing equation as follows.

$$\begin{aligned}
\Delta_{st} &= \int \left(\frac{P_{sit}}{P_{st}}\right)^{-\theta_s} di \\
&= \lambda_s \int \left(\frac{P_{sit-1}}{P_{st}}\right)^{-\theta_s} di + (1-\lambda_s) \int \left(\frac{P_{sit}(0)}{P_{st}}\right)^{-\theta_s} di \\
&= \lambda_s \left(\frac{P_{st-1}}{P_{st}}\right)^{-\theta_s} \int \left(\frac{P_{sit-1}}{P_{st-1}}\right)^{-\theta_s} di + (1-\lambda_s) \left(\frac{1-\lambda_s \left(\frac{P_{s,t-1}}{P_{s,t}}\right)^{1-\theta_s}}{1-\lambda_s}\right)^{\frac{-\theta_s}{1-\theta_s}} \\
&= \lambda_s \left(\frac{P_{st-1}}{P_{st}}\right)^{-\theta_s} \Delta_{s,t-1} + (1-\lambda_s) \left(\frac{1-\lambda_s \left(\frac{P_{s,t-1}}{P_{s,t}}\right)^{1-\theta_s}}{1-\lambda_s}\right)^{\frac{-\theta_s}{1-\theta_s}} \\
&= \lambda_s \left(\frac{P_{st}}{P_{st-1}}\right)^{\theta_s} \Delta_{s,t-1} + (1-\lambda_s) \left(\frac{1-\lambda_s \left(\frac{P_{s,t}}{P_{s,t-1}}\right)^{\theta_s-1}}{1-\lambda_s}\right)^{\frac{\theta_s}{\theta_s-1}}.
\end{aligned}$$

A.2 Proofs and Derivations for Section 1.3

A.2.1 Planner's solution

Given $\left\{\frac{Q_{Mt}^*}{P_t^*}, \frac{P_{Xt}^*}{P_t^*}, \mathcal{M}_{0,t}^*\right\}_{t=0}^{\infty}, \Lambda$, the planner maximizes

$$\max_{D_0, \left\{(C_{st}, M_{st}, X_{st}, L_{st})_{s\in S}\right\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{\left(\prod_{s\in S} C_{st}^{\psi_s}\right)^{1-\sigma}}{1-\sigma} - \frac{\left(\sum_{s\in S} L_{st}\right)^{1+\phi}}{1+\phi}\right] + \Lambda D_0,$$

$$s.t. \begin{cases} Z_{s,t} M_{st}^{\alpha_{sm}} L_{st}^{\alpha_{sl}} = C_{st} + X_{st} \ \forall s \in S & [\lambda_{st}] \\ E_0 \sum_{t=0}^{\infty} \left[\mathcal{M}_{0,t}^* P_t^* \sum_{s \in S} \left(X_{st}^{\frac{\theta_s^*-1}{\theta_s^*}} {X_{st}^*}^{\frac{1}{\theta_s^*}} \frac{P_{st}^*}{P_t^*} - \frac{Q_{st}^*}{P_t^*} M_{st} \right) \right] = D_0 & [\lambda_D] \end{cases}.$$

The first-order conditions are

$$\begin{cases} [C_{st}] & \beta^t C_t^{1-\sigma} \frac{\psi_s}{C_{st}} = \lambda_{st} \\ [M_{st}] & \alpha_{sm} \frac{Y_{st}}{M_{st}} \lambda_{st} = \mathcal{M}_{0,t}^* Q_{st}^* \lambda_D \\ [X_{st}] & \lambda_{st} = \mathcal{M}_{0,t}^* \frac{P_{st}^X}{\mathcal{E}_t} \frac{\theta_s^*-1}{\theta_s^*} \lambda_D \\ [L_{st}] & \beta^t L_t^{\phi} = \alpha_{sl} \frac{Y_{st}}{L_{st}} \lambda_{st} \\ [D_0] & \Lambda = \lambda_D \end{cases}$$

From the first-order conditions, we obtain aggregate consumption and the consumption price index. Rearranging the FOC with respect to C_{st},

$$C_{st} = \beta^t C_t^{1-\sigma} \frac{\psi_s}{\lambda_{st}}.$$

Plugging this into the consumption aggregator $C_t = \prod_{s \in S} C_{st}^{\psi_s}$, we obtain

$$C_t = \beta^t C_t^{1-\sigma} \prod_{s \in S} \psi_s^{\psi_s} \prod_{s \in S} \left(\frac{1}{\lambda_{st}} \right)^{\psi_s}.$$

Multiplying both sides by $\mathcal{M}_{0,t}^* \lambda_D = \mathcal{M}_{0,t}^* \Lambda$, we have

$$\mathcal{M}_{0,t}^* \Lambda = \beta^t C_t^{-\sigma} \prod_{s \in S} \psi_s^{\psi_s} \prod_{s \in S} \left(\frac{\mathcal{M}_{0,t}^* \Lambda}{\lambda_{st}} \right)^{\psi_s} = \beta^t C_t^{-\sigma} \left(\frac{P_t}{\mathcal{E}_t} \right)^{-1},$$

where $\frac{P_t}{\mathcal{E}_t} := \prod_{s\in S}\psi_s^{-\psi_s}\prod_{s\in S}\left(\frac{\lambda_{st}}{\mathcal{M}^*_{0,t}\Lambda}\right)^{\psi_s} = \prod_{s\in S}\psi_s^{-\psi_s}\prod_{s\in S}\left(\frac{\beta^t C_t^{1-\sigma}\frac{\psi_s}{C_{st}}}{\mathcal{M}^*_{0,t}\Lambda}\right)^{\psi_s}$ is defined as the shadow price of the aggregate consumption in terms of international currency. Combining this with the assumption on the relationship between $\mathcal{M}^*_{0,t}$, C^*_t and P^*_t, we obtain the risk sharing condition

$$\beta^t\frac{(C^*_t)^{-\sigma}/P^*_t}{(C^*_0)^{-\sigma}/P^*_0}\Lambda = \beta^t C_t^{-\sigma}\mathcal{E}_t P_t^{-1} \Rightarrow C_t = \xi C^*_t Q_t^{\frac{1}{\sigma}},$$

where ξ is the same constant as that in equation (1.8). The real exchange rate here is defined as $Q_t = \frac{\mathcal{E}_t P^*_t}{P_t}$ using the shadow price of the aggregate consumption defined above.

We can also obtain the intra-temporal conditions. Due to the assumption $\alpha_{sl} > 0$ for all $s \in S$, combining the FOC with respect to C_{st} and that with respect to L_{st} leads to

$$C_t\frac{\psi_s}{C_{st}}\alpha_{sl}\frac{Y_{st}}{L_{st}} = \frac{L_t^{\phi}}{C_t^{-\sigma}}.$$

For those sectors with $\alpha_{sm} > 0$, combining the FOC with respect to M_{st} and that with respect to L_{st},

$$\frac{\alpha_{sl}\frac{Y_{st}}{L_{st}}}{\alpha_{sm}\frac{Y_{st}}{M_{st}}} = \frac{\beta^t L_t^{\phi}}{\mathcal{M}^*_{0,t}Q^*_{st}\Lambda} = \frac{L_t^{\phi}/C_t^{-\sigma}}{Q_t\frac{Q^*_{st}}{P^*_t}}.$$

From this, we can calculate the aggregate labor productivity:

$$\begin{aligned}
Y_{st} &= Z_{s,t}M_{st}^{\alpha_{sm}}L_{st}^{\alpha_{sl}} \\
&= Z_{s,t}\left(\alpha_{sm}\frac{L_t^{\phi}/C_t^{-\sigma}}{Q_t\frac{Q^*_{st}}{P^*_t}}\frac{1}{\alpha_{sl}\frac{1}{L_{st}}}\right)^{\alpha_{sm}}L_{st}^{\alpha_{sl}} \\
&= Z_{s,t}\left(\frac{\alpha_{sm}}{\alpha_{sl}}\frac{L_t^{\phi}}{C_t^{-\sigma}}\frac{P^*_t}{Q_tQ^*_{st}}\right)^{\alpha_{sm}}L_{st}
\end{aligned}$$

For those sectors with positive exports, combining the FOC with respect to X_{st} and that with respect to C_{st}, we have

$$\frac{\theta_s^* - 1}{\theta_s^*} Q_t \frac{P_{st}^*}{P_t^*} \left(\frac{X_{st}}{X_{st}^*} \right)^{-\frac{1}{\theta_s^*}} = \frac{P_{st}^X}{P_t} \frac{\theta_s^* - 1}{\theta_s^*} = C_t \frac{\psi_s}{C_{st}}$$

Combining these, we obtain the conditions in equations (1.24)-(1.27).

A.2.2 Flexible price equilibrium

The household.

The first-order conditions are

$$\beta^t \psi_s \tfrac{C_t^{1-\sigma}}{C_{st}} = \tfrac{\mathcal{M}_{0t}^*}{\mathcal{E}_t} \lambda P_{st}$$

$$\beta^t L_t^{\phi} = \tfrac{\mathcal{M}_{0t}^*}{\mathcal{E}_t} \lambda W_{st}$$

$$\Lambda = \lambda$$

From the linearity of labor aggregator, we can immediately see that$W_{st} = W_s$ must hold in the equilibrium. From the first-order conditions, we can calculate aggregate consumption and price index.

$$C_{st} = \beta^t \psi_s \frac{C_t^{1-\sigma}}{\frac{\mathcal{M}_{0t}^*}{\mathcal{E}_t} \lambda P_{st}} \Rightarrow C_t = \prod_{s \in S} C_{st}^{\psi_s} = \beta^t \frac{C_t^{1-\sigma}}{\frac{\mathcal{M}_{0t}^*}{\mathcal{E}_t} \lambda} \prod_{s \in S} \left(\frac{\psi_s}{P_{st}} \right)^{\psi_s}$$

$$\Rightarrow \mathcal{M}_{0t}^* = \beta^t \mathcal{E}_t \frac{C_t^{-\sigma}}{\lambda} \prod_{s \in S} \left(\frac{\psi_s}{P_{st}} \right)^{\psi_s} = \beta^t \frac{\mathcal{E}_t}{P_t} \frac{C_t^{-\sigma}}{\Lambda},$$

where $P_t = \prod_{s \in S} \left(\frac{P_{st}}{\psi_s} \right)^{\psi_s}$ is the consumer price index. Combining this with the same sequence of C_t^* and P_t^*as in the planner's problem,

$$\beta^t \frac{\left(C_t^*\right)^{-\sigma}/P_t^*}{\left(C_0^*\right)^{-\sigma}/P_0^*}\Lambda = \beta^t C_t^{-\sigma}\mathcal{E}_t P_t^{-1} \Rightarrow C_t = \xi C_t^* Q_t^{\frac{1}{\sigma}},$$

where $\xi = \left(\frac{\Lambda}{\left(C_0^*\right)^{-\sigma}/P_0^*}\right)^{-\frac{1}{\sigma}}$ is the same constant as the planner's problem as long as the marginal utility $\Lambda$of the initial debt is the same.

We also get intra-temporal conditions

$$\begin{array}{c} \psi_s C_t = \frac{P_{st}}{P_t} C_{st} \\ \frac{L_t^\phi}{C_t^{-\sigma}} = \frac{W_t}{P_t} \end{array}.$$

The aggregator firm.

There are two aggregator firms in each sector: one for domestically consumed goods and the other for exported goods. The variables related to exports are indicated by the superscript X. The sectoral aggregator firm's cost minimization for domestic use is for each $s \in S$,

$$\min_{\{Y_{sit}\}_i} \int P_{sit} Y_{sit} di \ s.t. \ Y_{st} = \left(\int Y_{sit}^{\frac{\theta_s - 1}{\theta_s}} di\right)^{\frac{\theta_s}{\theta_s - 1}}$$

$$\Rightarrow Y_{sit} = \left(\frac{P_{sit}}{P_{st}}\right)^{-\theta_s} Y_{st}, \ P_{st} = \left(\int P_{sit}^{1-\theta_s} di\right)^{\frac{1}{1-\theta_s}}$$

and for export goods,

$$\min_{\left\{Y_{sit}^X\right\}_i} \int P_{sit}^X Y_{sit}^X di \ s.t. \ Y_{st}^X = \left(\int \left(Y_{sit}^X\right)^{\frac{\theta_s - 1}{\theta_s}} di\right)^{\frac{\theta_s}{\theta_s - 1}}$$

$$\Rightarrow Y_{sit}^{X} = \left(\frac{P_{sit}^{X}}{P_{st}^{X}}\right)^{-\theta_s} Y_{st}^{X},\ P_{st}^{X} = \left(\int \left(P_{sit}^{X}\right)^{1-\theta_s} di\right)^{\frac{1}{1-\theta_s}}.$$

The Individual Firm.

The individual firm in sector $s \in S$ takes wage W_t, import price $\mathcal{E}_t Q_{st}^{*}$, the demand function derived above, production function and tax rates for domestic salesτ_s and foreign sales τ_s^X as given. I allow the firm to set different prices for domestic consumers P_{sit} and for foreign buyers P_{sit}^X (pricing to market). As we will see later, this is necessary for the flexible price equilibrium to be efficient.

$$\max_{P_{sit}, P_{sit}^X, L_{sit}, M_{sit}, Y_{sit}, Y_{sit}^X} (1-\tau_s) P_{sit} Y_{sit} + \left(1-\tau_s^X\right) P_{sit}^X Y_{sit}^X - W_t L_{sit} - \mathcal{E}_t Q_{st}^{*} M_{sit}$$

$$s.t. \begin{cases} Y_{sit} = \left(\frac{P_{sit}}{P_{st}}\right)^{-\theta_s} Y_{st} \\ Y_{sit}^X = \left(\frac{P_{sit}^X}{P_{st}^X}\right)^{-\theta_s} Y_{st}^X \\ Y_{sit} + Y_{sit}^X = Z_{s,t} M_{sit}^{\alpha_{sm}} L_{sit}^{\alpha_{sl}} \end{cases}.$$

Solving the cost minimization problem as its sub-problem, the marginal cost can be calculated as $\left(\frac{\mathcal{E}_t Q_{st}^{*}}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{st}^{-1}$ and the factor demand should satisfy

$$M_{sit} = \frac{\alpha_{sm}}{\alpha_{sl}} \frac{W_t}{\mathcal{E}_t Q_{st}^{*}} L_{sit}.$$

Thus,

$$\Rightarrow \max_{P_{sit},P_{sit}^X} (1-\tau_s) P_{sit} \left(\frac{P_{sit}}{P_{st}}\right)^{-\theta_s} Y_{st} + \left(1-\tau_s^X\right) P_{sit}^X \left(\frac{P_{sit}^X}{P_{st}^X}\right)^{-\theta_s} Y_{st}^X$$
$$- \left(\frac{\mathcal{E}_t Q_{st}^*}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{st}^{-1} \left\{\left(\frac{P_{sit}}{P_{st}}\right)^{-\theta_s} Y_{st} + \left(\frac{P_{sit}^X}{P_{st}^X}\right)^{-\theta_s} Y_{st}^X\right\}$$

The first-order conditions are

$$\begin{cases} (1-\theta_s)(1-\tau_s) P_{sit} \frac{Y_{sit}}{P_{sit}} + \theta_s \left(\frac{\mathcal{E}_t Q_{st}^*}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{st}^{-1} \frac{Y_{sit}}{P_{sit}} = 0 \\ (1-\theta_s)\left(1-\tau_s^X\right) P_{sit}^X \frac{Y_{sit}^X}{P_{sit}^X} + \theta_s \left(\frac{\mathcal{E}_t Q_{st}^*}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{st}^{-1} \frac{Y_{sit}^X}{P_{sit}^X} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} P_{sit} = (1-\tau_s)^{-1} \frac{\theta_s}{\theta_s-1} \left(\frac{\mathcal{E}_t Q_{st}^*}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{st}^{-1} \\ P_{sit}^X = \left(1-\tau_s^X\right)^{-1} \frac{\theta_s}{\theta_s-1} \left(\frac{\mathcal{E}_t Q_{st}^*}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{st}^{-1} \end{cases}$$

With flexible prices, all firms are symmetric within a sector. Thus, subscript i can be dropped. In summary, we have

$$\begin{cases} P_{st} = (1-\tau_s)^{-1} \frac{\theta_s}{\theta_s-1} \left(\frac{\mathcal{E}_t Q_{st}^*}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{st}^{-1} \\ P_{st}^X = \left(1-\tau_s^X\right)^{-1} \frac{\theta_s}{\theta_s-1} \left(\frac{\mathcal{E}_t Q_{st}^*}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{st}^{-1} \end{cases}$$

$$\Rightarrow \begin{cases} P_{st} = \underbrace{(1-\tau_s)^{-1} \frac{\theta_s}{\theta_s-1}}_{=:\chi_s^{-1}} \left(\frac{\mathcal{E}_t Q_{st}^*}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{st}^{-1} \\ P_{st}^X = \underbrace{\left(1-\tau_s^X\right)^{-1} (1-\tau_s) \frac{\theta_s^*-1}{\theta_s^*}}_{=:\nu_s^{-1}} \frac{\theta_s^*}{\theta_s^*-1} P_{st} \end{cases}$$

Combining firms' pricing equations and factor demand equations with the household's optimization condition,

$$\begin{cases} \psi_s C_t = \frac{P_{st}}{P_t} C_{st} \\ \frac{L_t^{\phi}}{C_t^{-\sigma}} = \frac{W_t}{P_t} \\ P_{st} = \chi_s^{-1} \left(\frac{\mathcal{E}_t Q_{st}^*}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl}}\right)^{\alpha_{sl}} Z_{st}^{-1} \\ P_{st}^X = \nu_s^{-1} \frac{\theta_s^*}{\theta_s^* - 1} P_{st} \\ M_{st} = \frac{\alpha_{sm}}{\alpha_{sl}} \frac{W_t}{\mathcal{E}_t Q_{st}^*} L_{st} \end{cases} \Leftrightarrow \begin{cases} C_t \frac{\psi_s}{C_{st}} \alpha_{sl} \frac{Z_{st} M_{st}^{\alpha_{sm}} L_{st}^{\alpha_{sl}}}{L_{st}} = \chi_s^{-1} \frac{L_t^{\phi}}{C_t^{-\sigma}} \\ \frac{\alpha_{sl}}{L_{st}} \frac{M_{st}}{\alpha_{sm}} = \frac{L_t^{\phi}/C_t^{-\sigma}}{Q_t \frac{Q_{st}^*}{P_t^*}} \\ \frac{\theta_s^* - 1}{\theta_s^*} Q_t \frac{P_{st}^*}{P_t^*} \frac{P_{st}^X}{\mathcal{E}_t P_{st}^*} = \nu_s^{-1} C_t \frac{\psi_s}{C_{st}} \end{cases}$$

Recall the assumption on the foreign demand for exports

$$X_{st} = \left(\frac{P_{st}^X}{\mathcal{E}_t P_{st}^*}\right)^{-\theta_s^*} X_{st}^*.$$

Then, the third condition can be equivalently written as

$$\frac{\theta_s^* - 1}{\theta_s^*} Q_t \frac{P_{st}^*}{P_t^*} \left(X_{st}^*\right)^{\frac{1}{\theta_s^*}} = \nu_s^{-1} X_{st}^{\frac{1}{\theta_s^*}} C_t \frac{\psi_s}{C_{st}}.$$

Finally, using production technology and the market clearing condition, $X_{st} = Z_{st} M_{st}^{\alpha_{sm}} L_{st}^{\alpha_{sl}} - C_{st}$. Thus,

$$\frac{\theta_s^* - 1}{\theta_s^*} Q_t \frac{P_{st}^*}{P_t^*} \left(X_{st}^*\right)^{\frac{1}{\theta_s^*}} = \nu_s^{-1} \left(Z_{st} M_{st}^{\alpha_{sm}} L_{st}^{\alpha_{sl}} - C_{st}\right)^{\frac{1}{\theta_s^*}} C_t \frac{\psi_s}{C_{st}}.$$

Combining these leads to equations (1.28)-(1.31).

A.2.3 Definition of optimal steady state

The optimal steady state is defined as follows.

Definition A.1. The optimal steady state is the solution to the following problem. Given constant $\left(\left\{\frac{Q^*_{st}}{P^*_t}, \frac{P^*_{st}}{P^*_t}, Z_{st}, X^*_{st}\right\}_{s\in S}, \mathcal{M}^*_{t+1}, P^*_t\right) = \left(\{Q^*_s, P^*_s, Z_s, X^*_s\}_{s\in S}, \beta, 1\right)$, tax $\left(\tau_s, \tau^X_s\right)_{s\in S}$, and initial state variables $\left(P_{-1}, \mathcal{E}_{-1}, \left\{\Delta_{s,-1}, \Delta^X_{s,-1}\right\}_{s\in S}\right) = \left(1, 1, \{1, 1\}_{s\in S}\right)$, the central bank maximizes

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\phi}}{1+\phi}\right] + \Lambda D_0$$

s.t.

$$\begin{cases}
\psi_s C_t = \frac{P_{st}}{P_t} C_{st} \\
\frac{L_t^{\phi}}{C_t^{-\sigma}} = \frac{W_t}{P_t} \\
C_t = \xi C^*_t Q_t^{\frac{1}{\sigma}} \\
f_s\left(\frac{P^u_{s,t}}{P_t}, \Pi_t; \frac{P^u_{s,t-1}}{P_{t-1}}\right) \tilde{K}^u_{s,t} = \tilde{F}^u_{s,t} \\
\tilde{K}^u_{s,t} = (1-\tau^u_s)^{-1} \frac{\theta_s}{\theta_s - 1} C_t^{-\sigma} \left(\frac{Q_t Q^*_{s,t}}{\alpha_{sm} P^*_t}\right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl} P_t}\right)^{\alpha_{sl}} \frac{1}{Z_{st}} Y^u_{s,t} + \lambda_s \beta E_t \left(\Pi^u_{s,t+1}\right)^{\theta_s} \tilde{K}^u_{s,t+1} \\
\tilde{F}^u_{s,t} = C_t^{-\sigma} \frac{P^u_{s,t}}{P_t} Y^u_{s,t} + \lambda_s \beta E_t \left(\Pi^u_{s,t+1}\right)^{\theta_s - 1} \tilde{F}^u_{s,t+1} \\
\Delta^u_{st} = \lambda_s \left(\frac{P^u_{st}}{P^u_{st-1}}\right)^{\theta_s} \Delta^u_{s,t-1} + (1-\lambda_s)\left(f_s\left(\frac{P^u_{s,t}}{P_t}, \Pi_t; \frac{P^u_{s,t-1}}{P_{t-1}}\right)\right)^{\theta_s} \\
Z_{st} L_{st} = \left(\frac{\alpha_{sl}}{\alpha_{sm}} \frac{Q_t Q^*_{st}}{W_t/P_t}\right)^{\alpha_{sm}} \left\{\Delta_{st} C_{st} + \Delta^X_{st} \left(\frac{P^X_{st}/P_t}{Q_t P^*_s}\right)^{-\theta^*_s} X^*_s\right\} \\
M_{st} = \frac{\alpha_{sm}}{\alpha_{sl}} \frac{W_t/P_t}{Q_t Q^*_s} L_{st} \\
E_0 \sum_{t=0}^{\infty} \left[\beta^t \sum_{s\in S} \left(\left(\frac{P^X_{st}/P_t}{Q_t P^*_s}\right)^{-\theta^*_s} X^*_s \frac{P^X_{st}}{Q_t P_t} - M_{st} Q^*_s\right)\right] = D_0
\end{cases}$$

where

$$Y_{s,t} = C_{st},\ Y^X_{s,t} = \left(\frac{P^X_{st}/P_t}{Q_t P^*_s}\right)^{-\theta^*_s} X^*_s$$

and $C_t = \prod_{s\in S} C_{st}^{\psi_s},\ L_t = \sum_{s\in S} L_{st}$.

A.2.4 The solution and properties of the optimal steady state

A.2.4.1 The solution

Before solving this, solve out C_{st}, M_{st} as functions of prices and aggregate consumption.

