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Smooth ℓ-modular representations of unramified p-adic $\mathrm{U}(2,1)(E/F)$

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Abstract. Let E/F be an unramified quadratic extension of p-adic fields and G be the unitary group $\mathrm{U}(2,1)(E/F)$. In this thesis we construct all ℓ-modular irreducible cuspidal representations of G by compact induction from irreducible representations of compact open subgroups of G. Under an assumption on the possible cuspidal subquotients of representations parabolically induced from an irreducible positive level representation, we show that the supercuspidal support of an irreducible ℓ-modular representation of G is unique up to conjugacy.

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Introduction

In this thesis, we begin the study of the irreducible smooth ℓ-modular representations of unitary groups defined over a locally compact non-archimedean local field of residual characteristic different from ℓ. The irreducible ℓ-modular representations of the general linear group were classified in [**Vig96**]. Here, the majority of the general theory of ℓ-modular representations was introduced. This has subsequently been developed; notably in [**Vig98**], [**Dat05**] and [**Dat09**]. Recently, the irreducible ℓ-modular representations of $\mathrm{GL}_m(D)$ have been classified in [**MS11b**] and [**MS11a**]. The theory is less developed than the complex or ℓ-adic theory and the non-semisimplicity of representations of compact open subgroups can lead to striking differences. An example of this is the appearance of subquotients of parabolically induced representations with trivial Jacquet module.

The theory of Langlands has greatly motivated the need to understand the ℓ-adic representations of reductive p-adic groups. An area of study with great potential is to develop an ℓ-modular Langlands theory. In this direction it is shown in [**Vig01a**] that the semisimple local Langlands correspondence for general linear groups is compatible, in some sense, with decomposition modulo-ℓ. This is done by first restricting to supercuspidal representations. There is an ℓ-adic local Langlands correspondence for $\mathrm{U}(2,1)(E/F)$, due to [**Rog90**]. While we make little progress towards an ℓ-modular Langlands correspondence for $\mathrm{U}(2,1)(E/F)$ this does provide motivation for our study.

Eventually, we specialise to unramified unitary groups in three variables $\mathrm{U}(2,1)(E/F)$ defined over a p-adic field of odd residual characteristic and $\ell \neq 2,3$. Our specialisation is in steps, so many of our results apply in much more generality.

Firstly, our construction of all positive level cuspidal representations follows the general theory of [**Ste08**] where the residual characteristic is assumed to be odd. Furthermore, we only show that this construction produces all positive level cuspidal ℓ-modular representations in the case of $\mathrm{U}(2,1)(E/F)$. We classify the irreducible cuspidal level zero ℓ-modular representations of $\mathrm{U}(2,1)(E/F)$. Similarly for level zero representations, while the construction we follow is more general, we only know that we have constructed all level zero cuspidal ℓ-modular representations in the case of $\mathrm{U}(2,1)(E/F)$. However, it should be possible to remove this specialisation to $\mathrm{U}(2,1)(E/F)$ by adapting the general arguments of [**Ste08**] and [**Mor99**]. Our reason for specialising here was because it vastly simplified the arguments involved and because we needed to make this specialisation later. In contrast to irreducible cuspidal ℓ-modular representations of $\mathrm{GL}_n(F)$, we find that there are irreducible cuspidal ℓ-modular representations of $\mathrm{U}(2,1)(E/F)$ which do not lift to ℓ-adic representations. This essentially follows from the analagous result for finite groups.

The next step after describing the cuspidal representations is to describe the decomposition of the parabolically induced representations. Intricately connected to the unitary group we study are the finite unitary groups which appear as quotients of the compact open maximal parahoric subgroups by their pro-p unipotent radicals. This is where our specialisation becomes necessary. For finite general linear groups it is known that the supercuspidal support of an ℓ-modular representation is unique up to conjugacy. However this is not known, in general, for finite unitary groups. In these cases, this is only known for finite unitary groups in two or three variables. Our further specialisation to $\mathrm{U}(2,1)(E/F)$ with E/F an extension of p-adic fields is necessary to apply results of [**Dat05**]. Under these hypotheses we show that the supercuspidal support of an irreducible level zero ℓ-modular representation is unique up to conjugacy. We then make an assumption which we see as an analogue of the level zero result that parabolic induction preserves level zero representations. Under this assumption, we show that the supercuspidal support of an irreducible positive level ℓ-modular representation is unique up to conjugacy.

CHAPTER 1

Representations of p-adic groups

In this chapter we review the definitions of the reductive p-adic groups we study and then review the ℓ-adic and ℓ-modular representation theory of reductive p-adic groups.

1. Notation

Let p, ℓ be distinct prime numbers. Let F be a non-archimedean local field of residual characteristic p. We denote the ring of integers of F by $\mathcal{O}_F$, the multiplicative valuation associated to F by $|\ \cdot\ |_F$, the additive valuation associated to $|\ \cdot\ |_F$ by ν_F, a chosen uniformiser by ϖ_F, the unique maximal ideal (ϖ_F) by $\mathcal{P}_F$, the residue field $\mathcal{O}_F/\mathcal{P}_F$ by k_F, and the cardinality of the residue field by q_F (hence $q_F = p^r$ for some $r \in \mathbb{N}$). We assume $p \neq 2$.

Let $\mathbb{G}$ be a connected reductive group defined over F and $G = \mathbb{G}(F)$ the F-points of $\mathbb{G}$. Let R be a commutative ring with identity, of characteristic zero or ℓ.

An R-representation of G is a pair $(\pi, \mathcal{V})$ where $\mathcal{V}$ is a left R-module and $\pi : G \to \mathrm{GL}(\mathcal{V})$ is a homomorphism of groups. An R-representation $(\pi, \mathcal{V})$ is called smooth if, for all $v \in \mathcal{V}$, the stabilizer of v

$$\mathrm{stab}_G(v) = \{g \in G : \pi(g)v = v\}$$

is an open subgroup of G. Let $\mathfrak{R}_R(G)$ denote the category of smooth R-representations of G; a morphism from $(\pi_1, \mathcal{V}_1) \in \mathfrak{R}_R(G)$ to $(\pi_2, \mathcal{V}_2) \in \mathfrak{R}_R(G)$ is an R-module homomorphism $\Phi : \mathcal{V}_1 \to \mathcal{V}_2$ such that, for all $g \in G$,

$$\pi_2(g) \circ \Phi = \Phi \circ \pi_1(g).$$

We denote the set of all such morphisms $\mathcal{V}_1 \to \mathcal{V}_2$ by $\mathrm{Hom}_G(\mathcal{V}_1, \mathcal{V}_2)$ or by $\mathrm{Hom}_G(\pi_1, \pi_2)$. By [**Vig96**, Chapter 1, §4.2], $\mathfrak{R}_R(G)$ is abelian with direct sums and direct products.

An R-representation $(\pi, \mathcal{V})$ is called irreducible if there are no proper G-stable R-submodules of $\mathcal{V}$. We denote by $\mathrm{Irr}_R(G)$ the set of isomorphism classes of irreducible R-representations of G. The Grothendieck group of finite length R-representations of G is the free abelian group with $\mathbb{Z}$-basis $\mathrm{Irr}_R(G)$, which we denote by $\mathfrak{Gr}_R(G)$. Given a finite length R-representation π of G we denote by $[\pi]$ its semisimplification in $\mathfrak{Gr}_R(G)$.

We are interested in three different cases of coefficient ring R for a representation:

(1) ℓ-adic representations when $R = \overline{\mathbb{Q}}_\ell$;
(2) ℓ-integral representations when $R = \overline{\mathbb{Z}}_\ell$;
(3) ℓ-modular representations when $R = \overline{\mathbb{F}}_\ell$;

and the connections between these. Complex representations of G, when $R = \mathbb{C}$, have been well studied and in general the complex theory and the ℓ-adic theory coincide. Let Λ_ℓ denote the unique maximal ideal of $\overline{\mathbb{Z}}_\ell$.

We say that ℓ is banal for G if it does not divide the pro-order of any maximal compact open subgroup of G. These cases deviate less from the established ℓ-adic theory and our main interest is the non-banal primes for G.

2. Classical groups

In this section, let F be any field and let E/F be a separable quadratic or trivial extension of F. Let $\overline{}$ denote the generator of the cyclic group $\mathrm{Gal}(E/F)$.

If E/F is an extension of non-archimedean local fields we suppose that we have normalised the additive valuation ν_E to have image $\mathbb{Z}$. If E/F is ramified quadratic we choose ϖ_E so that $\overline{\varpi_E} = -\varpi_E$, if E/F is unramified quadratic we choose $\varpi_E \in F$. When E/F is quadratic, by local class field theory, there is a quadratic character $\omega_{E/F}$ of $F^\times$ associated to E/F.

Let $\varepsilon = \pm 1$. An ε-hermitian form h on a finite dimensional E-vector space V is a nondegenerate form

$$h : V \times V \to E$$

which is linear in the first variable, $\overline{}$-linear in the second variable and such that, for all $v_1, v_2 \in V$,

$$h(v_1, v_2) = \varepsilon\overline{h(v_2, v_1)}.$$

A pair (V, h) consisting of a finite dimensional E-vector space and an ε-hermitian form on V is called an ε-hermitian space. An ε-hermitian space (V, h) is called anisotropic if, for all nonzero $v \in V$, $h(v, v) \neq 0$ and is called totally isotropic if $h(v, v) = 0$ for all $v \in V$.

Example 2.1.

(1) Let $a \in E$ such that $a = \varepsilon\overline{a}$ and let $E(a)$ be a one dimensional E-vector with E-basis $\{e_0\}$ equipped with the ε-hermitian form h defined by: if $v, w \in E(a)$ such that $v = v_0 e_0$ and $w = w_0 e_0$ then,

$$h(v, w) = \overline{v_0} a w_0.$$

The form h is anisotropic.

(2) Let H be a two dimensional E-vector space with E-basis $\{e_{-1}, e_1\}$ equipped with the ε-hermitian form h defined by: if $v, w \in H$ such that $v = v_{-1}e_{-1} + v_1 e_1$ and $w = w_{-1}e_{-1} + w_1 e_1$ then,

$$h(v, w) = \overline{v_{-1}} w_1 + \varepsilon \overline{v_1} w_{-1}.$$

The form h is totally isotropic. We call the ε-hermitian space (H, h) the hyperbolic plane.

An ε-hermitian space (V, h) is the orthogonal sum of two subspaces V_1 and V_2 of V if $V = V_1 \oplus V_2$ and $h(v_1, v_2) = 0$, for all $v_1 \in V_1$, $v_2 \in V_2$. By restriction h defines ε-hermitian forms on V_1 and V_2 and we write $V = V_1 \perp V_2$. Given two ε-hermitian spaces (V_1, h_1) and (V_2, h_2) the orthogonal sum of V_1 and V_2 is the direct sum $V_1 \oplus V_2$ equipped with the ε-hermitian form $h_1 \oplus h_2$, defined in the obvious way.

Let (V, h_V) and (W, h_W) be ε-hermitian spaces. A bijective linear map $f : V \to W$ is called an isometry and V and W are called isometric if, for all $v_1, v_2 \in V$,

$$h_V(v_1, v_2) = h_W(f(v_1), f(v_2)).$$

We denote the subgroup of isometries of $\mathrm{GL}(V)$ by $\mathrm{U}(V, h_V)$, i.e.

$$\mathrm{U}(V, h_V) = \{g \in \mathrm{GL}(V) : h_V(gv_1, gv_2) = h_V(v_1, v_2), \text{ for all } v_1, v_2 \in V\}.$$

We denote the subgroup of isometries of $\mathrm{GL}(V)$ of determinant 1 by $\mathrm{SU}(V, h_V)$ and the subgroup of similitudes by $\mathrm{GU}(V, h_V)$, i.e.

$$\mathrm{GU}(V, h_V) = \{g \in \mathrm{GL}(V) : \exists \lambda_g \in F^\times \text{ with } h(gv_1, gv_2) = \lambda_g h(v_1, v_2), \text{ for all } v_1, v_2 \in V\}.$$

We fix an ε-hermitian space (V, h_V). If $X \in \mathrm{End}_E(V)$ then there exists a unique $X^\sigma \in \mathrm{End}_E(V)$ such that, for all $v_1, v_2 \in V$,

$$h_V(Xv_1, v_2) = h_V(v_1, X^\sigma v_2).$$

The map $(\ )^\sigma : X \mapsto X^\sigma$ is an anti-involution of $\mathrm{End}_E(V)$ and

$$\begin{aligned} \mathrm{U}(V, h_v) &= \{g \in \mathrm{GL}(V) : h_V(gv_1, gv_2) = h_V(v_1, v_2), \text{ for all } v_1, v_2 \in V\} \\ &= \{g \in \mathrm{GL}(V) : h_V(v_1, g^\sigma g v_2) = h_V(v_1, v_2), \text{ for all } v_1, v_2 \in V\} \\ &= \{g \in \mathrm{GL}(V) : g^\sigma g = 1\}. \end{aligned}$$

Let $\{e_0, e_1, ..., e_n\}$ be a basis of V and define $J \in \mathrm{GL}_n(E)$ by defining the (i, j)-th entry of J to be $h_V(e_i, e_j)$. Then with respect to this choice of basis, for all $v, w \in E^n$

$$h_V(v, w) = v^T J \overline{w},$$

and if $X \in M_n(E)$

$$h_V(Xv, w) = (Xv)^T J\overline{w} = v^T J(J^{-1}X^T J)\overline{w} = h_V(v, \overline{J^{-1}X^T J}w).$$

Hence $X^\sigma = \overline{J^{-1}X^T J}$.

When $E = F$, and $\varepsilon = 1$ the group of isometries $\mathrm{U}(V, h_V)$ is called the orthogonal group of V. When $E = F$ and $\varepsilon = -1$ the group of isometries $\mathrm{U}(V, h_V)$ is called the symplectic group of V. When E/F is quadratic and $\varepsilon = 1$ the group of isometries $\mathrm{U}(V, h_V)$ is called the unitary group of V.

The group of isometries $\mathrm{U}(V, h_V)$ is the group of F-points of a reductive algebraic group defined over F. We denote the algebraic group by $\mathbb{U}(V, h_V)$. By a classical group, we mean the F-points of the connected component of such an algebraic group $\mathbb{U}(V, h_V)$. Thus the orthogonal group is not a classical group, but the special orthogonal group is a classical group.

2.1. Unitary groups with E/F quadratic. We assume that E/F is quadratic and $\varepsilon = 1$. If E/F is an extension of non-archimedean local fields, we call $\mathrm{U}(V, h_V)$ an unramified unitary group if E/F is unramified and a ramified unitary group if E/F is ramified.

Let mH denote the orthogonal sum of m copies of the hyperbolic plane H.

THEOREM 2.2 ([**MVW87**, Chapter 1, §8]). Let (V, h) be an ε-hermitian space. Then there exists $m \in \mathbb{Z}$ such that V is isometric to $mH \perp V^0$ with V^0 anisotropic.

The number of hyperbolic planes which appear in a decomposition of V of the above form is called the Witt index $w(V)$ of V.

If E/F is an extension of finite fields then an anisotropic space, is zero or, has dimension one and is isometric to $E(a)$ with $a \in F^\times$, see Example 2.1. Furthermore, all the spaces $E(a)$ are isometric. Hence, for all $n \in \mathbb{N}$, there is a single isomorphism class of unitary group of dimension n.

If E/F is an extension of non-archimedean local fields then an anisotropic space has dimension less than or equal to two. In dimension one the anisotropic spaces are of the form $E(a)$ with $a \in F^\times$ and $E(a)$ is isometric to $E(b)$, $b \in F^\times$, if and only if a and b represent the same coset in the quotient $F^\times / N_{E/F}(E^\times)$ which is of order two. There is a single isometry class of two dimensional anisotropic spaces. If n is odd, the unitary groups of the two different isometry classes of hermitian space are isomorphic.

Let $a(V)$ be the anisotropic dimension of V, i.e. the dimension of V^0. We denote the unitary group $U(V, h_V)$ by

$$\mathrm{U}(a(V) + w(V), w(V))(E/F).$$

If E/F is a finite field, as there is a single isomorphism class of unitary group of dimension n, we will also use the notation $\mathrm{U}_n(E/F)$.

EXAMPLE 2.3. Let V be a three dimensional E-vector space, and $\{e_{-1}, e_0, e_1\}$ be the standard basis for V. Define $h_V : V \times V \to E$

$$h_V(v, w) = v_{-1}\overline{w_1} + v_0\overline{w_0} + v_1\overline{w_{-1}},$$

if $v = (v_{-1}, v_0, v_1)$ and $w = (w_{-1}, w_0, w_1)$ with respect to the standard basis. Then J is the matrix with one's on the anti-diagonal and zeroes elsewhere and

$$\mathrm{U}(2, 1)(E/F) = \{g \in \mathrm{GL}_3(E) : J\overline{g}^T Jg = 1\}.$$

2.2. Parabolic subgroups of G. Let (V, h) be an n-dimensional ε-hermitian space. A self dual flag in V is a flag of subspaces of V

$$V = V_{-r} \supsetneq V_{-r+1} \supsetneq \cdots \supsetneq V_{-1} \supsetneq V_0 \supseteq V_0 \supsetneq V_1 \supsetneq \cdots \supsetneq V_r = \{0\}$$

such that, for all $i \in \{0, 1, \ldots, r\}$,

$$V_{-i} = \{v \in V : h_V(v, w) = 0, \text{ for all } w \in V_i\}.$$

The stabilisers of the self dual flags in $\mathrm{U}(V, h_V)$ are the parabolic subgroups of $\mathrm{U}(V, h_V)$. A parabolic subgroup in a reductive group G has a Levi decomposition $P = L \ltimes N$ where N is the unipotent radical of P and L is reductive. We fix a maximal F-split torus T_0 and a minimal parabolic subgroup B in G with Levi decomposition $B = T \ltimes N_0$ such that T contains T_0. A parabolic subgroup of G containing B is called standard. The relative Weyl group W of G is the quotient group $N_G(T_0)/T_0$.

EXAMPLE 2.4. In $\mathrm{U}(2, 1)(E/F)$ there is one conjugacy class of proper self dual flags in V. Choosing the standard basis $\{e_{-1}, e_0, e_1\}$, as in Example 2.3, then a representative is

$$V = \langle e_{-1}, e_0, e_1\rangle \supsetneq \langle e_0, e_1\rangle \supsetneq \langle e_0\rangle \supsetneq \{0\}.$$

This gives rise to the standard Borel subgroup

$$B = \begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & 0 & \star \end{pmatrix} \cap G.$$

Letting

$$T = \left\{\mathrm{diag}(x, y, \overline{x}^{-1}) : x \in E^\times,\ y \in E^1\right\}$$

and

$$N_0 = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & \overline{x} \\ 0 & 0 & 1 \end{pmatrix} : x, y \in E,\ y + \overline{y} = x\overline{x} \right\}$$

then B has Levi decomposition $B = T \ltimes N_0$. The maximal F-split torus T_0 contained in T is

$$T_0 = \left\{\mathrm{diag}(x, 1, x^{-1}) : x \in F^\times\right\}.$$

The relative Weyl group of $\mathrm{U}(2,1)(E/F)$ is cyclic of order 2 and the (absolute) Weyl group is isomorphic to the symmetric group $\mathfrak{S}_3$.

2.3. Parahoric subgroups of G. Let E/F be an extension of non-archimedean local fields and let (V,h) be an n-dimensional ε-hermitian space. The Lie algebra of $\mathrm{U}(V,h)$ is

$$\mathfrak{g} = \{X \in \mathrm{End}_E(V) : X + X^\sigma = 0\},$$

and we can decompose $\mathrm{End}_E(V)$ into a direct sum

$$\mathrm{End}_E(V) = \mathfrak{g} \oplus \mathfrak{g}^+,$$

where $\mathfrak{g}^+ = \{X \in \mathrm{End}_E(V) : X - X^\sigma = 0\}$.

An $\mathcal{O}_E$-lattice in V is a compact open $\mathcal{O}_E$-submodule of V. Equivalently, an $\mathcal{O}_E$-lattice in V is the $\mathcal{O}_E$-span of an E-basis of V. Let L be an $\mathcal{O}_E$-lattice in V and let $\mathrm{Latt}_{\mathcal{O}_E} V = \{\mathcal{O}_E\text{-lattices in } V\}$. The lattice

$$L^\sharp = \{v \in V : h(v,L) \subseteq \mathcal{P}_E\},$$

defined relative to h, is called the dual lattice of L. An $\mathcal{O}_E$-order in a ring A is a subring of A with unit which is also an $\mathcal{O}_E$-lattice in A.

An $\mathcal{O}_E$-lattice sequence is a function $\Lambda : \mathbb{Z} \to \mathrm{Latt}_{\mathcal{O}_E} V$ which is decreasing and periodic, i.e.

(1) for all $n \in \mathbb{Z}$, $\Lambda(n+1) \subseteq \Lambda(n)$;
(2) there exists $e(\Lambda) \in \mathbb{N}$, called the period of Λ, such that, for all $n \in \mathbb{Z}$,

$$(\varpi_E)\Lambda(n) = \Lambda(n + e(\Lambda)).$$

An $\mathcal{O}_E$-lattice sequence is called an $\mathcal{O}_E$-lattice chain if it is strictly decreasing. Let Λ be an $\mathcal{O}_E$-lattice sequence. The dual $\mathcal{O}_E$-lattice sequence $\Lambda^\sharp$ of Λ is the $\mathcal{O}_E$-lattice sequence defined by

$$\Lambda^\sharp(n) = (\Lambda(-n))^\sharp$$

for all $n \in \mathbb{Z}$. We call Λ self dual if there exists $k \in \mathbb{Z}$ such that $\Lambda(n) = \Lambda^\sharp(n+k)$ for all $n \in \mathbb{Z}$. If Λ is self dual then we can always consider a translate Λ_k of Λ, i.e. $k \in \mathbb{Z}$ and Λ_k is defined by $\Lambda_k(n) = \Lambda(n+k)$ for all $n \in \mathbb{Z}$, such that either $\Lambda_k(0) = \Lambda_k^\sharp(0)$ or $\Lambda_k(1) = \Lambda_k^\sharp(0)$.

Let Λ be an $\mathcal{O}_E$-lattice sequence on V. For $n \in \mathbb{Z}$ define

$$\mathfrak{A}_n(\Lambda) = \{x \in \mathrm{End}_E(V) : x\Lambda(m) \subset \Lambda(m+n),\ \text{for all } m \in \mathbb{Z}\},$$

which is an $\mathcal{O}_E$-lattice in $\mathrm{End}_E(V)$. We let $\mathfrak{A}_n(\Lambda)^- = \mathfrak{A}_n(\Lambda) \cap \mathfrak{g}$.

If Λ is self dual then the groups $\mathfrak{A}_n(\Lambda)$ are stable under the anti-involution h induces on $\mathrm{End}_E(V)$. Define compact open subgroups of G by

$$\mathbf{P}(\Lambda) = \mathfrak{A}_0(\Lambda)^\times \cap G;$$

and

$$\mathbf{P}_m(\Lambda) = (1 + \mathfrak{A}_m(\Lambda)) \cap G,\ m \in \mathbb{N}.$$

The pro-unipotent radical of $\mathbf{P}(\Lambda)$ is isomorphic to $\mathbf{P}_1(\Lambda)$. The sequence $(\mathbf{P}_m(\Lambda))_{m\in\mathbb{N}}$ is a fundamental system of neighbourhoods of the identity in G and forms a decreasing filtration of $\mathbf{P}(\Lambda)$ by normal compact open subgroups. The quotient $\mathbf{P}(\Lambda)/\mathbf{P}_1(\Lambda)$ is a reductive group over k_F, but this may not be connected. We denote the connected component of $\mathbf{P}(\Lambda)/\mathbf{P}_1(\Lambda)$

by $\mathbf{M}(\Lambda)$ and denote the inverse image in $\mathbf{P}(\Lambda)$ of $\mathbf{M}(\Lambda)$ by $\mathbf{P}^0(\Lambda)$ and call this a parahoric subgroup of G.

In Appendix C we describe a model for the building $\mathcal{B}(G)$ of G in terms of lattice functions. The building $\mathcal{B}(G)$ of G can be used to study the geometry of the parahoric subgroups which are related to the stabilisers of points in the building. We denote the parahoric subgroup of G corresponding to $z \in \mathcal{B}(G)$ by G_z, its pro-unipotent radical by G_z^1 and the quotient G_z/G_z^1 by M_z. If the lattice sequence Λ corresponds to $z \in \mathcal{B}(G)$ then $\mathbf{P}^0(\Lambda) = G_z$, $\mathbf{P}_1(\Lambda) = G_z^1$ and $\mathbf{M}(\Lambda) = M_z$.

LEMMA 2.5. Let Λ be a self dual $\mathcal{O}_E$-lattice sequence and let $n, r \in \mathbb{Z}$ be such that $r \geqslant n > \frac{r}{2} > 0$. Then the map $x \mapsto 1 + x$ induces an isomorphism

$$\mathfrak{A}_{n,\Lambda}^-/\mathfrak{A}_{r,\Lambda}^- \xrightarrow{\ \sim\ } \mathbf{P}_n(\Lambda)/\mathbf{P}_r(\Lambda).$$

EXAMPLE 2.6. Let E/F be an unramified quadratic extension, V a three dimensional E-vector space and $G = \mathrm{U}(2,1)(E/F)$, as in Example 2.3. The parahoric subgroups of G will appear often in this thesis and we fix our notation here.

We have three self dual lattice chains in V up to conjugacy and three conjugacy classes of parahoric subgroups, two of which are maximal. In all three cases $\mathbf{P}^0(\Lambda) = \mathbf{P}(\Lambda)$. For a lattice chain Λ we let $e(\Lambda)$ denote its period.

(1) Lattice chain Λ_1, $e(\Lambda_1) = 1$

$$\Lambda_1(0) = \mathcal{O}_E \oplus \mathcal{O}_E \oplus \mathcal{O}_E;$$
$$\mathfrak{A}_i(\Lambda_1) = \varpi_E^i \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \end{pmatrix};$$
$$\mathbf{M}(\Lambda_1) \simeq \mathrm{U}(2,1)(k_E/k_F).$$

(2) Lattice chain Λ_2, $e(\Lambda_2) = 2$

$$\Lambda_2(0) = \mathcal{O}_E \oplus \mathcal{O}_E \oplus \mathcal{P}_E;$$
$$\Lambda_2(1) = \mathcal{O}_E \oplus \mathcal{P}_E \oplus \mathcal{P}_E \left(= (\Lambda_2(0))^\sharp\right);$$
$$\mathfrak{A}_i(\Lambda) = \begin{cases} \varpi_E^{\left[\frac{i}{2}\right]} \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{P}_E^{-1} \\ \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \end{pmatrix} & \text{if } i \equiv 0(2); \\ \varpi_E^{\left[\frac{i-1}{2}\right]} \begin{pmatrix} \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \\ \mathcal{P}_E^2 & \mathcal{P}_E & \mathcal{P}_E \end{pmatrix} & \text{if } i \equiv 1(2); \end{cases}$$
$$\mathbf{M}(\Lambda_2) \simeq \mathrm{U}(1,1)(k_E/k_F) \times \mathrm{U}(1)(k_E/k_F).$$

(3) The non-maximal case. Lattice chain Λ_3, $e(\Lambda_3) = 3$

$$\Lambda_3(0) = \mathcal{O}_E \oplus \mathcal{O}_E \oplus \mathcal{O}_E;$$
$$\Lambda_3(1) = \mathcal{O}_E \oplus \mathcal{O}_E \oplus \mathcal{P}_E;$$
$$\Lambda_3(2) = \mathcal{O}_E \oplus \mathcal{P}_E \oplus \mathcal{P}_E;$$

$$\mathfrak{A}_i(\Lambda_3) = \begin{cases} \varpi_E^{[\frac{i}{3}]} \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \end{pmatrix} & \text{if } i \equiv 0(3); \\ \varpi_E^{[\frac{i-1}{3}]} \begin{pmatrix} \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \end{pmatrix} & \text{if } i \equiv 1(3); \\ \varpi_E^{[\frac{i-2}{3}]} \begin{pmatrix} \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \\ \mathcal{P}_E^2 & \mathcal{P}_E & \mathcal{P}_E \end{pmatrix} & \text{if } i \equiv 2(3); \end{cases}$$

$$\mathbf{M}(\Lambda_3) \simeq \left\{\mathrm{diag}(x, y, \overline{x}^{-1}) : x \in k_E^\times,\ y \in k_E^1\right\}$$

a maximal torus in $\mathrm{U}(2,1)(k_E/k_F)$.

We let $x \in \mathcal{B}(G)$ be the point corresponding to Λ_1, $y \in \mathcal{B}(G)$ be the point corresponding to Λ_2 and write $\mathfrak{I} = \mathbf{P}(\Lambda_3)$. Let T^0 be the subgroup of T generated by all of its compact subgroups. The affine Weyl group W_{aff} of $\mathrm{U}(2,1)(E/F)$ is the quotient group $N_G(T)/T^0$. We have a short exact sequence

$$1 \to T/T^0 \to W_{\mathrm{aff}} \to W \to 1$$

hence W_{aff} is generated by the cosets represented by the elements

$$w_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } w_2 = \begin{pmatrix} 0 & 0 & \varpi_E^{-1} \\ 0 & 1 & 0 \\ \varpi_E & 0 & 0 \end{pmatrix}.$$

Furthermore, $G_x = \mathbf{P}(\Lambda_1) = \mathfrak{I} \cup \mathfrak{I} w_1 \mathfrak{I}$ and $G_y = \mathbf{P}(\Lambda_2) = \mathfrak{I} \cup \mathfrak{I} w_2 \mathfrak{I}$.

3. Representation theory of p-adic groups

Let $(\pi, \mathcal{V}) \in \mathfrak{R}_R(G)$. We say that $(\pi, \mathcal{V})$ is finitely generated if there exists a finite subset $\Sigma = \{v_1, v_2, \ldots, v_n\}$ of $\mathcal{V}$ such that $\mathcal{V}$ is generated by Σ as an RG-module, i.e.

$$\mathcal{V} = \sum_{i=1}^{n} RGv_i.$$

If $(\pi, \mathcal{V})$ is of finitely generated then there exists an irreducible quotient of $(\pi, \mathcal{V})$.

Let μ be a fixed left R-Haar measure on G. Let $\mathcal{H}_R(G)$ be the global Hecke algebra of G formed by the R-module of locally constant compactly supported functions $f : G \to R$ with the convolution product defined by, if $f_1, f_2 \in \mathcal{H}_R(G)$ and $h \in G$,

$$f_1 \star f_2(h) = \int_G f_1(g) f_2(g^{-1}h) d\mu(g).$$

Let $(\pi, \mathcal{V}) \in \mathfrak{R}_R(G)$, we can define a left $\mathcal{H}_R(G)$-module structure on $\mathcal{V}$ by, if $f \in \mathcal{H}_R(G)$ and $v \in \mathcal{V}$, $f \cdot v = \pi(f)v$ where

$$\pi(f) = \int_G f(g)\pi(g) d\mu(g).$$

A left $\mathcal{H}_R(G)$-module M is called nondegenerate if $M = \mathcal{H}_R(G)M$. Let $\mathcal{H}_R(G)$-mod denote the category of nondegenerate left $\mathcal{H}_R(G)$-modules. The categories $\mathfrak{R}_R(G)$ and $\mathcal{H}_R(G)$-mod are equivalent, [**Vig96**, Chapter 1, §4.4].

3.1. Smoothness and the smooth dual. Let $\mathfrak{R}_R^{ns}(G)$ denote the category of all, not necessarily smooth, R-representations of G. Define a functor

$$\begin{aligned}(\cdot)^\infty : \mathfrak{R}_R^{ns}(G) &\to \mathfrak{R}_R(G)\\ (\pi, \mathcal{V}) &\mapsto (\pi^\infty, \mathcal{V}^\infty)\\ F : \mathcal{V} \to W &\mapsto F\mid_{\mathcal{V}^\infty} : \mathcal{V}^\infty \to W^\infty\end{aligned}$$

where

$$\mathcal{V}^\infty = \{v \in \mathcal{V} : \mathrm{stab}_G(v) \text{ is an open subgroup of } G\}$$

and π^∞ is the restriction of π to the subspace $\mathcal{V}^\infty$ of V. We say $(\pi^\infty, \mathcal{V}^\infty)$ is the smooth part of $(\pi, \mathcal{V})$. This functor is left exact, but not necessarily right exact [**Vig96**, Chapter 1, §4.3].

Let $(\pi, \mathcal{V})$ be a smooth R-representation of G. We can define an R-representation π^* of G on $\mathcal{V}^* = \mathrm{Hom}_R(V, R)$ by

$$\langle \pi^*(g)v^*, v\rangle = \langle v^*, \pi(g^{-1})v\rangle;$$

where $\langle\ ,\ \rangle$ is the natural pairing on $\mathcal{V}^* \times \mathcal{V}$ given by evaluation. The contragredient representation $(\widetilde{\pi}, \widetilde{\mathcal{V}})$ of (π, V) is the smooth part of the R-representation $(\pi^*, \mathcal{V}^*)$.

THEOREM 3.1 ([**Vig96**, Chapter 1, §4.18]). Suppose R is a field. The functor $\mathfrak{R}_R(G) \to \mathfrak{R}_R(G)$ given by $\pi \mapsto \widetilde{\pi}$ is contravariant and exact.

3.2. Inflation and its adjoints. Let G be a reductive p-adic group and H a closed subgroup of G. If H is normal in G then we have the inflation functor

$$\mathrm{infl}_H : \mathfrak{R}_R(G/H) \to \mathfrak{R}_R(G),$$

given by composing representations with the natural projection $G \to G/H$.

Let $(\pi, \mathcal{V})$ be a smooth R-representation of a connected reductive p-adic group G, and let H be a closed subgroup of G. The H-invariants $\mathcal{V}^H$ of $(\pi, \mathcal{V})$ is the largest subrepresentation on which H acts trivially, i.e.

$$\mathcal{V}^H = \{v \in \mathcal{V} : \pi(h)v = v \text{ for all } h \in H\}.$$

Suppose H is normal in G, let $g \in G$ and $v \in \mathcal{V}^H \backslash \{0\}$. For all $h \in H$,

$$\pi(h)(\pi(g)v) = \pi(hg)v = \pi(gg^{-1}hg)v = \pi(g)(\pi(g^{-1}hg)v) = \pi(g)v$$

since $g^{-1}hg \in H$ by normality. Hence there is an action of G on $\mathcal{V}^H$ and thus an action of G/H on $\mathcal{V}^H$. In this case the H-invariants is the right adjoint of the inflation functor, i.e. for smooth representations π of G, and σ of G/H we have

$$\mathrm{Hom}_G(\mathrm{infl}_H\, \sigma, \pi) \simeq \mathrm{Hom}_{G/H}(\sigma, \pi^H).$$

We can identify $\mathcal{V}^H$ with $\mathrm{Hom}_H(1_H, \pi)$ via

$$\begin{array}{ccc}\mathrm{Hom}_H(1_H, \pi) & \xrightarrow{\sim} & \mathcal{V}^H\\ \varphi & \longmapsto & \varphi(1)\\ \begin{array}{c}\varphi : R \to \mathcal{V}\\ \varphi(r) = rv\end{array} & \longleftarrow & v.\end{array}$$

The H-coinvariants $\mathcal{V}_H$ of $(\pi, \mathcal{V})$ is the largest quotient of $(\pi, \mathcal{V})$ on which H acts trivially. We let $\mathcal{V}(H) = \langle \pi(h)v - v : v \in \mathcal{V}, h \in H \rangle$; then

$$\mathcal{V}_H = \mathcal{V}/\mathcal{V}(H).$$

If H is normal in G, there is an action of G on $\mathcal{V}_H$ by normality and thus an action of G/H on $\mathcal{V}_H$. In this case the H-coinvariants is the left adjoint of the inflation functor, i.e. for smooth representations π of G, and σ of G/H we have

$$\mathrm{Hom}_G(\pi, \mathrm{infl}_H \sigma) \simeq \mathrm{Hom}_{G/H}(\pi_H, \sigma).$$

LEMMA 3.2 ([**Vig96**, Chapter 1, §4.6 and §4.9]). Let K be a compact open subgroup of G with pro-order invertible in R. Then the functors $\mathcal{V} \to \mathcal{V}^K$ and $\mathcal{V} \to \mathcal{V}_K$ are exact and

$$\mathcal{V} \simeq V^K \oplus \mathcal{V}(K).$$

If K is a compact open subgroup with pro-order invertible in R then, by Lemma 3.2, $\mathcal{V}^K \simeq \mathcal{V}_K$; the invariants and coinvariants are isomorphic.

We also use inv_H to denote the H-invariants and coinv_H to denote the H-coinvariants.

3.3. Admissibility and Schur's lemma. Let $(\pi, \mathcal{V})$ be a smooth R-representation of a connected reductive p-adic group G. We call $(\pi, \mathcal{V})$ admissible if, for all open subgroups H of G, the subspace of H-invariants $\mathcal{V}^H$ of $\mathcal{V}$ is of finite dimension.

THEOREM 3.3 ([**Vig96**, Chapter 1 §4.18, Chapter 2, §2.8]). Let $(\pi, \mathcal{V}) \in \mathfrak{R}_R(G)$, π is admissible if and only if $\tilde{\tilde{\pi}} \simeq \pi$. Furthermore, suppose R is algebraically closed and π is irreducible, then π is admissible.

THEOREM 3.4 (Schur's Lemma). Suppose R is an algebraically closed field and let $(\pi, \mathcal{V})$ be an admissible representation of G then

$$\mathrm{Hom}_G(\mathcal{V}, \mathcal{V}) \simeq R.$$

If $\mathrm{Hom}_G(\mathcal{V}, \mathcal{V}) \simeq R$ then the centre Z of G must act as a character via π, i.e. there exists a character $\omega_\pi : Z \to R^\times$ such that, for all $z \in Z$, $\pi(z) = \omega_\pi(z)$. We call the character ω_π, when it exists, the central character of π.

3.4. Restriction and its adjoints. Let H be a closed subgroup of G. Then we have a restriction functor

$$\mathrm{Res}^G_H : \mathfrak{R}_R(G) \to \mathfrak{R}_R(H),$$

given by restriction of representations and morphisms to H. The restriction functor is clearly exact and transitive, i.e. if H_1 and H_2 are a closed subgroups of G such that $H_1 \subset H_2 \subset G$ then we have an isomorphism of functors

$$\mathrm{Res}^G_{H_1} \simeq \mathrm{Res}^{H_2}_{H_1} \circ \mathrm{Res}^G_{H_2}.$$

The restriction functor Res^G_H has a right adjoint, the Induction functor Ind^G_H, [**Vig96**, Chapter 1, §5.7]. There is a useful model for induction in terms of functions on G. Let $(\sigma, W) \in \mathfrak{R}_R(H)$. The induced representation $(\mathrm{Ind}^G_H \sigma, \mathrm{Ind}^G_H(W))$ is the space of all functions $f : G \to W$ which satisfy

(1) there exists a compact open subgroup K of G such that $f(gk) = f(g)$ for all $g \in G$, $k \in K$,
(2) $f(hg) = \sigma(h)f(g)$, for all $h \in H$, $g \in G$,

with the action of G given by the right regular action, i.e. for all $f \in \operatorname{Ind}_H^G W$, $x, g \in G$, $\operatorname{Ind}_H^G(g)f(x) = f(xg)$.

When H is open in G the restriction functor Res_H^G has a left adjoint [**Vig96**, Chapter 1, §5.7], compact induction ind_H^G. In terms of our model for induction, the compactly-induced representation $(\operatorname{ind}_H^G \sigma, \operatorname{ind}_H^G W)$, is the subspace of $\operatorname{Ind}_H^G \sigma$ of all functions with compact support modulo H, again with the right regular action of G. This also allows us to define compact induction when H is not open in G; however it may not be adjoint to restriction. Clearly, when G/H is compact the induction functors coincide.

The modulus character $\delta_P : P \to R^\times$ is defined by

$$\delta_P(g) = [gKg^{-1} : K]$$

where K is any compact open subgroup of P. This is a well defined character, independent of the choice of K, which is trivial on all compact subgroups of P, [**Vig96**, Chapter 2, 2.7].

THEOREM 3.5.

(1) **Exactness,** [**Vig96**, Chapter 1, §5.10]**:** Induction and compact induction are exact functors.
(2) **Transitivity,** [**Vig96**, Chapter 1, §5.3]**:** Induction and compact induction are transitive, i.e. if H_1 and H_2 are a closed subgroups of G such that $H_1 \subset H_2 \subset G$ then we have isomorphisms of functors
$$\operatorname{ind}_{H_2}^G \circ \operatorname{ind}_{H_1}^G \simeq \operatorname{ind}_{H_1}^G, \qquad \operatorname{Ind}_{H_2}^G \circ \operatorname{Ind}_{H_1}^G \simeq \operatorname{Ind}_{H_1}^G.$$
(3) **Restriction-induction formula,** [**Vig96**, Chapter 1, §5.5]**:** Let H and K be closed subgroups of G. Let $\sigma \in \mathfrak{R}_R(H)$. Then we have an isomorphism
$$\operatorname{Res}_K^G \operatorname{Ind}_H^G \sigma \simeq \prod_{H\backslash G/K} \operatorname{Ind}_{K\cap {}^gH}^K \operatorname{Res}_{K\cap {}^gH}^{{}^gH} {}^g\sigma.$$
Furthermore, suppose that the double cosets HgK, $g \in G$, are open in G. Then we have an isomorphism
$$\operatorname{Res}_K^G \operatorname{ind}_H^G \sigma \simeq \bigoplus_{H\backslash G/K} \operatorname{ind}_{K\cap {}^gH}^K \operatorname{Res}_{K\cap {}^gH}^{{}^gH} {}^g\sigma.$$
(4) **Contragredient of compact induction,** [**Vig96**, Chapter 1, §5.11]**:** Let H be a closed subgroup of G and let $\sigma \in \mathfrak{R}_R(H)$ then
$$(\operatorname{ind}_H^G \sigma)^\sim = \operatorname{Ind}_H^G \delta_G^{-1}\delta_H\widetilde{\sigma}.$$

3.5. Tensor product. In this section by $\otimes$ we mean $\otimes_R$. If $(\pi_1, \mathcal{V}_1), (\pi_2, \mathcal{V}_2) \in \mathfrak{R}_R(G)$ we define the internal tensor product of $(\pi_1, \mathcal{V}_1)$ and $(\pi_2, \mathcal{V}_2)$ to be $(\pi_1 \otimes \pi_2, \mathcal{V}_1 \otimes \mathcal{V}_2) \in \mathfrak{R}_R(G)$ where $\pi_1 \otimes \pi_2$ is defined by its action on the elements $v_1 \otimes v_2 \in \mathcal{V}_1 \otimes \mathcal{V}_2$, which generate $\mathcal{V}_1 \otimes \mathcal{V}_2$, by

$$\pi_1 \otimes \pi_2(g)(v_1 \otimes v_2) = \pi_1(g)v_1 \otimes \pi_2(g)v_2,$$

for all $g \in G$.

If $(\pi_1, \mathcal{V}_1) \in \mathfrak{R}_R(G)$ and $(\pi_2, \mathcal{V}_2) \in \mathfrak{R}_R(H)$ we define the external tensor product of $(\pi_1, \mathcal{V}_1)$ and $(\pi_2, \mathcal{V}_2)$ to be $(\pi_1 \otimes \pi_2, \mathcal{V}_1 \otimes \mathcal{V}_2) \in \mathfrak{R}_R(G \times H)$ where $\pi_1 \otimes \pi_2$ is defined by its action on the elements $v_1 \otimes v_2 \in \mathcal{V}_1 \otimes \mathcal{V}_2$ by

$$\pi_1 \otimes \pi_2(g,h)(v_1 \otimes v_2) = \pi_1(g)v_1 \otimes \pi_2(h)v_2,$$

for all $(g,h) \in G \times H$.

3.6. Parabolic induction. Let $P = L \ltimes N$ be a parabolic subgroup of G and $\sigma \in \mathfrak{R}_R(L)$. We define a representation $i_P^G(\sigma)$ of G by inflating σ to P and then inducing to G, this composite of functors is called parabolic induction:

$$i_P^G : \mathfrak{R}_R(L) \xrightarrow{\mathrm{infl}_N} \mathfrak{R}_R(P) \xrightarrow{\mathrm{Ind}_P^G} \mathfrak{R}_R(G).$$

Both inflation and induction have left adjoints and by composition we obtain a left adjoint r_P^G of parabolic induction i_P^G called parabolic restriction or the Jacquet functor. Thus the Jacquet functor is composed of first restricting to P, then taking the N-coinvariants:

$$r_P^G : \mathfrak{R}_R(G) \xrightarrow{\mathrm{Res}_P^G} \mathfrak{R}_R(P) \xrightarrow{\mathrm{coinv}_N} \mathfrak{R}_R(L).$$

Theorem 3.6 ([**Vig96**, Chapter 2, §2.1 and §5.13]). Parabolic induction and the Jacquet functor are exact, transitive, preserve admissibility and take finite length (resp. finitely generated) representations to finite length (resp. finitely generated) representations.

We fix a choice of square root of q in R. It can be useful to normalise parabolic induction by twisting by the character $\delta_P^{\frac{1}{2}}$ and considering the composite:

$$i_P^G\delta_P^{\frac{1}{2}} : \mathfrak{R}_R(L) \xrightarrow{\otimes\delta_P^{\frac{1}{2}}} \mathfrak{R}_R(L) \xrightarrow{\mathrm{infl}_N} \mathfrak{R}_R(P) \xrightarrow{\mathrm{Ind}_P^G} \mathfrak{R}_R(G).$$

which has left adjoint the normalised Jacquet functor $\delta_P^{-\frac{1}{2}} r_P^G$ given by:

$$\delta_P^{-\frac{1}{2}} r_P^G : \mathfrak{R}_R(G) \xrightarrow{\mathrm{Res}_P^G} \mathfrak{R}_R(P) \xrightarrow{\mathrm{coinv}_N} \mathfrak{R}_R(L) \xrightarrow{\otimes\delta_P^{-\frac{1}{2}}} \mathfrak{R}_R(L).$$

3.7. Cuspidal and supercuspidal representations.

Definition 3.7.

(1) An irreducible R-representation is called cuspidal if it is not a subrepresentation of any representation parabolically induced from an irreducible R-representation of the Levi factor of a proper standard parabolic subgroup of G.
(2) An irreducible R-representation is called supercuspidal if it is not a subquotient of any representation parabolically induced from an irreducible R-representation of the Levi factor of a proper standard parabolic subgroup of G.

For ℓ-adic or complex representations, a representation is supercuspidal if and only if it is cuspidal. However, for ℓ-modular representations, the two properties can be different.

Lemma 3.8. Let π be an irreducible R-representation of G. The following are equivalent:

(1) π is cuspidal.
(2) For all proper standard parabolic subgroups P of G the Jacquet module $r_P^G\pi$ is trivial.

(3) π is not a subrepresentation of a representation parabolically induced from any R-representation of the Levi factor of a proper standard parabolic subgroup of G.

PROOF: (2) $\Rightarrow$ (3): Let $P = L \ltimes N$ be a standard parabolic subgroup of G and assume there exists $\sigma \in \mathfrak{R}_R(G)$ such that π is a subrepresentation of $i_P^G\sigma$. Then by reciprocity

$$\mathrm{Hom}_L(r_P^G\pi, \sigma) \simeq \mathrm{Hom}_G(\pi, i_P^G\sigma) \neq \{0\}$$

and we have a nontrivial Jacquet module.

(1) $\Rightarrow$ (2): Suppose there exists a standard parabolic subgroup $P = L \ltimes N$ of G such that $r_P^G\pi$ is not trivial. Let σ be an irreducible quotient of $r_P^G\pi$ which exists because $r_P^G\pi$ is of finite length by Theorem 3.6. By reciprocity

$$\mathrm{Hom}_G(\pi, i_P^G\sigma) \simeq \mathrm{Hom}_L(r_P^G\pi, \sigma) \neq \{0\}$$

and π is a subrepresentation of $i_P^G\sigma$. The implication (3) $\Rightarrow$ (1) is clear. □

EXAMPLE 3.9. Let $G = \mathrm{U}(2,1)(E/F)$. Then all Levi factors of the proper parabolic subgroups of G are conjugate to T. There are two proper parabolic subgroups of G which contain T. The subgroup of upper triangular matrices B and its opposite $\overline{B}$. As $\overline{B} = {}^{w_1}B$, we have

$$i_B^G\chi \simeq i_{\overline{B}}^G \, {}^{w_1}\chi.$$

Hence we can remove the requirement that the parabolic subgroups considered are standard in the definitions of cuspidal and supercuspidal, i.e.

(1) An irreducible R-representation of G is cuspidal if and only if it is not a subrepresentation of any representation parabolically induced from an irreducible R-representation of the Levi factor of a proper parabolic subgroup of G.
(2) An irreducible R-representation of G is called supercuspidal if and only if it is not a subquotient of any representation parabolically induced from an irreducible R-representation of the Levi factor of a proper parabolic subgroup of G.

Let $(\pi, \mathcal{V}) \in \mathfrak{R}_R(G)$. The matrix coefficient of π associated to $v \in \mathcal{V}$ and $\widetilde{v} \in \widetilde{\mathcal{V}}$ is the function

$$\varphi_{v,\widetilde{v}} : G \to R$$
$$\varphi_{v,\widetilde{v}} : g \to \langle \widetilde{v}, \pi(g)v \rangle,$$

Let Z denote the centre of G. We say that π is Z-compact if all matrix coefficients of π are compactly supported modulo Z.

THEOREM 3.10 ([**Vig96**, Chapter 2, §2.7]). Let π be an irreducible R-representation of G then π is cuspidal if and only if π is Z-compact.

LEMMA 3.11. An irreducible R-representation is cuspidal if and only if it is not a quotient of a representation parabolically induced from an irreducible representation of the Levi factor of a proper standard parabolic subgroup of G.

PROOF: An irreducible representation π is cuspidal if and only if its contragredient is cuspidal by Theorems 3.10 and 3.3. If $P = L \ltimes N$ is a proper standard parabolic subgroup of G then $\pi \in \mathfrak{R}_R(L)$ is a quotient of $i_P^G\sigma$ if and only if the contragredient representation $\widetilde{\pi}$ is a subrepresentation of $i_P^G\delta_P\widetilde{\sigma}$, by Theorems 3.5 and 3.1. □

For standard Levi subgroups M_i, $i=1,2$, of G we let W_{M_i}, $i=1,2$, denote the Weyl group of M_i and let $W(M_1,M_2)=\{w\in W_G : {}^wM_1=M_2\}$.

LEMMA 3.12 ([**Vig96**, Chapter 2, §2.19]). Suppose R is an algebraically closed field. Let P_i, $i=1,2$, be standard parabolic subgroups of G with Levi decompositions $P_i=M_i\ltimes N_i$, $i=1,2$. Let σ_1 be an irreducible cuspidal representation of M_1.

(1) If M_1 and M_2 are not conjugate in G then $\delta_{P_2}^{-\frac{1}{2}} r_{P_2}^G\left(i_{P_1}^G\delta_{P_1}^{\frac{1}{2}}(\sigma_1)\right)$ does not have any cuspidal subrepresentations or quotients.
(2) If M_1 and M_2 are conjugate in G then the irreducible subquotients of a composition series of $\delta_{P_2}^{-\frac{1}{2}} r_{P_2}^G\left(i_{P_1}^G\delta_{P_1}^{\frac{1}{2}}(\sigma_1)\right)$ are the conjugates ${}^w\sigma_1$ of σ_1 with $w\in W(M_1,M_2)/W_{M_1}$.

THEOREM 3.13 ([**Vig96**, Chapter 2, §2.4 and §2.20]). Suppose R is an algebraically closed field. Let (π,V) be an irreducible R-representation of G. There exist a standard Levi subgroup M of G and an irreducible cuspidal representation τ of M such that, for the standard parabolic subgroup P of G with Levi decomposition $P=M\ltimes N$, $(\pi,\mathcal{V})$ is a subrepresentation of the parabolically induced representation $i_P^G\tau$. Furthermore the pair (M,τ) is unique up to conjugacy.

PROOF: Put a partial order on the finite set of standard parabolic subgroups $\mathbf{P}(G)$ of G by inclusion. For an irreducible representation π of G, by transitivity of the Jacquet module, there exists an element $P\in\mathbf{P}(G)$ such that $r_P^G(\pi)\neq 0$, but $r_Q^G(\pi)=0$ for all $Q\in\mathbf{P}(G)$ properly contained in P. By transitivity of the Jacquet module, $r_P^G(\pi)$ is a cuspidal representation of the Levi factor of P which is of finite length and so has an irreducible quotient τ. By reciprocity, π is a subrepresentation of $i_P^G\tau$. The pair (M,τ) is unique up to conjugacy by Lemma 3.12. □

Suppose π is an irreducible R-representation of G. Let $\mathrm{cusp}(\pi)$ be the set of pairs (M,τ) such that M is a standard Levi subgroup of G, τ is an irreducible cuspidal R-representation of M and π is a subrepresentation of $i_P^G(\tau)$ where P is the standard parabolic of G with Levi factor M. We call the set $\mathrm{cusp}(\pi)$ the cuspidal support of π. When R is an algebraically closed field, by Theorem 3.13, $\mathrm{cusp}(\pi)$ is nonempty and consists of a single G-conjugacy class: we say that the cuspidal support exists and is unique up to conjugacy.

Let $\mathrm{scusp}(\pi)$ be the set of pairs (M,τ) such that M is a standard Levi subgroup of G, τ is an irreducible supercuspidal R-representation of M and π is a subquotient of $i_P^G(\tau)$ where P is the standard parabolic of G with Levi factor M. We call the set $\mathrm{scusp}(\pi)$ the supercuspidal support of π. The next lemma shows that $\mathrm{scusp}(\pi)$ exists.

LEMMA 3.14 ([**Vig96**, Chapter 2, §2.6]). Let π be an irreducible R-representation of G. Then $\mathrm{scusp}(\pi)$ is non-empty.

