First year mathematics undergraduates' settled images of tangent line

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Abstract

This study concerns 182 first year mathematics undergraduates' perspectives on the tangent line of function graph in the light of a previous study on Year 12 pupils' perspectives. The aim was the investigation of tangency images that settle after undergraduates' distancing from the notion for a few months and after their participation in university admission examination. To this end we related the performances of the undergraduates and the pupils in the same questions of a questionnaire; we classified, the undergraduates in distinct groups through Latent Class Analysis; and, we examined this classification according to the *Analytical Local*, *Geometrical Global* and *Intermediate Local* perspectives on tangency we had identified among pupils. The findings suggest that more undergraduates than pupils demonstrated *Intermediate* perspectives on tangency. Also, the undergraduates' settled images were influenced by persistent images about tangency and their prior experience in the context of preparation for and participation in the examination.

Keywords

Analysis, tangent line, concept image, concept definition, generalization, transition to university studies, university entrance examination

1. Introduction

Functions and other concepts in Analysis (e.g., limit, derivative, integral etc.) have an important role not only in mathematical theory but also through their wide field of applications in other disciplines. Introductory courses in Analysis constitute a basic part of the mathematics curriculum of secondary education and Analysis is an essential part in many post-secondary mathematics related studies. Furthermore, in many countries good student performance in these courses is necessary for university entry. But the transition from secondary to tertiary mathematics studies involves many adjustments in content, teaching and learning style, as well as, other personal and interpersonal factors (Wood, 2001). Even students who succeeded in entrance examinations or selection processes “may not be able to reach their academic potential if they are unable to adjust” (ibid, p. 88). In this adjustment we believe that it is of great importance to know what students, who are on their way into post-secondary education, bring with them from previous educational experience.

The notion of tangent line of function graph has an important role in the instruction and applications of other concepts of Analysis (e.g., limiting processes, geometrical interpretation of derivative, linear approximation of a curve etc.). Moreover, the tangent line can be met in several mathematical contexts and educational levels (e.g., Euclidean Geometry, Analytic Geometry, Analysis etc.), can be defined in several ways and can be expressed through several representational systems (e.g., figural, symbolic etc.). Thus, a study on students’ understanding of tangent line in the course of their transition between mathematical contexts can inform us about the role of definitions and representations in students’ mathematical learning in general. Indicatively, previous research has revealed that prior knowledge about the circle tangent influences students' understanding regarding the more general curve tangent (see Biza, Christou, & Zachariades, 2008; Castela, 1995; Downs & Mamona-Downs, 2000; Tall, 1987; Vinner 1982, 1991).

The results presented in this paper are part of a doctoral study (Biza, 2008) that aimed to explore the complexity of the influence that prior educational experience has on students' understanding of tangency and make relevant teaching recommendations. In the first stage of the study we examined Year 12 pupils' understanding of tangent line when they were in the middle of their involvement with the notion of derivative and its applications, and we identified a spectrum of these pupils' perspectives on tangency (Biza et al., 2008). In this paper we report the second stage of the study in which, in the light of above Year 12 pupil results, we investigated first year mathematics undergraduates' perspectives on tangency. These undergraduates were at the very beginning of their university studies and four months earlier they had completed the same program as the pupils and had participated in the National University Admission Examination. The aims of this part of the study were, firstly, to examine what perspectives on tangency the undergraduates bring with them from their secondary education, and after a distancing – of at least four months – from systematic involvement with mathematics; and, secondly, how these perspectives are related to those of the pupils who were right in the middle of their involvement with relevant notions.

In what follows, first, we provide the theoretical background and a brief account of Year 12 pupils' perspectives on tangent line which informed the stage of the study we report here. Then, we describe the methodological approach and the results. Finally, we discuss the results in the light of the fact that the undergraduates participated in the study after their preparation for and participation in the National University Admission Examination and we conclude with some recommendations for the teaching of Analysis at the undergraduate level.

2. Theoretical framework

Tall and Vinner (1981) introduced the term *concept image* in order to describe the total cognitive structure in the individual's mind that is associated with a given concept. They distinguished this structure from the formal *concept definition* used by the mathematical community to specify this concept. Students, as part of their mathematical education, study some concepts in secondary school or at university that they have already been taught at an elementary level and probably only for specific cases. Furthermore, these concept images that have been developed during students' early studies, very often, are not appropriate for applying the concept in a broader context. This transition that, many times, requires a generalization of the concept in terms of mathematical theory is not obvious as students' knowledge acquisition is not a straightforward and accumulative process (Sierpinska, 1994). Harel and Tall (1991) proposed that there are different kinds of generalization which depend on the individual's mental construction. They refer to the *expansive generalization* that extends the student's existing cognitive structure without requiring changes in their current ideas and the *reconstructive generalization* as one which requires reconstruction of the existing cognitive structure. In the reconstructive generalization students have to change radically the old concept images so as to be applicable in a broader context.

In our research we investigate students' understanding on tangent line that is approached in several contexts and different levels of generalization: firstly in Euclidean Geometry with the tangent to a circle; then in Analytic Geometry with the tangent to a conic section; and, finally in Analysis with the tangent to a function graph.

Tall (1987) and Vinner (1982, 1991) observed that early experiences of the tangent to a circle contribute to the creation of an image of the tangent as a line that touches the graph at one point only and does not cross it. Tall (1987) defined *generic tangent* as a line touching the graph at only one point, even where this is inappropriate. In addition, Fischbein (1987) regarded the tangent to a circle as a *paradigmatic model* of the tangent line. These images contain coercive elements inappropriate when more extreme cases are considered, such as when the tangent line "crosses" the curve (e.g. at an inflection point); is vertical to the $x'x$ axis (e.g. at a cusp point); coincides with the curve or part of it (e.g. when the graph is a line); or, when the tangent does not exist (Biza et al. 2008; Castela, 1995; Tall, 1987; Tsamir & Ovodenko 2004; Tsamir, Rasslan, & Dreyfus, 2006; Vinner, 1982, 1991). These images of tangency may be related with the range of examples of tangents that students have met. Niss (1999) noticed that, very often, the set of domains in which a concept has been exemplified and embedded determines students' concept images. Even if students have been presented the formal definition of a concept by their teachers or have read it in their textbooks, their "actual notions and concept images will be shaped and limited by the examples, problems, and tasks on which they are actually set to work" (ibid, p. 16).