Define the Lagrangian

$$
\begin{aligned}
\mathcal{L} = \sum_{t=0}^{\infty} \beta^t & \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{\left(\sum_{s\in S} L_{st}\right)^{1+\phi}}{1+\phi} \right. \\
& + \sum_{s\in S} \Xi_{1t}^{s} \left\{ \left(\Delta_{st} \frac{\psi_s C_t}{P_{st}/P_t} + \Delta_{st}^{X} \left(\frac{P_{st}^{X}/P_t}{Q_t P_s^*} \right)^{-\theta_s^*} X_s^* \right) \bar{Z}_s^{-1} \left(\frac{\alpha_{sl}}{\alpha_{sm}} \frac{Q_t Q_s^*}{W_t/P_t} \right)^{\alpha_{sm}} - L_{st} \right\} \\
& + \Xi_{2t} \left\{ \left(\sum_{s\in S} L_{st} \right)^{\phi} - \frac{W_t}{P_t} C_t^{-\sigma} \right\} + \Xi_{3t} \left(\frac{P_{st}/P_t}{\psi_s} \right)^{\psi_s} \\
& + \sum_{(s,u)} \Xi_{4t}^{(s,u)} \left\{ f_s \left(\frac{P_{s,t}^{u}}{P_t}, \Pi_t; \frac{P_{s,t-1}^{u}}{P_{t-1}} \right) \tilde{K}_{s,t}^{u} - \tilde{F}_{s,t}^{u} \right\} \\
& + \sum_{s\in S} \Xi_{5t}^{s} \left\{ \psi_s C_t^{1-\sigma} + \lambda_s \beta E_t \left(\Pi_{s,t+1} \right)^{\theta_s - 1} \tilde{F}_{s,t+1} - \tilde{F}_{s,t} \right\} \\
& + \sum_{s\in S} \Xi_{5t}^{s,X} \left\{ C_t^{-\sigma} \frac{P_{s,t}^{X}}{P_t} \left(\frac{P_{st}^{X}/P_t}{Q_t P_s^*} \right)^{-\theta_s^*} X_s^* + \lambda_s \beta E_t \left(\Pi_{s,t+1}^{X} \right)^{\theta_s - 1} \tilde{F}_{s,t+1}^{X} - \tilde{F}_{s,t}^{X} \right\} \\
& + \sum_{s\in S} \Xi_{6t}^{s} \left\{ (1-\tau_s)^{-1} \frac{\theta_s}{\theta_s - 1} C_t^{1-\sigma} \left(\frac{Q_t Q_s^*}{\alpha_{sm}} \right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl} P_t} \right)^{\alpha_{sl}} \frac{1}{Z_{st}} \frac{\psi_s}{P_{st}/P_t} \right. \\
& \left. + \lambda_s \beta E_t \left(\Pi_{s,t+1} \right)^{\theta_s} \tilde{K}_{s,t+1} - \tilde{K}_{s,t} \right\} \\
& + \sum_{s\in S} \Xi_{6t}^{s,X} \left\{ \left(1 - \tau_s^{X} \right)^{-1} \frac{\theta_s}{\theta_s - 1} C_t^{-\sigma} \left(\frac{Q_t Q_s^*}{\alpha_{sm}} \right)^{\alpha_{sm}} \left(\frac{W_t}{\alpha_{sl} P_t} \right)^{\alpha_{sl}} \frac{1}{Z_{st}} \left(\frac{P_{st}^{X}/P_t}{Q_t P_s^*} \right)^{-\theta_s^*} X_s^*. \right. \\
& \left. + \lambda_s \beta E_t \left(\Pi_{s,t+1}^{X} \right)^{\theta_s} \tilde{K}_{s,t+1}^{X} - \tilde{K}_{s,t}^{X} \right\} \\
& + \sum_{(s,u)} \Xi_{7t}^{(s,u)} \left\{ \lambda_s \left(\frac{P_{st}^{u}}{P_{st-1}^{u}} \right)^{\theta_s} \Delta_{s,t-1}^{u} + (1-\lambda_s) \left(f_s \left(\frac{P_{s,t}^{u}}{P_t}, \Pi_t; \frac{P_{s,t-1}^{u}}{P_{t-1}} \right) \right)^{\theta_s} - \Delta_{st}^{u} \right\} \\
& \left. + \Xi_{8t} \left\{ \xi C^* Q_t^{\frac{1}{\sigma}} - C_t \right\} \right] + \Lambda D_0 \\
& + \Xi_9 \left\{ \sum_{t=0}^{\infty} \left[\beta^t \sum_{s\in S} \left(\left(\frac{P_{st}^{X}/P_t}{Q_t P_s^*} \right)^{-\theta_s^*} X_s^* \frac{P_{st}^{X}}{Q_t P_t} - \frac{\alpha_{sm}}{\alpha_{sl}} \frac{W_t/P_t}{Q_t} L_{st} \right) \right] - D_0 \right\}.
\end{aligned}
$$

By taking the first-order condition with respect to C_t , Q_t, L_{st}, W_t/P_t, Δ_{st}, P_{st}/P_t, P_{Xt}^{X}/P_t, Π_t, $\tilde{F}_{s,t}^{u}$, $\tilde{K}_{s,t}^{u}$, it can be shown that there exists a solution to this system of first-order conditions that satisfies $\Pi_t = \Pi_{s,t}^{u} = 1$, $\Delta_t^{(s,t)} = 1$, $C_t = \bar{C}$, $L_t = \bar{L}$,$Q_t = \bar{Q}$, $W_t/P_t = \bar{W}$,

$P^u_{st}/P_t = \bar{P}^u_s$, $\tilde{F}^u_{s,t} = \bar{F}^u_s$ and $\tilde{K}^u_{s,t} = \bar{K}^u_s$ with constant Lagrange multipliers. To do this, use the following relationships: $f_s(P_s, 1; P_s) = 1$, $f_{s1}(P_s, 1; P_s) = \frac{-\lambda_s}{1-\lambda_s} P_s^{-1}$, $f_{s2}(P_s, 1; P_s) = \frac{-\lambda_s}{1-\lambda_s}$ and $f_{s3}(P_s, 1; P_s) = \frac{\lambda_s}{1-\lambda_s} P_s^{-1}$ to see that the first-order conditions reduce to 10 linear equations with respect to $\left(\Xi_{1t}, \Xi_{2t}, \Xi_{3t}, \Xi^M_{6t}, \Xi^X_{6t}, \Xi^{XX}_{6t}, \Xi^M_{7t}, \Xi^X_{7t}, \Xi^{XX}_{7t}, \Xi_{8t}\right)$. Thus, generically, we can solve the system given C, $\{L_s\}$,Q, W, P^u_s, and F^u_s . The values for C, $\{L_s\}$,Q, W, P^u_s, F^u_s are the solutions to the constraints with zero inflation. Thus we have shown that the optimal steady state is characterized by the following.

$$\begin{cases}
\psi_s C = P_s C_s \\
\frac{L^\phi}{C^{-\sigma}} = W \\
C = \xi C^* Q^{\frac{1}{\sigma}} \\
\chi_s^{-1} \left(\frac{QQ^*_s}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W}{\alpha_{sl}}\right)^{\alpha_{sl}} \frac{1}{Z_s} = P_s \\
\chi_s^{-1} \nu_s^{-1} \frac{\theta^*_s}{\theta^*_s - 1} \left(\frac{QQ^*_s}{\alpha_{sm}}\right)^{\alpha_{sm}} \left(\frac{W}{\alpha_{sl}}\right)^{\alpha_{sl}} \frac{1}{Z_s} = P^X_s \\
Z_s L_s = \left(\frac{\alpha_{sl}}{\alpha_{sm}} \frac{QQ^*_s}{W}\right)^{\alpha_{sm}} \left(C_s + \left(\frac{P^X_s}{QP^*_s}\right)^{-\theta^*_s} X^*_s\right) \\
M_s = \frac{\alpha_{sm}}{\alpha_{sl}} \frac{W}{QQ^*_s} L_s = \left(\frac{W}{\alpha_{sl}}\right)^{\alpha_{sl}} \left(\frac{QQ^*_s}{\alpha_{sm}}\right)^{\alpha_{sm}-1} (C_s + X_s) \frac{1}{Z_s} = \alpha_{sm} \chi_s \frac{P_s}{QQ^*_s} (C_s + X_s) \\
\sum_{t=0}^{\infty} \left[\beta^t \sum_{s \in S} \left(\left(\frac{P^X_s}{QP^*_s}\right)^{-\theta^*_s} X^*_s \frac{P^X_s}{Q} - M_s Q^*_s\right)\right] = D_0
\end{cases}$$

A.2.4.2 Properties

Note that by the definition of $\xi := \left(\frac{\Lambda}{\left(C^*_0\right)^{-\sigma} / P^*_0}\right)^{-\frac{1}{\sigma}}$ and the assumption that $P^* = 1$, we have

$$C = \xi C^* Q^{\frac{1}{\sigma}} \Leftrightarrow C^\sigma \frac{\Lambda}{\left(C^*_0\right)^{-\sigma} / P^*_0} = C^{*\sigma} Q \Leftrightarrow \frac{\Lambda}{Q} = C^{-\sigma}.$$

Note also that by the definition of μ_s, ξ_s,

$$\mu_s = \frac{M_s Q_s^*}{\sum_{s \in S} \left(\left(\frac{P_s^X}{QP_s^*} \right)^{-\theta_s^*} X_s^* \frac{P_s^X}{Q} - M_s Q_s^* \right)}, \xi_s = \frac{\left(\frac{P_s^X}{QP_s^*} \right)^{-\theta_s^*} X_s^* \frac{P_s^X}{Q}}{\sum_{s \in S} \left(\left(\frac{P_s^X}{QP_s^*} \right)^{-\theta_s^*} X_s^* \frac{P_s^X}{Q} - M_s Q_s^* \right)}.$$

Thus,

$$\mu_s \left(1 - \beta\right) D_0 = M_s Q_s^*, \xi_s \left(1 - \beta\right) D_0 = \left(\frac{P_s^X}{QP_s^*} \right)^{-\theta_s^*} X_s^* \frac{P_s^X}{Q}.$$

Let us first show the following relationships since these appear a few times.

$$\begin{cases} \Lambda \bar{D}_0 \left(1 - \beta\right) \xi_s \left(1 - \theta_s^*\right) = -\chi_s^{-1} \nu_s^{-1} \theta_s^* \phi_{sx} \frac{\phi_{ls}}{\alpha_{sl}} L^{1+\phi} \\ C^{1-\sigma} = \sum_{s \in S} \chi_s^{-1} \phi_{sc} \frac{\phi_{ls}}{\alpha_{sl}} L^{1+\phi} \\ \Lambda \bar{D}_0 \left(1 - \beta\right) \mu_s = \alpha_{sm} \frac{\phi_{ls}}{\alpha_{sl}} L^{1+\phi} \\ M_w M_l^{-1} = L^{1+\phi} \boldsymbol{\phi}_l' d \left(\boldsymbol{\alpha}_l\right)^{-1} \\ \psi_s = \frac{\chi_s^{-1} \phi_{sc} \frac{\phi_{ls}}{\alpha_{sl}}}{\sum_{s' \in S} \chi_{s'}^{-1} \phi_{s'c} \frac{\phi_{ls'}}{\alpha_{s'l}}} \end{cases}$$

First,

$$\Lambda \bar{D}_0 \left(1 - \beta\right) \xi_s \left(1 - \theta_s^*\right) = -\chi_s^{-1} \nu_s^{-1} \theta_s^* \phi_{sx} \frac{\phi_{ls}}{\alpha_{sl}} L^{1+\phi}$$

where I used the characterization of the steady state and

$$Z_s L_s = \left(\frac{\alpha_{sl}}{\alpha_{sm}} \frac{QQ_s^*}{W} \right)^{\alpha_{sm}} \left(C_s + X_s\right) \Rightarrow \left(\frac{\alpha_{sl}}{\alpha_{sm}} \frac{QQ_s^*}{W} \right)^{\alpha_{sm}} = \frac{Z_s L_s}{C_s + X_s}.$$

Next, $C^{1-\sigma}$ becomes

$$C^{1-\sigma} = \boldsymbol{\phi}_l' diag \boldsymbol{\chi}^{-1} diag \boldsymbol{\alpha}_l^{-1} \boldsymbol{\phi}_c L^{1+\phi},$$

where I again used the relationship derived from the resource constraint. Finally,$\Lambda \bar{D}_0 \left(1-\beta\right) \mu_s$ can be calculated as follows.

$$\Lambda \bar{D}_0 \left(1-\beta\right) \mu_s = L^{1+\phi} \alpha_{sm} \chi_s \frac{\psi_s}{\phi_{sc}} \boldsymbol{\phi}_l' diag \boldsymbol{\chi}^{-1} diag \boldsymbol{\alpha}_l^{-1} \boldsymbol{\phi}_c$$

Thus, recall that

$$\begin{aligned} \phi_{ls} &= \frac{L_s}{L} \\ &= \left(\sum_{s' \in S} \chi_{s'}^{-1} \phi_{sc'} \frac{\phi_{ls'}}{\alpha_{s'l}} \right) \frac{\psi_s}{\phi_{sc}} \alpha_{sl} \chi_s . \end{aligned}$$

Therefore, we have

$$\psi_s = \frac{\chi_s^{-1} \phi_{sc} \frac{\phi_{ls}}{\alpha_{sl}}}{\sum_{s' \in S} \chi_{s'}^{-1} \phi_{sc'} \frac{\phi_{ls'}}{\alpha_{s'l}}}.$$

A.2.5 Second-order approximated welfare function

Exact relationships In the following, I will use the following equilibrium relationships.

$$
\begin{cases}
\psi_s C_t = \frac{P_{st}}{P_t} C_{st} \\
\frac{L_t^{\phi}}{C_t^{-\sigma}} = \frac{W_t}{P_t} \\
C_t = \xi C_t^* Q_t^{\frac{1}{\sigma}} \\
M_{st} = \frac{\alpha_{sm}}{\alpha_{sl}} \frac{W_t/P_t}{Q_t Q_{st}^*/P_t^*} L_{st} \\
Y_{s,t} = C_{st},\ Y_{s,t}^X = \left(\frac{P_{st}^X/P_t}{Q_t P_{st}^*/P_t^*}\right)^{-\theta_s^*} X_{st}^* = X_{st} \\
C_t = \prod_{s\in S} C_{st}^{\psi_s}
\end{cases}
$$

$$
\Rightarrow \begin{cases}
p_{st} = c_t - c_{st} = \sum_{s'\in S} \psi_{s'} c_{s't} - c_{st} & (a) \\
w_t = \phi l_t + \sigma c_t = \phi l_t + \sigma \sum_{s'\in S} \psi_{s'} c_{s't} & (b) \\
q_t = \sigma\left(c_t - c_t^*\right) = \sigma\left(\sum_{s'\in S} \psi_{s'} c_{s't} - c_t^*\right) & (c) \\
m_{st} = w_t - q_t - q_{st}^* + l_{st} = \phi l_t + l_{st} + \sigma c_t^* - q_{st}^* & (d) \\
y_{st} = c_{st},\ y_{st}^X = -\theta_s^*\left(p_{st}^X - q_t - p_{st}^*\right) + x_{st}^* = x_{st} & (e) \\
c_t = \sum_{s\in S} \psi_s c_{st} & (f)
\end{cases} \tag{A.1}
$$

One can see that the $\left\{p_{st}, p_{st}^X, m_{st}, y_{st}, y_{st}^X\right\}_{s\in S}, c_t, w_t, q_t$ can be written as linear functions of $\{c_{st}, x_{st}, l_{st}\}_{s\in S}$ and l_t. The rest of the equations are the resource constraint

$$
Z_{st} L_{st} = \left(\frac{\alpha_{sl}}{\alpha_{sm}} \frac{Q_t Q_{st}^*/P_t^*}{W_t/P_t}\right)^{\alpha_{sm}} \left\{\Delta_{st} C_{st} + \Delta_{st}^X \left(\frac{P_{st}^X/P_t}{Q_t P_{st}^*/P_t^*}\right)^{-\theta_s^*} X_{st}^*\right\}
$$

and the initial level of debt

$$E_0 \sum_{t=0}^{\infty} \left[\beta^t \frac{(C_t^*)^{-\sigma}}{(C_0^*)^{-\sigma} / P_0^*} \sum_{s \in S} \left(\left(\frac{P_{st}^X / P_t}{Q_t P_{st}^* / P_t^*} \right)^{-\theta_s^*} X_{st}^* \frac{P_{st}^X}{Q_t P_t} - M_{st} \frac{Q_{st}^*}{P_t^*} \right) \right] = D_0.$$

Naive Welfare Since welfare is

$$\mathcal{W} = E_0 \sum_{t=0}^{\infty} \beta^t \left[\underbrace{\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\phi}}{1+\phi}}_{=:U_t} \right] + \Lambda D_0,$$

denote the steady-state value of the welfare by

$$\bar{\mathcal{W}} = \frac{1}{1-\beta} U + \Lambda \bar{D}_0.$$

Subtracting this from welfare can still serve as our welfare criterion.

$$\mathcal{W} - \bar{\mathcal{W}} = E_0 \sum_{t=0}^{\infty} \beta^t \left[U_t - U \right] + \Lambda \bar{D}_0 \left(\frac{D_0 - \bar{D}_0}{\bar{D}_0} \right),$$

The second-order Taylor expansion of $U_t \left(C_t, L_t \right) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\phi}}{1+\phi}$ around the steady state (C, L) is

$$U_t - U \quad \approx \quad C^{1-\sigma} \left(c_t + \frac{1-\sigma}{2} c_t^2 \right) - L^{1+\phi} \left(l_t + \frac{1+\phi}{2} l_t^2 \right)$$

Using $L_t = \sum_{s \in S} L_{st}$,

$$l_t + \frac{1}{2} l_t^2 = \sum_{s \in S} \phi_{ls} l_{st} + \frac{1}{2} \sum_{s \in S} \phi_{ls} l_{st}^2$$

where

$$\phi_{ls} = \frac{L_s}{L}.$$

Plugging this into the above,

$$\begin{aligned} U_t - U &\approx C^{1-\sigma}\left(c_t + \frac{1-\sigma}{2}c_t^2\right) - L^{1+\phi}\left(\sum_{s\in S}\phi_{ls}\hat{l}_{st} + \frac{1}{2}\sum_{s\in S}\phi_{ls}\hat{l}_{st}^2 + \frac{\phi}{2}\hat{l}_t^2\right) \\ &= C^{1-\sigma}c_t - L^{1+\phi}\sum_{s\in S}\phi_{ls}l_{st} + \frac{1}{2}S_{Wt} \end{aligned}$$

where

$$S_{Wt} = C^{1-\sigma}\left(1-\sigma\right)c_t^2 - L^{1+\phi}\left(\sum_{s\in S}\phi_{ls}l_{st}^2 + \phi l_t^2\right).$$

Similarly to the standard closed economy NK models, we can use the approximated resource constraint to derive the relationship between l_t and c_t. First, take the second-order approximation as follows. Since Δ_{st} is of second order or higher,

$$Z_{st}L_{st} = \left(\frac{\alpha_{sl}}{\alpha_{sm}}\frac{Q_tQ_{st}^*/P_t^*}{W_t/P_t}\right)^{\alpha_{sm}}\left\{\Delta_{st}C_{st} + \Delta_{st}^X\left(\frac{P_{st}^X/P_t}{Q_tP_{st}^*/P_t^*}\right)^{-\theta_s^*}X_{st}^*\right\}$$

$$\begin{aligned} &\Rightarrow z_{st} + l_{st} - \alpha_{sm}\left(q_t + q_{st}^* - w_t\right) + \frac{1}{2}\left\{z_{st} + l_{st} - \alpha_{sm}\left(q_t + q_{st}^* - w_t\right)\right\}^2 \\ &\quad = \phi_{sc}\left(\Delta_{st} + c_{st} + \frac{1}{2}c_{st}^2\right) + \phi_{sx}\left(\Delta_{st}^X + x_{st} + \frac{1}{2}x_{st}^2\right), \end{aligned}$$

where

$$\phi_{sc} = \frac{C_s}{C_s + X_s},\ \phi_{sx} = \frac{X_s}{C_s + X_s}.$$

Utilize equation A.1-(b),(c), and

$$l_t + \frac{1}{2}l_t^2 = \sum_{s\in S}\phi_{ls}l_{st} + \frac{1}{2}\sum_{s\in S}\phi_{ls}l_{st}^2,$$

$$\begin{aligned}
&z_{st} + l_{st} - \alpha_{sm}\left(-\sigma c_t^* + q_{st}^* - \phi\left(\sum_{s'\in S}\phi_{ls'}l_{s't} + \frac{1}{2}\sum_{s'\in S}\phi_{ls'}l_{s't}^2 - \frac{1}{2}l_t^2\right)\right)\\
&+\frac{1}{2}\left\{z_{st} + l_{st} - \alpha_{sm}\left(-\sigma c_t^* + q_{st}^* - \phi l_t\right)\right\}^2\\
=&\phi_{sc}\left(\Delta_{st} + c_{st} + \frac{1}{2}c_{st}^2\right) + \phi_{sx}\left(\Delta_{st}^X + x_{st} + \frac{1}{2}x_{st}^2\right),
\end{aligned}$$

Solving for the linear term in l_{st}, and gathering the quadratic terms together,

$$\begin{aligned}
&l_{st} + \alpha_{sm}\phi\left(\sum_{s'\in S}\phi_{ls'}l_{s't}\right)\\
=&\phi_{sc}c_{st} + \phi_{sx}x_{st} + \alpha_{sm}\left(-\sigma c_t^* + q_{st}^*\right) - \alpha_{sm}\phi\left(\frac{1}{2}\sum_{s'\in S}\phi_{ls'}l_{s't}^2 - \frac{1}{2}l_t^2\right)\\
&+\phi_{sc}\left(\Delta_{st} + \frac{1}{2}c_{st}^2\right) + \phi_{sx}\left(\Delta_{st}^X + \frac{1}{2}x_{st}^2\right) - z_{st} - \frac{1}{2}\left\{z_{st} + l_{st} - \alpha_{sm}\left(-\sigma c_t^* + q_{st}^* - \phi l_t\right)\right\}^2
\end{aligned}$$

In matrix,

$$\begin{aligned}
&\underbrace{\left[I + \phi d\left(\boldsymbol{\alpha}_m\right)\mathbf{1}_{S\times 1}\boldsymbol{\phi}_l'\right]}_{=:M_l}\boldsymbol{l}_t\\
=&d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t + d\left(\boldsymbol{\alpha}_m\right)\left(-\mathbf{1}_{S\times 1}\sigma c_t^* + \boldsymbol{q}_t^*\right) - \boldsymbol{z}_t\\
&-\frac{1}{2}\boldsymbol{\alpha}_m\phi\boldsymbol{l}_t'\left(d\left(\boldsymbol{\phi}_l\right) - \boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)\boldsymbol{l}_t + \frac{1}{2}d\left(\boldsymbol{\phi}_c\right)\left(2\boldsymbol{\Delta}_t + d\left(\boldsymbol{c}_t\right)\boldsymbol{c}_t\right) + \frac{1}{2}d\left(\boldsymbol{\phi}_x\right)\left(2\boldsymbol{\Delta}_t^X + d\left(\boldsymbol{x}_t\right)\boldsymbol{x}_t\right)\\
&-\frac{1}{2}d\left(d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t\right)\left(d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t\right)
\end{aligned}$$

Thus, up to first order,

$$\boldsymbol{l}_t = M_l^{-1} \left\{ d\left(\boldsymbol{\phi}_c\right) \boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right) \boldsymbol{x}_t + d\left(\boldsymbol{\alpha}_m\right) \left(-\mathbf{1}_{S\times 1}\sigma c_t^* + \boldsymbol{q}_t^*\right) - \boldsymbol{z}_t \right\}.$$

Furthermore, noticing that $\sum_{t=0}^{\infty} \beta^t E_0 \Delta_{st}^u = \sum_{t=0}^{\infty} \beta^t E_0 \frac{\theta_s}{2\kappa_s} \left(\pi_{s,t}^u\right)^2$,where $\kappa_s = \frac{(1-\lambda_s)(1-\beta\lambda_s)}{\lambda_s}$ the infinite sum becomes

$$\sum_{t=0}^{\infty} \beta^t E_0 \hat{\boldsymbol{l}}_t = M_l^{-1} \sum_{t=0}^{\infty} \beta^t E_0 \left[d\left(\boldsymbol{\phi}_c\right) \boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right) \boldsymbol{x}_t \right] + \frac{1}{2} \sum_{t=0}^{\infty} \beta^t E_0 \boldsymbol{S}_{Rt} + t.i.p.,$$

where

$$\begin{aligned} \boldsymbol{S}_{Rt} &= M_l^{-1} \left[d\left(\boldsymbol{\phi}_c\right) d\left(\boldsymbol{c}_t\right) \boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{x}_t\right) \boldsymbol{x}_t - \boldsymbol{\alpha}_m \phi \boldsymbol{l}_t' \left(d\left(\boldsymbol{\phi}_l\right) - \boldsymbol{\phi}_l \boldsymbol{\phi}_l' \right) \boldsymbol{l}_t \right] \\ &- M_l^{-1} \left[d\left(d\left(\boldsymbol{\phi}_c\right) \boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right) \boldsymbol{x}_t \right) \left(d\left(\boldsymbol{\phi}_c\right) \boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right) \boldsymbol{x}_t \right) \right] \\ &+ M_l^{-1} \left[d\left(\boldsymbol{\theta}\right) d\left(\boldsymbol{\kappa}\right)^{-1} \left\{ d\left(\boldsymbol{\phi}_c\right) d\left(\boldsymbol{\pi}_t\right) \boldsymbol{\pi}_t + d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\pi}_t^X\right) \boldsymbol{\pi}_t^X \right\} \right] \end{aligned}$$