PROOF: Either π is supercuspidal and $\mathrm{scusp}(\pi)=\pi$ or the set

$$\Sigma=\left\{(M,\sigma): P=M\ltimes N \text{ standard},\ \sigma\in\mathrm{Irr}_R(M),\ \pi\in\left[i_P^G(\sigma)\right]\right\}$$

is nonempty. Choose an element $(M,\sigma)\in\Sigma$ with M minimal under the partial order of inclusion of standard Levi subgroups of G. Suppose σ is not supercuspidal then there exists an irreducible representation of a standard Levi subgroup M' of M such that $\sigma\in\left[i_{P'}^M\sigma'\right]$ for some standard parabolic subgroup $P'=M'\ltimes N'$ and some $\sigma'\in\mathrm{Irr}_R(M')$. By transitivity of induction,

$$i_P^G(i_{P'}^M(\sigma')) = i_{P'N}^G(\sigma')$$

and, by exactness of induction, π is a subquotient of $i_{P'N}^G\sigma'$. Hence M was not minimal. □

In general it is not known if $\mathrm{scusp}(\pi)$ is always a single G-conjugacy class, i.e. whether the supercuspidal support of an ℓ-modular representation is unique up to conjugacy. It is unique up to conjugacy in the cases of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(D)$, [**Vig96**] and [**MS11b**], and it is conjectured to be unique up to conjugacy in general, [**Vig96**, Chapter 2, §2.6].

Example 3.15. Let $G = \mathrm{GL}_2(F)$. Let T be the diagonal torus in G, and B the upper triangular Borel subgroup containing T. Consider the parabolically induced representation $i_B^G 1$. The space of constant functions with trivial G action form an irreducible subrepresentation, equivalent to $(1_G, R)$ and the quotient representation of $i_B^G 1$ by this is called the Steinberg representation. In [**Vig89**] it is shown that the Steinberg is reducible if and only if $\ell \mid q+1$, and in this case has a unique subrepresentation (σ, W) and a unique quotient (ν, R) where $\nu(g) = (-1)^{\nu_F(\det(g))}$. By Lemma 3.12, $r_P^G(i_B^G 1)$ has length 2. Thus, by exactness of the Jacquet functor, σ is cuspidal non-supercuspidal.

4. Integral representations and decomposition modulo-ℓ

Definition 4.1. A finite length admissible $\overline{\mathbb{Q}}_\ell$-representation $(\pi, \mathcal{V})$ of G is called integral if there exists a free G-stable $\overline{\mathbb{Z}}_\ell$-submodule L of $\mathcal{V}$ which contains a $\overline{\mathbb{Q}}_\ell$-basis of $\mathcal{V}$. The $\overline{\mathbb{Z}}_\ell$-module L is called a lattice, or integral structure, in $\mathcal{V}$.

Lemma 4.2 ([**Vig96**, Chapter 1, §9.3]). A subquotient of an integral $\overline{\mathbb{Q}}_\ell$-representation is integral.

Given a finite length integral ℓ-adic representation $(\pi, \mathcal{V})$, with lattice L, we can define a finite length ℓ-modular representation

$$L/\Lambda_\ell L \simeq L \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell,$$

called the reduction modulo ℓ of L. Note that this depends on the choice of L. However by the Brauer–Nesbitt principle [**Vig04**, Theorem 1], its semisimplification in the Grothendieck group of finite length ℓ-modular representations is independent of the lattice chosen and we define the decomposition modulo-ℓ of $(\pi, \mathcal{V})$ by

$$d_\ell(\pi) = \left[L \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell\right].$$

The decomposition modulo-ℓ map extends by linearity to a group homomorphism between Grothendieck groups.

Theorem 4.3 ([**Vig96**, Chapter 2, §4.12 and §4.13]). Let $\pi \in \mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(G)$ be an irreducible representation Then the following are equivalent:

(1) π is integral;
(2) the cuspidal support of π is integral;
(3) the central character of the cuspidal support of π is integral.

5. Cuspidal representations and decomposition modulo-ℓ

We examine the relationship between cuspidality and decomposition modulo ℓ.

Theorem 5.1 ([**Vig96**, Chapter 1, §9.3]). Let H be a closed subgroup of G. Let $\sigma \in \mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(H)$ be admissible and integral with lattice L. Then $\operatorname{ind}_H^G L$ is a lattice in $\operatorname{ind}_H^G \sigma$. Furthermore, if $\operatorname{Ind}_H^G \sigma$ is admissible then $\operatorname{Ind}_H^G L$ is a lattice in $\operatorname{Ind}_H^G \sigma$.

Corollary 5.2. Let H be a closed subgroup of G and $\sigma \in \mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(H)$ be admissible, integral and of finite length. Suppose $\operatorname{ind}_H^G \sigma$ is of finite length then $d_\ell(\operatorname{ind}_H^G \sigma) = [\operatorname{ind}_H^G d_\ell(\sigma)]$. Suppose $\operatorname{Ind}_H^G \sigma$ is admissible and of finite length then $d_\ell(\operatorname{Ind}_H^G \sigma) = [\operatorname{Ind}_H^G d_\ell(\sigma)]$.

Proof: The following proof was suggested by Alberto Minguez. Let L be a lattice in σ. We have a short exact sequence of $\overline{\mathbb{Z}}_\ell H$-modules

$$0 \to \Lambda_\ell L \to L \to L/\Lambda_\ell L \to 0.$$

Induction, $\operatorname{Ind}_H^G : \mathfrak{R}_{\overline{\mathbb{Z}}_\ell}(H) \to \mathfrak{R}_{\overline{\mathbb{Z}}_\ell}(G)$, is exact thus

$$0 \to \operatorname{Ind}_H^G(\Lambda_\ell L) \to \operatorname{Ind}_H^G(L) \to \operatorname{Ind}_H^G(L/\Lambda_\ell L) \to 0$$

is an exact sequence of $\overline{\mathbb{Z}}_\ell G$-modules. Furthermore, $\operatorname{Ind}_H^G(\Lambda_\ell L) \simeq \Lambda_\ell \operatorname{Ind}_H^G(L)$ hence

$$\operatorname{Ind}_H^G(L/\Lambda_\ell L) \simeq \operatorname{Ind}_H^G(L)/\Lambda_\ell \operatorname{Ind}_H^G(L).$$

This depends on the choice of lattice L. However $\operatorname{Ind}_H^G L$ is a lattice in $\operatorname{Ind}_H^G \sigma$, by Theorem 5.1, hence the semisimplification of $\operatorname{Ind}_H^G(L)/\Lambda_\ell \operatorname{Ind}_H^G(L)$ is independent of L by the Brauer–Nesbitt principle for integral finite length representations of G. Furthermore $L/\Lambda_\ell L$ is canonically an ℓ-modular representation of G and the functor $\operatorname{Ind}_H^G : \mathfrak{R}_{\overline{\mathbb{F}}_\ell}(H) \to \mathfrak{R}_{\overline{\mathbb{F}}_\ell}(G)$ is naturally isomorphic to the functor $\operatorname{Ind}_H^G : \mathfrak{R}^{\Lambda_\ell}_{\overline{\mathbb{Z}}_\ell}(H) \to \mathfrak{R}^{\Lambda_\ell}_{\overline{\mathbb{Z}}_\ell}(G)$ where $\mathfrak{R}^{\Lambda_\ell}_{\overline{\mathbb{Z}}_\ell}(H)$ consists of all representations in $W \in \mathfrak{R}_{\overline{\mathbb{Z}}_\ell}(H)$ which satisfy $\lambda w = 0$ for all $w \in W$, $\lambda \in \Lambda_\ell$. Thus

$$d_\ell\left(\operatorname{Ind}_H^G \sigma\right) \simeq [\operatorname{Ind}_H^G d_\ell(\sigma)].$$

The same argument works for compact induction. □

It is more difficult to show that the Jacquet functor commutes with decomposition modulo-ℓ. The difficulty is in showing that if L is a lattice in an integral ℓ-adic representation π of G then $r_P^G(L)$ is a lattice in $r_P^G(\pi)$. For classical groups this is proved in [**Dat05**].

Lemma 5.3 ([**Dat05**, Proposition 1.4]). Let G be a classical group. Let P be a proper parabolic subgroup of G and π an integral ℓ-adic representation of G. Then $d_\ell(r_P^G(\pi)) = [r_P^G(d_\ell(\pi))]$.

Corollary 5.4. Let G be a classical group. Let $\pi \in \mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(G)$ be an integral irreducible representation such that

$$d_\ell(\pi) = \overline{\pi}_1 \oplus \overline{\pi}_2 \oplus \cdots \oplus \overline{\pi}_n$$

with $\overline{\pi}_i \in \mathfrak{R}_{\overline{\mathbb{F}}_\ell}(G)$ irreducible, $i \in \{1, 2, \ldots, n\}$. Then π is cuspidal if and only if, for all $i \in \{1, 2, \ldots, n\}$, the representation $\overline{\pi}_i$ is cuspidal.

Proof: The ℓ-adic representation π is cuspidal if and only if $r_P^G(\pi) = 0$ for all proper parabolic subgroups P of G. By Lemma 5.3, $d_\ell(r_P^G(\pi)) = [r_P^G(d_\ell(\pi))] = 0$ and the Jacquet functor is exact. □

Remark. Corollary 5.4 does not give any indication on the reducibility of the decomposition modulo-ℓ of an irreducible cuspidal representation. Indeed we shall see that the decomposition modulo-ℓ of an integral irreducible ℓ-adic cuspidal can be reducible.

Supercuspidal ℓ-modular representations are more elusive and difficult to describe via decomposition modulo-ℓ arguments. By Corollary 5.2, if $\overline{\pi}$ is a supercuspidal ℓ-modular representation of G it cannot appear in $d_\ell(i_P^G\sigma)$ for any proper parabolic P of G and any irreducible integral ℓ-adic representation σ of the Levi factor of P.

6. Constructing cuspidals

The following theorem suggests an effective way of constructing cuspidal representations of G.

Theorem 6.1 ([**Car84**, §1]). Let K be a compact modulo centre subgroup of G, let (σ, W) be an irreducible representation of K and let $\pi = \operatorname{ind}_K^G \sigma$. If π is irreducible then it is cuspidal.

Proof: Let $\mathcal{V}$ be the space of π. By admissibility the contragredient $\widetilde{\pi}$ of π is irreducible. Fix $v_0 \in \mathcal{V}$ and $\widetilde{v}_0 \in \widetilde{\mathcal{V}}$. By irreducibility, if $v \in \mathcal{V}$ and $\widetilde{v} \in \widetilde{\mathcal{V}}$ then there exist $g_i, \widetilde{g}_i \in G$ such that $v = \sum_{i=1}^n r_i\pi(g_i)v_0$ and $\widetilde{v} = \sum_{i=1}^n \widetilde{r}_i\widetilde{\pi}(\widetilde{g}_i)\widetilde{v}_0$. Then

$$\operatorname{supp}(\varphi_{v,\widetilde{v}}) \subseteq \bigcup_i \widetilde{g}_i \operatorname{supp}(\varphi_{v_0,\widetilde{v}_0})g_i^{-1}$$

and thus $\varphi_{v,\widetilde{v}}$ has compact support modulo Z if $\varphi_{v_0,\widetilde{v}_0}$ has compact support modulo Z. Futhermore, by the same argument reversing the roles of v and v_0, $\varphi_{v_0,\widetilde{v}_0}$ has compact support modulo Z if $\varphi_{v,\widetilde{v}}$ has compact support modulo Z. Therefore we only need to check that one matrix coefficient of π has compact support modulo Z. Let $w \in W$, $\widetilde{w} \in \widetilde{W}$ and let $f_w \in \pi$ be defined by

$$f_w(g) = \begin{cases} \rho(g)w & \text{if } g \in K \\ 0 & \text{otherwise.} \end{cases}$$

Let $\widetilde{f}_{\widetilde{w}} \in \widetilde{\pi}$ be defined by

$$\widetilde{f}_{\widetilde{w}}(f) = \langle \widetilde{w}, f(1)\rangle$$

for all $f \in \pi$. Then

$$\varphi_{f_w,\widetilde{f}_{\widetilde{w}}}(g) = \begin{cases} \langle \widetilde{w}, \rho(g)w\rangle & \text{if } g \in K \\ 0 & \text{otherwise,} \end{cases}$$

is a matrix coefficient of π which is compactly supported modulo Z. Hence π is cuspidal by Theorem 3.10. □

Our first candidate pairs $(K, (\sigma, W))$ are the parahoric subgroups K of G and the irreducible cuspidal representations σ of K/K^1 which we inflate to K.

CHAPTER 2

ℓ-ADIC REPRESENTATIONS OF FINITE REDUCTIVE GROUPS

In this chapter we classify the ℓ-adic representations of some finite reductive groups which appear as quotients of certain compact open subgroups of the p-adic groups we study. In particular we are interested in the finite reductive groups related to the parahoric subgroups of the unramified unitary group in three variables.

In this chapter and the next, to simplify notation, we let F denote a finite field with q elements and E a quadratic extension of F.

1. Finite groups of Lie type

Let G be a connected reductive linear algebraic group over $\overline{\mathbb{F}}_p$ with a Frobenius morphism $\mathrm{Fr} : G \to G$; see [**Sri79**, Chapter 2]. The subgroup of fixed points G^{Fr} of G under Fr is called a finite group of Lie type.

Example 1.1. Let $G = \mathrm{GL}_n\left(\overline{\mathbb{F}}_p\right)$. We can define two Frobenius morphisms on G:

(1) Let Fr denote the standard Frobenius map defined by $\mathrm{Fr} : (x_{ij}) \to (x_{ij}^q)$, $q = p^r$, then the fixed points of G under Fr form a finite general linear group:
$$G^{\mathrm{Fr}} = \mathrm{GL}_n(F).$$
(2) Let $\widetilde{\mathrm{Fr}}$ denote the twisted Frobenius map defined by $\widetilde{\mathrm{Fr}} : (x_{ij}) \to (x_{ji}^q)^{-1}$, then the fixed points of G under $\widetilde{\mathrm{Fr}}$ form a finite unitary group:
$$G^{\widetilde{\mathrm{Fr}}} = \mathrm{U}_n(E/F).$$

We will denote a finite group of Lie type also by G when it is not easily confused with the underlying algebraic group.

Lemma 1.2 ([**Sri79**, Corollary 2.8]). Let (G, Fr) be a pair consisting of a connected reductive group over $\overline{\mathbb{F}}_p$ and a Frobenius morphism $\mathrm{Fr} : G \to G$. There exists a pair (T_0, B_0), unique up to G^{Fr}-conjugacy, such that:

(1) B_0 is a Fr-stable Borel subgroup;
(2) T_0 is a Fr-stable maximal torus contained in B_0.

We fix a choice of (T_0, B_0). The Fr-stable proper parabolic subgroups $P \supseteq B_0$ are called the standard parabolic subgroups. For each standard parabolic subgroup P we have a standard Levi decomposition
$$P = L \ltimes N,$$
where L is Fr-stable, contains T_0 and is called the standard Levi factor; N is the unipotent radical of P.

Let $W(T_0) = N_G(T_0)/T_0$ be the absolute Weyl group of G. The Frobenius morphism Fr acts on $W(T_0)$ because T_0 is Fr-stable. Two elements $w_1, w_2 \in W(T_0)$ are called Fr-conjugate if
$$w w_1 (\mathrm{Fr}\, w)^{-1} = w_2$$
for some $w \in W(T_0)$.

Lemma 1.3 ([**Sri79**, Corollary 2.8]). The G^{Fr}-conjugacy classes of maximal tori in G^{Fr} are in bijection with Fr-conjugacy classes of $W(T_0)$.

It follows from Lemma 1.3 that one can obtain the maximal tori in G^{Fr} from T_0 by twisting by elements representing the Fr-conjugacy classes of $W(T_0)$.

Remark. For convenience when $G^{\widetilde{\mathrm{Fr}}} = \mathrm{U}_n(E/F)$ we will always consider a conjugate of $G^{\widetilde{\mathrm{Fr}}}$, by a representative of the element in the Weyl group of maximal length. This has the advantage of making the standard parabolic subgroups upper triangular.

2. Structure of finite unitary groups

Let $G = \mathrm{U}_n(E/F)$ denote the finite unitary group in n-variables

$$\mathrm{U}_n(E/F) = \{g \in \mathrm{GL}_n(E) : wgw\overline{g}^t = 1\},$$

where w is the n by n matrix with ones on the antidiagonal and zeroes elsewhere and $\bar{\ }$ denotes the involution induced on g by the Frobenius morphism of $\mathrm{Gal}(E/F)$. We have a natural surjective homomorphism of groups

$$\det : \mathrm{U}_n(E/F) \longrightarrow \mathrm{U}_1(E/F).$$

The normal subgroup of G defined by the kernel of this homomorphism is called the special unitary group and denoted $\mathrm{SU}_n(E/F)$. The order of the special unitary groups in n variables is

$$|\,\mathrm{SU}_n(E/F)| = |\,\mathrm{U}_n(E/F)|/(q+1).$$

The order of the finite unitary group in n variables is

$$|\,\mathrm{U}_n(E/F)| = (q^n + q^{n-1})(q^n - q^{n-2}) \cdots (q^n - (-1)^n).$$

The special unitary group in two variables $\mathrm{SU}_2(E/F)$ is conjugate to $\mathrm{SL}_2(F)$ in $\mathrm{GL}_2(E)$. Assuming q is odd, and choosing an element $\sqrt{\varepsilon} \in E\backslash F$ we have

$$\begin{aligned}\mathrm{SU}_2(E/F) &= \left\{ \begin{pmatrix} a & b\sqrt{\varepsilon} \\ c\sqrt{\varepsilon}^{-1} & d \end{pmatrix} : a,b,c,d \in F,\ ad - bc = 1 \right\} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\varepsilon}^{-1} \end{pmatrix} \mathrm{SL}_2(F) \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\varepsilon} \end{pmatrix}.\end{aligned}$$

The centre $Z_{\mathrm{SU}_{n,q}}$ of $\mathrm{SU}_n(E/F)$ is equal to the centre $Z_{\mathrm{U}_{n,q}}$ of $\mathrm{U}_n(E/F)$ intersected with $\mathrm{SU}_n(E/F)$. And we have

$$Z_{\mathrm{U}_{n,q}} = \left\{\lambda \mathrm{Id} : \lambda \in E,\ \lambda^{q+1} = 1\right\}; \qquad Z_{\mathrm{SU}_{n,q}} = \left\{\lambda \mathrm{Id} : \lambda \in E,\ \lambda^{q+1} = 1, \lambda^n = 1\right\}.$$

Example 2.1.

(1) When $n = 2$ there are two cases:
  (a) If q is even then $Z_{\mathrm{SU}_{2,q}} = \{\mathrm{Id}\}$.
  (b) If q is odd then $Z_{\mathrm{SU}_{2,q}} = \{\pm \mathrm{Id}\}$, which is cyclic of order 2.
(2) When $n = 3$ there are two cases:
  (a) If $q + 1 \not\equiv 0 \bmod 3$ then $Z_{\mathrm{SU}_{3,q}} = \{\mathrm{Id}\}$.
  (b) If $q + 1 \equiv 0 \bmod 3$ then $Z_{\mathrm{SU}_{3,q}} = \left\{\lambda \mathrm{Id} : \lambda^3 = 1\right\}$, which is cyclic of order 3.

We let $d_{n,q}$, or more simply d, denote the order of the cyclic group $Z_{\mathrm{SU}_{n,q}}$. When the centre $Z_{\mathrm{SU}_{n,q}}$ is trivial we have a direct product decomposition:

$$\mathrm{U}_n(E/F) \simeq \mathrm{SU}_n(E/F) \times Z_{U_{n,q}}.$$

3. Parabolic induction

Let G be a reductive group over $\overline{\mathbb{F}}_p$ with a Frobenius morphism $\mathrm{Fr} : G \to G$. Let $P = L \ltimes N$ be a standard parabolic subgroup of G and φ an irreducible representation of L^{Fr}. Let $\mathrm{infl}_N(\varphi)$ be the representation of P^{Fr} obtained by inflating φ to P^{Fr}, defining $\mathrm{infl}_N(\varphi)$ to be trivial on N^{Fr}. Parabolic induction, or Harish-Chandra induction, $i_P^G\varphi$ is the composite of this inflation followed by induction from P^{Fr} to G^{Fr}:

$$i_P^G : \mathfrak{R}_R(L^{\mathrm{Fr}}) \xrightarrow{\mathrm{infl}_N} \mathfrak{R}_R(P^{\mathrm{Fr}}) \xrightarrow{\mathrm{Ind}_{P^{\mathrm{Fr}}}^{G^{\mathrm{Fr}}}} \mathfrak{R}_R(G^{\mathrm{Fr}}).$$

Analogously to the p-adic case, parabolic induction has a natural left adjoint, composed of first restricting to P^{Fr} then taking the N^{Fr}-coinvariants, which we denote by r_P^G and call the Jacquet functor or parabolic restriction:

$$r_P^G : \mathfrak{R}_R(G^{\mathrm{Fr}}) \xrightarrow{\mathrm{Res}_{P^{\mathrm{Fr}}}^{G^{\mathrm{Fr}}}} \mathfrak{R}_R(P^{\mathrm{Fr}}) \xrightarrow{\mathrm{coinv}_N} \mathfrak{R}_R(L^{\mathrm{Fr}}).$$

An irreducible representation ρ of G^{Fr} is called cuspidal if ρ is not a subrepresentation of $i_P^G(\varphi)$, for all irreducible representations φ of the Frobenius fixed points of the Levi factors of all standard proper Fr-stable parabolic subgroups of G. If ρ is not a subquotient of $i_P^G(\varphi)$, for all irreducible representations φ of the Frobenius fixed points of the Levi factors of all standard proper Fr-stable parabolic subgroups of G then ρ is called supercuspidal. Analogously to the p-adic setting the cuspidal support of an irreducible representation exists and is unique up to conjugacy, see Chapter 1 Lemma 3.13. This reduces the classification of irreducible representations of G^{Fr} to completing two steps:

(1) Constructing all cuspidal representations of every Levi factor of G^{Fr}.
(2) The decomposition of $i_P^G(\rho)$ with ρ a cuspidal representation of L^{Fr}.

4. Twisted induction

Let L be a Fr-stable Levi subgroup of G. Then L is contained in a parabolic subgroup of G with Levi decomposition $P = L \ltimes N$: however P is not necessarily Fr-stable. In [**DL76**] a map, called Deligne-Lusztig induction,

$$R_{L\subset P}^G : \mathfrak{Gr}_K(L^{\mathrm{Fr}}) \to \mathfrak{Gr}_K(G^{\mathrm{Fr}})$$

is defined, which behaves as a "twisted" generalisation of parabolic induction seemingly inducing through a parabolic subgroup which is invisible in G^{Fr}. The definition is complicated: see [**Sri79**, Chapter 6] and [**CE04**, Chapter 7]. The functor $R_{L\subset P}^G$ has an adjoint, Deligne-Lusztig restriction, denoted ${}^*R_{L\subset P}^G$. We write $R_{L,P}^G$ for the corresponding map between K-class functions, defined via the $\mathbb{Z}$-bases $\mathrm{Irr}(L^{\mathrm{Fr}})$ and $\mathrm{Irr}(G^{\mathrm{Fr}})$,

$$R_{L,P}^G : \mathrm{CF}(L^{\mathrm{Fr}}, K) \to \mathrm{CF}(G^{\mathrm{Fr}}, K).$$

Similarly, we define

$${}^*R_{L,P}^G : \mathrm{CF}(G^{\mathrm{Fr}}, K) \to \mathrm{CF}(L^{\mathrm{Fr}}, K).$$

When T is a torus, [**Sri79**, Proposition 6.18], $R_{T\subset P}^G$ is independent of the choice of Borel subgroup B containing T. Hence, when T is a torus, we have a well defined map R_T^G : $\mathfrak{Gr}_K(T^{\mathrm{Fr}}) \to \mathfrak{Gr}_K(G^{\mathrm{Fr}})$. Furthermore, for any irreducible representation σ of G^{Fr}, there exist a Fr-stable torus T and an irreducible representation θ of T^{Fr} such that σ is a constituent of $R_T^G\theta$, [**Sri79**, Theorem 6.23]. The constituents of $R_T^G 1$, as T ranges over Fr-stable maximal tori in G, are called the unipotent representations of G^{Fr}.

Let T be a torus, then $T = T_d T_a$ where T_d is the maximal F-split part of T. Let $\sigma(T) = \dim(T_d)$, $\varepsilon_T = (-1)^{\sigma(T)}$ and $\varepsilon_G = (-1)^{\sigma(T_0)}$. For a finite set A we can write $|A| = p^s b$ with $(b,p) = 1$ and we set $|A|_{p'} = b$.

(1) (Dimension Formula, [**Sri79**, Theorem 6.21]): Let θ be an irreducible K-representation of a maximal torus T^{Fr} in G^{Fr}. Then
$$\dim R_T^G\theta = \varepsilon_G\varepsilon_T \frac{|G^{\mathrm{Fr}}|_{p'}}{|T^{\mathrm{Fr}}|}.$$

Let T_1 and T_2 be Fr-stable maximal tori in G. Let $N(T_1,T_2) = \{g \in G : gT_1g^{-1} = T_2\}$ and $W(T_1,T_2) = \{gT_1 : g \in N(T_1,T_2)\}$ then Fr acts on $W(T_1,T_2)$ because T_1 and T_2 are Fr-stable. Furthermore, [**Sri79**, Page 79], $W(T_1,T_2)^{\mathrm{Fr}} \simeq N(T_1,T_2)^{\mathrm{Fr}}/T_1^{\mathrm{Fr}}$.

(2) (Weak orthogonality, [**Sri79**, Proposition 6.14]): Let θ_i, $i = 1,2$, be ℓ-adic characters of T_i^{Fr}; then
$$(R_{T_1}^G(\theta_1), R_{T_2}^G(\theta_2)) = \left|\{w \in W(T_1,T_2)^{\mathrm{Fr}} : {}^w\theta_2 = \theta_1\}\right|$$
where ${}^w\theta_2(x) = \theta_2(gxg^{-1})$ for $w = gT_1$.

Let T be a Fr-stable maximal torus in G. A character χ of T^{Fr} is said to be in general position if the orbit under the Weyl group $W(T)^{\mathrm{Fr}}$ of χ is maximal, i.e. of order of $W(T)^{\mathrm{Fr}}$.

(3) (General position implies irreducible, [**Sri79**, Theorem 6.17]): If θ is in general position then either $R_T^G\theta$ or $-R_T^G\theta$ is an irreducible representation of G^{Fr}.

The representations in general position are the irreducible representations equal to $R_T^G\theta$ or $-R_T^G\theta$ which are induced from characters in general position.

(4) (Transitivity, [**Sri79**, Proposition 8.6]): Suppose T is a Fr-stable maximal torus contained in a Fr-stable Levi subgroup L of G, then $R_{L\subset P}^G R_T^L = R_T^G$.
(5) (Generalisation of parabolic induction, [**Sri79**, Theorem 6.24]): Suppose T is contained in an Fr-stable Levi subgroup L which is contained in an Fr-stable parabolic subgroup P; then
$$R_T^G\theta = i_P^G\left(R_T^L\theta\right).$$
(6) (Compatibility with characters and the determinant, [**CE04**, §8.20].) Let $G = \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, Fr a Frobenius morphism of G, and T a Fr-stable maximal torus in G. Let χ be a character of G^{Fr} that factors through the determinant map; then
$$R_T^G(\theta)\otimes\chi = R_T^G(\theta\otimes(\chi\mid_{T^{\mathrm{Fr}}})).$$

A torus T in G is called minisotropic if it is not contained in any proper Fr-stable parabolic subgroup of G.

(8) (Supercuspidals in general position): If T is minisotropic and θ is in general position then either $R_T^G\theta$ or $-R_T^G\theta$ is an irreducible supercuspidal representation.

By (5) to determine the other supercuspidals of G^{Fr} it remains to decompose $R_T^G\theta$ when T is minisotropic and θ is not in general position. In these cases the irreducible constituents may or may not be supercuspidal.

5. The ℓ-adic representations of $\mathrm{GL}_2(F)$

In this section we realize the irreducible representations of $\mathrm{GL}_2(F)$ as factors of induced representations from maximal tori in $\mathrm{GL}_2(F)$. Similar computations are made for $\mathrm{GL}_2(F)$ and $\mathrm{SL}_2(F)$ in [**DM91**, §15.9].

Let $G^{\mathrm{Fr}} = \mathrm{GL}_2(F)$. There are two maximal tori in G^{Fr} up to G^{Fr}-conjugacy. Let T_0^{Fr} be the diagonal torus in G^{Fr}

$$T_0^{\mathrm{Fr}} = \left\{\mathrm{diag}(x, y) : x, y \in F^\times\right\}.$$

The torus T_0^{Fr} represents the G^{Fr}-conjugacy class of maximal Fr-split tori in G^{Fr}. Choose a representative T_1^{Fr} of the G^{Fr}-conjugacy class of non Fr-split maximal tori in G^{Fr}. The torus T_1^{Fr} is conjugate in $\mathrm{GL}_2(E)$ to

$$\left\{\mathrm{diag}(x, x^q) : x \in E^\times\right\}.$$

5.1. Parabolic induction from T_0^{Fr}. Let θ be a character of T_0^{Fr}. Define characters χ_i, $i = 1, 2$ of $F^\times$ by

$$\chi_1(x) = \theta(\mathrm{diag}(x, 1)), \quad \chi_2(x) = \theta(\mathrm{diag}(1, x)).$$

We identify θ with $\chi_1 \otimes \chi_2$. The character θ is in general position if and only if $\chi_1 \neq \chi_2$. In this case the induced representation $R_{T_0}^G \theta = i_{B_0}^G \theta$ is irreducible. By the dimension formula

$$\dim(R_{T_0}^G \theta) = (q + 1).$$

If θ is not in general position, by weak orthogonality,

$$(R_{T_0}^G \theta, R_{T_0}^G \theta) = 2.$$

Suppose $\chi = \chi_1 = \chi_2$ then θ extends to the character $\chi \circ \det$ of G^{Fr}. Hence $R_{T_0}^G \theta = (i_{B_0}^G(1))(\chi \circ \det)$ which is of dimension $q+1$ with a one dimensional subrepresentation $1_G(\chi) = 1_G(\chi \circ \det)$ and a q-dimensional quotient $\mathrm{St}_G(\chi) = \mathrm{St}_G(\chi \circ \det)$.

5.2. Deligne-Lusztig induction from T_1^{Fr}. Let $g \in \mathrm{GL}_2(E)$ such that

$$(T_1^{\mathrm{Fr}})^g = \left\{\mathrm{diag}(x, x^q) : x \in E^\times\right\}.$$

Define a character $\widetilde{\chi}$ of $E^\times$ by

$$\widetilde{\chi}(x) = \theta^g(\mathrm{diag}(x, x^q)).$$

We identify the character θ with $\widetilde{\chi}$. By the dimension formula

$$\dim(R_{T_1}^G \theta) = -(q - 1).$$

The character θ is in general position if and only if $\theta^{q-1} \neq 1$. In this case, $-R_{T_1}^G \theta$ is an irreducible supercuspidal representation of G^{Fr}. We let

$$\sigma_{T_1,\theta} = -R_{T_1}^G \theta.$$

By counting we find that we have found all irreducible representations of G. Thus the representations $R_{T_1}^G \theta$, where θ is a character not in general position of T_1^{Fr}, must already occur in our list. By weak orthogonality and the dimension formula for Deligne-Lusztig representations we have

$$R_{T_1}^G 1 = 1_G - \mathrm{St}_G.$$

The character θ of T_1^{Fr} is not in general position if and only if $\widetilde{\chi}$ factors through the norm map $\xi_{q+1} : x \mapsto x^{q+1}$ and identifies with a character χ of $F^\times$

$$\widetilde{\chi} = \chi \circ \xi_{q+1}.$$

Then

$$\begin{aligned} R_{T_1}^G \theta &= R_{T_1}^G 1(\chi \circ \det) \\ &= 1_G(\chi) - \mathrm{St}_G(\chi). \end{aligned}$$

Deligne-Lusztig Representation	Decomposition in $\mathfrak{Gr}_{\overline{\mathbb{Q}}_\ell}(G^{\mathrm{Fr}})$	Parameters	Degree	Number
$R_{T_0}^G\theta$	$1_G(\chi)+\mathrm{St}_G(\chi)$	χ a character of $F^\times$ $\theta=\chi\otimes\chi$	$1, q$	$2(q-1)$
	$i_{B_0}^G(\theta)$	χ_1,χ_2 characters of $F^\times$ $\chi_1\neq\chi_2$ $\theta=\chi_1\otimes\chi_2$ $i_{B_0}^G(\theta)=i_{B_0}^G(\chi_2\otimes\chi_1)$	$q+1$	$\frac{q^2-3q}{2}$
$R_{T_1}^G\theta$	$1_G(\chi)-\mathrm{St}_G(\chi)$	χ a character of $F^\times$ $\theta=\chi\circ\xi_{q+1}$	$1, q$	$2(q-1)$
	$-\sigma_{T_1,\theta}$	θ a character of $E^\times$ $\theta^{q-1}\neq 1$ $R_{T_1}^G\theta=R_{T_1}^G\theta^q$	$q-1$	$\frac{q^2-q-2}{2}$

6. The ℓ-adic representations of $\mathrm{SL}_2(F)$

Let $G^{\mathrm{Fr}}=\mathrm{SL}_2(F)$. In [**Bon11**] there is a complete description of the complex and modular representations of G^{Fr}. There are two maximal tori in G^{Fr} up to conjugation by G^{Fr}. We can choose a representative of the G^{Fr}-conjugacy class of maximal Fr-split tori in G^{Fr} to be the diagonal torus $T_0^{\mathrm{Fr}}=\left\{\mathrm{diag}(x,x^{-1}) : x\in F^\times\right\}$.

We choose a representative T_1^{Fr} of the maximal nonsplit tori in G^{Fr} and note that T_1^{Fr} is isomorphic to E^1, being conjugate in $\mathrm{GL}_2(E)$ to $\left\{\mathrm{diag}(x,x^q) : x\in E^1\right\}$. A character θ of T_0^{Fr} identifies with a character χ of $F^\times$ and is in general position if $\theta^2\neq 1$. A character θ of T_1^{Fr} identifies with a character χ of E^1 and is in general position if $\theta^2\neq 1$. The following table was extracted from [**Bon11**, Chapter 5].

Deligne-Lusztig Representation	Decomposition in $\mathfrak{Gr}_{\overline{\mathbb{Q}}_\ell}(G^{\mathrm{Fr}})$	Parameters	Degree	Number
$R^G_{T_0}\theta$	$1_G + \mathrm{St}_G$	θ a character of $F^\times$ $\theta = 1$	$1, q$	2
	$i^+\theta + i^-\theta$	θ a character of $F^\times$ $\theta \neq 1$ $\theta^2 = 1$	$\frac{q+1}{2}, \frac{q+1}{2}$	2
	$i^G_{B_0}(\theta)$	θ a character of $F^\times$ $\theta^2 \neq 1$ $i^G_{B_0}(\theta) = i^G_{B_0}(\theta^{-1})$	$q+1$	$\frac{q-3}{2}$
$R^G_{T_1}\theta$	$1_G - \mathrm{St}_G$	θ a character of E^1 $\theta = 1$	$1, q$	2
	$-R^+\theta - R^-\theta$	θ a character of E^1 $\theta \neq 1$ $\theta^2 = 1$	$\frac{q-1}{2}, \frac{q-1}{2}$	2
	$-\sigma_{T_1,\theta}$	θ a character of E^1 $\theta^2 \neq 1$ $R^G_{T_1}\theta = R^G_{T_1}\theta^{-1}$	$q-1$	$\frac{q-1}{2}$

7. The ℓ-adic representations of $\mathrm{U}_2(E/F)$ and $\mathrm{SU}_2(E/F)$

Let $G^{\mathrm{Fr}} = \mathrm{U}_2(E/F)$. Let T_0^{Fr} be the maximal diagonal torus in G^{Fr}

$$T_0^{\mathrm{Fr}} = \left\{ \mathrm{diag}(x, x^{-q}) : x^{q^2-1} = 1 \right\}.$$

We choose a representative T_1^{Fr} of the other G^{Fr}-conjugacy class of maximal tori in G^{Fr}. Then T_1^{Fr} is conjugate in $\mathrm{GL}_2(E)$ to

$$\left\{ \mathrm{diag}(x, y) : x^{q+1} = y^{q+1} = 1 \right\}.$$

7.1. Parabolic induction from T_0^{Fr}. Let θ be a character of T_0^{Fr}. Define a character χ_1 of $E^\times$ by

$$\chi_1(x) = \theta(\mathrm{diag}(x, x^{-q})).$$

We identify θ with χ_1. The character θ is in general position if $\chi_1^{q+1} \neq 1$. In this case the induced representation $i^G_{B_0}\theta$ is irreducible. By the dimension formula

$$\dim(i^G_{B_0}\theta) = q + 1.$$

If $\chi_1^{q+1} = 1$ then χ_1 factors through the map $\xi_{q-1} : x \mapsto x^{q-1}$, and corresponds to a character χ of E^1

$$\chi_1 = \chi \circ \xi_{q-1}.$$

Then θ extends to the character $\chi \circ \det$ of G^{Fr}, hence

$$i^G_{B_0}(\theta) = i^G_{B_0}(1)(\chi \circ \det).$$

By Frobenius reciprocity $i^G_{B_0}(1)$ contains 1_G and the irreducible quotient denoted by St_G of $i^G_{B_0}(1)$ by 1_G is q-dimensional. Thus

$$i^G_{B_0}(\theta) = 1_G(\chi \circ \det) + \mathrm{St}_G(\chi \circ \det).$$

We denote $1_G(\chi \circ \det)$ by $1_G(\chi)$ and $\mathrm{St}_G(\chi \circ \det)$ by $\mathrm{St}_G(\chi)$.

7.2. Deligne-Lusztig induction from T_1^{Fr}. Let $g \in \mathrm{GL}_2(E)$ such that $(T_1^{\mathrm{Fr}})^g$ is equal to the set of diagonal matrices in $\mathrm{GL}_2(E)$ with entries in E^1. Let θ be a character of T_1^{Fr}. Define characters χ_i, $i = 1, 2$, of E^1 by

$$\chi_1(x) = \theta^g(\mathrm{diag}(x, 1)), \quad \chi_2(x) = \theta^g(\mathrm{diag}(1, x)).$$

We can identify θ with $\chi_1 \otimes \chi_2$. The character θ is in general position if and only if $\chi_1 \neq \chi_2$. By the dimension formula

$$\dim(R_{T_1}^G \theta) = -(q-1).$$

If θ is in general position $-R_{T_1}^G \theta$ is an irreducible cuspidal representation of G^{Fr}. We let

$$\sigma_{T_1,\theta} = -R_{T_1}^G \theta.$$

If θ is not in general position, by weak orthogonality,

$$(R_{T_1}^G \theta, R_{T_1}^G \theta) = 2.$$

Hence $R_{T_1}^G \theta$ contains two irreducible representations each with multiplicity ± 1. Moreover $R_{T_1}^G \theta = (R_{T_1}^G 1)(\chi_1 \circ \det)$ and we know $R_{T_1}^G 1$ contains 1_G with multiplicity 1. Thus, by comparing with $i_{B_0}^G(1)$ via weak orthogonality, $R_{T_1}^G 1$ contains St_G with multiplicity -1 and

$$R_{T_1}^G \theta = 1_G(\chi) - \mathrm{St}_G(\chi)$$

Deligne-Lusztig Representation	Decomposition in $\mathfrak{Gr}_{\overline{\mathbb{Q}}_\ell}(G^{\mathrm{Fr}})$	Parameters	Degree	Number
$R_{T_0}^G \theta$	$1_G(\chi) + \mathrm{St}_G(\chi)$	χ a character of E^1 $\theta = \chi \circ \xi_{q-1}$	$1, q$	$2(q+1)$
	$i_{B_0}^G(\theta)$	θ a character of $E^\times$ $\theta^{q+1} \neq 1$ $i_{B_0}^G(\theta) = i_{B_0}^G(\theta^{-q})$	$q+1$	$\frac{q^2-q-2}{2}$
$R_{T_1}^G \theta$	$1_G(\chi) - \mathrm{St}_G(\chi)$	χ a character of E^1 $\theta = \chi \otimes \chi$	$q, 1$	$2(q+1)$
	$-\sigma_{T_1,\theta}$	θ_i characters of E^1 $\theta = \chi_1 \otimes \chi_2$ $\chi_1 \neq \chi_2$ $R_{T_1}^G \theta = R_{T_1}^G(\chi_2 \otimes \chi_1)$	$q-1$	$\frac{q^2+q}{2}$

Let $H^{\mathrm{Fr}} = \mathrm{SU}_2(E/F)$. Then H^{Fr} is isomorphic to $\mathrm{SL}_2(F)$ by Section 2. The maximal F-split tori in H^{Fr} are isomorphic to $F^\times$ and $S_0^{\mathrm{Fr}} = T_0^{\mathrm{Fr}} \cap H^{\mathrm{Fr}}$ is a representative. The non-split maximal tori are isomorphic to E^1, and $S_1^{\mathrm{Fr}} = T_1^{\mathrm{Fr}} \cap H^{\mathrm{Fr}}$ is a representative.

Following the same methods we have used for the groups $\mathrm{GL}_2(F)$ and $\mathrm{U}_2(E/F)$ we produce the following table of induced representations.

Deligne-Lusztig Representation	Decomposition in $\mathfrak{Gr}_{\overline{\mathbb{Q}}_\ell}(H^{\mathrm{Fr}})$	Parameters	Degree	Number
$R^H_{S_0}\theta$	$1_H + \mathrm{St}_H$	$\theta = 1$	$1, q$	2
	$i^+\theta + i^-\theta$	θ a character of $F^\times$ $\theta^2 = 1$ $\theta \neq 1$	$\frac{q+1}{2}, \frac{q+1}{2}$	2
	$i^H_{B_0}(\chi)$	χ a character of $F^\times$ $\chi^2 \neq 1$ $i^H_{B_0}(\chi) = i^H_{B'_0}(\chi^{-1})$	$q+1$	$\frac{q-3}{2}$
$R^H_{S_1}\theta$	$1_H - \mathrm{St}_H$	$\theta = 1$	$q, 1$	2
	$-\sigma^+_{S_1,\theta} - \sigma^-_{S_1,\theta}$	θ a character of E^1 $\theta^2 = 1$ $\theta \neq 1$	$\frac{q-1}{2}, \frac{q-1}{2}$	2
	$-\sigma_{S_1,\theta}$	θ a character of E^1 $\theta^2 \neq 1$ $R^H_{S_1}\theta = R^H_{S_1}\theta^{-1}$	$q-1$	$\frac{q-1}{2}$

8. The ℓ-adic representations of $\mathrm{U}_3(E/F)$

Let $G^{\mathrm{Fr}} = \mathrm{U}_3(E/F)$. The maximal diagonal torus in G^{Fr} is

$$T_0^{\mathrm{Fr}} = \left\{ \mathrm{diag}(x, y, x^{-q}) : x^{q^2-1} = y^{q+1} = 1 \right\},$$

and $W(T_0)^{\mathrm{Fr}} \simeq C_2$. Let

$$w_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad w_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which are representatives for the nontrivial Fr-conjugacy classes in $W(T_0)$. Let T_i^{Fr}, $i = 1, 2$, be representatives of the G^{Fr}-conjugacy class of maximal tori in G^{Fr} obtained by twisting T_0^{Fr} by w_i, $i = 1, 2$. Let F_3 be a cubic extension of F and E_3 a quadratic extension of F_3. The torus T_1^{Fr} is isomorphic to the kernel E_3^1 of the norm map N_{E_3/F_3}. It is conjugate in $\mathrm{GL}_3(\mathbb{F}_{q^6})$ to

$$\left\{ \mathrm{diag}(x, x^{q^2}, x^{-q}) : x^{q^3+1} = 1 \right\}$$

and $W(T_1)^{\mathrm{Fr}} \simeq C_3$. The torus T_2^{Fr} is conjugate in $\mathrm{GL}_3(E)$ to

$$\left\{ \mathrm{diag}(x, y, z) : x^{q+1} = y^{q+1} = z^{q+1} = 1 \right\}$$

and $W(T_2)^{\mathrm{Fr}} \simeq \mathfrak{S}_3$.

There is a Fr-stable Levi subgroup L of G

$$L^F = \begin{pmatrix} \star & 0 & \star \\ 0 & \star & 0 \\ \star & 0 & \star \end{pmatrix} \cap G^{\mathrm{Fr}}$$

which is isomorphic to $\mathrm{U}_1(E/F) \times \mathrm{U}_2(E/F)$. It is not contained in any proper Fr-stable parabolic subgroup of G, but contains both T_0^{Fr} and T_2^{Fr}. Thus when we are decomposing induced representations from T_0^{Fr} and T_2^{Fr}, by transitivity of induction, there will be two steps: first inducing to L^{Fr} then to G^{Fr}. Let $H^{\mathrm{Fr}} = \mathrm{U}_2(E/F)$.

8.1. Parabolic induction from T_0^{Fr}. Let χ_1 be a character of $E^\times$ and χ_2 be a character of E^1. Define a character θ of T_0^{Fr} by

$$\theta(\mathrm{diag}(x, y, x^{-q})) = \chi_1(x)\chi_2(xx^{-q}y).$$

All characters θ of T_0^{Fr} appear in this way and we can identify θ with the pair (χ_1, χ_2). The character θ is in general position if and only if $\chi_1^{q+1} \neq 1$. By the dimension formula

$$\dim(i_{B_0}^G(\theta)) = q^3 + 1.$$

If θ is not in general position, by weak orthogonality,

$$(i_{B_0}^G(\theta), i_{B_0}^G(\theta)) = 2.$$

Hence $i_{B_0}^G(\theta)$ is the sum of two irreducible representations. In this case, because $\chi_1^{q+1} = 1$, χ_1 factors through the map $\xi_{q-1} : x \mapsto x^{q-1}$, and identifies with a character χ of E^1,

$$\chi_1 = \chi \circ \xi_{q-1}.$$

We have two cases:

(1) If $\chi = 1$, then

$$\begin{aligned} i_{B_0}^G(\theta) &= i_{B_0}^G(\chi_2 \circ \det) \\ &= i_{B_0}^G(1)(\chi_2 \circ \det). \end{aligned}$$

By Frobenius reciprocity $i_{B_0}^G(1)$ contains 1_G and the irreducible quotient denoted by St_G of $i_{B_0}^G(1)$ by 1_G is q^3-dimensional. Thus

$$i_{B_0}^G(\theta) = 1_G(\chi_2 \circ \det) + \mathrm{St}_G(\chi_2 \circ \det).$$

We denote $1_G(\chi_2 \circ \det)$ by $1_G(\chi_2)$ and $\mathrm{St}_G(\chi_2 \circ \det)$ by $\mathrm{St}_G(\chi_2)$.

(2) Otherwise, when $\chi \neq 1$,

$$\begin{aligned} i_{B_0}^G(\theta) &= i_{B_0}^G(\chi_1 \otimes 1)(\chi_2 \circ \det) \\ &= R_L^G(i_{B_0 \cap L}^L(\chi \circ \xi_{q-1} \otimes 1))(\chi_2) \\ &= R_L^G(1_H(\chi \circ \det) \otimes 1 + \mathrm{St}_H(\chi \circ \det) \otimes 1)(\chi_2) \\ &= R_L^G(1_H(\chi) \otimes 1)(\chi_2) + R_L^G(\mathrm{St}_H(\chi) \otimes 1)(\chi_2). \end{aligned}$$

The dimension of $R_L^G(1_H(\chi) \otimes 1)(\chi_2)$ is $q^2 - q + 1$, and the dimension of $R_L^G(\mathrm{St}_H(\chi) \otimes 1)(\chi_2)$ is $q(q^2 - q + 1)$. By comparing with the complex character table, [**Enn63**], these are irreducible representations of G^{Fr}.

8.2. Deligne-Lusztig induction from T_1^{Fr}. Let θ be a character of T_1^{Fr}. The character θ is in general position if $\theta^{q+1} \neq 1$. By the dimension formula

$$\dim(R_{T_1}^G \theta) = -(q^2 - 1)(q + 1).$$

If θ is not in general position, by weak orthogonality,

$$(R_{T_1}^G \theta, R_{T_1}^G \theta) = 3.$$

Hence $R_{T_1}^G \theta$ contains three irreducible representations each with multiplicity ± 1. If θ is not in general position then θ factors through the map $\xi_{q^2-q+1} : x \mapsto x^{q^2-q+1}$ and hence can be identified with character χ of E^1

$$\theta = \chi \circ \xi_{q^2-q+1}.$$

Then $R^G_{T_1}\theta = R^G_{T_1}1(\chi\circ\det)$ and we are reduced to decomposing $R^G_{T_1}1$. We know $R^G_{T_1}1$ contains 1_G with multiplicity 1. Thus by comparing with $i^G_{B_0}(1)$, via weak orthogonality, $R^G_{T_1}1$ contains St_G with multiplicity -1. There is one other irreducible representation ν of dimension q^2-q and because it did not appear in any of the induced representations from T_0^{Fr} it is . Thus

$$R^G_{T_1}\theta = 1_G(\chi) - \nu(\chi\circ\det) - \mathrm{St}_G(\chi)$$

We let $\nu_\chi = \nu(\chi\circ\det)$.

If θ is in general position $-R^G_{T_1}\theta$ is an irreducible cuspidal representation of G^{Fr}. We let

$$\sigma_{T_1,\theta} = -R^G_{T_1}\theta.$$

8.3. Deligne-Lusztig induction from T_2^{Fr}. Let $g \in \mathrm{GL}_3(E)$ such that $(T_2^{\mathrm{Fr}})^g$ is equal to the set of diagonal matrices in $\mathrm{GL}_3(E)$ with entries in E^1. Let θ be a character of T_2^{Fr}. Define characters θ_i, $i=1,2,3$ of E^1 by

$$\chi_1(x) = \theta^g(\mathrm{diag}(x,1,1)),\ \chi_2(x) = \theta^g(\mathrm{diag}(1,x,1)),\ \chi_3(x) = \theta^g(\mathrm{diag}(1,1,x)).$$

We can thus identify θ with $\chi_1\otimes\chi_2\otimes\chi_3$. The character θ is in general position if the characters χ_i, $i=1,2,3$, are pairwise distinct. By the dimension formula

$$\dim(R^G_{T_2}\theta) = -(q-1)(q^2-q+1).$$

If θ is in general position $-R^G_{T_2}\theta$ is an irreducible cuspidal representation of G^{Fr}, we let

$$\sigma_{T_2,\theta} = -R^G_{T_2}\theta.$$

When θ is not in general position we have two cases:

(1) If $\chi=\chi_1=\chi_2=\chi_3$, then by weak orthogonality

$$(R^G_{T_2}\theta, R^G_{T_2}\theta) = 6.$$

Thus either $R^G_{T_2}\theta$ contains six irreducible representations each with multiplicity ± 1, or it contains three irreducible representations, two of which have multiplicity ±1 and one of which has multiplicity ±2. Because $R^G_{T_2}\theta = R^G_{T_2}1(\chi\circ\det)$, we are reduced to decomposing $R^G_{T_2}1$. By comparing with $i^G_{B_0}(1)$ and $R^G_{T_1}1$ via weak orthogonality, we have

$$R^G_{T_2}1 = 1_G - 2\nu_1 - \mathrm{St}_G\,.$$

(2) If $\chi=\chi_1=\chi_2\neq\chi_3$, then by weak orthogonality

$$(R^G_{T_2}\theta, R^G_{T_2}\theta) = 2.$$

Thus $R^G_{T_2}\theta$ contains two irreducible representations each with multiplicity ±1. In fact by counting we have already found all irreducible representations of G^{Fr}, and by comparing dimensions we must have

$$R^G_{T_2} = R^G_L(1_H(\chi)\otimes 1)(\chi_3) - R^G_L(1_H(\chi)\otimes 1)(\chi_3).$$

TABLE 1. Decomposition of induced representations of $\mathrm{U}_3(E/F)$. There are q^3+2q^2+3q+2 irreducible representations of G^{Fr}.

Deligne-Lusztig Representation	Decomposition in $\mathfrak{Gr}_{\overline{\mathbb{Q}}_\ell}(G^{\mathrm{Fr}})$	Parameters	Degree	Number
$R^G_{T_0}\theta$	$1_G(\chi)+\mathrm{St}_G(\chi)$	χ a character of E^1 $\chi_1=\chi\circ\xi_{q-1}$ $\theta(\mathrm{diag}(x,1,x^{-q}))=\chi_1(x)\chi(xx^{-q}y)$	$1, q^3$	$2(q+1)$
	$R^G_L(1_H(\chi)\otimes 1)(\chi_2)+R^G_L(\mathrm{St}_H(\chi)\otimes 1)(\chi_2)$	χ,χ_2 characters of E^1 $\chi\neq\chi_2$ $\chi_1=\chi\circ\xi_{q-1}$ $\theta(\mathrm{diag}(x,1,x^{-q}))=\chi_1(x)\chi_2(xx^{-q}y)$	q^2-q+1, $q(q^2-q+1)$	$2(q^2+q)$
	$i^G_{B_0}(\theta)$	χ_1 a character of $E^\times$ χ_2 a character of E^1 $\chi_1^{q+1}\neq 1$ $\theta(\mathrm{diag}(x,1,x^{-q}))=\chi_1(x)\chi_2(xx^{-q}y)$	q^3+1	$\frac{q^3-3q-2}{2}$
$R^G_{T_1}\theta$	$1_G(\chi)+\nu_\chi-\mathrm{St}_G(\chi)$	χ a character of E^1 $\theta=\chi\circ\xi_{q^2-q+1}$	$1, q^2-q, q^3$	$3(q+1)$
	$-\sigma_{T_1,\theta}$	θ a character of E^1_3 $\theta^{q+1}\neq 1$ $R^G_{T_1}\theta=R^G_{T_1}\theta^q$	$(q^2-1)(q+1)$	$\frac{(q+1)q(q-1)}{3}$
$R^G_{T_2}\theta$	$1_G(\chi)-2\nu_\chi-\mathrm{St}_G(\chi)$	χ a character of E^1 $\theta=\chi\otimes\chi\otimes\chi$	$1, q^2-q, q^3$	$3(q+1)$
	$R^G_L(1_H(\chi)\otimes 1)(\chi_2)-R^G_L(\mathrm{St}_H(\chi)\otimes 1)(\chi_2)$	χ,χ_2 characters of E^1 $\chi\neq\chi_2$ $\theta=\chi\otimes\chi\otimes\chi_2$	q^2-q+1, $q(q^2-q+1)$	$2(q^2+q)$
	$-\sigma_{T_2,\theta}$	χ_i characters of E^1 χ_i pairwise distinct $\theta=\chi_1\otimes\chi_2\otimes\chi_3$ $R^G_{T_2}(\chi_1\otimes\chi_2\otimes\chi_3)=R^G_{T_2}(\chi_{\sigma(1)}\otimes\chi_{\sigma(2)}\otimes\chi_{\sigma(3)})$ $\sigma\in\mathfrak{S}_3$	$(q-1)(q^2-q+1)$	$\frac{(q+1)q(q-1)}{6}$

9. The ℓ-adic representations of $\mathrm{GL}_3(F)$

Let $G^{\mathrm{Fr}} = \mathrm{GL}_3(F)$. The maximal F-split diagonal torus in G^{Fr} is

$$T_0^{\mathrm{Fr}} = \left\{\mathrm{diag}(x, y, z) : x, y, z \in F^\times\right\},$$

and $W(T_0)^F \simeq \mathfrak{S}_3$. Let

$$w_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } w_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

be representatives of the two nontrivial Fr-conjugacy classes in $W(T_0)$. Let T_i^F, $i = 1, 2$, be representatives of the G^{Fr}-conjugacy class of maximal tori in G^{Fr} obtained by twisting T_0^{Fr} by w_i, $i = 1, 2$. The torus T_1^{Fr} is conjugate in $\mathrm{GL}_3(F_3)$ to

$$\left\{\mathrm{diag}(x, x^{q^2}, x^q) : x \in F_3^\times\right\}.$$

and $W(T_1)^F \simeq C_3$. The torus T_2^{Fr} is conjugate in $\mathrm{GL}_3(E)$ to

$$\left\{\mathrm{diag}(x, x^q, y) : x \in E^\times,\ y \in F^\times\right\},$$

and $W(T_2)^F \simeq C_2$. The tori T_0 and T_2 are contained in the Fr-stable parabolic subgroup

$$P = \begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ 0 & 0 & \star \end{pmatrix}.$$

Let $H^{\mathrm{Fr}} = \mathrm{GL}_2(F)$ and S_1^{Fr} be a representative of the H^{Fr}-conjugacy class of non split maximal tori in H^{Fr}. Following the same methods we have used for the groups $\mathrm{GL}_2(F)$, $\mathrm{U}_2(E/F)$ and $\mathrm{U}_3(E/F)$ we produce the following table of induced representations.