The emphasis, described above, in specific examples in each mathematical context could be related not only with the teaching practice but also with the epistemological differences between these domains. For example, one of the main differences that have been highlighted in research between the tangent line in Geometry and Analysis is the point of view under which the notion and its properties are examined. In Analysis, the existence of a tangent line at a point is a property of the curve at this point and the points close to it (e.g., the tangent line is the limiting position of secant lines). Thus the point of view is *local.* However, in Geometry, in the case of the circle, the tangent line can be defined through properties characterizing the line and the entire curve (e.g., the tangent line to a circle is a line that has only one common point with it). Thus the point of view is *global*. This diversity in approaches could be connected with students' perspectives on functions and function graph in general (Maschietto, 2008). For instance, according to a global point of view, a function and its properties are considered within its domain whereas, according to a local point of view, attention focuses on what happens at a specific value and the points close to it. We distinguish the local point of view from the *point-wise* consideration of the function in which only specific values/points are considered (Bell & Janvier, 1981; Even, 1998). The transition from the global to the local point of view and the consideration of both approaches (*global/local game*) are very important for an introduction to Analysis (Maschietto, 2008). This transition may demand not only an expansive but also a reconstructive generalization in terms of Harel and Tall (1991), namely an entirely reconstruction of students' point of view. The difficulties of students to deal with this transition may origin their misinterpretation of tangents (e.g.: Biza et al., 2008; Castela, 1995).

In Biza et al. (2008) where we reported the part of the study that concerned Year 12 pupils, the pupils had a previous experience of tangent line to a circle in Euclidean

Geometry and to other conics in Analytic Geometry and were working on tangents to function graph in the context of an introductory course to Analysis. We used Latent Class Analysis (LCA, Muthen, 2001) to classify pupils in distinct groups according to their responses to a questionnaire regarding the tangent line of function graph. The pupils of each group were characterized by different perspectives on tangent line and its relationship with the curve: the *Analytical Local*; the *Intermediate Local*; and, the *Geometrical Global* perspective. The Analytical Local perspective fits the definition and uses of the tangent line in Analysis (e.g., limiting position of secant lines, slope of the graph, and derivative) and demonstrates a general view on tangency that does not consider geometrical properties. In the Geometrical Global perspective, in contrast, the tangent preserves geometrical properties applied globally on the entire curve. For example, a tangent line, according to this perspective, could be a line that has only one common point with the curve. The influence of the paradigmatic model of the tangent to the circle (Fischbein, 1987) is very strong in this group of pupils. Finally, the Intermediate Local perspective lies between the other two and is characterized by the application of geometrical properties locally at a neighborhood of the tangency point. For example, pupils with this perspective often believe that the tangent line *could have more than one common point with the curve under the condition that there is only one common point in a neighborhood of the tangency point and keeps the curve in the same semi-plane in this neighborhood* (Biza et al., 2008). In the Intermediate Local perspective, as in the Geometrical Global, the influence of the geometrical properties of circle tangent is strong. However the pupils had *expanded* their use in the case of function graph by applying them at a neighborhood of the tangency point. This observation led us to the claim that these pupils had achieved an expansive generalization whereas the pupils with the Analytical Local a reconstructive generalization (Harel and Tall, 1991).

These perspectives of tangency drive pupils towards erroneous or contradictory responses in critical cases where the tangent line has more than one common point or when it coincides with the curve as well as in cases of *inflection* and *edge points*. In this paper we use the terms *inflection* and *edge point* as they are used in the Greek educational context in which the study was conducted. Accordingly to this context, we will call *inflection point* of a function graph a point in which the concavity of the curve changes and there is a tangent line at this point; and, *edge point* a point in which the function is continuous and the derivative from the left and the right exist without being equal.

The variation of the Year 12 pupil responses regarding the above critical cases in combination with their choices in *simple* cases in which the graph was a conic section, and their answers in questions in which they were asked to provide the formula of tangent line in general and in specific cases, were summarized in a model of seven influential factors (Biza et al., 2008, p.67, see also Table 1). These factors were connected with specific questions of the questionnaire (see Figure 1 in the methods section). This model was confirmed through Confirmatory Factor Analysis (CFA, Marcoulides & Schumacker, 1996). As we present in Table 1, these factors and the related questions of the questionnaire were connected with: the number of common points between the tangent line and the curve (F1, F2 and F3); the tangency at an inflection point (F4); the non-existence of the tangent at an edge point (F5); the symbolic manipulation of the formula of tangent line (F6); and, the tangent to conic sections (F7). Pupils with different perspectives on tangent line (Analytical Local, Intermediate Local, Geometrical Global) performed differently with regard these seven factors (see Biza et al., 2008, p. 67). This model and the corresponding factors are the lenses through which we conducted the stage of the study that we present in this paper.

[Insert Table 1 about here]

Similar difficulties with tangency have been identified amongst secondary mathematics in-service or prospective teachers (Biza, Nardi, & Zachariades, 2009; Potari et al., 2007; Tsamir & Ovodenko, 2004; Tsamir et al., 2006). Most of the participants in the above studies had a solid mathematical background (e.g., first degree in mathematics, computer science etc.). For example, in Greece or Cyprus where some of these studies were conducted (Biza et al., 2009; Potari et al., 2007) only mathematics graduates can teach mathematics at the secondary level. However, these teachers' prior mathematical studies do not necessarily imply that the participants have considered the notion of the tangent line to a function graph at the tertiary level. In many cases university lecturers – especially in mathematics departments – consider the tangent line as an already taught pre-Calculus topic at the secondary level and thus they do not revisit it in their lectures. In the light of the above results we aimed to examine what perspectives on tangency settle in students' minds after the end of their secondary studies, after their participation and success in the university entrance examination and after some distancing from systematic involvement with the notion. Namely, we intended to examine what perspectives on tangency first year mathematics undergraduates bring with them from previous educational experience. To this aim we distributed a questionnaire – similar to the one we had used with Year 12 pupils – to mathematics undergraduates at the very beginning of their university studies.

3. Methods

The participants of the study were 182 first year undergraduates from two Greek mathematics departments. The study was conducted in the light of the results of another group that consists of 196 Year 12 pupils from nine Greek secondary schools (Biza et al., 2008).