By approximating the lifetime international budget condition, we can approximate the initial debt $\frac{D_0 - \bar{D}_0}{\bar{D}_0} \approx \hat{d}_0 + \frac{1}{2}\hat{d}_0^2$ as

$$d_0 + \frac{1}{2} d_0^2 = (1-\beta) E_0 \sum_{t=0}^{\infty} \beta^t \left[\sum_{s\in S} \xi_s \frac{\theta_s^* - 1}{\theta_s^*} x_{st} - \boldsymbol{\mu}' M_m \hat{\boldsymbol{l}}_t + \frac{1}{2} \frac{S_{Dt}}{1-\beta} \right] + t.i.p.$$

$$
\begin{aligned}
\frac{\tilde{S}_{Dt}}{1-\beta} := {} & \boldsymbol{x}_t' d\left(\boldsymbol{\theta}^* - 1\right) d\left(\boldsymbol{\theta}^*\right)^{-1} d\left(\boldsymbol{\xi}\right) d\left(\boldsymbol{\theta}^* - 1\right) d\left(\boldsymbol{\theta}^*\right)^{-1} \boldsymbol{x}_t \\
& + 2\left(-\sigma \hat{c}_t^* \mathbf{1}_{S\times 1} + d\left(\boldsymbol{\theta}^*\right)^{-1} \boldsymbol{x}_t^* + \boldsymbol{p}_t^*\right)' d\left(\boldsymbol{\xi}\right) d\left(\boldsymbol{\theta}^* - 1\right) d\left(\boldsymbol{\theta}^*\right)^{-1} \boldsymbol{x}_t \\
& - \boldsymbol{l}_t' M_m' d\left(\boldsymbol{\mu}\right) M_m \boldsymbol{l}_t
\end{aligned}
$$

$$
\begin{aligned}
\frac{S_{Dt}}{1-\beta} = {} & \frac{\tilde{S}_{Dt}}{1-\beta} - \phi \boldsymbol{\mu}' \mathbf{1}_{S\times 1} \boldsymbol{l}_t' \left(d\left(\boldsymbol{\phi}_l\right) - \boldsymbol{\phi}_l \boldsymbol{\phi}_l'\right) \boldsymbol{l}_t \\
= {} & \boldsymbol{x}_t' d\left(\boldsymbol{\theta}^* - 1\right) d\left(\boldsymbol{\theta}^*\right)^{-1} d\left(\boldsymbol{\xi}\right) d\left(\boldsymbol{\theta}^* - 1\right) d\left(\boldsymbol{\theta}^*\right)^{-1} \boldsymbol{x}_t \\
& + 2\left(-\sigma \hat{c}_t^* \mathbf{1}_{S\times 1} + d\left(\boldsymbol{\theta}^*\right)^{-1} \boldsymbol{x}_t^* + \boldsymbol{p}_t^*\right)' d\left(\boldsymbol{\xi}\right) d\left(\boldsymbol{\theta}^* - 1\right) d\left(\boldsymbol{\theta}^*\right)^{-1} \boldsymbol{x}_t \\
& - \boldsymbol{l}_t' \left(M_m' d\left(\boldsymbol{\mu}\right) M_m + \phi \boldsymbol{\mu}' \mathbf{1}_{S\times 1} \left(d\left(\boldsymbol{\phi}_l\right) - \boldsymbol{\phi}_l \boldsymbol{\phi}_l'\right)\right) \boldsymbol{l}_t
\end{aligned}
$$

$$
M_m = \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l' + I
$$

Plugging the expressions for $U_t - U$, $\sum_{t=0}^{\infty} \beta^t E_0\left[\hat{l}_t\right]$, and $\frac{D_0 - \bar{D}_0}{\bar{D}_0}$ obtained above into the equation for $\mathcal{W} - \bar{\mathcal{W}}$, we obtain the following welfare criterion

$$
\begin{aligned}
U_t - U & \approx C^{1-\sigma}\left(c_t + \frac{1-\sigma}{2} c_t^2\right) - L^{1+\phi}\left(\sum_{s\in S} \phi_{ls} \hat{l}_{st} + \frac{1}{2} \sum_{s\in S} \phi_{ls} \hat{l}_{st}^2 + \frac{\phi}{2} \hat{l}_t^2\right) \\
& = C^{1-\sigma} c_t - L^{1+\phi} \sum_{s\in S} \phi_{ls} l_{st} + \frac{1}{2} S_{Wt}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{W} - \bar{\mathcal{W}} &= \sum_{t=0}^{\infty} \beta^t E_0 \left[C^{1-\sigma} c_t \right] - L^{1+\phi} \boldsymbol{\phi}_l' \sum_{t=0}^{\infty} \beta^t E_0 \boldsymbol{l}_t \\
&+ \Lambda \bar{D}_0 \left(1-\beta\right) E_0 \sum_{t=0}^{\infty} \beta^t \left[\boldsymbol{\xi}' d\left(\boldsymbol{\theta}^* - 1\right) d\left(\boldsymbol{\theta}^*\right)^{-1} \boldsymbol{x}_t - \boldsymbol{\mu}' M_m \boldsymbol{l}_t \right] \\
&+ \frac{1}{2} \sum_{t=0}^{\infty} \beta^t E_0 \left[S_{Wt} + \Lambda \bar{D}_0 S_{Dt} \right] + t.i.p. \\
&= \sum_{t=0}^{\infty} \beta^t E_0 \underbrace{\left[C^{1-\sigma} \boldsymbol{\psi}' - M_w M_l^{-1} d\left(\boldsymbol{\phi}_c\right) \right]}_{=:L^{1+\phi} f_c(\chi,\nu)} \boldsymbol{c}_t \\
&+ \sum_{t=0}^{\infty} \beta^t E_0 \underbrace{\left[\Lambda \bar{D}_0 \left(1-\beta\right) \boldsymbol{\xi}' d\left(\boldsymbol{\theta}^* - 1\right) d\left(\boldsymbol{\theta}^*\right)^{-1} - M_w M_l^{-1} d\left(\boldsymbol{\phi}_x\right) \right]}_{=:L^{1+\phi} f_x(\chi,\nu)} \boldsymbol{x}_t \\
&+ \frac{1}{2} \sum_{t=0}^{\infty} \beta^t E_0 \left[S_{Wt} + \Lambda \bar{D}_0 S_{Dt} - M_w \boldsymbol{S}_{Rt} \right] + t.i.p.
\end{aligned}
$$

Finally, I show that $f_c\left(\chi,\nu\right)$ and $f_x\left(\chi,\nu\right)$ can be simplified as

$$
f_c\left(\chi,\nu\right) = \boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_c\right) \left(d\left(\boldsymbol{\chi}\right)^{-1} - I \right)
$$

$$
f_x\left(\chi,\nu\right) = \boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right) \left(d\left(\boldsymbol{\chi}\right)^{-1} d\left(\boldsymbol{\nu}\right)^{-1} - I \right).
$$

To see this, first note the following.

$$
\begin{aligned}
M_w M_l^{-1} &= \left(L^{1+\phi} \boldsymbol{\phi}_l' + \Lambda \bar{D}_0 \left(1-\beta\right) \boldsymbol{\mu}' \left[I + \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l' \right] \right) \left[I + \phi d\left(\boldsymbol{\alpha}_m\right) \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l' \right]^{-1} \\
&= L^{1+\phi} \boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1}
\end{aligned}
$$

Using the properties derived in Appendix A.2.4, the desired relationships hold as follows;

$$\begin{aligned}
f_c(\chi,\nu) &= \frac{C^{1-\sigma}}{L^{1+\phi}}\boldsymbol{\psi}' - \frac{1}{L^{1+\phi}}M_w M_l^{-1} d(\boldsymbol{\phi}_c) \\
&= \sum_{s\in S}\chi_s^{-1}\phi_{sc}\frac{\phi_{ls}}{\alpha_{sl}}\boldsymbol{\psi}' - \boldsymbol{\phi}_l' d(\boldsymbol{\alpha}_l)^{-1} d(\boldsymbol{\phi}_c) \\
&= \boldsymbol{\phi}_l' d(\boldsymbol{\alpha}_l)^{-1} d(\boldsymbol{\phi}_c)\left(d(\boldsymbol{\chi})^{-1} - I\right)
\end{aligned}$$

$$\begin{aligned}
f_x(\chi,\nu) &= \frac{1}{L^{1+\phi}}\Lambda\bar{D}_0(1-\beta)\boldsymbol{\xi}' d(\boldsymbol{\theta}^*-1) d(\boldsymbol{\theta}^*)^{-1} - \frac{1}{L^{1+\phi}}M_w M_l^{-1} d(\boldsymbol{\phi}_x) \\
&= \boldsymbol{\phi}_l' d(\boldsymbol{\alpha}_l)^{-1} d(\boldsymbol{\phi}_x)\left(d(\boldsymbol{\chi})^{-1} d(\boldsymbol{\nu})^{-1} - I\right).
\end{aligned}$$

A.2.6 Natural rate under the efficient steady state

When the steady state is efficient, $\chi_M = \chi_X = \chi_T = 1$, and all the f are zeros. Thus, recalling $M_w M_l^{-1} = L^{1+\phi}\boldsymbol{\phi}_l' diag\left(\boldsymbol{\alpha}_l\right)^{-1}, M_m = \phi \mathbf{1}_{S\times 1}\boldsymbol{\phi}_l' + I$,

$$
\begin{aligned}
& L^{-(1+\phi)}\left(\mathcal{W}-\bar{\mathcal{W}}\right)|_{\text{efficient}} \\
=& \frac{L^{-(1+\phi)}}{2}\sum_{t=0}^{\infty}\beta^t E_0\left[S_{Wt} - M_w \boldsymbol{S}_{Rt} + \Lambda \bar{D}_0 S_{Dt}\right] + t.i.p. \\
=& \frac{1}{2}\sum_{t=0}^{\infty}\beta^t E_0\left[(1-\sigma)\boldsymbol{\phi}_l' d\left(\boldsymbol{\chi}\right)^{-1} d\left(\boldsymbol{\alpha}_l\right)^{-1}\boldsymbol{\phi}_c \boldsymbol{c}_t'\boldsymbol{\psi}\boldsymbol{\psi}'\boldsymbol{c}_t - \boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_c\right) d\left(\boldsymbol{c}_t\right)\boldsymbol{c}_t\right. \\
& -\boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{x}_t\right)\boldsymbol{x}_t + \boldsymbol{x}_t' d\left(\boldsymbol{\theta}^*-1\right) d\left(\boldsymbol{\theta}^*\right)^{-1} d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\chi}\right)^{-1} d\left(\boldsymbol{\nu}\right)^{-1} d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t \\
& +\boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t\right)\left(d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t\right) \\
& -\boldsymbol{l}_t'\left(d\left(\boldsymbol{\phi}_l\right) + \phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)\boldsymbol{l}_t + \boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1}\boldsymbol{\alpha}_m \phi \boldsymbol{l}_t'\left(d\left(\boldsymbol{\phi}_l\right) - \boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)\boldsymbol{l}_t \\
& -\boldsymbol{l}_t'\left(M_m' d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_m\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} M_m + \phi\mathbf{1}_{1\times S} d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_m\right) d\left(\boldsymbol{\alpha}_l\right)^{-1}\mathbf{1}_{S\times 1}\left(d\left(\boldsymbol{\phi}_l\right) - \boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)\right)\boldsymbol{l}_t \\
& +2\left(-\sigma\hat{c}_t^*\mathbf{1}_{S\times 1} + d\left(\boldsymbol{\theta}^*\right)^{-1}\boldsymbol{x}_t^* + \boldsymbol{p}_t^*\right)' d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\chi}\right)^{-1} d\left(\boldsymbol{\nu}\right)^{-1} d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t \\
& \left.-\boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1}\left[d\left(\boldsymbol{\theta}\right) d\left(\boldsymbol{\kappa}\right)^{-1}\left\{d\left(\boldsymbol{\phi}_c\right) d\left(\boldsymbol{\pi}_t\right)\boldsymbol{\pi}_t + d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\pi}_t^X\right)\boldsymbol{\pi}_t^X\right\}\right]\right] + t.i.p.
\end{aligned}
$$

where

$$
\boldsymbol{l}_t = M_l^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t + d\left(\boldsymbol{\alpha}_m\right)\left(-\mathbf{1}_{S\times 1}\sigma c_t^* + \boldsymbol{q}_t^*\right) - \boldsymbol{z}_t\right\}.
$$

Collecting terms, and recalling $\boldsymbol{\chi} = \boldsymbol{\nu} = \mathbf{1}_{S\times 1}$,

$$
\begin{aligned}
& L^{-(1+\phi)} \left(\mathcal{W} - \bar{\mathcal{W}}\right)|_{\text{efficient}} \\
=& \frac{1}{2}\sum_{t=0}^{\infty}\beta^t E_0 \Big[\boldsymbol{c}_t' \left((1-\sigma)\,\boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1}\boldsymbol{\phi}_c \boldsymbol{\psi}\boldsymbol{\psi}'\right)\boldsymbol{c}_t \\
& - \boldsymbol{c}_t' d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t \\
& - \boldsymbol{x}_t' \left(d\left(\boldsymbol{\theta}^*\right)^{-1} - d\left(\boldsymbol{\phi}_x\right)\right) d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t \\
& + 2\boldsymbol{c}_t' d\left(\boldsymbol{\phi}_c\right) d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t \\
& - \boldsymbol{l}_t' M_l' d\left(\boldsymbol{\alpha}_l\right)^{-1}\left(d\left(\boldsymbol{\phi}_l\right) + \phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)\boldsymbol{l}_t \\
& + 2\left(-\sigma\hat{c}_t^*\mathbf{1}_{S\times 1} + d\left(\boldsymbol{\theta}^*\right)^{-1}\boldsymbol{x}_t^* + \boldsymbol{p}_t^*\right)' d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t \\
& - \boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1}\left[d\left(\boldsymbol{\theta}\right) d\left(\boldsymbol{\kappa}\right)^{-1}\left\{d\left(\boldsymbol{\phi}_c\right) d\left(\boldsymbol{\pi}_t\right)\boldsymbol{\pi}_t + d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\pi}_t^X\right)\boldsymbol{\pi}_t^X\right\}\right] + t.i.p.
\end{aligned}
$$

Using

$$
\boldsymbol{l}_t = M_l^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t + d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t + d\left(\boldsymbol{\alpha}_m\right)\left(-\mathbf{1}_{S\times 1}\sigma c_t^* + \boldsymbol{q}_t^*\right) - \boldsymbol{z}_t\right\},
$$

$$
\begin{aligned}
& L^{-(1+\phi)}\left(\mathcal{W}-\bar{\mathcal{W}}\right)|_{\text{efficient}} \\
=& \frac{1}{2}\sum_{t=0}^{\infty}\beta^t E_0\Big[\boldsymbol{c}_t'\left(\left(1-\sigma\right)\boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1}\boldsymbol{\phi}_c\boldsymbol{\psi}\boldsymbol{\psi}'\right)\boldsymbol{c}_t \\
& -\boldsymbol{c}_t'\left\{d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\phi}_l\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}+d\left(\boldsymbol{\phi}_c\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}\left(d\left(\boldsymbol{\phi}_l\right)+\phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)M_l^{-1}\right\}d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t \\
& -\boldsymbol{x}_t'\left\{\left(d\left(\boldsymbol{\theta}^*\right)^{-1}-d\left(\boldsymbol{\phi}_x\right)\right)d\left(\boldsymbol{\phi}_l\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}+d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}\left(d\left(\boldsymbol{\phi}_l\right)+\phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)M_l^{-1}\right\}d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t \\
& +2\boldsymbol{c}_t' d\left(\boldsymbol{\phi}_c\right)\left\{d\left(\boldsymbol{\phi}_l\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}-d\left(\boldsymbol{\alpha}_l\right)^{-1}\left(d\left(\boldsymbol{\phi}_l\right)+\phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)M_l^{-1}\right\}d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t \\
& -2\left\{d\left(\boldsymbol{\alpha}_m\right)\left(-\mathbf{1}_{S\times 1}\sigma c_t^*+\boldsymbol{q}_t^*\right)-\boldsymbol{z}_t\right\}' d\left(\boldsymbol{\alpha}_l\right)^{-1}\left(d\left(\boldsymbol{\phi}_l\right)+\phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)M_l^{-1}d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t \\
& -2\Big[\Big[\left\{d\left(\boldsymbol{\alpha}_m\right)\left(-\mathbf{1}_{S\times 1}\sigma c_t^*+\boldsymbol{q}_t^*\right)-\boldsymbol{z}_t\right\}' d\left(\boldsymbol{\alpha}_l\right)^{-1}\left(d\left(\boldsymbol{\phi}_l\right)+\phi\boldsymbol{\phi}_l\boldsymbol{\phi}_l'\right)M_l^{-1}\Big] \\
& -\left(-\sigma\hat{c}_t^*\mathbf{1}_{S\times 1}+d\left(\boldsymbol{\theta}^*\right)^{-1}\boldsymbol{x}_t^*+\boldsymbol{p}_t^*\right)' d\left(\boldsymbol{\phi}_l\right)d\left(\boldsymbol{\alpha}_l\right)^{-1}\Big]d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t \\
& -\boldsymbol{\phi}_l' d\left(\boldsymbol{\alpha}_l\right)^{-1}\left[d\left(\boldsymbol{\theta}\right)d\left(\boldsymbol{\kappa}\right)^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)d\left(\boldsymbol{\pi}_t\right)\boldsymbol{\pi}_t+d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\pi}_t^X\right)\boldsymbol{\pi}_t^X\right\}\right]+t.i.p.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& L^{-(1+\phi)}\left(\mathcal{W}-\bar{\mathcal{W}}\right) \\
& = E_0\sum_{t=0}^{\infty}\beta^t\left[\frac{1}{2}v_t'\Gamma_{v2}v_t+\xi_t'\Gamma_{\xi v}v_t+\sum_{s\in S}\frac{\theta_s}{2\kappa_s}\left(\Gamma_{\pi s}\pi_{s,t}^2+\Gamma_{\pi s}^X\left(\pi_{s,t}^X\right)^2\right)\right] \\
& +t.i.p.,
\end{aligned}
$$

where

$$
\Gamma_{v2}=\begin{bmatrix}\Gamma_{c2} & \Gamma_{cx}\\ \Gamma_{cx}' & \Gamma_{x2}\end{bmatrix},
$$

$$\begin{cases} \Gamma_{c2} = & (1-\sigma)\,\boldsymbol{\phi}_l' d(\boldsymbol{\alpha}_l)^{-1}\,\boldsymbol{\phi}_c \boldsymbol{\psi}\boldsymbol{\psi}' \\ & -\left\{ d(\boldsymbol{\phi}_x)\, d(\boldsymbol{\phi}_l)\, d(\boldsymbol{\alpha}_l)^{-1} + d(\boldsymbol{\phi}_c)\, d(\boldsymbol{\alpha}_l)^{-1} \left(d(\boldsymbol{\phi}_l) + \phi \boldsymbol{\phi}_l \boldsymbol{\phi}_l' \right) M_l^{-1} \right\} d(\boldsymbol{\phi}_c) \\ \Gamma_{cx} = & -\phi d(\boldsymbol{\phi}_c)\, \boldsymbol{\phi}_l \boldsymbol{\phi}_l' M_l^{-1} d(\boldsymbol{\phi}_x) \\ \Gamma_{x2} = & -\left\{ \left(d(\boldsymbol{\theta}^*)^{-1} - d(\boldsymbol{\phi}_x) \right) d(\boldsymbol{\phi}_l)\, d(\boldsymbol{\alpha}_l)^{-1} \right\} d(\boldsymbol{\phi}_x) \\ & -\left\{ d(\boldsymbol{\phi}_x)\, d(\boldsymbol{\alpha}_l)^{-1} \left(d(\boldsymbol{\phi}_l) + \phi \boldsymbol{\phi}_l \boldsymbol{\phi}_l' \right) M_l^{-1} \right\} d(\boldsymbol{\phi}_x) \end{cases}.$$

To obtain the expression for $\Gamma_{\xi v}$, note

$$d(\boldsymbol{\alpha}_m)\left(-\mathbf{1}_{S\times 1}\sigma c_t^* + \boldsymbol{q}_t^*\right) - \boldsymbol{z}_t = \left[\begin{array}{ccccc} -\sigma\boldsymbol{\alpha}_m & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S} & d(\boldsymbol{\alpha}_m) & -I \end{array}\right] \xi_t$$

$$-\sigma \hat{c}_t^* \mathbf{1}_{S\times 1} + d(\boldsymbol{\theta}^*)^{-1} \boldsymbol{x}_t^* + \boldsymbol{p}_t^* = \left[\begin{array}{ccccc} -\sigma\mathbf{1}_{S\times 1} & d(\boldsymbol{\theta}^*)^{-1} & I & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S} \end{array}\right] \xi_t.$$

Thus,

$$\Gamma_{\xi v} = \left[\begin{array}{cc} \Gamma_{\xi c} & \Gamma_{\xi x} \end{array}\right],$$

$$\begin{cases} \Gamma_{\xi c} = & \left[\begin{array}{ccccc} \sigma\boldsymbol{\alpha}_m & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S} & -d(\boldsymbol{\alpha}_m) & I \end{array}\right]' d(\boldsymbol{\alpha}_l)^{-1} \left(d(\boldsymbol{\phi}_l) + \phi \boldsymbol{\phi}_l \boldsymbol{\phi}_l' \right) M_l^{-1} d(\boldsymbol{\phi}_c) \\ \Gamma_{\xi x} = & \left[\begin{array}{ccccc} \sigma\boldsymbol{\alpha}_m & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S} & -d(\boldsymbol{\alpha}_m) & I \end{array}\right]' d(\boldsymbol{\alpha}_l)^{-1} \left(d(\boldsymbol{\phi}_l) + \phi \boldsymbol{\phi}_l \boldsymbol{\phi}_l' \right) M_l^{-1} d(\boldsymbol{\phi}_x) \\ & + \left[\begin{array}{ccccc} -\sigma\mathbf{1}_{S\times 1} & d(\boldsymbol{\theta}^*)^{-1} & I & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S} \end{array}\right]' d(\boldsymbol{\phi}_l)\, d(\boldsymbol{\alpha}_l)^{-1} d(\boldsymbol{\phi}_x) \end{cases}$$

Now, calculate the flexible price equilibrium to simplify the above expression. The flexible price equilibrium is characterized by

$$\begin{cases} d\left(\boldsymbol{\alpha}_m\right)\left(\mathbf{1}_{S\times 1}q_t^F+\boldsymbol{q}_t^*\right)+\boldsymbol{\alpha}_l w_t^F-\boldsymbol{z}_t-\boldsymbol{p}_t^F=0 \\ d\left(\boldsymbol{\alpha}_m\right)\left(\mathbf{1}_{S\times 1}q_t^F+\boldsymbol{q}_t^*\right)+\boldsymbol{\alpha}_l w_t^F-\boldsymbol{z}_t-\boldsymbol{p}_t^{XF}=0 \end{cases}$$

where

$$\begin{cases} q_t^F=\sigma\left(\boldsymbol{\psi}'\boldsymbol{c}_t^F-c_t^*\right) \\ w_t^F=\phi\boldsymbol{\phi}_l'\boldsymbol{l}_t^F+\sigma\boldsymbol{\psi}'\boldsymbol{c}_t^F \\ \boldsymbol{l}_t^F=M_l^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t^F+d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t^F+d\left(\boldsymbol{\alpha}_m\right)\left(-\mathbf{1}_{S\times 1}\sigma c_t^*+\boldsymbol{q}_t^*\right)-\boldsymbol{z}_t\right\} \\ \boldsymbol{p}_t^F=\mathbf{1}_{S\times 1}\boldsymbol{\psi}'\boldsymbol{c}_t^F-\boldsymbol{c}_t^F \\ \boldsymbol{p}_t^{XF}=-d\left(\boldsymbol{\theta}^*\right)^{-1}\left(\boldsymbol{x}_t^F-\boldsymbol{x}_t^*\right)+\mathbf{1}_{S\times 1}\sigma\left(\boldsymbol{\psi}'\boldsymbol{c}_t^F-c_t^*\right)+\boldsymbol{p}_t^* \end{cases}.$$