TABLE 2. Decomposition of induced representations for $\mathrm{GL}_3(F)$. There are $q(q^2-1)$ irreducible representations.

Deligne-Lusztig Representation	Decomposition in $\mathfrak{Gr}_{\overline{\mathbb{Q}}_\ell}(G^{\mathrm{Fr}})$	Parameters	Degree	Number
$R^G_{T_0}\theta$	$1_G(\chi)+2\nu_G(\chi)+\mathrm{St}_G(\chi)$	χ a character of $F^\times$ $\theta=\chi\otimes\chi\otimes\chi$	$1,,q^2+q,q^3$	$3(q-1)$
	$i_P^G(1_H(\chi)\otimes\chi_3)+i_P^G(\mathrm{St}_H(\chi)\otimes\chi_3)$	χ,χ_3 characters of $F^\times$ $\chi\neq\chi_3$ $\theta=\chi\otimes\chi\otimes\chi_3$	$q^2+q+1,q(q^2+q+1)$	$2(q-1)(q-2)$
	$i^G_{B_0}(\theta)$	χ_i characters of $F^\times$ χ_i pairwise distinct $\theta=\chi_1\otimes\chi_2\otimes\chi_3$ $i^G_B(\theta)=i^G_B(\chi_{\sigma(1)}\otimes\chi_{\sigma(2)}\otimes\chi_{\sigma(3)})$ $\sigma\in\mathfrak{S}_3$	$(q+1)(q^2+q+1)$	$\frac{(q-1)(q-2)(q-3)}{6}$
$R^G_{T_1}\theta$	$1_G(\chi)-\nu(\chi)+\mathrm{St}_G(\chi)$	χ a character of $F^\times$ $\theta=\chi\circ\xi_{q^2+q+1}$	$1,q^2+q,q^3$	$3(q-1)$
	$\sigma_{T_1,\theta}$	θ a character of $F_3^\times$ $\theta^{q-1}\neq 1$ $R^G_{T_1}\theta=R^G_{T_1}\theta^q$	$(q^2-1)(q-1)$	$\frac{(q+1)q(q-1)}{3}$
$R^G_{T_2}\theta$	$1_G(\chi)-\mathrm{St}_G(\chi)$	χ a character of $F^\times$ $\chi_1=\chi\circ\xi_{q+1}$ $\theta=\chi_1\otimes\chi$	$1,q^3$	$2(q-1)$
	$i_P^G(1_H(\chi)\otimes\chi_2)-i_P^G(\mathrm{St}_H(\chi)\otimes\chi_2)$	χ,χ_2 characters of $F^\times$ $\chi\neq\chi_2$ $\chi_1=\chi\circ\xi_{q+1}$ $\theta=\chi_1\otimes\chi_2$	$q^2+q+1,q(q^2+q+1)$	$2(q-1)(q-2)$
	$-i_P^G(\sigma_{S_1,\chi_1}\otimes\chi_2)$	χ_1 a character of $E^\times$ χ_2 a character of $F^\times$ $\chi_1^{q-1}\neq 1$ $\theta=\chi_1\otimes\chi_2$ $R^G_{T_2}\chi_1\otimes\chi_2=R^G_{T_2}\chi_1^q\otimes\chi_2$	(q^3-1)	$\frac{q(q-1)^2}{2}$

CHAPTER 3

ℓ-MODULAR REPRESENTATIONS OF FINITE REDUCTIVE GROUPS

In this chapter we study the relationship between ℓ-modular and ℓ-adic representations of finite groups of Lie type. We then classify the ℓ-modular representations of certain finite reductive groups which appear as quotients of the parahoric subgroups of the p-adic unitary group in three variables.

In this chapter, as in Chapter 2, F is a finite field with q-elements and E a quadratic extension of F.

1. Decomposition modulo-ℓ

Let G be a finite group of Lie type. Let $(K, \mathcal{O}_K, k)$ be an ℓ-modular splitting system for G. Recall that this means

(1) $\mathcal{O}_K$ is a discrete valuation ring in characteristic 0 with maximal ideal $\wp$;
(2) K is the quotient field of $\mathcal{O}_K$ which is sufficiently large; containing all $|G|$-th roots of unity;
(3) $k = \mathcal{O}_K/\wp$ is a finite field of characteristic ℓ.

The representations of G over K are called ordinary and those over k are called ℓ-modular. In analogy with Chapter 1 Section 4 for p-adic groups we define a decomposition modulo-ℓ map for representations of G.

Let $(\rho, \mathcal{V})$ be an integral ordinary representation of G. Let L be an $\mathcal{O}_K G$-lattice in $\mathcal{V}$; define the reduction modulo-ℓ of $(\rho, \mathcal{V})$, with respect to L, to be the induced representation of G on the k-vector space

$$\overline{L} = L \otimes_{\mathcal{O}_K} k.$$

The Brauer-Nesbitt Principle, [**CR81**, Theorem 16.16], states that the composition factors that appear are independent of the choice of lattice in $\mathcal{V}$. Thus we define the decomposition modulo ℓ of $\mathcal{V}$ by

$$d_\ell(\mathcal{V}) = [L \otimes_{\mathcal{O}_K} k].$$

We extend d_ℓ by linearity to the Grothendieck group $\mathfrak{Gr}_K(G)$ of virtual ordinary representations of G. Then, [**CR81**, Theorem 17.17], d_ℓ is a surjective homomorphism between Grothendieck groups

$$d_\ell : \mathfrak{Gr}_K(G) \to \mathfrak{Gr}_k(G),$$

which is compatible with extension of K, [**CR81**, Proposition 16.23]. The decomposition matrix (d_{ij}) of G is the matrix formed by indexing the rows by the irreducible ordinary representations $(\pi_i, \mathcal{V}_i)$, the columns by the irreducible ℓ-modular representations (ρ_j, W_j) and the (i,j)-th entry d_{ij} is the multiplicity of (ρ_j, W_j) in $d_\ell(\mathcal{V}_i)$.

1.1. Brauer Characters.

Let G be a finite group and $(K, \mathcal{O}_K, k)$ an ℓ-modular splitting system for G.

The field k contains all $|G|_{\ell'}$-th roots of unity, where $|G|_{\ell'}$ denotes the ℓ-regular part of $|G|$. These form a cyclic group of order $|G|_{\ell'}$ under multiplication. Fix an isomorphism of cyclic groups

$$\Psi : \left\{ |G|_{\ell'}\text{-th roots of unity in } k^\times \right\} \to \left\{ |G|_{\ell'}\text{-th roots of unity in } K^\times \right\}.$$

For (ψ, M) a d-dimensional ℓ-modular representation of G, we define a function on the ℓ-regular elements of G by

$$\chi_M(g) = \sum_{i=1}^{d} \Psi(\lambda_i),$$

where $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of $\psi(g)$ counted with multiplicity.

Hence we have a map

$$\chi_M : \{\text{conjugacy classes of } \ell\text{-regular elements of } G\} \to K.$$

The map χ_M is called a Brauer character of (ψ, M) and we let

$$\mathrm{IBr}(G) = \{\chi_M : M \text{ is an irreducible } \ell\text{-modular representation of } G\}.$$

The definition of a Brauer character involves a choice of ℓ-modular system and a choice of isomorphism Ψ. For a given G we assume we have fixed such choices so that a finite dimensional ℓ-modular representation of G defines a unique Brauer character.

Theorem 1.1 ([**CR81**, Chapter 17]).

(1) A Brauer character is a class function on the ℓ-regular classes of G.
(2) The set $\mathrm{IBr}(G)$ is a basis for the K-vector space of ℓ-regular class functions on G. (Hence $|\mathrm{IBr}(G)|$ is equal to the number of conjugacy classes of ℓ-regular elements of G.)
(3) If $\ell \nmid |G|$ then $\mathrm{Irr}(G) = \mathrm{IBr}(G)$, where $\mathrm{Irr}(G)$ denotes the set of ordinary characters of G.

Let H be a subgroup of G, (ρ, W) an ℓ-modular representation of H, and $\psi \in \mathrm{IBr}(G)$ be the Brauer character of (ρ, W), then ψ^G given by

$$\psi^G(x) = \frac{1}{|H|_{\ell'}} \sum_{g \in G} \psi(gxg^{-1}),$$

is the Brauer character of the induced representation $i_H^G \rho$, [**Nav98**, Theorem 8.2].

A $\mathbb{Z}$-linear combination of Brauer characters is called a virtual Brauer character. The set of all irreducible ℓ-modular representations of G is a $\mathbb{Z}$-basis of $\mathfrak{Gr}_k(G)$, and we have an isomorphism of rings $\Omega_k : \mathfrak{Gr}_k(G) \to \mathbb{Z}\,\mathrm{IBr}(G)$ given by

$$\Omega_k : \sum_i a_i \mathcal{V}_i \mapsto \sum_i a_i \chi_{\mathcal{V}_i}.$$

Similarly there is an isomorphism of rings Ω_K between $\mathfrak{Gr}_K(G)$ and $\mathbb{Z}\,\mathrm{Irr}(G)$. Define a map $d^1 : \mathrm{CF}(G, K) \to \mathrm{CF}(G, K)$ on the set of K-valued class functions of G by

$$d^1\chi(g) = \begin{cases} \chi(g) & \text{if } g \text{ is } \ell\text{-regular;} \\ 0 & \text{otherwise.} \end{cases}$$

Under the usual inner product on K-valued class functions of G,

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\chi_2(g^{-1}),$$

d^1 is a self adjoint functor, i.e. for all $\chi_1, \chi_2 \in \mathrm{CF}(G, K)$ we have $\langle d^1\chi_1, \chi_2 \rangle = \langle \chi_1, d^1\chi_2 \rangle$.

Lemma 1.2 ([**CR81**, Proposition 17.15]). Let $\rho \in \mathfrak{Gr}_K(G)$ then

$$\Omega_k \circ d_\ell(\rho) = d^1 \circ \Omega_K(\rho).$$

We define a decomposition matrix using Brauer characters. The rows are indexed by the irreducible ordinary characters χ_i of G, the columns are indexed by the irreducible Brauer characters ψ_j of G, the (i, j)-th entry is the multiplicity of ψ_j in $d^1(\chi_i)$. By Lemma 1.2 this decomposition matrix is equal to the decomposition matrix of G given by d_ℓ.

1.2. Theory of ℓ-blocks. Let $(\rho_i, \mathcal{V}_i)$, $i = 1, 2$, be irreducible ordinary representations of G. We say that ρ_1 and ρ_2 are in the same ℓ-block if there exists a sequence of irreducible ordinary representations σ_i, $i = 1, ..., n$, such that

(1) $\rho_1 = \sigma_1$ and $\rho_2 = \sigma_n$;
(2) For all $i \in \{1, ..., n-1\}$, $d_\ell(\sigma_i)$ and $d_\ell(\sigma_{i+1})$ share a composition factor.

With this equivalence relation we partition the irreducible ordinary representations $\mathrm{Irr}(G)$ of G into ℓ-blocks. We also consider irreducible ℓ-modular representations, ordinary characters and Brauer characters as associated to a unique ℓ-block in the obvious way.

Given an ℓ-block $\mathbf{B}$ one can associate a conjugacy class of subgroups of G, [**Nav98**, Chapter 4], called the defect group of the block. The defect group gives an indication of the complexity of the structure of $\mathbf{B}$. A block with trivial defect group is called of defect zero. An ℓ-block $\mathbf{B}$ is of defect zero if and only if there are only one ordinary character χ and one Brauer character $d^1(\chi)$ associated to the block, [**Nav98**, Theorem 3.18].

When the defect group of a block is cyclic, the structure of $\mathbf{B}$ is particularly nice: One associates to $\mathbf{B}$ a graph with N vertices, where $N-1$ vertices correspond to distinct ordinary characters in $\mathbf{B}$, and the N-th exceptional vertex corresponds to a collection of ordinary characters in $\mathbf{B}$ with equal decomposition modulo ℓ. A vertex corresponding to an ordinary character χ_1 is joined to a vertex corresponding to ordinary character χ_2 if and only if $d^1(\chi_1)$ and $d^1(\chi_2)$ have a common constituent. The graph defined in this way is a tree, [**Nav98**, Page 271], called the Brauer tree of $\mathbf{B}$.

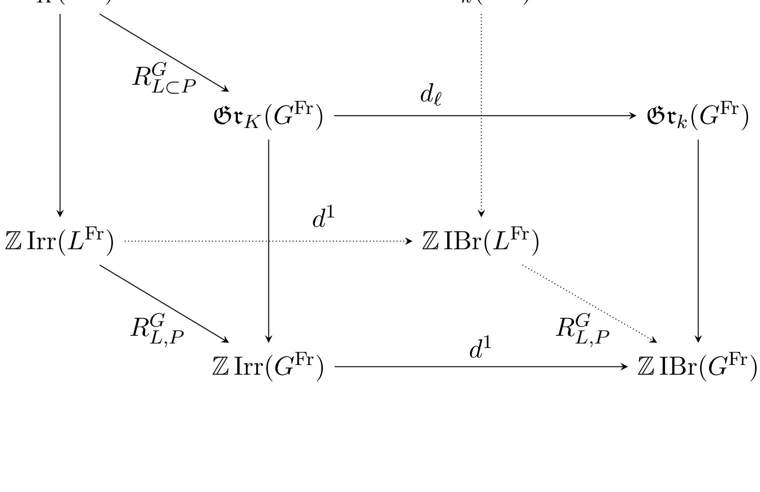
1.3. R_T^G and ℓ-modular Representations. Let G^{Fr} be a finite group of Lie type, L be an Fr-stable Levi subgroup of G and P be an Fr-stable parabolic subgroup of G. In analogy with Chapter 1 Corollary 5.2, parabolic induction and restriction commute with decomposition modulo-ℓ, i.e. $d_\ell(i_P^G\varphi) = \left[i_P^G(d_\ell(\varphi))\right]$ and $d_\ell(r_P^G(\sigma)) = \left[r_P^G(d_\ell(\sigma))\right]$. Thus the decomposition modulo ℓ of an irreducible cuspidal representation is a sum of irreducible cuspidal ℓ-modular representations.

LEMMA 1.3. Let θ_1, θ_2 be representations of L^{Fr} such that $d_\ell(\theta_1) = d_\ell(\theta_2)$. Then $d_\ell(R_L^G\theta_1) = d_\ell(R_L^G\theta_2)$.

PROOF: ${}^*R_{L,P}^G$ commutes with decomposition maps d^1, i.e.

$$d^1 \circ {}^*R_{L,P}^G = {}^*R_{L,P}^G \circ d^1$$

[**CE04**, Definition 5.7, Theorem 21.4]. Hence $R_{L,P}^G$ commutes with d^1 because d^1 is a self-adjoint functor. This gives the base square on the following commutative diagram, the vertical arrows being the isomorphisms of rings introduced earlier:



The lemma follows by passing from the top incomplete square to the base commutative square, using the commutativity there and then passing back. ☐

REMARK. We can lift $\overline{\rho} \in \mathfrak{Gr}_k(L^{\mathrm{Fr}})$ to a virtual representation $\rho \in \mathfrak{Gr}_K(L^{\mathrm{Fr}})$, thus we can define an ℓ-modular Deligne-Lusztig induction

$$\begin{aligned} R^G_{L\subset P} : \mathfrak{Gr}_k(L^{\mathrm{Fr}}) &\to \mathfrak{Gr}_k(G^{\mathrm{Fr}}) \\ \overline{\rho} &\mapsto d_\ell(R^G_{L\subset P}\rho) \end{aligned}$$

which completes the commutative cube.

Define an equivalence relation on $\mathrm{Irr}(G^{\mathrm{Fr}})$ by $\chi \sim \psi$ if there exists a sequence $(\chi_i)_{i=1}^n \in \mathrm{Irr}(G^{\mathrm{Fr}})$ such that

(1) $\chi_1 = \chi$ and $\chi_2 = \psi$,
(2) For $i = 1, ..., n-1$ there exist a Fr-stable maximal torus T of G and a character θ of T^{Fr} such that
$$(R^G_T\theta, \chi_i) \neq 0 \qquad \text{and} \qquad (R^G_T\theta, \chi_{i+1}) \neq 0.$$

The equivalence classes are called geometric conjugacy classes and can be parametrized by semisimple conjugacy classes in the dual group $G^{*\mathrm{Fr}^*}$, [**CE04**, Section 8.4]

LEMMA 1.4. Let θ be a character of T^{Fr} in general position. Assume that $d_\ell(\theta)$ is also in general position, i.e. $w \in W(T)^{\mathrm{Fr}}$ and $d_\ell(\theta) = d_\ell(\theta)^w$ implies that $w = 1$. Then either $d_\ell R^G_T(\theta)$ or $-d_\ell R^G_T(\theta)$ is an irreducible representation.

PROOF: The following proof was suggested by Gunter Malle. Let $\overline{\theta}$ be an ℓ-modular character of T^{Fr} such that for all $w \in W$, $\theta^w \neq \theta$. By [**CE04**, 9.12],

$$\{R^G_T\theta : d_\ell(\theta) = \overline{\theta}\},$$

is both a union of ℓ-blocks and a union of geometric conjugacy classes. By Lemma 1.3 the decomposition modulo ℓ of any of the representations $R^G_T\theta$ in this set are equal. These ordinary representations are in general position and thus are all irreducible of the same dimension. Hence, because decomposition modulo-ℓ is a surjective map between Grothendieck groups, $d_\ell R^G_T(\theta)$ or $-d_\ell R^G_T(\theta)$ is an irreducible representation. ☐

2. Decomposition matrices of $\mathrm{GL}_2(F)$

We use the notation of Chapter 2 Section 5 to describe the decomposition matrices of $\mathrm{GL}_2(F)$ which are extracted from [**Jam90**] in Appendix A.

Notation in Appendix A	Notation in Chapter 2 Section 5
$S_K(s,(1^2))$	$1_G(\chi)$
$S_K(s,(2))$	$\mathrm{St}_G(\chi)$
$\mathrm{Ind}(S_K(s_1,(1)) \otimes S_K(s_2,(1)))$	$i^G_{B_0}(\theta)$
$S_K(s,(1))$	$\sigma_{T_1,\theta}$

2.1. Decomposition matrices if $\ell \neq 2$ and $\ell \mid q-1$.

Let $\ell^a \mid\mid q-1$. There are three types of ℓ-blocks.

(1) **The ℓ-blocks $\mathbf{B}_1(\overline{\chi})$.** Let $\overline{\chi}$ be an irreducible ℓ-modular character of $F^\times$. Associated to $\overline{\chi}$ we have an ℓ-block $\mathbf{B}_1(\overline{\chi})$ with decomposition matrix:

			Conditions	Number
$1_G(\chi)$	1	0	$d_\ell(\chi) = \overline{\chi}$	ℓ^a
$\mathrm{St}_G(\chi)$	0	1	$d_\ell(\chi) = \overline{\chi}$	ℓ^a
$i_{B_0}^G(\theta)$	1	1	$d_\ell(\theta) = \overline{\chi} \otimes \overline{\chi}$	$\frac{\ell^a(\ell^a - 1)}{2}$

There are $\frac{q-1}{\ell^a}$ irreducible ℓ-modular characters of $F^\times$ hence $\frac{q-1}{\ell^a}$ distinct ℓ-blocks $\mathbf{B}_1(\overline{\chi})$.

(2) **The ℓ-blocks $\mathbf{B}_2(\overline{\chi}_1, \overline{\chi}_2)$.** Let $\overline{\chi}_i$, $i = 1, 2$ be irreducible ℓ-modular characters of $F^\times$ such that $\overline{\chi}_1 \neq \overline{\chi}_2$. Associated to the pair $(\overline{\chi}_1, \overline{\chi}_2)$ we have an ℓ-block $\mathbf{B}_2(\overline{\chi}_1, \overline{\chi}_2)$ with decomposition matrix:

		Conditions	Number
$i_{B_0}^G(\theta)$	1	$d_\ell(\theta) = \overline{\chi}_1 \otimes \overline{\chi}_2$	ℓ^{2a}

There are $(q-1)(q-2)$ distinct ℓ-blocks $\mathbf{B}_2(\overline{\chi}_1, \overline{\chi}_2)$.

(3) **ℓ-blocks of defect zero.** The irreducible supercuspidal representations $\sigma_{T_1,\theta}$ are in ℓ-blocks of defect zero. Thus there are $\frac{q^2-q+2}{2}$ cuspidal ℓ-modular representations, all of which are supercuspidal.

2.2. Decomposition matrices if $\ell \neq 2$ and $\ell \mid q+1$.

Let $\ell^a \mid\mid q+1$. There are three types of ℓ-blocks.

(1) **The ℓ-blocks $\mathbf{B}_1(\overline{\chi})$.** Let $\overline{\chi}$ be an irreducible ℓ-modular character of $F^\times$. Associated to $\overline{\chi}$ we have an ℓ-block $\mathbf{B}_1(\overline{\chi})$ with decomposition matrix:

			Conditions	Number
$1_G(\chi)$	1	0	$d_\ell(\chi) = \overline{\chi}$	1
$\mathrm{St}_G(\chi)$	1	1	$d_\ell(\chi) = \overline{\chi}$	1
$\sigma_{T_1,\theta}$	0	1	$d_\ell(\theta) = \overline{\chi} \circ \xi_{q+1}$	$\frac{(\ell^a - 1)}{2}$

There are $q-1$ distinct ℓ-blocks $\mathbf{B}_1(\overline{\chi})$

(2) **The ℓ-blocks $\mathbf{B}_2(\overline{\chi})$.** Let $\overline{\chi}$ be an irreducible ℓ-modular character of $E^\times$ such that $\overline{\chi}^{q-1} \neq 1$. Associated to $\overline{\chi}$ we have an ℓ-block $\mathbf{B}_2(\overline{\chi})$ with decomposition matrix:

		Conditions	Number
$\sigma_{T_1,\theta}$	1	$d_\ell(\theta) = \overline{\chi}$	ℓ^a

There are $\frac{q^2-1}{\ell^a} - (q-1)$ distinct ℓ-blocks $\mathbf{B}_2(\overline{\chi})$

(3) **ℓ-blocks of defect zero.** The irreducible principal series representations $i_B^G\theta$ are in ℓ-blocks of defect zero.

3. Decomposition matrices of $\mathrm{GL}_3(F)$

We use the notation of Chapter 2 Section 9 to describe the decomposition matrices of $\mathrm{GL}_3(F)$ which are extracted from [**Jam90**] in Appendix A.

Notation in Appendix A	Notation in Chapter 2 Section 9
$S_K(s,(1^3))$	$1_G(\chi)$
$S_K(s,(21))$	$\nu_G(\chi)$
$S_K(s,(3))$	$\mathrm{St}_G(\chi)$
$\mathrm{Ind}(S_K(s_1,(1^2))\otimes S_K(s_2,(1)))$	$i_P^G(1_H(\chi_1)\otimes\chi_2)$
$\mathrm{Ind}(S_K(s_1,(2))\otimes S_K(s_2,(1)))$	$i_P^G(\mathrm{St}_H(\chi_1)\otimes\chi_2)$
$\mathrm{Ind}(S_K(s_1,(1))\otimes S_K(s_2,(1))\otimes S_K(s_3,(1)))$	$i_{B_0}^G(\theta)$
$\mathrm{Ind}(S_K(s_1^2,(1))\otimes S_K(s_2,(1)))$	$i_P^G(\sigma_{S_1,\chi_1}\otimes\chi_2)$
$S_K(s^3,(1))$	$\sigma_{T_1,\theta}$

3.1. Decomposition matrices if $\ell\neq 2,3$ and $\ell\mid q-1$.
Let $\ell^a \mid\mid q-1$. There are four types of ℓ-blocks.

(1) **The ℓ-blocks $\mathbf{B}_1(\overline{\chi})$.** Let $\overline{\chi}$ be an irreducible ℓ-modular character of $F^\times$. Associated to $\overline{\chi}$ we have an ℓ-block $\mathbf{B}_1(\overline{\chi})$ with decomposition matrix:

				Conditions	Number
$1_G(\chi)$	1	0	0	$d_\ell(\chi)=\overline{\chi}$	ℓ^a
ν_χ	0	1	0	$d_\ell(\chi)=\overline{\chi}$	ℓ^a
$\mathrm{St}_G(\chi)$	0	1	1	$d_\ell(\chi)=\overline{\chi}$	ℓ^a
$i_P^G(1_H(\chi_1)\otimes\chi_2)$	1	1	0	$d_\ell(\chi_i)=\overline{\chi}$	$\ell^a(\ell^a-1)$
$i_P^G(\mathrm{St}_H(\chi_1)\otimes\chi_2)$	0	1	1	$d_\ell(\chi_i)=\overline{\chi}$	$\ell^a(\ell^a-1)$
$i_B^G\theta$	1	2	1	$d_\ell(\theta)=\overline{\chi}\otimes\overline{\chi}\otimes\overline{\chi}$	$\frac{\ell^a(\ell^a-1)(\ell^a-2)}{6}$

There are $\frac{q-1}{\ell^a}$ distinct ℓ-blocks $\mathbf{B}_1(\overline{\chi})$.

(2) **The ℓ-blocks $\mathbf{B}_2(\overline{\chi}_1,\overline{\chi}_2)$.** Let $\overline{\chi}_i$, $i=1,2$ be distinct irreducible ℓ-modular characters of $F^\times$. Associated to the pair $(\overline{\chi}_1,\overline{\chi}_2)$ we have an ℓ-block $\mathbf{B}_2(\overline{\chi}_1,\overline{\chi}_2)$ with decomposition matrix:

			Conditions	Number
$i_P^G(1_H(\chi_1)\otimes\chi_2)$	1	0	$d_\ell(\chi_i)=\overline{\chi}_i$	ℓ^{2a}
$i_P^G(\mathrm{St}_H(\chi_1)\otimes\chi_2)$	0	1	$d_\ell(\chi_i)=\overline{\chi}_i$	ℓ^{2a}
$i_B^G\theta$	1	1	$d_\ell(\theta)=\overline{\chi}_1\otimes\overline{\chi}_1\otimes\overline{\chi}_2$	$\frac{\ell^{2a}(\ell^a-1)}{2}$

There are $\frac{q-1}{\ell^a}\left(\frac{q-1}{\ell^a}-1\right)$ distinct ℓ-blocks $\mathbf{B}_2(\overline{\chi}_1,\overline{\chi}_2)$.

(3) **The ℓ-blocks $\mathbf{B}_3(\overline{\chi}_1,\overline{\chi}_2,\overline{\chi}_3)$.** Let $\overline{\chi}_i$, $i=1,2,3$ be distinct irreducible ℓ-modular characters of $F^\times$. Associated to the triple $(\overline{\chi}_1,\overline{\chi}_2,\overline{\chi}_3)$ we have an ℓ-block $\mathbf{B}_3(\overline{\chi}_1,\overline{\chi}_2,\overline{\chi}_3)$ with decomposition matrix:

		Conditions	Number
$i_{B_0}^G(\theta)$	1	$d_\ell(\theta)=\overline{\chi}_1\otimes\overline{\chi}_2\otimes\overline{\chi}_3$	ℓ^{3a}

There are $\frac{q-1}{\ell^a}\left(\frac{q-1}{\ell^a}-1\right)\left(\frac{q-1}{\ell^a}-2\right)$ distinct ℓ-blocks $\mathbf{B}_3(\overline{\chi}_1,\overline{\chi}_2,\overline{\chi}_3)$.

(4) **ℓ-blocks of defect zero.** All other ℓ-modular representations are in ℓ-blocks of defect zero.

3.2. Decomposition matrices if $\ell \neq 2$ and $\ell \mid q+1$.
Let $\ell^a \mid\mid q+1$. There are four types of ℓ-blocks.

(1) **The ℓ-blocks $\mathbf{B}_1(\overline{\chi})$.** Let $\overline{\chi}$ be an irreducible ℓ-modular character of $F^\times$. Associated to $\overline{\chi}$ we have an ℓ-block $\mathbf{B}_1(\overline{\chi})$ with decomposition matrix:

			Conditions	Number
$1_G(\chi)$	1	0	$d_\ell(\chi) = \overline{\chi}$	1
$\mathrm{St}_G(\chi)$	1	1	$d_\ell(\chi) = \overline{\chi}$	1
$i_P^G(\sigma_{S_1,\chi_1} \otimes \chi_2)$	0	1	$d_\ell(\theta) = (\overline{\chi} \circ \xi_{q+1}) \otimes \overline{\chi}$	$\ell^a - 1$

There are $q-1$ distinct ℓ-blocks $\mathbf{B}_1(\overline{\chi})$.

(2) **The ℓ-blocks $\mathbf{B}_2(\overline{\chi}_1, \overline{\chi}_2)$.** Let $\overline{\chi}_i$, $i = 1, 2$ be distinct irreducible ℓ-modular characters of $F^\times$. Associated to the pair $(\overline{\chi}_1, \overline{\chi}_2)$ we have an ℓ-block $\mathbf{B}_2(\overline{\chi}_1, \overline{\chi}_2)$ with decomposition matrix:

			Conditions	Number
$i_P^G(1_H(\chi_1) \otimes \chi_2)$	1	0	$d_\ell(\chi_i) = \overline{\chi}_i$	1
$i_P^G(\mathrm{St}_H(\chi_1) \otimes \chi_2)$	1	1	$d_\ell(\chi_i) = \overline{\chi}_i$	1
$i_P^G(\sigma_{S_1,\chi_1} \otimes \chi_2)$	0	1	$d_\ell(\theta) = (\overline{\chi}_1 \circ \xi_{q+1}) \otimes \overline{\chi}_2$	$\ell^a - 1$

There are $(q-1)(q-2)$ distinct ℓ-blocks $\mathbf{B}_2(\overline{\chi}_1, \overline{\chi}_2)$.

(3) **The ℓ-blocks $\mathbf{B}_3(\overline{\chi}_1, \overline{\chi}_2)$.** Let $\overline{\chi}_1$ be an irreducible ℓ-modular character of $E^\times$ such that $\overline{\chi}_1^{q-1} \neq 1$; and let $\overline{\chi}_2$ be an irreducible ℓ-modular character of $F^\times$. Associated to the pair $(\overline{\chi}_1, \overline{\chi}_2)$ we have an ℓ-block $\mathbf{B}_3(\overline{\chi}_1, \overline{\chi}_2)$ with decomposition matrix:

		Conditions	Number
$i_P^G(\sigma_{S_1,\chi_1} \otimes \chi_2)$	1	$d_\ell(\theta) = \overline{\chi}_1 \otimes \overline{\chi}_2$	ℓ^a

There are $\left(\frac{q+1}{\ell^a} - 1\right)(q-1)^2$ distinct ℓ-blocks $\mathbf{B}_3(\overline{\chi}_1, \overline{\chi}_2)$.

(4) **ℓ-blocks of defect zero.** All other ℓ-modular representations are in ℓ-blocks of defect zero.

3.3. Decomposition matrices if $\ell \neq 2, 3$ and $\ell \mid q^2+q+1$.
Let $\ell^a \mid\mid q^2+q+1$. There are three types of ℓ-blocks.

(1) **The ℓ-blocks $\mathbf{B}_1(\overline{\chi})$.** Let $\overline{\chi}$ be an irreducible ℓ-modular character of $F^\times$. Associated to $\overline{\chi}$ we have an ℓ-block $\mathbf{B}_1(\overline{\chi})$ with decomposition matrix:

				Conditions	Number
$1_G(\chi)$	1	0	0	$d_\ell(\chi) = \overline{\chi}$	1
$\nu_G(\chi)$	1	1	0	$d_\ell(\chi) = \overline{\chi}$	1
$\mathrm{St}_G(\chi)$	0	1	1	$d_\ell(\chi) = \overline{\chi}$	1
$\sigma_{T_1,\theta}$	0	0	1	$d_\ell(\theta) = \overline{\chi} \circ \xi_{q^2+q+1}$	$\frac{\ell^a - 1}{2}$

There are $q-1$ distinct ℓ-blocks $\mathbf{B}_1(\overline{\chi})$.

(2) **The ℓ-blocks $\mathbf{B}_2(\overline{\chi})$.** Let $\overline{\chi}$ be an irreducible ℓ-modular character of $F_3^\times$ such that $\overline{\chi}^{q-1} \neq 1$. Associated to $\overline{\chi}$ we have an ℓ-block $\mathbf{B}_2(\overline{\chi})$ with decomposition matrix:

		Conditions	Number
$\sigma_{T_1,\theta}$	1	$d_\ell(\theta) = \overline{\chi}$	ℓ^a

There are $\frac{q^3-1}{\ell^a} - (q-1)$ distinct ℓ-blocks $\mathbf{B}_2(\overline{\chi})$.

(3) **ℓ-blocks of defect zero.** All other ℓ-modular representations are in ℓ-blocks of defect zero.

4. Decomposition matrices of $\mathrm{U}_2(E/F)$

The decomposition matrices of $\mathrm{SU}_2(E/F)$ coincide with the decomposition matrices of $\mathrm{SL}_2(F)$ because the groups are isomorphic. Using Clifford theory for Brauer Characters, see Appendix B Section 1, we can work out the decomposition matrices for $\mathrm{U}_2(E/F)$. In Appendix B Section 3 we explain in detail how to apply Clifford theory in this context, but with the more difficult case of $\mathrm{SU}_3(E/F)$ and $\mathrm{U}_3(E/F)$.

4.1. Decomposition matrices if $\ell \neq 2$ and $\ell \mid q-1$.

Let $\ell^a \mid\mid q-1$. There are three types of ℓ-block.

(1) **The ℓ-blocks $\mathbf{B}_1(\overline{\chi})$.** Let $\overline{\chi}$ be an irreducible ℓ-modular character of E^1. Associated to $\overline{\chi}$ we have an ℓ-block $\mathbf{B}_1(\overline{\chi})$ with decomposition matrix:

			Conditions	Number
$1_G(\chi)$	1	0	$d_\ell(\chi) = \overline{\chi}$	1
$\mathrm{St}_G(\chi)$	0	1	$d_\ell(\chi) = \overline{\chi}$	1
$i_{B_0}^G(\theta)$	1	1	$d_\ell(\theta) = \overline{\chi} \circ \xi_{q-1}$	$\frac{(\ell^a - 1)}{2}$

There are $q+1$ irreducible ℓ-modular characters of E^1, hence $q+1$ distinct ℓ-blocks $\mathbf{B}_1(\overline{\chi})$.

(2) **The ℓ-blocks $\mathbf{B}_2(\overline{\chi})$.** Let $\overline{\chi}$ be an irreducible ℓ-modular character of $E^\times$ such that $\overline{\chi}^{q+1} \neq 1$. Associated to $\overline{\chi}$ we have an ℓ-block $\mathbf{B}_2(\overline{\chi})$ with decomposition matrix:

		Conditions	Number
$i_{B_0}^G(\theta)$	1	$d_\ell(\theta) = \overline{\chi}$	ℓ^a

There are $(q+1)\left(\frac{q-1}{\ell^a} - 1\right)$ distinct ℓ-blocks $\mathbf{B}_2(\overline{\chi})$.

(3) **ℓ-blocks of defect zero.** The supercuspidal representations $\sigma_{T_1,\theta}$ are in ℓ-blocks of defect zero. We write $\overline{\sigma}_{T_1,\overline{\theta}}$ for $d_\ell(\sigma_{T_1,\theta})$.

4.2. Decomposition matrices if $\ell \neq 2$ and $\ell \mid q+1$.

Let $\ell^a \mid\mid q+1$. There are three types of ℓ-block.

(1) **The ℓ-blocks $\mathbf{B}_1(\overline{\chi})$.** Let $\overline{\chi}$ be an irreducible ℓ-modular character of E^1. Associated to $\overline{\chi}$ we have an ℓ-block $\mathbf{B}_1(\overline{\chi})$ with decomposition matrix:

			Conditions	Number
$1_G(\chi)$	1	0	$d_\ell(\chi) = \overline{\chi}$	ℓ^a
$\mathrm{St}_G(\chi)$	1	1	$d_\ell(\chi) = \overline{\chi}$	ℓ^a
$\sigma_{T_1,\theta}$	0	1	$d_\ell(\theta) = \overline{\chi} \otimes \overline{\chi}$	$\frac{\ell^a(\ell^a-1)}{2}$

There are $\frac{q+1}{\ell^a}$ irreducible ℓ-modular characters of E^1, hence $\frac{q+1}{\ell^a}$ distinct ℓ-blocks $\mathbf{B}_1(\overline{\chi})$.

(2) **The ℓ-blocks $\mathbf{B}_2(\overline{\chi}_1, \overline{\chi}_2)$.** Let $\overline{\chi}_i$, $i = 1, 2$ be irreducible ℓ-modular characters of E^1 such that $\overline{\chi}_1 \neq \overline{\chi}_2$. Associated to the pair $(\overline{\chi}_1, \overline{\chi}_2)$ we have an ℓ-block $\mathbf{B}_2(\overline{\chi}_1, \overline{\chi}_2)$ with decomposition matrix:

		Conditions	Number
$\sigma_{T_1,\theta}$	1	$d_\ell(\theta) =$ $\overline{\chi}_1 \otimes \overline{\chi}_2$	ℓ^{2a}

There are $\frac{q+1}{\ell^a}\left(\frac{q+1}{\ell^a} - 1\right)$ distinct ℓ-blocks $\mathbf{B}_2(\overline{\chi}_1, \overline{\chi}_2)$.

(3) **ℓ-blocks of defect zero.** The irreducible principal series representations $i_{B_0}^G(\theta)$ are in ℓ-blocks of defect zero.

5. Decomposition matrices of $\mathrm{U}_3(E/F)$

We use the notation of Chapter 2 Section 8 to describe the decomposition matrices of $\mathrm{U}_3(E/F)$ which are extracted from [**Gec90**] in Appendix B Section 3.

5.1. Decomposition matrices if $\ell \neq 2$ and $\ell \mid q-1$.

Let $\ell^a \mid\mid q-1$. There are four types of ℓ-blocks.

(1) **The ℓ-blocks $\mathbf{B}_1(\overline{\chi})$.** Let $\overline{\chi}$ be an irreducible ℓ-modular character of E^1. Associated to $\overline{\chi}$ we have an ℓ-block $\mathbf{B}_1(\overline{\chi})$ with decomposition matrix:

			Conditions	Number
$1_G(\chi)$	1	0	$d_\ell(\chi) = \overline{\chi}$	1
$\mathrm{St}_G(\chi)$	0	1	$d_\ell(\chi) = \overline{\chi}$	1
$i_{B_0}^G(\theta)$	1	1	$d_\ell(\theta) =$ $(\overline{\chi} \circ \xi_{q-1}) \otimes \overline{\chi}$	$\frac{(\ell^a-1)}{2}$

There are $q+1$ distinct ℓ-blocks $\mathbf{B}_1(\overline{\chi})$.

(2) **The ℓ-blocks $\mathbf{B}_2(\overline{\chi}_1, \overline{\chi}_2)$.** Let $\overline{\chi}_i$, $i = 1, 2$ be distinct irreducible ℓ-modular characters of E^1. Associated to the pair $(\overline{\chi}_1, \overline{\chi}_2)$ we have an ℓ-block $\mathbf{B}_2(\overline{\chi}_1, \overline{\chi}_2)$ with decomposition matrix:

			Conditions	Number
$R_L^G(1_H(\chi_1) \otimes 1)(\chi_2)$	1	0	$d_\ell(\chi_i) = \overline{\chi}_i$	1
$R_L^G(\mathrm{St}_H(\chi_1) \otimes 1)(\chi_2)$	0	1	$d_\ell(\chi_i) = \overline{\chi}_i$	1
$i_{B_0}^G(\theta)$	1	1	$d_\ell(\theta(\mathrm{diag}(x, y, x^{-q}))) =$ $(\overline{\chi}_1 \circ \xi_{q-1}(x))\overline{\chi}_2(xx^{-q}y)$	$\frac{(\ell^a-1)}{2}$

There are $q(q+1)$ distinct ℓ-blocks $\mathbf{B}_2(\overline{\chi}_1, \overline{\chi}_2)$.

(3) **The ℓ-blocks $\mathbf{B}_3(\overline{\chi}_1, \overline{\chi}_2)$.** Let $\overline{\chi}_1$ be an irreducible ℓ-modular character of $E^\times$ such that $\overline{\chi}_1^{q+1} \neq 1$; and let $\overline{\chi}_2$ be an irreducible ℓ-modular character of E^1. Associated to the pair $(\overline{\chi}_1, \overline{\chi}_2)$ we have an ℓ-block $\mathbf{B}_3(\overline{\chi}_1, \overline{\chi}_2)$ with decomposition matrix:

		Conditions	Number
$i_{B_0}^G(\theta)$	1	$d_\ell(\theta) = \overline{\chi}_1 \otimes \overline{\chi}_2$	ℓ^{2a}

There are $(q+1)^2\left(\frac{q-1}{\ell^a} - 1\right)$ distinct ℓ-blocks $\mathbf{B}_3(\overline{\chi}_1, \overline{\chi}_2)$.

(4) **ℓ-blocks of defect zero.** All other ℓ-modular representations are in ℓ-blocks of defect zero.

5.2. Decomposition matrices if $\ell \neq 2, 3$ and $\ell \mid q^2 - q + 1$.

Let $\ell^a \mid\mid q^2 - q + 1$. There are three types of ℓ-blocks.

(1) **The ℓ-blocks $\mathbf{B}_1(\overline{\chi})$.** Let $\overline{\chi}$ be an irreducible ℓ-modular character of E^1. Associated to $\overline{\chi}$ we have an ℓ-block $\mathbf{B}_1(\overline{\chi})$ with decomposition matrix:

				Conditions	Number
$1_G(\chi)$	1	0	0	$d_\ell(\chi) = \overline{\chi}$	1
$\mathrm{St}_G(\chi)$	1	1	0	$d_\ell(\chi) = \overline{\chi}$	1
ν_χ	0	0	1	$d_\ell(\chi) = \overline{\chi}$	1
$\sigma_{T_1,\theta}$	0	1	1	$d_\ell(\theta) = \overline{\chi} \circ \xi_{q^2-q+1}$	$\ell^a - 1$

There are $q+1$ distinct ℓ-blocks $\mathbf{B}_1(\overline{\chi})$. We write $\overline{\nu}_{\overline{\chi}}$ for $d_\ell(\nu_\chi)$ and

$$d_\ell(\sigma_{T_1,\theta}) = \overline{\nu}_{\overline{\chi}} + \overline{\sigma}^-_{T_1,\overline{\theta}}.$$

The representation $\overline{\sigma}^-_{T_1,\overline{\theta}}$ is an example of a cuspidal ℓ-modular representation which does not lift. It is non-supercuspidal because it is a subquotient of $i_B^G(\overline{\chi} \otimes \overline{\chi} \otimes \overline{\chi})$ appearing in the reduction of $\mathrm{St}_G(\chi)$.

(2) **The ℓ-blocks $\mathbf{B}_2(\overline{\chi})$.** Let $\overline{\chi}$ be an irreducible ℓ-modular character of E_3^1 such that $\overline{\chi}^{q+1} \neq 1$. Associated to $\overline{\chi}$ we have an ℓ-block $\mathbf{B}_2(\overline{\chi})$ with decomposition matrix:

		Conditions	Number
$\sigma_{T_1,\theta}$	1	$d_\ell(\theta) = \overline{\chi}$	ℓ^a

There are $\frac{q^3+1}{\ell^a} - (q+1)$ distinct ℓ-blocks $\mathbf{B}_2(\overline{\chi})$.

(3) **ℓ-blocks of defect zero.** All other ℓ-modular representations are in ℓ-blocks of defect zero.

5.3. Decomposition matrices if $\ell \neq 2, 3$ and $\ell \mid q + 1$.

Let $\ell^a \mid\mid q + 1$. There are five types of ℓ-blocks.

(1) **The ℓ-blocks $\mathbf{B}_1(\overline{\chi})$.** Let $\overline{\chi}$ be an irreducible ℓ-modular character of E^1. Associated to $\overline{\chi}$ we have an ℓ-block $\mathbf{B}_1(\overline{\chi})$ with decomposition matrix:

				Conditions	Number
$1_G(\chi)$	1	0	0	$d_\ell(\chi) = \overline{\chi}$	ℓ^a
ν_χ	0	1	0	$d_\ell(\chi) = \overline{\chi}$	ℓ^a
$\mathrm{St}_G(\chi)$	1	2	1	$d_\ell(\chi) = \overline{\chi}$	ℓ^a
$R_L^G(1_H(\chi_1) \otimes 1)(\chi_2)$	1	1	0	$d_\ell(\chi_i) = \overline{\chi}$	$\ell^a(\ell^a - 1)$
$R_L^G(\mathrm{St}_H(\chi_1) \otimes 1)(\chi_2)$	1	1	1	$d_\ell(\chi_i) = \overline{\chi}$	$\ell^a(\ell^a - 1)$
$\sigma_{T_2,\theta}$	0	0	1	$d_\ell(\theta) = \overline{\chi} \otimes \overline{\chi} \otimes \overline{\chi}$	$\frac{\ell^a(\ell^a-1)(\ell^a-2)}{6}$

There are $\frac{q+1}{\ell^a}$ distinct ℓ-blocks $\mathbf{B}_1(\overline{\chi})$.

(2) **The ℓ-blocks $\mathbf{B}_2(\overline{\chi}_1, \overline{\chi}_2)$.** Let $\overline{\chi}_i$, $i = 1, 2$ be distinct irreducible ℓ-modular characters of E^1. Associated to the pair $(\overline{\chi}_1, \overline{\chi}_2)$ we have an ℓ-block $\mathbf{B}_2(\overline{\chi}_1, \overline{\chi}_2)$ with decomposition matrix:

			Conditions	Number
$R_L^G(1_H(\chi_1) \otimes 1)(\chi_2)$	1	0	$d_\ell(\chi_i) = \overline{\chi}_i$	ℓ^{2a}
$R_L^G(\mathrm{St}_H(\chi_1) \otimes 1)(\chi_2)$	1	1	$d_\ell(\chi_i) = \overline{\chi}_i$	ℓ^{2a}
$\sigma_{T_2,\theta}$	0	1	$d_\ell(\theta) = \overline{\chi}_1 \otimes \overline{\chi}_1 \otimes \overline{\chi}_2$	$\frac{\ell^{2a}(\ell^a-1)}{2}$

There are $\frac{q+1}{\ell^a}\left(\frac{q+1}{\ell^a} - 1\right)$ distinct ℓ-blocks $\mathbf{B}_2(\overline{\chi}_1, \overline{\chi}_2)$.

(3) **The ℓ-blocks $\mathbf{B}_3(\overline{\chi}_1, \overline{\chi}_2, \overline{\chi}_3)$.** Let $\overline{\chi}_i$, $i = 1, 2, 3$ be pairwise distinct irreducible ℓ-modular characters of E^1. Associated to the triple $(\overline{\chi}_1, \overline{\chi}_2, \overline{\chi}_3)$ we have an ℓ-block $\mathbf{B}_3(\overline{\chi}_1, \overline{\chi}_2, \overline{\chi}_3)$ with decomposition matrix:

		Conditions	Number
$\sigma_{T_2,\theta}$	1	$d_\ell(\theta) = \overline{\theta}_1 \otimes \overline{\theta}_2 \otimes \overline{\theta}_3$	ℓ^{3a}

There are $\frac{q+1}{\ell^a}\left(\frac{q+1}{\ell^a} - 1\right)\left(\frac{q+1}{\ell^a} - 2\right)$ distinct ℓ-blocks $\mathbf{B}_3(\overline{\chi}_1, \overline{\chi}_2, \overline{\chi}_3)$.

(4) **The ℓ-blocks $\mathbf{B}_4(\overline{\chi})$.** Let $\overline{\chi}$ be an irreducible ℓ-modular character of E^1. Associated to $\overline{\chi}$ we have an ℓ-block $\mathbf{B}_4(\overline{\chi})$ with decomposition matrix:

		Conditions	Number
$i_{B_0}^G \theta$	1	$d_\ell(\theta) = (\overline{\chi} \circ \xi_{q-1}) \otimes \overline{\chi}$	ℓ^{2a}

There are $\frac{q+1}{\ell^a}$ distinct ℓ-blocks $\mathbf{B}_4(\overline{\chi})$.

(5) **ℓ-blocks of defect zero.** All other ℓ-modular representations are in ℓ-blocks of defect zero.

CHAPTER 4

Level zero representations

In this chapter we study irreducible representations which have nontrivial invariants under the pro-unipotent radical of a parahoric subgroup. These are called level zero representations.

Our initial observations apply to a general reductive p-adic group. Then we specialise to unramified unitary groups. This has the advantage that for all parahoric subgroups G_z we have $G_z^+ = G_z$ and the affine Weyl group is a Coxeter group rather than an extended Coxeter group. We then specialise to an unramified unitary group in three variables $\mathrm{U}(2,1)(E/F)$. First, because both maximal parahoric subgroups of $\mathrm{U}(2,1)(E/F)$ admit Iwasawa decompositions. Then because, from Chapters 2 and 3, we understand the ℓ-adic and ℓ-modular representations of the finite reductive groups M_z which appear as quotients of the maximal parahoric subgroups. For example, it is important to know that the supercuspidal support of an ℓ-modular representation of M_z is unique up to conjugacy. Finally we specialise to an unramified p-adic unitary group in three variables $\mathrm{U}(2,1)(E/F)$, where F is of characteristic zero, so that we can apply results of [**Dat05**].

We partition the irreducible level zero ℓ-modular representations of an unramified p-adic unitary group in three variables $\mathrm{U}(2,1)(E/F)$ by supercuspidal support. We do this in two steps:

(1) Giving a complete list of the irreducible cuspidal level zero ℓ-modular representations with each representation explicitly produced by compact induction from an irreducible ℓ-modular representation of a compact open subgroup.
(2) Describing the decomposition of the ℓ-modular representations which are parabolically induced from an irreducible ℓ-modular representation of the standard torus and which have irreducible cuspidal level zero subquotients.

1. Irreducible level zero representations

An irreducible representation $(\pi, \mathcal{V})$ of G has level zero if there exists a parahoric subgroup G_x of G such that

$$\mathcal{V}^{G_x^1} \neq \{0\}.$$

It is equivalent to ask for a maximal parahoric subgroup G_x with $\pi^{G_x^1} \neq \{0\}$, because $G_y \subseteq G_x$ implies that $G_x^1 \subseteq G_y^1$.

Suppose $(\pi, \mathcal{V})$ is an admissible representation such that $\mathcal{V}^{G_x^1} \neq \{0\}$. By normality, G_x acts on $\mathcal{V}^{G_x^1}$. Thus the finite reductive group M_x acts on $\mathcal{V}^{G_x^1}$. By admissibility of π, this is a finite dimensional representation of M_x. Let σ be an irreducible M_x-subrepresentation of $\mathcal{V}^{G_x^1}$ then

$$\begin{aligned}\{0\} \neq \operatorname{Hom}_{M_x}(\sigma, (\operatorname{Res}^G_{G_x} \pi)^{G_x^1}) &\simeq \operatorname{Hom}_{G_x}(\operatorname{infl}^{G_x}_{M_x} \sigma, \operatorname{Res}^G_{G_x} \pi) \\ &\simeq \operatorname{Hom}_G(\operatorname{ind}^G_{G_x} \circ \operatorname{infl}^{G_x}_{M_x} \sigma, \pi),\end{aligned}$$

by reciprocity: we say that π contains (G_x, σ). Suppose that π is irreducible. Then π is a quotient of the induced representation $\operatorname{ind}^G_{G_x} \circ \operatorname{infl}^{G_x}_{M_x} \sigma$. In general the representation $\operatorname{ind}^G_{G_x} \circ \operatorname{infl}^{G_x}_{M_x} \sigma$ is not irreducible.

Fix a chamber in the reduced building $\mathcal{B}(G)$. The standard parahoric subgroups are the parahoric subgroups that only fix points in the closure of this chamber. If π contains (G_x, σ), then ${}^g\pi$ contains $(G_{gx}, {}^g\sigma)$. Thus, because $\pi \simeq {}^g\pi$, it is enough to consider only the standard parahoric subgroups.