The pupils were candidates for science, medical or polytechnic studies and had mathematics as a major subject with various levels of performance. The undergraduates, only a few months earlier, had been at the same educational level as the pupils and thus had followed similar with the pupils' courses in mathematics based on the same curriculum content. At this point we have to note that in Greece the secondary mathematics curriculum is the same across the country and the same textbook is distributed by the Ministry of Education to the pupils at each level of education and for each discipline. According to the Greek mathematics syllabus, pupils are taught about the tangent line to the circle in Euclidean Geometry (Year 10); to other conic sections (parabola, hyperbola and ellipse) in Analytical Geometry (Year 11); and, to a function graph (Year 12) in introductory Analysis courses.

We scrutinized the Year 12 textbook of the introductory Analysis course regarding the introduction, uses and representations of tangent line. We found twenty nine graphical representations in examples and applications of the tangent line to a function graph.

The vast majority of them (23) represent a tangent line which has only one common point with the curve. Two of these represent a tangent to a circle and almost all others represent a tangent to a parabola. Of the remaining six, there are four graphical representations in which the tangent line and the curve have more than one common point but they have only one common point in a neighborhood of the tangency point. Nevertheless, in three of these four cases the other common points are not sketched in the figure. Also, there are two graphical representations that illustrate a tangent line at an inflection point. However, even in these cases, there is no reference to the tangent line in the text. The first one concerns the function $f(x) = x^3$ and the fact that $f'(0)=0$ without this point being a local extreme. The second demonstrates the property that a point in which the second derivative does not exit is a potential inflection point. Almost all exercises in this textbook concerning tangent lines require only the use of symbolic manipulations of this concept. Finally, there is only one exercise that refers to the case in which a tangent line has infinite number of common points with the curve in any neighborhood of the tangency point.

Of course, although both groups of students had been taught similar curricular content about tangents we cannot claim that these students had exactly the same educational experience as they came from different schools and they had different teachers.

At the time of the study, the undergraduates had already succeeded in the National University Admission Examination and they had been admitted in mathematics departments. This examination takes place at the end of the academic year (June) and the tests are the same across the country. Pupils are trained for this examination during Year 12 and this preparation focuses mainly on symbolic manipulation tasks. By the time the pupils participated to the research – it the middle of the spring term (April) – they had been introduced to the tangent to a function graph. Also they had been working on that on several applications of derivative with a focus on symbolic manipulations of the formula. The undergraduates had followed a similar process in their preparation for the examination and had not attended any relevant courses at tertiary level as the research took place at the very beginning of their university studies (October).

We chose for our study this group of undergraduates in order to examine what images of tangency settle after a *distancing* from the notion, what perspectives on tangency these undergraduates bring with them in their university studies and how these perspectives are related with what had emerged from our study with the pupils.

The undergraduates – after their participation in the examination and because of the summer break – had not been working in a systematic way with mathematics for at least four months. Also, they had succeeded in the exams, namely their mathematical performance had been assessed and regarded as high according to the official system of examination. The pupils on the other hand were in the course of preparing for the examination and had a variation of performance in mathematics. We, therefore, examine the relationship between the perspectives of the two groups in full awareness that the two groups, are not necessarily of the same level. We believe that this difference does not affect the outcomes of the study as we do not aim to compare the general performance of these two groups of the students but examine what images on tangency the undergraduates bring with them in the light of the results from the pupils.

The questionnaire (Figure 1) that was used in both stages of the study (pupils and undergraduates) included tasks in which the participants had to: describe in their own words the tangent line and its properties; identify or construct the tangent line; provide definitions; and, write and apply the formula in general and specific cases (for more details about the questionnaire design see Biza et al., 2008). For the scope of this paper we will confine ourselves to the investigation of students' correct/incorrect responses in the common part of the two questionnaires (questions 3, 4, 5, 7 and 8). The responses in questions 1, 2 and 6 and students justifications were not used in the statistical analysis but were used towards qualitative explanations of the statistical results. What we regarded as a correct response is noted below each task in Figure 1. Especially, in the graph of question 5 there are three potential tangent lines passing through point *A* that had been coded differently: one that touches the curve only at the point of tangency (q5.1), one that intersects with the curve in its extension after the point of tangency (q5.2) and one that has to intersect with the curve in order to reach the point of tangency (q5.3).

[Insert Figure 1 about here]

Based on undergraduates' responses to the questionnaires, firstly we examined if the model of the seven factors derived from the sample of pupils (Biza et al., 2008) is valid in the sample of the undergraduates through Confirmatory Factor Analysis (CFA, Marcoulides & Schumacker, 1996). Then, we examined similarities and differences in undergraduates' and pupils' performances in the same questions at 5% level of significance and, as in the previous stage of the study (Biza et al., 2008) we used Latent Class Analysis (LCA, Muthen, 2001) to classify undergraduates in distinct groups. Finally, we scrutinized students' responses to the questionnaire to explicate the statistical results in the context of the educational system in which this study was conducted. For the above statistical analyses we used the structural equation modeling software MPLUSv4.21 (Muthen & Muthen, 2007).

4. Results from the statistical analysis: Undergraduates' perspectives on tangent line

In this section we will discuss similarities and differences in undergraduates' and pupils' performances and then we will examine the undergraduates' classification into distinct groups – derived through LCA – in the light of the corresponding classification of pupils. The CFA verified that the model of seven factors that emerged from the statistical analysis of the pupil data was valid in the case of the undergraduate data. We will, therefore, structure the presentation of results in this section in accordance with these factors and the corresponding questions.

4.1. Undergraduates' and pupils' performance in the same questions

In Table 2, we present the percentages of undergraduates' and pupils' correct responses, respectively. In the last column we indicate whether the difference of these performances in the corresponding question is statistically significant at 5% level of significance. At a glance we could say that both groups of participants had difficulties in certain questions and that none of the groups performed better than the other generally.

Some of the questions proved more challenging for the undergraduates, some other for the pupils and for some questions both groups had difficulties.