First, the pricing equation gives

$$\boldsymbol{p}_t^F=\boldsymbol{p}_t^{XF}$$

$$\mathbf{1}_{S\times 1}\boldsymbol{\psi}'\boldsymbol{c}_t^F-\boldsymbol{c}_t^F=-d\left(\boldsymbol{\theta}^*\right)^{-1}\left(\boldsymbol{x}_t^F-\boldsymbol{x}_t^*\right)+\mathbf{1}_{S\times 1}\sigma\left(\boldsymbol{\psi}'\boldsymbol{c}_t^F-\hat{c}_t^*\right)+\boldsymbol{p}_t^*$$

$$\boldsymbol{x}_t^F=-d\left(\boldsymbol{\theta}^*\right)\left(\left(1-\sigma\right)\mathbf{1}_{S\times 1}\boldsymbol{\psi}'-I\right)\boldsymbol{c}_t^F+\boldsymbol{x}_t^*-\boldsymbol{\theta}^*\sigma\hat{c}_t^*+d\left(\boldsymbol{\theta}^*\right)\boldsymbol{p}_t^*$$

Therefore,

$$\begin{aligned}
\boldsymbol{l}_t^F &= M_l^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)\boldsymbol{c}_t^F + d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t^F + d\left(\boldsymbol{\alpha}_m\right)\left(-\mathbf{1}_{S\times 1}\sigma c_t^* + \boldsymbol{q}_t^*\right) - \boldsymbol{z}_t\right\} \\
&= M_l^{-1}\left\{d\left(\boldsymbol{\phi}_c\right) - d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right)\left(\left(1-\sigma\right)\mathbf{1}_{S\times 1}\boldsymbol{\psi}' - I\right)\right\}\boldsymbol{c}_t^F \\
&+ M_l^{-1}\left[-\left(d\left(\boldsymbol{\phi}_x\right)\boldsymbol{\theta}^* + \boldsymbol{\alpha}_m\right)\sigma c_t^* + d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t^* + d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right)\boldsymbol{p}_t^* + d\left(\boldsymbol{\alpha}_m\right)\boldsymbol{q}_t^* - \boldsymbol{z}_t\right]
\end{aligned}$$

Thus,

$$d\left(\boldsymbol{\alpha}_m\right)\left(\mathbf{1}_{S\times 1}q_t^F + \hat{\boldsymbol{q}}_t^*\right) + \boldsymbol{\alpha}_l w_t^F - \hat{\boldsymbol{z}}_t = \boldsymbol{p}_t^F$$

$$\begin{aligned}
&d\left(\boldsymbol{\alpha}_m\right)\left(\mathbf{1}_{S\times 1}\sigma\left(\boldsymbol{\psi}'\boldsymbol{c}_t^F - \hat{c}_t^*\right) + \hat{\boldsymbol{q}}_t^*\right) + \boldsymbol{\alpha}_l\left(\phi\boldsymbol{\phi}_l'\boldsymbol{l}_t^F + \sigma\boldsymbol{\psi}'\boldsymbol{c}_t^F\right) - \hat{\boldsymbol{z}}_t \\
&= \mathbf{1}_{S\times 1}\boldsymbol{\psi}'\boldsymbol{c}_t^F - \boldsymbol{c}_t^F
\end{aligned}$$

$$\begin{aligned}
&d\left(\boldsymbol{\alpha}_m\right)\mathbf{1}_{S\times 1}\sigma\boldsymbol{\psi}'\boldsymbol{c}_t^F + \boldsymbol{\alpha}_l\sigma\boldsymbol{\psi}'\boldsymbol{c}_t^F - \left(\mathbf{1}_{S\times 1}\boldsymbol{\psi}' - I\right)\boldsymbol{c}_t^F \\
&= \boldsymbol{\alpha}_m\sigma\hat{c}_t^* - d\left(\boldsymbol{\alpha}_m\right)\hat{\boldsymbol{q}}_t^* - \boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'\boldsymbol{l}_t^F + \hat{\boldsymbol{z}}_t
\end{aligned}$$

$$\begin{aligned}
&d\left(\boldsymbol{\alpha}_m\right)\mathbf{1}_{S\times 1}\sigma\boldsymbol{\psi}'\boldsymbol{c}_t^F + \boldsymbol{\alpha}_l\sigma\boldsymbol{\psi}'\boldsymbol{c}_t^F - \left(\mathbf{1}_{S\times 1}\boldsymbol{\psi}' - I\right)\boldsymbol{c}_t^F \\
&+ \boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l' M_l^{-1}\left\{d\left(\boldsymbol{\phi}_c\right) - d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right)\left(\left(1-\sigma\right)\mathbf{1}_{S\times 1}\boldsymbol{\psi}' - I\right)\right\}\boldsymbol{c}_t^F \\
=&\boldsymbol{\alpha}_m\sigma\hat{c}_t^* - d\left(\boldsymbol{\alpha}_m\right)\hat{\boldsymbol{q}}_t^* + \hat{\boldsymbol{z}}_t - \boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l' M_l^{-1} \\
&\times\left[-\left(d\left(\boldsymbol{\phi}_x\right)\boldsymbol{\theta}^* + \boldsymbol{\alpha}_m\right)\sigma c_t^* + d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t^* + d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right)\boldsymbol{p}_t^* + d\left(\boldsymbol{\alpha}_m\right)\boldsymbol{q}_t^* - \boldsymbol{z}_t\right]
\end{aligned}$$

$$
\begin{aligned}
&\left[(\sigma-1)\mathbf{1}_{S\times 1}\boldsymbol{\psi}'+I\right]\boldsymbol{c}_t^F\\
&+\left[\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)-d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right)\left((1-\sigma)\mathbf{1}_{S\times 1}\boldsymbol{\psi}'-I\right)\right\}\right]\boldsymbol{c}_t^F\\
=&\boldsymbol{\alpha}_m\sigma\hat{c}_t^*-d\left(\boldsymbol{\alpha}_m\right)\hat{\boldsymbol{q}}_t^*+\hat{\boldsymbol{z}}_t-\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\\
&\times\left[-\left(d\left(\boldsymbol{\phi}_x\right)\boldsymbol{\theta}^*+\boldsymbol{\alpha}_m\right)\sigma c_t^*+d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t^*+d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right)\boldsymbol{p}_t^*+d\left(\boldsymbol{\alpha}_m\right)\boldsymbol{q}_t^*-\boldsymbol{z}_t\right]
\end{aligned}
$$

That is,

$$
\begin{aligned}
&\left[(\sigma-1)\mathbf{1}_{S\times 1}\boldsymbol{\psi}'+I+\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)-d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right)\left((1-\sigma)\mathbf{1}_{S\times 1}\boldsymbol{\psi}'-I\right)\right\}\right]\boldsymbol{c}_t^F\\
&=\left[\boldsymbol{\alpha}_m+\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\left(d\left(\boldsymbol{\phi}_x\right)\boldsymbol{\theta}^*+\boldsymbol{\alpha}_m\right)\right]\sigma c_t^*-\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}d\left(\boldsymbol{\phi}_x\right)\boldsymbol{x}_t^*\\
&-\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right)\boldsymbol{p}_t^*-\left[I+\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\right]d\left(\boldsymbol{\alpha}_m\right)\boldsymbol{q}_t^*+\left[I+\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\right]\boldsymbol{z}_t
\end{aligned}
$$

$$
\boldsymbol{c}_t^F=M_{cc}^{-1}M_{c\xi}\xi_t
$$

$$
M_{cc}=\left[(\sigma-1)\mathbf{1}_{S\times 1}\boldsymbol{\psi}'+I+\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\left\{d\left(\boldsymbol{\phi}_c\right)-d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right)\left((1-\sigma)\mathbf{1}_{S\times 1}\boldsymbol{\psi}'-I\right)\right\}\right]
$$

$$
M_{c\xi}=\begin{bmatrix} M_{cc*} & M_{cx*} & M_{cp*} & M_{cq*} & M_{cz}\end{bmatrix}.
$$

$$
\begin{cases}
M_{cc*}=\left[\boldsymbol{\alpha}_m+\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\left(d\left(\boldsymbol{\phi}_x\right)\boldsymbol{\theta}^*+\boldsymbol{\alpha}_m\right)\right]\sigma\\
M_{cx*}=-\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}d\left(\boldsymbol{\phi}_x\right)\\
M_{cp*}=-\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}d\left(\boldsymbol{\phi}_x\right)d\left(\boldsymbol{\theta}^*\right)\\
M_{cq*}=-\left[I+\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\right]d\left(\boldsymbol{\alpha}_m\right)\\
M_{cz}=\left[I+\boldsymbol{\alpha}_l\phi\boldsymbol{\phi}_l'M_l^{-1}\right]
\end{cases}
$$

$$
\begin{aligned}
\boldsymbol{x}_t^F &= -d\left(\boldsymbol{\theta}^*\right)\left(\left(1-\sigma\right)\mathbf{1}_{S\times 1}\boldsymbol{\psi}' - I\right)M_{cc}^{-1}M_{c\xi}\xi_t + \begin{bmatrix} -\boldsymbol{\theta}^*\sigma & I & d\left(\boldsymbol{\theta}^*\right) & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S}\end{bmatrix}\xi_t \\
&= \left\{-d\left(\boldsymbol{\theta}^*\right)\left(\left(1-\sigma\right)\mathbf{1}_{S\times 1}\boldsymbol{\psi}' - I\right)M_{cc}^{-1}M_{c\xi} + \begin{bmatrix} -\boldsymbol{\theta}^*\sigma & I & d\left(\boldsymbol{\theta}^*\right) & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S}\end{bmatrix}\right\}\xi_t
\end{aligned}
$$

In terms of v_t^F,

$$
v_t^F = \begin{bmatrix} \boldsymbol{c}_t^F \\ \boldsymbol{x}_t^F \end{bmatrix} = \underbrace{\begin{bmatrix} M_{cc}^{-1}M_{c\xi} \\ \left\{-d\left(\boldsymbol{\theta}^*\right)\left(\left(1-\sigma\right)\mathbf{1}_{S\times 1}\boldsymbol{\psi}' - I\right)M_{cc}^{-1}M_{c\xi} + \begin{bmatrix} -\boldsymbol{\theta}^*\sigma & I & d\left(\boldsymbol{\theta}^*\right) & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S}\end{bmatrix}\right\}\end{bmatrix}}_{=:F}\xi_t
$$

Defining

$$
F_c = M_{cc}^{-1}M_{c\xi},\; F_x = \left\{-d\left(\boldsymbol{\theta}^*\right)\left(\left(1-\sigma\right)\mathbf{1}_{S\times 1}\boldsymbol{\psi}' - I\right)M_{cc}^{-1}M_{c\xi} + \begin{bmatrix} -\boldsymbol{\theta}^*\sigma & I & d\left(\boldsymbol{\theta}^*\right) & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S}\end{bmatrix}\right\},
$$

$$
F = \left[F_c', F_x'\right]'.
$$

The following shows that the second-order approximated welfare can be expressed in the quadratic form of the gap from the flexible price equilibrium. That is,

$$
\begin{cases}
\Gamma_{c2}F_c + \Gamma_{cx}F_x + \Gamma_{\xi c}' = 0 & (a)' \\
\Gamma_{cx}'F_c + \Gamma_{x2}F_x + \Gamma_{\xi x}' = 0 & (b)'
\end{cases}.
$$

This set of equations is sufficient to see that when we express the real terms $\frac{1}{2}v_t'\Gamma_{v2}v_t + \xi_t'\Gamma_{\xi v}v_t$ in the deviations from the natural level as

$$
\frac{1}{2}\tilde{v}_t'\Gamma_{v2}\tilde{v}_t,
$$

where $\tilde{v}_t := v_t - v_t^{Nat}$, the natural level v_t^{Nat} coincides with the flexible price equilibrium since

$$\frac{1}{2}\tilde{v}_t'\Gamma_{v2}\tilde{v}_t = \frac{1}{2}\left(v_t - N\xi_t\right)'\Gamma_{v2}\left(v_t - N\xi_t\right) = \frac{1}{2}v_t'\Gamma_{v2}v_t - \xi_t'N'\Gamma_{v2}v_t + t.i.p.$$

Part (a)'

$$\begin{aligned}
&\Gamma_{c2}F_c + \Gamma_{cx}F_x \\
=& - d\left(\boldsymbol{\phi}_c\right) d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} \begin{bmatrix} M_{cc*} & M_{cx*} & M_{cp*} & M_{cq*} & M_{cz} \end{bmatrix} \\
& - \phi d\left(\boldsymbol{\phi}_c\right) \boldsymbol{\phi}_l \boldsymbol{\phi}_l' M_l^{-1} d\left(\boldsymbol{\phi}_x\right) \begin{bmatrix} -\boldsymbol{\theta}^*\sigma & I & d\left(\boldsymbol{\theta}^*\right) & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S} \end{bmatrix}
\end{aligned}$$

The second to $S+1$th columns of part (a)' is

$$\begin{aligned}
& - d\left(\boldsymbol{\phi}_c\right) d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} M_{cx*} - \phi d\left(\boldsymbol{\phi}_c\right) \boldsymbol{\phi}_l \boldsymbol{\phi}_l' M_l^{-1} d\left(\boldsymbol{\phi}_x\right) \\
=& d\left(\boldsymbol{\phi}_c\right) d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} \boldsymbol{\alpha}_l \phi \boldsymbol{\phi}_l' M_l^{-1} d\left(\boldsymbol{\phi}_x\right) - \phi d\left(\boldsymbol{\phi}_c\right) \boldsymbol{\phi}_l \boldsymbol{\phi}_l' M_l^{-1} d\left(\boldsymbol{\phi}_x\right) \\
=& d\left(\boldsymbol{\phi}_c\right) d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} \left(\boldsymbol{\alpha}_l \phi - \phi \boldsymbol{\alpha}_l\right) \boldsymbol{\phi}_l' M_l^{-1} d\left(\boldsymbol{\phi}_x\right) \\
=& 0
\end{aligned}$$

The $S+2$ to $2S+1$ columns of part (a)' are

$$\begin{aligned}
& - d\left(\boldsymbol{\phi}_c\right) d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} M_{cp*} - \phi d\left(\boldsymbol{\phi}_c\right) \boldsymbol{\phi}_l \boldsymbol{\phi}_l' M_l^{-1} d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\theta}^*\right) \\
=& d\left(\boldsymbol{\phi}_c\right) d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} \left(\boldsymbol{\alpha}_l \phi - \phi \boldsymbol{\alpha}_l\right) \boldsymbol{\phi}_l' M_l^{-1} d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\theta}^*\right) \\
=& 0
\end{aligned}$$

The $2S+2$ to $3S+1$ columns of part (a)' are

$$
\begin{aligned}
& -d(\boldsymbol{\phi}_c)\, d(\boldsymbol{\phi}_l)\, d(\boldsymbol{\alpha}_l)^{-1} M_{cq*} \\
& -d(\boldsymbol{\phi}_c) \left(M_l^{-1}\right)' (d(\boldsymbol{\phi}_l) + \phi \boldsymbol{\phi}_l \boldsymbol{\phi}_l')' \, d(\boldsymbol{\alpha}_l)^{-1} d(\boldsymbol{\alpha}_m) \\
& = d(\boldsymbol{\phi}_c) \left\{ \underbrace{d(\boldsymbol{\phi}_l)\, d(\boldsymbol{\alpha}_l)^{-1} (I + \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l') M_l^{-1}}_{(*)} \right\} d(\boldsymbol{\alpha}_m) \\
& - d(\boldsymbol{\phi}_c) \left\{ \underbrace{\left\{ d(\boldsymbol{\alpha}_l)^{-1} d(\boldsymbol{\phi}_l) (I + \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l') M_l^{-1} \right\}'}_{=(*)'} \right\} d(\boldsymbol{\alpha}_m)
\end{aligned}
$$

It suffices to show that $d(\boldsymbol{\phi}_l)\, d(\boldsymbol{\alpha}_l)^{-1} (I + \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l') M_l^{-1}$ is symmetric, but it is indeed symmetric.

The last S columns of part (a)' are

$$
\begin{aligned}
& - d(\boldsymbol{\phi}_c)\, d(\boldsymbol{\phi}_l)\, d(\boldsymbol{\alpha}_l)^{-1} M_{cz} + \left(d(\boldsymbol{\alpha}_l)^{-1} (d(\boldsymbol{\phi}_l) + \phi \boldsymbol{\phi}_l \boldsymbol{\phi}_l') M_l^{-1} d(\boldsymbol{\phi}_c) \right)' \\
= & - d(\boldsymbol{\phi}_c)\, d(\boldsymbol{\phi}_l)\, d(\boldsymbol{\alpha}_l)^{-1} \left(I + \boldsymbol{\alpha}_l \phi \boldsymbol{\phi}_l' M_l^{-1} \right) + d(\boldsymbol{\phi}_c) \left((I + \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l') M_l^{-1} \right)' d(\boldsymbol{\phi}_l)\, d(\boldsymbol{\alpha}_l)^{-1} \\
= & - d(\boldsymbol{\phi}_c) \left\{ \underbrace{d(\boldsymbol{\phi}_l)\, d(\boldsymbol{\alpha}_l)^{-1} (I + \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l') M_l^{-1}}_{=(*)} - \underbrace{\left\{ d(\boldsymbol{\phi}_l)\, d(\boldsymbol{\alpha}_l)^{-1} \left((I + \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l') M_l^{-1} \right) \right\}'}_{=(*)'} \right\}
\end{aligned}
$$

Part (b)'

Therefore,

$$
\begin{aligned}
&\Gamma_{cx}' F_c + \Gamma_{x2} F_x \\
=& -d(\boldsymbol{\phi}_l)\, d(\boldsymbol{\alpha}_l)^{-1}\, d(\boldsymbol{\phi}_x)\, M_{c\xi} \\
& - \left\{ d(\boldsymbol{\theta}^*)^{-1}\, d(\boldsymbol{\phi}_l)\, d(\boldsymbol{\alpha}_l)^{-1} + d(\boldsymbol{\phi}_x)\, d(\boldsymbol{\phi}_l)\, \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l' M_l^{-1} \right\} \\
& \times d(\boldsymbol{\phi}_x) \begin{bmatrix} -\boldsymbol{\theta}^* \sigma & I & d(\boldsymbol{\theta}^*) & \mathbb{O}_{S\times S} & \mathbb{O}_{S\times S} \end{bmatrix}
\end{aligned}
$$

The first column of part (b)' is

$$
\begin{aligned}
& -d(\boldsymbol{\phi}_l)\, d(\boldsymbol{\alpha}_l)^{-1}\, d(\boldsymbol{\phi}_x) \left[\boldsymbol{\alpha}_m + \boldsymbol{\alpha}_l \phi \boldsymbol{\phi}_l' M_l^{-1} \left(d(\boldsymbol{\phi}_x)\, \boldsymbol{\theta}^* + \boldsymbol{\alpha}_m \right) \right] \sigma \\
& + \left\{ d(\boldsymbol{\theta}^*)^{-1}\, d(\boldsymbol{\phi}_l)\, d(\boldsymbol{\alpha}_l)^{-1} + d(\boldsymbol{\phi}_x)\, d(\boldsymbol{\phi}_l)\, \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l' M_l^{-1} \right\} d(\boldsymbol{\phi}_x)\, \boldsymbol{\theta}^* \sigma \\
& + d(\boldsymbol{\phi}_x) \left(M_l^{-1} \right)' \left(d(\boldsymbol{\phi}_l) + \phi \boldsymbol{\phi}_l \boldsymbol{\phi}_l' \right) d(\boldsymbol{\alpha}_l)^{-1}\, \sigma \boldsymbol{\alpha}_m - d(\boldsymbol{\phi}_x)\, d(\boldsymbol{\alpha}_l)^{-1}\, d(\boldsymbol{\phi}_l)\, \mathbf{1}_{S\times 1} \sigma \\
=& -d(\boldsymbol{\phi}_x) \left\{ \underbrace{d(\boldsymbol{\phi}_l)\, d(\boldsymbol{\alpha}_l)^{-1} \left(I + \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l' \right) M_l^{-1}}_{=(*)} \right\} \sigma \boldsymbol{\alpha}_m \\
& + d(\boldsymbol{\phi}_x) \left\{ \underbrace{\left(d(\boldsymbol{\alpha}_l)^{-1}\, d(\boldsymbol{\phi}_l) \left(I + \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l' \right) M_l^{-1} \right)'}_{=(*)'} \right\} \sigma \boldsymbol{\alpha}_m \\
& - d(\boldsymbol{\phi}_x)\, d(\boldsymbol{\phi}_l) \left\{ \mathbf{1}_{S\times 1} \phi - \phi \mathbf{1}_{S\times 1} \right\} \boldsymbol{\phi}_l' M_l^{-1} d(\boldsymbol{\phi}_x)\, \boldsymbol{\theta}^* \sigma
\end{aligned}
$$

The 2nd to $S+1$'th columns of part (b)' are

$$
\begin{aligned}
& d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right) \boldsymbol{\alpha}_l \phi \boldsymbol{\phi}_l' M_l^{-1} d\left(\boldsymbol{\phi}_x\right) \\
& -\left\{d\left(\boldsymbol{\theta}^*\right)^{-1} d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1}+d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\phi}_l\right) \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l' M_l^{-1}\right\} d\left(\boldsymbol{\phi}_x\right) \\
& +d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\theta}^*\right)^{-1} \\
= & d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\phi}_l\right) \mathbf{1}_{S\times 1} \phi \boldsymbol{\phi}_l' M_l^{-1} d\left(\boldsymbol{\phi}_x\right) \\
& -d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\phi}_l\right) \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l' M_l^{-1} d\left(\boldsymbol{\phi}_x\right) \\
= & 0
\end{aligned}
$$

The $S+2$th to $2S+1$th columns of part (b)' are

$$
\begin{aligned}
& d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right) \boldsymbol{\alpha}_l \phi \boldsymbol{\phi}_l' M_l^{-1} d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\theta}^*\right) \\
& -\left\{d\left(\boldsymbol{\theta}^*\right)^{-1} d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1}+d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\phi}_l\right) \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l' M_l^{-1}\right\} d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\theta}^*\right) \\
& +d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_l\right) \\
= & d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\phi}_x\right) \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l' M_l^{-1} d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\theta}^*\right) \\
& -d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\phi}_l\right) \phi \mathbf{1}_{S\times 1} \boldsymbol{\phi}_l' M_l^{-1} d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\theta}^*\right) \\
= & 0
\end{aligned}
$$

The $2S+2$'th to $3S+1$'th columns of part (b)' are

$$\begin{aligned}
& d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right)\left[I+\boldsymbol{\alpha}_l \phi \boldsymbol{\phi}_l^{\prime} M_l^{-1}\right] d\left(\boldsymbol{\alpha}_m\right) \\
& -d\left(\boldsymbol{\phi}_x\right)\left(M_l^{-1}\right)^{\prime}\left(d\left(\boldsymbol{\phi}_l\right)+\phi \boldsymbol{\phi}_l \boldsymbol{\phi}_l^{\prime}\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\alpha}_m\right) \\
= & d\left(\boldsymbol{\phi}_x\right)\left\{\underbrace{d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_l\right)\left(I+\phi \mathbf{1}_{S \times 1} \boldsymbol{\phi}_l^{\prime}\right) M_l^{-1}}_{=(*)}\right\} d\left(\boldsymbol{\alpha}_m\right) \\
& -d\left(\boldsymbol{\phi}_x\right)\left\{\underbrace{\left(d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_l\right)\left(I+\phi \mathbf{1}_{S \times 1} \boldsymbol{\phi}_l^{\prime}\right) M_l^{-1}\right)^{\prime}}_{=(*)^{\prime}}\right\} d\left(\boldsymbol{\alpha}_m\right)
\end{aligned}$$

The last S columns of part (b)' are

$$\begin{aligned}
& -d\left(\boldsymbol{\phi}_l\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_x\right)\left(I+\boldsymbol{\alpha}_l \phi \boldsymbol{\phi}_l^{\prime} M_l^{-1}\right) \\
& +d\left(\boldsymbol{\phi}_x\right)\left(M_l^{-1}\right)^{\prime}\left(d\left(\boldsymbol{\phi}_l\right)+\phi \boldsymbol{\phi}_l \boldsymbol{\phi}_l^{\prime}\right) d\left(\boldsymbol{\alpha}_l\right)^{-1} \\
= & -d\left(\boldsymbol{\phi}_x\right)\left\{\underbrace{d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_l\right)\left(I+\phi \mathbf{1}_{S \times 1} \boldsymbol{\phi}_l^{\prime}\right) M_l^{-1}}_{=(*)}\right\} \\
& +d\left(\boldsymbol{\phi}_x\right)\left\{\underbrace{\left(d\left(\boldsymbol{\alpha}_l\right)^{-1} d\left(\boldsymbol{\phi}_l\right)\left(I+\phi \mathbf{1}_{S \times 1} \boldsymbol{\phi}_l^{\prime}\right) M_l^{-1}\right)^{\prime}}_{=(*)^{\prime}}\right\} .
\end{aligned}$$

A.2.7 Proof of Lemma 1.2

From Appendix A.2.6, we can see that, under the efficient steady state, the objective function is approximated purely quadratically by

$$\begin{aligned}
&\mathcal{W} - \bar{\mathcal{W}} \\
&\propto E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{2} \tilde{v}_t' \Gamma_{v2} \tilde{v}_t + \sum_{s \in S} \frac{\theta_s}{2\kappa_s} \left(\Gamma_{\pi s} \pi_{s,t}^2 + \Gamma_{\pi s}^X \left(\pi_{s,t}^X \right)^2 \right) \right] \\
&+ t.i.p.,
\end{aligned}$$

where $\tilde{v}_t = v_t - v_t^F = v_t - \left[\left(\boldsymbol{c}_t^F \right)', \left(\boldsymbol{x}_t^F \right)' \right]'$. It remains to show the form of the constraints.