2. Minimality

Let $(\pi, \mathcal{V})$ be an irreducible representation of G of level zero. Thus there exists a standard maximal parahoric subgroup G_x of G such that $\mathcal{V}^{G_x^1} \neq \{0\}$. Let

$$L_0^{G_x}(\pi) = \{\text{standard parahoric subgroups } G_y \subseteq G_x : \ \mathcal{V}^{G_y^1} \neq \{0\}\}.$$

We are interested in the minimal elements in $L_0^{G_x}(\pi)$ under the partial order of inclusion of parahoric subgroups.

Lemma 2.1. Let $G_z \in L_0^{G_x}(\pi)$, and let σ be an irreducible representation of M_z such that π contains (G_z, σ). If G_z is minimal under the partial order of inclusion on $L_0^{G_x}(\pi)$ then σ is cuspidal.

Proof: Assume that σ is not cuspidal. Then there exists a proper standard parabolic subgroup P of M_z with Levi decomposition $P = L \ltimes N$ such that $\operatorname{Res}^{M_z}_N \sigma$ contains the trivial representation 1_N of N, i.e.

$$\operatorname{Hom}_N(1_N, \operatorname{Res}^{M_z}_N \sigma) \neq \{0\}.$$

Let G_P be the parahoric subgroup of G equal to the preimage in G_z of P under the projection $G_z \to M_z$.

$$\begin{CD} 1 @>>> G_z^1 @>>> G_z @>>> M_z @>>> 1 \\ @. @| @AAA @AAA @. \\ 1 @>>> G_z^1 @>>> G_P @>>> P @>>> 1 \end{CD}$$

The preimage of N under the projection from G_P to P is the pro-p unipotent radical G_P^1 of G_P. By reciprocity

$$\begin{aligned} \{0\} \neq \mathrm{Hom}_N(1_N, \mathrm{Res}_N^{M_z} \sigma) &\simeq \mathrm{Hom}_N(1_N, \mathrm{inv}_{G_P^1} \circ \mathrm{Res}_{G_P^1}^{G_z}(\mathrm{infl}_{G_z^1} \sigma)) \\ &\simeq \mathrm{Hom}_{G_P^1}(1_{G_P^1}, \mathrm{Res}_{G_P^1}^{G_z}(\mathrm{infl}_{G_z^1} \sigma)). \end{aligned}$$

Because σ is contained in π

$$\{0\} \neq \mathrm{Hom}_{M_z}(\sigma, (\mathrm{Res}_{G_z}^{G} \pi)^{G_z^1}) \simeq \mathrm{Hom}_{G_z}(\mathrm{infl}_{G_z^1} \sigma, \mathrm{Res}_{G_z}^{G} \pi)$$

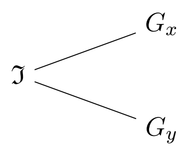
Thus, because G_P^1 is pro-p, $\mathrm{Res}_{G_P^1}^{G_z} \mathrm{infl}_{G_z^1} \sigma$ is a direct summand of $\mathrm{Res}_{G_P^1}^{G} \pi$. Hence

$$\mathcal{V}^{G_P^1} \simeq \mathrm{Hom}_{G_P^1}(1_{G_P^1}, \mathrm{Res}_{G_P^1}^{G} \pi) \neq \{0\}$$

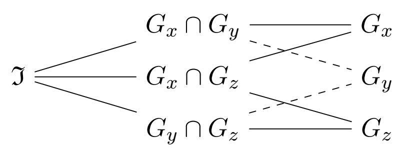
contradicting the minimality of G_z. □

When $G = \mathrm{GL}_n(F)$, all maximal parahoric subgroups are conjugate in G and can be identified with $\mathrm{GL}_n(\mathcal{O}_F)$. Thus it is enough to consider one set, $L_0^{\mathrm{GL}_n(\mathcal{O}_F)}(\pi)$.

When G is a unitary group of dimension greater than or equal to three, there are multiple maximal parahoric subgroups. The minimal parahoric subgroups, the Iwahori subgroups, in a reductive group G are always all conjugate in G. When $G = \mathrm{U}(2,1)(E/F)$, there are two maximal parahoric subgroups up to conjugacy. We fix a choice of Iwahori subgroup $\mathfrak{I}$ then there are two maximal parahorics containing $\mathfrak{I}$.



The situation is more complex in higher dimensions: for example when $G = \mathrm{U}(2,2)(E/F)$ and E/F is unramified there are three maximal parahoric subgroups up to conjugacy.



Not being able to work inside a single fixed maximal parahoric subgroup makes classification arguments for level zero representations more difficult.

3. Decomposition of categories

Let $\mathfrak{R}_R(G)$ denote the category of smooth R-representations of G. In this section we review briefly some of the ℓ-adic theory and then split off the subcategory of level zero representations from $\mathfrak{R}_R(G)$ in the ℓ-modular case. For ℓ-adic representations the idea is to try to split $\mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(G)$ by cuspidal support, but one finds that this is slightly too fine.

3.1. The ℓ-adic Bernstein decomposition. Let $X(M)^0$ be the set of unramified characters of M; that is the characters of M which are trivial on every compact subgroup of M. Let $(\pi,\mathcal{V})$ be a smooth R-representation of G with supercuspidal support $[M,\sigma]$. On these pairs we define an equivalence relation, called inertial equivalence, by $[M_1,\sigma_1]$ is equivalent to $[M_2,\sigma_2]$ if and only if there exist $g\in G$ and $\chi\in X(M_1)^0$ such that ${}^gM_2=M_1$ and ${}^g\sigma_2\simeq\sigma_1\otimes\chi$. The equivalence classes are called inertial classes and the set of inertial classes is called the Bernstein spectrum $\mathfrak{B}(G)$.

We say $\mathfrak{R}_R(G)$ is a direct product of subcategories $\mathfrak{R}^i_R(G)$, for $i\in I$,

$$\mathfrak{R}_R(G)=\prod_{i\in I}\mathfrak{R}^i_R(G),$$

if for all $(\pi,\mathcal{V}),(\sigma,W)\in\mathfrak{R}_R(G)$,

(1) there is a unique decomposition into subrepresentations

$$\mathcal{V}=\bigoplus_{i\in I}\mathcal{V}^i$$

with $\mathcal{V}^i$ an element of $\mathfrak{R}^i_R(G)$;

(2) if

$$W=\bigoplus_{i\in I}W^i$$

is the decomposition of W given by (1), then

$$\mathrm{Hom}_G(\mathcal{V},W)=\prod_{i\in I}\mathrm{Hom}_G(\mathcal{V}^i,W^i).$$

For ℓ-adic representations of p-adic groups the fundamental result is the optimal decomposition of Bernstein:

Theorem 3.1 (Bernstein Decomposition, [**DKV84**, Chapter 1]). *For $\mathfrak{s}\in\mathfrak{B}(G)$ let $\mathfrak{R}^{\mathfrak{s}}_{\overline{\mathbb{Q}}_\ell}(G)$ denote the full subcategory of $\mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(G)$ of smooth representations all of whose irreducible subquotients have inertial support $\mathfrak{s}$. Then*

$$\mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(G)=\prod_{\mathfrak{s}\in\mathfrak{B}(G)}\mathfrak{R}^{\mathfrak{s}}_{\overline{\mathbb{Q}}_\ell}(G).$$

Lemma 3.2. *Let G be a classical group and let π be an irreducible cuspidal representation of G. Then π is integral.*

Proof: This follows as the centre of G is compact. A character is integral if and only if it takes values in $\overline{\mathbb{Z}}_\ell^\times$. By Chapter 1 Theorem 3.4, π has a central character ω_π. As the centre of G is compact and the image of a compact group under a smooth homomorphism so compact, ω_π is integral. Therefore, by Chapter 1 Theorem 4.3, π is integral. □

A similar proof when $G=\mathrm{GL}_n(F)$ shows that every irreducible cuspidal representation is inertially equivalent to an integral representation.

3.2. The ℓ-modular decomposition by level. Optimal decompositions of $\mathfrak{R}_{\overline{\mathbb{F}}_\ell}(G)$ are not known in general. However it is possible to decompose $\mathfrak{R}_{\overline{\mathbb{F}}_\ell}(G)$ by normalised level. In particular it is possible to decompose $\mathfrak{R}_{\overline{\mathbb{F}}_\ell}(G)$ into a product of the level zero representations and the positive level representations.

A smooth R-representation $(\pi, \mathcal{V})$ of G has level zero if V is generated by the union of its G_x^1-invariant vectors as G_x runs over the maximal parahoric subgroups of G.

A smooth R-representation $(\pi, \mathcal{V})$ of G has positive level if, for any maximal parahoric subgroup G_x of G, $\mathcal{V}^{G_x^1} = \{0\}$.

Theorem 3.3 ([**Dat09**, Proposition 6.3]). Let G be a reductive p-adic group, $\mathfrak{R}_R^0(G)$ the full subcategory of $\mathfrak{R}_R(G)$ consisting of representations of level zero and $\mathfrak{R}_R^{>0}(G)$ the full subcategory of $\mathfrak{R}_R(G)$ consisting of representations of positive level. Then

$$\mathfrak{R}_R(G) = \mathfrak{R}_R^0(G) \times \mathfrak{R}_R^{>0}(G).$$

Furthermore the functors of parabolic induction and parabolic restriction respect the decomposition; i.e. take level zero (resp. positive level) representations to level zero (resp. positive level) representations.

4. Level zero R-types

In this section we introduce R-types.

Definition 4.1.

(1) An R-type of G is a pair (K, σ) consisting of a compact open subgroup K of G and an irreducible smooth R-representation σ of K.
(2) A smooth R-representation π of G is said to contain the R-type (K, σ) if
$$\mathrm{Hom}_K(\sigma, \mathrm{Res}_K^G(\pi)) \neq \{0\}.$$
(3) Two R-types (K_1, σ_1) and (K_2, σ_2) of G are called equivalent if $\mathrm{ind}_{K_1}^G \sigma_1 \simeq \mathrm{ind}_{K_2}^G \sigma_2$.

Definition 4.2.

(1) An R-type of level zero is an R-type of the form (G_z, σ) where G_z is a parahoric subgroup of G and σ is an irreducible representation of the finite reductive group M_z inflated to G_z.
(2) An R-type of level zero (G_z, σ) is called maximal if G_z is a maximal parahoric subgroup of G.
(3) An R-type of level zero (G_z, σ) is called cuspidal if σ is a cuspidal representation of M_z.
(4) An R-type of level zero (G_z, σ) is called supercuspidal if σ is a supercuspidal representation of M_z.

Thus, in terms of R-types, Lemma 2.1 implies:

Lemma 4.3. Let π be an irreducible representation of level zero. Then π contains a cuspidal R-type of level zero.

4.1. Spherical Hecke algebras. Let K be a compact open subgroup of G, R a commutative ring with unit, (σ, W) an irreducible smooth R-representation of K. The spherical Hecke algebra, $\mathcal{H}_R(G, K, \sigma)$, of σ is the R-module consisting of the set of functions $f : G \to \mathrm{End}_R(W)$ which satisfy:

(1) The support of f, $\mathrm{supp}(f)$, is a finite union of double cosets Kh_iK with $h_i \in G$.
(2) The function f transforms by σ on the left and the right, i.e. for all $k_1, k_2 \in K$ and all $g \in G$
$$f(k_1gk_2) = \sigma(k_1)f(g)\sigma(k_2).$$

The product $f_1 \star f_2$ of $f_1, f_2 \in \mathcal{H}_R(G, K, \sigma)$ is defined by convolution
$$f_1 \star f_2(h) = \sum_{G/K} f_1(g)f_2(g^{-1}h),$$
for all $h \in G$.

Let $\mathbf{I}_g(\sigma) = \mathrm{Hom}_K(\sigma, \mathrm{ind}_{K\cap {}^gK}^K\, {}^g\sigma)$ and
$$\mathbf{I}_G(\sigma) = \{g \in G : \mathbf{I}_g(\sigma) \neq \{0\}\}.$$
By reciprocity, $\mathbf{I}_g(\sigma) \simeq \mathrm{Hom}_{K\cap\, {}^gK}(\sigma, {}^g\sigma)$.

LEMMA 4.4 ([**Vig96**, Chapter 1, Section 8.10]). Let $f \in \mathcal{H}(G, K, \sigma)$ be supported on $Kg^{-1}K$. Then $f(g^{-1}) \in \mathbf{I}_g(\sigma)$ and for each $\chi \in \mathbf{I}_g(\sigma)$ there exists a unique $f \in \mathcal{H}_R(G, K, \sigma)$ supported on $Kg^{-1}K$ with $f(g^{-1}) = \chi$.

PROOF: Let $g \in G$, $k \in K \cap {}^gK$ and suppose there exists $f \in \mathcal{H}_R(G, K, \sigma)$ supported on $Kg^{-1}K$ then $f(g^{-1}) \neq 0$ and
$$f(g^{-1})\sigma(k) = f(g^{-1}kgg^{-1}) = {}^g\sigma(k)f(g^{-1}).$$
Hence $f(g^{-1}) \in \mathbf{I}_g(\sigma)$.

Let $\chi \in \mathbf{I}_g(\sigma)$. Define $f : G \to \mathrm{End}_R(W)$ by
$$f(h) = \begin{cases} 0 & \text{if } g \notin Kg^{-1}K; \\ \sigma(k_1)\chi\sigma(k_2) & \text{if } h = k_1g^{-1}k_2 \text{ with } k_1, k_2 \in K. \end{cases}$$
Then $f \in \mathcal{H}_R(G, K, \sigma)$, has support $Kg^{-1}K$ and is the unique element in $\mathcal{H}_R(G, K, \sigma)$ with support $Kg^{-1}K$ and $f(g^{-1}) = \chi$. □

The R-algebra $\mathcal{H}(G, K, \sigma)$ is isomorphic to $\mathrm{End}_G(\mathrm{ind}_K^G\, \sigma)$ by [**Vig96**, Section 8.5 and 8.6(f)] where $\mathrm{End}_G(\mathrm{ind}_K^G\, \sigma)$ is given its natural multiplication of composition.

4.2. G-Covers for R-types. In the framework of R-types G-covers are a tool for studying parabolic induction. Let P be a parabolic subgroup of G with Levi decomposition $P = M \ltimes N$, and let $\overline{P}$ be the opposite parabolic subgroup with Levi decomposition $\overline{P} = M \ltimes \overline{N}$. Let K be a compact open subgroup of G.

DEFINITION 4.5. An element z of the centre of M is called strongly (P, K)-positive if

(1) $zK^+z^{-1} \subset K^+$ and $zK^-z^{-1} \supset K^-$ where $K^- = K \cap \overline{N}$ and $K^+ = K \cap N$.
(2) For all compact subgroups H_1, H_2 of N (resp. $\overline{N}$) there exists a positive (resp. negative) integer m such that $z^mH_1z^{-m} \subset H_2$.

DEFINITION 4.6. Let (K_M, ρ_M) be an R-type of M. An R-type (K, ρ) of G is called a G-cover of (K_M, ρ_M) relative to the parabolic subgroup P of G if the following three properties are satisfied:

(1) We have an Iwahori decomposition:

$$K \cap M = K_M$$
$$K = K^- K_M K^+$$

where $K^- = K \cap \overline{N}$ and $K^+ = K \cap N$.

(2) $\mathrm{Res}^K_{K_M}(\rho) = \rho_M$, and $\mathrm{Res}^K_{K^+}(\rho)$ and $\mathrm{Res}^K_{K^-}(\rho)$ are both multiples of the trivial representation.

(3) There exists a strongly (P,K)-positive element z of the centre of M such that the double coset KzK supports an invertible element of $\mathcal{H}_R(G,K,\rho)$.

Let $G = \mathrm{U}(2,1)(E/F)$ be the unramified unitary group in three variables and B be the standard Borel subgroup of upper triangular matrices in G with Levi decomposition $B = T \ltimes N$. Let

$$\mathfrak{I} = \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \end{pmatrix} \cap G$$

be the standard Iwahori subgroup of G and $\mathfrak{I}_T = T \cap \mathfrak{I}$.

Let χ be a level zero character of T. Define a character $\widetilde{\chi}$ of $\mathfrak{I}$ by

$$\widetilde{\chi}(i^- i_T i^+) = \chi(i_T)$$

for all $i^- \in \mathfrak{I}^-$, $i^+ \in \mathfrak{I}^+$, and $i_T \in \mathfrak{I}_T$.

Lemma 4.7. The level zero R-type $(\mathfrak{I}, \widetilde{\chi})$ is a G-cover of $(\mathfrak{I}_T, \chi)$.

Proof: Properties (1) and (2) of Definition 4.6 are clear. Let

$$w_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad w_2 = \begin{pmatrix} 0 & 0 & \varpi_F^{-1} \\ 0 & 1 & 0 \\ \varpi_F & 0 & 0 \end{pmatrix}.$$

If $g \in \mathbf{I}_G(\chi)$,

$$\mathbf{I}_g(\widetilde{\chi}) \simeq \mathrm{Hom}_{\mathfrak{I}_T}(\chi, \chi) \simeq R,$$

because χ is an character. Hence, up to a scalar, there exists a unique function in $\mathcal{H}_R(G,K,\sigma)$ supported on $\mathfrak{I} g \mathfrak{I}$. For $x \in \mathbf{I}_G(\widetilde{\chi})$ and $a \in R$ define $f_{x,a} \in \mathcal{H}_R(G,\mathfrak{I},\widetilde{\chi})$ to be the unique function with support $\mathfrak{I} x \mathfrak{I}$ such that $f_x(x) = a$. If $a = 1$ we write $f_x = f_{x,1}$. The proof is split into two cases:

<u>If χ is regular:</u> let $\zeta = w_1 w_2$. Then

$$\begin{aligned} f_\zeta \star f_{\zeta^{-1}}(1_G) &= \sum_{G/\mathfrak{I}} f_\zeta(x) f_{\zeta^{-1}}(x^{-1}) \\ &= \sum_{G/\mathfrak{I}} f_\zeta(x) \\ &= [\mathfrak{I}\zeta\mathfrak{I} : \mathfrak{I}] \\ &= [\mathfrak{I} : \mathfrak{I} \cap \zeta \mathfrak{I} \zeta^{-1}] = q^4. \end{aligned}$$

The support of $f_\zeta \star f_{\zeta^{-1}}$ is contained in $\mathfrak{I}\zeta\mathfrak{I}\zeta^{-1}\mathfrak{I}$.

The double coset space $\mathfrak{I}\backslash\mathbf{I}_G(\widetilde{\chi})/\mathfrak{I}$ is contained in

$$\mathfrak{I}\backslash G/\mathfrak{I} \simeq \bigcup_{w\in W_{\text{aff}}} \mathfrak{I}w\mathfrak{I}.$$

An element $w \in W_{\text{aff}}$ of the form

$$w = \begin{pmatrix} 0 & 0 & \varpi^a \\ 0 & 1 & 0 \\ \varpi^{-a} & 0 & 0 \end{pmatrix}$$

intertwines $\widetilde{\chi}$ if and only if w_1 intertwines χ. But χ is regular, i.e. w_1 does not intertwine χ. Thus $\mathbf{I}_G(\widetilde{\chi}) = \mathfrak{I}T\mathfrak{I}$.

Hence $\mathfrak{I}\,\mathbf{I}_G(\chi)\mathfrak{I} \cap \mathfrak{I}\zeta\mathfrak{I}\zeta^{-1}\mathfrak{I} = \mathfrak{I}$ which by Lemma 4.4 implies that $f_\zeta \star f_{\zeta^{-1}}$ is supported on $\mathfrak{I}$. Thus f_ζ is invertible with inverse $\frac{1}{q^4} f_{\zeta^{-1}}$.

<u>If χ is not regular:</u> then $w_1, w_2 \in \mathbf{I}_G(\chi)$ and hence, $f_{w_1}, f_{w_2} \in \mathcal{H}(G,\mathfrak{I},\widetilde{\chi})$ by Lemma 4.4. Furthermore,

$$\begin{aligned} f_{w_1} \star f_{w_1}(1_G) &= \sum_{G/\mathfrak{I}} f_{w_1}(x) f_{w_1}(x^{-1}) \\ &= \sum_{G/\mathfrak{I}} f_{w_1}(x) f_{w_1}(x) \\ &= [\mathfrak{I}w_1\mathfrak{I} : \mathfrak{I}] \\ &= [\mathfrak{I} : \mathfrak{I} \cap w_1\mathfrak{I}w_1] = q^3. \end{aligned}$$

which is nonzero. Similarly, $f_{w_2} \star f_{w_2}(1_G) = q$ which is nonzero. The support of f_{w_1} is contained in the group $G_x = \mathfrak{I} \cup \mathfrak{I}w_1\mathfrak{I}$, thus

$$\operatorname{supp}(f_{w_1} \star f_{w_1}) \subseteq G_x G_x = G_x.$$

Similarly the support of $f_{w_2} \star f_{w_2}$ is contained in $\mathfrak{I} \cup \mathfrak{I}w_2\mathfrak{I}$. Hence

$$f_{w_1} \star f_{w_1} = r f_{w_1} + q^3 f_1$$

with $r \in R$, and f_{w_1} is invertible with inverse $f_{w_1}^{-1} = \frac{1}{q^3}(f_{w_1} - r f_1)$. Similarly, f_{w_2} is invertible with inverse $f_{w_2}^{-1} = \frac{1}{q}(f_{w_2} - s f_1)$ for some $s \in R$. Hence $f = f_{w_2} \star f_{w_1}$ is an invertible element of $\mathcal{H}_R(G,\mathfrak{I},\widetilde{\chi})$. Using the Iwahori decomposition of $\mathfrak{I}$, because

$$\begin{aligned} (\mathfrak{I}^-)^{w_1} &\subseteq \mathfrak{I}^+ \\ (\mathfrak{I}^+)^{w_2} &\subseteq \mathfrak{I}^- \\ (\mathfrak{I}_T)^{w_1} &= (\mathfrak{I}_T)^{w_2} = \mathfrak{I}_T, \end{aligned}$$

we have

$$\mathfrak{I}w_1\mathfrak{I}w_2\mathfrak{I} = \mathfrak{I}(w_1\mathfrak{I}^-w_1)w_1w_2(w_2\mathfrak{I}_Tw_2)(w_2\mathfrak{I}^+w_2)\mathfrak{I} = \mathfrak{I}w_1w_2\mathfrak{I}.$$

Thus the support of f is contained in the double coset

$$\mathfrak{I}w_1w_2\mathfrak{I},$$

and hence, as f is invertible, $f = c f_{w_1w_2}$ with $c \in R$ nonzero.

In both cases, the element w_1w_2 is a strongly $(B,\mathfrak{I})$-positive element of the centre of T and $f_{w_1w_2}$ is an invertible element of $\mathcal{H}_R(G,\mathfrak{I},\widetilde{\chi})$ supported on $\mathfrak{I}w_1w_2\mathfrak{I}$. $\square$

THEOREM 4.8. Let $R = \overline{\mathbb{F}}_\ell$ or $\overline{\mathbb{Q}}_\ell$ and suppose that χ is not regular. The Hecke algebra $\mathcal{H}_R(G,\mathfrak{I},\widetilde{\chi})$ is generated as an R-algebra by f_{w_1} and f_{w_2} and the relations

$$f_{w_1,a} \star f_{w_1,a} = (q^r - 1) f_{w_1,a} + q^r f_1;$$
$$f_{w_2} \star f_{w_2} = (q-1) f_{w_2} + q f_1.$$

where $r = 3$ and $a = 1$ if χ factors through the determinant; and $r = 1$ and $a = \frac{1}{q}$ otherwise.

PROOF: The algebra $\mathcal{H}_R(G,K,\sigma)$ is generated as an R-module by the functions which are supported on a single double coset.

As in the proof of Lemma 4.7, where we showed

$$(\star_1) \qquad f_{w_1} \star f_{w_2} = c f_{w_1 w_2}$$

with $c \in R$ nonzero, but using the Iwahori decomposition $\mathfrak{I} = \mathfrak{I}^+\mathfrak{I}_T\mathfrak{I}^-$ we have $\mathfrak{I}w_2\mathfrak{I}w_1\mathfrak{I} = \mathfrak{I}w_2w_1\mathfrak{I}$ and hence

$$(\star_2) \qquad f_{w_2} \star f_{w_1} = c f_{w_2 w_1}$$

with $c \in R$ nonzero. For any $w \in W_{\text{aff}}$, because w_1 and w_2 generate W_{aff}, we have $w = w_{i_1} w_{i_2} \cdots w_{i_n}$ with $w_{i_j} \in \{w_1, w_2\}$ such that $w_{i_{j+1}} \neq w_{i_j}$ and, by the method that gave us $(\star_1)$ and $(\star_2)$, we have

$$f_{w_{i_1}} \star f_{w_{i_2}} \star \cdots \star f_{w_{i_n}} = c f_w$$

with $c \in R$ nonzero. Thus, by Lemma 4.4 and the Bruhat decomposition,

$$\mathfrak{I}\backslash G/\mathfrak{I} \simeq \bigcup_{w \in W_{\text{aff}}} \mathfrak{I}w\mathfrak{I},$$

f_{w_1} and f_{w_2} generate $\mathcal{H}_R(G,K,\sigma)$. Hence it remains to calculate $f_{w_i} \star f_{w_i}$, $i = 1, 2$. We do this by restricting to the maximal parahoric subgroups $G_x = \mathfrak{I} \cup \mathfrak{I}w_1\mathfrak{I}$ and $G_y = \mathfrak{I} \cup \mathfrak{I}w_2\mathfrak{I}$. Let $\overline{B} = \overline{T} \ltimes \overline{N}$ denote the Borel subgroup in M_x and let $\overline{\chi}$ be the character of $\overline{T}$ given by $\operatorname{Res}^T_{\mathfrak{I}_T} \chi = \operatorname{infl}_{\mathfrak{I}^1_T} \overline{\chi}$. Then

$$\mathcal{H}_R(G_x, \mathfrak{I}, \widetilde{\chi}) \simeq \mathcal{H}_R(M_x, \overline{B}, \overline{\chi}).$$

If $R = \overline{\mathbb{F}}_\ell$, we choose a lift ζ of $\overline{\chi}$ such that $\zeta^{w_1} = \zeta$. If L is a lattice in ζ then ζ is a called reduction stable lift if $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(M_x, \overline{B}, \overline{\chi}) \simeq \overline{\mathbb{F}}_\ell \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{H}_{\overline{\mathbb{Z}}_\ell}(M_x, \overline{B}, L)$ and $\mathcal{H}_{\overline{\mathbb{Q}}_\ell}(M_x, \overline{B}, \zeta) \simeq \overline{\mathbb{Q}}_\ell \otimes_{\overline{\mathbb{Z}}_\ell} \mathcal{H}_{\overline{\mathbb{Z}}_\ell}(M_x, \overline{B}, L)$, [**DF92**, Page 64]. A basis of $\mathcal{H}_{\overline{\mathbb{Q}}_\ell}(M_x, \overline{B}, \zeta)$ is called reduction stable if it is a basis of $\mathcal{H}_{\overline{\mathbb{Z}}_\ell}(M_x, \overline{B}, L)$ and ζ is reduction stable . Then the image of this basis in $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(M_x, \overline{B}, \overline{\chi})$ is a basis of $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(M_x, \overline{B}, \overline{\chi})$. By [**GHM94**, Section 3.1] an ℓ-adic character ζ of $\overline{T}$ such that $d_\ell(\zeta) = \overline{\chi}$ and $\zeta^{w_1} = \zeta$ is a reduction stable lift and a basis of $\mathcal{H}_{\overline{\mathbb{Z}}_\ell}(M_x, \overline{B}, \zeta)$ is reduction stable.

By [**HL80**, Theorem 4.14], if $\operatorname{Ind}^G_B \zeta = \rho_1 \oplus \rho_2$ with $\dim \rho_1 \geqslant \dim \rho_2$ then $\mathcal{H}_{\overline{\mathbb{Q}}_\ell}(M_x, \overline{B}, \zeta)$ is generated by T_w which is supported on the double coset $\overline{B}w\overline{B}$ and satisfies the quadratic relation

$$T_w \star T_w = \left(\frac{\dim \rho_1}{\dim \rho_2} - 1\right) T_w + \frac{\dim \rho_1}{\dim \rho_2} T_1$$

where T_1 is the identity of $\mathcal{H}_{\overline{\mathbb{Q}}_\ell}(M_x, \overline{B}, \zeta)$. By Chapter 2 Section 8,

$$\frac{\dim \rho_1}{\dim \rho_2} = \begin{cases} q^3 & \text{if } \zeta \text{ factors through the determinant;} \\ q & \text{otherwise.} \end{cases}$$

By inflation, the element $T_w \in \mathcal{H}_R(M_x, \overline{B}, \widetilde{\chi})$ determines an element $f_{w_1,a} \in \mathcal{H}_R(G_x, \mathfrak{I}, \widetilde{\chi})$ supported on $\mathfrak{I}w_1\mathfrak{I}$. Furthermore in the proof of Lemma 4.7 we showed

$$f_{w_1} \star f_{w_1} = r' f_{w_1} + q^3 f_1,$$

with $r' \in R$, hence we can recover a. The same method, using the computations of Chapter 2 Section 7, shows that the element $f_{w_2} \in \mathcal{H}_R(G_y, \mathfrak{I}, \widetilde{\chi})$ supported on $\mathfrak{I}w_2\mathfrak{I}$ satisfies the quadratic relation

$$f_{w_2} \star f_{w_2} = (q-1) f_{w_2} + q f_1.$$

□

Corollary 4.9. There are four one dimensional $\overline{\mathbb{Q}}_\ell$-modules of the algebra $\mathcal{H}_{\overline{\mathbb{Q}}_\ell}(G, \mathfrak{I}, 1)$ which are determined by their values on the generators f_{w_1} and f_{w_2}.

Character χ of $\mathcal{H}_R(G, \mathfrak{I}, 1)$	$\chi(f_{w_1})$	$\chi(f_{w_2})$
χ_{sgn}	-1	-1
χ_{ind}	q^3	q
χ_1	q^3	-1
χ_2	-1	q

(1) If $q^3 \not\equiv -1 \bmod \ell$ then there are four one dimensional $\overline{\mathbb{F}}_\ell$-modules of $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, \mathfrak{I}, 1)$.
(2) If $q^3 \equiv -1 \bmod \ell$, but $q \not\equiv -1 \bmod \ell$, then there are two one dimensional $\overline{\mathbb{F}}_\ell$-modules of $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, \mathfrak{I}, 1)$.
(3) If $q \equiv -1 \bmod \ell$ then there is a unique one dimensional $\overline{\mathbb{F}}_\ell$-module of $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, \mathfrak{I}, 1)$.

Let $\mathcal{H}(G, K, \sigma)$-$\mathfrak{Mod}$ denote the right modules over $\mathcal{H}(G, K, \sigma)$. We have a functor

$$\begin{aligned} M_\sigma : \mathfrak{R}_R(G) &\to \mathcal{H}(G, K, \sigma)\text{-}\mathfrak{Mod} \\ \pi &\mapsto \operatorname{Hom}_G(\operatorname{ind}_K^G \sigma, \pi) \end{aligned}$$

where $\operatorname{Hom}_G(\operatorname{ind}_K^G \sigma, \pi)$ is a right $\mathcal{H}(G, K, \sigma)$-module by identifying $\mathcal{H}(G, K, \sigma)$ with $\operatorname{End}_G(\operatorname{ind}_K^G \sigma)$ and the action of $\operatorname{End}_G(\operatorname{ind}_K^G \sigma)$ is given by composition. By reciprocity,

$$M_\sigma(\pi) = \operatorname{Hom}_K(\sigma, \operatorname{Res}_K^G \pi)$$

the σ-invariants of π.

Let $P = M \ltimes N$ be a parabolic subgroup of G and let (K, σ) be a G-cover of (K_M, σ_M) relative to P. There is an injective homomorphism of algebras, [**Vig98**, , II 10.1(2)],

$$j_P : \mathcal{H}(M, K_M, \sigma_M) \to \mathcal{H}(G, K, \sigma).$$

This homomorphism induces a restriction functor

$$j_P^* : \mathcal{H}(G, K, \sigma)\text{-}\mathfrak{Mod} \to \mathcal{H}(M, K_M, \sigma_M)\text{-}\mathfrak{Mod}.$$

Theorem 4.10 ([**Vig98**, Section 2, 10.1(3)]). There is an isomorphism

$$j_P^*(M_\sigma(\pi)) \simeq M_{\sigma_M}(r_P^G(\pi)).$$

An immediate consequence of Theorem 4.10 is:

COROLLARY 4.11. If $\pi \in \mathfrak{R}_R(G)$ contains an R-type which is a G-cover relative to P then $r_P^G(\pi)$ is nonzero.

THEOREM 4.12 ([**Blo05**, Theorem 2]). There is an isomorphism

$$\operatorname{ind}_K^G \sigma \simeq \operatorname{ind}_P^G \operatorname{ind}_{K_M}^M \sigma_M.$$

5. Level zero parahoric functors

Let (G_x, σ) be a level zero R-type. The functor taking smooth R-representations of M_x to smooth R-representations of G defined by inflating σ to G_x and compactly inducing to G is called level zero parahoric induction.

$$\begin{aligned} \mathrm{I}_{M_x}^G : \mathfrak{R}_R(M_x) &\longrightarrow \mathfrak{R}_R(G_x) \longrightarrow \mathfrak{R}_R(G) \\ \sigma &\longmapsto \operatorname{infl}_{G_x^1} \sigma \longmapsto \operatorname{ind}_{G_x}^G(\operatorname{infl}_{G_x^1} \sigma) \end{aligned}$$

It has a right adjoint called level zero parahoric restriction.

$$\begin{aligned} \mathrm{R}_{M_x}^G : \mathfrak{R}_R(G) &\longrightarrow \mathfrak{R}_R(G_x) \longrightarrow \mathfrak{R}_R(M_x) \\ \pi &\longmapsto \operatorname{Res}_{G_x}^G \pi \longmapsto \operatorname{inv}_{G_x^1}(\operatorname{Res}_{G_x}^G \pi) \end{aligned}$$

If we fix a maximal parahoric subgroup G_z and consider the level zero parahoric functors $\mathrm{I}_{M_x}^G$ and $R_{M_x}^G$ for parahoric subgroups $G_x \subseteq G_z$ our viewpoint is that these functors mirror the parabolic functors inside the finite reductive group M_z.

REMARK. Let G_x be a maximal parahoric subgroup of a classical group G. Let

$$G_x^+ = N_G(G_x).$$

There are two short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_x^1 & \longrightarrow & G_x^+ & \longrightarrow & M_x^+ & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & G_x^1 & \longrightarrow & G_x & \longrightarrow & M_x & \longrightarrow & 1 \end{array}$$

where M_x^+ is a finite reductive group that contains M_x as a normal subgroup of finite index. Let σ be an irreducible cuspidal representation of M_x and let Σ be an irreducible quotient of $\operatorname{ind}_{M_x}^{M_x^+} \sigma$. When trying to construct all irreducible representations of a general p-adic group G it is more natural to define level zero parahoric induction making this initial step from σ to Σ before inflating to G_x^+ and inducing to G. We have chosen to define level zero parahoric induction without this step because we are going to specialize to unramified p-adic unitary groups, where $G_x = G_x^+$.

The following theorem reduces the classification of irreducible cuspidal level zero representations to the classification of maximal cuspidal level zero R-types.

THEOREM 5.1. Let G be an unramified unitary group in three variables, and $(\pi, \mathcal{V})$ an irreducible cuspidal level zero representation of G. Then π contains a maximal cuspidal level zero R-type and this R-type is unique up to conjugacy.

We follow [**Vig01b**] to prove Theorem 5.1.

Level zero R-types (G_x, σ_x) and (G_y, σ_y) are called associate, [**MP94**, Definition 5.1], if there exists $g \in G$ such that $G_x \cap G_{gy}$ surjects onto M_x and M_{gy} and $\sigma_x \simeq {}^g\sigma_y$.

THEOREM 5.2 ([**Vig01b**, Corollary 5.2]). Let G be an unramified unitary group. Let x, y be points in $\mathcal{B}(G)$ and let σ_x (respectively σ_y) be an irreducible cuspidal representation of M_x (respectively M_y). If

$$\mathrm{Hom}_G\left(\mathrm{I}_{M_x}^G(\sigma_x), \mathrm{I}_{M_y}^G(\sigma_y)\right) \neq \{0\},$$

then (G_x, σ_x) and (G_y, σ_y) are associate.

6. Level zero restriction-induction

In this section we give a proof of Theorem 5.2 due to [**Vig01b**]

6.1. Distinguished double coset representatives. Let G be an unramified unitary group, W be the affine Weyl group of G and S be a set of fundamental reflections for W. The choice of S corresponds to a choice of chamber in the building of G. This choice defines the standard Iwahori subgroup $\mathfrak{I}$ of G. The group W is a Coxeter group. If $J \subset S$, we let W_J be the subgroup generated by the reflections in J. The standard parahoric subgroups of G correspond to proper subsets J of S, where J maps to $G_J = \mathfrak{I} N_J \mathfrak{I}$, for N_J any set of representatives of W_J in G.

Let $J, K \subset S$. A set of double coset representatives $\overline{D}_{J,K}$ for $W_J \backslash W / W_K$ is called distinguished if each $d \in \overline{D}_{J,K}$ has minimal length in its double coset, [**Mor93**, 3.10]. A set of double coset representatives $D_{J,K}$ in $N_G(T)$ for $G_J \backslash G / G_K$ is called distinguished if the projection to W of the set $D_{J,K}$ is a set of distinguished double coset representatives for $W_J \backslash W / W_K$, [**Mor93**, 3.12].

For $w \in W$ the length of w is equal to the length of w^{-1}. Thus if $\overline{D}_{J,K}$ is a set of distinguished double coset representatives for $W_J \backslash W / W_K$ then $\overline{D}_{J,K}^{-1}$ is a set of distinguished double coset representatives for $W_K \backslash W / W_J$. If $n \in D_{J,K}$ has projection $w \in W_{J,K}$ then $n^{-1} \in N_G(T)$ has projection $w^{-1} \in W_{K,J}$. Thus if $D_{J,K}$ is a set of distinguished double coset representatives for $G_J \backslash G / G_K$, then $D_{J,K}^{-1}$ is a set of distinguished double coset representatives for $G_K \backslash G / G_J$.

6.2. Intersections of parahoric subgroups. Let $x, y \in \mathcal{B}(G)$. Fix a set of distinguished double coset representatives $D_{y,x}$ for $G_y \backslash G / G_x$.

THEOREM 6.1 ([**Mor93**, Corollary 3.20, Lemma 3.21]). Let $n \in D_{y,x}$ then

$$P_{x,ny} = G_x^1(G_x \cap G_{ny})/G_x^1$$

is a parabolic subgroup of G_x/G_x^1. Furthermore, the pro-p unipotent radical of $G_x^1(G_x \cap G_{ny})$ is $G_x^1(G_x \cap G_{ny}^1)$.

By Section 6.1, $D_{y,x}^{-1}$ is a set of distinguished double coset representatives for $G_x\backslash G/G_y$ hence

$$P_{y,n^{-1}x} = G_y^1(G_y \cap G_{n^{-1}x})/G_y^1$$

is a parabolic subgroup of G_y/G_y^1. Furthermore, the pro-p unipotent radical of $G_y^1(G_y \cap G_{n^{-1}x})$ is $G_y^1(G_y \cap G_{n^{-1}x}^1)$.

Suppose $P_{y,n^{-1}x}$ has Levi decomposition $P_{y,n^{-1}x} = M_{y,n^{-1}x} \ltimes N_{y,n^{-1}x}$ and $P_{x,ny}$ has Levi decomposition $P_{x,ny} = M_{x,ny} \ltimes N_{x,ny}$.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G_y^1 & \longrightarrow & G_y & \longrightarrow & M_y & \longrightarrow & 1 \\
 & & \Vert & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & G_y^1 & \longrightarrow & G_y^1\left(G_y \cap G_{n^{-1}x}\right) & \longrightarrow & P_{y,n^{-1}x} & \longrightarrow & 1 \\
 & & \downarrow & & \Vert & & \uparrow & & \\
1 & \longrightarrow & G_y^1\left(G_y \cap G_{n^{-1}x}^1\right) & \longrightarrow & G_y^1\left(G_y \cap G_{n^{-1}x}\right) & \longrightarrow & M_{y,n^{-1}x} & \longrightarrow & 1
\end{array}$$

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G_x^1\left(G_x \cap G_{ny}^1\right) & \longrightarrow & G_x^1\left(G_x \cap G_{ny}\right) & \longrightarrow & M_{x,ny} & \longrightarrow & 1 \\
 & & \uparrow & & \Vert & & \downarrow & & \\
1 & \longrightarrow & G_x^1 & \longrightarrow & G_x^1\left(G_x \cap G_{ny}\right) & \longrightarrow & P_{x,ny} & \longrightarrow & 1 \\
 & & \Vert & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & G_x^1 & \longrightarrow & G_x & \longrightarrow & M_x & \longrightarrow & 1
\end{array}$$

6.3. Level zero parahoric restriction-induction. Following [**Vig01b**] we recover a convenient formula for the composition of level zero parahoric restriction and induction. The first step is an application of the usual restriction-induction formula for compact induction.

LEMMA 6.2. Let $x, y \in \mathcal{B}(G)$ and σ_y be a representation of M_y. Then

$$\mathrm{R}_{M_x}^G \circ \mathrm{I}_{M_y}^G(\sigma_y) \simeq \bigoplus_{g \in G_y\backslash G/G_x} \left(\mathrm{ind}_{G_x \cap G_{gy}}^{G_x} \mathrm{Res}_{G_x \cap G_{gy}}^{G_{gy}} {}^g(\mathrm{infl}_{G_y^1} \sigma_y)\right)^{G_x^1}.$$

PROOF: We have

$$\mathrm{R}_{M_x}^G \circ \mathrm{I}_{M_y}^G(\sigma_y) = \left(\mathrm{Res}_{G_x}^G \circ \mathrm{ind}_{G_y}^G \circ \mathrm{infl}_{G_y^1} \sigma_y\right)^{G_x^1}$$

and we apply the restriction-induction formula, Chapter 1 Lemma 3.5, to $\mathrm{Res}_{G_x}^G \circ \mathrm{ind}_{G_y}^G$. □

Let G be a group, H be a subgroup of G and $g \in G$. We let $\mathrm{conj}(g) : \mathfrak{R}_R(H) \to \mathfrak{R}_R({}^gH)$ be the functor that takes $\pi \in \mathfrak{R}_R(H)$ to ${}^g\pi \in \mathfrak{R}_R({}^gH)$.

Let G be a group and H a normal subgroup of G. In the following proof we identify the category $\mathfrak{R}_R^H(G)$ of R-representations of G trivial on H with $\mathfrak{R}_R(G/H)$. This allows us to act on $\pi \in \mathfrak{R}_R(G/H)$ by conjugation by elements of a group which contains G as a normal subgroup.

LEMMA 6.3 (Level zero restriction induction formula). Let $x, y \in \mathcal{B}(G)$ and $D_{y,x}$ a set of distinguished double coset representatives for $G_y\backslash G/G_x$. Let σ be a representation of M_y.

There is an isomorphism

$$\mathrm{R}^G_{M_x} \circ \mathrm{I}^G_{M_y}(\sigma) \simeq \bigoplus_{n \in D_{y,x}} i^{M_x}_{P_{x,ny}} {}^n\Big(r^{M_y}_{P_{y,n^{-1}x}}(\sigma)\Big).$$

where $r^{M_y}_{P_{y,n^{-1}x}}$ denotes the Jacquet functor associated to the parabolic subgroup $P_{y,n^{-1}x}$ of the finite reductive group M_y and $i^{M_x}_{P_{x,ny}}$ denotes the parabolic induction functor associated to the parabolic subgroup $P_{x,ny}$ of the finite reductive group M_x.

Proof: We start with the isomorphism of Lemma 6.2, choosing $D_{y,x}$ as a set of double coset representatives for $G_y\backslash G/G_x$, and consider the functor associated to one of the summands with $n \in D_{y,x}$

$$\Psi_n(\sigma) = \mathrm{inv}_{G^1_x} \circ \mathrm{ind}^{G_x}_{G_x \cap G_{ny}} \circ \mathrm{Res}^{G_{ny}}_{G_x \cap G_{ny}} \circ \mathrm{conj}(n) \circ \mathrm{infl}_{G^1_y}(\sigma).$$

Thus $\Psi_n : \mathfrak{R}_R(M_y) \to \mathfrak{R}_R(M_x)$ which we also consider as a functor $\mathfrak{R}^{G^1_y}_R(G_y) \to \mathfrak{R}^{G^1_x}_R(G_x)$. Let $(\pi, \mathcal{V}) \in \mathfrak{R}_R(G^1_x(G_x \cap G_{ny}))$. By normality of G^1_x in $G^1_x(G_x \cap G_{ny})$ and in G_x, any $f \in \mathrm{inv}_{G^1_x} \circ \mathrm{ind}^{G_x}_{G^1_x(G_x \cap G_{ny})} \pi$ has image in $\mathrm{inv}_{G^1_x} \pi$. Moreover, again by normality of G^1_x, if $f \in \mathrm{ind}^{G_x}_{G^1_x(G_x\cap G_{ny})} \circ \mathrm{inv}_{G^1_x} \pi$ and $g \in G^1_x$ then $g \cdot f = f$. Hence, letting $\Sigma = \mathrm{Res}^{G_{ny}}_{G_x \cap G_{ny}} \circ \mathrm{conj}(n) \circ \mathrm{infl}_{G^1_y}(\sigma)$, by transitivity of induction

$$\begin{aligned}\Psi_n(\sigma) &= \mathrm{inv}_{G^1_x} \circ \mathrm{ind}^{G_x}_{G_x \cap G_{ny}} \Sigma \\ &= \mathrm{inv}_{G^1_x} \circ \mathrm{ind}^{G_x}_{G^1_x(G_x \cap G_{ny})} \circ \mathrm{ind}^{G^1_x(G_x \cap G_{ny})}_{G_x \cap G_{ny}} \Sigma \\ &= \mathrm{ind}^{G_x}_{G^1_x(G_x \cap G_{ny})} \circ \mathrm{inv}_{G^1_x} \circ \mathrm{ind}^{G^1_x(G_x \cap G_{ny})}_{G_x \cap G_{ny}} \Sigma.\end{aligned}$$

By normality the map $f \mapsto f(1)$ induces an isomorphism of representations of $G^1_x(G_x \cap G_{ny})$

$$\mathrm{inv}_{G^1_x} \circ \mathrm{ind}^{G^1_x(G_x \cap G_{ny})}_{G_x \cap G_{ny}} \Sigma \simeq \mathrm{infl}_{G^1_x} \circ \mathrm{inv}_{G^1_x \cap G_{ny}} \Sigma$$

where $\mathrm{infl}_{G^1_x}$ denotes the functor $\mathfrak{R}^{G^1_x \cap G_{ny}}_R(G_x \cap G_{ny}) \to \mathfrak{R}_R(G^1_x(G_x \cap G_{ny}))$ by letting G^1_x act trivially. Hence

$$\Psi_n(\sigma) \simeq \mathrm{ind}^{G_x}_{G^1_x(G_x \cap G_{ny})} \circ \mathrm{infl}_{G^1_x} \circ \mathrm{inv}_{G^1_x \cap G_{ny}} \circ \mathrm{Res}^{G_{ny}}_{G_x \cap G_{ny}} \circ \mathrm{conj}(n) \circ \mathrm{infl}_{G^1_y} \sigma.$$

The conjugation commutes with restriction thus

$$\Psi_n(\sigma) \simeq \mathrm{ind}^{G_x}_{G^1_x(G_x \cap G_{ny})} \circ \mathrm{infl}_{G^1_x} \circ \mathrm{inv}_{G^1_x \cap G_{ny}} \circ \mathrm{conj}(n) \circ \mathrm{Res}^{G_y}_{G_{n^{-1}x} \cap G_y} \circ \mathrm{infl}_{G^1_y} \sigma.$$

A subgroup H of G acts trivially on π if and only if ${}^nH$ acts trivially on ${}^n\pi$ hence

$$\Psi_n(\sigma) = \mathrm{ind}^{G_x}_{G^1_x(G_x \cap G_{ny})} \circ \mathrm{infl}_{G^1_x} \circ \mathrm{conj}(n) \circ \mathrm{inv}_{G^1_{n^{-1}x} \cap G_y} \circ \mathrm{Res}^{G_y}_{G_{n^{-1}x} \cap G_y} \circ \mathrm{infl}_{G^1_y} \sigma.$$

Restriction is transitive, as functors $\mathfrak{R}_R(G_y) \to \mathfrak{R}_R(G_{n^{-1}x} \cap G_y)$,

$$\mathrm{Res}^{G_y}_{G_{n^{-1}x} \cap G_y} = \mathrm{Res}^{G^1_y(G_{n^{-1}x} \cap G_y)}_{G_{n^{-1}x} \cap G_y} \circ \mathrm{Res}^{G_y}_{G^1_y(G_{n^{-1}x} \cap G_y)}.$$

By normality of G^1_y in G_y and because G^1_y acts trivially, the largest $(G_{n^{-1}x} \cap G_y)$-submodule on which $(G^1_{n^{-1}x} \cap G_y)$ acts trivially is equal to the largest $G^1_y(G_{n^{-1}x} \cap G_y)$-submodule on which $G^1_y(G^1_{n^{-1}x} \cap G_y)$ acts trivially. Furthermore, by normality of G^1_y in G_y,

$$(G_{n^{-1}x} \cap G_y)/(G^1_y \cap G_{n^{-1}x})(G^1_{n^{-1}x} \cap G_y) \simeq G^1_y(G_{n^{-1}x} \cap G_y)/G^1_y(G^1_{n^{-1}x} \cap G_y).$$

Hence we can identify the $(G_{n^{-1}x}\cap G_y)$-submodule on which $(G^1_y\cap G_{n^{-1}x})(G^1_{n^{-1}x}\cap G_y)$ acts trivially with the $G^1_y(G_{n^{-1}x}\cap G_y)$-submodule on which $G^1_y(G^1_{n^{-1}x}\cap G_y)$ acts trivially. Thus

$$\Psi_n(\sigma)\simeq \mathrm{ind}^{G_x}_{G^1_x(G_x\cap G_{ny})}\circ \mathrm{infl}_{G^1_x}\circ \mathrm{conj}(n)\circ \mathrm{inv}_{G^1_y(G^1_{n^{-1}x}\cap G_y)}\circ \mathrm{Res}^{G_y}_{G^1_y(G_{n^{-1}x}\cap G_y)}\circ \mathrm{infl}_{G^1_y}\sigma.$$

Invariants and coinvariants under a pro-p group are isomorphic, thus

$$\Psi_n(\sigma)\simeq \mathrm{ind}^{G_x}_{G^1_x(G_x\cap G_{ny})}\circ \mathrm{infl}_{G^1_x}\circ \mathrm{conj}(n)\circ \mathrm{coinv}_{G^1_y(G^1_{n^{-1}x}\cap G_y)}\circ \mathrm{Res}^{G_y}_{G^1_y(G_{n^{-1}x}\cap G_y)}\circ \mathrm{infl}_{G^1_y}\sigma.$$

By Theorem 6.1 we identify

$$P_{x,ny}=G^1_x(G_x\cap G_{ny})/G^1_x,\quad P_{y,n^{-1}x}=G^1_y(G_y\cap G_{n^{-1}x})/G^1_y,$$

$M_x=G_x/G^1_x$ and $M_y=G_y/G^1_y$. Hence

$$\Psi_n(\sigma)\simeq \mathrm{ind}^{M_x}_{P_{x,ny}}\circ \mathrm{infl}_{N_{x,ny}}\circ \mathrm{conj}(n)\circ \mathrm{coinv}_{N_{y,n^{-1}x}}\circ \mathrm{Res}^{M_y}_{P_{y,n^{-1}x}}\sigma.$$

Therefore

$$\mathrm{R}^G_{M_x}\circ \mathrm{I}^G_{M_y}(\sigma)\simeq \bigoplus_{n\in D_{y,x}} i^{M_x}_{P_{x,ny}}\,{}^{n}\!\left(r^{M_y}_{P_{y,n^{-1}x}}(\sigma)\right).$$

□

Proof: [Proof of Theorem 5.2] By Lemma 6.3,

$$\mathrm{Hom}_G\left(\mathrm{I}^G_{M_x}(\sigma_x),\mathrm{I}^G_{M_y}(\sigma_y)\right)=\bigoplus_{n\in D_{y,x}}\mathrm{Hom}_{M_x}\left(\sigma_x, i^{M_x}_{P_{x,ny}}\,{}^{n}\!\left(r^{M_y}_{P_{y,n^{-1}x}}(\sigma_y)\right)\right).$$

Hence

$$\mathrm{Hom}_G\left(\mathrm{I}^G_{M_x}(\sigma_x),\mathrm{I}^G_{M_y}(\sigma_y)\right)\neq\{0\}$$

if and only if there exists $n\in D_{y,x}$ such that

$$\mathrm{Hom}_{M_x}\left(\sigma_x, i^{M_x}_{P_{x,ny}}\,{}^{n}\!\left(r^{M_y}_{P_{y,n^{-1}x}}(\sigma_y)\right)\right)\neq\{0\}.$$

Assume there exists $n\in D_{y,x}$ such that

$$\mathrm{Hom}_{M_x}\left(\sigma_x, i^{M_x}_{P_{x,ny}}\,{}^{n}\!\left(r^{M_y}_{P_{y,n^{-1}x}}(\sigma_y)\right)\right)\neq\{0\}.$$

By cuspidality of σ_y, $P_{y,n^{-1}x}=M_y$; hence $G^1_y(G_y\cap G_{n^{-1}x})/G^1_y=M_y$. By cuspidality of σ_x, $P_{x,ny}=M_x$; hence $G^1_x(G_x\cap G_{ny})/G^1_x=M_y$. Thus

$$\mathrm{Hom}_{M_x}\left(\sigma_x,{}^{n}\sigma_y\right)\neq\{0\}.$$

Therefore (G_x,σ_x) and (G_y,σ_y) are associate. □

Corollary 6.4. Let G be an unramified unitary group, G_x (respectively G_y) be a maximal parahoric subgroup of G, σ_x (respectively σ_y) be an irreducible cuspidal representation of M_x (respectively M_y). If

$$\mathrm{Hom}_G\left(\mathrm{I}^G_{M_x}(\sigma_x),\mathrm{I}^G_{M_y}(\sigma_y)\right)\neq\{0\}$$

then (G_x,σ_x) and (G_y,σ_y) are conjugate.