With regard to factors (F1-F7) we could say that, in relation to factor F1 (q5.1), in which the tangent has no other common point with the function graph, both groups (undergraduates and pupils) performed very well (95% and 94%, respectively). Regarding factor F2 (q3.1, q3.2, q5.2 and q5.3), according to which the tangent line could have more than one common point with the curve but only one in a neighborhood of the tangency point, the undergraduates scored lower than the pupils. Especially in q3.1, q3.2 and q5.2 this difference is statistically significant. In relation to factor F3 (q4.6 and q4.7), according to which the tangent line can have infinite number of common points with the curve in any neighborhood of the tangency point, the performance was low for both groups. Nevertheless, significantly fewer undergraduates than pupils accepted that the tangent line of a line was itself in question q4.7. Regarding factor F4 (q3.4 and q4.4) that concerns tangency at an inflection point, the undergraduates' performance was lower than the pupils (but not significantly). In relation to factors F5 (q3.3, q3.5 and q4.5) and F7 (q4.1, q4.2 and q4.3) that concern *tangency* at an edge point and tangent line to conic sections respectively, the undergraduates performed better than the pupils. Especially for questions q3.5, q4.2 and q4.5 the difference is statistically significant. Finally, regarding factor F6 (q7 and q8) which concerns symbolic manipulation of the tangent line, the results of both groups were almost the same.

4.2. Distinct groups of undergraduates

Through LCA, the undergraduates were classified into three groups: group A with 47 undergraduates, group B with 103 undergraduates and group C with 32 undergraduates.

These three groups are distinctly different, since the probability that an undergraduate belongs to the group where s/he was classified (the diagonal numbers of Table 3) is more than 0.9.

[Insert Table 3 about here]

We present the performance (%) in each question for each group of the undergraduates in Table 4a. For reference reasons in Table 4b we quote the corresponding classification and performances derived from the LCA analysis we had done for the sample of pupils (Biza et al., 2008). In both Tables 4a and 4b the numbers above the bold zigzag line relate to the questions that more than 50% of the students in the corresponding group answered correctly. We consider a group's performance in a question to be *satisfactory* when this question is above the zigzag line.

[Insert Table 4a and 4b about here]

The 47 undergraduates classified in group A performed satisfactorily in the questions of factors F1, F2, F4, F5, F6 and F7. In relation to factor F3, the group performed satisfactorily only in question q4.6 but not in q4.7. We could say that most of these undergraduates demonstrated an *Analytical Local* perspective (Biza et al., 2008) on tangency based on a local approach mainly free from geometrical characteristics.
The 103 undergraduates classified in Group B performed satisfactorily in the questions of factors F1, F2, F5, F6 and F7 but not in question q5.3 (of F2) and in factors F3 and

F4. In the previous stage in our study we supposed that pupils in question q5 probably did not draw the line corresponding to q5.3 not because they did not accept it as a tangent line, but because they did not see it (Biza et al., 2008). We could say that the undergraduates in this group acted under the influence of the circle tangent properties but applied them locally. The dominant concept image of the tangent line in group B is a line that *touches* the graph at the tangency point (in which the graph is smooth), can have more than one common point with the graph, given that there is a neighborhood in which the tangency point is the only common point and cannot split the graph at this point. This is what we called *Intermediate Local* perspective (Biza et al., 2008) as the undergraduates with such a perspective apply locally the geometrical properties of the one common point and the remaining of the graph in the same semi-plane.

The 32 undergraduates classified in group C performed satisfactorily only in the questions of factors F1, F6 and F7. Also, 66% of this group sketched the correct line in question q5.2 (F2), although it had another common point at its extension, probably without realizing the existence of this point. We could say that the undergraduates in this group were strongly influenced by the circle tangent properties – one common point and/or remaining in one semi-plane – and they apply them globally on the entire graph. The latter implied that these undergraduates had what we generally called the *Geometrical Global* perspective (Biza et al., 2008) that characterizes the tangent line in Euclidean Geometry.

4.3. Undergraduates' classification in the light of pupils' classification

The groups derived by the classification of both the undergraduates and the pupils (Tables 4a and 4b, respectively) demonstrated a gradation in students' abilities to complete the questionnaire. Thus, satisfactory performance in any question by more than half of the students in a group is associated with satisfactory performance by more than half of the students in all subsequent groups. Nevertheless, some differences were observed between the group profiles in each sample of participants:

- Group A of undergraduates performed better than the corresponding group of pupils in question q4.6 (the tangent line partly coincides with the graph). However, the group A of undergraduates did not perform satisfactorily in question q4.7 (tangent to a line) unlike the corresponding group of pupils.
- Although both groups B performed satisfactorily in the same questions, the results in questions q3.1, q3.2 and q5.2 (the tangent has another common point) were lower for undergraduates than these of pupils.
- In contrast with group C of pupils, group C of undergraduates performed satisfactorily in the symbolic manipulation questions (q7 and q8).
- Less than half of the undergraduates in group C (47%) accepted a tangent that had another common point with the graph at its extension (q3.1), whereas more than half of the pupils in group C (84%) answered the same question correctly.

The three perspectives that had emerged in the case of pupils (Analytical Local, Intermediate Local and Geometrical Global, Biza et al., 2008) were identified in the case of undergraduates as well. However the pupils and the undergraduates were not distributed in the same way to the three groups as we present in Table 5. Although group A – connected with the most sophisticated Analytical Local perspective on tangent line – is the most populated group in the pupil sample, it decreases to one fourth of participants in the undergraduate sample. In addition, although in both samples of participants group C has the fewest students, in the sample of undergraduates it is even smaller. Finally, many more undergraduates than pupils have been classified in group B than in groups A and C (see Table 5).

[Insert Table 5 about here]

Also, there is higher internal variation in group B of undergraduates than in group B of pupils. For example, the marginal performance of 53% of undergraduates in question q3.2 made us think that there are members in this group who cannot accept the tangent line when it has another common point with the curve, especially when this point is clearly sketched in the figure, as in question q3.2, and not being in the extension of the line, as in question q3.1. Given that these undergraduates performed very well to the questions with an edge point, these undergraduates were not classified in group C. This result made us think that in the group of undergraduates there is greater variation of perspectives within the *Intermediate Local* perspective. Indeed, further qualitative analysis of the undergraduates' responses to the entire questionnaire showed a spectrum of *intermediate* perspectives (Biza, thesis, chapter 6, 2008). For example, according to one of those intermediate perspectives, a *tangent* is a line that *could have more than one common point with the curve under the condition that all of them are tangency points.*

In the next section we will exemplify from undergraduate responses and discuss their use of analytic definition of tangent line and how this is related to their perspectives about tangency in each one of the three groups.