All the constraints in Definition 1.1 except for pricing equations (1.13)-(1.18), (1.21), and (1.22) are already used to substitute out auxiliary endogenous variables. The linear approximations of these pricing equations reduce to the Phillips curve for each sector.

$$\begin{aligned}
&\kappa_s^{-1} \left(\pi_{s,t} - \beta E_t \left[\pi_{s,t+1} \right] \right) = \alpha_{sm} \left(q_t + q_{st}^* \right) + \alpha_{sl} w_t - z_{st} - p_{st} \\
&\kappa_s^{-1} \left(\pi_{s,t}^X - \beta E_t \left[\pi_{s,t+1}^X \right] \right) = \alpha_{sm} \left(q_t + q_{st}^* \right) + \alpha_{sl} w_t - z_{st} - p_{st}^X \\
&\pi_{s,t} = \pi_t + p_{st} - p_{st-1} \\
&\pi_{s,t}^X = \pi_t + p_{st}^X - p_{st-1}^X
\end{aligned}$$

In matrix,

$$d\left(\boldsymbol{\kappa}\right)^{-1}\left(\boldsymbol{\pi}_t-\beta E_t\left[\boldsymbol{\pi}_{t+1}\right]\right)=d\left(\boldsymbol{\alpha}_m\right)\left(\mathbf{1}_{S\times1}q_t+\boldsymbol{q}_t^*\right)+\boldsymbol{\alpha}_l w_t-\boldsymbol{z}_t-\boldsymbol{p}_t$$

$$d\left(\boldsymbol{\kappa}\right)^{-1}\left(\boldsymbol{\pi}_t^X-\beta E_t\left[\boldsymbol{\pi}_{t+1}^X\right]\right)=d\left(\boldsymbol{\alpha}_m\right)\left(\mathbf{1}_{S\times1}q_t+\boldsymbol{q}_t^*\right)+\boldsymbol{\alpha}_l w_t-\boldsymbol{z}_t-\boldsymbol{p}_t^X$$

$$\boldsymbol{\pi}_t=\mathbf{1}_{S\times1}\pi_t+\boldsymbol{p}_t-\boldsymbol{p}_{t-1}$$

$$\boldsymbol{\pi}_t^X=\mathbf{1}_{S\times1}\pi_t+\boldsymbol{p}_t^X-\boldsymbol{p}_{t-1}^X.$$

Comparing this with the condition of the flexible price equilibrium:

$$d\left(\boldsymbol{\alpha}_m\right)\left(\mathbf{1}_{S\times1}q_t^F+\boldsymbol{q}_t^*\right)+\boldsymbol{\alpha}_l w_t^F-\boldsymbol{z}_t-\boldsymbol{p}_t^F=0$$

$$d\left(\boldsymbol{\alpha}_m\right)\left(\mathbf{1}_{S\times1}q_t^F+\boldsymbol{q}_t^*\right)+\boldsymbol{\alpha}_l w_t^F-\boldsymbol{z}_t-\boldsymbol{p}_t^{XF}=0,$$

the Phillips curves can be rewritten as

$$d\left(\boldsymbol{\kappa}\right)^{-1}\left(\boldsymbol{\pi}_t-\beta E_t\left[\boldsymbol{\pi}_{t+1}\right]\right)=d\left(\boldsymbol{\alpha}_m\right)\mathbf{1}_{S\times1}\tilde{q}_t+\boldsymbol{\alpha}_l\tilde{w}_t-\tilde{\boldsymbol{p}}_t \tag{A.2}$$

$$d\left(\boldsymbol{\kappa}\right)^{-1}\left(\boldsymbol{\pi}_t^X-\beta E_t\left[\boldsymbol{\pi}_{t+1}^X\right]\right)=d\left(\boldsymbol{\alpha}_m\right)\mathbf{1}_{S\times1}\tilde{q}_t+\boldsymbol{\alpha}_l\tilde{w}_t-\tilde{\boldsymbol{p}}_t^X. \tag{A.3}$$

Since the linear approximation of other equilibrium conditions that map $q_t, w_t, \boldsymbol{p}_t, \boldsymbol{p}_t^X$ into $\boldsymbol{c}_t, \boldsymbol{x}_t$ hold both in the sticky price equilibrium and in the flexible price equilibrium, the gap on the right hand side is linear in $\tilde{\boldsymbol{c}}_t$ and $\tilde{\boldsymbol{x}}_t$

$$\begin{cases} \tilde{q}_t = \sigma \boldsymbol{\psi}' \tilde{\boldsymbol{c}}_t \\ \tilde{w}_t = \phi \boldsymbol{\phi}_l' \tilde{\boldsymbol{l}}_t + \sigma \boldsymbol{\psi}' \tilde{\boldsymbol{c}}_t \\ \tilde{\boldsymbol{l}}_t = M_l^{-1} \left\{ d\left(\boldsymbol{\phi}_c\right) \tilde{\boldsymbol{c}}_t + d\left(\boldsymbol{\phi}_x\right) \tilde{\boldsymbol{x}}_t \right\} \\ \tilde{\boldsymbol{p}}_t = \mathbf{1}_{S\times 1} \boldsymbol{\psi}' \tilde{\boldsymbol{c}}_t - \tilde{\boldsymbol{c}}_t \\ \tilde{\boldsymbol{p}}_t^X = -d\left(\boldsymbol{\theta}^*\right)^{-1} \tilde{\boldsymbol{x}}_t + \mathbf{1}_{S\times 1} \sigma \boldsymbol{\psi}' \tilde{\boldsymbol{c}}_t \end{cases}.$$

Plugging these into the Phillips curve (A.2) and (A.3), we can find γ_v^P and γ_{Xv}^P in the following expressions.

$$d\left(\boldsymbol{\kappa}\right)^{-1} \left(\boldsymbol{\pi}_t - \beta E_t \left[\boldsymbol{\pi}_{t+1}\right]\right) = \gamma_v^P \tilde{v}_t$$
$$d\left(\boldsymbol{\kappa}\right)^{-1} \left(\boldsymbol{\pi}_t^X - \beta E_t \left[\boldsymbol{\pi}_{t+1}^X\right]\right) = \gamma_{Xv}^P \tilde{v}_t.$$

For the identity, we can rewrite

$$\boldsymbol{\pi}_t = \mathbf{1}_{S\times 1} \pi_t + \tilde{\boldsymbol{p}}_t - \tilde{\boldsymbol{p}}_{t-1} + \boldsymbol{p}_t^F - \boldsymbol{p}_{t-1}^F$$
$$\boldsymbol{\pi}_t^X = \mathbf{1}_{S\times 1} \pi_t + \tilde{\boldsymbol{p}}_t^X - \tilde{\boldsymbol{p}}_{t-1}^X + \boldsymbol{p}_t^{XF} - \boldsymbol{p}_{t-1}^{XF}.$$

The gaps $\tilde{\boldsymbol{p}}_t$ and $\tilde{\boldsymbol{p}}_t^X$ can be similarly rewritten in terms of $\tilde{\boldsymbol{c}}_t$ and $\tilde{\boldsymbol{x}}_t$. This gives the expressions for γ_v^I and γ_{vX}^I. Regarding the flexible price equilibrium objects, $\boldsymbol{p}_t^F$ and $\boldsymbol{p}_t^{XF}$, substitute the solutions as functions of exogenous variables. This gives the expressions for ϵ_t^I and ϵ_t^{IX}.

A.2.8 Solution in the long-run expectation

This section derives the RPI as the index whose long-run expectation remains constant under the optimal monetary policy. The argument parallels that in Woodford (2010). To this end, I take the first-order condition of the approximated Ramsey problem given in Lemma 1.2.

$$\begin{cases} [\tilde{v}_t] & \Gamma_{v2}\tilde{v}_t - \left(\gamma_v^P\right)'\boldsymbol{\varphi}_t - \left(\gamma_{Xv}^P\right)'\left(\boldsymbol{\varphi}_t^X\right) - \left(\gamma_v^I\right)'\left(\boldsymbol{\psi}_t - \beta E_t\boldsymbol{\psi}_{t+1}\right) - \left(\gamma_{vX}^I\right)'\left(\boldsymbol{\psi}_t^X - \beta E_t\boldsymbol{\psi}_{t+1}^X\right) = 0 \\ [\boldsymbol{\pi}_t] & \Gamma_\pi d\left(\boldsymbol{\theta}\right) d\left(\boldsymbol{\kappa}\right)^{-1} d\left(\boldsymbol{\psi}\right)\boldsymbol{\pi}_t + d\left(\boldsymbol{\kappa}\right)^{-1}\left(\boldsymbol{\varphi}_t - \boldsymbol{\varphi}_{t-1}\right) + \boldsymbol{\psi}_t = 0 \\ \left[\boldsymbol{\pi}_t^X\right] & \Gamma_\pi d\left(\boldsymbol{\theta}\right) d\left(\boldsymbol{\kappa}\right)^{-1} d\left(\boldsymbol{\psi}\right) d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\phi}_c\right)\boldsymbol{\pi}_t^X + d\left(\boldsymbol{\kappa}\right)^{-1}\left(\boldsymbol{\varphi}_t^X - \boldsymbol{\varphi}_{t-1}^X\right) + \boldsymbol{\psi}_t^X = 0 \\ [\pi_t] & \mathbf{1}_{1\times S}\boldsymbol{\psi}_t + \mathbf{1}_{1\times S}\boldsymbol{\psi}_t^X = 0 \end{cases} \tag{A.4}$$

where $\boldsymbol{\varphi}_t, \boldsymbol{\varphi}_t^X, \boldsymbol{\psi}_t, \boldsymbol{\psi}_t^X$ are S dimensional Lagrange multipliers for the Phillips curves and the identity.

I first focus on the long-run expectation. Assuming the existence of long-run expectations of $\tilde{v}_t = v_t - N\xi_t$ denoted by $\tilde{v}_t^\infty := \lim_{T\to\infty} E_t\tilde{v}_T$, Lagrange multipliers also have long-run expectations $\boldsymbol{\varphi}_t^\infty, \boldsymbol{\varphi}_t^{X\infty}, \boldsymbol{\psi}_t^\infty, \boldsymbol{\psi}_t^{X\infty}$.

$$\begin{cases} [\tilde{v}_t] & \Gamma_{v2}\tilde{v}_t^\infty - \left(\gamma_v^P\right)'\boldsymbol{\varphi}_t^\infty - \left(\gamma_{Xv}^P\right)'\left(\boldsymbol{\varphi}_t^{X\infty}\right) - (1-\beta)\left(\gamma_v^I\right)'\boldsymbol{\psi}_t^\infty - (1-\beta)\left(\gamma_{vX}^I\right)'\boldsymbol{\psi}_t^{X\infty} = 0 \\ [\boldsymbol{\pi}_t] & \Gamma_\pi d\left(\boldsymbol{\theta}\right) d\left(\boldsymbol{\kappa}\right)^{-1} d\left(\boldsymbol{\psi}\right)\boldsymbol{\pi}_t^\infty + \boldsymbol{\psi}_t^\infty = 0 \\ \left[\boldsymbol{\pi}_t^X\right] & \Gamma_\pi d\left(\boldsymbol{\theta}\right) d\left(\boldsymbol{\kappa}\right)^{-1} d\left(\boldsymbol{\psi}\right) d\left(\boldsymbol{\phi}_x\right) d\left(\boldsymbol{\phi}_c\right)\boldsymbol{\pi}_t^{X\infty} + \boldsymbol{\psi}_t^{X\infty} = 0 \\ [\pi_t] & \mathbf{1}_{1\times S}\boldsymbol{\psi}_t + \mathbf{1}_{1\times S}\boldsymbol{\psi}_t^X = 0 \end{cases}$$

Combining this with the conditions implied by the constraints in Lemma 1.2,

$$\begin{cases} (1-\beta)\, d\left(\boldsymbol{\kappa}\right)^{-1} \boldsymbol{\pi}_t^{\infty} = \gamma_v^P \tilde{v}_t^{\infty} \\ (1-\beta)\, d\left(\boldsymbol{\kappa}\right)^{-1} \boldsymbol{\pi}_t^{X\infty} = \gamma_{Xv}^P \tilde{v}_t^{\infty} \\ \boldsymbol{\pi}_t^{\infty} = \mathbf{1}_{S\times 1} \pi_t^{\infty} \\ \boldsymbol{\pi}_t^{X\infty} = \mathbf{1}_{S\times 1} \pi_t^{\infty} \end{cases},$$

the long-run expectation of the Lagrange multipliers for the Phillips curves $\varphi_t^{\infty}, \varphi_t^{X\infty}$ can be shown to be zeros.

Specifically, from the last three equations of the first-order conditions and the third and fourth equations of the constraints, we have $\pi_{s,t}^{\infty} = \pi_{s,t}^{X\infty} = \pi_t^{\infty} = \psi_{st}^{\infty} = \psi_{st}^{X\infty} = 0\ \forall s \in S$. Thus, the system simplifies to

$$\Gamma_{v2}\tilde{v}_t^{\infty} - \left(\gamma_v^P\right)' \boldsymbol{\varphi}_t^{\infty} - \left(\gamma_{Xv}^P\right)' \left(\boldsymbol{\varphi}_t^{X\infty}\right) + \psi_t^{N\infty} \left(\gamma_v^I\right)' \boldsymbol{\psi} = 0$$

and

$$\begin{cases} \gamma_v^P \tilde{v}_t^{\infty} = 0 & \left[\boldsymbol{\varphi}_t\right] \\ \gamma_{Xv}^P \tilde{v}_t^{\infty} = 0 & \left[\boldsymbol{\varphi}_t^X\right] \end{cases}.$$

That is,

$$\tilde{v}_t^{\infty} = 0$$

and

$$\left(\gamma_v^P\right)' \boldsymbol{\varphi}_t^\infty + \left(\gamma_{Xv}^P\right)' \boldsymbol{\varphi}_t^{X\infty} - \psi_t^{N\infty} \left(\gamma_v^I\right)' \boldsymbol{\psi} = \Gamma_{v2} \tilde{v}_t^\infty$$

$$\begin{bmatrix} \left(\gamma_v^P\right)' & \left(\gamma_{Xv}^P\right)' & \left(\gamma_v^I\right)' \boldsymbol{\psi} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi}_t^\infty \\ \boldsymbol{\varphi}_t^{X\infty} \\ -\psi_t^{N\infty} \end{bmatrix} = 0$$

Next, by summing the first-order conditions (A.4) with respect to sectoral inflation rates $\boldsymbol{\pi}_t, \boldsymbol{\pi}_t^X$, I obtain

$$\Gamma_\pi \sum_{s \in S} \frac{\theta_s}{\kappa_s} \psi_s \left[\pi_{s,t} + \frac{\phi_{sx}}{\phi_{sc}} \pi_{s,t}^X\right] + \sum_{s \in S} \left(\kappa_s^{-1} \left(\varphi_{st} - \varphi_{st-1}\right) + \kappa_s^{-1} \left(\varphi_{st}^X - \varphi_{st-1}^X\right)\right) = 0.$$

Recalling the definitions of $\pi_{s,t}$ and $\pi_{s,t}^X$,

$$\Gamma_\pi \sum_{s \in S} \frac{\theta_s}{\kappa_s} \psi_s \left[\left(\log P_{st} - \log P_{st-1}\right) + \frac{\phi_{sx}}{\phi_{sc}} \left(\log P_{st}^X - \log P_{st-1}^X\right)\right]$$
$$+ \sum_{s \in S} \left(\kappa_s^{-1} \left(\varphi_{st} - \varphi_{st-1}\right) + \kappa_s^{-1} \left(\varphi_{st}^X - \varphi_{st-1}^X\right)\right) = 0.$$

By rearranging, we can see that for any t,

$$\sum_{s \in S} \left[\Gamma_\pi \frac{\theta_s}{\kappa_s} \psi_s \left(\log P_{st} + \frac{\phi_{sx}}{\phi_{sc}} \log P_{st}^X\right) + \kappa_s^{-1} \varphi_{st} + \kappa_s^{-1} \varphi_{st}^X\right] = const.$$

This also holds in long-run expectation.

Since the long-run expectations of the Phillips curve Lagrange multipliers are zero, we obtain

$$\lim_{T\to\infty} E_t \sum_{s\in S} \frac{\theta_s}{\kappa_s}\psi_s \left[\log P_{sT} + \frac{\phi_{sx}}{\phi_{sc}} \log P_{sT}^X\right] = \overline{\log \mathbb{P}},$$

where $\overline{\log \mathbb{P}}$ is a constant.

A.3 Appendix to Section 1.4

A.3.1 Detailed welfare evaluation procedure

For each country-specific calibration of these parameters, we can solve for the equilibrium characterized by the Phillips curves, identities relating inflation rates and relative prices, and the normalization of CPI and monetary policy.

$$\begin{cases} \frac{\lambda_s}{(1-\lambda_s)(1-\lambda_s\beta)}\left(\pi_{s,t} - \beta E_t\left[\pi_{s,t+1}\right]\right) = \alpha_{sm}\left(q_t + q_{st}^*\right) + \alpha_{sl} w_t - z_{st} - p_{st} & \forall s \\ \frac{\lambda_s}{(1-\lambda_s)(1-\lambda_s\beta)}\left(\pi_{s,t}^X - \beta E_t\left[\pi_{s,t+1}^X\right]\right) = \alpha_{sm}\left(q_t + q_{st}^*\right) + \alpha_{sl} w_t - z_{st} - p_{st}^X & \forall s \\ \pi_{s,t} = \pi_t + p_{st} - p_{st-1} & \forall s \\ \pi_{s,t}^X = \pi_t + p_{st}^X - p_{st-1}^X & \forall s \\ Monetary\ Policy & \end{cases} .$$

The normalization equation comes from all nominal variables being expressed relative to CPI.

To consider the optimal policy, denote $y_t = \left[c_{1t}, ..., c_{St}, x_{1t}, ..., x_{St}, \pi_{1t}, ..., \pi_{St}, \pi_{1t}^X, ..., \pi_{St}^X, \pi_t\right]$, $\xi_t = [c_t^*, x_{1t}^*, ..., x_{St}^*, p_{1t}^*, ..., p_{St}^*, q_{1t}^*, ..., q_{St}^*, z_{1t}..., , z_{St}]'$. Define $\Gamma_{y2}, \Gamma_{\xi y}, \gamma_y^P, \gamma_{yp}^P, \gamma_x^P, \gamma_y^I, \gamma_{ym}^I$ so that

$$L^{-(1+\phi)}\left(\mathcal{W}-\bar{\mathcal{W}}\right)|_{\text{efficient}} = E_0\sum_{t=0}^{\infty}\left[y_t'\Gamma_{y2}y_t+\xi_t'\Gamma_{\xi y}y_t\right]$$

$$\begin{bmatrix} \left(-\frac{\lambda_s}{(1-\lambda_s)(1-\lambda_s\beta)}\left(\pi_{s,t}-\beta E_t\left[\pi_{s,t+1}\right]\right)+\alpha_{sm}\left(q_t+q_{st}^*\right)+\alpha_{sl}w_t-z_{st}-p_{st}\right)_{s\in S} \\ \left(-\frac{\lambda_s}{(1-\lambda_s)(1-\lambda_s\beta)}\left(\pi_{s,t}^X-\beta E_t\left[\pi_{s,t+1}^X\right]\right)+\alpha_{sm}\left(q_t+q_{st}^*\right)+\alpha_{sl}w_t-z_{st}-p_{st}^X\right)_{s\in S}\end{bmatrix} = \gamma_y^P y_t+\gamma_{yp}^P E_t y_{t+1}+\gamma_x^P\xi_t$$

$$\begin{bmatrix} -\pi_{s,t}+\pi_t+p_{st}-p_{st-1} \\ -\pi_{s,t}^X+\pi_t+p_{st}^X-p_{st-1}^X\end{bmatrix} = \gamma_y^I y_t+\gamma_{ym}^I y_{t-1}.$$

Consider

$$\max E_0\sum_{t=0}^{\infty}\left[y_t'\Gamma_{y2}y_t+\xi_t'\Gamma_{\xi y}y_t\right]$$

$$s.t.\begin{cases}\gamma_y^P y_t+\gamma_{yp}^P E_t y_{t+1}+\gamma_x^P\xi_t=0 & \varphi_t^P \\ \gamma_y^I y_t+\gamma_{ym}^I y_{t-1}=0 & \varphi_t^I\end{cases}$$

The first-order condition is

$$2\Gamma_{y2}y_t+\Gamma_{\xi y}'\xi_t+\gamma_y^{P\prime}\varphi_t^P+\gamma_y^{I\prime}\varphi_t^I+\gamma_{yp}^{P\prime}\varphi_{t-1}^P+\gamma_{ym}^{I\prime}E_t\varphi_{t+1}^I=0.$$

Thus, assuming an exogenous process $\xi_{t+1}=\rho\xi_t+u_t$, I solve the dynamics

$$\begin{cases}\gamma_y^P y_t+\gamma_{yp}^P E_t y_{t+1}+\gamma_x^P\xi_t=0 \\ \gamma_y^I y_t+\gamma_{ym}^I y_{t-1}=0 \\ E_t\xi_{t+1}-\rho\xi_t=0 \\ 2\Gamma_{y2}y_t+\Gamma_{\xi y}'\xi_t+\gamma_y^{P\prime}\varphi_t^P+\gamma_y^{I\prime}\varphi_t^I+\gamma_{yp}^{P\prime}\varphi_{t-1}^P+\gamma_{ym}^{I\prime}E_t\varphi_{t+1}^I=0\end{cases}$$

and evaluate welfare at the solution.

It is convenient to define $\tilde{y}_t = \left[y_t', \left(\varphi_t^P\right)', \left(\varphi_t^I\right)'\right]'$ and $x_t = \left[\xi_t', (p_{t-1})', \left(p_{t-1}^X\right)', \left(\varphi_{t-1}^P\right)'\right]$, $\tilde{u}_t = [u_t', \mathbb{O}_{1\times 3S}]'$. Then, the solution takes the form

$$\tilde{y}_t = G_x x_t$$

$$x_{t+1} = H_x x_t + \tilde{u}_t.$$

Note that H_x consists of two parts, one without the Lagrange multipliers and the Lagrange multipliers.

$$\begin{bmatrix} \begin{bmatrix} \xi_{t+1} \\ \hat{p}_t \\ \hat{p}_t^X \end{bmatrix} \\ \varphi_t^P \end{bmatrix} = \begin{bmatrix} H_{xx} & \mathbb{O} \\ H_{\varphi x} & H_{\varphi\varphi} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \xi_t \\ p_{t-1} \\ p_{t-1}^X \end{bmatrix} \\ \varphi_{t-1}^P \end{bmatrix} + \begin{bmatrix} I & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} u_{t+1}.$$

A.3.1.1 Alternative policies

Alternative policies can be solved for by replacing the first-order condition with the monetary policy rule considered.