Proof: By Theorem 5.2 the R-types (G_x,σ_x) and (G_y,σ_y) are associate. If G_x and G_y are not conjugate then for all $g\in G$, in particular $n\in D_{y,x}$, the group $G_x\cap G_{gy}$ must stabilise an edge in the building and hence is not maximal. Thus it cannot surject onto either M_x or M_y. Hence there exists $n\in D_{y,x}$ such that $G_x=G_{ny}$ and

$$\mathrm{Hom}_{M_x}(\sigma_x,{}^{n}\sigma_y)\neq\{0\},$$

i.e. σ_x and σ_y are conjugate. □

6.4. Maximal parahoric subgroups and irreducibility.

LEMMA 6.5 ([**Vig01b**, Lemma 4.2]). Let (K, σ) be an R-type. For all irreducible ℓ-modular representations π of G, suppose that σ is a subrepresentation of $\operatorname{Res}_K^G \pi$ implies that σ is a quotient of $\operatorname{Res}_K^G \pi$ and that $\dim_R \operatorname{Hom}_G(\operatorname{ind}_K^G \sigma, \operatorname{ind}_K^G \sigma) = 1$. Then $\operatorname{ind}_K^G \sigma$ is irreducible.

PROOF: Let π be an irreducible quotient of $\operatorname{ind}_K^G \sigma$ then

$$\{0\} \neq \operatorname{Hom}_G(\operatorname{ind}_K^G \sigma, \pi) \simeq \operatorname{Hom}_K(\sigma, \operatorname{Res}_K^G \pi)$$

by reciprocity. Hence, because σ is irreducible, σ is a subrepresentation of $\operatorname{Res}_K^G \pi$ and thus, by our hypotheses, a quotient of $\operatorname{Res}_K^G \pi$. Hence

$$\{0\} \neq \operatorname{Hom}_K(\operatorname{Res}_K^G \pi, \sigma) \simeq \operatorname{Hom}_G(\pi, \operatorname{Ind}_K^G \sigma)$$

by reciprocity. Thus we have nonzero G-morphisms $\varphi_1 : \pi \to \operatorname{Ind}_K^G \sigma$ and $\varphi_2 : \operatorname{ind}_K^G \sigma \to \pi$. The composite $\varphi = \varphi_1 \circ \varphi_2$ gives a nonzero G-morphism $\operatorname{ind}_K^G \sigma \to \operatorname{Ind}_K^G \sigma$ with image isomorphic to π.

By reciprocity and Chapter 1 Lemma 3.5

$$\begin{aligned}\operatorname{Hom}_G(\operatorname{ind}_K^G \sigma, \operatorname{ind}_K^G \sigma) &\simeq \prod_{K\backslash G/K} \operatorname{Hom}_K(\sigma, \operatorname{ind}_{K\cap {}^gK}^K \operatorname{Res}_{K\cap {}^gK}^{{}^gK}\, {}^g\sigma) \\ &\simeq \operatorname{Hom}_G(\operatorname{ind}_K^G \sigma, \operatorname{Ind}_K^G \sigma).\end{aligned}$$

Thus $\dim_R(\operatorname{Hom}_G(\operatorname{ind}_K^G \sigma, \operatorname{Ind}_K^G \sigma)) = 1$, and for any $\psi \in \operatorname{Hom}_G(\operatorname{ind}_K^G \sigma, \operatorname{Ind}_K^G \sigma)$ there exists $r \in R$ such that $\psi = r\varphi$. But the embedding

$$(e : \operatorname{ind}_K^G \sigma \hookrightarrow \operatorname{Ind}_K^G \sigma) \in \operatorname{Hom}_G(\operatorname{ind}_K^G \sigma, \operatorname{Ind}_K^G \sigma),$$

hence $\pi \simeq \operatorname{ind}_K^G \sigma$ and $\operatorname{ind}_K^G \sigma$ is irreducible. □

LEMMA 6.6 ([**Vig01b**, Proposition 7.1]). Let G be an unramified unitary group, G_x be a maximal parahoric subgroup of G and σ be an irreducible cuspidal representation of M_x. Then $\mathrm{R}_{M_x}^G(\mathrm{I}_{M_x}^G(\sigma)) \simeq \sigma$ and $\mathrm{I}_{M_x}^G(\sigma)$ is irreducible. Furthermore, if G_y is a maximal parahoric subgroup of G not conjugate to G_x then $\mathrm{R}_{M_y}^G(\mathrm{I}_{M_x}^G(\sigma)) = \{0\}$.

PROOF: By Lemma 6.3, because G_x is equal to the full stabilizer of the vertex x hence $G_x\backslash G/G_x \simeq 1$,

$$\mathrm{R}_{M_x}^G(\mathrm{I}_{M_x}^G(\sigma)) \simeq \sigma.$$

Let π be an irreducible quotient of $I_{M_x}^G(\sigma)$ then

$$\{0\} \neq \operatorname{Hom}_G(\mathrm{I}_{M_x}^G(\sigma), \pi) = \operatorname{Hom}_{M_x}(\sigma, \mathrm{R}_{M_x}^G(\pi))$$

and σ is a subrepresentation of $\mathrm{R}_{M_x}^G(\pi)$.

If G_y is a maximal parahoric subgroup of G not conjugate to G_x then, by Lemma 6.3,

$$\mathrm{R}_{M_y}^G(\mathrm{I}_{M_x}^G(\sigma)) = \{0\}$$

because $P_{x,n^{-1}y}$ is properly contained in M_x and σ is cuspidal thus the Jacquet modules $r_{P_{x,n^{-1}y}}^{M_x}(\sigma)$ vanish.

By Theorem 3.3, π is level zero hence there exists a maximal parahoric subgroup G_z of G such that $\mathrm{R}_{M_z}^G(\pi)$ is non-zero. Assume $\mathrm{R}_{M_y}^G(\pi) \neq \{0\}$ then, by exactness, $\mathrm{R}_{M_y}^G(I_{M_x}^G(\sigma)) \neq \{0\}$, a contradiction.Hence $\mathrm{R}^G_{M_x}(\pi) \neq \{0\}$ and, by exactness,

$$\mathrm{R}^G_{M_x}(\pi) \simeq \sigma.$$

Because G_x^1 is pro-p and normal in G_x, by Chapter 1 Lemma 3.2, as representations of G_x

$$\pi \simeq \mathrm{R}^G_{M_x}(\pi) \oplus \pi(G_x^1).$$

Thus $\mathrm{infl}_{G_x^1} \sigma$ is a direct factor of $\mathrm{Res}^G_{G_x} \pi$.

Furthermore

$$\begin{aligned}\dim(\mathrm{Hom}_G(\mathrm{I}^G_{M_x}(\sigma), \mathrm{I}^G_{M_x}(\sigma)) &= \dim(\mathrm{Hom}_{M_x}(\sigma, \mathrm{R}^G_{M_x} \circ I^G_{M_x}(\sigma)) \\ &= \dim(\mathrm{Hom}_{M_x}(\sigma,\sigma)) = 1.\end{aligned}$$

Therefore, by Lemma 6.5, $\mathrm{I}^G_{M_x}(\sigma)$ is irreducible. □

6.5. Cuspidal level zero representations.

LEMMA 6.7. Let G be an unramified unitary group and $(\pi, \mathcal{V})$ be an irreducible representation of level zero of G. Suppose π contains a maximal and cuspidal level zero R-type (G_x, σ). Then π is cuspidal.

PROOF: As $\pi \simeq \mathrm{I}^G_{M_x}(\sigma)$, by Chapter 1 Theorem 6.1, π is cuspidal. □

REMARK. By Chapter 3 Sections 4 and 5, σ appears in the decomposition modulo ℓ of a cuspidal ℓ-adic representation λ. In this case we can adapt the lifting argument used for $\mathrm{GL}_n(F)$, [**Vig96**, Chapter 3, 3.3], to give a proof of Lemma 6.7. In the ℓ-adic case the representation $\mathrm{I}^G_{M_x} \lambda$ is irreducible and cuspidal, [**Mor99**, Section 2]. Decomposition modulo ℓ commutes with compact induction, by Chapter 1 Corollary 5.2,

$$d_\ell(\mathrm{I}^G_{M_x} \lambda) = [\mathrm{I}^G_{M_x} d_\ell \lambda],$$

and $\mathrm{I}^G_{M_x} \sigma$ is contained in $d_\ell(\mathrm{I}^G_{M_x} \lambda)$. By Chapter 1 Lemma 5.4, $d_\ell(\mathrm{ind}^G_{M_x} \lambda)$ is a sum of irreducible cuspidal representations. Hence $\mathrm{I}^G_{M_x} \sigma$ is cuspidal and $\pi \simeq \mathrm{I}^G_{M_x} \sigma$.

The next two lemmas are applications of the theory of G-covers. Hence we specialise to an unramified unitary group in three variables. Let $\mathfrak{I}$ be the standard Iwahori subgroup of G and $\mathfrak{I}_T = T \cap \mathfrak{I}$.

LEMMA 6.8. Let G be an unramified unitary group in three variables. Let (G_x, σ) be a maximal cuspidal level zero R-type in G with σ a cuspidal subquotient of $i^{M_x}_{\overline{B}} \chi$ where $\overline{B}$ is the standard Borel subgroup of M_x. Then $\pi \simeq \mathrm{I}^G_{M_x} \sigma$ is a subquotient of $i^G_B(\mathrm{ind}^T_{\mathfrak{I}_T} \chi)$.

PROOF: By exactness, $\mathrm{I}^G_{M_x} \sigma$ is a subquotient of $\mathrm{I}^G_{M_x}(i^{M_x}_{\overline{B}} \chi)$ and we have

$$\begin{aligned}\mathrm{I}^G_{M_x}(i^{M_x}_{\overline{B}} \chi) = \mathrm{ind}^G_{G_x} \mathrm{infl}_{G_x^1} i^{M_x}_{\overline{B}} \chi &\simeq \mathrm{ind}^G_{G_x} \mathrm{ind}^{G_x}_{\mathfrak{I}} \mathrm{infl}_{\mathfrak{I}^1} \chi \\ &\simeq \mathrm{ind}^G_{\mathfrak{I}} \mathrm{infl}_{\mathfrak{I}^1} \chi.\end{aligned}$$

By Lemma 4.7, $(\mathfrak{I}, \chi)$ is a G-cover of $(\mathfrak{I}_T, \chi)$. Hence, by Theorem 4.12, $\mathrm{ind}^G_{\mathfrak{I}} \mathrm{infl}_{\mathfrak{I}^1} \chi \simeq i^G_B(\mathrm{ind}^T_{\mathfrak{I}_T} \chi)$. □

LEMMA 6.9. Let G be an unramified unitary group in three variables. Let $(\pi, \mathcal{V})$ be an irreducible cuspidal representation of level zero of G. Let (K, σ) be a cuspidal level zero R-type contained in π. Then (K, σ) is maximal.

PROOF: Suppose that (K,σ) is a cuspidal level zero R-type contained in π which is not maximal. Thus $K = \mathfrak{I}$, and by Lemma 4.7 $(\mathfrak{I},\sigma)$ is a G-cover of $(\mathfrak{I}_T,\sigma_T)$. The representation $r_B^G\pi \neq 0$ by Corollary 4.11 . Thus π is not cuspidal. □

REMARK. For ℓ-adic representations the analogous theorem is true for a general p-adic reductive group, [**Mor99**] . Any non-maximal cuspidal ℓ-adic type is shown to be a G-cover relative to some parabolic subgroup and Corollary 4.11 then implies the analogous theorem.

Assume that the Jacquet functor commutes with d_ℓ, i.e. $[r_P^G \circ d_\ell] = d_\ell \circ r_P^G$ for all parabolic subgroups P of G. By Chapter 1 Lemma 5.3, this is known for classical groups. Then if σ lifts to an ℓ-adic representation, which is true if $G = \mathrm{U}(2,1)(E/F)$ and K is a non-maximal parahoric subgroup, we can use the ℓ-adic result and a decomposition modulo ℓ argument to give an alternative proof of Lemma 6.9.

LEMMA 6.10. Let G be an unramified unitary group. Let G_z be a maximal parahoric subgroup of G for which we have an Iwasawa decomposition $G = BG_z$. Let $\overline{B} = \overline{T} \ltimes \overline{N}$ be the standard Borel subgroup of the finite group M_z. Let χ be an irreducible level zero character of T and σ be a cuspidal representation of M_z. Then

$$\mathrm{R}_{M_z}^G(i_B^G\chi) = i_{\overline{B}}^{M_z}(\mathrm{inv}_{G_z^1\cap T}(\mathrm{Res}_{T\cap G_z}^T(\chi))).$$

Furthermore, if $\mathrm{I}_{M_z}^G\,\sigma$ is a subquotient of $i_B^G\chi$ then

$$\sigma \in \left[i_{\overline{B}}^{M_z}(\mathrm{inv}_{G_z^1\cap T}(\mathrm{Res}_{T\cap G_z}^T(\chi)))\right]$$

and hence is not supercuspidal.

PROOF: By the restriction-induction formula, Chapter 1 Lemma 3.5, and the Iwasawa decomposition $G = BG_z$ we have

$$\begin{aligned}\mathrm{R}_{M_z}^G(i_B^G\chi) = \left(\mathrm{Res}_{G_z}^G(i_B^G\chi)\right)^{G_z^1} &\simeq \left(\prod_{B\backslash G/G_z}\mathrm{Ind}_{{}^gB\cap G_z}^{G_z}(\mathrm{Res}_{{}^gB\cap G_z}^B\ {}^g(\chi))\right)^{G_z^1}\\ &\simeq \left(\mathrm{Ind}_{B\cap G_z}^{G_z}(\mathrm{Res}_{B\cap G_z}^B(\chi))\right)^{G_z^1}.\end{aligned}$$

We proceed with similar arguments as those given in the proof of Lemma 6.3. We identify the categories $\mathfrak{R}_R(M_z)$ and $\mathfrak{R}_R^{G_z^1}(G_z)$ representations of G_z trivial on G_z^1. Because G_z^1 is normal in G_z and $N\cap G_z$ acts trivially on $\mathrm{infl}_N(\chi)$

$$\begin{aligned}(\mathrm{Ind}_{B\cap G_z}^{G_z}\chi)^{G_z^1} &\simeq \mathrm{Ind}_{(B\cap G_z)G_z^1}^{G_z}\circ\mathrm{infl}_{G_z^1}\circ\mathrm{inv}_{G_z^1\cap B}\circ\mathrm{Res}_{B\cap G_z}^B\circ\mathrm{infl}_N(\chi)\\ &\simeq \mathrm{Ind}_{(B\cap G_z)G_z^1}^{G_z}\circ\mathrm{infl}_{G_z^1}\circ\mathrm{infl}_{N\cap G_z}\circ\mathrm{inv}_{G_z^1\cap T}\circ\mathrm{Res}_{T\cap G_z}^T(\chi)\end{aligned}$$

The character $\mathrm{inv}_{G_z^1\cap T}\circ\mathrm{Res}_{T\cap G_z}^T(\chi)$ of $T\cap G_z$ is trivial on $T\cap G_z^1$ and identifies with a character of $\overline{T}$ and we have

$$(\mathrm{Ind}_{B\cap G_z}^{G_z}\chi)^{G_z^1} \simeq i_{\overline{B}}^{M_z}\,\mathrm{inv}_{G_z^1\cap T}\circ\mathrm{Res}_{T\cap G_z}^T(\chi).$$

Thus, by exactness of level zero parahoric restriction and because $\mathrm{R}_{M_z}^G(\mathrm{I}_{M_z}^G\,\sigma) = \sigma$, σ is a subquotient of $i_{\overline{B}}^{M_z}\,\mathrm{inv}_{G_z^1\cap T}\circ\mathrm{Res}_{T\cap G_z}^T(\chi)$. □

7. Level zero types for unramified $\mathrm{U}(2,1)(E/F)$

Let G be the unramified unitary group $\mathrm{U}(2,1)(E/F)$. Then G has two classes of maximal parahoric subgroups, Chapter 1 Section 2.3. We fix representatives

$$G_x = M_3(\mathcal{O}_E) \cap G, \qquad G_y = \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{P}_E^{-1} \\ \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \end{pmatrix} \cap G,$$

which both contain the standard Iwahori subgroup

$$\mathfrak{I} = \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \end{pmatrix} \cap G.$$

Both G_x and G_y are equal to their normalizers in G, $M_x \simeq \mathrm{U}(2,1)(k_E/k_F)$ and $M_y \simeq \mathrm{U}(1,1)(k_E/k_F) \times \mathrm{U}(1)(k_E/k_F)$.

7.1. The level zero cuspidal representations of unramified $\mathrm{U}(2,1)(E/F)$. The results of the preceding sections, in particular Lemmas 6.6 and 6.9 together with the decomposition matrices of Chapter 3, allow us to completely list the irreducible cuspidal level zero ℓ-modular representations of $\mathrm{U}(2,1)(E/F)$.

Theorem 7.1. Let G be an unramified p-adic unitary group in three variables and suppose $\ell \neq 2, 3$. Every irreducible cuspidal ℓ-modular level zero representation of G appears in the decomposition modulo ℓ of an irreducible cuspidal ℓ-adic level zero representation of G. When $\ell \nmid q^2 - q + 1$ all irreducible cuspidal ℓ-modular level zero representations lift. If $\ell \mid q^2 - q + 1$ there are $q+1$ irreducible cuspidal ℓ-modular level zero representations $\mathrm{I}_{M_x}^G\, \overline{\sigma}^-_{T_1,\overline{\theta}}$ that do not lift. All irreducible ℓ-modular and ℓ-adic cuspidal level zero representations of G are listed in the table; for the conditions on χ and θ see Chapter 3 Sections 4 and 5.

Irreducible ℓ-adic	Decomposition modulo ℓ		
level zero cuspidal	$\ell \mid q-1$	$\ell \mid q+1$	$\ell \mid q^2-q+1$
$\mathrm{I}_{M_x}^G\, \nu_\chi$	$\mathrm{I}_{M_x}^G\, \overline{\nu}_{\overline{\chi}}$	$\mathrm{I}_{M_x}^G\, \overline{\nu}_{\overline{\chi}}$	$\mathrm{I}_{M_x}^G\, \overline{\nu}_{\overline{\chi}}$
$\mathrm{I}_{M_x}^G\, \sigma_{T_1,\theta}$	$\mathrm{I}_{M_x}^G\, \overline{\sigma}_{T_1,\overline{\theta}}$	$\mathrm{I}_{M_x}^G\, \overline{\sigma}_{T_1,\overline{\theta}}$	$\mathrm{I}_{M_x}^G\, \overline{\nu}_{\overline{\chi}} \oplus \mathrm{I}_{M_x}^G\, \overline{\sigma}^-_{T_1,\overline{\theta}}$
$\mathrm{I}_{M_x}^G\, \sigma_{T_2,\theta}$	$\mathrm{I}_{M_x}^G\, \overline{\sigma}_{T_2,\overline{\theta}}$	$\mathrm{I}_{M_x}^G\, \overline{\sigma}_{T_2,\overline{\theta}}$	$\mathrm{I}_{M_x}^G\, \overline{\sigma}_{T_2,\overline{\theta}}$
$\mathrm{I}_{M_y}^G(\sigma_{T_1,\theta} \otimes \sigma)$	$\mathrm{I}_{M_y}^G(\overline{\sigma}_{T_1,\overline{\theta}} \otimes \overline{\sigma})$	$\mathrm{I}_{M_y}^G(\overline{\sigma}_{T_1,\overline{\theta}} \otimes \overline{\sigma})$	$\mathrm{I}_{M_y}^G(\overline{\sigma}_{T_1,\overline{\theta}} \otimes \overline{\sigma})$

Proof: By Corollary 6.4 all cuspidal level zero ℓ-modular representations of G are of the form $\mathrm{I}_{M_z}^G\, \sigma$ where G_z is a maximal parahoric subgroup of G and σ an irreducible cuspidal ℓ-modular representation of M_z. Furthermore these are all cuspidal and irreducible by Lemmas 6.6 and 6.7, and are distinct by Corollary 6.4. Suppose σ is an irreducible cuspidal ℓ-adic representation of M_z with

$$d_\ell(\sigma) = \bigoplus_{i=1}^n \overline{\sigma}_i.$$

By Chapter 1 Corollary 5.2 decomposition modulo ℓ commutes with compact induction. Thus

$$d_\ell\left(\mathrm{I}_{M_z}^G\, \sigma\right) = \left[\mathrm{ind}_{G_z}^G\, d_\ell \sigma\right]$$

$$= \left[\operatorname{ind}_{G_z}^G \bigoplus_{i=1}^n \overline{\sigma}_i\right]$$
$$= \bigoplus_{i=1}^n \mathrm{I}_{M_z}^G \, \overline{\sigma}_i.$$

The remaining statements follow from the decomposition matrices, Chapter 3 Sections 4 and 5, of the finite groups M_x and M_y. □

8. Quasi-projective representations

A representation $(\pi_1, \mathcal{V}_1)$ of G is called quasi-projective if, for all representations $(\pi_2, \mathcal{V}_2)$ of G and all surjective morphisms

$$\varphi \in \mathrm{Hom}_G(\mathcal{V}_1, \mathcal{V}_2),$$

the homomorphism

$$\mathrm{End}_G(\mathcal{V}_1) \to \mathrm{Hom}_G(\mathcal{V}_1, \mathcal{V}_2)$$
$$\alpha \mapsto \varphi \circ \alpha$$

is surjective.

Theorem 8.1 ([**Vig98**, Appendix, Theorem 10]). Let $(\pi, \mathcal{V})$ be quasi-projective and finitely generated. Let $(\sigma, \mathcal{W})$ be a be a representation of G such that $\mathrm{Hom}_G(\mathcal{V}, \mathcal{W}) \neq 0$. The map taking $\mathcal{W}$ to $\mathrm{Hom}_G(\mathcal{V}, \mathcal{W})$ induces a bijection between the irreducible quotients of $\mathcal{V}$ and the simple $\mathrm{End}_G(\mathcal{V})$-modules.

$$\left\{ \begin{array}{c} \text{Irreducible representations } W \\ \text{such that } \mathrm{Hom}_G(\mathcal{V}, W) \neq \{0\} \end{array} \right\} \longleftrightarrow \{\text{Simple right } \mathrm{End}_G(\mathcal{V})\text{-modules}\}$$

Let K be a closed subgroup of G, $(\pi, \mathcal{V}) \in \mathfrak{R}_R(G)$ and $\sigma \in \mathrm{Irr}_R(K)$. The σ-isotypic component $(\pi^\sigma, \mathcal{V}^\sigma)$ of $(\pi, \mathcal{V})$ is the largest subrepresentation of $\mathrm{Res}_K^G(\sigma)$ which is semisimple and all of whose irreducible subquotients are isomorphic to σ. Hence

$$\pi^\sigma = \sum_{m \in \mathrm{Hom}_K(\sigma, \mathrm{Res}_K^G(\pi))} m(\sigma)$$

is equal to the σ-isotypic component of π.

Lemma 8.2 ([**Vig01b**, Lemma 3.1]). Let (K, σ) be an R-type of G and $\mathcal{V} = \operatorname{ind}_K^G \sigma$. If there is a decomposition

$$\mathcal{V} = \mathcal{V}^\sigma \oplus \mathcal{V}_\sigma$$

as representations of K such that no subquotient of $\mathcal{V}_\sigma$ is isomorphic to σ then $\mathcal{V}$ is quasi-projective.

Lemma 8.3 ([**Vig01b**, Proposition 6.1]). Let (G_z, σ) be a cuspidal level zero R-type. Then $\mathrm{I}_{M_z}^G \, \sigma$ is quasi-projective and finitely generated.

Proof: As compact induction sends finitely generated representations to finitely generated representations, $\mathrm{I}_{M_z}^G \, \sigma$ is finitely generated. Let $\mathcal{V}$ be the space of $\mathrm{I}_{M_z}^G \, \sigma$. As G_z^1 is pro-p, by Chapter 1 Lemma 3.2,

$$\mathcal{V} = \mathcal{V}^{G_z^1} \oplus \mathcal{V}(G_z^1).$$

No irreducible subquotient of $\mathrm{Res}^G_{G_z}(\mathcal{V}(G_z^1))$ is trivial on G_z^1 hence no irreducible subquotient can be isomorphic to σ. Because σ is cuspidal, by Lemma 6.3,

$$\mathcal{V}^{G_z^1} \simeq \bigoplus_{D_{z,z}} {}^n\sigma.$$

Therefore $\mathcal{V}^{G_z^1} \simeq \mathcal{V}^\sigma$ and we have a direct sum decomposition $\mathcal{V} = \mathcal{V}_\sigma \oplus \mathcal{V}^\sigma$. □

LEMMA 8.4. Let G be an unramified unitary group in three variables and $B = T \ltimes N$ be the standard Borel subgroup of G. Suppose $\ell \neq 2, 3$ and $\ell \mid q+1$. Let χ be an unramified character of T such that $\mathrm{Ind}_B^G \chi$ is reducible. Then 1_G is an irreducible quotient of $\mathrm{Ind}_B^G \chi$.

PROOF: By Theorem 8.3, $\mathrm{ind}_{\mathfrak{I}}^G 1_{\mathfrak{I}}$ is quasi-projective and of finite type. Thus, by Theorem 8.1, the map:

$$M_{1_{\mathfrak{I}}} : \mathfrak{R}_R(G) \to \mathcal{H}(G, \mathfrak{I}, 1_{\mathfrak{I}})\text{-}\mathfrak{Mod}$$
$$\pi \mapsto \mathrm{Hom}_G(\mathrm{ind}_{\mathfrak{I}}^G 1_{\mathfrak{I}}, \pi)$$

induces a bijection between irreducible quotients of $\mathrm{ind}_{\mathfrak{I}}^G 1_{\mathfrak{I}}$ and simple $\mathcal{H}(G, \mathfrak{I}, 1_{\mathfrak{I}})$-modules. Let $\mathfrak{I}_T = T \cap \mathfrak{I}$. By Theorem 4.12

$$\mathrm{ind}_{\mathfrak{I}}^G 1_{\mathfrak{I}} \simeq \mathrm{Ind}_B^G \mathrm{ind}_{\mathfrak{I}_T}^T 1_{\mathfrak{I}_T}.$$

Thus, an irreducible quotient of $\mathrm{Ind}_B^G \chi$ is an irreducible quotient of $\mathrm{ind}_{\mathfrak{I}}^G 1_{\mathfrak{I}}$ as χ is a quotient of $\mathrm{ind}_{\mathfrak{I}_T}^T 1_{\mathfrak{I}_T}$.

By Theorem 4.8

$$\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, \mathfrak{I}, 1_{\mathfrak{I}}) \simeq \overline{\mathbb{F}}_\ell \left[f_1, f_2 : (f_1+1)^2 = (f_2+1)^2 = 0\right].$$

The Hecke algebra $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(T, \mathfrak{I}_T, 1_{\mathfrak{I}_T}) \simeq \overline{\mathbb{F}}_\ell[X]$, and the inclusion j_B is induced by mapping X to $f_1 f_2$. Suppose σ is a quotient of $\mathrm{Ind}_B^G \chi$. By Theorem 4.10

$$j_B^*(M_{1_{\mathfrak{I}}}(\sigma)) \simeq M_{1_{\mathfrak{I}_T}}(r_B^G(\sigma)).$$

If $\mathrm{Ind}_B^G \chi$ is reducible and σ is a proper quotient, then by the geometric lemma $r_B^G(\sigma)$ is a character. Hence $M_{1_{\mathfrak{I}}}(\sigma)$ is a simple one dimensional $\overline{\mathbb{F}}_\ell$-module. By Corollary 4.9 when $\ell \mid q+1$ there is a unique one dimensional $\overline{\mathbb{F}}_\ell$-module of $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, \mathfrak{I}, 1_{\mathfrak{I}})$. By reciprocity 1_G is always an irreducible quotient of $\mathrm{ind}_{\mathfrak{I}}^G 1_{\mathfrak{I}}$, hence $\sigma \simeq 1_G$. □

LEMMA 8.5.

(1) Suppose $\ell \neq 2, 3$ and $\ell \mid q^2 - q + 1$. Then

$$\left|\left\{\text{Irreducible quotients of } \mathrm{Ind}_B^G \chi : \chi \text{ is unramified and } \mathrm{Ind}_B^G \chi \text{ is reducible}\right\}\right| \leqslant 2$$

(2) Suppose $\ell \neq 2, 3$ and $\ell \mid q - 1$ or ℓ is banal. Then

$$\left|\left\{\text{Irreducible quotients of } \mathrm{Ind}_B^G \chi : \chi \text{ is unramified and } \mathrm{Ind}_B^G \chi \text{ is reducible}\right\}\right| \leqslant 4$$

PROOF: The proof is similar to the proof of Lemma 8.4. Suppose $\ell \neq 2, 3$ and $\ell \mid q^2 - q + 1$. Then by Corollary 4.9 there are two distinct characters of $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, \mathfrak{I}, 1_{\mathfrak{I}})$, hence if the unramified principal series representation $\mathrm{Ind}_B^G \chi$ is reducible then there are two possible quotients. One of these is 1_G. Similarly if $\ell \neq 2$ and ℓ is banal or $\ell \mid q - 1$, there are four possible quotients. □

9. Reducibility points of unramified representations of $\mathrm{U}(2,1)(E/F)$

Let $G = \mathrm{U}(2,1)(E/F)$ be the unramified unitary group in three variables. In the ℓ-adic case the reducibility points of the parabolic induction of G are worked out in [**Key84**], which are the same as the reducibility points of $\mathrm{SU}(2,1)(E/F)$. Recall, T_0 denotes the diagonal maximal F-split torus

$$T_0 = \left\{\mathrm{diag}(x, 1, x^{-1}) : x \in F^\times\right\},$$

isomorphic to $F^\times$, T the centraliser of T_0 in G,

$$T = \left\{\mathrm{diag}(x, y, \overline{x}^{-1}) : x \in E^\times,\ y \in E^1\right\}$$

isomorphic to $E^\times \times E^1$, $B = T \ltimes N$ the standard Borel subgroup containing T

$$B = \begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & 0 & \star \end{pmatrix} \cap G.$$

The Weyl group $W \simeq C_2$ and if $w \in W$ is the nontrivial element, then

$$\mathrm{diag}(x, y, \overline{x}^{-1})^w = \mathrm{diag}(\overline{x}^{-1}, y, x).$$

Let χ_1 be a character of $E^\times$ and χ_2 be a character of E^1. Let χ be the character of T defined by

$$\chi\left(\mathrm{diag}(x, y, \overline{x}^{-1})\right) = \chi_1(x)\chi_2(x\overline{x}^{-1}y)$$

which is well defined because $x \to x\overline{x}^{-1}$ is a surjective map $E^\times \to E^1$. Every character of T appears in this way; we can recover χ_1 and χ_2 from χ

$$\chi_1(x) = \chi(\mathrm{diag}(x, \overline{x}/x, \overline{x}^{-1})), \quad \chi_2(y) = \chi(\mathrm{diag}(1, y, 1)).$$

The character χ_2 factors through the determinant and

$$i_B^G(\chi) = i_B^G(\chi_1 \otimes 1)(\chi_2 \circ \det)$$

where $\chi_1 \otimes 1$ is defined by $\chi_1 \otimes 1(\mathrm{diag}(x, y, \overline{x}^{-1})) = \chi_1(x)$. Hence the reducibility of $i_B^G(\chi)$ is completely determined by that of $i_B^G(\chi_1 \otimes 1)$. The character χ is regular when $\chi_1(x) \neq \chi_1(\overline{x})^{-1}$, i.e. when χ_1 is non-trivial on the norm 1 elements E^1 of E.

The modulus character $\delta_B : T \to R^\times$ is given by

$$\delta_B \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & \overline{x}^{-1} \end{pmatrix} = q^{-4val_F(x)}.$$

Thus δ_B is trivial when $\ell \mid q-1$ and when $\ell \mid q+1$.

9.1. Harish-Chandra j-functions. We suppose that G has discrete co-compact subgroups. By a p-adic field we mean a non-archimedean local field of characteristic zero By [**BH78**, Theorem A], if F is a p-adic field and G is the F points of a reductive group defined over F then G has discrete co-compact subgroups. Thus, while it may not be necessary, we specialise to when G is an unramified p-adic unitary group in three variables.

Fix a character $\chi \simeq \chi_1 \otimes 1$ of T. The reducibility of the representation $\mathrm{Ind}_B^G\, \delta_B^{\frac{1}{2}} \otimes \chi$ is related to the order of the zero at 1 of the associated j-function j_χ, [**Dat05**].

By [**Dat05**, Proposition 8.2], given an irreducible ℓ-modular representation σ of T which lifts to an ℓ-adic representation $\widetilde{\sigma}$, the ℓ-modular j-function j_σ is given by restriction of the ℓ-adic j-function $j_{\widetilde{\sigma}}$. All irreducible ℓ-modular representations of T lift. Let $o_1(j_\chi)$ denote the order of vanishing of j_χ at 1.

THEOREM 9.1 ([**Dat05**, Proposition 8.4]). Suppose $\ell \neq 2$. Let G be an unramified p-adic unitary group in three variables and let χ be an irreducible ℓ-modular representation of T,

(1) If $\chi^w \neq \chi$, then $i_B^G(\delta_B^{\frac{1}{2}} \otimes \chi)$ is reducible if and only if $o_1(j_\chi) \geqslant 1$.
(2) If $\chi^w = \chi$, then $o_1(j_\chi) \geqslant -2$ is even, and:
    (a) If $o_1(j_\chi) = -2$, then $i_B^G(\delta_B^{\frac{1}{2}} \otimes \chi)$ is irreducible.
    (b) If $o_1(j_\chi) = 0$, then $i_B^G(\delta_B^{\frac{1}{2}} \otimes \chi)$ is reducible and semisimple.
    (c) If $o_1(j_\chi) \geqslant 2$, then $i_B^G(\delta_B^{\frac{1}{2}} \otimes \chi)$ is reducible.

In fact, in our case, there is a necessary and sufficient condition for a cuspidal representation appearing in the composition series of $i_B^G(\delta_B^{\frac{1}{2}} \otimes \chi)$.

THEOREM 9.2 ([**Dat05**, Proposition 8.6]). Suppose $\ell \neq 2$ prime. Let G be an unramified p-adic unitary group in three variables and χ be an irreducible ℓ-modular representation of T. Then $o(j_\chi(1)) \geqslant 2$ if and only if $i_B^G(\delta_B^{\frac{1}{2}} \otimes \chi)$ has a cuspidal subquotient.

REMARK. Theorems 9.1 and 9.2 are special cases of the propositions given in [**Dat05**] which we have not stated in full generality. However we have to specialise to a group with discrete cocompact subgroups to apply Theorems 9.1 and 9.2.

Let $\omega_{E/F}$ be the unique unramified character of $E^\times$ whose restriction to $F^\times$ is the unique character related by class field theory to the quadratic unramified extension of E/F. Assume $\ell \neq 2, 3$. Using the computations of [**Key84**, Section 5] and Theorems 9.1 and 9.2, we can calculate the reducibility points of $\mathrm{Ind}_B^G(\chi)$:

THEOREM 9.3. Let G be an unramified p-adic unitary group in three variables. Let $\chi = \chi_1 \otimes 1$ be an unramified character of T. Then $i_B^G(\chi)$ is irreducible unless χ is one of the four characters 1_T, δ_B^{-1}, $\omega_{E/F} \otimes \delta_B^{-\frac{1}{4}}$ and $\omega_{E/F} \otimes \delta_B^{-\frac{3}{4}}$. Furthermore, if $i_B^G\chi$ is reducible then it has length two if $q^3 \not\equiv -1 \bmod \ell$ and length greater than or equal to three if $q^3 \equiv -1 \bmod \ell$.

PROOF: By [**Key84**, Section 5],

$$j_1(\chi) = q^k \frac{(q^2 - \chi_1(\varpi_E))(q + \chi_1(\varpi_E))}{(1 - \chi_1(\varpi_E))(1 + \chi_1(\varpi_E))} \frac{(q^{-2} - \chi_1(\varpi_E))(q^{-1} + \chi_1(\varpi_E))}{(1 - \chi_1(\varpi_E))(1 + \chi_1(\varpi_E))}$$

with $k \in \mathbb{Z}$. Thus $i_B^G(\delta_B^{\frac{1}{2}} \otimes \chi)$ is irreducible unless $\chi_1(\varpi_E) = -q^{\pm 1}, q^{\pm 2}$, by Theorem 9.1, which corresponds to $\chi = \delta_B^{\pm\frac{1}{2}}$, $\omega_{E/F} \otimes \delta_B^{\pm\frac{1}{4}}$.

If $q^3 = -1 \bmod \ell$, then $q^{\pm 2} = q^{\mp 1} \bmod \ell$ and there are only two cases. Equivalently, notice that when $q^3 = -1 \bmod \ell$ we have $\delta_B^{\pm\frac{1}{2}} = \omega_{E/F} \otimes \delta_B^{-\mp\frac{1}{4}}$. Applying Theorem 9.2:

(1) If $\ell \mid q - 1$,
    (a) In the case $\chi = \delta_B^{\pm\frac{1}{2}}$, $o_1(j_\chi) = 0$ and the length of $i_B^G(\delta_B^{\frac{1}{2}} \otimes \chi)$ is two.
    (b) In the case, $\chi = \omega_{E/F} \otimes \delta_B^{\pm\frac{1}{4}}$, $o_1(j_\chi) = 0$ and the length of $i_B^G(\delta_B^{\frac{1}{2}} \otimes \chi)$ is two.

(2) If $\ell \mid q+1$, then $o_1(j_\chi) = 2$, and the length of $i_B^G(\delta_B^{\frac{1}{2}} \otimes \chi)$ is greater than or equal to three.
(3) If $\ell \mid q^2 - q + 1$, then $o_1(j_\chi) = 2$, and the length of $i_B^G(\delta_B^{\frac{1}{2}} \otimes \chi)$ is greater than or equal to three.

Finally twist by the character $\delta_B^{-\frac{1}{2}}$. □

10. Decomposition of $i_B^G(1_T)$

In this section we do not appeal to Theorems 9.1 and 9.2, hence the decompositions we obtain also apply when F is of positive characteristic.

Theorem 10.1. Let $G = \mathrm{U}(2,1)(E/F)$ be the unramified unitary group in three variables, and assume $\ell \neq 2, 3$. Then

(1) If $\ell \mid q-1$ or if ℓ is banal, then $i_B^G 1$ has length two with an irreducible subrepresentation isomorphic to 1_G and an irreducible countably infinite dimensional quotient St_G.

$$0 \longrightarrow 1_G \longrightarrow i_B^G 1_T \longrightarrow \mathrm{St}_G \longrightarrow 0$$

Furthermore, if $\ell \mid q-1$ there is a direct sum decomposition:

$$i_B^G 1_T = 1_G \oplus \mathrm{St}_G\,.$$

(2) If $\ell \mid q+1$, then $i_B^G 1$ has length six with a unique irreducible subrepresentation isomorphic to 1_G, a unique irreducible quotient isomorphic to 1_G, and four cuspidal subquotients isomorphic to

$$\mathrm{I}_{M_x}^G\, \overline{\nu}_{\overline{1}},\ \mathrm{I}_{M_x}^G\, \overline{\sigma}_{T_2,\overline{\theta}},\ \mathrm{I}_{M_y}^G\, \overline{\sigma}_{T_1,\overline{\theta}}$$

with $\mathrm{I}_{M_x}^G\, \overline{\nu}_{\overline{1}}$ appearing with with multiplicity 2.

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & \pi & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & 1_G & \longrightarrow & i_B^G 1_T & \longrightarrow & \mathrm{St}_G & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & 1_G & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

The quotient St_G is reducible with subrepresentation π with $\pi \simeq \mathrm{I}_{M_y}^G\, \overline{\sigma}_{T_1,\overline{\theta}} \oplus \widehat{\pi}$,

$$[\widehat{\pi}] \simeq \mathrm{I}_{M_x}^G\, \overline{\nu}_{\overline{1}} \oplus \mathrm{I}_{M_x}^G\, \overline{\nu}_{\overline{1}} \oplus \mathrm{I}_{M_x}^G\, \overline{\sigma}_{T_2,\overline{\theta}}.$$

(3) If $\ell \mid q^2 - q + 1$, then $i_B^G 1$ has length three with a unique irreducible subrepresentation isomorphic to 1_G, a unique irreducible quotient η with $r_B^G(\eta) = \delta_B$, and one cuspidal subquotient isomorphic to $\mathrm{I}_{M_x}^G\, \overline{\sigma}_{T_1,\overline{\theta}}$.

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & \mathrm{I}_{M_x}^G\,\overline{\sigma}_{T_1,\overline{\theta}} & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & 1_G & \longrightarrow & i_B^G 1_T & \longrightarrow & \mathrm{St}_G & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & \eta & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

Proof: The space of constant functions is an irreducible subrepresentation in all three cases isomorphic to 1_G. We denote by St_G the quotient of $i_B^G 1_T$ by 1_G.

$$0 \longrightarrow 1_G \longrightarrow i_B^G 1_T \longrightarrow \mathrm{St}_G \longrightarrow 0$$

By the Chapter 1 Lemma 3.12, $r_B^G(i_B^G 1_T)$ has length two and its semisimplification is

$$[r_B^G(i_B^G 1_T)] = 1_T \oplus \delta_B.$$

By exactness of the Jacquet functor, knowing that 1_G is a subrepresentation of $i_B^G 1_T$ with $r_B^G(1_G) = 1_T$, $r_B^G(\mathrm{St}_G) \simeq \delta_B$.

$$0 \longrightarrow 1_T \longrightarrow r_B^G(i_B^G 1_T) \longrightarrow \delta_B \longrightarrow 0$$

Because the Jacquet functor of a quotient of a parabolically induced representation is non-zero, Chapter 1 Lemma 3.11, and a quotient of St_G is a quotient of $i_B^G 1_T$ there is an irreducible quotient η of St_G which is a quotient of $i_B^G 1_T$ with $r_B^G(\eta) = \delta_B$. By exactness of the Jacquet functor all other irreducible subquotients of St_G must be cuspidal. Hence either $i_B^G 1_T$ has a unique irreducible subrepresentation and a unique irreducible quotient η, which could be St_G, or 1_G is a direct factor. If 1_G is a direct factor then $i_B^G 1_T \simeq 1_G \oplus \mathrm{St}_G$ and St_G must be irreducible; otherwise the irreducible subrepresentation of St_G would be a subrepresentation of $i_B^G 1_T$ and not cuspidal.

Let G_z be one of the two standard maximal parahoric subgroups G_x and G_y of G. The next step is a slight simplification of the proof of Lemma 6.10. By the restriction-induction formula, Lemma 3.5, and the Iwasawa decomposition $G = BG_z$ we have

$$\begin{aligned} \mathrm{R}_{M_z}^G(i_B^G 1) = \left(\mathrm{Res}_{G_x}^G(i_B^G 1)\right)^{G_x^1} &\simeq \left(\prod_{B\backslash G/G_x} \mathrm{Ind}_{{}^gB\cap G_x}^{G_x}(\mathrm{Res}_{{}^gB\cap G_x}^B\ {}^g(1))\right)^{G_x^1} \\ &\simeq \left(\mathrm{Ind}_{B\cap G_x}^{G_x}(\mathrm{Res}_{B\cap G_x}^B(1))\right)^{G_x^1}. \end{aligned}$$

Because G_z^1 is normal in G_z

$$(\mathrm{Ind}_{B\cap G_z}^{G_z} 1)^{G_z^1} \simeq \mathrm{Ind}_{(B\cap G_z)G_z^1}^{G_z} 1.$$

Inflation and induction commute hence

$$\mathrm{Ind}_{(B\cap G_z)G_z^1}^{G_z} 1 \simeq \mathrm{infl}_{G_z^1}\, i_{\overline{B}}^{M_z} 1_{\overline{T}}.$$

By Theorem 3.3, all subquotients of $i_B^G 1$ are level zero. Thus every irreducible subquotient must have non-trivial invariants under the pro-unipotent radical of one of the two maximal parahoric subgroups G_x and G_y. Furthermore, we know that

$$\mathrm{R}_{M_z}^G(i_B^G 1) \simeq \mathrm{infl}_{G_z^1}\, i_B^{M_z} 1$$

and we listed the subquotients of $i_B^{M_z} 1$ in Chapter 3. Let St_{M_z} denote the, not necessarily irreducible, quotient of $i_B^{M_z} 1$ by 1_{M_z}.

$$0 \longrightarrow 1_{M_z} \longrightarrow i_B^{M_z} 1 \longrightarrow \mathrm{St}_{M_z} \longrightarrow 0$$

By exactness of G_z^1-invariants

$$(\star) \qquad \mathrm{R}_{M_z}^G(\mathrm{St}_G) \simeq \mathrm{St}_{M_z}\,.$$

All cuspidal level zero representations of G are listed in Theorem 7.1. These cuspidal subquotients are isomorphic to $\mathrm{I}_{M_z}^G\,\sigma$ where G_z is one of the two maximal parahoric subgroups of G and σ is a cuspidal representation of M_z.

Let G_z be one of the two maximal parahoric subgroups of G, and σ a cuspidal representation of M_z. By Lemma 6.3, because G_z is equal to the full stabilizer of the vertex z,

$$\mathrm{R}_{M_z}^G \circ \mathrm{I}_{M_z}^G(\sigma) \simeq \sigma.$$

Furthermore if $(\pi, \mathcal{V})$ is a level zero irreducible representation of G, then $(\mathcal{V})^{G_z^1} \simeq \sigma$ implies that $\pi \simeq \mathrm{ind}_{G_z}^G\,\sigma$. Note that this requires σ to be cuspidal.

By Lemma 6.6, for all cuspidal representations σ_x of M_x,

$$(\dagger_1) \qquad \mathrm{R}_{M_y}^G \circ \mathrm{I}_{M_x}^G(\sigma_x) = \{0\}.$$

Similarly, for all cuspidal representations σ_y of M_y,

$$(\dagger_2) \qquad \mathrm{R}_{M_x}^G \circ \mathrm{I}_{M_y}^G(\sigma_y) = \{0\}.$$

Using these properties of invariance under the pro-p unipotent radicals of the maximal parahoric subgroups of G we can identify all subquotients of the induced representation $\mathrm{Ind}_B^G 1$:

If $\ell \mid q-1$, or ℓ is banal for G then, by $(\star)$ and Chapter 3 Sections 4.1 and 5.1,

$$\mathrm{R}_{M_x}^G(\mathrm{St}_G) = \mathrm{St}_{U(2,1)(k_E/k_F)} \text{ and } \mathrm{R}_{M_y}^G(\mathrm{St}_G) = \mathrm{St}_{U(1,1)(k_E/k_F)}$$

where St_{M_z} are irreducible ℓ-modular representations of M_z. Because there are no cuspidal representations in either the G_x^1-invariants or the G_y^1-invariants St_G cannot have any cuspidal subquotients, and by exactness of the Jacquet functor $i_B^G 1_T$ has length 2.

By Chapter 1 Theorem 3.5,

$$\left(i_B^G 1_T\right)^{\sim} \simeq i_B^G \delta_B.$$

The contragredient is a contravariant and exact functor, Chapter 1 Theorem 3.1, thus we have an exact sequence of representations of G:

$$0 \longrightarrow \widetilde{\mathrm{St}}_G \longrightarrow i_B^G \delta_B \longrightarrow 1_G \longrightarrow 0$$

If $\ell \mid q \pm 1$, then $\delta_B = 1_T$. Hence 1_G appears as a quotient of $i_B^G 1_T$ in these cases. If $\ell \mid q-1$ because $i_B^G 1_T$ is of length two, 1_G is a direct factor of $i_B^G 1_T$:

$$i_B^G 1_T = 1_G \oplus \mathrm{St}_G\,.$$

If $\ell \mid q+1$ then, by $(\star)$ and Chapter 3 Sections 4.2 and 5.3,

$$\left[\mathrm{R}_{M_x}^G(\mathrm{St}_G)\right] = \overline{1}_{M_x} \oplus \overline{\nu}_{\overline{1}} \oplus \overline{\nu}_{\overline{1}} \oplus \overline{\sigma}_{T_2,\overline{\theta}}.$$

The representation $\mathrm{R}_{M_x}^G(\mathrm{St}_G)$ has a cuspidal subrepresentation ζ. If 1_G was a direct factor of $i_B^G 1_T$ then, by exactness, $\mathrm{R}_{M_x}^G(\mathrm{St}_G)$ would be a subrepresentation of $\mathrm{R}_{M_x}^G(i_B^G 1_T) = \mathrm{Ind}_{\overline{B}}^{M_x} 1$ contradicting the cuspidality of ζ. Hence 1_G appears twice in the composition series of $i_B^G 1_T$, as the unique irreducible subrepresentation and as the unique irreducible quotient. Furthermore, $i_B^G 1_T$ has length greater than equal to three and there is a proper subrepresentation π of St_G. All irreducible subquotients of π are cuspidal by exactness of the Jacquet functor.

Thus 1_G is a quotient of St_G with $\mathrm{R}_{M_x}^G(1_G) = 1_{M_x}$. By exactness

$$\left[\mathrm{R}_{M_x}^G(\pi)\right] = \overline{\nu}_{\overline{1}} \oplus \overline{\nu}_{\overline{1}} \oplus \overline{\sigma}_{T_2,\overline{\theta}}$$

is cuspidal. Therefore, by reciprocity,

$$\left(\mathrm{I}_{M_x}^G\,\overline{\nu}_{\overline{1}} \oplus \mathrm{I}_{M_x}^G\,\overline{\nu}_{\overline{1}} \oplus \mathrm{I}_{M_x}^G\,\overline{\sigma}_{T_2,\overline{\theta}}\right) \in [\pi].$$

Similarly,

$$[\mathrm{R}_{M_y}^G(\mathrm{St}_G)] = \overline{1}_{M_y} \oplus \overline{\sigma}_{T_1,\overline{\theta}}.$$

Therefore, by exactness,

$$\mathrm{R}_{M_x}^G(\pi) = \overline{\sigma}_{T_1,\overline{\theta}}$$

and by reciprocity

$$\mathrm{Hom}_G(\mathrm{I}_{M_y}^G\,\overline{\sigma}_{T_1,\overline{\theta}}, \pi) \simeq \mathrm{Hom}_{M_y}(\overline{\sigma}_{T_1,\overline{\theta}}, \mathrm{R}_{M_x}^G(\pi)) \neq \{0\}.$$

Thus $\mathrm{I}_{M_y}^G\,\overline{\sigma}_{T_1,\overline{\theta}}$ is an irreducible subrepresentation of π. In fact, $\mathrm{I}_{M_y}^G\,\overline{\sigma}_{T_1,\overline{\theta}}$ is a direct factor of π by applying the same reciprocity argument to $\mathrm{R}_{M_x}^G(\pi)$. Hence, because every irreducible subquotient must have nontrivial invariants under one of the two maximal parahoric subgroups,

$$[\pi] = \mathrm{I}_{M_x}^G\,\overline{\nu}_{\overline{1}} \oplus \mathrm{I}_{M_x}^G\,\overline{\nu}_{\overline{1}} \oplus \mathrm{I}_{M_x}^G\,\overline{\sigma}_{T_2,\overline{\theta}} \oplus \mathrm{I}_{M_y}^G\,\overline{\sigma}_{T_1,\overline{\theta}}.$$

If $\ell \mid q^2-q+1$ then, by $(\star)$ and Chapter 3 Sections 4 and 5.2,

$$[\mathrm{R}_{M_x}^G(\mathrm{St}_G)] = \overline{1}_{M_x} \oplus \overline{\sigma}_{T_1,\overline{\theta}} \text{ and } [\mathrm{R}_{M_y}^G(\mathrm{St}_G)] = \mathrm{St}_{U(1,1)(k_E/k_F)}$$

where $\overline{\sigma}_{T_1,\overline{\theta}}$ is irreducible and cuspidal and $\mathrm{St}_{U(1,1)(k_E/k_F)}$ is irreducible. By cuspidality of $\overline{\sigma}_{T_1,\overline{\theta}}$ it cannot be a quotient of $\mathrm{St}_{M_x} \simeq \mathrm{R}_{M_x}^G(\mathrm{St}_G)$. Hence $\overline{\sigma}_{T_1,\overline{\theta}}$ is an irreducible subrepresentation of $\mathrm{R}_{M_x}^G(\mathrm{St}_G)$ and, by reciprocity,

$$\mathrm{Hom}_G(\mathrm{I}_{M_x}^G\,\overline{\sigma}_{T_1,\overline{\theta}}, \mathrm{St}_G) \simeq \mathrm{Hom}_{M_x}(\overline{\sigma}_{T_1,\overline{\theta}}, \mathrm{R}_{M_x}^G(\mathrm{St}_G)) \neq \{0\}.$$

Thus $\mathrm{I}_{M_x}^G\,\overline{\sigma}_{T_1,\overline{\theta}}$ is a subrepresentation of St_G and $i_B^G 1_T$ has length three. By exactness of level zero parahoric restriction the irreducible quotient η of St_G has $\mathrm{R}_{M_x}^G\,\eta = \overline{1}_{M_x}$ and $\mathrm{R}_{M_y}^G\,\eta = \mathrm{St}_{U(1,1)(k_E/k_F)}$. □

11. Decomposition of $i_B^G(\omega_{E/F} \otimes \delta_B^{-\frac{1}{4}})$

In this section we finish our description of the decomposition of the unramified principal series representations of the unramified unitary group in three variables. It remains to describe the decomposition of $i_B^G(\omega_{E/F} \otimes \delta_B^{-\frac{1}{4}})$ and its contragredient $i_B^G(\omega_{E/F} \otimes \delta_B^{-\frac{3}{4}})$ when ℓ is banal or $\ell \mid q-1$. The length of $i_B^G(\omega_{E/F} \otimes \delta_B^{-\frac{1}{4}})$ is two because there are no irreducible cuspidal representations in $\mathrm{R}_{M_z}^G(i_B^G(\omega_{E/F} \otimes \delta_B^{-\frac{1}{4}}))$, by the same proof as Theorem 10.1 when ℓ is banal or $\ell \mid q-1$.