5. Undergraduates’ uses of the analytic definition of the tangent line

Regardless of the sophistication of their perspectives on tangency all groups of undergraduates responded satisfactorily to the request to write the formula of the tangent line in general and in specific cases (questions q7 and q8). Although the undergraduates’ total performance in these questions was similar to this of pupils (Table 2), these questions were not the cause of distinguishing groups A, B and C of the undergraduates as they were for the pupils. From the above results we could say that the knowledge of the formula and its applications does not necessarily imply that an undergraduate has a rich and accurate image of the tangent line, its properties and its relationship with the curve. Moreover, as the following excerpts from undergraduates’ responses exemplify, sometimes they use argumentation based on the concept definition in order to support insufficient concept images they have about tangency.

An undergraduate classified in group A replied in question q1: “Tangent line at point *A* of a function graph C_f is the limiting position of a line passing through the point *A* and *M*, when *M* tends infinitely to *A*” and she sketched the graph in Figure 2.

[Insert Figure 2 about here]

Her responses to the questionnaire were correct in all tasks except those in questions q4.2 and q4.4. In question q4.2 she wrote: “there isn’t a tangent at [point] *A* of the parabola because ε will be vertical”. Similarly, in question q4.4, she sketched the right line and she wrote “ε isn’t tangent at *A* because it is vertical”. In the same spirit, another undergraduate of this group wrote in question q4.2 “[the tangent] does not exist as the derivative is not defined at point *A*". At this point we have to note that pupils at Year 12 do not discuss the case a tangent line that is vertical to the *xx*' axis when the graph's slope (and the derivative) is not defined. These undergraduates had met the tangent line to conics in their Analytic Geometry courses but they had not negotiated this case in examples of function graphs. The curve in question q4.2 is not a function graph and, in cases of questions q4.2 and q4.4, the *x* and *y* axes were not sketched. These questions could have been answered by using previous knowledge from Geometry. However, the above undergraduates were not able to recall this knowledge even though they demonstrated very good knowledge of the analytic concept definitions and rich concept images of tangent line of function graph in all other cases.

Many of the undergraduates classified in group B demonstrated a good knowledge of the concept definition. However, they adapted this definition to their restricted images of tangent line. For example, an undergraduate who had responded to question q1 that "[tangent] is a line that touches the curve at a point in which it has the same derivative" and used this argument in most of her responses, she responded to question q3.4 that "*ε* isn't a tangent because the curve doesn't have derivative at point *A*". Another undergraduate from group B wrote about the tangent line and its properties in question q2: "Point *A* is the common point of the tangent and the figure [graph] and, if the figure has a function $f(x)$, then $f'(x_0)$ (where $A(x_0,y_0)$) will be equal to the slope of the line". In the questionnaire, she accepted the sketched lines in questions q3.1 and q3.2 but she rejected the line in q3.4 and she explained:

"If we consider the figure as two figures with point *A* as a common point, then the line *ε* is the tangent line of both figures. If we consider the figure as one, then [*ε*] is not a tangent to the figure and point *A* is an inflection point".

Some undergraduates demonstrated very good knowledge not only of the definition but also of other properties of the tangent line. Nevertheless, they were not able to integrate this knowledge to the image they had about tangents. For example, an undergraduate wrote in question q1: "The tangent line is a line of the form $y-f(x_0)=f'(x_0)(x-x_0)$ for the point $(x_0,f(x_0))$, where $f'(x_0)=\lim_{x\to x_0}\frac{f(x_0+h)-f(x_0)}{h}$. This line touches the graph/circle at only one point, the tangency point". And later, in question q2 she added: "If the function is concave, the tangent is above the graph, whereas if it is convex, [the tangent] is below [the graph]. The tangent expresses the slope that is expressed by the derivative at this point". Later on, in question q3.4 she rejected the line because: "[the line] cuts the graph at *A* and [the line] doesn't touch it but splits it". We could say that the mere formulation of a definition is not enough to reveal her understanding about the notion. Another undergraduate from group B wrote about the tangent in question q1: "A function that is differentiable at a point $A(x_0,f(x_0))$ has a tangent at this point. The tangent is a line that has one common point with the function graph and its formula is $y-f(x_0) = f'(x_0)(x-x_0)$." This undergraduate demonstrated good knowledge of the formula and the incorrect statement that "the tangent is a line that has one common point with the function graph" could be regarded as an accidental omission of "*at least*" before "common point". Nevertheless, the same undergraduate rejected the line in question q3.1 for which she wrote: "Although if the function is differentiable at *A*, then it has a tangent, the [line] *ε* passing through *A* will have another common point with the function at its extension so it isn't a tangent". With this argument she rejected all the tangents that had another common point. In question q3.4 – in which point *A* is an inflection point – she wrote: "*ε* is a tangent because the function is differentiable at *A* and it [the function] has only one common point with *ε*". Whereas in question q3.5 – in which point *A* is an edge point – she wrote: "*ε* isn't a tangent because the function isn't differentiable at *A* then it doesn't have a tangent at point *A*". This undergraduate recognized correctly from the graph the edge point and claimed that the function is not differentiable at this point (for this reason she was classified in group B and not in group C). Year 12 pupils often encounter examples of functions of modulus (e.g., $f(x)=|x|$) with an edge point – or a *corner point* as it is known in Greek classroom mathematical jargon – in their graphs as characteristic examples of a function that is not non-differentiable at a point. This undergraduate, as many others, had kept these images in her mind and answered correctly the questions in which point *A* was an edge point.

This is not the case for the majority of undergraduates who were classified in group C. For example, an undergraduate from this group wrote in question q2 that: "i) Point *A* is the only one in common between the graph and the tangent, ii) The derivative at point *A* equals the slope of the tangent" and later in questions q3.1 and q3.2 she rejected the lines because "*ε* has more than one common point with the curve". Similarly to the previous example, this undergraduate, under the influence of the image of the graph of the modulus function that is known to the students at this age, rejected the line in question q3.3. She explained that "the derivative of the function is not defined". However, this influence wasn't strong enough in every case. Later, in question q3.5, she accepted the line as "*A* is the only common point between the graph and *ε*".