I also track the Lagrange multipliers φ_t^P, φ_t^I defined as in the optimal dynamics as auxiliary variables that do not affect the system (that is, defined by the state variable x_t and do not appear in any of the other equations). To do so, I solve

$$\begin{cases} \gamma_y^P y_t + \gamma_{yp}^P E_t y_{t+1} + \gamma_x^P \xi_t = 0 \\ \gamma_y^I y_t + \gamma_{ym}^I y_{t-1} = 0 \\ E_t \xi_{t+1} - \rho \xi_t = 0 \\ \pi_t = 0 \text{ or } \quad \sum_{s\in S} \mathbb{I}_{s\in Core} \psi_s \pi_{st} = 0 \\ -\varphi_t^P + H_\varphi x_t = 0 \end{cases}.$$

In this way, the solution takes the same form

$$\tilde{y}_t = G_x x_t$$

$$x_{t+1} = H_x x_t + \tilde{u}_t.$$

Note that the difference in the policy is reflected in the coefficients G_x and H_x.

A.3.1.2 Calculation of welfare

The unconditional expectation of welfare

$$E\left[L^{-(1+\phi)}\left(\mathcal{W} - \bar{\mathcal{W}}\right)|_{\text{efficient}}\right] = E\sum_{t=0}^{\infty}\left[y_t'\Gamma_{y2}y_t + \xi_t'\Gamma_{\xi y}y_t\right] = E\sum_{t=0}^{\infty}\left[\tilde{y}_t'\tilde{\Gamma}_{y2}\tilde{y}_t + x_t'\tilde{\Gamma}_{\xi y}\tilde{y}_t\right]$$

under any solution

$$\tilde{y}_t = G_x x_t$$

$$x_{t+1} = H_x x_t + \tilde{u}_t$$

can be calculated as follows by assuming $E\tilde{u}_t\tilde{u}_t' = \Sigma_u$,$E\tilde{u}_t\tilde{u}_s' = 0\ \forall t \neq s$. Define

$$V = \frac{\beta}{1-\beta}\Sigma_u + \underbrace{Ex_0x_0'}_{=:\Sigma_x} + \beta H_x V H_x',$$

then

$$E\left[L^{-(1+\phi)}\left(\mathcal{W} - \bar{\mathcal{W}}\right)|_{\text{efficient}}\right] = tr\left[\left(G_x'\Gamma_{2y}G_x + 2\Gamma_{yx}G_x\right)V\right].$$

The choice of Σ_x depends on the type of policy experiment.

I consider two types of policy experiment. The first type is that in which the economy starts form the stationary distribution obtained under headline inflation targeting as an approximation of the current policy. Then, this experiment compares switching from the current headline inflation targeting to different policies. To obtain the variance-covariance matrix, I use H_x obtained under the headline targeting policy H_x^{Head}. By solving

$$x_{t+1} = H_x^{Head} x_t + u_{t+1},$$

we obtain

$$\Sigma_x = H_x^{Head}\Sigma_x\left(H_x^{Head}\right)' + \Sigma_u.$$

The second type of policy experiment compares different worlds each of which starts from the steady state under the policy considered and continues the policy. In this case, Σ_x is the solution to

$$\Sigma_x = H_x\Sigma_x\left(H_x\right)' + \Sigma_u$$

where H_x is the solution to the equilibrium system under each policy.

A.3.1.3 Conversion to units of consumption

To interpret the welfare loss in units of consumption, the following procedure calculates the consumption equivalent of the welfare loss relative to the optimal policy. Compare the welfare at the optimal

$$W^O := E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{\left(C_t^O\right)^{1-\sigma}}{1-\sigma} - \frac{\left(L_t^O\right)^{1+\phi}}{1+\phi} \right] + \Lambda D_0^O$$

with sub-optimal

$$W^S := E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{\left(C_t^S\right)^{1-\sigma}}{1-\sigma} - \frac{\left(L_t^S\right)^{1+\phi}}{1+\phi} \right] + \Lambda D_0^S.$$

Consider discounting C_t^O by a fraction γ^S to make them equal.

$$W^S = E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{\left(\left(1-\gamma^S\right) C_t^O\right)^{1-\sigma}}{1-\sigma} - \frac{\left(L_t^O\right)^{1+\phi}}{1+\phi} \right] + \Lambda D_0^O$$

Using the approximation, $U_t \approx U + C^{1-\sigma}\left(\hat{c}_t + \frac{1-\sigma}{2}\hat{c}_t^2\right) - L^{1+\phi}\left(\hat{l}_t + \frac{1+\phi}{2}\hat{l}_t^2\right)$

$$W^S = E_0 \sum_{t=0}^{\infty} \beta^t C^{1-\sigma} \left(\log\left(1-\gamma^S\right) + \frac{1-\sigma}{2}\left(\log\left(1-\gamma^S\right)\right)^2 + (1-\sigma)\log\left(1-\gamma^S\right)\hat{c}_t^O \right) + W^O$$

Under the stationarity of exogenous variables, $E_0 \hat{c}_t = 0$. Thus,

$$\log\left(1-\gamma^S\right) + \frac{1-\sigma}{2}\left(\log\left(1-\gamma^S\right)\right)^2 = \frac{(1-\beta)}{\sum_{s\in S} \phi_{sc} \frac{\phi_{ls}}{\alpha_{sl}}} \left(\frac{W^S}{L^{1+\phi}} - \frac{W^O}{L^{1+\phi}} \right)$$

Table A.1: Concordance between WIOT, NS2008, and BW2006

WIOT	description	ISIC	NS2008
1	Agriculture, Hunting, Forestry and Fishing	01, 02, 05	Farm products
2	Mining and Quarrying	10-14	(Note 1)
3	Food, Beverages and Tobacco	15,16	Processed foods and feeds
4	Textiles and Textile Products	17,18	Textile products and apparel
5	Leather, Leather and Footwear	19	Hides, skins, leather, and related products
6	Wood and Products of Wood and Cork	20	Lumber and wood products
7	Pulp, Paper, Paper , Printing and Publishing	21,22	Pulp, paper, and allied products
8	Coke, Refined Petroleum and Nuclear Fuel	23	Fuels and related products and power
9	Chemicals and Chemical Products	24	Chemicals and allied products
10	Rubber and Plastics	25	Rubber and plastic products
11	Other Non-Metallic Mineral	26	Nonmetallic mineral products
12	Basic Metals and Fabricated Metal	27,28	Metals and metal products
13	Machinery, Nec	29	Machinery and equipment
14	Electrical and Optical Equipment	30-33	Machinery and equipment
15	Transport Equipment	34,35	Transportation equipment
16	Manufacturing, Nec; Recycling	36,37	Miscellaneous products
17	Electricity, Gas and Water Supply	40,41	Fuels and related products and power
18	Construction	45	Services (excl. travel)
19	Sale, Maintenance and Repair of Motor Vehicles and Motorcycles; Retail Sale of Fuel	50	(Note 2)
20	Wholesale Trade and Commission Trade, Except of Motor Vehicles and Motorcycles	51	Services (excl. travel)
21	Retail Trade, Except of Motor Vehicles and Motorcycles; Repair of Household Goods	52	Services (excl. travel)
22	Hotels and Restaurants	55	Services (excl. travel)
23	Inland Transport	60	Travel
24	Water Transport	61	Travel
25	Air Transport	62	Travel
26	Other Supporting and Auxiliary Transport Activities; Activities of Travel Agencies	63	Travel
27	Post and Telecommunications	64	Services (excl. travel)
28	Financial Intermediation	65-67	Services (excl. travel)
29	Real Estate Activities	70	Services (excl. travel)
30	Renting of M and Eq and Other Business Activities	71-74	Services (excl. travel)
31	Public Admin and Defence; Compulsory Social Security	75	Services (excl. travel)
32	Education	80	Services (excl. travel)
33	Health and Social Work	85	Services (excl. travel)
34	Other Community, Social and Personal Services	90-93	Services (excl. travel)
35	Private Households with Employed Persons	95	Services (excl. travel)

$$\Rightarrow \gamma^S = 1 - \exp\left\{ \frac{-1 + \sqrt{1 + 2\left(1-\sigma\right) \frac{(1-\beta)}{\sum_{s\in S} \phi_{sc} \frac{\phi_{ls}}{\alpha_{sl}}} \left(\frac{W^S}{L^{1+\phi}} - \frac{W^O}{L^{1+\phi}} \right)}}{1-\sigma} \right\}.$$

A.3.2 Concordance of sectors across the World Input-Output Table, Nakamura and Steinsson (2008) and Broda and Weinstein (2006)

Table A.1 is the concordance table created by the author.

A.3.3 Input-output adjustment

By aggregating the input-output table, I can obtain the following matrix.

$$\begin{bmatrix} P_1 Y_{11} & \cdots & P_1 Y_{1S} & P_1 C_1 & P_1^X X_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ P_S Y_{S1} & \cdots & P_S Y_{SS} & P_S C_S & P_S^X X_S \\ \mathcal{E} Q_1^* M_1 & \cdots & \mathcal{E} Q_S^* M_S & n.a. & n.a. \\ W L_1 & \cdots & W L_S & n.a. & n.a. \end{bmatrix}$$

Define

$$Totc_s = \sum_{s'} P_{s'} Y_{s's} + \mathcal{E} Q_s^* M_s + W L_s$$

$$\tilde{\alpha}_{ls} = \frac{W L_s}{Totc_s},\ \tilde{\alpha}_{ms} = \frac{\mathcal{E} Q_s^* M_s}{Totc_s},\ A = \begin{bmatrix} \frac{P_1 Y_{11}}{Totc_1} & \cdots & \frac{P_1 Y_{1S}}{Totc_S} \\ \vdots & \ddots & \vdots \\ \frac{P_S Y_{S1}}{Totc_1} & \cdots & \frac{P_S Y_{SS}}{Totc_S} \end{bmatrix}$$

Then, if we count all indirect usage of labor and imported goods,

$$\begin{bmatrix} \alpha_{l1} \\ \vdots \\ \alpha_{lS} \end{bmatrix} = \begin{bmatrix} \tilde{\alpha}_{l1} \\ \vdots \\ \tilde{\alpha}_{lS} \end{bmatrix} + A' \begin{bmatrix} \tilde{\alpha}_{l1} \\ \vdots \\ \tilde{\alpha}_{lS} \end{bmatrix} + (A')^2 \begin{bmatrix} \tilde{\alpha}_{l1} \\ \vdots \\ \tilde{\alpha}_{lS} \end{bmatrix} + ... = (I - A')^{-1} \begin{bmatrix} \tilde{\alpha}_{l1} \\ \vdots \\ \tilde{\alpha}_{lS} \end{bmatrix}.$$

$$\begin{bmatrix} \alpha_{m1} \\ \vdots \\ \alpha_{mS} \end{bmatrix} = \begin{bmatrix} \tilde{\alpha}_{m1} \\ \vdots \\ \tilde{\alpha}_{mS} \end{bmatrix} + A' \begin{bmatrix} \tilde{\alpha}_{m1} \\ \vdots \\ \tilde{\alpha}_{mS} \end{bmatrix} + (A')^2 \begin{bmatrix} \tilde{\alpha}_{m1} \\ \vdots \\ \tilde{\alpha}_{mS} \end{bmatrix} + ... = (I - A')^{-1} \begin{bmatrix} \tilde{\alpha}_{m1} \\ \vdots \\ \tilde{\alpha}_{mS} \end{bmatrix}.$$

Similarly, define

$$P_s Y_s = \sum_{s'} P_s Y_{ss'} + P_s C_s + \frac{\theta_s^* - 1}{\theta_s^*} P_s^X X_s$$

$$\tilde{\phi}_{sc} = \frac{P_s C_s}{P_s Y_s},\ \tilde{\phi}_{sx} \left(= \frac{P_s}{P_s^X} \frac{P_s^X X_s}{P_s Y_s} \right) = \frac{\theta^* - 1}{\theta^*} \frac{P_s^X X_s}{P_s Y_s},\ \Phi = \begin{bmatrix} \frac{P_1 Y_{11}}{P_1 Y_1} & \cdots & \frac{P_1 Y_{1S}}{P_1 Y_1} \\ \vdots & \ddots & \vdots \\ \frac{P_S Y_{S1}}{P_S Y_S} & \cdots & \frac{P_S Y_{SS}}{P_S Y_S} \end{bmatrix}$$

Then, I count all indirect demand from domestic and foreign consumers,

$$\begin{bmatrix} \phi_{1c} \\ \vdots \\ \phi_{Sc} \end{bmatrix} = \begin{bmatrix} \tilde{\phi}_{1c} \\ \vdots \\ \tilde{\phi}_{Sc} \end{bmatrix} + \Phi \begin{bmatrix} \tilde{\phi}_{1c} \\ \vdots \\ \tilde{\phi}_{Sc} \end{bmatrix} + \Phi^2 \begin{bmatrix} \tilde{\phi}_{1c} \\ \vdots \\ \tilde{\phi}_{Sc} \end{bmatrix} ... = (I - \Phi)^{-1} \begin{bmatrix} \tilde{\phi}_{1c} \\ \vdots \\ \tilde{\phi}_{Sc} \end{bmatrix}$$

$$\begin{bmatrix} \phi_{1x} \\ \vdots \\ \phi_{Sx} \end{bmatrix} = \begin{bmatrix} \tilde{\phi}_{1x} \\ \vdots \\ \tilde{\phi}_{Sx} \end{bmatrix} + \Phi \begin{bmatrix} \tilde{\phi}_{1x} \\ \vdots \\ \tilde{\phi}_{Sx} \end{bmatrix} + \Phi^2 \begin{bmatrix} \tilde{\phi}_{1x} \\ \vdots \\ \tilde{\phi}_{Sx} \end{bmatrix} ... = (I - \Phi)^{-1} \begin{bmatrix} \tilde{\phi}_{1x} \\ \vdots \\ \tilde{\phi}_{Sx} \end{bmatrix}$$

Appendix B

Appendix to Chapter 2

B.1 Results using e-commerce intensity using Rakuten

B.1.1 E-commerce and National Prices

Table B.1: Relative Price Changes

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Dependent Variable	Δp_{ict}	Δp_{ict}	Δp_{ict}	Δp_{ict}	Δp_{ict}	Δp_{ict}	Δp_{ict}
D_t	-0.0014**	0.0002	0.0002	0.0047***	0.0064***	0.0064***	0.0030***
	(0.0005)	(0.0006)	(0.0006)	(0.0005)	(0.0005)	(0.0005)	(0.0004)
Internet Intensity	0.0047	-0.0156***	-0.0155***	-0.0211***	-0.0421***	-0.0421***	-0.0279***
$\times D_t$	(0.0035)	(0.0050)	(0.0050)	(0.0018)	(0.0043)	(0.0043)	(0.0040)
Constant	-0.0040***	-0.0121***	-0.0097**	-0.0041***	-0.0166***	-0.0164***	-0.0138***
	(0.0004)	(0.0040)	(0.0045)	(0.0004)	(0.0027)	(0.0030)	(0.0028)
Sample	Goods	Goods	Goods	Goods	Goods	Goods	Goods and Service
Fixed Effects		Product	Product and City		Product	Product and City	Product and City
t	1992-2001	1992-2001	1992-2001	1992-2016	1992-2016	1992-2016	1992-2016
Observations	152,958	152,958	152,958	394,663	394,663	394,663	459,279
R^2	0.000	0.040	0.040	0.001	0.038	0.038	0.038

B.1.2 Gains Due to Price Arbitrage

Table B.2: Estimates Over Alternative Periods

Dependent Variable	(1) Δp_{ict}	(2) Δp_{ict}	(3) Δp_{ict}	(4) Δp_{ict}	(5) Δp_{ict}
Lagged Price	-0.399***	-0.466***	-0.391***	-0.401***	-0.169***
	(0.005)	(0.005)	(0.005)	(0.003)	(0.001)
Internet Intensity	0.562***	0.816***	0.359***	0.416***	0.189***
× Lagged Price	(0.073)	(0.078)	(0.072)	(0.073)	(0.025)
Internet Intensity	-0.950***	-1.863***	-1.142***	-1.011***	-0.348***
× Lagged Price × Post Rakuten	(0.091)	(0.091)	(0.085)	(0.078)	(0.026)
t	{1996,2006}	{1996,2011}	{1996,2016}	{1996,2001,	Annual
				2006,2016}	1992-2016
k	5	5	5	5	1
Observations	52,017	43,388	42,683	87,818	394,663
R^2	0.55	0.60	0.64	0.61	0.46

Table B.3: Robustness Check Using All Goods and Services

Dependent Variable	(1) Δp_{ict}	(2) Δp_{ict}	(3) Δp_{ict}
Lagged Price	-0.297***	-0.353***	-0.141***
	(0.004)	(0.003)	(0.001)
Internet Intensity	-0.052	0.145**	0.025
× Lagged Price	(0.065)	(0.070)	(0.024)
Internet Intensity	-1.061***	-1.013***	-0.329***
× Lagged Price × Post Rakuten	(0.083)	(0.076)	(0.025)
t	{1996,2001}	{1996,2001	Annual
		,2006,2016}	1992-2016
Observations	59,466	102,102	459,279
R^2	0.52	0.60	0.45

Appendix C

Appendix to Chapter 3

C.1 Proof of Theorem 3.1

The proof of the theorem is by comparison of coefficients. At the end, the system of coefficients boils down to a cubic polynomial of β. The second order condition for the dealer then selects the unique negative root. All other coefficients are uniquely determined once β is obtained. Since a cubic polynomial equation has a closed form solution, all the equilibrium coefficients can be written in closed forms.With additional calculations, we can also derive welfare of agents in closed forms. We first prove existence and uniqueness. $BC\beta_I\gamma_I\beta_U\beta\gamma \neq 0$ is assumed until it is proven at the end.

Existence and uniqueness

Proof is by guessing and verifying $p(z, s) = A + B(z + Cs)$.

Traders' problem

For each $i = I, U$, by log normality we get

$$\arg\max_x E_i\left[-e^{-\theta(d-p)x}\right] = \arg\max_x \left(E_i d - p\right) x - \frac{\theta}{2}\left(V_i d\right) x^2 = \frac{E_i d - p}{\theta V_i d}.$$

By the joint normality of X and $p(z,s) = A + B(z + Cs)$, the moments of return d conditional on the traders' information are

$$\begin{aligned}
E_I d &= E[d|z] = \frac{\kappa_d \bar{d} + \kappa_\epsilon z}{\kappa_d + \kappa_\epsilon} \\
V_I d &= V[d|z] = \frac{1}{\kappa_d + \kappa_\epsilon} \\
E_U d &= E[d|z + Cs] = \bar{d} + \frac{\kappa_d^{-1}}{\kappa_d^{-1} + \kappa_\epsilon^{-1} + C^2\kappa_s^{-1}}\left(\frac{p - A}{B} - C\bar{s} - \bar{d}\right) \\
V_U d &= \kappa_d^{-1} - \frac{\kappa_d^{-2}}{\kappa_d^{-1} + \kappa_\epsilon^{-1} + C^2\kappa_s^{-1}}.
\end{aligned}$$

Hence, the informed agent's best response is $x_I^B(z,p) = \alpha_I + \beta_I p + \gamma_I z$ where

$$\alpha_I = \frac{\kappa_d \bar{d}}{\theta}, \beta_I = -\frac{\kappa_d + \kappa_\epsilon}{\theta}, \gamma_I = \frac{\kappa_\epsilon}{\theta}. \tag{C.1}$$

The uninformed agent's best response is $x_U^B(p) = \alpha_U + \beta_U p$ where

$$\alpha_U = \frac{\left(\kappa_\epsilon^{-1} + C^2\kappa_s^{-1}\right)\bar{d} - \kappa_d^{-1}\left(\frac{A}{B} + C\bar{s}\right)}{\theta\kappa_d^{-1}\left(\kappa_\epsilon^{-1} + C^2\kappa_s^{-1}\right)}, \beta_U = \frac{\frac{1}{B}\kappa_d^{-1} - \left(\kappa_d^{-1} + \kappa_\epsilon^{-1} + C^2\kappa_s^{-1}\right)}{\theta\kappa_d^{-1}\left(\kappa_\epsilon^{-1} + C^2\kappa_s^{-1}\right)}. \tag{C.2}$$

The total demand is $x^B(z,p) = \alpha + \beta p + \gamma z$ where

$$\alpha = \lambda\alpha_I + (1-\lambda)\alpha_U, \beta = \lambda\beta_I + (1-\lambda)\beta_U, \gamma = \lambda\gamma_I. \tag{C.3}$$

Dealer's problem

Given the total demand, the dealer can infer z, and therefore the dealer's problem is

$$
\begin{aligned}
&\arg\max_p E_D\left[-e^{-\theta_D\left\{\left(s-x^B(z,p)\right)d+px^B(z,p)\right\}}\right] \\
=&\arg\max_p \left(s-\alpha-\beta p-\gamma z\right)E\left[d|z\right]+p\left(\alpha+\beta p+\gamma z\right)-\frac{\theta_D}{2}\left(s-\alpha-\beta p-\gamma z\right)^2 V\left[d|z\right].
\end{aligned}
$$

The first order condition with respect to p gives $p\left(z,s\right)=A+B\left(z+Cs\right)$ where

$$
\begin{aligned}
[A]:\ A &= \frac{\alpha\left(\kappa_d+\kappa_\epsilon\right)-\beta\left(\theta_D\alpha+\kappa_d\bar{d}\right)}{\theta_D\beta^2-2\beta\left(\kappa_d+\kappa_\epsilon\right)} \\
[B]:\ B &= \frac{\gamma\left(\kappa_d+\kappa_\epsilon\right)-\beta\left(\theta_D\gamma+\kappa_\epsilon\right)}{\theta_D\beta^2-2\beta\left(\kappa_d+\kappa_\epsilon\right)} \\
[C]:\ C &= \frac{\theta_D\beta}{\gamma\left(\kappa_d+\kappa_\epsilon\right)-\beta\left(\theta_D\gamma+\kappa_\epsilon\right)}.
\end{aligned}
\tag{C.4}
$$

The second order condition is

$$
\beta\left(\beta-\frac{2\left(\kappa_d+\kappa_\epsilon\right)}{\theta_D}\right)>0. \tag{C.5}
$$

Fixed point

Recall $\left(\theta,\theta_D,\bar{d},\bar{s},\kappa_d,\kappa_s,\kappa_\epsilon,\lambda\right)$ is exogenous. Existence and uniqueness of an equilibrium is equivalent to those of the eleven parameters $\left(\alpha_I,\beta_I,\gamma_I,\alpha_U,\beta_U,\alpha,\beta,\gamma,A,B,C\right)$ that satisfy (C.1), (C.2), (C.3), (C.4) and (C.5).

We can reduce this problem to finding a root of the equation that only contains β. To see this, note that $\left(\alpha_I,\beta_I,\gamma_I\right)$ and therefore γ are already functions of exogenous parameters.

By substituting (C.2) into (C.3), the problem reduces to finding (α, β, A, B, C) satisfying

$$
\begin{aligned}
[\alpha]:\ \alpha &= \lambda\alpha_I + (1-\lambda)\frac{\left(\kappa_\epsilon^{-1} + C^2\kappa_s^{-1}\right)\bar{d} - \kappa_d^{-1}\left(\frac{A}{B} + C\bar{s}\right)}{\theta\kappa_d^{-1}\left(\kappa_\epsilon^{-1} + C^2\kappa_s^{-1}\right)} \\
[\beta]:\ \beta &= \lambda\beta_I + (1-\lambda)\frac{\frac{1}{B}\kappa_d^{-1} - \left(\kappa_d^{-1} + \kappa_\epsilon^{-1} + C^2\kappa_s^{-1}\right)}{\theta\kappa_d^{-1}\left(\kappa_\epsilon^{-1} + C^2\kappa_s^{-1}\right)}
\end{aligned}
$$

$$
\begin{aligned}
[A]:\ A &= \frac{\alpha\left(\kappa_d + \kappa_\epsilon\right) - \beta\left(\theta_D\alpha + \kappa_d\bar{d}\right)}{\theta_D\beta^2 - 2\beta\left(\kappa_d + \kappa_\epsilon\right)} \\
[B]:\ B &= \frac{\gamma\left(\kappa_d + \kappa_\epsilon\right) - \beta\left(\theta_D\gamma + \kappa_\epsilon\right)}{\theta_D\beta^2 - 2\beta\left(\kappa_d + \kappa_\epsilon\right)} \\
[C]:\ C &= \frac{\theta_D\beta}{\gamma\left(\kappa_d + \kappa_\epsilon\right) - \beta\left(\theta_D\gamma + \kappa_\epsilon\right)}
\end{aligned}
$$

$$
[SOC]:\ \beta\left(\beta - 2\frac{\kappa_d + \kappa_\epsilon}{\theta_D}\right) > 0.
$$

By substituting $[B]$ and $[C]$ into $[\beta]$, we can obtain an equation that contains only β. Once β that satisfies both this equation and $[SOC]$ is obtained, (C, B) can be uniquely determined by $[C]$ and $[B]$. (α, A) is then the unique solution of a system of linear equations, $[\alpha]$ and $[A]$.