We let $\mathcal{U}$ denote an irreducible subrepresentation of $i_B^G(\omega_{E/F} \otimes \delta_B^{-\frac{1}{4}})$ with quotient $\mathcal{V}$. Thus the contragredient $i_B^G(\omega_{E/F} \otimes \delta_B^{-\frac{3}{4}})$ has $\widetilde{\mathcal{V}}$ as an irreducible subrepresentation and $\widetilde{\mathcal{U}}$ as an irreducible quotient.

$$0 \longrightarrow \mathcal{U} \longrightarrow i_B^G(\omega_{E/F} \otimes \delta_B^{-\frac{1}{4}}) \longrightarrow \mathcal{V} \longrightarrow 0$$

$$0 \longrightarrow \widetilde{\mathcal{V}} \longrightarrow i_B^G(\omega_{E/F} \otimes \delta_B^{-\frac{3}{4}}) \longrightarrow \widetilde{\mathcal{U}} \longrightarrow 0$$

When $\ell \mid q-1$ the character δ_B is trivial. If E/F is p-adic, by Theorem 9.1, when $\ell \mid q-1$, $i_B^G(\omega_{E/F})$ is semisimple.

12. Reducibility points; ramified characters

Using Theorem 9.1 we find the reducibility points not yet considered where the induced representation has cuspidal subquotients. Then we decompose the level zero induced representations with this property.

Theorem 12.1. *Assume $\ell \neq 2,3$. Let G be an unramified p-adic unitary group in three variables. Let $\chi = \chi_1 \otimes 1$ be a ramified character of T. Then $i_B^G(\chi)$ is semisimple unless $\ell \mid q+1$ and $\chi_1 \mid F^\times$ is trivial. Furthermore when $\ell \mid q+1$ and $\chi_1 \mid F^\times$ is trivial, $i_B^G(\chi)$ has length greater than or equal to three.*

Proof: Assume $\ell \neq 2,3$. When χ_1 is ramified if $\chi_1 |_{F^\times}$ is nontrivial then the j-function is a power of q, by [**Key84**, Section 5], hence, by Theorem 9.1, $i_B^G(\delta_B^{\frac{1}{2}} \otimes \chi)$ is irreducible. By [**Key84**, Section 5], if $\chi_1 |_{F^\times}$ is trivial, or equivalently ${}^w\chi = \chi$, then
$$j_\chi(1) = q^r \frac{(q+\chi_1(\varpi_E))(q^{-1}+\chi_1(\varpi_E))}{(1+\chi_1(\varpi_E))(1+\chi_1(\varpi_E))}$$
for some $r \in \mathbb{Z}$. Thus, by Theorem 9.1, $i_B^G(\delta_B^{\frac{1}{2}} \otimes \chi)$ is semisimple unless $o_1(j_\chi) = 2$ which occurs if and only if $\ell \mid q+1$ because $\chi_1(\varpi_E) = 1$. Because $\ell \mid q+1$, the character δ_B is trivial. By Theorem 9.2, when $\ell \mid q+1$ the length of $i_B^G(\chi)$ is greater than or equal to three. □

We consider two cases:

(1) <u>The level zero case:</u> If χ_1 is trivial on $1+\mathcal{P}_E$. Then $\chi_1 = \mathrm{infl}_{\langle\varpi_E\rangle}\,\chi_{\mathrm{ram}}$ where χ_{ram} is a character of $\mathcal{O}_E^\times$, because $E^\times = \mathcal{O}_E^\times\langle\varpi_E\rangle$. The character χ_{ram} is trivial on $1+\mathcal{P}_E$. Thus χ_{ram} identifies with a character $\overline{\chi}_{\mathrm{ram}}$ of $k_E^\times$. Furthermore $\overline{\chi}_{\mathrm{ram}}$ is trivial on $k_F^\times$ because χ_1 is trivial on $F^\times$ hence on $\mathcal{O}_F^\times$ and $1+\mathcal{P}_F$.

(2) <u>The positive level case:</u> If χ_1 is not trivial on $1+\mathcal{P}_E$. By smoothness there exists $i \in \mathbb{N}$ such that χ_1 is trivial on $1+\mathcal{P}_E^i$. Similarly $\chi_1 = \operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}}$ with χ_{ram} a character of $\mathcal{O}_E^\times$ trivial on $1+\mathcal{P}_E^i$.

In this chapter we only consider case (1) when $\ell \mid q+1$ and the induced representations are of the form $i_B^G(\operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1)$.

THEOREM 12.2. Let $\ell \mid q+1$. The representation $i_B^G(\operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1)$ has length four with cuspidal subquotients isomorphic to $\mathrm{I}_{M_x}^G(\overline{\sigma}_{T_2,\overline{\theta}})$ and $\mathrm{I}_{M_y}^G(\overline{\sigma}_{T_1,\overline{\theta}} \otimes \overline{1})$. Furthermore, $i_B^G(\operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1)$ has a unique irreducible subrepresentation and a unique irreducible quotient, the irreducible quotient is isomorphic to the irreducible subrepresentation and we have the following exact diagram:

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & \mathrm{I}_{M_x}^G(\overline{\sigma}_{T_2,\overline{\theta}}) \oplus \mathrm{I}_{M_y}^G(\overline{\sigma}_{T_1,\overline{\theta}} \otimes \overline{1}) & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & \zeta & \longrightarrow & i_B^G(\operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1) & \longrightarrow & \pi_2 & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & \zeta & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

PROOF: By Theorem 12.1, $i_B^G(\operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1)$ is reducible with length greater than or equal to three. By Chapter 3 Sections 4 and 5

$$\left[\mathrm{R}_{M_x}^G(i_B^G(\operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1))\right] \simeq \overline{1}_{M_x}(\overline{\chi}_{\mathrm{ram}}) \oplus \overline{1}_{M_x}(\overline{\chi}_{\mathrm{ram}}) \oplus \overline{\sigma}_{T_2,\overline{\theta}},$$
$$\left[\mathrm{R}_{M_y}^G(i_B^G(\operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1))\right] \simeq (\overline{1}_{M_y}(\overline{\chi}_{\mathrm{ram}}) \otimes \overline{1}) \oplus (\overline{1}_{M_y}(\overline{\chi}_{\mathrm{ram}}) \otimes \overline{1}) \oplus (\overline{\sigma}_{T_1,\overline{\theta}} \otimes \overline{1}).$$

By cuspidality of $\overline{\sigma}_{T_2,\overline{\theta}}$, $\mathrm{R}_{M_x}^G(i_B^G(\operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1)) \simeq \operatorname{Ind}_B^{M_x}(\overline{\chi}_{\mathrm{ram}} \otimes 1)$ has a unique composition series

$$0 \subsetneq \overline{1}_{M_x}(\overline{\chi}_{\mathrm{ram}}) \subsetneq \mathcal{V} \subsetneq \mathrm{R}_{M_x}^G(i_B^G(\operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1))$$

with $\mathrm{R}_{M_x}^G(i_B^G(\operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1))/\mathcal{V} \simeq \overline{1}_{M_x}(\overline{\chi}_{\mathrm{ram}})$ and $\mathcal{V}/\overline{1}_{M_x}(\overline{\chi}_{\mathrm{ram}}) \simeq \overline{\sigma}_{T_2,\overline{\theta}}$.

Similarly, $\mathrm{R}_{M_y}^G(i_B^G(\operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1))$ has a unique composition series

$$0 \subsetneq \overline{1}_{M_y}(\overline{\chi}_{\mathrm{ram}}) \otimes \overline{1} \subsetneq \mathcal{W} \subsetneq \mathrm{R}_{M_y}^G(i_B^G(\operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1))$$

with $\mathrm{R}_{M_y}^G(i_B^G(\operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1))/\mathcal{W} \simeq \overline{1}_{M_y}(\overline{\chi}_{\mathrm{ram}}) \otimes \overline{1}$ and $\mathcal{W}/\overline{1}_{M_y}(\overline{\chi}_{\mathrm{ram}}) \otimes \overline{1} \simeq \overline{\sigma}_{T_1,\overline{\theta}} \otimes \overline{1}$.

By the Chapter 1 Lemma 3.12, $r_B^G(i_B^G(\operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1)))$ has length two with

$$\left[r_B^G(i_B^G(\operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1)))\right] = \operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1 \oplus \operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1.$$

By exactness of the Jacquet functor, either $i_B^G(\operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1)$ has a unique irreducible subrepresentation π_1 and a unique irreducible quotient ζ, or is semisimple. However, as the length of $i_B^G(\operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1))$ is greater than or equal to three, it cannot be semisimple. We let π_2 denote the quotient of $i_B^G(\operatorname{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1))$ by π_1. Because ζ and π_1 are not cuspidal, by exactness of the Jacquet functor, $r_B^G(\zeta)$ and $r_B^G(\pi_1)$ are non-zero and irreducible.

By Theorem 4.12 and Lemma 4.7,

$$\mathrm{I}_{M_{\mathfrak{I}}}^{G}(\chi_{\mathrm{ram}} \otimes 1) \simeq i_B^G(\mathrm{ind}_{\mathfrak{I}_T}^T(\chi_{\mathrm{ram}} \otimes 1)).$$

By reciprocity, ζ is a quotient of $\mathrm{I}_{M_{\mathfrak{I}}}^{G}(\chi_{\mathrm{ram}} \otimes 1)$.

By Theorem 4.8, $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, \mathfrak{I}, (\chi_{ram} \otimes 1))$ is generated by $f_{w_1,\frac{1}{q}}$ and f_{w_2} and the quadratic relations

$$(f_{w_2} + 1) \star (f_{w_2} + 1) = (f_{w_1,\frac{1}{q}} + 1) \star (f_{w_1,\frac{1}{q}} + 1) = 0.$$

Thus $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, \mathfrak{I}, (\chi_{ram} \otimes 1)$ has a unique simple one-dimensional module M. Because $(\mathfrak{I}, \mathrm{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1)$ is quasi-projective by Lemma 8.3, a simple module of $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, \mathfrak{I}, (\chi_{\mathrm{ram}} \otimes 1)$ corresponds to ζ by the bijection of Theorem 8.1. By Theorem 4.10, because $r_B^G(\zeta)$ is irreducible, ζ must correspond to the unique simple one dimensional module of $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, \mathfrak{I}, (\chi_{\mathrm{ram}} \otimes 1)$. Hence, if $\mathcal{V}$ is the space of ζ, $\mathcal{V}^{\mathfrak{I}^1}$ is one dimensional and the action of $\mathfrak{I}$ is given by $\chi_{ram} \otimes 1$.

By Chapter 1 Theorem 3.5,

$$\left(i_B^G(\mathrm{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1)\right)^{\sim} \simeq i_B^G\left((\mathrm{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1)^{\sim}\right),$$

as δ_B is trivial. Furthermore

$$(\mathrm{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1)^{\sim} = (\mathrm{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}}^{-1} \otimes 1)$$

where χ_{ram}^{-1} is the character of $\mathcal{O}_E^{\times}$ defined by

$$\chi_{\mathrm{ram}}^{-1}(x) = \chi_{\mathrm{ram}}(x^{-1})$$

for all $x \in \mathcal{O}_E^{\times}$.

Similar arguments, given for $i_B^G(\mathrm{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1)$, apply to $i_B^G(\mathrm{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}}^{-1} \otimes 1)$. We find:

(1) $i_B^G(\mathrm{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}}^{-1} \otimes 1)$ has a unique irreducible subrepresentation and a unique irreducible quotient η;
(2) η corresponds to the unique simple one dimensional module of $\mathcal{H}_{\overline{\mathbb{F}}_\ell}(G, \mathfrak{I}, (\chi_{ram}^{-1} \otimes 1))$ under the bijection of Theorem 8.1.

Let $\mathcal{V}$ be the space of η. By Chapter 1 Lemma 3.2,

$$\mathcal{V} = \mathcal{V}^{\mathfrak{I}^1} \oplus \mathcal{V}(\mathfrak{I}^1)$$

and $\mathrm{Res}_{\mathfrak{I}}^{G}(\mathrm{inv}_{\mathfrak{I}^1} \eta)$ is one dimensional, hence isomorphic to $(\chi_{ram}^{-1} \otimes 1)$. By Chapter 1 Theorem 3.1, $\widetilde{\eta}$ is a subrepresentation of $i_B^G(\mathrm{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1)$. Because

$$\left(\mathcal{V}^{\mathfrak{I}^1}\right)^{\sim} \simeq \widetilde{\mathcal{V}}^{\mathfrak{I}^1}$$

$\widetilde{\eta}^{\mathfrak{I}^1}$ is one dimensional hence must be isomorphic to $(\chi_{ram} \otimes 1)$. Hence $\widetilde{\eta}$ must be irreducible and isomorphic to ζ. Thus ζ appears twice in the composition series of $i_B^G(\mathrm{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1)$ as the unique irreducible quotient and as the unique irreducible subrepresentation. Let π_3 denote the subrepresentation of π_2 such that the quotient of π_2 by π_3 is ζ. By exactness of level zero parahoric restriction, $\mathrm{R}_{M_x}^G(\pi_3) \simeq \overline{\sigma}_{T_2,\overline{\theta}}$ and $\mathrm{R}_{M_y}^G(\pi_3) \simeq (\overline{\sigma}_{T_1,\overline{\theta}} \otimes \overline{1})$. Therefore, by reciprocity, $\mathrm{I}_{M_x}^G(\overline{\sigma}_{T_2,\overline{\theta}})$ and $\mathrm{I}_{M_y}^G(\overline{\sigma}_{T_1,\overline{\theta}} \otimes \overline{1})$ are subrepresentations of π_3. Every irreducible subquotient of π_3 must have nontrivial invariants under a maximal parahoric subgroup hence $\pi_3 = \mathrm{I}_{M_x}^G(\overline{\sigma}_{T_2,\overline{\theta}}) \oplus \mathrm{I}_{M_y}^G(\overline{\sigma}_{T_1,\overline{\theta}} \otimes \overline{1})$. □

Remark. Choose a lift of $\mathrm{infl}_{\langle \varpi_E \rangle} \chi_{\mathrm{ram}} \otimes 1$ to an integral ℓ-adic character ρ of T. By [**Key84**], the induced ℓ-adic representation $i_B^G(\rho)$ is semisimple of length two. By Chapter 1 Theorem

5.1, $i_B^G(\rho)$ is an integral ℓ-adic representation. We let L be a lattice in $\mathrm{infl}_{\langle\varpi_E\rangle}\chi_{\mathrm{ram}}\otimes 1$ and $i_B^G(L)$ the induced lattice in $i_B^G(\rho)$. Note that, by a slight adaption to the proof of Chapter 1 Corollary 5.2, because $L/\Lambda_\ell L\simeq \mathrm{infl}_{\langle\varpi_E\rangle}\chi_{\mathrm{ram}}\otimes 1$ is irreducible we have

$$i_B^G(\mathrm{infl}_{\langle\varpi_E\rangle}\chi_{\mathrm{ram}}\otimes 1)\simeq i_B^G(L)/\Lambda_\ell(i_B^G(L)).$$

Hence there is a lattice in the semisimple representation $i_P^G(\rho)$ whose reduction modulo ℓ is $i_B^G(\mathrm{infl}_{\langle\varpi_E\rangle}\chi_{\mathrm{ram}}\otimes 1)$. This lattice $i_B^G(L)$ cannot be semisimple.

13. The cuspidal subquotients of $i_B^G(\chi)$

Let χ a level zero character of T. By Section 9, there exist characters χ_1 of $E^\times$ and χ_2 of E^1 such that

$$i_B^G(\chi)\simeq i_B^G(\chi_1\otimes 1)(\chi_2\circ\det).$$

Therefore the length of $i_B^G(\chi)$ is equal to length of $i_B^G(\chi_1\otimes 1)$. By Theorems 10.1 and 12.2, we have described the decomposition of $i_B^G(\chi_1\otimes 1)$ in the cases where the length is greater than or equal to three. Let $T_{\mathfrak{J}}=T\cap\mathfrak{J}$ and $T_{\mathfrak{J}}^1=T\cap\mathfrak{J}^1$. By Theorem 6.10,

$$\mathrm{R}_{M_z}^G(i_B^G(\chi))\simeq i_{\overline{B}}^{M_z}(\mathrm{inv}_{T_{\mathfrak{J}^1}}(\mathrm{Res}_{T_{\mathfrak{J}}}^T\chi)).$$

The character χ is level zero if and only if both χ_1 and χ_2 are level zero and

$$\mathrm{R}_{M_z}^G(i_B^G(\chi))\simeq i_{\overline{B}}^{M_z}(\mathrm{inv}_{T_{\mathfrak{J}^1}}(\mathrm{Res}_{T_{\mathfrak{J}}}^T(\chi_1\otimes 1)))(\mathrm{inv}_{T_{\mathfrak{J}^1}}(\mathrm{Res}_{T_{\mathfrak{J}}}^T(\chi_2))\circ\det).$$

By Theorems 10.1 and 12.2, we have described all parabolically induced representations with χ_2 trivial which have irreducible cuspidal subquotients. Thus, the remaining parabolically induced representations with cuspidal subquotients have χ_2 nontrivial and are a twist of a case already considered. In these cases the cuspidal subquotients are twists of the cuspidal subquotients of the parabolically induced representation with χ_2 trivial by the same character.

14. Supercuspidal support

Let $G=\mathrm{U}(2,1)(E/F)$ be the unramified p-adic unitary group in three variables.

Theorem 14.1. Let π be an irreducible smooth level zero ℓ-modular representation of G. Then $\mathrm{scusp}(\pi)$ exists and is unique up to conjugacy.

Proof: By Chapter 1 Lemma 3.14, $\mathrm{scusp}(\pi)$ exists. Suppose π is not cuspidal then $\mathrm{cusp}(\pi)$ is supercuspidal and, by Chapter 1 Theorem 3.13, $\mathrm{cusp}(\pi)$ exists and is unique up to conjugacy. Hence $\mathrm{scusp}(\pi)$ exists and is unique up to conjugacy.

If π is a level zero cuspidal representation it is in the list given in Theorem 7.1 and we have seen by the decomposition of the level zero principal series, Theorems 10.1 and 12.2 and Section 13, that $\mathrm{scusp}(\pi)$ exists and is unique up to conjugacy. □

Let $\mathrm{Irr}^0_{\overline{\mathbb{F}}_\ell}(G)$ denote the set of isomorphism classes of irreducible level zero ℓ-modular representations of G. Thus, by Theorem 14.1, we can partition $\mathrm{Irr}^0_{\overline{\mathbb{F}}_\ell}(G)$ by supercuspidal support. For (L,σ) a pair consisting of a Levi subgroup L of G and an irreducible supercuspidal representation σ of L we let Ω denote the conjugacy class of (L,σ) and let $\mathrm{Irr}(\Omega)$ denote the set of all irreducible representations of G with supercuspidal support Ω. By Theorem 3.3, σ is a level zero representation of L if and only if $\mathrm{Irr}(\Omega)\subset\mathrm{Irr}^0_{\overline{\mathbb{F}}_\ell}(G)$. Let $\Sigma^0=\{\Omega=(L,\sigma):\sigma \text{ is level zero}\}$.

By Theorem 14.1,

$$\mathrm{Irr}^0_{\overline{\mathbb{F}}_\ell}(G) = \bigsqcup_{\Omega\in\Sigma^0} \mathrm{Irr}(\Omega).$$

LEMMA 14.2. Let (G_w, σ_w) and (G_z, σ_z) be supercuspidal standard level zero $\overline{\mathbb{F}}_\ell$-types in G. Then $\mathrm{I}^G_{M_w}(\sigma_w)$ and $\mathrm{I}^G_{M_z}(\sigma_z)$ have an irreducible subquotient in common if and only if they are isomorphic.

PROOF: Suppose G_w and G_z are maximal then the result follows from Corollary 6.4. Suppose G_z is a standard maximal parahoric subgroup and G_w is the standard Iwahori subgroup of G. Then $\mathrm{I}^G_{M_z}(\sigma_z)$ is irreducible, by Lemma 6.6. Hence $\mathrm{I}^G_{M_w}(\sigma_w)$ and $\mathrm{I}^G_{M_z}(\sigma_z)$ have an irreducible subquotient in common if and only if $\mathrm{I}^G_{M_z}(\sigma_z)$ is a subquotient of $\mathrm{I}^G_{M_w}(\sigma_w)$. Suppose $\mathrm{I}^G_{M_z}(\sigma_z)$ is a subquotient of $\mathrm{I}^G_{M_w}(\sigma_w)$ then, by exactness, $\mathrm{R}^G_{M_z}(\mathrm{I}^G_{M_z}(\sigma_z))$ is a subquotient of $\mathrm{R}^G_{M_z}(\mathrm{I}^G_{M_w}(\sigma_w))$. By Lemma 6.6, $\mathrm{R}^G_{M_z}(\mathrm{I}^G_{M_z}(\sigma_z)) \simeq \sigma_z$ and, by Lemma 6.3,

$$(\star) \qquad \mathrm{R}^G_{M_z}(\mathrm{I}^G_{M_w}(\sigma_w)) \simeq \bigoplus_{n\in D_{w,z}} i^{M_z}_{P_{z,nw}}\, {}^n\!\left(r^{M_w}_{P_{w,n^{-1}z}}(\sigma_w)\right).$$

Because M_w does not have any proper parabolic subgroups, $M_w = P_{w,n^{-1}z}$ and $P_{z,nw}$ is a proper parabolic subgroup of M_z. Hence σ_z is a subquotient of $i^{M_z}_{P_{z,nw}}({}^n\sigma_w)$ for some proper parabolic subgroup of M_z contradicting the supercuspidality of σ_z.

Suppose G_z and G_w are both equal to the standard Iwahori subgroup $\mathfrak{I}$ of G. If $\mathrm{I}^G_{M_\mathfrak{I}}(\sigma_1)$ and $\mathrm{I}^G_{M_\mathfrak{I}}(\sigma_2)$ share a common subquotient X then, because X is level zero, there exists a standard maximal parahoric subgroup G_v of G such that $\mathrm{R}^G_{M_v}(X) \neq 0$. By exactness, $\mathrm{R}^G_{M_v}(\mathrm{I}^G_{M_\mathfrak{I}}(\sigma_1))$ and $\mathrm{R}^G_{M_v}(\mathrm{I}^G_{M_\mathfrak{I}}(\sigma_2))$ share a common subquotient. Thus, by $(\star)$, there exist $n_1, n_2 \in D_{w,v}$ such that $i^{M_v}_B({}^{n_1}\sigma_1)$ and $i^{M_v}_B({}^{n_2}\sigma_2)$ share a common subquotient where B is the standard Borel subgroup of M_v. Thus ${}^{n_1}\sigma_1$ is conjugate to ${}^{n_2}\sigma_2$ in M_v by unicity of supercuspidal support in M_v. Hence σ_1 is conjugate to σ_2 in G and $\mathrm{I}^G_{M_\mathfrak{I}}(\sigma_1) \simeq \mathrm{I}^G_{M_\mathfrak{I}}(\sigma_2)$ □

By Lemma 14.2, we can partition $\mathrm{Irr}^0_{\overline{\mathbb{F}}_\ell}(G)$ by supercuspidal standard level zero $\overline{\mathbb{F}}_\ell$-types. Let (G_z, σ_z) be a supercuspidal level zero $\overline{\mathbb{F}}_\ell$-type. We write $\mathrm{Irr}(G_z, \sigma_z)$ as the subset of $\mathrm{Irr}^0_{\overline{\mathbb{F}}_\ell}(G)$ of irreducible subquotients of $\mathrm{I}^G_{M_z}(\sigma_z)$. Let Θ^0 be the set of all supercuspidal standard level zero $\overline{\mathbb{F}}_\ell$-types, up to equivalence, then

$$\mathrm{Irr}^0_{\overline{\mathbb{F}}_\ell}(G) = \bigsqcup_{(G_z,\sigma_z)\in\Theta^0} \mathrm{Irr}(G_z, \sigma_z).$$

LEMMA 14.3. The partition of $\mathrm{Irr}^0_{\overline{\mathbb{F}}_\ell}(G)$ by supercuspidal support is a refinement of the partition of $\mathrm{Irr}^0_{\overline{\mathbb{F}}_\ell}(G)$ by conjugacy classes of supercuspidal standard level zero $\overline{\mathbb{F}}_\ell$-types.

PROOF: Let (G_w, σ_w) be a supercuspidal standard level zero $\overline{\mathbb{F}}_\ell$-type. If G_w is maximal then $\mathrm{I}^G_{M_w}(\sigma_w)$ is irreducible and supercuspidal. Furthermore all irreducible supercuspidal representations of G appear in this way. Thus we can suppose G_w is the standard Iwahori subgroup of G.

Suppose σ is a subquotient of $i^G_B(\chi)$ for some level zero character χ of T. At least one of $\mathrm{R}^G_{M_x}(\sigma)$ and $\mathrm{R}^G_{M_y}(\sigma)$ is nonzero. Without loss of generality we assume $\mathrm{R}^G_{M_x}(\sigma)$ is nonzero. By Lemma 6.10, $\mathrm{R}^G_{M_x}(\sigma)$ is a subquotient of $i^{M_x}_{\overline{B}}(\overline{\chi})$.

(1) If σ is cuspidal then $\overline{\sigma} = \mathrm{R}^G_{M_x}(\sigma)$ is irreducible and cuspidal and $\sigma \simeq \mathrm{I}^G_{M_x}(\overline{\sigma})$ which is a subquotient of $\mathrm{I}^G_{M_\mathfrak{I}}(\overline{\chi})$ by exactness of level zero parahoric induction.
(2) If σ is a quotient of $i^G_B(\chi)$ then it is a quotient of $\mathrm{I}^G_{M_\mathfrak{I}}(\overline{\chi})$ by reciprocity.

(3) If σ is a subrepresentation of $i_B^G(\chi)$ then it is a quotient of $(i_B^G(\chi))^{\sim} \simeq i_B^G(\chi^{-1})$ and thus a quotient of $\mathrm{I}_{M_\mathfrak{I}}^G(\overline{\chi}^{-1})$ by reciprocity.

□

Remark. Let $\overline{\Sigma}^0$ be the subset of $\mathfrak{B}(G)$ consisting of the level zero inertial classes. Let $\overline{\Omega} \in \overline{\Sigma}^0$ and let $\mathfrak{R}_{\overline{\mathbb{F}}_\ell}(\overline{\Omega})$ denote the full abelian subcategory of level zero representations of G all of whose irreducible subquotients have inertial support $\overline{\Omega}$. Then the results of this section should be a first step towards establishing a decomposition of the category of level zero ℓ-modular representations into a product of indecomposable subcategories,

$$\mathfrak{R}^0_{\overline{\mathbb{F}}_\ell}(G) = \prod_{\Omega \in \overline{\Sigma}^0} \mathfrak{R}_{\overline{\mathbb{F}}_\ell}(\overline{\Omega}).$$

15. Level zero supercuspidal base change

We show that the level zero stable base change map, defined in [**AL05**],

$$\mathrm{BC} : \mathfrak{R}^0_{\overline{\mathbb{Q}}_\ell}(\mathrm{U}(2,1)(E/F)) \to \mathfrak{R}^0_{\overline{\mathbb{Q}}_\ell}(\mathrm{GL}_3(E))$$

from irreducible level zero ℓ-adic representations of $\mathrm{U}(2,1)(E/F)$ to irreducible level zero ℓ-adic representations of $\mathrm{GL}_3(E)$, when restricted to the irreducible representations of $\mathrm{U}(2,1)(E/F)$ whose image is supercuspidal, is compatible with decomposition modulo ℓ.

Lemma 15.1. Assume $\ell \neq 2, 3$. Let π_i, $i = 1, 2$, be irreducible integral level zero ℓ-adic representations of $\mathrm{U}(2,1)(E/F)$ such that $\mathrm{BC}(\pi_i)$ is supercuspidal, $i = 1, 2$. If $d_\ell(\mathrm{BC}(\pi_1)) = d_\ell(\mathrm{BC}(\pi_2))$ then $d_\ell(\pi_1) = d_\ell(\pi_2)$.

Proof: Let $\widetilde{\sigma}_{T_1,\widetilde{\theta}}$ be a supercuspidal ℓ-adic representation of $\mathrm{GL}_3(k_E)$, Chapter 2 Section 9. By [**Vig96**, Chapter 3 3.3],

$$\mathrm{ind}^G_{E^\times \mathrm{GL}_3(\mathcal{O}_E)} \mathrm{infl}\, \widetilde{\sigma}_{T_1,\widetilde{\theta}}$$

is supercuspidal. By [**AL05**, Table 1], these are the only irreducible supercuspidal representations appearing in the image of BC.

Let $\sigma_{T_1,\theta}$ be the ℓ-adic representation of $\mathrm{U}(2,1)(k_E/k_F)$ given in Chapter 2 Section 8. By Theorem 7.1, the ℓ-adic representation $\mathrm{I}^G_{M_x}(\sigma_{T_1,\theta})$ is irreducible and cuspidal.

By [**AL05**, Table 1],

$$\mathrm{BC} : \mathrm{I}^G_{M_x}(\sigma_{T_1,\theta}) \mapsto \mathrm{ind}^G_{E^\times \mathrm{GL}_3(\mathcal{O}_E)} \mathrm{infl}\, \widetilde{\sigma}_{T_1,\widetilde{\theta}}$$

where $\widetilde{\theta}$ is the Shintani lift, see [**AL05**, Section 2.2], of θ. The Shintani lift of θ is $\widetilde{\theta} = \theta \circ \xi_{q-1}$ where $\xi_{q-1} : x \mapsto x^{q-1}$.

By Chapter 3 Sections 3 and 5, if $\ell \mid q+1$ or $\ell \mid q-1$ then $\widetilde{\sigma}_{T_1,\widetilde{\theta}}$ is in an ℓ-block of $\mathrm{GL}_3(k_E)$ of defect zero and $\sigma_{T_1,\theta}$ is in an ℓ-block of $\mathrm{U}_3(k_E/k_F)$ of defect zero.

Suppose $\ell \mid q^2 - q + 1$. By Chapter 3 Section 3, $d_\ell(\widetilde{\sigma}_{T_1,\widetilde{\theta}})$ is supercuspidal if and only if $d_\ell(\widetilde{\theta})^{q^2-1} \neq 1$ and is irreducible in this case. Furthermore, $d_\ell(\widetilde{\theta})^{q^2-1} \neq 1$ if and only if $d_\ell(\sigma_{T_1,\theta})$ is irreducible and, by Chapter 3 Section 5, if $d_\ell(\theta)^{q+1} \neq 1$ then $d_\ell(\sigma_{T_1,\theta})$ is irreducible and supercuspidal. Comparing the structures of the ℓ-blocks, $d_\ell(\widetilde{\sigma}_{T_1,\widetilde{\theta}_1}) = d_\ell(\widetilde{\sigma}_{T_1,\widetilde{\theta}_2})$ if and only if $d_\ell(\sigma_{T_1,\theta_1}) = d_\ell(\sigma_{T_1,\theta})$. □

By Lemma 15.1 we can define a level zero ℓ-modular base change map $\overline{\mathrm{BC}}$ from certain supercuspidal level zero ℓ-modular representations of $\mathrm{U}(2,1)(E/F)$ to supercuspidal level zero ℓ-modular representations of $\mathrm{GL}_3(E)$. An interesting question is: is it possible to extend $\overline{\mathrm{BC}}$ to all level zero representations of $\mathrm{U}(2,1)(E/F)$ in a natural way?

CHAPTER 5

Positive level representations

Let G be an unramified unitary group in three variables. In this chapter we construct the positive level irreducible cuspidal ℓ-modular representations of G. We show that the supercuspidal support of the irreducible representations of G is unique up to conjugacy under an assumption on the possible R-types which the subquotients of $i_B^G(\chi)$ can contain.

1. Introduction

Let $G = \mathrm{U}(V,h)$, $(\pi,\mathcal{V}) \in \mathfrak{R}_R(G)$ and Λ be a self dual lattice sequence in $\mathcal{V}$. Then, by smoothness, $\mathcal{V} = \bigcup_{n\geqslant 1} \mathcal{V}^{\mathbf{P}_n(\Lambda)}$. Thus there exists $n \in \mathbb{N}$ such that $\mathcal{V}^{\mathbf{P}_n(\Lambda)}$ is non-zero. In this section we assume π is not of level zero, hence $n > 1$. Because $\mathbf{P}_n(\Lambda)$ is normal in $\mathbf{P}_{n-1}(\Lambda)$ we get a representation of $\mathbf{P}_{n-1}(\Lambda)$ on $\mathcal{V}^{\mathbf{P}_n(\Lambda)}$. The quotient $\mathbf{P}_{n-1}(\Lambda)/\mathbf{P}_n(\Lambda)$ is abelian, as $n > 1$, hence $\mathrm{Res}^G_{\mathbf{P}_{n-1}(\Lambda)}(\pi)$ contains a character. This is the starting point of the construction of the positive level irreducible smooth cuspidal representations of G. One attempts to refine groups and consider irreducible representations of these new groups which contain this character on restriction; the goal being to find an R-type (K,σ) contained in π such that $\mathrm{ind}_K^G\,\sigma$ is irreducible and hence isomorphic to π. The first part of the complex construction for classical groups of [**Ste08**] involves pro-p compact open subgroups of G. This is where we start; however we eventually specialise to unramified unitary groups in three variables.

Let $G = \mathrm{U}(2,1)(E/F)$ be an unramified unitary group in three variables. For complex representations of G, under a similar construction to that of [**Ste08**], it is shown in [**Bla02**] that every irreducible positive level cuspidal representation of G is compactly induced. We follow the construction of [**Ste08**] for G together with adaptations to this construction introduced in [**Vig01b**] and [**Vig96**] when dealing with generalising the construction of all cuspidal complex representations of $\mathrm{GL}_n(F)$ in [**BK93**] to the construction of all cuspidal ℓ-modular representations of $\mathrm{GL}_n(F)$. We show that every irreducible positive level cuspidal ℓ-modular representation of G is compactly induced.

2. Semisimple characters

We let $\widetilde{G} = \mathrm{GL}(V)$, $G = \mathrm{U}(V,h)$, $\widetilde{A}$ be the Lie algebra of $\widetilde{G}$ and $\mathfrak{g}$ be the Lie algebra of G. The filtration $\mathfrak{A}_n(\Lambda)$ induces a valuation ν_Λ on $\widetilde{A}$ by

$$\nu_\Lambda(x) = \begin{cases} \sup\{n \in \mathbb{Z} : x \in \mathfrak{A}_n(\Lambda)\} & \text{if } x \in \widetilde{A}\backslash\{0\}, \\ \infty & \text{otherwise.}\end{cases}$$

2.1. Simple strata. A stratum in $\widetilde{A}$ is a quadruple $[\Lambda, n, r, \beta]$ with Λ an $\mathcal{O}_E$-lattice sequence in $\mathcal{V}$, r, $n \in \mathbb{Z}$ such that $n \geqslant r \geqslant 0$ and $\beta \in \mathfrak{A}_{-n}(\Lambda)$. Two strata $[\Lambda_1, n_1, r_1, \beta_1]$ and $[\Lambda_2, n_2, r_2, \beta_2]$ are called equivalent if $n_1 = n_2$, $r_1 = r_2$, and $\beta_1 - \beta_2 \in \mathfrak{A}_{-r_1}(\Lambda)$. A stratum is called null if $n = r$ and $\beta = 0$.

The level of a stratum $[\Lambda, n, r, b]$ is the rational number $\frac{n}{e(\Lambda)}$ where $e(\Lambda)$ is the $\mathcal{O}_E$-period of Λ. Let $g = (n, e(\Lambda))$. The characteristic polynomial $\varphi_\beta \in k_E[X]$ of $[\Lambda, n, r, \beta]$ is the reduction modulo (ϖ_E) of the characteristic polynomial of $\varpi_E^{\frac{n}{g}}\beta^{\frac{e(\Lambda)}{g}}$. The characteristic polynomial and level of a stratum are invariant under equivalence of strata.

A stratum $[\Lambda, n, r, \beta]$ is called self dual if Λ is a self dual $\mathcal{O}_E$-lattice sequence and $\beta \in \mathfrak{A}_n^-(\Lambda)$. For a locally compact abelian group H, we let $H^\wedge$ denote the Pontryagin dual of H.

Theorem 2.1. If $n > r \geqslant \frac{n}{2} > 0$ and Λ is self dual we have a $\mathbf{P}(\Lambda)$-equivariant isomorphism

$$\begin{aligned}\mathfrak{A}^-_{1-n}(\Lambda)/\mathfrak{A}^-_{1-r}(\Lambda) &\to (\mathbf{P}_r(\Lambda)/\mathbf{P}_n(\Lambda))^\wedge \\ \beta + \mathfrak{A}^-_{1-r}(\Lambda) &\mapsto \psi_\beta : (1+x) \mapsto \psi_E \circ \mathrm{Tr}_{\widetilde{A}/E}(\beta x).\end{aligned}$$

Thus, if $n > r \geqslant \frac{n}{2} > 0$ then an equivalence class of a skew stratum $[\Lambda, n, r, \beta]$ corresponds to a character of $\mathbf{P}_r(\Lambda)$ trivial on $\mathbf{P}_n(\Lambda)$.

Let $[\Lambda, n, r, \beta]$ be a stratum in $\widetilde{A}$. Suppose $D = E[\beta]$ is a field, then the action of D on V gives V the structure of a D-vector space. We call $[\Lambda, n, r, \beta]$ pure if D is a field, Λ is an $\mathcal{O}_D$-lattice sequence and $\nu_\Lambda(\beta) = -n$.

Let $[\Lambda, n, r, \beta]$ be a pure stratum in $\widetilde{A}$. Let

$$\widetilde{B} = \{\phi \in \widetilde{A} : \phi d = d\phi \text{ for all } d \in D\}$$

and put $\mathfrak{B}_0(\Lambda) = \mathfrak{A}_0(\Lambda) \cap \widetilde{B}$. Let $d = [D : E]$ then V is an N-dimensional E-vector space hence it is an $\frac{N}{d}$-dimensional D-vector space. Let $\widetilde{G}_D = \widetilde{B}^\times$, the D-automorphisms of V. Choosing an E-basis for V identifies $\widetilde{G}$ with $\mathrm{GL}_N(E)$ and choosing a D-basis for V identifies $\widetilde{G}_D$ with $\mathrm{GL}_{\frac{N}{d}}(D)$. We let $G_D = \widetilde{G}_D \cap G$ and $\mathbf{P}(\Lambda_D) = \mathbf{P}(\Lambda) \cap G_D$.

For $k \in \mathbb{Z}$, we define

$$\mathfrak{n}_k(\beta, \Lambda) = \{x \in \mathfrak{A}_0(\Lambda) : \beta x - x\beta \in \mathfrak{A}_k(\Lambda)\},$$

which is an $\mathcal{O}_E$-lattice in $\widetilde{A}$. Let

$$k_0(\beta, \Lambda) = \max\left\{-n, \max\{k \in \mathbb{Z} : \mathfrak{n}_k(\beta, \Lambda) \not\subset \mathfrak{B}_0(\Lambda) + \mathfrak{A}_1(\Lambda)\}\right\}.$$

This is a integer greater than or equal to $-n$. If $k_0(\beta, \Lambda) = -n$ we call β minimal. A pure stratum $[\Lambda, n, r, \beta]$ is called simple if $k_0(\beta, \Lambda) < -r$. A stratum is called simple if it is either a null stratum or it is a pure stratum which is simple.

Let $[\Lambda, n, 0, \beta]$ be a simple stratum in $\widetilde{A}$. If β is minimal over E, define $\mathcal{O}_E$-orders

$$\mathfrak{H}(\beta, \Lambda) = \mathfrak{B}_0(\Lambda) + \mathfrak{A}_{[n/2]+1}(\Lambda)$$
$$\mathfrak{J}(\beta, \Lambda) = \mathfrak{B}_0(\Lambda) + \mathfrak{A}_{[(n+1)/2]}(\Lambda).$$

If β is not minimal over E put $r = -k_0(\beta, \Lambda)$ and suppose that $r < n$. Let $[\Lambda, n, r, \gamma]$ be a simple stratum equivalent to $[\Lambda, n, r, \beta]$, which exists by [**BK93**, Theorem 2.4.1], define $\mathcal{O}_E$-orders

$$\mathfrak{H}(\beta, \Lambda) = \mathfrak{B}_0(\Lambda) + (\mathfrak{H}(\gamma, \Lambda) \cap \mathfrak{A}_{[n/2]+1}(\Lambda))$$
$$\mathfrak{J}(\beta, \Lambda) = \mathfrak{B}_0(\Lambda) + (\mathfrak{J}(\gamma, \Lambda) \cap \mathfrak{A}_{[(n+1)/2]}(\Lambda)).$$

The $\mathcal{O}_E$-orders $\mathfrak{H}(\beta, \Lambda)$ and $\mathfrak{J}(\beta, \Lambda)$ are well defined, independent of the choice of simple stratum $[\Lambda, n, r, \gamma]$, [**BK93**, Proposition 3.1.9]. For $i \geqslant 0$, let $\widetilde{H}^i(\beta, \Lambda) = \mathfrak{H}(\beta, \Lambda) \cap \mathbf{P}_i(\Lambda)$ and $\widetilde{J}^i(\beta, \Lambda) = \mathfrak{J}(\beta, \Lambda) \cap \mathbf{P}_i(\Lambda)$. If $[\Lambda, n, 0, \beta]$ is skew then the groups are stable under the involution h induces on $\widetilde{G}$. We let $H^i(\beta, \Lambda) = \widetilde{H}^i(\beta, \Lambda) \cap G$ and $J^i(\beta, \Lambda) = \widetilde{J}^i(\beta, \Lambda) \cap G$ and put $J(\beta, \Lambda) = J^0(\beta, \Lambda)$. These are compact open subgroups of G and $J^1(\beta, \Lambda)$ is normal in $J(\beta, \Lambda)$ with

$$J(\beta, \Lambda)/J^1(\beta, \Lambda) \simeq \mathbf{P}(\Lambda_D)/\mathbf{P}_1(\Lambda_D).$$

2.2. Semisimple strata. Let $[\Lambda, n, r, \beta]$ be a stratum in $\widetilde{A}$. A decomposition $V = \bigoplus_{i=1}^{l} V_i$ of V is called a splitting for $[\Lambda, n, r, \beta]$ if $\Lambda(k) = \bigoplus_{i=1}^{l} (\Lambda(k) \cap V_i)$ and $\beta = \sum_{i=1}^{l} \mathbf{1}_i \beta \mathbf{1}_i$ where $\mathbf{1}_i : V \to V_i$ is the projection map. We let $\beta_i = \mathbf{1}_i \beta \mathbf{1}_i$, $\Lambda_i = \Lambda \cap V_i$ and set

$$q_i = \begin{cases} r & \text{if } \beta_i = 0, \\ -\nu_{\Lambda_i}(\beta_i) & \text{otherwise.} \end{cases}$$

Definition 2.2 ([**Ste05**, Definition 3.2]). A stratum $[\Lambda, n, r, \beta]$ in $\widetilde{A}$ is called semisimple if it is null or $\nu(\beta) = -n$ and there exists a splitting $V = \bigoplus_{i=1}^{l} V_i$ of V such that for $1 \leqslant i, \leqslant l$ the stratum $[\Lambda_i, q_i, r, \beta_i]$ is either simple or null and, for all $1 \leqslant i < j \leqslant l$, the stratum $[\Lambda_i \oplus \Lambda_j, \max\{q_i, q_j\}, r, \beta_i + \beta_j]$ is not equivalent to a simple or null stratum.

A semisimple stratum $[\Lambda, n, r, \beta]$ is called skew if Λ is self dual, $\beta \in \mathfrak{g}$ and the splitting $V = \bigoplus_{i=1}^{l} V_i$ is orthogonal with respect to h, i.e. if $1 \leqslant i < j \leqslant l$, $v_i \in V_i$ and $v_j \in V_j$ then $h(v_i, v_j) = 0$. If $[\Lambda, n, r, \beta]$ is a skew semisimple stratum then, for all $1 \leqslant i \leqslant l$, the stratum $[\Lambda_i, q_i, r, \beta_i]$ is a skew simple stratum in $\mathrm{End}_E(V_i)$.

Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum with associated splitting $V = \bigoplus_{i=1}^{l} V_i$. We define $D = E[\beta] \simeq \bigoplus_{i=1}^{l} D_i$ to be the sum of field extensions given by β. We let $\widetilde{B} = \bigoplus_{i=1}^{l} \widetilde{B}_i$,

$$G_D = \left(\prod_{i=1}^{l} \widetilde{B}_i^{\times} \right) \cap G$$

and set $\mathbf{P}(\Lambda_D) = \mathbf{P}(\Lambda) \cap G_D$.

2.3. Semisimple characters. Fix $[\Lambda, n, 0, \beta]$ a non-null skew semisimple stratum in G. Define $k_0(\beta, \Lambda)$ to be the least r such that $[\Lambda, n, r, \beta]$ is not semisimple. By [**Ste05**, Lemma 3.5], $[\Lambda, n, r, \beta]$ is equivalent to a semisimple stratum $[\Lambda, n, r, \gamma]$ and [**Ste05**, Page 143] defines orders $\mathfrak{H}(\beta, \Lambda)$ and $\mathfrak{J}(\beta, \Lambda)$ inductively analogously to the simple case; this definition is independent of the choice of γ by [**Ste05**, Lemma 3.9].

We have compact open subgroups $J(\beta, \Lambda) = \mathfrak{J}(\beta, \Lambda)^{\times} \cap G$ and $H(\beta, \Lambda) = \mathfrak{H}(\beta, \Lambda)^{\times} \cap G$ of G, such that $H(\beta, \Lambda) \subseteq J(\beta, \Lambda)$, with decreasing filtrations by pro-p subgroups

$$H^i(\beta, \Lambda) = H(\beta, \Lambda) \cap \mathbf{P}^i(\Lambda) \text{ and } J^i(\beta, \Lambda) = J(\beta, \Lambda) \cap \mathbf{P}^i(\Lambda),$$

$i \geqslant 1$. We have $J(\beta, \Lambda) = \mathbf{P}(\Lambda_D) J^1$ and $J(\beta, \Lambda)/J^1(\beta, \Lambda) \simeq \mathbf{P}(\Lambda_D)/\mathbf{P}_1(\Lambda_D)$. Let $J^0(\beta, \Lambda) = \mathbf{P}^0(\Lambda_D) J^1(\beta, \Lambda)$ which is the inverse image of the connected component of $J(\beta, \Lambda)/J^1(\beta, \Lambda)$ in $J(\beta, \Lambda)$. Associated to $[\Lambda, n, 0, \beta]$ is a set of characters $\mathcal{C}_-(\Lambda, \beta)$ of $H^1(\beta, \Lambda)$ which are intertwined by all of G_D.

Theorem 2.3 ([**Ste05**, Proposition 3.27 and Theorem 5.1]). Let π be an irreducible positive level cuspidal representation of G. Then there exists a skew semisimple stratum $[\Lambda, n, 0, \beta]$ such that π contains a semisimple character $\theta \in \mathcal{C}_-(\Lambda, \beta)$. Furthermore,

$$\mathbf{I}_G(\theta) = J^1(\beta, \Lambda) G_D J^1(\beta, \Lambda).$$

This is our starting point for constructing all irreducible positive level cuspidal representations of G. Let $\pi \in \mathfrak{R}_R(G)$. We say that π contains the stratum $[\Lambda, n, 0, \beta]$ if π contains some semisimple character $\theta \in \mathcal{C}_-(\Lambda, \beta)$.

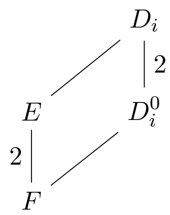
Let $[\Lambda_i, n_i, 0, \beta]$, $i = 1, 2$, be skew semisimple strata in G and $\theta \in \mathcal{C}_-(\Lambda_i, \beta)$, $i = 1, 2$.

Theorem 2.4 ([**Ste08**, Proposition 3.2]). Let $\theta_1 \in \mathcal{C}_-(\Lambda_1, \beta)$. There exists a unique character $\theta_2 \in \mathcal{C}_-(\Lambda_2, \beta)$ such that $G_D \cap \mathbf{I}_G(\theta_1, \theta_2)$ is non-empty.

In the setting of Theorem 2.4, we say that θ_2 is the transfer of θ_1 and write $\tau_{\Lambda_1, \Lambda_2, \beta}$ for the bijection induced $\mathcal{C}_-(\Lambda_1, \beta) \to \mathcal{C}_-(\Lambda_2, \beta)$.

3. Skew semisimple strata in U(2, 1)(E/F)

Let $G = \mathrm{U}(2,1)(E/F)$ be an unramified unitary group in three variables considered as the group of isometries, with respect to a hermitian form h, of a three dimensional E-vector space V. A skew semisimple stratum $[\Lambda, n, 0, \beta]$ in V defines a sum of field extensions $D = \bigoplus_{i=1}^{n} D_i$ of E. The involution defined by h on $\mathrm{End}_E(V_i)$ restricts to an involution σ on D_i and we let D_i^0 denote the fixed field of this involution. The extension D_i/D_i^0 is of degree two, otherwise the involution would fix E, and is unramified because E/F is unramified.



We consider the different classes of skew semisimple strata in G:

(SS-1) Skew simple strata, $[\Lambda, n, 0, \beta]$.

(a) If $\beta \in E$ then $|\beta|_E = -n$ and

$$J(\beta, \Lambda)/J^1(\beta, \Lambda) \simeq \mathbf{P}(\Lambda)/\mathbf{P}_1(\Lambda),$$

which is isomorphic to $\mathrm{U}(2,1)(k_E/k_F)$, or $\mathrm{U}(1,1)(k_E/k_F) \times \mathrm{U}(1)(k_E/k_F)$, or $\mathrm{GL}_1(k_E) \times \mathrm{U}(1)(k_E/k_F)$. These strata include the skew strata, $[\Lambda, 0, 0, 0]$; in other words the level zero case which we have considered in Chapter 4.

(b) If D/E is cubic then

$$J(\beta, \Lambda)/J^1(\beta, \Lambda) \simeq \mathbf{P}(\Lambda_D)/\mathbf{P}_1(\Lambda_D) \simeq \mathrm{U}_1(k_D/k_{D^0})$$

is a cyclic group of order $q_{D^0} + 1$ where q_{D^0} is either q_F or q_F^3 depending on whether D/E is ramified or unramified.

(SS-2) Skew semisimple $(2,1)$-strata, $[\Lambda, n, 0, \beta]$, which are not equivalent to a skew simple stratum and for which we have a splitting $V = V_1 \perp V_2$ orthogonal with respect to h such that

$$[\Lambda, n, 0, \beta] = \bigoplus_{i=1}^{2} [\Lambda_i, q_i, 0, \beta_i]$$

with $[\Lambda_i, q_i, 0, \beta_i]$ skew simple strata in $\mathrm{End}_F(V_i)$, $i = 1, 2$. We suppose V_1 is one-dimensional and V_2 is two-dimensional. Let $D_2 = E[\beta_2]$. We have two cases

(a) If $\beta_2 \in E$ then

$$J(\beta, \Lambda)/J^1(\beta, \Lambda) \simeq \mathbf{P}(\Lambda_{D_2})/\mathbf{P}_1(\Lambda_{D_2}) \times \mathbf{P}(\Lambda_{D_1})/\mathbf{P}_1(\Lambda_{D_1}),$$

which is isomorphic to $\mathrm{U}(1,1)(k_E/k_F) \times \mathrm{U}(1)(k_E/k_F)$ or $\mathrm{GL}_1(k_E) \times \mathrm{U}(1)(k_E/k_F)$ if $G_{D_2} \simeq \mathrm{U}(1,1)(E/F)$, or isomorphic to $\mathrm{U}(1)(k_E/k_F) \times \mathrm{U}(1)(k_E/k_F) \times \mathrm{U}(1)(k_E/k_F)$ if $G_{D_2} \simeq \mathrm{U}(2)(E/F)$. The case $\mathrm{GL}_1(k_E) \times \mathrm{U}(1)(k_E/k_F)$ corresponds to when $\mathbf{P}(\Lambda_{D_2})$ is an Iwahori subgroup of $\mathrm{U}(1,1)(E/F)$.

(b) If D_2/E is quadratic. Suppose D_2/E is unramified then D_2^0 would be equal to E because there is a unique unramified extension of F in each degree hence E would be fixed by σ, a contradiction. Thus D_2/E is ramified and

$$J(\beta, \Lambda)/J^1(\beta, \Lambda) \simeq \mathrm{U}(1)(k_E/k_F) \times \mathrm{U}(1)(k_E/k_F).$$

(SS-3) Skew semisimple $(1,1,1)$-strata, $[\Lambda, n, 0, \beta]$ which are not equivalent to a skew simple stratum or a skew semisimple $(2,1)$-stratum. Thus we have a splitting $V = V_1 \perp V_2 \perp V_3$ which is orthogonal with respect to h with V_i, $i = 1,2,3$, one-dimensional. We have

$$[\Lambda, n, 0, \beta] = \bigoplus_{i=1}^{3} [\Lambda_i, q_i, 0, \beta_i]$$

with $[\Lambda_i, q_i, 0, \beta_i]$ skew simple strata in $\mathrm{End}_F(V_i)$, $i = 1,2,3$. We have

$$J(\beta, \Lambda)/J^1(\beta, \Lambda) \simeq \mathrm{U}(1)(k_E/k_F) \times \mathrm{U}(1)(k_E/k_F) \times \mathrm{U}(1)(k_E/k_F).$$

Lemma 3.1. Let π be an irreducible representation of G. Suppose π contains a skew simple stratum $[\Lambda, n, 0, \beta]$ such that $\beta \in E$. Then there exists an irreducible character χ of E^1 such that $\pi \otimes (\chi \circ \det)$ is level zero.

Proof: By definition π contains a simple character $\theta \in \mathcal{C}_-(\Lambda, \beta)$. By [**BK93**, Definition 3.23], $\theta |_{\mathbf{P}_{[n/2]+1}(\Lambda)} = \psi_\beta$ and $\theta = \chi \circ \det$ for some character χ of $\mathbf{P}_1(\Lambda)$. The character χ extends to a character $\widetilde{\chi}$ of E^1 and $\pi \otimes (\widetilde{\chi}^{-1} \circ \det)$ contains $\theta \otimes (\chi^{-1} \circ \det) = 1$ on $\mathbf{P}_1(\Lambda)$. Hence $\pi \otimes (\widetilde{\chi}^{-1} \circ \det)$ is level zero. □

Using Chapter 4 and Lemma 3.1 we can construct by compact induction the irreducible cuspidal representations of G which contain a skew simple stratum with $\beta \in E$.