In all above examples the undergraduates demonstrated adequate knowledge of the formula of tangent line and its application in specific cases and many times they offered sufficient responses regarding the concept definition. However their perspectives about tangent line were inconsistent with the formal concept definition. Below we discuss these responses in the light of the fact that the undergraduates participated in the study soon after their preparation for and participation in the National University Admission Examination.

6. Discussion of the results and conclusions

In this paper we investigated 182 first year mathematics undergraduates' perspectives on tangency through their responses to a questionnaire about tangent line to a curve. We examined these perspectives in the light of the results that emerged from the analysis of 196 Year 12 pupils' responses to the same questions (reported in Biza et al. 2008). Both groups of students had been taught similar curricular content about tangents. Pupils had been taught the tangent line to a function graph and had been working on this notion close to when the research took place. The undergraduates, on the other hand, had succeeded in the National University Admission Examination, they had been admitted in mathematics departments and they had not worked in a systematic way on the topic of tangents for a few months. With this study we aimed to investigate what perspectives about tangency the undergraduates bring with them in the beginning of their university studies and after some distancing from systematic involvement with mathematics.

The results proposed that the undergraduates not only worked on tangents under the influence of geometrical properties – as pupils had done – but also, with regard to some properties, this influence was stronger than in the case of the pupils. Tangent line at an inflection point and tangent line that coincides with the graph were among the most challenging cases for both groups of students. Moreover, many more undergraduates rejected erroneously a tangent that had more than one common point with the graph. On the other hand, fewer undergraduates made the mistake of accepting or sketching a tangent line at an edge point. These results indicate that the tangency images of the undergraduates somewhat lagged behind those of the pupils with regard to considering of the geometrical property of one common point. At the same time they were a step ahead with regard to considering of the smoothness of the curve. Both above observations led to fewer undergraduates – in comparison to pupils – being classified in group A (Analytical Local perspective, closer to the notion of tangent line in the context of Analysis). Also, for similar reasons, fewer undergraduates were classified in group C (Geometrical Global perspective according to which the tangent line is examined through its geometrical properties). As a result, more undergraduates than pupils were classified in group B (intermediate perspectives between the Analytical Local and the Geometrical Global perspectives). Contrary to the pupils, all groups of undergraduates had a satisfactory performance in the questions that asked for the formula of tangent line and its application.

We could say that the undergraduates had not constructed an appropriate image about tangency in their Year 12 studies or, even if they had managed one, that image was not very solid and regressed to inadequate images when the undergraduates did not work with tangents for a few months. That means that the undergraduates in their previous mathematical experiences had not achieved a stable reconstructive generalization

(Harel & Tall, 1991) of their images about tangency. This result did not surprise us as we expected that, in the absent of engagement with the concept, the undergraduates might forget what they had learnt a few months ago. To us a pertinent question was what perspectives on tangency had settled in undergraduates' minds over this period and if these perspectives could be interpreted in the context of their prior educational experiences and, especially, their preparation for the examination.

We note that we consider the role of the preparation for examinations as very important in the construction of students' knowledge and learning approaches. Studies have shown the influence of assessment on the learning of mathematics (e.g. Niss, 1993). Sometimes the design of the exams influences students' skills and attitudes (Bergqvist, 2007). Also students' effort to perform in these exams can make their behavior more conducive to *imitative reasoning*, namely reasoning founded on recalling answers or remembering algorithms (Lithner, 2008), and not *creative reasoning* (Bergqvist, 2007). Later, students with the habit of using imitative reasoning still pass the exams and then continue their tertiary studies in the same limited vein (ibid).

The questions used in the National University Admission Examination usually focus on the knowledge of formulas and their applications and graphs are used only in a few cases. As a result, students' preparation for this examination focuses on the development of the appropriate algorithmic skills as formulae recall and use in specific cases. These skills ask for remembering and citing concept definitions but not necessarily in connection with the related properties of the concept. In addition, usually the emphasis is given only on the analytic aspect of mathematical definitions and it is disconnected from any visualization. Especially for the tangent line, the textbook offers examples that do not challenge students' understanding on crucial cases as the tangency at an inflection point or tangent with more than one common point with the curve. In this context students' attention is on algorithmic manipulations and rather away from other fundamental and conceptual aspects of mathematical notions (Niss, 1999). This approach may create obstacles in the mind of a student who does not readily see the geometrical aspects (e.g., the geometrical interpretation of derivative through the tangent slope), and connect them with the analytical interpretation (Habre & Abboud, 2006).

The undergraduates, who participated in this study, after some distancing from the systematic involvement with the notion, had an adequate knowledge of the formula (and sometimes the definition) of tangent line often in fusion with inappropriate images. These images, in many cases, were related with the examples, representations and applications we found in the textbook and used commonly in school practice. Even undergraduates with rich perspective about tangency, as the Analytical Local perspective, failed in questions that were simple (e.g. a tangent to parabola out of the context of coordinate system) that they had not seen in the context of Analysis. This observation agrees with the *domain specificity* mentioned by Niss (1999) according to which the set of domains in which a mathematical concept has been concretely exemplified and embedded determines the specific nature, content and range of this concept that a student is acquiring.

A teaching practice that is informed by the complexity of students' understandings and does not confine itself in the development of algorithmic skills has to engage students with more representations and richer set of examples in order to facilitate students firstly to reveal their understandings and then, if necessary, to reconstruct their images. A system of assessment that looks for evidence of algorithmic skills is not sufficient in ensuring students' understandings being brought from one level of education to another. As a result, teaching mathematics at university level should not be based on the assumption that the knowledge which has been acquired at secondary level does not need revisiting; nor on the assumption that university studies start from a clean slate and that it suffices to redefine everything afresh. Students come to the university with understandings that may need reconstruction. A teaching approach that ignores these persistent understandings that have been settling in students' minds over a long period of time may in fact keep these understandings alive until the end of the students' university studies. This could be detrimental if these particularly graduates become teachers of mathematics with restricted perspective that can be replicated through their teaching.