Now we show such β exists uniquely. By substituting $[B]$ and $[C]$ into $[\beta]$,

$$
b(\beta) := b_0 + b_1\beta + b_2\beta^2 + b_3\beta^3 = 0, \tag{C.6}
$$

where coefficients (b_0, b_1, b_2, b_3) are

$$\begin{cases} b_0 & = \gamma^2 \kappa_s \left(\kappa_d + \kappa_\epsilon\right)^3 \\ b_1 & = -\frac{\kappa_\epsilon^2 \kappa_s \lambda^2}{\theta} \left(\kappa_d + \kappa_\epsilon\right)^2 \left(1 + 2\frac{\theta_D}{\theta}\right) \\ b_2 & = \kappa_s \left(\kappa_d + \kappa_\epsilon\right) \left\{\left(\gamma\theta_D + \kappa_\epsilon\right)^2 - 2\left(\theta\gamma + (1-\lambda)\kappa_\epsilon\right)\left(\gamma\theta_D + \kappa_\epsilon\right) - (1-\lambda)\kappa_\epsilon\gamma\theta_D\right\} \cdot \\ & \quad + \kappa_\epsilon \lambda \theta_D^2 \left(\kappa_d + \kappa_\epsilon\right) + \kappa_d \kappa_\epsilon \theta_D^2 (1-\lambda) \\ b_3 & = \kappa_s \gamma \theta \theta_D \left(\gamma\theta_D + 2\kappa_\epsilon\right) + \kappa_\epsilon \kappa_s \gamma \theta_D^2 (1-\lambda) + \kappa_s \kappa_\epsilon^2 \left\{(1-\lambda)\theta_D + \theta\right\} + \kappa_\epsilon \theta \theta_D^2 \end{cases}$$

$b_0 = b(0) > 0$ and $b_3 > 0$ implies that there is a $\beta < 0$ that satisfies $b(\beta) = 0$. Since $\beta < 0$ satisfies $[SOC]$, we obtain existence. For the uniqueness of β, it suffices to show $b'(0) = b_1 < 0$, $b\left(2\frac{\kappa_d+\kappa_\epsilon}{\theta_D}\right) > 0$ and $b'\left(2\frac{\kappa_d+\kappa_\epsilon}{\theta_D}\right) > 0$. Indeed,

$$b\left(2\frac{\kappa_d + \kappa_\epsilon}{\theta_D}\right) = \frac{\kappa_\epsilon \left(\kappa_d + \kappa_\epsilon\right)^3}{\theta^2 \theta_D^3} \left\{\kappa_s \kappa_\epsilon \left(2\theta + \theta_D\right)\left(2\theta + \lambda\theta_D\right)^2 + 4\theta^2 \theta_D^3 \frac{\kappa_d + \lambda\kappa_\epsilon}{\kappa_d + \kappa_\epsilon} + 8\theta^3 \theta_D^2\right\} > 0$$

$$b'\left(2\frac{\kappa_d + \kappa_\epsilon}{\theta_D}\right) = \frac{\kappa_\epsilon \left(\kappa_d + \kappa_\epsilon\right)^2}{\theta^2 \theta_D^2} \left\{\kappa_s \kappa_\epsilon \left(3\theta + 2\theta_D\right)\left(2\theta + \lambda\theta_D\right)^2 + 4\theta^2 \theta_D^3 \frac{\kappa_d + \lambda\kappa_\epsilon}{\kappa_d + \kappa_\epsilon} + 12\theta^3 \theta_D^2\right\} > 0.$$

Signs and Regions

We next show $\beta_I < \beta < \beta_U < 0 < \gamma < \gamma_I$, $C < 0 < B$ and $BC\beta_I\gamma_I\beta_U\beta\gamma \neq 0$.

$0 < \gamma = \lambda\gamma_I < \gamma_I$ follows by (C.1) and (C.3). We know $\beta < 0$ by the above argument. $C < 0 < B$ follows from $[C]$ and $[B]$. $\beta_I < \beta < \beta_U < 0$ comes from the fact that β is a convex combination of (β_I, β_U) and

$$b\left(\lambda\beta_I\right) = \frac{(1-\lambda)\kappa_d \kappa_\epsilon \lambda^2 \theta_D^2 \left(\kappa_d + \kappa_\epsilon\right)^2}{\theta^2} > 0$$

$$b\left(\beta_I\right)=-\frac{1-\lambda}{\theta^3}\kappa_\epsilon\left(\kappa_d+\kappa_\epsilon\right)^2\left\{\kappa_s\kappa_\epsilon^2\left(2\theta+\theta_D\right)+\kappa_\epsilon\theta\theta_D^2\right.$$
$$\left.+\kappa_d\kappa_s\kappa_\epsilon\left(2\theta+\theta_D\right)+\gamma\left(\kappa_d+\kappa_\epsilon\right)\kappa_s\left(2\theta+\theta_D\right)\left(\theta+\theta_D\right)\right\}<0.$$

Finally, we show $BC\beta_I\gamma_I\beta_U\beta\gamma\neq 0$. $\gamma\beta_I\gamma_I\neq 0$ follows from the closed form solutions (C.1) and (C.3). $BC\beta\neq 0$ is proven by contradiction next. $\beta_U\neq 0$ follows from $\beta\neq 0$.

To show $\beta\neq 0$, suppose $\beta=0$. Then, the dealer's objective is

$$\max_p\left(s-\alpha-\gamma z\right)E\left[d|z\right]+p\left(\alpha+\gamma z\right)-\frac{\theta_D}{2}\left(s-\alpha-\gamma z\right)^2V\left[d|z\right].$$

For equilibrium price to exist for all $z\in\mathbb{R}$, $\alpha=\gamma=0$, a contradiction to $\gamma>0$.

To show $B\neq 0$, suppose $B=0$. Then, the price function tells nothing about the signal z. Then the aggregate demand becomes

$$x^B\left(z,p\right)=\frac{\kappa_d}{\theta}\bar{d}-\frac{\kappa_d+\lambda\kappa_\epsilon}{\theta}p+\frac{\lambda\kappa_\epsilon}{\theta}z.$$

Given this demand function, the dealer's optimization implies

$$\begin{cases}B&=\frac{\gamma(\kappa_d+\kappa_\epsilon)-\beta(\theta_D\gamma+\kappa_\epsilon)}{\theta_D\beta^2-2\beta(\kappa_d+\kappa_\epsilon)}=0\\ \beta&=-\frac{\kappa_d+\lambda\kappa_\epsilon}{\theta}\end{cases}\Rightarrow 0=\gamma\left(\kappa_d+\kappa_\epsilon\right)+\frac{\kappa_d+\lambda\kappa_\epsilon}{\theta}\left(\theta_D\gamma+\kappa_\epsilon\right)>0,$$

a contradiction.

To see $C \neq 0$, suppose $C = 0$. Since $B \neq 0$, the price function fully reveals z.

$$\begin{cases} x_I^B(z,p) &= \frac{\kappa_d}{\theta}\bar{d} - \frac{\kappa_d+\kappa_\epsilon}{\theta}p + \frac{\kappa_\epsilon}{\theta}z \\ x_U^B(p) &= \frac{1}{\theta}\left(\kappa_d\bar{d} - \frac{A}{B}\right) + \frac{1}{\theta}\left\{\frac{\kappa_\epsilon}{B} - (\kappa_d+\kappa_\epsilon)\right\}p \end{cases}$$

$$\Rightarrow x^B(z,p) = \frac{1}{\theta}\left\{\kappa_d\bar{d} - (1-\lambda)\frac{A}{B}\right\} + \frac{1}{\theta}\left\{\frac{1-\lambda}{B}\kappa_\epsilon - (\kappa_d+\kappa_\epsilon)\right\}p + \frac{\lambda\kappa_\epsilon}{\theta}z.$$

Given $x^B(z,p) = \alpha + \beta p + \gamma z$, $\beta \neq 0$, and $B \neq 0$, three possibilities have to be considered. If $[SOC]$ is met, the optimal pricing is given by (C.4). Hence,

$$\frac{\theta_D\beta}{\gamma(\kappa_d+\kappa_\epsilon) - \beta(\theta_D\gamma+\kappa_\epsilon)} = 0 \Rightarrow \beta = 0,$$

a contradiction to $\beta \neq 0$. If $\beta\left(\beta - \frac{2(\kappa_d+\kappa_\epsilon)}{\theta_D}\right) < 0$, there is no optimal price. If $\beta\left(\beta - \frac{2(\kappa_d+\kappa_\epsilon)}{\theta_D}\right) = 0$, $\beta = 2\frac{\kappa_d+\kappa_\epsilon}{\theta_D}$. The dealer's objective function becomes

$$\begin{aligned} &\arg\max_p (s-\alpha-\beta p-\gamma z)E[d|z] + p(\alpha+\beta p+\gamma z) - \frac{\theta_D}{2}(s-\alpha-\beta p-\gamma z)^2 V[d|z] \\ =&\arg\max_p \{\alpha+\gamma z + 2(\kappa_d+\kappa_\epsilon)(s-\alpha-\gamma z-E[d|z])\}p. \end{aligned}$$

There is no way for the coefficient of p to be 0 identically for all s and z, implying there is no optimal price in this case.

C.2 Proof of Proposition 3.1

Since $s-x^B(z,p)$ is observable to the dealer, noting d being independent of ξ, we get

$$\begin{aligned}
& -E_D e^{-\theta\left\{\left(s-x^B(z,p)\right)d+px^B(z,p)-\frac{\alpha}{2}\left(s-x^B(z,p)+\xi\right)^2\right\}} \\
= & -e^{-\theta p x^B(z,p)} E_D e^{-\theta\left(s-x^B(z,p)\right)d} E_D e^{-\theta\frac{\alpha}{2}\left(s-x^B(z,p)+\xi\right)^2}.
\end{aligned}$$

Each part can be calculated in closed form. The second term becomes

$$E_D e^{-\theta\left(s-x^B(z,p)\right)d} = e^{-\theta\left(s-x^B(z,p)\right)E_D d+\frac{\theta^2}{2}\left(s-x^B(z,p)\right)^2 V_D d}$$

The third term is

$$E_D e^{-\theta\frac{\alpha}{2}\left(s-x^B(z,p)+\xi\right)^2} = e^{-\theta\frac{\alpha}{2}\left(s-x^B(z,p)\right)^2} E_D e^{-\theta\alpha\left(s-x^B(z,p)\right)\xi-\theta\frac{\alpha}{2}\xi^2}$$

$$\begin{aligned}
& E_D e^{-\theta\alpha\left(s-x^B(z,p)\right)\xi-\theta\frac{\alpha}{2}\xi^2} \\
= & \frac{1}{\sqrt{2\pi\sigma_\xi^2}}\int e^{-\theta\alpha\left(s-x^B(z,p)\right)\xi-\theta\frac{\alpha}{2}\xi^2} e^{-\frac{\xi^2}{2\sigma_\xi^2}} d\xi \\
= & \exp\left(\frac{1}{2}\frac{(\theta\alpha)^2\left(s-x^B(z,p)\right)^2}{\frac{1}{\sigma_\xi^2}+\theta\alpha}\right)\frac{\sqrt{2\pi\left(\frac{1}{\sigma_\xi^2}+\theta\alpha\right)^{-1}}}{\sqrt{2\pi\sigma_\xi^2}} \\
& \times \underbrace{\frac{1}{\sqrt{2\pi\left(\frac{1}{\sigma_\xi^2}+\theta\alpha\right)^{-1}}}\int \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_\xi^2}+\theta\alpha\right)\left(\xi+\frac{\theta\alpha\left(s-x^B(z,p)\right)}{\frac{1}{\sigma_\xi^2}+\theta\alpha}\right)^2\right\}d\xi}_{=1} \\
= & \frac{1}{\sqrt{1+\theta\alpha\sigma_\xi^2}}\exp\left(\frac{1}{2}\frac{(\theta\alpha)^2}{\frac{1}{\sigma_\xi^2}+\theta\alpha}\left(s-x^B(z,p)\right)^2\right)
\end{aligned}$$

Combining these results, the objective function related to p is

$$\left(s - x^B(z,p)\right) E_D d + p x^B(z,p) - \left(\frac{\theta}{2} V_D d + \frac{1}{2}\frac{\alpha}{1+\theta\alpha\sigma_\xi^2}\right)\left(s - x^B(z,p)\right)^2.$$

Comparing with Eq.(3.14) completes the proof.

$$\frac{\theta_D}{2} V_D d = \frac{\theta}{2} V_D d + \frac{1}{2}\frac{\alpha}{1+\theta\alpha\sigma_\xi^2}.$$

C.3 Proof of Proposition 3.2

Note that the explicit form of $x(z,s) = x^B(z, p(z,s))$ is

$$x(z,s) = \alpha + \beta A + (\beta B + \gamma) z + \beta B C s.$$

For the first result $\lim_{\theta_D \to \infty} x(z,s) = s$, recall the system of characterizing Eq. $[A]$, $[B]$, and $[C]$. By taking $\theta_D \to \infty$,

$$[A]: \; A = \frac{-\alpha}{\beta}, \; [B]: \; B = \frac{-\gamma}{\beta}, \; [C]: \; C = -\frac{1}{\gamma}.$$

In other words, $\beta BC = 1$, $\beta B + \gamma = 0$, and $\alpha + \beta A = 0$. Hence, as $\theta_D \to \infty$, $x(z,s) \to s$.

For the second result $\lim_{\theta_D \to 0} x(z,s) = 0$, note that from $[C]$ and $\beta_I < \beta < 0$, $C \to 0$ as $\theta_D \to 0$. By substituting $C = 0$ into $\{[\alpha], [\beta], [A], [B]\}$,

$$[\alpha]: \;\; \alpha = \frac{\kappa_d \bar{d}}{\theta} - (1-\lambda)\frac{\kappa_\epsilon}{\theta}\frac{A}{B}$$

$$[\beta]: \ \beta = -\frac{\kappa_d + \kappa_\epsilon}{\theta} + (1-\lambda)\frac{\kappa_\epsilon}{\theta}\frac{1}{B}$$

$$[A]: \ A = -\frac{\alpha}{2\beta} + \frac{\kappa_d \bar{d}}{2(\kappa_d + \kappa_\epsilon)}$$

$$[B]: \ B = -\frac{\gamma}{2\beta} + \frac{\kappa_\epsilon}{2(\kappa_d + \kappa_\epsilon)}.$$

One can check

$$B = \frac{\kappa_\epsilon}{\kappa_d + \kappa_\epsilon}, \ \beta = -\lambda\frac{\kappa_d + \kappa_\epsilon}{\theta}, \ \alpha = \lambda\frac{\kappa_d \bar{d}}{\theta}, \ A = \frac{\kappa_d \bar{d}}{\kappa_d + \kappa_\epsilon}$$

solve the system and $\alpha + \beta A = \beta B + \gamma = \beta B C = 0$. Hence, $x(z,s) \to 0$ as $\theta_D \to 0$.

C.4 Proof of Theorem 3.2

We prove each of the four claims in order.

Price informativeness

By theorem 3.1 and $[C]$, C is negative and satisfies

$$\frac{1}{C} = \frac{\gamma(\kappa_d + \kappa_\epsilon) - \beta(\theta_D \gamma + \kappa_\epsilon)}{\theta_D \beta} = \frac{\gamma(\kappa_d + \kappa_\epsilon)}{\theta_D \beta} - \gamma - \frac{\kappa_\epsilon}{\theta_D}.$$

To show Q decreases, it suffices to show $\beta < 0$ is decreasing in θ_D. The range of Q comes from the range of C, $\left(-\frac{1}{\gamma}, 0\right)$.

We show β is decreasing by the implicit function theorem. Denote $b(\beta)$ in Eq. (C.6) by

$b(\beta, \theta_D)$. The goal is to show

$$\frac{\partial \beta}{\partial \theta_D} = -\frac{\partial b(\beta, \theta_D)/\partial \theta_D}{\partial b(\beta, \theta_D)/\partial \beta} < 0.$$

Since $b(\beta)$ crosses horizontal line from below at the equilibrium $\beta < 0$, $\frac{\partial b(\beta, \theta_D)}{\partial \beta}$ evaluated at the equilibrium $\beta < 0$ is always strictly positive. Hence, by the implicit function theorem, it suffices to check $\frac{\partial b}{\partial \theta_D}$ evaluated at the equilibrium $\beta < 0$ is positive. By using $\beta^3 = -\frac{1}{b_3}(b_0 + b_1\beta + b_2\beta^2)$,

$$\begin{aligned}\frac{\partial b}{\partial \theta_D} &= \frac{\partial b_0}{\partial \theta_D} + \frac{\partial b_1}{\partial \theta_D}\beta + \frac{\partial b_2}{\partial \theta_D}\beta^2 + \frac{\partial b_3}{\partial \theta_D}\beta^3 \\ &= \frac{\partial b_0}{\partial \theta_D} - \frac{\partial b_3}{\partial \theta_D}\frac{b_0}{b_3} + \left(\frac{\partial b_1}{\partial \theta_D} - \frac{\partial b_3}{\partial \theta_D}\frac{b_1}{b_3}\right)\beta + \left(\frac{\partial b_2}{\partial \theta_D} - \frac{\partial b_3}{\partial \theta_D}\frac{b_2}{b_3}\right)\beta^2.\end{aligned}$$

By direct calculation, with $\Psi := \theta^2\theta_D^2 + \kappa_s\kappa_\epsilon\left(\theta^2 + (1+\lambda)\theta\theta_D + \lambda\theta_D^2\right) > 0$,

$$\frac{\partial b_0}{\partial \theta_D} - \frac{\partial b_3}{\partial \theta_D}\frac{b_0}{b_3} = -\frac{\kappa_\epsilon^2\kappa_s\lambda^2(\kappa_d + \kappa_\epsilon)^3\left[2\theta^2\theta_D + \kappa_\epsilon\kappa_s\left\{(1+\lambda)\theta + 2\lambda\theta_D\right\}\right]}{\theta^2\Psi} < 0,$$

$$\begin{aligned}\frac{\partial b_2}{\partial \theta_D} - \frac{\partial b_3}{\partial \theta_D}\frac{b_2}{b_3} =& \frac{\kappa_\epsilon^2\kappa_s}{\theta\Psi}\Big[\kappa_s\kappa_\epsilon(\kappa_d + \kappa_\epsilon)\Big\{\lambda^2(\theta + \theta_D)^2 + (\lambda\theta_D + \theta)^2\Big\} \\ &+ 2\kappa_\epsilon\theta\theta_D\Big\{(1+\lambda)\theta^2 + \lambda\theta\theta_D\Big\} + \kappa_d\theta^2\theta_D^2\left\{1 + \lambda(2-\lambda)\right\} + 4\kappa_d\theta^3\theta_D\Big] > 0,\end{aligned}$$

$$\frac{\partial b(\lambda\beta_I, \theta_D)}{\partial \theta_D} = \frac{\kappa_d\kappa_\epsilon^2\kappa_s\lambda^2\theta_D(\kappa_d + \kappa_\epsilon)^2(1-\lambda)\left\{2\theta + (1+\lambda)\theta_D\right\}}{\theta\Psi} > 0.$$

These results imply that the quadratic function $\frac{\partial b(\beta)}{\partial \theta_D}$ is decreasing and positive on $[\beta_I, \lambda\beta_I]$. Since the equilibrium β is in this region by theorem 3.1, the proof completes.

Price volatility

By direct calculation,

$$\begin{aligned}
&\lim_{\theta_D\to\infty} V\left(p\left(z,s\right)\right)-\lim_{\theta_D\to 0} V\left(p\left(z,s\right)\right)\\
=&\left(\lim_{\theta_D\to\infty} B^2-\lim_{\theta_D\to 0} B^2\right)\left(\kappa_d^{-1}+\kappa_\epsilon^{-1}\right)+\left(\lim_{\theta_D\to\infty}\left(BC\right)^2-\lim_{\theta_D\to 0}\left(BC\right)^2\right)\kappa_s^{-1}\\
=&\frac{\theta^2\kappa_\epsilon^2\lambda^2\left(2\lambda-1\right)\left(\kappa_d+\kappa_\epsilon\right)\kappa_s^2+\theta^4\kappa_\epsilon\left\{\left(2\kappa_\epsilon+\kappa_d\right)\lambda^2+\kappa_d\left(2\lambda-1\right)\right\}\kappa_s+\theta^6\left(\kappa_d+\kappa_\epsilon\right)}{\kappa_s\left(\kappa_d+\kappa_\epsilon\right)\left\{\kappa_s\kappa_\epsilon\lambda^2\left(\kappa_d+\kappa_\epsilon\right)+\left(\kappa_d+\lambda\kappa_\epsilon\right)\theta^2\right\}^2}.
\end{aligned}$$

Note that the denominators of the last expression is positive. From the second equality, if $2\lambda-1\geq 0$, there is no restriction on κ_s. If $2\lambda-1<0$, the numerator of the ratio after the last equality is a concave quadratic function of κ_s with a positive intercept. Hence, by taking small enough κ_s, the numerator is positive.

Welfare of the traders

Fix a positive $\theta_D=\bar{\theta}_D>0$ and its associated equilibrium price $p\left(z,s\right)$. Note that the traders can always choose demand functions to be identically 0 by selecting $\alpha_I=\beta_I=\gamma_I=\alpha_U=\beta_U=0$. Hence, the equilibrium interim utilities satisfy

$$E\left[-e^{-\theta(d-p(z,s))x_I^B(z,p(z,s))}|p\left(z,s\right),z\right]\geq E\left[-e^{-\theta(d-p(z,s))\times 0}|p\left(z,s\right),z\right]=-1$$
$$E\left[-e^{-\theta(d-p(z,s))x_U^B(p(z,s))}|p\left(z,s\right)\right]\geq E\left[-e^{-\theta(d-p(z,s))\times 0}|p\left(z,s\right)\right]=-1.$$

The right sides of the inequalities are the ex-ante welfare when $\theta_D\to 0$ by proposition 3.2. By taking expectation on both sides and using the tower property, the proof completes.

Welfare of the dealer

When Y is an $n \times 1$ vector distributed $N(\mu, \Sigma)$ and A is an $n \times n$ matrix,

$$EY'AY = Etr(Y'AY) = tr(AEYY') = tr\{A(\Sigma + \mu\mu')\}.$$

Since the cash flow of the dealer is a quadratic form of uncertainty X, by using this property, we can calculate the expected cash flow of the dealer in terms of coefficients $(\alpha, \beta, \gamma, A, B, C)$. By substituting coefficient values that correspond to $\theta_D \to \infty$ and $\theta_D \to 0$,

$$\lim_{\theta_D \to 0} u_D - \lim_{\theta_D \to \infty} u_D = \frac{\theta\left(\bar{s}^2 \kappa_s \left(\kappa_\epsilon \kappa_s \lambda^2 + \theta^2\right) + \kappa_\epsilon \kappa_s \lambda + \theta^2\right)}{\kappa_s\left(\kappa_d + \kappa_\epsilon\right)\left\{\kappa_s \kappa_\epsilon \lambda + \theta^2\left(1 + \lambda\right)\right\}} > 0.$$

C.5 Proof of Proposition 3.3

The proof is by guess and verify, and is composed of three parts: the trader's problem, the dealer's problem, and coefficient matching. Guess the equilibrium price is linear

$$p_t = p\left(z_t, s_t, x_{t-1}^D\right) = A_0 + A_x x_{t-1}^D + B\left(z_t + C's_t\right).$$

Note that x_{t-1}^D is the endogenous state.