4. Heisenberg extensions

We return to a general $G = \mathrm{U}(V, h)$. Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum in G and $\theta \in \mathcal{C}_-(\Lambda, \beta)$. Continuing on from Theorem 2.3, the next step is to extend θ to an irreducible representation η of $J^1(\beta, \Lambda)$ called a Heisenberg extension. As $J^1(\beta, \Lambda)$ is pro-p, the analogous result holds for ℓ-modular representations.

Theorem 4.1 ([**Ste05**, Corollary 3.29 and Proposition 3.31]). There is a unique irreducible representation η of $J^1(\beta, \Lambda)$ which contains θ. The dimension of η is $(J^1(\beta, \Lambda) : H^1(\beta, \Lambda))^{\frac{1}{2}}$. Furthermore,

$$\dim_R(\mathbf{I}_g(\eta)) = \begin{cases} 1 & \text{if } g \in J^1(\beta, \Lambda) G_D J^1(\beta, \Lambda), \\ 0 & \text{otherwise.} \end{cases}$$

Let $[\Lambda^i, n_i, 0, \beta]$, $i = 1,2,3$, be skew semisimple strata in G with the same splitting such that $\mathbf{P}(\Lambda^1) \subseteq \mathbf{P}(\Lambda^2) \subseteq \mathbf{P}(\Lambda^3)$. Let $\theta_i \in \mathcal{C}_-(\Lambda^i, \beta)$, $i = 1,2,3$, such that $\theta_1 = \tau_{\Lambda^2, \Lambda^1, \beta}(\theta_2)$ and $\theta_3 = \tau_{\Lambda^2, \Lambda^3, \beta}(\theta_2)$. Let η_i be the unique irreducible representation of $J^1(\beta, \Lambda^i)$ which contains θ_i, $i = 1,2,3$. Set $J^1_{1,3} = \mathbf{P}_1(\Lambda^1_D) J^1(\beta, \Lambda^3)$.

Theorem 4.2 ([**Ste08**, Proposition 3.7]). There exists a unique irreducible representation $\hat{\eta}$ of $J^1_{1,3}$ such that $\mathrm{Res}^{J^1_{1,3}}_{J^1(\beta, \Lambda^3)} \widehat{\eta} = \eta_3$ and $\mathrm{ind}^{\mathbf{P}_1(\Lambda^1)}_{J^1_{1,3}} \widehat{\eta} \simeq \mathrm{ind}^{\mathbf{P}_1(\Lambda^1)}_{J^1(\beta, \Lambda^1)} \eta_1$. Furthermore

$$\dim_R(\mathbf{I}_g(\widehat{\eta})) = \begin{cases} 1 & \text{if } g \in J^1_{1,3} G_D J^1_{1,3}, \\ 0 & \text{otherwise.} \end{cases}$$

5. β-extensions

By Theorems 2.3 and 4.1, for any irreducible positive level ℓ-modular cuspidal representation π of G there exist a pro-p subgroup $J^1(\beta,\Lambda)$ of G and an irreducible representation η of $J^1(\beta,\Lambda)$ such that π contains η. We suppose that $\mathbf{M}(\Lambda_D)=\mathbf{P}(\Lambda_D)/\mathbf{P}_1(\Lambda_D)$ is connected; this is the case for all skew semisimple strata $[\Lambda,n,0,\beta]$ when G is an unramified unitary group.

The next step is to extend η to an irreducible representation κ of $J(\beta,\Lambda)$ for which $\mathbf{I}_G(\kappa\otimes\mathrm{infl}_{J/J^1}(\sigma))=J$ for all irreducible cuspidal representations σ of $\mathbf{M}(\Lambda_D)$. To that end, ideally we would like to define a β-extension of η to be an irreducible representation κ of $J(\beta,\Lambda)$ such that $\mathrm{Res}^J_{J^1}(\kappa)=\eta$ and $\mathbf{I}_G(\kappa)=\mathbf{I}_G(\eta)$. This is the definition given in [**Bla02**]. This is different to the definition of [**Ste08**, Section 4]; the difficulty in defining β-extensions in this way is showing that such an extension always exists. It is not known for classical groups in general whether such extensions, with the maximal intertwining possible, do exist. However, in the maximal case, by [**Ste08**, Section 4, Proposition 6.18] for complex representations, extensions κ of η such that the representations $\kappa\otimes\mathrm{infl}_{J/J^1}(\sigma)$ have the minimal intertwining possible exist; this is the property of the extensions which is needed to show that $\mathrm{ind}_J^G(\kappa\otimes\mathrm{infl}_{J/J^1}(\sigma))$ is irreducible.

Let $[\Lambda,n,0,\beta]$ be a skew semisimple stratum in G. Let $[\Lambda_{\min},n,0,\beta]$ be a skew semisimple stratum in G with the same splitting such that $\mathbf{P}(\Lambda_{\min,D})$ is minimal and such that $\mathbf{P}(\Lambda_{\min})\subseteq\mathbf{P}(\Lambda)$. Let $\theta\in\mathcal{C}_-(\Lambda,\beta)$ and $\widehat{J}^1=\mathbf{P}_1(\Lambda_{\min,D})J^1(\beta,\Lambda)$. Then $\widehat{J}^1$ is a Sylow p-subgroup of $J(\beta,\Lambda)$. Let $\widehat{\eta}$ be the unique irreducible representation of $\widehat{J}^1$ given by Theorem 4.2. We write a subscript $\overline{\mathbb{Q}}_\ell$ for ℓ-adic representations and a subscript $\overline{\mathbb{F}}_\ell$ for ℓ-modular representations.

Lemma 5.1. There exists an irreducible representation κ of $J(\beta,\Lambda)$ which extends $\widehat{\eta}$.

Proof: As $J^1(\beta,\Lambda)$ and $\widehat{J}^1$ are pro-p, decomposition modulo-ℓ defines a bijection from ℓ-adic to ℓ-modular representations. For $\widehat{\eta}_{\overline{\mathbb{F}}_\ell}$ an ℓ-modular representation of $\widehat{J}^1$ we let $\widehat{\eta}_{\overline{\mathbb{Q}}_\ell}$ be the lift of $\widehat{\eta}_{\overline{\mathbb{F}}_\ell}$. Then, by [**Ste08**, Theorem 4.1], $\widehat{\eta}_{\overline{\mathbb{Q}}_\ell}$ extends to a representation $\kappa_{\overline{\mathbb{Q}}_\ell}$ of $J(\beta,\Lambda)$ and $\kappa_{\overline{\mathbb{F}}_\ell}=d_\ell(\kappa_{\overline{\mathbb{Q}}_\ell})$ is an extension of $\widehat{\eta}_{\overline{\mathbb{F}}_\ell}$ to $J(\beta,\Lambda)$. □

In the case where $\mathbf{P}(\Lambda_D)$ is a maximal parahoric subgroup of G_D we call an extension as in Lemma 5.1 a β-extension.

Remark. Let G be an unramified unitary group in three variables and let $[\Lambda,n,0,\beta]$ be any skew semisimple stratum such that $\mathbf{P}(\Lambda_D)$ is maximal and which is not a simple stratum with $\beta\in E$. It is possible to show that there exist extensions κ of η which are extensions of $\widehat{\eta}$ and such that $\mathbf{I}_G(\kappa)=J^1(\beta,\Lambda)G_DJ^1(\beta,\Lambda)$. Either G_D is abelian and contained in $J(\beta,\Lambda)$ and all extensions are intertwined by G_D or $G_D\simeq\mathrm{U}(V_2,h_2)\times\mathrm{U}(V_1,h_1)$ with V_2 two-dimensional and V_1 one-dimensional. In the ℓ-adic case it is shown in [**Bla02**, Lemma 5.8] that such extensions exist and for ℓ-modular representations we can obtain such extensions by decomposition modulo-ℓ from the ℓ-adic case.

Let $[\Lambda,n,0,\beta]$ be a skew semisimple stratum in $\widetilde{A}$. If $\mathbf{P}(\Lambda_D)$ is maximal we set $J_{\max}(\beta,\Lambda)=J(\beta,\Lambda)$. If $\mathbf{P}(\Lambda_D)$ is not maximal we choose a maximal parahoric subgroup $\mathbf{P}(\Lambda_D^{\max})$ of G_D such that

$$\mathbf{P}(\Lambda_D^{\max})\supsetneq\mathbf{P}(\Lambda_D)\supsetneq\mathbf{P}_1(\Lambda_D)\supsetneq\mathbf{P}_1(\Lambda_D^{\max})$$

and let $J_{\max}=J(\beta,\Lambda_D^{\max})$.

We define β-extensions in the case when $\mathbf{P}(\Lambda_D)$ is not maximal and $\mathbf{P}(\Lambda^{\max}) \supseteq \mathbf{P}(\Lambda)$. This is enough for an unramified unitary group in three variables. For the general case see [**Ste08**, Section 4].

LEMMA 5.2. Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum, $\theta \in \mathcal{C}_{-}(\Lambda, \beta)$, η a Heisenberg extension of θ. Choose $\Lambda_{\max}$ such that $\mathbf{P}(\Lambda_D^{\max})$ is maximal and $\mathbf{P}(\Lambda^{\max}) \supseteq \mathbf{P}(\Lambda)$. Let $\theta_{\max} = \tau_{\Lambda,\Lambda^{\max},\beta}(\theta)$, $\eta_{\max}$ a Heisenberg extension of $\theta_{\max}$ and $\kappa_{\max}$ a β-extension of $\theta_{\max}$. There exists a unique irreducible representation κ of $J(\beta, \Lambda)$ such that:

1. κ extends η;
2. κ and $\operatorname{Res}^{J(\beta,\Lambda^{\max})}_{\mathbf{P}(\Lambda_D)J^1(\beta,\Lambda^{\max})}(\kappa_{\max})$ induce equivalent irreducible representations of $\mathbf{P}(\Lambda_D)\mathbf{P}^1(\Lambda)$.

PROOF: Of course, if $\mathbf{P}(\Lambda_D)$ is maximal then $\kappa_{\max} = \kappa$ and there is nothing to prove. If $\mathbf{P}(\Lambda_D)$ is not maximal then $[\Lambda, n, 0, \beta]$ is a skew semisimple $(2, 1)$-stratum, in the notation of Section 3, and we have $\mathbf{P}(\Lambda_D) = \mathbf{P}(\Lambda_{D_1}) \times \mathbf{P}(\Lambda_{D_2})$ with $\mathbf{P}(\Lambda_{D_2})$ an Iwahori subgroup in $\mathrm{U}(1, 1)(E/F)$ and $\mathbf{P}(\Lambda_{D_1}) \simeq G_{D_1} \simeq E^1$. In the ℓ-adic case, by [**Ste08**, Lemma 4.3], there exists a unique irreducible representation $\widetilde{\kappa}$ of $J(\beta, \Lambda)$ which satisfies (1) and (2). By decomposition modulo-ℓ, we have an irreducible representation $\kappa = d_\ell(\widetilde{\kappa})$ which satisfies (1) and such that

$$\left[\operatorname{ind}^{\mathbf{P}(\Lambda_D)\mathbf{P}^1(\Lambda)}_{J(\beta,\Lambda)} \kappa\right] = \left[\operatorname{ind}^{\mathbf{P}(\Lambda_D)\mathbf{P}^1(\Lambda)}_{\mathbf{P}(\Lambda_D)J^1(\beta,\Lambda^{\max})} \operatorname{Res}^{J(\beta,\Lambda^{\max})}_{\mathbf{P}(\Lambda_D)J^1(\beta,\Lambda^{\max})}(\kappa_{\max})\right].$$

Furthermore, $\mathbf{I}_{\mathbf{P}(\Lambda_D)\mathbf{P}^1(\Lambda)}(\kappa) \subseteq \mathbf{I}_{\mathbf{P}(\Lambda_D)\mathbf{P}^1(\Lambda)}(\eta) = J$ hence by Clifford Theory $\operatorname{ind}^{\mathbf{P}(\Lambda_D)\mathbf{P}^1(\Lambda)}_{J(\beta,\Lambda)} \kappa$ is irreducible. □

6. κ-INDUCTION

Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum. Let $\theta \in \mathcal{C}_{-}(\Lambda, \beta)$, η be the unique irreducible representation of $J^1(\beta, \Lambda)$ containing θ and κ a β-extension of η to $J(\beta, \Lambda)$. Let $J = J(\beta, \Lambda)$, $J^1 = J^1(\beta, \Lambda)$ and $M = \mathbf{M}(\Lambda_D)$.

As $M = J/J^1$, we have an inflation functor $\operatorname{infl}_{J^1} : \mathfrak{R}_R(M) \to \mathfrak{R}_R(J)$. Let $\kappa\text{-}\mathrm{I}^G_M : \mathfrak{R}_R(M) \to \mathfrak{R}_R(G)$ be defined by

$$\kappa\text{-}\mathrm{I}^G_M(\sigma) = \operatorname{ind}^G_J(\kappa \otimes \operatorname{infl}_{J^1} \sigma),$$

for all $\sigma \in M$. When $R = \overline{\mathbb{F}}_\ell$ or $\overline{\mathbb{Q}}_\ell$, the functor of κ-induction is exact as it is composed of exact functors.

Let $\kappa\text{-}\mathrm{R}^G_M : \mathfrak{R}_R(G) \to \mathfrak{R}_R(M)$ be defined by

$$\kappa\text{-}\mathrm{R}^G_M(\pi) = \operatorname{Hom}_{J^1}(\operatorname{Res}^J_{J^1} \kappa, \operatorname{Res}^G_{J^1} \pi),$$

for all $\pi \in \mathfrak{R}_R(G)$, where action of M on $\kappa\text{-}\mathrm{R}^G_{M_x}(\pi)$ is given by if $m \in M$, $f \in \operatorname{Hom}_{J^1}(\operatorname{Res}^J_{J^1} \kappa, \operatorname{Res}^G_{J^1} \pi)$ and $g \in J$ is a representative of the coset m of J/J^1 then

$$m \cdot f = \pi(g) \circ f \circ \kappa(g^{-1}).$$

In this chapter we often omit the restriction in our notation and write this, more simply, as

$$\kappa\text{-}\mathrm{R}^G_M(\pi) = \operatorname{Hom}_{J^1}(\kappa, \pi).$$

LEMMA 6.1. Let $\pi \in \mathfrak{R}_R(G)$ and $\sigma \in \mathfrak{R}_R(M)$ then

$$\operatorname{Hom}_J(\kappa \otimes \operatorname{infl}_{J^1} \sigma, \pi) \simeq \operatorname{Hom}_M(\sigma, \operatorname{Hom}_{J^1}(\kappa, \pi)).$$

PROOF: Let

$$\Psi : \operatorname{Hom}_J(\kappa \otimes \operatorname{infl}_{J^1} \sigma, \operatorname{Res}^G_J \pi) \to \operatorname{Hom}_M(\sigma, \operatorname{Hom}_{J^1}(\operatorname{Res}^G_{J^1} \kappa, \operatorname{Res}^G_{J^1} \pi))$$
$$\varphi \mapsto \Psi(\varphi) : w \mapsto \psi_w$$

where $\psi_w(v) = \varphi(v \otimes w)$. Let $m \in M$ and j be a representative of the coset in J/J^1 corresponding to m. Because $\varphi(v \otimes \sigma(j)w) = \pi(j)\varphi(\kappa(j^{-1})v \otimes w)$, we have

$$\Psi(\varphi)(\sigma(j)w)(v) = \pi(j) \circ \Psi(\varphi)(w) \circ \kappa(j^{-1})(v).$$

Hence $\Psi(\varphi) \in \operatorname{Hom}_M(\sigma, \operatorname{Hom}_{J^1}(\operatorname{Res}^G_{J^1} \kappa, \operatorname{Res}^G_{J^1} \pi))$. Let

$$\Phi : \operatorname{Hom}_M(\sigma, \operatorname{Hom}_{J^1}(\operatorname{Res}^G_{J^1} \kappa, \operatorname{Res}^G_{J^1} \pi)) \to \operatorname{Hom}_J(\kappa \otimes \operatorname{infl}_{J^1} \sigma, \operatorname{Res}^G_J \pi)$$
$$\psi \mapsto \Phi(\psi) : v \otimes w \mapsto (\psi(w))(v).$$

We have

$$\begin{aligned}\Phi(\psi)(\kappa(j)v \otimes \sigma(j)w) &= \psi(\sigma(j)w)(\kappa(j)v) \\ &= \pi(j)\psi(w)(\kappa(j^{-1})\kappa(j)v) = \pi(j)\Phi(\psi)(v \otimes w).\end{aligned}$$

Hence $\Phi(\psi) \in \operatorname{Hom}_J(\kappa \otimes \operatorname{infl}_{J^1} \sigma, \operatorname{Res}^G_J \pi)$. It is easy to check $\Phi(\Psi(\varphi)) = \varphi$ and $\Psi(\Phi(\psi)) = \psi$. □

By Lemma 6.1 and reciprocity, κ-restriction is right adjoint to κ-induction.

Let K be a pro-p subgroup of G. For $\rho \in \operatorname{Irr}(K)$, define $e_\rho \in \mathcal{H}_R(G)$ by

$$e_\rho(x) = \begin{cases} \mu(K)^{-1} \dim(\rho) \operatorname{Tr}(\rho(x^{-1})) & \text{if } x \in K, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 6.2. Let $(\pi, \mathcal{V}) \in \mathfrak{R}_R(G)$. The functor $\mathcal{V} \to \mathcal{V}^\rho$ is exact and $\mathcal{V}^\rho$ is a direct factor of $\operatorname{Res}^G_J(\mathcal{V})$.

PROOF: Let $(\pi_i, \mathcal{V}_i) \in \mathfrak{R}_R(G)$, $i = 1, 2, 3$ and suppose

$$0 \to \mathcal{V}_1 \xrightarrow{\varphi} \mathcal{V}_2 \xrightarrow{\psi} \mathcal{V}_3 \to 0$$

is a short exact sequence of representations of G. We define a sequence

$$\mathcal{V}_1^\rho \xrightarrow{\operatorname{Res}(\varphi)} \mathcal{V}_2^\rho \xrightarrow{\operatorname{Res}(\psi)} \mathcal{V}_3^\rho,$$

where the morphisms are given by restriction. It is routine to show this sequence is left exact. Let $\rho_1, \rho_2 \in \operatorname{Irr}_R(K)$, we have

$$\begin{aligned} e_{\rho_1} \star e_{\rho_2}(x) &= \int_G e_{\rho_1}(g) e_{\rho_2}(g^{-1}x) d\mu(g) \\ &= \begin{cases} \displaystyle\int_K \mu(K)^{-2} \dim(\rho_1) \dim(\rho_2) \operatorname{Tr}(\rho_1(g^{-1})) \operatorname{Tr}(\rho_2(x^{-1}g)) d\mu(g) & \text{if } x \in K^1 \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

There exists a normal compact open subgroup $K_1 \subset K$ such that ρ_1 and ρ_2 are trivial on K and $e_{\rho_1}, e_{\rho_2} \in \mathcal{H}_R(K \backslash G / K)$. Thus if $x \in K$

$$e_{\rho_1} \star e_{\rho_2}(x) = \mu(K)^{-2} \dim(\rho_1) \dim(\rho_2) \mu(K_1) \int_{K/K_1} \operatorname{Tr}(\rho_1(g^{-1})) \operatorname{Tr}(\rho_2(x^{-1}g)) d\mu(g).$$

This integral is a finite sum and, if we identify $\operatorname{Tr}(\rho_1)$ and $\operatorname{Tr}(\mathbf{L}(x)\rho_2)$ with trace characters $\overline{\rho}_1$ and $\mathbf{L}(x)\overline{\rho}_2$, where $\mathbf{L}(x)$ denotes the left translation by x^{-1}, of the finite group K/K_1, we can write

$$e_{\rho_1} \star e_{\rho_2}(x) = \mu(K)^{-1} \dim(\rho_1) \dim(\rho_2) \langle \mathbf{L}(x)\overline{\rho_2}, \overline{\rho_1} \rangle$$

where $\langle \ , \ \rangle$ is the inner product on the space of R-valued class functions

$$\langle \mathbf{L}(x)\overline{\rho_2}, \overline{\rho_1} \rangle = \frac{1}{|K/K_1|} \sum_{g \in K/K_1} \overline{\rho_2}(x^{-1}g)\overline{\rho_1}(g^{-1}).$$

By the generalised orthogonality relation for characters of a finite group [**Isa06**, Theorem 2.13],

$$\langle \mathbf{L}(x)\overline{\rho_2}, \overline{\rho_1} \rangle = \begin{cases} \dim(\rho_1)^{-1} \operatorname{Tr}(\rho_1(x^{-1})) & \text{if } \rho_1 = \rho_2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$e_{\rho_1} \star e_{\rho_2} = \begin{cases} e_{\rho_1} & \text{if } \rho_1 = \rho_2, \\ 0 & \text{otherwise.} \end{cases}$$

Hence e_ρ is an idempotent of $\mathcal{H}_R(G)$. Thus we can write

$$\mathcal{V}_i = \pi_i(e_\rho)\mathcal{V}_i \oplus (1 - \pi_i(e_\rho))\mathcal{V}_i,$$

$i = 1, 2, 3$. Furthermore, $\pi_i(e_\rho)\mathcal{V}_i^\rho \simeq \mathcal{V}_i^\rho$ and, because

$$\mathcal{V} = \bigoplus_{\sigma \in \operatorname{Irr}(J^1)} \mathcal{V}^\sigma$$

and $\pi_i(e_{\rho_1})\pi_i(e_{\rho_2}) = 0$ if $\rho_1 \neq \rho_2$, we have that $\pi_i(e_\rho)$ is a projection onto $\mathcal{V}_i^\rho$, $i = 1, 2, 3$ and $\mathcal{V}^\rho$ is a direct factor of $\operatorname{Res}_K^G(\pi)$.

The proof is now clear. Suppose $v_3 \in \mathcal{V}_3^\rho$ then there exists $v_2 \in \mathcal{V}$ such that $\psi(v_2) = v_3$. Then $\pi_2(e_\rho)v_2 \in \mathcal{V}_2^\rho$ and

$$\psi(\pi_2(e_\rho)v_2) = \pi_3(e_\rho)\psi(v_2) = v_3.$$

Hence $\operatorname{Res}(\psi)$ is surjective. $\square$

Lemma 6.3. When $R = \overline{\mathbb{F}}_\ell$ or $\overline{\mathbb{Q}}_\ell$, the functor of κ-restriction is exact.

Proof: Let $(\pi, \mathcal{V}) \in \mathfrak{R}_R(G)$ and let $\mathcal{W}$ be the space of η. We have an isomorphism $\operatorname{Hom}_{J^1}(\eta, \pi) \otimes \mathcal{W} \simeq \mathcal{V}^\eta$ induced by the map $f \otimes w \mapsto f(w)$. Thus, by Lemma 6.2, the functor $\pi \mapsto \kappa\text{-}\operatorname{R}_M^G(\pi) \otimes \mathcal{W}$ is exact and if follows that κ-restriction is exact. $\square$

6.1. $\kappa_{\max}$-induction. Lemma 5.2 allows us to relate certain κ-induction functors. We do this by defining $\kappa_{\max}$-induction. Suppose we are in the setting of Lemma 5.2 with $[\Lambda, n, 0, \beta]$ a skew semisimple stratum, $\theta \in \mathcal{C}_-(\Lambda, \beta)$, η a Heisenberg extension of θ and a chosen $\Lambda_{\max}$ with $\mathbf{P}(\Lambda_D^{\max})$ maximal and $\mathbf{P}(\Lambda^{\max}) \supseteq \mathbf{P}(\Lambda)$.. Let σ be an irreducible representation of M and $\operatorname{infl}_{J^1}(\sigma)$ denote the inflation of σ to J. We have $\mathbf{P}_1(\Lambda)\mathbf{P}(\Lambda_D)/\mathbf{P}_1(\Lambda) \simeq M$ and $J^1_{\max}\mathbf{P}(\Lambda_D)/J^1_{\max}\mathbf{P}_1(\Lambda_D) \simeq M$. We denote the inflation of σ to $\mathbf{P}_1(\Lambda)\mathbf{P}(\Lambda_D)$ by $\operatorname{infl}_{\mathbf{P}_1(\Lambda)}(\sigma)$ and the inflation of σ to $J^1_{\max}\mathbf{P}(\Lambda_D)$ by $\operatorname{infl}_{J^1_{\max}\mathbf{P}_1(\Lambda_D)}(\sigma)$. Thus $\operatorname{infl}_{J^1}(\sigma)$ extends by inflation to $\operatorname{infl}_{J^1_{\max}\mathbf{P}_1(\Lambda_D)}(\sigma)$ and to $\operatorname{infl}_{\mathbf{P}_1(\Lambda)}(\sigma)$. Thus, using transitivity of compact induction and Lemma 5.2,

$$\begin{aligned} \kappa\text{-}\operatorname{I}_M^G(\sigma) &= \operatorname{ind}_J^G(\kappa \otimes \operatorname{infl}_{J^1}(\sigma)) \\ &\simeq \operatorname{ind}^G_{\mathbf{P}_1(\Lambda)\mathbf{P}(\Lambda_D)} \operatorname{ind}_J^{\mathbf{P}_1(\Lambda)\mathbf{P}(\Lambda_D)}(\kappa \otimes \operatorname{infl}_{J^1}(\sigma)) \\ &\simeq \operatorname{ind}^G_{\mathbf{P}_1(\Lambda)\mathbf{P}(\Lambda_D)}(\operatorname{ind}_J^{\mathbf{P}_1(\Lambda)\mathbf{P}(\Lambda_D)}(\kappa) \otimes \operatorname{infl}_{\mathbf{P}_1(\Lambda)}(\sigma)) \end{aligned}$$

$$\begin{aligned}
&\simeq \mathrm{ind}_{\mathbf{P}_1(\Lambda)\mathbf{P}(\Lambda_D)}^{G}(\mathrm{ind}_{J^1_{\max}\mathbf{P}(\Lambda_D)}^{\mathbf{P}_1(\Lambda)\mathbf{P}(\Lambda_D)}(\mathrm{Res}_{J^1_{\max}\mathbf{P}(\Lambda_D)}^{J_{\max}}(\kappa_{\max}))\otimes \mathrm{infl}_{\mathbf{P}_1(\Lambda)}(\sigma))\\
&\simeq \mathrm{ind}_{\mathbf{P}_1(\Lambda)\mathbf{P}(\Lambda_D)}^{G}(\mathrm{ind}_{J^1_{\max}\mathbf{P}(\Lambda_D)}^{\mathbf{P}_1(\Lambda)\mathbf{P}(\Lambda_D)}(\mathrm{Res}_{J^1_{\max}\mathbf{P}(\Lambda_D)}^{J_{\max}}(\kappa_{\max})\otimes \mathrm{infl}_{J^1_{\max}\mathbf{P}_1(\Lambda_D)}(\sigma)))\\
&\simeq \mathrm{ind}_{J^1_{\max}\mathbf{P}(\Lambda_D)}^{G}(\mathrm{Res}_{J^1_{\max}\mathbf{P}(\Lambda_D)}^{J_{\max}}(\kappa_{\max})\otimes \mathrm{infl}_{J^1_{\max}\mathbf{P}_1(\Lambda_D)}(\sigma)).
\end{aligned}$$

If $\mathbf{P}(\Lambda_D)$ is maximal then this is just $\kappa_{\max}\text{-}\,\mathrm{I}_M^G(\sigma)$. In the non-maximal case we define $\kappa_{\max}$-induction

$$\kappa_{\max}\text{-}\,\mathrm{I}_M^G : \mathfrak{R}_R(M)\to\mathfrak{R}_R(G)$$

by

$$\kappa_{\max}\text{-}\,\mathrm{I}_M^G(\sigma) = \mathrm{ind}_{J^1_{\max}\mathbf{P}(\Lambda_D)}^{G}(\mathrm{Res}_{J^1_{\max}\mathbf{P}(\Lambda_D)}^{J_{\max}}(\kappa_{\max})\otimes \mathrm{infl}_{J^1_{\max}\mathbf{P}_1(\Lambda_D)}(\sigma)),$$

for all $\sigma\in\mathfrak{R}_R(M)$. As we have seen, if we are in the setting of Lemma 5.2 and in particular for unramified $\mathrm{U}(2,1)(E/F)$, this is isomorphic to $\kappa\text{-}\,\mathrm{I}_M^G(\sigma)$. However, choosing to work with $\kappa_{\max}$-induction allows us to make some comparisons between the κ-induced representations from the different subgroups J which are contained in a fixed maximal group $J_{\max}$. We define $\kappa_{\max}$-restriction $\kappa_{\max}\text{-}\,\mathrm{R}_M^G$ to be the right adjoint of $\kappa_{\max}$-induction $\kappa_{\max}\text{-}\,\mathrm{I}_M^G$. By Lemma 6.1 and reciprocity, for all $\pi\in\mathfrak{R}_R(G)$, we have

$$\kappa_{\max}\text{-}\,\mathrm{R}_M^G(\pi)\simeq \mathrm{Hom}_{J^1_{\max}\mathbf{P}_1(\Lambda_D)}(\kappa_{\max},\pi).$$

7. κ-RESTRICTION-INDUCTION FOR $\mathrm{U}(2,1)(E/F)$

From now on we specialise to $G=\mathrm{U}(2,1)(E/F)$. We show that κ-induction and restriction in G is related to level zero induction and restriction in G_D.

THEOREM 7.1. Let $[\Lambda^i,n_i,0,\beta]$, $i=1,2$, be semisimple strata and let $(J_{\max},\Lambda^{\max},\kappa_{\max})$ be a triple as defined in Lemma 5.2 such that $\mathbf{P}(\Lambda_D^{\max})\supseteq\mathbf{P}(\Lambda_D^i)$, $i=1,2$. Let $M_i=\mathbf{P}(\Lambda^i_D)/\mathbf{P}_1(\Lambda^i_D)$, $i=1,2$ and σ_1 be a finite length representation of M_1. Then

$$\kappa_2\text{-}\,\mathrm{R}_{M_2}^G\circ\kappa_1\text{-}\,\mathrm{I}_{M_1}^G(\sigma_1)\simeq \mathrm{R}_{M_2}^{G_D}\circ \mathrm{I}_{M_1}^{G_D}(\sigma_1).$$

LEMMA 7.2. Let $\widetilde{\sigma}_1=\mathrm{infl}_{J^1_{\max}\mathbf{P}_1(\Lambda_D)}(\sigma)$. Then

$$\kappa_2\text{-}\,\mathrm{R}_{M_2}^G\left(\kappa_1\text{-}\,\mathrm{I}_{M_1}^G(\sigma_1)\right)\simeq\bigoplus_{g\in\Sigma}\mathrm{Hom}_{J^1_{\max}\mathbf{P}_1(\Lambda^2_D)}\left(\kappa_{\max},\Phi(\sigma_1,g)\right)$$

where

$$\begin{aligned}
\Sigma &= J^1_{\max}\mathbf{P}(\Lambda^1_D)\backslash J^1_{\max}G_DJ^1_{\max}/J^1_{\max}\mathbf{P}_1(\Lambda^2_D),\\
\Phi(\sigma_1,g) &= \mathrm{ind}_{J^1_{\max}\mathbf{P}_1(\Lambda^2_D)\cap\, {}^g(J^1_{\max}\mathbf{P}(\Lambda^1_D))}^{J^1_{\max}\mathbf{P}_1(\Lambda^2_D)}\left(\mathrm{Res}_{J^1_{\max}\mathbf{P}_1(\Lambda^2_D)\cap\, {}^g(J^1_{\max}\mathbf{P}(\Lambda^1_D))}^{{}^g(J^1_{\max}\mathbf{P}(\Lambda^1_D))}\,{}^g(\kappa_{\max}\otimes\widetilde{\sigma}_1)\right).
\end{aligned}$$

PROOF: We use the relation between the κ_i-induction functors, $i=1,2$, and the $\kappa_{\max}$-induction functors,

$$\kappa_2\text{-}\,\mathrm{R}_{M_2}^G\left(\kappa_1\text{-}\,\mathrm{I}_{M_1}^G(\sigma_1)\right)\simeq\kappa_{\max}\text{-}\,\mathrm{R}_{M_2}^G\left(\kappa_{\max}\text{-}\,\mathrm{I}_{M_1}^G(\sigma_1)\right).$$

By Lemma 6.1, exactness of κ-restriction and the restriction-induction formula, Chapter 1 Lemma 3.5, we have

$$\kappa_{\max}\text{-}\,\mathrm{R}_{M_2}^G\left(\kappa_{\max}\text{-}\,\mathrm{I}_{M_1}^G(\sigma_1)\right)\simeq\bigoplus_{J^1_{\max}\mathbf{P}(\Lambda^1_D)\backslash G/J^1_{\max}\mathbf{P}_1(\Lambda^2_D)}\mathrm{Hom}_{J^1_{\max}\mathbf{P}_1(\Lambda^2_D)}\left(\kappa_{\max},\Phi(\sigma_1,g)\right).$$

Let $g \in G$ be such that

$$\mathrm{Hom}_{J^1_{\max}\mathbf{P}_1(\Lambda^2_D)}\left(\kappa_{\max}, \Phi(\sigma_1, g)\right) \neq \{0\}.$$

Because $J^1_{\max}\mathbf{P}_1(\Lambda^2_D)/\left(J^1_{\max}\mathbf{P}_1(\Lambda^2_D) \cap {}^g(J^1_{\max}\mathbf{P}(\Lambda^1_D))\right)$ is compact, compact induction is right adjoint to restriction, hence

$$\mathrm{Hom}_{J^1_{\max}\mathbf{P}_1(\Lambda^2_D)\cap {}^g(J^1_{\max}\mathbf{P}(\Lambda^1_D))}\left(\kappa_{\max}, {}^g(\kappa_{\max} \otimes \widetilde{\sigma}_1)\right) \neq \{0\}.$$

We have

$$J^1_{\max} \cap {}^g(J^1_{\max}) \subseteq J^1_{\max}\mathbf{P}_1(\Lambda^2_D) \cap {}^g(J^1_{\max}\mathbf{P}(\Lambda^1_D)).$$

Thus, restricting to $J^1_{\max} \cap {}^g(J^1_{\max})$, we have

$$\mathrm{Hom}_{J^1_{\max}\cap {}^g(J^1_{\max})}\left(\kappa_{\max}, {}^g(\kappa_{\max} \otimes \widetilde{\sigma}_1)\right) \neq \{0\}.$$

The tensor product satisfies ${}^g(\kappa_{\max} \otimes \widetilde{\sigma}_1) = {}^g\kappa_{\max} \otimes {}^g\widetilde{\sigma}_1$. Furthermore, by transitivity of restriction and as the restriction commutes with the conjugation,

$$\mathrm{Res}^{{}^g(J^1_{\max}\mathbf{P}(\Lambda^{\max}_D))}_{J^1_{\max}\cap {}^g(J^1_{\max})}({}^g\kappa_{\max}) = \mathrm{Res}^{{}^g(J^1_{\max})}_{J^1_{\max}\cap {}^g(J^1_{\max})}({}^g\eta_{\max})$$

and

$$\mathrm{Res}^{{}^g(J^1_{\max}\mathbf{P}(\Lambda^{\max}_D)))}_{J^1_{\max}\cap {}^g(J^1_{\max})}({}^g\mathrm{infl}_{J^1_{\max}}\ \sigma_1) = \dim(\sigma_1)1_{J^1_{\max}\cap {}^g(J^1_{\max})},$$

a sum of copies of the trivial representation. Thus

$$\begin{aligned}\mathrm{Hom}_{J^1_{\max}\cap {}^g(J^1_{\max})}\left(\kappa_{\max}, {}^g(\kappa_{\max} \otimes \widetilde{\sigma}_1)\right) &= \mathrm{Hom}_{J^1_{\max}\cap {}^g(J^1_{\max})}\left(\eta_{\max}, \bigoplus_{i=1}^{\dim(\sigma_1)} {}^g(\eta_{\max})\right)\\ &\simeq \bigoplus_{i=1}^{\dim(\sigma_1)} \mathrm{Hom}_{J^1_{\max}\cap {}^g(J^1_{\max})}\left(\eta_{\max}, {}^g(\eta_{\max})\right).\end{aligned}$$

However, $\mathbf{I}_G(\eta_{\max}) = J^1_{\max}G_DJ^1_{\max}$ by Lemma 4.2. Hence if

$$\mathrm{Hom}_{J^1_{\max}\cap {}^g(J^1_{\max})}\left(\kappa_{\max}, {}^g(\kappa_{\max} \otimes \widetilde{\sigma}_1)\right) \neq \{0\}$$

then $g \in J^1_{\max}G_DJ^1_{\max}$. □

LEMMA 7.3. Let Λ^1_D and Λ^2_D be self dual $\mathcal{O}_D$-lattice sequences such that $\mathbf{P}(\Lambda^1_D), \mathbf{P}(\Lambda^2_D) \subseteq \mathbf{P}(\Lambda^{\max}_D)$. Then the map

$$\begin{aligned}\mathbf{P}(\Lambda^1_D)\backslash G_D/\mathbf{P}_1(\Lambda^2_D) &\to J^1_{\max}\mathbf{P}(\Lambda^1_D)\backslash J^1_{\max}G_DJ^1_{\max}/J^1_{\max}\mathbf{P}_1(\Lambda^2_D)\\ X &\mapsto J^1_{\max}XJ^1_{\max}\end{aligned}$$

is a bijection.

PROOF: Let $g \in G_D$. We have $\mathbf{P}_1(\Lambda^{\max}) \supseteq J^1_{\max}$ hence

$$J^1_{\max}\left(\mathbf{P}(\Lambda^1_D)g\mathbf{P}_1(\Lambda^2_D)\right)J^1_{\max} \cap G_D \subseteq \mathbf{P}_1(\Lambda^{\max})\left(\mathbf{P}(\Lambda^1_D)g\mathbf{P}_1(\Lambda^2_D)\right)\mathbf{P}_1(\Lambda^{\max}) \cap G_D.$$

We choose a set of representatives for the finite double coset space

$$\mathbf{P}_1(\Lambda^{\max}_D)\backslash\mathbf{P}(\Lambda^1_D)g\mathbf{P}_1(\Lambda^2_D)/\mathbf{P}_1(\Lambda^{\max}_D).$$

For each representative Φ of this double coset space we apply the semisimple intersection property [**Ste08**, Lemma 2.6] to get $\mathbf{P}_1(\Lambda^{\max})\Phi\mathbf{P}_1(\Lambda^{\max}) \cap G_D = \mathbf{P}_1(\Lambda^{\max}_D)\Phi\mathbf{P}_1(\Lambda^{\max}_D)$. Hence

$$\mathbf{P}_1(\Lambda^{\max})\left(\mathbf{P}(\Lambda^1_D)g\mathbf{P}_1(\Lambda^2_D)\right)\mathbf{P}_1(\Lambda^{\max}) \cap G_D = \mathbf{P}(\Lambda^1_D)g\mathbf{P}_1(\Lambda^2_D).$$

Therefore

$$\mathbf{P}(\Lambda_D^1)g\mathbf{P}_1(\Lambda_D^2) = J^1_{\max}(\mathbf{P}(\Lambda_D^1)g\mathbf{P}_1(\Lambda_D^2))J^1_{\max} \cap G_D$$

and the map is injective. □

LEMMA 7.4. Let $g \in G_D$ then

$$\mathrm{Hom}_{J^1_{\max}\mathbf{P}_1(\Lambda_D^2)\cap {}^g(J^1_{\max}\mathbf{P}(\Lambda_D^1))}\left(\kappa_{\max}, {}^g(\kappa_{\max}\otimes\widetilde{\sigma}_1)\right) \simeq \mathrm{Hom}_{\mathbf{P}_1(\Lambda_D^2)\cap {}^g(\mathbf{P}(\Lambda_D^1))}(1, {}^g(\mathrm{infl}_{\mathbf{P}_1(\Lambda_D^1)}(\sigma_1))).$$

PROOF: The argument is essentially the same as [**BK93**, Proposition 5.3.2]. We identify $\kappa_{\max}$ with $\kappa_{\max}\otimes 1$. We showed at the start of the proof of Lemma 7.2,

$$\mathrm{Hom}_{J^1_{\max}\mathbf{P}_1(\Lambda_D^2)\cap {}^g(J^1_{\max}\mathbf{P}(\Lambda_D^1))}\left(\kappa_{\max}\otimes 1, {}^g(\kappa_{\max}\otimes\widetilde{\sigma}_1)\right) \neq \{0\}$$

if and only if $g \in J^1_{\max}G_DJ^1_{\max}$. Let $\phi \in \mathrm{Hom}_{J^1_{\max}\mathbf{P}_1(\Lambda_D^2)\cap {}^g(J^1_{\max}\mathbf{P}(\Lambda_D^1))}\left(\kappa_{\max}\otimes 1, {}^g(\kappa_{\max}\otimes\widetilde{\sigma}_1)\right)$ be non-zero. We write

$$\phi = \sum_k S_k \otimes T_k$$

with $S_k \in \mathrm{Hom}_R(\mathcal{V},\mathcal{V})$ and $T_k \in \mathrm{Hom}_R(R,\mathcal{W})$ where $\mathcal{V}$ is the space of $\kappa_{\max}$ and $\mathcal{W}$ is the space of $\widetilde{\sigma}_1$. We further assume that $\{T_k\}$ are linearly independent. Let $h \in J^1_{\max} \cap {}^g(J^1_{\max})$ and $v \in \mathcal{V}$ then

$$\phi((\kappa_{\max}\otimes 1)(h)v) = {}^g(\kappa_{\max}\otimes\widetilde{\sigma}_1)(h)\phi(v)$$

and, as $\widetilde{\sigma}_1$ is trivial on $J^1_{\max} \cap {}^g(J^1_{\max})$, we have

$$\sum_k (S_k\kappa_{\max}(h) - ({}^g\kappa_{\max}(h))S_k)\otimes T_k = 0.$$

Hence, by linear independence of T_k, we have $S_k \in \mathrm{Hom}_{J^1_{\max}\cap {}^g(J^1_{\max})}(\kappa_{\max}, {}^g\kappa_{\max})$. By Lemma 4.1

$$\mathrm{Hom}_{J^1_{\max}\cap {}^g(J^1_{\max})}(\kappa_{\max}, {}^g\kappa_{\max}) \simeq R.$$

Furthermore,

$$\mathrm{Hom}_{J^1_{\max}\mathbf{P}_1(\Lambda_D^2)\cap {}^g(J^1_{\max}\mathbf{P}(\Lambda_D^1))}(\kappa_{\max}, {}^g\kappa_{\max}) \simeq R$$

because $J^1_{\max}\mathbf{P}_1(\Lambda_D^2) \cap {}^g(J^1_{\max}\mathbf{P}(\Lambda_D^1))$ is contained in ${}^g(\widetilde{J})$ for $\widetilde{J}$ a Sylow p-subgroup of $J^1_{\max}\mathbf{P}(\Lambda_D^1)$ and, by Lemma 4.2, we have

$$\mathrm{Hom}_{{}^g(\widetilde{J})}(\kappa_{\max}, {}^g\kappa_{\max}) \simeq R$$

Thus $S_k \in \mathrm{Hom}_{J^1_{\max}\mathbf{P}_1(\Lambda_D^2)\cap {}^g(J^1_{\max}\mathbf{P}(\Lambda_D^1))}(\kappa_{\max}, {}^g\kappa_{\max})$ is a scalar multiple of $S = S_1$ and we can write $\phi = S\otimes T$ with $T \in \mathrm{Hom}_R(1,\widetilde{\sigma}_1)$. Furthermore, for $h \in J^1_{\max}\mathbf{P}_1(\Lambda_D^2) \cap {}^g(J^1_{\max}\mathbf{P}(\Lambda_D^1))$ and $v \in \mathcal{V}$,

$$(S\otimes T)((\kappa_{\max}\otimes 1)(h)v) = ({}^g(\kappa_{\max})(h)S \otimes {}^g(\widetilde{\sigma}_1)(h)T)(v)$$

and

$$(S\otimes T)((\kappa_{\max}\otimes 1)(h)v) = (S\circ\kappa_{\max}(h)\otimes T)(v) = ({}^g(\kappa_{\max})(h)\circ S\otimes T)(v).$$

Therefore $T \in \mathrm{Hom}_{J^1_{\max}\mathbf{P}_1(\Lambda_D^2)\cap {}^g(J^1_{\max}\mathbf{P}(\Lambda_D^1))}(1, {}^g(\widetilde{\sigma}_1))$ and the map $T \mapsto S\otimes T$ defines an isomorphism from $\mathrm{Hom}_{J^1_{\max}\mathbf{P}_1(\Lambda_D^2)\cap {}^g(J^1_{\max}\mathbf{P}(\Lambda_D^1))}(1, {}^g(\widetilde{\sigma}_1))$ to

$$\mathrm{Hom}_{J^1_{\max}\mathbf{P}_1(\Lambda_D^2)\cap {}^g(J^1_{\max}\mathbf{P}(\Lambda_D^1))}\left(\kappa_{\max}, {}^g(\kappa_{\max}\otimes\widetilde{\sigma}_1)\right).$$

Furthermore, we have an isomorphism

$$\mathrm{Hom}_{J^1_{\max}\mathbf{P}_1(\Lambda_D^2)\cap {}^g(J^1_{\max}\mathbf{P}(\Lambda_D^1))}(1, {}^g(\widetilde{\sigma}_1)) \simeq \mathrm{Hom}_{\mathbf{P}_1(\Lambda_D^2)\cap {}^g(\mathbf{P}(\Lambda_D^1))}(1, {}^g(\mathrm{infl}_{\mathbf{P}_1(\Lambda_D^1)}(\sigma_1))).$$

□

PROOF: [Proof of Theorem 7.1] By Lemmas 7.2, 7.3 and 7.4 we have

$$\kappa_2\text{-}\,\mathrm{R}^G_{M_2}\circ\kappa_1\text{-}\,\mathrm{I}^G_{M_1}(\sigma_1)\simeq\bigoplus_{\mathbf{P}(\Lambda^1_D)\backslash G_D/\mathbf{P}_1(\Lambda^2_D)}\mathrm{Hom}_{\mathbf{P}_1(\Lambda^2_D)\cap\,{}^g(\mathbf{P}(\Lambda^1_D))}(1,{}^g(\mathrm{infl}_{\mathbf{P}_1(\Lambda^1_D)}(\sigma_1))).$$

As $\mathbf{P}_1(\Lambda^2_D)/\mathbf{P}_1(\Lambda^2_D)\cap\,{}^g(\mathbf{P}(\Lambda^1_D))$ is compact, compact induction is right adjoint to restriction, and we have

$$\begin{aligned}\kappa_2\text{-}\,\mathrm{R}^G_{M_2}\circ\kappa_1\text{-}\,\mathrm{I}^G_{M_1}(\sigma_1)&\simeq\bigoplus_{\mathbf{P}(\Lambda^1_D)\backslash G_D/\mathbf{P}_1(\Lambda^2_D)}\mathrm{Hom}_{\mathbf{P}_1(\Lambda^2_D)}(1,\Psi(g))\\&\simeq\mathrm{Hom}_{\mathbf{P}_1(\Lambda^2_D)}\left(1,\bigoplus_{\mathbf{P}(\Lambda^1_D)\backslash G_D/\mathbf{P}_1(\Lambda^2_D)}\Psi(g)\right)\end{aligned}$$

where

$$\Psi(g)=\mathrm{ind}^{\mathbf{P}_1(\Lambda^2_D)}_{\mathbf{P}_1(\Lambda^2_D)\cap\,{}^g(\mathbf{P}(\Lambda^1_D))}\mathrm{Res}^{{}^g(\mathbf{P}(\Lambda^1_D))}_{\mathbf{P}_1(\Lambda^2_D)\cap\,{}^g(\mathbf{P}(\Lambda^1_D))}{}^g(\mathrm{infl}_{\mathbf{P}_1(\Lambda^1_D)}(\sigma_1)).$$

By the restriction-induction formula, Chapter 1 Lemma 3.5, we have

$$\mathrm{Res}^{G_D}_{\mathbf{P}_1(\Lambda^2_D)}\circ\mathrm{I}^{G_D}_{M_1}(\sigma_1)\simeq\bigoplus_{\mathbf{P}(\Lambda^1_D)\backslash G_D/\mathbf{P}_1(\Lambda^2_D)}\Psi(g).$$

Therefore

$$\begin{aligned}\kappa_2\text{-}\,\mathrm{R}^G_{M_2}\circ\kappa_1\text{-}\,\mathrm{I}^G_{M_1}(\sigma_1)&\simeq\mathrm{Hom}_{\mathbf{P}_1(\Lambda^2_D)}\left(1,\mathrm{Res}^{G_D}_{\mathbf{P}_1(\Lambda^2_D)}\circ\mathrm{I}^{G_D}_{M_1}(\sigma_1)\right)\\&\simeq\mathrm{R}^{G_D}_{M_2}\circ\mathrm{I}^{G_D}_{M_1}(\sigma_1).\end{aligned}$$

□

REMARK. It should be possible to generalise Theorem 7.1 in two ways.

(1) With a more general definition of β-extensions in the non-maximal case, together with compatibility with the definitions in the maximal case, the proof would apply to a general classical group. Note that at the moment it does apply if one has $\kappa_1=\kappa_2=\kappa_{\max}$.
(2) Let $[\Lambda^i,n_i,0,\beta]$, $i=1,2$, be skew semisimple strata in G. Let $\theta_i\in\mathcal{C}_-(\Lambda^i,\beta)$, $i=1,2$, such that $\theta_1=\tau_{\Lambda^2,\Lambda^1,\beta}(\theta_2)$. Let η_i be Heisenberg extensions of θ_i to H^1_i and κ_i be β-extensions of η_i to J_i, $i=1,2$. Then, an analogous proof to that of Theorem 7.1 would show that

$$\kappa_2\text{-}\,\mathrm{R}^G_{M_2}\circ\kappa_1\text{-}\,\mathrm{I}^G_{M_1}(\sigma)\simeq\mathrm{R}^{G_D}_{M_2}\circ\mathrm{I}^{G_D}_{M_1}(\sigma)$$

if we assume that

$$\dim_R(\mathrm{Hom}_{J^1_2\cap\,{}^gJ^1_1}(\eta_2,\,{}^g\eta_1))=\begin{cases}1&\text{if }g\in J_2G_DJ_1,\\0&\text{otherwise.}\end{cases}$$

8. CUSPIDAL POSITIVE LEVEL REPRESENTATIONS

Combining Theorem 7.1 and the results of Chapter 4 we can construct cuspidal positive level representations.

THEOREM 8.1. Let $[\Lambda,n,0,\beta]$ be a skew semisimple stratum. Let $\theta\in\mathcal{C}_-(\Lambda,\beta)$, η be the unique irreducible representation of J^1 containing θ and κ be a β-extension of η to J. Suppose $J=\mathbf{P}(\Lambda_D)J^1$ with $\mathbf{P}(\Lambda_D)$ a maximal parahoric subgroups of G_D. Let σ be an irreducible cuspidal representation of $\mathbf{M}(\Lambda_D)$. Then

$$\kappa\text{-}\,\mathrm{R}^G_{\mathbf{M}(\Lambda_D)}\left(\kappa\text{-}\,\mathrm{I}^G_{\mathbf{M}(\Lambda_D)}(\sigma)\right)\simeq\sigma$$

and

$$\left(\kappa\text{-}\,\mathrm{I}^G_{\mathbf{M}(\Lambda_D)}(\sigma)\right)^\eta \simeq \kappa\otimes\sigma.$$

Furthermore, $\kappa\text{-}\,\mathrm{I}^G_{\mathbf{M}(\Lambda_D)}(\sigma)$ is irreducible, cuspidal and quasi-projective.