Acknowledgements

We would like to thank the three anonymous reviewers and the Research in Mathematics Education Group at the University of East Anglia, particularly Elena Nardi, for their helpful comments on this paper.

References

Bell, A., & Janvier, C. (1981). The interpretation of graphs representing situations. *For the Learning of Mathematics, 2*(1), 34-42.

Bergqvist, E. (2007). Types of reasoning required in university exams in mathematics. *The Journal of Mathematical Behavior, 26*, 348–370.

Biza, I. (2008). *[Students' intuitive perspectives on the tangent line in the context of Analysis].* Unpublished PhD in Mathematics Education, University of Athens, Athens, Greece.

Biza, I., Christou, C., & Zachariades, T. (2008). Student perspectives on the relationship between a curve and its tangent in the transition from Euclidean Geometry to Analysis. *Research in Mathematics Education, 10*(1), 53-70.

Biza, I., Nardi, E., & Zachariades, T. (2009). Teacher beliefs and the didactic contract on visualisation. *For the Learning of Mathematics, 29*(3), 31-36.
Castela, C. (1995). Apprendre avec et contre ses connaissances antérieures: Un example concret, celui de la tangente. *Recherches en Didactiques des mathématiques, 15*(1), 7-47.
Downs, M., & Mamona - Downs, J. (2000). On graphic representation of differentiation of real functions. *Themes in Education, 1*(2), 173–198.
Even, R. (1998). Factors involved in linking representations of functions. *The Journal of Mathematical Behavior, 17*(1), 105-121.
Fischbein, E. (1987). *Intuition in Science and Mathematics: An Educational Approach*. Dordrecht, The Netherlands: Reidel.
Habre, S., & Abboud, M. (2006). Students' conceptual understanding of a function and its derivative in an experimental calculus course. *The Journal of Mathematical Behavior, 25*(1), 57-72.
Harel, G., & Tall, D. O. (1991). The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics, 11*(1), 38-42.
Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics, 67*, 255-276.
Marcoulides, G. A., & Schumacker, R. E. (1996). *Advanced structural equation modeling: Issues and techniques*. Mahwah, NJ: Lawrence Erlbaum Associates.
Maschietto, M. (2008). Graphic Calculators and Micro-Straightness: Analysis of a Didactic Engineering. *International Journal of Computers for Mathematical Learning, 13*, 207-230.
Muthén, B. O. (2001). Latent variable mixture modeling. In G. A. Marcoulides & R. E. Schumacher (Eds.), *New Developments and Techniques in Structural Equation Modeling* (pp. 1-33). Mahwah, NJ: Lawrence Erlbaum Associates.
Muthén, L. K., & Muthén, B. O. (2007). *Mplus User's Guide. Fourth Edition*. Los Angeles, CA: Muthen & Muthen.
Niss, M. (1999). Aspects of the Nature and State of Research in Mathematics Education. *Educational Studies in Mathematics, 40*(1), 1-24.
Niss, M. (Ed.). (1993). *Investigations into assessment in mathematics education - An ICMI Study*. Dordrecht, The Netherlands: Kluwer Academic Publisher.
Potari, D., Zachariades, T., Christou, C., Kyriazis, G., & Pitta-Pantazi, D. (2007). Teachers' mathematical knowledge and pedagogical practices in the teaching of derivative. In D. Pitta & G. Philippou (Eds.), *Proceedings of the 4th Conference on European Research in Mathematics Education* (pp. 1955-1964). Larnaca, Cyprus. Available at: http://ermeweb.free.fr/CERME5b/WG12.pdf.
Sierpinska, A. (1994). *Understanding in mathematics*. London: The Falmer Press.
Tall, D. (1987). Constructing the concept image of a tangent. In J. C. Bergeron, N. Herscovics & C. Kieran (Eds.), *Proceedings of the 11th PME International Conference* (Vol. 3, pp. 69-75). Montréal, Canada.
Tall, D., & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with particular reference to Limits and Continuity. *Educational Studies in Mathematics, 12*, 151–169.
Tsamir, P., & Ovodenko, R. (2004). Prospective teachers' images and definitions: The case of inflection points. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of*

the 28th Conference of the International Group for the Psychology of Mathematics Education (Vol. 4, pp. 337–344). Bergen, Norway.

Tsamir, P., Rasslan, S., & Dreyfus, T. (2006). Prospective teachers' reactions to Right-or-Wrong tasks: The case of derivatives of absolute value functions. *The Journal of Mathematical Behavior, 25*, 240-251.

Vinner, S. (1982). Conflicts between definitions and intuitions: the case of the tangent. . In A. Vermandel (Ed.), *Proceedings of the 6th Conference of the International Group for the Psychology of Mathematics Education* (pp. 24-28). Antewerp, Belgium.

Vinner, S. (1991). The role of definitions in the teaching and learning of Mathematics. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 65-81). Dordrecht, The Netherlands: Kluwer.

Wood, L. (2001). The secondary-tertiary interface. In D. A. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 87-98). Dordrecht, The Netherlands: Kluwer Academic Publishers.

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Table 1: Influential factors on students' thinking about tangent lines that emerged from the analysis of the pupil's data (Biza et al., 2008, p.67).

Influential Factor	Description and related questions
F1	The tangent line could have only one common point with the curve (q5.1)
F2	The tangent line could have only one common point in a neighbourhood of the tangency point (q3.1, q3.2, q5.2, q5.3).
F3	In any neighbourhood of the tangency point the tangent line could have an infinite number of common points with the curve (q4.6, q4.7).
F4	There exists a tangent line at an inflection point (q3.4, q4.4).
F5	There is no tangent line at an edge point (q3.3, q3.5, q4.5).
F6	Symbolic manipulation of the tangent line (q7, q8).
F7	Tangent to conic sections (q4.1, q4.2, q4.3).

Table 2. Percentages of undergraduates' and pupils' correct responses in the questionnaire.

"✓": the difference is statistically significant at 5% level of significance.