Traders' problem

Guess the form of value function of the informed traders is, for some nonzero constant $\alpha_I \in \mathbb{R}$ and symmetric matrix Q_I,

$$J\left(\boldsymbol{x}_{t-1}, m_t^I; z_t, s_t\right) = -e^{-\alpha_I\left(m_t^I + p_t x_{t-1}^I\right) - q_I\left(z_t, s_t, x_{t-1}^D\right)},\ q_I\left(z_t, s_t, x_{t-1}^D\right) = \begin{bmatrix} 1 \\ z_t \\ s_t \\ x_{t-1}^D \end{bmatrix}' Q_I \begin{bmatrix} 1 \\ z_t \\ s_t \\ x_{t-1}^D \end{bmatrix}$$

where the bold letter $\boldsymbol{x}_{t-1} = \left(x_{t-1}^I, x_{t-1}^D\right)$ and p_t are functions of (z_t, s_t) and q_I is a quadratic form. Note that although informed traders cannot observe s_t^D, J depends on s_t^D through p_t. To be more precise, J depends on all past shocks $\left(s_\tau^D\right)_{\tau \leq t}$ through equilibrium $\left(p_\tau, x_{\tau-1}^D\right)_{\tau \leq t}$. The equilibrium x_t^D is measurable with respect to $\mathcal{F}_t^I$ due to market clearing $x_t^I + x_t^D = \bar{x}$ in equilibrium, so is a constant for informed agents. Given $\left(x_{t-1}^I, x_t^D, s_t^I, p_t\right)$, informed trader solves

$$\max_{x_t, y_t, \pi_t} -e^{-\theta_I \pi_t} - \beta E_t^I\left[e^{-\alpha_I\left(m_{t+1}^I + p_{t+1} x_t\right) - q_I\left(z_{t+1}, s_{t+1}, x_t^D\right)}\right]\ s.t. \begin{cases} m_{t+1}^I = d_{t+1} x_t + R y_t + s_t^I d_{t+1}^I \\ \pi_t + y_t + p_t x_t = p_t x_{t-1}^I + m_t^I \end{cases}$$

Since there are three control variables, we solve the problem in three steps. The first step is to substitute out risk-free bond y_t. Note that m_{t+1}^I can be written as

$$m_{t+1}^I = \left(d_{t+1} - R p_t\right) x_t + R p_t x_{t-1}^I + s_t^I d_{t+1}^I + R m_t^I - R c_t$$

so that the second term of the objective function can be written as a quadratic function of shocks

$$E_t^I\left[e^{-\alpha_I\left\{(p_{t+1}+d_{t+1}-Rp_t)x_t+Rp_tx_{t-1}^I+s_t^Id_{t+1}^I+Rm_t^I-Rc_t\right\}-q_I\left(z_{t+1},s_{t+1},x_t^D\right)}\right]$$

$$= E_t^I \exp\left(\begin{bmatrix}1\\ d_{t+1}\\ d_{t+1}^I\\ z_{t+1}\\ s_{t+1}\end{bmatrix}'\begin{bmatrix}n_{I0} & \frac{1}{2}n_I'\\ \frac{1}{2}n_I & \frac{1}{2}N_I\end{bmatrix}\begin{bmatrix}1\\ d_{t+1}\\ d_{t+1}^I\\ z_{t+1}\\ s_{t+1}\end{bmatrix}\right)$$

where N_I is a matrix function of Q_I and (n_{I0}, n_I) are linear functions of other non-random variables such as x_t. Second, given the normality assumption, this term is an increasing transformation of a quadratic function of x_t, so x_t can be maximized out

$$\begin{aligned}&\min_{x_t} n_{I0} + n_I'\mu_I(z_t) + \frac{1}{2}\mu_I(z_t)' N_I\mu_I(z_t) + \frac{1}{2}\left(n_I + N_I\mu_I(z_t)\right)'\left(\Sigma_I^{-1} - N_I\right)^{-1}\left(n_I + N_I\mu_I(z_t)\right)\\ =&-\alpha_I R\left(m_t^I + p_tx_{t-1}^I - c_t\right) + \tilde{q}_t^I\end{aligned}$$

where (μ_I, Σ_I) are conditional moments

$$\mu_I(z_t) = E\left[\left[d_{t+1}, d_{t+1}^I, z_{t+1}, s_{t+1}\right]' | z_t\right],\ \Sigma_I = V\left[d_{t+1}, d_{t+1}^I, z_{t+1}, s_{t+1} | z_t\right]$$

and $\tilde{q}_t^I$ is a quadratic form of $\left[1, p_t, x_t^D, z_t, s_t^I\right]$ independent of π_t. This step also yiels the asset demand

$$x_t^I = q_{xp} p_t + q_{xz} z_t + q_{xs} s_t^I + q_{x0}$$

where $(q_{xp}, q_{xz}, q_{xs}, q_{x0})$ are functions of (A_0, A_x, B, C, Q_I). The third step is to maximize over consumption π_t the objective function

$$\max_{\pi_t} -e^{-\theta_I \pi_t} - \beta \left| I - \Sigma_I N_I \right|^{-\frac{1}{2}} e^{-\alpha_I R \left(m_t^I + p_t x_{t-1}^I - \pi_t \right) + \tilde{q}_t^I}$$

By taking first order condition with respect to π_t,

$$\pi_t = \frac{1}{\theta_I + \alpha_I R} \ln \left(\frac{\theta_I}{\beta \alpha_I R \left| I - \Sigma_I N_I \right|^{-\frac{1}{2}}} \right) + \frac{\alpha_I R}{\theta_I + \alpha_I R} \left(m_t^I + p_t x_{t-1}^I \right) - \frac{1}{\theta_I + \alpha_I R} \tilde{q}_t^I.$$

Substituting π_t into the objective function leads to

$$\begin{aligned}
& J\left(x_{t-1}, m_t^I; z_t, s_t\right) \\
= & -\beta \left| I - \Sigma_I N_I \right|^{-\frac{1}{2}} \left(\frac{\alpha_I R + \theta_I}{\theta_I} \right) \left(\frac{\theta_I}{\beta \alpha_I R \left| I - \Sigma_I N_I \right|^{-\frac{1}{2}}} \right)^{\frac{\alpha_I R}{\theta_I + \alpha_I R}} \\
& \times \exp \left(-\frac{\theta_I \alpha_I R}{\theta_I + \alpha_I R} \left(m_t^I + p_t x_{t-1}^I \right) + \frac{\theta_I}{\theta_I + \alpha_I R} \tilde{q}_t^I \right)
\end{aligned}$$

This has to be identical to the original guess so that

$$\frac{\theta_I \alpha_I R}{\theta_I + \alpha_I R} = \alpha_I \Rightarrow \alpha_I = \frac{R-1}{R} \theta_I.$$

To get the equation that characterizes Q_I, note that $\tilde{q}_t^I = \tilde{q}^I\left(p_t, x_t^D, z_t, s_t^I\right)$ can be written as a quadratic form with respect to the same variables as $q^I\left(z_t, s_t, x_{t-1}^D\right)$.

$$\begin{cases} x_t^I = q_{xp}p_t + q_{xz}z_t + q_{xs}s_t^I + q_{x0} \\ x_t^I + x_t^D = \bar{x} \\ p_t = A_0 + A_x x_{t-1}^D + B\left(z_t + C's_t\right) \end{cases} \Rightarrow \tilde{q}^I\left(p_t, x_t^D, z_t, s_t^I\right) = \begin{bmatrix} 1 \\ z_t \\ s_t \\ x_{t-1}^D \end{bmatrix}' \tilde{Q}_I \begin{bmatrix} 1 \\ z_t \\ s_t \\ x_{t-1}^D \end{bmatrix}$$

Substituting α_I into the objective function and comparing it with the original guess lead to

$$Q_I = -\left[\frac{1}{R}\tilde{Q}_I + \left\{\frac{1}{R}\ln\left(\left|I - \Sigma_I N_I\right|^{-\frac{1}{2}}\beta\left(R-1\right)\right) + \ln\frac{R}{R-1}\right\} i_{11}\right]$$

where i_{11} is 5×5 matrix with $(1,1)$ element being 1 and all other elements being 0. Since $\tilde{Q}_I$ is a function of Q_I, this equation pins down the fixed point.

Dealer's problem

Guess the value function of the dealer to be, for some nonzero constant $\alpha_D \in \mathbb{R}$ and symmetric matrix Q_D,

$$J\left(x_{t-1}^D, m_t^D; s_t^D, x_t^I\left(z_t, s_t^I, \cdot\right)\right) = -e^{-\alpha_D m_t^D - q_D\left(z_t, s_t, x_{t-1}^D\right)},\ q_D\left(z_t, s_t, x_{t-1}^D\right) = \begin{bmatrix} 1 \\ z_t \\ s_t \\ x_{t-1}^D \end{bmatrix}' Q_D \begin{bmatrix} 1 \\ z_t \\ s_t \\ x_{t-1}^D \end{bmatrix}$$

Dealers do not observe s_t^I, but it affects value through the demand schedule

$$x_t^I = q_{xp}p_t + q_{xz}z_t + q_{xs}s_t^I + q_{x0} = q_{xp}p_t + q_{xz}\underbrace{\left(z_t + \frac{q_{xs}}{q_{xz}}s_t^I\right)}_{\hat{z}_t} + q_{x0}$$

where $\hat{z}_t$ is observable to the dealer. The dealer's problem is then

$$\max_{x_t, y_t, \pi_t, p_t} -e^{-\theta_D \pi_t} - \beta E_t^D\left[e^{-\alpha_D m_{t+1}^D - q_D(z_{t+1}, s_{t+1}x_t)}\right] \ s.t. \begin{cases} m_{t+1}^D = d_{t+1}x_t + Ry_t + s_t^D d_{t+1}^D \\ \pi_t + y_t + p_t x_t = p_t x_{t-1}^D + m_t^D \\ x_t = \bar{x} - \left(q_{xp}p_t + q_{xz}\hat{z}_t + q_{x0}\right) \end{cases}$$

The steps to solve the problem is similar to those in the traders' problem, except that the choice variable contains price. First, by deleting risk-free bond,

$$m_{t+1}^D = \left(d_{t+1} - Rp_t\right)x_t + Rp_t x_{t-1}^D + s_t^D d_{t+1}^D + Rm_t^D - R\pi_t$$

The second term of the objective function becomes

$$
\begin{aligned}
& E_t^D e^{-\alpha_D\left\{(d_{t+1}-Rp_t)x_t+Rp_t x_{t-1}^D+s_t^D d_{t+1}^D+Rm_t^D-R\pi_t\right\}-q_D(z_{t+1},s_{t+1},x_t)} \\
= & E_t^D \exp\left(\begin{bmatrix} 1 \\ d_{t+1} \\ d_{t+1}^D \\ z_{t+1} \\ s_{t+1} \end{bmatrix}' \begin{bmatrix} n_{D0} & \frac{1}{2}n_D' \\ \frac{1}{2}n_D & \frac{1}{2}N_D \end{bmatrix} \begin{bmatrix} 1 \\ d_{t+1} \\ d_{t+1}^D \\ z_{t+1} \\ s_{t+1} \end{bmatrix}\right) \\
\propto & n_{D0}+n_D'\mu_D\left(\hat{z}_t\right)+\frac{1}{2}\mu_D\left(\hat{z}_t\right)' N_D\mu_D\left(\hat{z}_t\right)+\frac{1}{2}\left(n_D+N_D\mu_D\left(\hat{z}_t\right)\right)'\left(\Sigma_D^{-1}-N_D\right)^{-1}\left(n_D+N_D\mu_D\left(\hat{z}_t\right)\right),
\end{aligned}
$$

where N_D is a matrix function of Q_D, n_{D0} is a linear function of $(x_t, x_t^2, p_t x_t)$, n_D is a linear function of x_t,

$$
\mu_D\left(\hat{z}_t\right) \;=\; E\left[\left[d_{t+1}, d_{t+1}^D, z_{t+1}, s_{t+1}\right]' |\hat{z}_t\right],\ \Sigma_D = V\left[d_{t+1}, d_{t+1}^D, z_{t+1}, s_{t+1}|\hat{z}_t\right].
$$

By substituting the demand schedule, the problem reduces to a minimization of a quadratic function of p_t, which is the second step. Hence, the second term of the objective function is an increasing function of

$$\min_{p_t} n_{D0} + n_D' \mu\left(\hat{z}_t\right) + \frac{1}{2}\mu\left(\hat{z}_t\right)' N_D \mu\left(\hat{z}_t\right) + \frac{1}{2}\left(n_D + N_D \mu\left(\hat{z}_t\right)\right)' \left(\Sigma_D^{-1} - N_D\right)^{-1} \left(n_D + N_D \mu\left(\hat{z}_t\right)\right)$$

$$= -\alpha_D R\left(m_t^D - \pi_t\right) + \underbrace{\begin{bmatrix} 1 \\ z_t \\ s_t \\ x_{t-1}^D \end{bmatrix}' \tilde{Q}_D \begin{bmatrix} 1 \\ z_t \\ s_t \\ x_{t-1}^D \end{bmatrix}}_{\tilde{q}_t^D}$$

and the optimal pricing is

$$p_t = q_{p0} + q_{pz}\hat{z}_t + q_{ps}s_t^D + q_{px}x_{t-1}^D$$

and $(q_{p0}, q_{pz}, q_{ps}, q_{px})$ are functions of $(q_{xp}, q_{xz}, q_{xs}, q_{x0}, Q_D)$. The third step is to choose π_t to solve

$$\max_{\pi_t} -e^{-\theta_D \pi_t} - \beta\left|I - \Sigma_D N_D\right|^{-\frac{1}{2}} e^{-\alpha_D R\left(m_t^D - \pi_t\right) + \tilde{q}_t^D}.$$

By taking the first order condition,

$$\pi_t = \frac{1}{\theta_D + \alpha_D R}\ln\left(\frac{\theta_D}{\beta\alpha_D R\left|I - \Sigma_D N_D\right|^{-\frac{1}{2}}}\right) + \frac{\alpha_D R}{\theta_D + \alpha_D R}m_t^D - \frac{1}{\theta_D + \alpha_D R}\tilde{q}_t^D.$$

Since the implied value function has to coincide with the original guess,

$$\alpha_D = \frac{R-1}{R}\theta_D,\ Q_D = -\left[\frac{1}{R}\tilde{Q}_D + \left\{\frac{1}{R}\ln\left(\left|I - \Sigma_D N_D\right|^{-\frac{1}{2}}\beta\left(R-1\right)\right) + \ln\frac{R}{R-1}\right\} i_{11}\right].$$

Once (α_D, Q_D) is obtained, the consumption becomes

$$\pi_t^D = -\frac{1}{R\theta_D} \ln\left(\beta\left(R-1\right)\left|I - \Sigma_D N_D\right|^{-\frac{1}{2}}\right) + \frac{R-1}{R} m_t^D - \frac{1}{R\theta_D} \tilde{q}_t^D$$

and the coefficients of the pricing decision becomes a function of $(q_{xp}, q_{xz}, q_{xs}, q_{x0})$ and has to coincide with the initial guess

$$p_t = q_{p0} + q_{pz}\hat{z}_t + q_{ps}s_t^D + q_{px}x_{t-1}^D = A_0 + A_x x_{t-1}^D + B\left(z_t + C's_t\right).$$

Fixed point

Given the price coefficients (A_0, A_x, B, C), the informed agents' problem gives a fixed point equation of Q_I. (A_0, A_x, B, C, Q_I) determines coefficients of x^I, $(q_{xp}, q_{xz}, q_{xs}, q_{x0})$. Given $(q_{xp}, q_{xz}, q_{xs}, q_{x0})$, the dealer's problem gives a fixed point equation for Q_D. $(q_{xp}, q_{xz}, q_{xs}, q_{x0}, Q_D)$ then determines (A_0, A_x, B, C). One can solve the fixed point $(Q_I, Q_D, A_0, A_x, B, C, q_{xp}, q_{xz}, q_{xs}, q_{x0})$ numerically by repeating the loop until convergence. Once the fixed point is obtained, the transition of x_t^D can be obtained from

$$p_t = A_0 + A_x x_{t-1}^D + B\left(z_t + C's_t\right), \;\; x_t^I = q_{xp}p_t + q_{xz}z_t + q_{xs}s_t^I + q_{x0}, \;\; x_t^D = \bar{x} - x_t^I.$$

By substituting x_t^I and p_t out, the transition of $x_t^D$is obtained by

$$x_t^D = \bar{x} - q_{x0} - q_{xp}A_0 - \left(q_{xp}B + q_{xz}\right)\bar{d} - q_{xp}A_x x_{t-1}^D - \left(q_{xp}B + q_{xz}\right)\left(z_t - \bar{d}\right) - q_{xp}BC's_t - q_{xs}s_t^I.$$

One can define

$$\rho_0 := \bar{x} - q_{x0} - q_{xp}A_0 - (q_{xp}B + q_{xz})\,\bar{d},\ \ \rho_1 := -q_{xp}A_x,\ \ \epsilon_t^D := -\,(q_{xp}B + q_{xz})\left(z_t - \bar{d}\right) - q_{xp}BC's_t - q_{xs}s_t^I.$$

C.6 Proof of Theorem 3.3

C.6.1 Proof of the first claim

We know from Appendix C.1 that the demand takes the following form

$$x_i = \frac{E_i d - p}{\theta V_i d},\ \ i = I, U.$$

The welfare of the informed agents can be written as

$$\begin{aligned}
u_I &= e^{\theta c} E\left[-e^{-(d-p(z,s))\frac{E_I d - p(z,s)}{V_I d}}\right] \\
&= e^{\theta c} E\left[E_I\left[-e^{-(d-p(z,s))\frac{E_I d - p(z,s)}{V_I d}}\right]\right] \\
&= e^{\theta c} E\left[-e^{-(E_I d - p(z,s))\frac{E_I d - p(z,s)}{V_I d} + \frac{V_I d}{2}\left(\frac{E_I d - p(z,s)}{V_I d}\right)^2}\right] \\
&= e^{\theta c} E\left[-e^{-\frac{1}{2}\left(\frac{E_I d - p(z,s)}{\sqrt{V_I d}}\right)^2}\right] \\
&= e^{\theta c} E\left[E_U\left[-e^{-\frac{1}{2}\frac{V_U(E_I d)}{V_I d}\left(\frac{E_I d - p(z,s)}{\sqrt{V_U(E_I d)}}\right)^2}\right]\right] \\
&= e^{\theta c}\frac{1}{\sqrt{\frac{V_U(E_I d)}{V_I d} + 1}} E\left[-\exp\left(-\frac{\frac{1}{2}\frac{V_U(E_I d)}{V_I d}\left(\frac{E_U d - p(z,s)}{\sqrt{V_U(E_I d)}}\right)^2}{\frac{V_U(E_I d)}{V_I d} + 1}\right)\right]
\end{aligned}$$

where we apply the following formula in the last step

$$X \sim N(\mu, \sigma) \Rightarrow Ee^{-tX^2} = \frac{1}{\sqrt{2\sigma^2 t + 1}} \exp\left(-\frac{t\mu^2}{2\sigma^2 t + 1}\right).$$

By the law of total variance and the homoskedasticity of $V_I d$,

$$V_U(E_I d) = V_U d - E_U V_I d = V_U d - V_I d \Leftrightarrow V_U(E_I d) + V_I d = V_U d.$$

Hence, the welfare is

$$u_I = e^{\theta c}\sqrt{\frac{V_I d}{V_U d}} E\left[-e^{-\frac{1}{2}\frac{(E_U d - p(z,s))^2}{V_U d}}\right].$$

Similarly, one can also derive

$$u_U = E\left[-e^{-\frac{1}{2}\frac{(E_U d - p(z,s))^2}{V_U d}}\right].$$

The welfare ratio is then

$$\frac{u_I}{u_U} = e^{\theta c}\sqrt{\frac{V_I d}{V_U d}}.$$

We know from Appendix C.1

$$V_I d = \frac{1}{\kappa_d + \kappa_\epsilon},\ V_U d = \kappa_d^{-1} - \frac{\kappa_d^{-2}}{\kappa_d^{-1} + \kappa_\epsilon^{-1} + C^2 \kappa_s^{-1}}$$

where C is the coefficient of the price $p(z,s) = A + B(z + Cs)$. Hence, for $u_I = u_U$, C has to be constant over θ_D, and therefore the price informativeness does not change after the regulation. We can obtain the closed form solution of C

$$e^{\theta c}\sqrt{\frac{V_I d}{V_U d}} = 1 \Leftrightarrow C = -\sqrt{\kappa_s \frac{\kappa_d^{-1}+\kappa_\epsilon^{-1}}{\left(e^{2\theta c}-1\right)^{-1}\kappa_\epsilon\kappa_d^{-1}-1}}.$$

For this expression to be meaningful, c has to be bounded by

$$\kappa_\epsilon\kappa_d^{-1} > e^{2\theta c}-1 \Rightarrow c < \frac{1}{2\theta}\left(\kappa_\epsilon\kappa_d^{-1}+1\right).$$

C.6.2 Proof of the second claim

We know from Appendix C.1, (β, B, C, λ) is characterized by

$$\begin{aligned}
[\beta]:\ \beta &= \lambda\beta_I + (1-\lambda)\frac{\frac{1}{B}\kappa_d^{-1} - \left(\kappa_d^{-1}+\kappa_\epsilon^{-1}+C^2\kappa_s^{-1}\right)}{\theta\kappa_d^{-1}\left(\kappa_\epsilon^{-1}+C^2\kappa_s^{-1}\right)} \\
[B]:\ B &= \frac{\gamma\left(\kappa_d+\kappa_\epsilon\right)-\beta\left(\theta_D\gamma+\kappa_\epsilon\right)}{\theta_D\beta^2-2\beta\left(\kappa_d+\kappa_\epsilon\right)} \\
[C]:\ C &= \frac{\theta_D\beta}{\gamma\left(\kappa_d+\kappa_\epsilon\right)-\beta\left(\theta_D\gamma+\kappa_\epsilon\right)} \\
[\lambda]:\ C &= -\sqrt{\kappa_s\frac{\kappa_d^{-1}+\kappa_\epsilon^{-1}}{\left(e^{2\theta c}-1\right)^{-1}\kappa_\epsilon\kappa_d^{-1}-1}}
\end{aligned}$$

Let $C\left(\lambda, \theta_D\right)$ denote the equilibrium coefficient as the function of λ and θ_D. We already know from Theorem 3.2 that, for each $\lambda \in (0,1)$, $C<0$ is a decreasing function of θ_D. By the implicit function theorem $\partial\lambda/\partial\theta_D = -\frac{\partial C/\partial\theta_D}{\partial C/\partial\lambda}$, it suffices to show

$$\frac{\partial C}{\partial \lambda} > 0.$$

By differentiating $[C]$ with respect to λ and rearranging the numerator, one can see that this is equivalent to

$$\frac{\partial \beta}{\partial \lambda}\lambda - \beta + \beta^2 \frac{\theta_D}{\kappa_d + \kappa_\epsilon} > 0.$$

To show this is indeed the case for large enough θ_D, it suffices to show $\lim_{\theta_D \to \infty} \frac{\partial \beta}{\partial \lambda}$ is bounded and $\lim_{\theta_D \to \infty} |\beta| > 0$. Suppose the limits are exchangeable. When $\theta_D \to \infty$, the system is characterized by

$$\begin{aligned}
[\beta] &: \ \beta = \lambda \beta_I + (1-\lambda) \frac{\frac{1}{B}\kappa_d^{-1} - \left(\kappa_d^{-1} + \kappa_\epsilon^{-1} + C^2 \kappa_s^{-1}\right)}{\theta \kappa_d^{-1} \left(\kappa_\epsilon^{-1} + C^2 \kappa_s^{-1}\right)} \\
[B] &: \ B = -\frac{\gamma}{\beta} \\
[C] &: \ C = -\frac{1}{\gamma} \\
[\lambda] &: \ C = constant
\end{aligned}$$

The system is linear in β, so can be solved in closed form

$$\beta = -\frac{\kappa_s \kappa_\epsilon^2 \lambda^2 + \kappa_d \kappa_s \kappa_\epsilon \lambda^2 + \kappa_\epsilon \lambda \theta^2 + \kappa_d \theta^2}{\theta^3 + \kappa_\epsilon \kappa_s \lambda \theta}.$$

For the boundedness of β, one can set $\lambda = 1$ in the denominator and $\lambda = 0$ in the numerator to find the strictly negative number that is independent of λ and bounds β from above.

$$\beta \leq -\frac{\kappa_d \theta}{\theta^2 + \kappa_\epsilon \kappa_s} < 0.$$

For the boundedness of $\frac{\partial \beta}{\partial \lambda}$, the same argument gives

$$\begin{aligned}
\left|\frac{\partial \beta}{\partial \lambda}\right| &= \frac{\kappa_\epsilon}{\theta} \frac{\left|\left(\kappa_\epsilon \kappa_s^2 \lambda^2 + 2\kappa_s \lambda \theta^2\right)\left(\kappa_\epsilon + \kappa_d\right) - \kappa_d \kappa_s \theta^2 + \theta^4\right|}{\left(\theta^2 + \kappa_s \kappa_\epsilon \lambda\right)^2} \\
&\leq \kappa_\epsilon \frac{\left(\kappa_\epsilon \kappa_s^2 + 2\kappa_s \theta^2\right)\left(\kappa_\epsilon + \kappa_d\right) + \kappa_d \kappa_s \theta^2 + \theta^4}{\theta^3}.
\end{aligned}$$