Question	Undergraduates %	Pupils %	SS
q3.1	70	94	✓
q3.2	58	76	✓
q3.3	82	75	
q3.4	46	54	
q3.5	85	72	✓
q4.1	95	90	
q4.2	96	87	✓
q4.3	90	87	
q4.4	28	33	
q4.5	88	58	✓
q4.6	31	32	
q4.7	22	32	✓
q5.1	95	94	
q5.2	81	91	✓
q5.3	43	50	
q7	74	75	
q8	69	65	

Table 3: Average group probabilities for the three undergraduate groups

Group of classification \ Group	A	B	C
A	0.939	0.060	0.001
B	0.023	0.935	0.042
C	0.000	0.009	0.991

Table 4: Groups of undergraduates' and pupils' correct responses (%) in each of the questions.

Table 4a: Groups of undergraduates' correct responses (%)			
UNDERGRADUATES			
	Correct responses (%)		
Quest.	Group A (47)	Group B (103)	Group C (32)
q4.2	94	97	94
q5.1	100	93	91
q4.1	98	95	88
q7	83	65	88
q4.3	94	92	75
q5.2	100	78	66
q8	83	65	63
q4.5	96	100	41
q3.5	98	99	22
q3.3	91	94	31
q3.1	94	67	47
q3.2	96	53	16
q3.4	100	26	31
q4.4	85	9	6
q4.6	74	18	9
q5.3	70	39	16
q4.7	40	19	3

Table 4b: Groups of pupils' correct responses (%)			
PUPILS			
	Correct responses (%)		
Quest.	Group A (78)	Group B (60)	Group C (58)
q5.1	94	97	93
q4.1	99	83	86
q3.1	99	97	84
q5.2	95	98	79
q4.2	99	82	78
q4.3	100	80	78
q3.5	88	98	24
q3.3	91	97	31
q4.5	73	85	10
q3.2	95	82	43
q7	96	73	48
q8	87	70	31
q3.4	94	20	36
q5.3	85	47	7
q4.4	64	3	21
q4.7	56	17	16
q4.6	53	12	24

Table 5: Comparison of undergraduates' and pupils' classifications

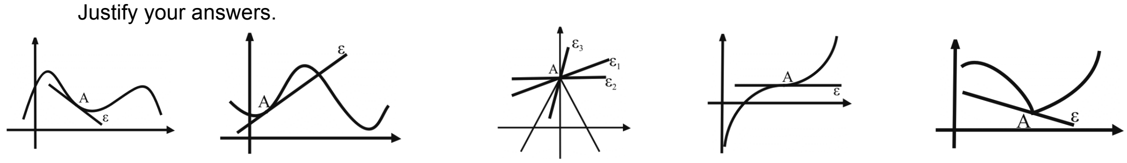
Group	Undergraduates	Pupils
A	25,8%	39,8%
B	56,6%	30,6%
C	17,6%	29,6%

List of Figures

Question 1 (q1): Try to explain, in simple word, what you are thinking when you hear the term “tangent line”.

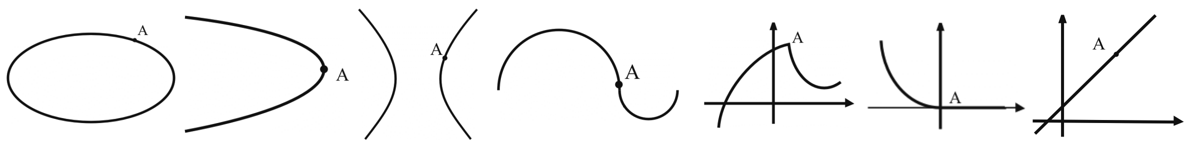
Question 2 (q2): Write as many properties as you can think of about the relationship between a curve and its line at a point *A*.

Question 3 (q3): Which of the lines that are drawn in the following figures are tangent lines of the corresponding graph at point *A*? Justify your answers.



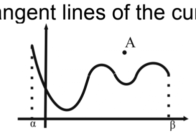
	q3.1	q3.2	q3.3	q3.4	q3.5
Correct answer:	ε is a tangent	ε is a tangent	none of the lines is a tangent	ε is a tangent	ε isn’t a tangent

Question 4 (q4): Sketch the tangent lines of the following curves at point A, if they exist. Justify your answers.

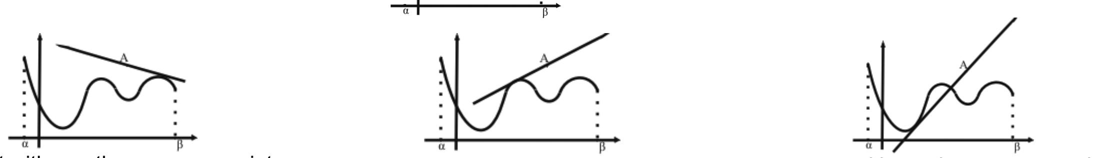


	q4.1	q4.2	q4.3	q4.4	q4.5	q4.6	q4.7
Correct answer:	tangent exists	tangent exists	tangent exists	tangent exists	tangent doesn’t exist	tangent exists	tangent exists

Question 5 (q5): In the following figure, draw as many tangent lines of the curve as you can passing through point *A*



Correct answer:



q5.1	q5.2	q5.3
tangent with no other common point	tangent with another common point at its extension	tangent with another common point

Question 6 (q6): What is the definition of the tangent line of a function graph at its point *A*?

Question 7 (q7): Let *f* be a function and a point $A(x_0, f(x_0))$ of its graph; write the equation of the tangent line of the graph of *f* at the point *A*, if it exists.

Correct answer:

$$y - f(x_0) = f'(x_0)(x - x_0)$$

Question 8 (q8): Calculate the equation of the tangent line of the graph of the function *f* defined by $f(x) = (x-2)^3+3$ at the point $A(2, f(2))$.

Correct answer: Calculation of $f(2) = 3$ and $f'(2) = 0$, application of the formula, correct equation: $y=3$

Figure 1. Common part of the pupils’ and undergraduates’ questionnaires.

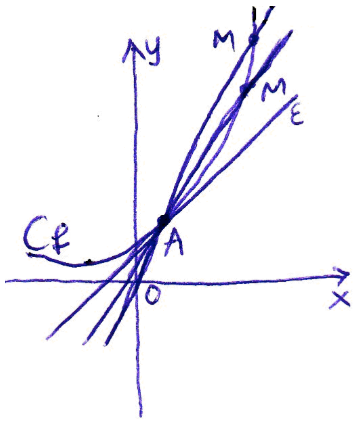


Figure 2. Undergraduate in group A, question 1