Proof: By Theorem 7.1 and Chapter 4 Lemma 6.6, we have

$$\begin{aligned}\kappa\text{-}\,\mathrm{R}^G_{\mathbf{M}(\Lambda_D)}\left(\kappa\text{-}\,\mathrm{I}^G_{\mathbf{M}(\Lambda_D)}(\sigma)\right) &\simeq \mathrm{R}^{G_D}_{\mathbf{M}(\Lambda_D)}\left(\mathrm{I}^{G_D}_{\mathbf{M}(\Lambda_D)}(\sigma)\right).\\ &\simeq \sigma.\end{aligned}$$

Hence

$$\left(\kappa\text{-}\,\mathrm{I}^G_{\mathbf{M}(\Lambda_D)}(\sigma)\right)^\eta \simeq \kappa\otimes\sigma.$$

By reciprocity,

$$\mathrm{Hom}_G(\kappa\text{-}\,\mathrm{I}^G_{\mathbf{M}(\Lambda_D)}(\sigma_{x,D}),\kappa\text{-}\,\mathrm{I}^G_{\mathbf{M}(\Lambda_D)}(\sigma))\simeq \mathrm{Hom}_{\mathbf{M}(\Lambda_D)}(\sigma,\sigma)\simeq R.$$

If π is an irreducible quotient of $\kappa\text{-}\,\mathrm{I}^G_{\mathbf{M}(\Lambda_D)}(\sigma)$ then, by reciprocity, $\kappa\otimes\sigma$ is a subrepresentation of $\mathrm{Res}^G_J(\pi)$. Furthermore, by Lemma 6.2,

$$\pi\simeq\pi^\eta\oplus\pi(\eta).$$

Because J^1 is pro-p, ℓ-modular representations of J^1 are semisimple, and no irreducible subquotient of $\pi(\eta)$ is isomorphic to η. Thus, as representations of J, no irreducible subquotient of $\pi(\eta)$ is isomorphic to $\kappa\otimes\sigma$. As $\kappa\otimes\sigma$ is a direct factor of $\mathrm{Res}^G_{J_{\max}}(\pi)$, by reciprocity, π is a subrepresentation of $\kappa\text{-}\,\mathrm{I}^G_{\mathbf{M}(\Lambda_D)}(\sigma)$. Therefore, by Chapter 4 Lemma 6.5, $\kappa\text{-}\,\mathrm{I}^G_{\mathbf{M}(\Lambda_D)}(\sigma)$ is irreducible and, by Chapter 4 Lemma 8.2, $\kappa\text{-}\,\mathrm{I}^G_{\mathbf{M}(\Lambda_D)}(\sigma)$ is quasi-projective. □

9. The non-maximal semisimple case for U(2,1)(E/F)

In this section let G be an unramified unitary group in three variables. Recall that B denotes the parabolic subgroup of G given by

$$B=\begin{pmatrix}\star&\star&\star\\0&\star&\star\\0&0&\star\end{pmatrix}\cap G,$$

with Levi decomposition $B=T\ltimes N$ Let $[\Lambda,n,0,\beta]$ be a skew semisimple stratum which is not a simple stratum with $\beta\in E$. The only such semisimple stratum of G which does not give rise to a maximal $\mathbf{P}(\Lambda_D)$ is the case of a skew semisimple $(2,1)$-strata, see Section 3,

$$[\Lambda,n,0,\beta]=[\Lambda_1,n_1,0,\beta_1]\oplus[\Lambda_2,n_2,0,\beta_2]$$

with $\beta_2\in E$, $G_{D_2}\simeq \mathrm{U}(1,1)(E/F)$ and $\mathbf{P}(\Lambda_{2,D})$ an Iwahori subgroup. We have

$$\mathfrak{J}(\beta,\Lambda)=\begin{pmatrix}\mathcal{O}_E&\mathcal{P}_E^\alpha&\mathcal{O}_E\\\mathcal{P}_E^\beta&\mathcal{O}_E&\mathcal{P}_E^\alpha\\\mathcal{P}_E&\mathcal{P}_E^\beta&\mathcal{O}_E\end{pmatrix}$$

with $\alpha\geqslant 0$ and $\beta\geqslant 1$. Furthermore,

$$J(\beta,\Lambda)=\mathfrak{J}(\beta,\Lambda)^\times\cap G;\qquad J^1(\beta,\Lambda)=J(\beta,\Lambda)\cap\begin{pmatrix}1+\mathcal{P}_E&\mathcal{O}_E&\mathcal{O}_E\\\mathcal{P}_E&1+\mathcal{P}_E&\mathcal{O}_E\\\mathcal{P}_E&\mathcal{P}_E&1+\mathcal{P}_E\end{pmatrix}.$$

Write $J = J(\beta,\Lambda)$. Let $\theta \in \mathcal{C}_-(\Lambda,\beta)$, η be the unique irreducible representation of $J^1(\beta,\Lambda)$ containing θ, κ a β-extension of η to $J(\beta,\Lambda)$ and σ an irreducible representation of J/J^1. For $i \geqslant 0$, we have

$$\mathfrak{A}_i(\Lambda) = \begin{cases} \varpi^{[\frac{i}{4}]}\begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \end{pmatrix} & \text{if } i \equiv 0 \mod 4; \\ \varpi^{[\frac{i-1}{4}]}\begin{pmatrix} \mathcal{P}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \end{pmatrix} & \text{if } i \equiv 1 \mod 4; \\ \varpi^{[\frac{i-2}{4}]}\begin{pmatrix} \mathcal{P}_E & \mathcal{P}_E & \mathcal{O}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \end{pmatrix} & \text{if } i \equiv 2 \mod 4; \\ \varpi^{[\frac{i-3}{4}]}\begin{pmatrix} \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \\ \mathcal{P}_E & \mathcal{P}_E & \mathcal{P}_E \\ \mathcal{P}_E^2 & \mathcal{P}_E & \mathcal{P}_E \end{pmatrix} & \text{if } i \equiv 3 \mod 4. \end{cases}$$

Therefore $J^1 = H^1$. In [**Ste08**], G-covers relative to a parabolic subgroup P for a general classical group are defined and from an R-type (J,λ) one forms a group $J_P = H^1(J \cap P)$ on which the cover lives. However, in our case, $J_B = J$ which makes things a bit simpler. Let $J_T = J \cap T$, this is the group we denoted $\mathfrak{I}_T$ in Chapter 4.

LEMMA 9.1. The R-type $(J, \kappa\otimes\sigma)$ is a G-cover of the R-type $(J_T, \mathrm{Res}^J_{J_T}(\kappa\otimes\sigma))$.

PROOF: In the ℓ-adic case this is a special case of the general results of [**Ste08**, Propositions 7.10 and 7.13]. Thus, in the ℓ-modular case, $(J,\kappa\otimes\sigma)$ satisfies properties (1) and (2) of Chapter 4 Definition 4.6. It remains to show that there exists a strongly (B,J)-positive element z of the centre of T such that JzJ supports an invertible element of $\mathcal{H}(G,J,\kappa\otimes\sigma)$. The proof is similar to the proof of Chapter 4 Lemma 4.7. Let

$$w_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad w_2 = \begin{pmatrix} 0 & 0 & \varpi_F^{-1} \\ 0 & 1 & 0 \\ \varpi_F & 0 & 0 \end{pmatrix}$$

and $\zeta = w_1w_2$. For $x \in \mathbf{I}_G(\kappa\otimes\sigma)$, as in the proof of Chapter 4 Lemma 4.7, because $\kappa\otimes\sigma$ is a character there is a unique function in $f_x \in \mathcal{H}(G,J,\kappa\otimes\sigma)$ with support JxJ such that $f_x(x) = 1$. By [**Ste05**, Lemma 2.1] $\zeta,\zeta^{-1} \in \mathbf{I}_G(\kappa\otimes\sigma)$ hence $f_\zeta, f_{\zeta^{-1}} \in \mathcal{H}(G,J,\kappa\otimes\sigma)$. Suppose that $w_1 \notin \mathbf{I}_G(\kappa\otimes\sigma)$. Then $f_\zeta \star f_{\zeta^{-1}}(1_G) = q^4$. Furthermore, $\mathrm{supp}(f_\zeta \star f_{\zeta^{-1}}) = J$ by [**Ste08**, Corollary 7.12]. Hence f_ζ is an invertible element of $\mathcal{H}(G,J,\kappa\otimes\sigma)$ supported on J.

Now, suppose that $w_1 \in \mathbf{I}_G(\kappa\otimes\sigma)$. Then, because w_1 normalises J_T, we have w_1 normalises $\mathrm{Res}^J_{J_T}(\kappa\otimes\sigma)$. For all $x \in J_T$ we have $w_1xw_1 = w_2xw_2$, hence w_2 normalises $\mathrm{Res}^J_{J_T}(\kappa\otimes\sigma)$. Let $j \in J \cap w_2Jw_2$ such that $j = w_2j'w_2$. Using the Iwahori decomposition of J we have $j = j_{\overline{N}}j_Tj_N$ and $j' = j'_Nj'_Tj'_{\overline{N}}$ with j_N, j'_N upper triangular unipotent, $j_{\overline{N}}, j'_{\overline{N}}$ lower triangular unipotent and j_T, j'_T in T. Thus

$$j = w_2j'w_2^{-1} = (w_2j'_Nw_2)(w_2j'_Tw_2)(w_2j'_{\overline{N}}w_2)$$

and, by uniqueness of the Iwahori decomposition, $j_{\overline{N}} = w_2j'_Nw_2$, $j_T = w_2j'_Tw_2$ and $j_N = w_2j'_{\overline{N}}w_2$. Therefore $w_2 \in \mathbf{I}_G(\kappa\otimes\sigma)$. Hence $f_{w_1}, f_{w_2} \in \mathcal{H}(G,J,\kappa\otimes\sigma)$. We have $f_{w_i}\star f_{w_i}(1_G) =$

$[J : J \cap w_i J w_i]$ is a power of q, $i = 1, 2$. By [**Ste08**, Lemma 5.12], $\mathbf{I}_G(\eta) = J G_D J$, thus the support of $\mathcal{H}(G, J, \kappa \otimes \sigma)$ is contained in $J G_D J$. Hence, $\mathrm{supp}(f_{w_i} \star f_{w_i}) \subseteq (J \cup J w_i J) \cap J G_D J = J(K \cap G_D)J$ where K is a maximal parahoric subgroup of G. Therefore $\mathrm{supp}(f_{w_i} \star f_{w_i}) = J \cup J w_i J$ and f_{w_i}, $i = 1, 2$, are invertible elements of $\mathcal{H}(G, J, \kappa \otimes \sigma)$. By [**Ste08**, Lemma 7.11] we have $(J \cap N)^{w_1} \subseteq J \cap N$ and $(J \cap \overline{N})^{w_2} \subseteq J \cap \overline{N}$. Thus, by the Iwahori decomposition of J,

$$J w_1 J w_2 J = J(w_1(J \cap \overline{N})w_1) w_1 w_2 (w_2(J \cap T)w_2)(w_2(J \cap N)w_2)J = J w_1 w_2 J.$$

Hence $f_{w_2} \star f_{w_1}$ is an invertible element of $\mathcal{H}(G, J, \kappa \otimes \sigma)$ supported on the single double coset $J \zeta J$. □

LEMMA 9.2. Let G be an unramified unitary group in three variables. Suppose that $(\pi, \mathcal{V})$ is an irreducible cuspidal representation of G which contains the R-type $(J, \kappa \otimes \sigma)$. Then π is not cuspidal.

PROOF: Suppose that π contains $(J, \kappa \otimes \sigma)$. By Lemma 9.1, $(J, \kappa \otimes \sigma)$ is a G-cover of $(J_T, \mathrm{Res}^J_{J_T}(\kappa \otimes \sigma))$. Hence, by Chapter 4 Corollary 4.11, $r^G_B \pi \neq 0$ and π is not cuspidal. □

By Theorem 2.3, every positive level cuspidal representation of G contains a semisimple character. Every cuspidal positive level representation π is either a twist of a cuspidal level zero representation and for these we refer to Chapter 4; or contains a positive level $\overline{\mathbb{F}}_\ell$-type $(J, \kappa \otimes \sigma)$. If $(J, \kappa \otimes \sigma)$ is not maximal then, by Lemma 9.2, π is not cuspidal. Hence π contains a positive level $\overline{\mathbb{F}}_\ell$-type $(J, \kappa \otimes \sigma)$ of the form given in Theorem 8.1 and is compactly induced. Therefore, we have constructed all irreducible cuspidal representations of G by compact induction.

9.1. κ-induction and parabolic induction for U(2, 1)(E/F). Recall that $\overline{B}$ the parabolic subgroup of $M = J/J^1$ given by

$$\overline{B} = \begin{pmatrix} \star & \star & \star \\ 0 & \star & \star \\ 0 & 0 & \star \end{pmatrix} \cap M,$$

with Levi decomposition $\overline{B} = \overline{T} \ltimes \overline{N}$.

LEMMA 9.3. Let $\pi \in \mathfrak{R}_{\overline{\mathbb{F}}_\ell}(T)$ be an irreducible representation and let $\widetilde{\pi} \in \mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(T)$ be a lift of π. Suppose we are in the situation of Lemma 9.1 with $(J_T, \kappa_T \otimes \sigma)$ an R-type contained in $\widetilde{\pi}$ and $(J, \kappa \otimes \sigma)$ an ℓ-adic G-cover of $(J_T, \kappa_T \otimes \sigma)$ which has associated skew semisimple stratum $[\Lambda, n, 0, \beta]$ together with a semisimple character $\theta \in \mathcal{C}_-(\Lambda, \beta)$, a β-extension κ and an irreducible representation σ of J/J^1. associated to a semisimple character $\theta \in \mathcal{C}_-(\Lambda, \beta)$. Choose $\Lambda^{\max}$ such that $\mathbf{P}(\Lambda^{\max}_D)$ is maximal and define $\theta_{\max}$, $\eta_{\max}$ and $\kappa_{\max}$ as in Lemma 5.2. Let $\overline{\kappa}_{\max} = d_\ell(\kappa_{\max})$, $\overline{\kappa}_T = d_\ell(\kappa_T)$ and $M_{\max} = J_{\max}/J^1_{\max}$. Then

$$\left[\overline{\kappa}_{\max}\text{-}\,\mathrm{R}^G_{M_{\max}} \circ i^G_B(\pi)\right] \simeq \left[i^{M_{\max}}_{\overline{B}} \circ \overline{\kappa}_T\text{-}\,\mathrm{R}^T_{\overline{T}}(\pi)\right].$$

PROOF: In the ℓ-adic case a stronger result for $\mathrm{GL}_n(F)$ is shown in [**SZ99**, Proposition 7]. First we give references to adapt this proof in the ℓ-adic case for U(2, 1)(E/F). Recall, from Lemma 9.1, that $\kappa_T = \mathrm{Res}^J_{J \cap T}(\kappa)$ and let $\rho = \mathrm{ind}^T_{J \cap T}(\kappa \otimes \sigma)$, $\Omega_T = [T, \rho]_T$ and $\Omega = [T, \rho]_G$. Recall that $\mathfrak{R}(\Omega)$ denotes the full subcategory of $\mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(G)$ of representations all of whose irreducible subquotients have inertial support Ω; $\mathfrak{R}(\Omega_T)$ denotes the full subcategory of $\mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(T)$ of representations all of whose irreducible subquotients have inertial support Ω_T. Let ω denote the $M_{\max}$-conjugacy class of σ and ω_T denote the $\overline{T}$-conjugacy class of σ.

We let $\mathfrak{R}(\omega)$ be the full subcategory of $\mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(M)$ of representations all of whose irreducible subquotients have supercuspidal support in ω and let $\mathfrak{R}(\omega_T)$ be the full subcategory of $\mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(\overline{T})$ of representations all of whose irreducible subquotients have supercuspidal support in ω_T. Let $\mathcal{H}(T, \kappa_T \otimes \sigma) = \mathcal{H}(T, J_T, \kappa_T \otimes \sigma)$, $\mathcal{H}(G, \kappa \otimes \sigma) = \mathcal{H}(G, J, \kappa \otimes \sigma)$, $\overline{\mathcal{H}}(M_{\max}, \sigma) = \overline{\mathcal{H}}(M_{\max}, \overline{B}, \sigma)$ and $\overline{\mathcal{H}}(\overline{T}, \sigma) = \overline{\mathcal{H}}(\overline{T}, \overline{T}, \sigma)$ denote the spherical Hecke algebras as defined in Chapter 4 Section 4.1. The strategy of [**SZ99**] is to show that the following diagram is commutative where the horizontal arrows are equivalences of categories:

$$
\begin{array}{ccccccc}
\mathfrak{R}(\omega) & \xrightarrow[M_\omega]{\simeq} & \overline{\mathcal{H}}(M_{\max}, \sigma)\text{-}\mathfrak{Mod} & = & \overline{\mathcal{H}}(M_{\max}, \sigma)\text{-}\mathfrak{Mod} & \xleftarrow[M_\omega]{\simeq} & \mathfrak{R}(\omega) \\
\kappa_{\max}\text{-}\mathrm{R}^G_{M_{\max}} \uparrow & & \uparrow \mathrm{Res} & & (t_{\overline{B}})_* \uparrow & & \uparrow i^{M_{\max}}_{\overline{B}} \\
\mathfrak{R}(\Omega) & \xrightarrow[M_{\kappa\otimes\sigma}]{\simeq} & \mathcal{H}(G, \kappa \otimes \sigma)\text{-}\mathfrak{Mod} & & \overline{\mathcal{H}}(\overline{T}, \sigma)\text{-}\mathfrak{Mod} & \xleftarrow[M_{\omega_T}]{\simeq} & \mathfrak{R}(\omega_T) \\
i^G_B \uparrow & & \uparrow (t_B)_* & & \mathrm{Res} \uparrow & & \uparrow \kappa_T\text{-}\mathrm{R}^G_{\overline{T}} \\
\mathfrak{R}(\Omega_T) & \xrightarrow[M_{\kappa_T\otimes\sigma}]{\simeq} & \mathcal{H}(T, \kappa_T \otimes \sigma)\text{-}\mathfrak{Mod} & = & \mathcal{H}(T, \kappa_T \otimes \sigma)\text{-}\mathfrak{Mod} & \xleftarrow[M_{\kappa_T\otimes\sigma}]{\simeq} & \mathfrak{R}(\Omega_T)
\end{array}
$$

The bottom left commutative square, $M_{\kappa\otimes\sigma} \circ i^G_B \simeq (t_B)_* \circ M_{\kappa_T\otimes\sigma}$, where $(t_B)_*$ is an induction functor given by an injective homomorphism of algebras t_B in [**BK98**, Theorem 7.2], follows from the general result of [**BK98**, Corollary 8.4] using Lemma 9.1. The top right commutative square, $M_\omega \circ i^M_{\overline{B}} \simeq (t_{\overline{B}})_* \circ M_{\omega_T}$, is the analogue of this result for finite reductive groups.

By [**Ste08**, Proposition 7.1], we have a support preserving isomorphism

$$\mathcal{H}(G, J, \kappa \otimes \sigma) \simeq \mathcal{H}(G, \mathbf{P}(\Lambda_D)J^1_{\max}, \mathrm{Res}^{J_{\max}}_{\mathbf{P}(\Lambda_D)J^1_{\max}}(\kappa_{\max}) \otimes \sigma)$$

where σ is considered as a representation of $\mathbf{P}(\Lambda_D)J^1_{\max}$ by inflation; this makes sense as $\mathbf{P}(\Lambda_D)J^1_{\max}$ has $\mathbf{P}(\Lambda_D)/\mathbf{P}_1(\Lambda_D)$ as a quotient. By [**Ste08**, Proposition 7.2], we have a support preserving isomorphism

$$\mathcal{H}(J_{\max}, \mathbf{P}(\Lambda_D)J^1_{\max}, \mathrm{Res}^{J_{\max}}_{\mathbf{P}(\Lambda_D)J^1_{\max}}(\kappa_{\max}) \otimes \sigma) \simeq \mathcal{H}(\mathbf{P}(\Lambda^{\max}_D), \mathbf{P}(\Lambda_D), \sigma).$$

Furthermore, we have a support preserving isomorphism of Hecke algebras

$$\overline{\mathcal{H}}(M_{\max}, \sigma) \simeq \mathcal{H}(\mathbf{P}(\Lambda^{\max}_D), \mathbf{P}(\Lambda_D), \sigma).$$

Hence we have a support preserving injective map of algebras

$$\overline{\mathcal{H}}(M_{\max}, \sigma) \to \mathcal{H}(G, \kappa \otimes \sigma)$$

given by the injection

$$\mathcal{H}(J_{\max}, \mathbf{P}(\Lambda_D)J^1_{\max}, \mathrm{Res}^{J_{\max}}_{\mathbf{P}(\Lambda_D)J^1_{\max}}(\kappa_{\max}) \otimes \sigma) \to \mathcal{H}(G, \mathbf{P}(\Lambda_D)J^1_{\max}, \mathrm{Res}^{J_{\max}}_{\mathbf{P}(\Lambda_D)J^1_{\max}}(\kappa_{\max}) \otimes \sigma).$$

Therefore we can view $\overline{\mathcal{H}}(M_{\max}, \sigma)$ as a subalgebra of $\mathcal{H}(J, \kappa \otimes \sigma)$; similarly we view $\overline{\mathcal{H}}(\overline{T}, \sigma)$ as a subalgebra of $\mathcal{H}(J_T, \kappa_T \otimes \sigma)$. Hence we have restriction functors, denoted in the diagram by Res, between the categories of modules over these algebras. The top left commutative square, $M_\omega \circ \kappa_{\max}\text{-}\mathrm{R}^G_{M_{\max}} \simeq \mathrm{Res} \circ M_{\kappa\otimes\sigma}$, and the bottom right commutative square, $M_{\omega_T} \circ \kappa_T\text{-}\mathrm{R}^G_{\overline{T}} \simeq \mathrm{Res} \circ M_{\kappa_T\otimes\sigma}$, follow from [**SZ99**, Lemma 4] whose proof applies to $\mathrm{U}(2,1)(E/F)$. Consider the inclusions of algebras

$$\begin{array}{ccc} \mathcal{H}(T, \kappa_T \otimes \sigma) & \longrightarrow & \mathcal{H}(G, \kappa \otimes \sigma) \\ \Big\uparrow & & \Big\uparrow \\ \overline{\mathcal{H}}(\overline{T}, \sigma) & \longrightarrow & \overline{\mathcal{H}}(M_{\max}, \sigma) \end{array}$$

All the maps are injective homomorphisms of algebras hence must send the identity to the identity. But $\overline{\mathcal{H}}(\overline{T}, \sigma)$ is one-dimensional hence the diagram must commute. Thus the middle square commutes in our initial diagram, $\mathrm{Res} \circ (t_B)_* \simeq (t_{\overline{B}})_* \circ \mathrm{Res}$. Therefore we have the ℓ-adic result: For all finite length representations $\rho \in \mathfrak{R}(\Omega_T)$

$$\kappa_{\max}\text{-}\,\mathrm{R}^G_{M_{\max}} \circ i^G_B(\rho) \simeq i^{M_{\max}}_{\overline{B}} \circ \kappa_T\text{-}\,\mathrm{R}^T_{\overline{T}}(\rho).$$

Let ξ be a finite length integral ℓ-adic representation of G. By [**MS11b**, Lemma 5.14],

$$d_\ell(\kappa_{\max}\text{-}\,\mathrm{R}^G_{M_{\max}}(\xi)) \simeq \left[\kappa_{\max}\text{-}\,\mathrm{R}^G_{M_{\max}}(d_\ell(\xi))\right],$$

i.e. $\kappa_{\max}$-restriction commutes with decomposition modulo-ℓ. Similarly, for finite length integral representations of T, κ_T-restriction commutes with decomposition modulo-ℓ. Thus, by the ℓ-adic result, we have

$$\kappa_{\max}\text{-}\,\mathrm{R}^G_{M_{\max}} \circ i^G_B(\widetilde{\pi}) \simeq i^{M_{\max}}_{\overline{B}} \circ \kappa_T\text{-}\,\mathrm{R}^T_{\overline{T}}(\widetilde{\pi}).$$

Hence, by decomposition modulo-ℓ,

$$\left[\overline{\kappa}_{\max}\text{-}\,\mathrm{R}^G_{M_{\max}} \circ i^G_B(\pi)\right] \simeq \left[i^{M_{\max}}_{\overline{B}} \circ \overline{\kappa}_T\text{-}\,\mathrm{R}^T_{\overline{T}}(\pi)\right].$$

□

10. The positive level principal series of U(2, 1)(E/F)

In this section let G be an unramified unitary group in three variables. We describe the decomposition of the the positive level induced representations which have cuspidal subquotients. These are of the form:

(1) Twists $i^G_B(\chi_1 \otimes 1)(\chi_2 \circ \det)$ with χ_1 level zero and χ_2 positive level and for the description of the decomposition in this case we refer back to Chapter 4 Theorems 10.1 and 12.2.
(2) Of the form $i^G_B(\chi_1 \otimes 1)(\chi_2 \circ \det)$ with χ_1 a positive level character of $E^\times$.

We describe the decomposition of $i^G_B(\chi_1 \otimes 1)$ when χ_1 has positive level and when $\ell \mid q + 1$.

Let $(J_T, \mathrm{Res}^J_{J_T}(\kappa \otimes \sigma))$ be an R-type contained in $(\chi_1 \otimes 1)$ such that $(J, \kappa \otimes \sigma)$ is a G-cover of $(J_T, \mathrm{Res}^J_{J_T}(\kappa \otimes \sigma))$ as in Lemma 9.1. Then $J = \mathbf{P}(\Lambda) J^1$ with $\mathbf{P}(\Lambda) \simeq \mathfrak{I} \times E^1$ where $\mathfrak{I}$ is an Iwahori subgroup of $\mathrm{U}(1, 1)(E/F)$. Suppose $(\kappa_{\max}, \Lambda^{\max})$ is chosen to be compatible with (κ, Λ), as in Lemma 5.2. There are two such possible choices corresponding to the two non-conjugate maximal parahoric subgroups of $\mathrm{U}(1, 1)(E/F)$ which contain $\mathfrak{I}$. We denote these by $(\kappa^i_{\max}, \Lambda^{\max}_i)$, $i = 1, 2$. We have $M_{\max,i} = \mathbf{M}(\Lambda^{\max}_i) \simeq \mathrm{U}(1, 1)(k_E/k_F) \times \mathrm{U}(1)(k_E/k_F)$. For our proof of the next theorem we require an assumption:

Assumption 1: Let π be an irreducible subquotient of $i^G_B(\chi_1 \otimes 1)$. Then at least one of the two representations $\kappa^i_{\max}\text{-}\,\mathrm{R}^G_{M^i_{\max}}(\pi)$, $i = 1, 2$, is nonzero.

We consider this assumption as a generalisation of the result stated in Chapter 4 Theorem 3.3: parabolic induction preserves level zero representations.

THEOREM 10.1. Let G be an unramified p-adic unitary group in three variables. Suppose $\ell \neq 2$ or 3 and $\ell \mid q+1$. Let χ_1 be a positive level character of $E^\times$ trivial on $F^\times$ and let $\chi_1 \otimes 1$ be the positive level of character of T given by $\chi_1 \otimes 1(\mathrm{diag}(x, y, \overline{x}^{-1})) = \chi_1(x)$. Let $\overline{\chi}$ be the character of $\overline{T}$ given by $\mathrm{inv}_{1+\mathcal{P}_E}(\mathrm{Res}^{E^\times}_{\mathcal{O}_E^\times}(\chi_1)) \otimes 1$. Let σ_i denote the cuspidal subquotient of $i_{\overline{B}}^{M_{\max,i}}(\overline{\chi})$. The representation $i_B^G(\chi_1 \otimes 1)$ has the following composition series

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & \pi & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & \pi_1 & \longrightarrow & i_B^G(\chi_1 \otimes 1) & \longrightarrow & \pi_2 & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & \pi_3 & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

with π semisimple and either irreducible and isomorphic to $\kappa^1_{\max}\text{-}\,\mathrm{I}^G_{M^1_{\max}}(\sigma_1)$ or of length two with

$$\pi \simeq \kappa^1_{\max}\text{-}\,\mathrm{I}^G_{M^1_{\max}}(\sigma_1) \oplus \kappa^2_{\max}\text{-}\,\mathrm{I}^G_{M^2_{\max}}(\sigma_2).$$

PROOF: By Chapter 4 Theorem 12.1, the length of $i_B^G(\chi_1 \otimes 1)$ is greater than or equal to three. By Chapter 1 Theorem 3.12, $r_B^G(i_B^G(\chi_1 \otimes 1))$ has length two. Hence, as in the proof of Chapter 4 Theorem 10.1, $i_B^G(\chi_1 \otimes 1)$ has a unique irreducible subrepresentation, a unique irreducible quotient and at least one cuspidal subquotient. Let π_1 be the unique irreducible subrepresentation of $i_B^G(\chi_1 \otimes 1)$, π_2 the quotient of $i_B^G(\chi_1 \otimes 1)$ by π_1 and π_3 the unique irreducible subquotient.

By Lemmas 8.1 and 9.3, and Chapter 3 Section 4

$$\left[\kappa^i_{\max}\text{-}\,\mathrm{R}^G_{M_{\max,i}}(i_B^G(\chi_1 \otimes 1))\right] \simeq 1_{M_{\max,i}}(\overline{\chi}) \oplus 1_{M_{\max,i}}(\overline{\chi}) \oplus \sigma_i.$$

Thus, by Assumption 1, every cuspidal subquotient must be of the form $\kappa^i_{\max}\text{-}\,\mathrm{I}^G_{M^i_{\max}}(\sigma_i)$. Hence either we have two non-isomorphic cuspidal subquotients

$$\kappa^1_{\max}\text{-}\,\mathrm{I}^G_{M^1_{\max}}(\sigma_1) \text{ and } \kappa^2_{\max}\text{-}\,\mathrm{I}^G_{M^2_{\max}}(\sigma_2)$$

appearing in the composition series of $i_B^G(\chi_1 \otimes 1)$ or, without loss of generality, $\kappa^1_{\max}\text{-}\,\mathrm{I}^G_{M^1_{\max}}(\sigma_1)$ is the only cuspidal subquotient. By reciprocity, as in the proof of Chapter 4 Theorem 10.1, if $i_B^G(\chi_1 \otimes 1)$ is of length four then $\kappa^i_{\max}\text{-}\,\mathrm{I}^G_{M^i_{\max}}(\sigma_i)$ are both subrepresentations of π and hence π is semisimple. □

REMARK. We conjecture that the length of $i_B^G(\chi_1 \otimes 1)$ in Theorem 10.1 is four. A generalisation of Theorem 7.1, of the form remarked after our proof of Theorem 7.1, would imply that the length was greater than or equal to four.

10.1. Supercuspidal Support.

THEOREM 10.2. Let G be an unramified p-adic unitary group in three variables and π be an irreducible smooth ℓ-modular representation of G. Then $\mathrm{scusp}(\pi)$ exists and is unique up to conjugacy.

PROOF: This follows from Theorem 10.1 and Chapter 4 Theorem 14.1. $\square$

APPENDIX A

ℓ-MODULAR REPRESENTATIONS OF FINITE $\mathrm{GL}_n(F)$

In this appendix we extract the decomposition matrices for $\mathrm{GL}_2(F)$ and $\mathrm{GL}_3(F)$ from [**Jam90**].

1. Decomposition numbers of $\mathrm{GL}_2(F)$ and $\mathrm{GL}_3(F)$

Let $G = \mathrm{GL}_n(\mathbb{F}_q)$, and $(K, \mathcal{O}, k)$ be an ℓ-modular splitting system for G.

Let d, w be positive integers such that $dw = n$. For every partition λ of w, and every element s of degree d over $\mathbb{F}_q$, in [**DJ89**, Section 3] certain representations of G are defined:

(1) $S_K(s, \lambda)$ an irreducible ordinary representation of G;
(2) $S_k(s, \lambda)$ a reduction modulo ℓ of $S_K(s, \lambda)$ relative to a chosen lattice in $S_K(s, \lambda)$.
(3) $D_k(s, \lambda)$ an irreducible ℓ-modular representation of G; (in fact equal to the quotient of $S_k(s, \lambda)$ by its unique maximal submodule).

When $s = 1$ these representations are called unipotent. For λ_1, λ_2 partitions of w if $S_K(s, \lambda_1) \simeq S_K(s, \lambda_2)$ then $\lambda_1 = \lambda_2$ (similarly for S_k, and for D_k if $D_k \neq 0$) . We will use the shorthand $\lambda \vdash w$ for λ is a partition of w.

Lemma 1.1 ([**DJ89**, Section 3 (v)]). For s an element of degree d over $\mathbb{F}_q$, and $\lambda \vdash w$,

$$[S_k(s, \lambda)] = \bigoplus_{\mu \vdash w} d_{\lambda\mu} D_k(s, \mu).$$

For a fixed s, in [**DJ89**] a matrix $\Delta(s, w)$ of decomposition numbers is defined

$$\Delta(s, w) = (d_{\lambda\mu}).$$

This is part of the ℓ-modular decomposition matrix of G, but it is not immediately clear how these matrices overlap. The problem is split into two: first choosing a uniform notation for all the irreducible complex and irreducible ℓ-modular representations in terms of induced representations involving the S_k and D_k, then calculating the necessary decomposition matrices. Finally we can align these matrices to find the full decomposition matrix.

Lemma 1.2 ([**DJ89**, Theorem 6.2]). Let s be an element of degree d over $\mathbb{F}_q$. The matrix $\Delta(s, w)$ coincides with the matrix $\Delta'(1, w)$ of decomposition matrices for unipotent representations of $\mathrm{GL}_n(\mathbb{F}_{q^d})$.

Let $e(a)$ be the least positive integer such that $\ell \mid 1 + q^a + \cdots q^{a(e-1)}$, and $e = e(1)$.

Lemma 1.3 ([**Jam90**, Theorem 6.4]). When $n < e$, $\Delta(1, w)$ is the identity matrix.

In [**Jam90**, Appendix 1] the matrices $\Delta(1, w)$ are listed for $e = 2, 3$, $n \leqslant 10$, and a procedure for working out the matrices $n \leqslant 10$ for higher e is given.

We first recall the parametrisation of [**DJ89**] of the irreducible complex and ℓ-modular representations of G and then in the case of $n = 2, 3$ follow an algorithm for gluing the matrices $\Delta(s, w)$ in a coherent way to get the full ℓ-modular decomposition matrix of G.

Lemma 1.4 ([**DJ89**, Section 7.2]).

(1) If s, t are roots of the same irreducible polynomial over $\mathbb{F}_q$, then $S_K(s, \lambda) \simeq S_K(t, \lambda)$.
(2) If s, t are roots of irreducible polynomials of the same degree over $\mathbb{F}_q$ and their ℓ-regular parts are roots of the same irreducible polynomial over $\mathbb{F}_q$ then $D_k(s, \lambda) \simeq D_k(t, \lambda)$ and $S_k(s, \lambda) \simeq S_k(t, \lambda)$.

(3) Let $\mathcal{C}'$ be a complete set of roots from every irreducible monic polynomial over $\mathbb{F}_q$ of degree at most n, i.e. $\mathcal{C}' = \bigcup_{m \leqslant n} \mathbb{F}_{q^m}$. Define an equivalence relation $\sim$ on $\mathcal{C}'$ by two roots are equivalent if and only if they are roots of the same irreducible monic polynomial over $\mathbb{F}_q$ and let $\mathcal{C} = \mathcal{C}'/\sim$. Put a total order $<$ on the finite set $\mathcal{C}$. The following classes of representations give complete sets of irreducible inequivalent ordinary and ℓ-modular representations of G:

(a) Let $s_1 < s_2 < \cdots < s_r$ be classes of $\mathcal{C}$, with $d_i = \deg(s_i)$, such that

$$d_1 w_1 + d_2 w_2 + \cdots d_r w_r = n,$$

and for each $w_i \neq 0$ let $\lambda_i \vdash w_i$. Then

$$\mathrm{Ind}\left(S_K\left(s_1, \lambda_1\right) \otimes S_K\left(s_2, \lambda_2\right) \otimes \cdots \otimes S_K\left(s_r, \lambda_r\right)\right),$$

is an irreducible ordinary representation of G, where we consider the representation $\bigotimes S_K(s_i, \lambda_i)$ as a representation of the Levi subgroup $\mathrm{GL}_{w_1}(\mathbb{F}_q) \times \cdots \times \mathrm{GL}_{w_2}(\mathbb{F}_q)$. The induction, Ind, is Harish-Chandra induction.

(b) Let $s_1 < s_2 < \cdots < s_r$ be ℓ-regular elements of $\mathcal{C}$, λ_i as above, then

$$\mathrm{Ind}\left(D_k\left(s_1, \lambda_1\right) \otimes D_k\left(s_2, \lambda_2\right) \otimes \cdots \otimes D_k\left(s_r, \lambda_r\right)\right)$$

is an irreducible ℓ-modular representation of G.

Remark. This classification originates in [**DJ86**]. They call certain sets of data,

$$\begin{pmatrix} s_i & \lambda_i \\ d_i & w_i \end{pmatrix},$$

(n, ℓ)-indices. These give induced representations as in Lemma 1.4 part (3). The "head" (n, ℓ)-indices are the ones that satisfy the conditions in (3)(b) (or (a) if $\ell = \infty$). Thus this classification is referred to as a classification in terms of head (n, ℓ)-indices. In their paper they also construct a bijection from head (n, ℓ)-indices to "special foot" (n, ℓ)-indices where some of the properties required on the s_i in a head (n, ℓ)-index are weakened and replaced with stronger properties on the λ_i.

Our goal is to write the ℓ-modular decomposition of the representations in (a) in terms of the representations in (b) in the cases $n = 2, 3$. Thankfully it is explained how to do this in the penultimate section of [**DJ89**]. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of n, the dual partition λ' of n is the partition $(\lambda_1', \lambda_2', \ldots, \lambda_s')$ where $\lambda_i' = |\{j : \lambda_j \geqslant i\}|$. We write s^ℓ for the ℓ-regular part of s. We require two more Lemmas to apply the algorithm:

Lemma 1.5 ([**DJ89**, Section 3.5]).] Let s be an element of degree d over $\mathbb{F}_q$, $\lambda_i \vdash w_i$, $i = 1, 2$, $d(w_1 + w_2) = n$. Then

$$[\mathrm{Ind}(S_K(s, \lambda_1) \otimes S_K(s, \lambda_2))] = \left[\sum_{\tau \vdash w_1 + w_2} a_{\lambda_1 \lambda_2 \tau} S_K(s, \tau)\right],$$

where $a_{\lambda_1 \lambda_2 \tau}$ is calculated using the Littlewood-Richardson rule.

Lemma 1.6 ([**DJ89**, Section 7.3]). Let s be an element of degree d over $\mathbb{F}_q$, $\lambda \vdash w$. Let a be the degree of s^ℓ. Then $a \mid d$, and defining $\mu \vdash \frac{dw}{a}$ to be the partition given via its dual partition $\mu' = (\mu_1, \mu_2, ..., \mu_r)$ with $\mu_i' = \frac{d}{a}\lambda_i'$, we have

$$D_k(s, \lambda) = D_k(s^\ell, \mu).$$

1.1. Decomposition modulo ℓ when $n = 2$. The order of $GL_2(\mathbb{F}_q)$ is $(q+1)(q-1)^2q$. In non-describing characteristic, as $\gcd(q+1, q-1) = 2$, there are three cases to consider

(A-1) $\ell \neq 2$ and $\ell \mid q-1$;
(A-2) $\ell \neq 2$ and $\ell \mid q+1$;
(A-3) $\ell = 2$.

Case (A-1) corresponds to $e \geqslant 3$, cases (A-2) and (A-3)t to $e = 2$. The classification gives three classes of ordinary representation of G:

(B-1) $S_K\left(s, \left(1^2\right)\right)$ and $S_K\left(s, (2)\right)$ with s of degree 1 over $\mathbb{F}_q$;
(B-2) $\mathrm{Ind}(S_K(s_1, 1) \otimes S_K(s_2, 1))$ with s_i both of degree 1 over $\mathbb{F}_q$, and $s_1 \neq s_2$;
(B-3) $S_K(s, 1)$ with s of degree 2 over $\mathbb{F}_q$.

Class (B-1) constitutes the characters and special representations of G that appear as quotients and subquotients of the reducible principal series; class (B-2) are the irreducible principal series; class (B-3) are the supercuspidals.

We compute the decomposition of each of these three classes:

In [**Jam90**] we find the matrices $\Delta(1, 2)$ for $GL_2(F)$, $e = 2, 3$ are the following:

	$D_k(1, (1^2))$	$D_k(1, (2))$
$S_K(1, (1^2))$	1	0
$S_K(1, (2))$	1	1

$e = 2$

	$D_k(1, (1^2))$	$D_k(1, (2))$
$S_K(1, (1^2))$	1	0
$S_K(1, (2))$	0	1

$e = 3$

We will see that the circled ℓ-modular representation is cuspidal non-supercuspidal, the appearance of such representations being one of the key differences between the ordinary and ℓ-modular theory of representations of finite reductive groups.

(B-1) Applying Lemmas 1.1 - 1.3, and looking at the matrices $\Delta(1,2)$ we find

$$[S_k(s,(1^2))] = D_k(s^\ell,(1^2)) \quad \text{always;}$$

$$[S_k(s,(2))] = \begin{cases} D_k(s^\ell,(1^2)) + D_k(s^\ell,(2)) & \text{if } e=2; \\ D_k(s^\ell,(2)) & \text{otherwise.} \end{cases}$$

(B-2) Applying Lemma 1.4,

$$\begin{aligned}[\mathrm{Ind}(S_k(s_1,(1)) \otimes S_k(s_2,(1)))] &= [\mathrm{Ind}(D_k(s_1,(1)) \otimes D_k(s_2,(1)))] \\ &= [\mathrm{Ind}(D_k(s_1^\ell,(1)) \otimes D_k(s_2^\ell,(1)))].\end{aligned}$$

Either

(a) $s_1^\ell \neq s_2^\ell$ in $\mathcal{C}$ then $\mathrm{Ind}(D_k(s_1^\ell,(1)) \otimes D_k(s_2^\ell,(1)))$ is irreducible and in the classification; or

(b) $s_1^\ell = s_2^\ell$ in $\mathcal{C}$, then the idea is to show that the representation is reducible. The plan is to work with the S_k. Then Lemma 1.5 applies, because reduction modulo ℓ commutes with induction. For $\mathrm{GL}_2(F)$ this is all very simple; applying Lemma 1.5

$$[\mathrm{Ind}(S_k(s_1,(1)) \otimes S_k(s_2,(1)))] = S_k(s_1^\ell,(1^2)) + S_k(s_1^\ell,(2)).$$

Further $d=1$, hence by case (B-1):

$$[\mathrm{Ind}(S_k(s_1,(1)) \otimes S_k(s_2,(1)))] = \begin{cases} 2D_k(s_1^\ell,(1^2)) + D_k(s_1^\ell,(2)) & \text{if } e=2; \\ D_k(s_1^\ell,(1^2)) + D_k(s_1^\ell,(2)) & \text{otherwise.} \end{cases}$$

Remark. It is worth noting that a necessary condition for $s_1^\ell \equiv s_2^\ell$ in $\mathcal{C}$, but $s_1 \not\equiv s_2$ in $\mathcal{C}$ is that ℓ divides $q-1$. The only ℓ which has $e=2$ and divides $q-1$ is $l=2$. Hence unless $l=2$, the $e=2$ case of (b) is empty.

(B-3) Applying Lemmas 1.1

$$[S_k(s,(1))] = D_k(s,(1))$$

There are two possibilities:

(a) s^ℓ is of degree 2 over $\mathbb{F}_q$. Then, by Lemma 1.6, $D_k(s,(1)) = D_k(s^\ell,(1))$ is in the classification. These correspond to the cuspidal ordinary representations of $\mathrm{GL}_2(F)$ which on reduction are supercuspidal.

(b) s^ℓ is of degree 1 over $\mathbb{F}_q$. In this case $D_k(s^\ell,(1))$ is not in the classification and we need to apply Lemma 1.6. We have $d=2$, $a=1$, $e(a)=2$, and find

$$\begin{aligned}[S_k(s,(1))] &= D_k(s^\ell,(1)) \\ &= D_k(s^\ell,(2)).\end{aligned}$$

These ℓ-modular representations occur in the decomposition of ordinary special representations, and are cuspidal non-supercuspidal.

An element α is of degree 1 over $\mathbb{F}_q$ if and only if $\alpha^q = \alpha$, i.e. the order of α divides $q-1$. We consider the different cases for ℓ, where we can have an element s of degree 2 over $\mathbb{F}_q$ such that s^ℓ is of degree 1 over $\mathbb{F}_q$.

Assume that s^ℓ is of degree 1 over $\mathbb{F}_q$ then $(s^\ell)^{q-1} = 1$, hence $s^{q-1} = (s_\ell)^{q-1}$. Thus s has the same degree as s_ℓ the ℓ-part of s. Furthermore, as s_ℓ is at most of degree 2 over $\mathbb{F}_q$, the order of s_ℓ has to divide $q^2 - 1$. When $\ell \neq 2$ we have two cases:

(i) $\ell \mid q-1$. As $\gcd(q-1, q+1) = 2$, the order of s_ℓ and hence of s must divide $q-1$. Therefore no cuspidal non-supercuspidal representations appear.

(ii) $\ell \mid q+1$. Any nontrivial ℓ-element has degree 2 over $\mathbb{F}_q$. Let $\ell^a || q+1$, then there are $(\ell^a - 1)$ nontrivial ℓ-elements, $(q-1)$ ℓ-regular elements of degree one over $\mathbb{F}_q$ and two roots of each irreducible polynomial of degree two over $\mathbb{F}_q$. Hence

$$\text{Number of cuspidal subquotients} = \frac{(\ell^a - 1)(q-1)}{2}.$$

Finally when $l = 2$, let $2^b || q-1$ and $2^a || q+1$. The 2-elements x of degree two over $\mathbb{F}_q$ are the ones that satisfy $2^{b+1} \leqslant o(x)$, where $o(x)$ denotes the order of x. Because $o(x) \mid q^2 - 1$ we have $2^{b+1} \leqslant o(x) \leqslant 2^{a+b}$. Thus there are 2^a 2-elements of degree two over $\mathbb{F}_q$ and $\frac{(q-1)}{2^b}$ 2-regular elements of degree one over $\mathbb{F}_q$. Hence

$$\text{Number of cuspidal subquotients} = \frac{2^a(q-1)}{2^{b+1}}.$$

1.2. Decomposition modulo ℓ when $n = 3$. The order of $\mathrm{GL}_3(\mathbb{F}_q)$ is $q^3(q-1)^3(q+1)(q^2+q+1)$. In non-describing characteristic, as $\gcd(q+1, q-1) \mid 2$, $\gcd(q-1, q^2+q+1) \mid 3$, and $\gcd(q+1, q^2+q+1) = 1$ there are five cases to consider:

(A-1) $\ell \neq 2, 3$ and $\ell \mid q-1$,
(A-2) $\ell \neq 2$ and $\ell \mid q+1$,
(A-3) $\ell \neq 2, 3$ and $\ell \mid q^2+q+1$,
(A-4) $\ell = 3$ and $\ell \mid q-1$,
(A-5) $\ell = 2$.

Case (A-2) has $e = 2$, case (A-1) has $e > 3$, case (A-3) has $e = 3$, case (A-4) has $e = 3$, case (A-5) has $e = 2$. By Lemma 1.3 and [**Jam90**], we know the matrices $\Delta(1, 3)$ in each of these cases. Note that, to follow the same procedure for $\mathrm{GL}_n(\mathbb{F}_q)$ when $n \geqslant 4$ one has to use an algorithm in [**Jam90**] to compute the matrices $\Delta(1, w)$ for the values of e between 3 and n and w dividing n.

The classification gives five classes of ordinary representation of G:

(B-1) $S_K\left(s, (1^3)\right)$, $S_K(s, (21))$, and $S_K(s, (3))$ with s is of degree 1 over $\mathbb{F}_q$;
(B-2) $\mathrm{Ind}(S_K(s_1, 2) \otimes S_K(s_2, 1))$, $\mathrm{Ind}(S_K(s_1, (1^2)) \otimes S_K(s_2, 1))$ with s_i both of degree 1 over $\mathbb{F}_q$;
(B-3) $\mathrm{Ind}(S_K(s_1, 1) \otimes S_K(s_2, 1))$ with s_1 of degree 2, and s_2 of degree 1 over $\mathbb{F}_q$;
(B-4) $\mathrm{Ind}(S_K(s_1, 1) \otimes S_K(s_2, 1) \otimes S_K(s_3, 1))$ with s_i all of degree 1 over $\mathbb{F}_q$;
(B-5) $S_K(s, 1)$ with s of degree 3 over $\mathbb{F}_q$.

Class (B-1) constitutes the unipotent representations of $\mathrm{GL}_3(\mathbb{F}_q)$, the characters , the generalized Steinberg representations, and the special representations (also generalized Steinberg representations). Class (B-4) and, classes (B-2) and (B-3), the irreducible representations induced from characters of the diagonal torus and of the Levi subgroup isomorphic to $\mathrm{GL}_2(\mathbb{F}_q) \times \mathrm{GL}_1(\mathbb{F}_q)$. Class (B-5) are the supercuspidal representations of $\mathrm{GL}_3(\mathbb{F}_q)$.

In [**Jam90**] we find the matrices $\Delta(1,3)$ for $\mathrm{GL}_3(F)$, $e = 2, 3$ are the following:

$e = 2$:

	$D_k(1,(1^3))$	$D_k(1,(21))$	$D_k(1,(3))$
$S_K(1,(1^3))$	1	0	0
$S_K(1,(21))$	0	1	0
$S_K(1,(3))$	1	0	1

$e = 3$:

	$D_k(1,(1^3))$	$D_k(1,(21))$	$D_k(1,(3))$
$S_K(1,(1^3))$	1	0	0
$S_K(1,(21))$	1	1	0
$S_K(1,(3))$	0	1	1

We apply the algorithm we have followed already for $\mathrm{GL}_2(F)$ to each of the classes of representations of $\mathrm{GL}_3(F)$

(B-1) Applying Lemmas 1.1 - 1.3 and looking at the decomposition matrices $\Delta(1,3)$ we find

$$[S_k(s,(1^3))] = D_k(s^\ell,(1^3)) \quad \text{always;}$$

$$[S_k(s,(21))] = \begin{cases} D_k(s^\ell,(1^3)) + D_k(s^\ell,(21)) & \text{if } e = 3; \\ D_k(s^\ell,(21)) & \text{otherwise.} \end{cases}$$

$$[S_k(s,(3))] = \begin{cases} D_k(s^\ell,(1^3)) + D_k(s^\ell,(3)) & \text{if } e=2; \\ D_k(s^\ell,(21)) + D_k(s^\ell,(3)) & \text{if } e=3; \\ D_k(s^\ell,(3)) & \text{otherwise.} \end{cases}$$

(B-2) If $s_1^\ell \neq s_2^\ell$ in $\mathcal{C}$, $\mathrm{Ind}(D_k(s_1^\ell,(2)) \otimes D_k(s_2^\ell,(1)))$ and $\mathrm{Ind}(D_k(s_1^\ell,(1^2)) \otimes D_k(s_2^\ell,(1)))$ are irreducible and in the classification. By the decomposition matrices $\Delta(1,2)$ we have:

$$\mathrm{Ind}(S_k(s_1,(1^2)) \otimes S_k(s_2,(1))) = \mathrm{Ind}(D_k(s_1^\ell,(1^2)) \otimes D_k(s_2^\ell,(1))) \quad \text{always;}$$

$$\mathrm{Ind}(S_k(s_1,(2)) \otimes S_k(s_2,(1)))$$
$$= \begin{cases} \mathrm{Ind}(D_k(s_1^\ell,(1^2)) \otimes D_k(s_2^\ell,(1))) + \mathrm{Ind}(D_k(s_1^\ell,(2)) \otimes D_k(s_2^\ell,(1))) & \text{if } e=2; \\ \mathrm{Ind}(D_k(s_1^\ell,(2)) \otimes D_k(s_2^\ell,(1))) & \text{otherwise.} \end{cases}$$

Now assume $s_1^\ell = s_2^\ell$ in $\mathcal{C}$. Applying Lemmas 1.5 and 1.4 and using the matrices $\Delta(1,3)$:

$$[\mathrm{Ind}(S_k(s_1,(2)) \otimes S_k(s_2,(1)))] = S_k(s_1^\ell,(3)) + S_k(s_1^\ell,(21))$$
$$= \begin{cases} D_k(s_1^\ell,(3)) + D_k(s_1^\ell,(21)) + D_k(s_1^\ell,(1^3)) & \text{if } e=2; \\ D_k(s_1^\ell,(3)) + 2D_k(s_1^\ell,(21)) + D_k(s_1^\ell,(1^3)) & \text{if } e=3; \\ D_k(s_1^\ell,(3)) + D_k(s_1^\ell,(21)) & \text{if } e>3. \end{cases}$$

$$[\mathrm{Ind}(S_k(s_1,(1^2)) \otimes S_k(s_2,(1)))] = S_k(s_1^\ell,(21)) + S_k(s_1^\ell,(1^3))$$
$$= \begin{cases} D_k(s_1^\ell,(21)) + D_k(s_1^\ell,(1^3)) & \text{if } e=2; \\ D_k(s_1^\ell,(21)) + 2D_k(s_1^\ell,(1^3)) & \text{if } e=3; \\ D_k(s_1^\ell,(21)) + D_k(s_2^\ell,(1^3)) & \text{if } e>3. \end{cases}$$

(B-3) Applying Lemma 1.4,

$$\begin{aligned}[\mathrm{Ind}(S_k(s_1,(1)) \otimes S_k(s_2,(1)))] &= [\mathrm{Ind}(D_k(s_1,(1)) \otimes D_k(s_2,(1)))] \\ &= [\mathrm{Ind}(D_k(s_1^\ell,(1)) \otimes D_k(s_2^\ell,(1)))].\end{aligned}$$

Either

(a) s_1^ℓ is of degree 2 over $\mathbb{F}_q$, then $\mathrm{Ind}(D_k(s_1^\ell,(1)) \otimes D_k(s_2^\ell,(1)))$ is irreducible and in the classification.

(b) s_1^ℓ is of degree 1 over $\mathbb{F}_q$, then by Lemma 1.6 $[\mathrm{Ind}(D_k(s_1,(1)) \otimes D_k(s_2^\ell,(1)))] = [\mathrm{Ind}(D_k(s_1^\ell,(2)) \otimes D_k(s_2^\ell,(1)))]$ and we have two further subcases:

  (i) If $s_1^\ell \not\equiv s_2^\ell$ in $\mathcal{C}$. Then $\mathrm{Ind}(D_k(s_1^\ell,(2)) \otimes D_k(s_2^\ell,(1)))$ is irreducible and in the classification.

  (ii) If $s_1^\ell \equiv s_2^\ell$ in $\mathcal{C}$. We use the matrices $\Delta(1,2)_e$ to write $D_k(s_1^\ell,(2))$ in terms of S_k, then apply Lemma 1.5, and the decomposition matrices $\Delta(1,3)_e$:

$$[\mathrm{Ind}(D_k(s_1^\ell,(2)) \otimes D_k(s_1^\ell))]$$
$$= \begin{cases} [\mathrm{Ind}((S_k(s_1^\ell,(2)) - S_k(s_1^\ell,(1^2))) \otimes S_k(s_1^\ell,(1))] & \text{if } e=2; \\ [\mathrm{Ind}(S_k(s_1^\ell,(2)) \otimes S_k(s_1^\ell,(1))] & \text{otherwise.} \end{cases}$$
$$= \begin{cases} [\mathrm{Ind}(S_k(s_1^\ell,(2)) \otimes S_k(s_1^\ell,(1)) - \mathrm{Ind}(S_k(s_1^\ell,(1^2)) \otimes S_k(s_1^\ell,(1))] & \text{if } e=2; \\ [\mathrm{Ind}(S_k(s_1^\ell,(2)) \otimes S_k(s_1^\ell,(1))] & \text{otherwise.} \end{cases}$$

$$= \begin{cases} [(S_k(s_1^\ell,(3)) + S_k(s_1^\ell,(21))) - (S_k(s_1^\ell,(21)) + S_k(s_1^\ell,(1^3)))] & \text{if } e=2; \\ [S_k(s_1^\ell,(3)) + S_k(s_1^\ell,(21))] & \text{otherwise.} \end{cases}$$

$$= \begin{cases} D_k(s_1^\ell,(3)) & \text{if } e=2; \\ D_k(s_1^\ell,(1^3)) + 2D_k(s_1^\ell,(21)) + D_k(s_1^\ell,(3)) & \text{if } e=3; \\ D_k(s_1^\ell,(3)) + D_k(s_1^\ell,(21)) & \text{otherwise.} \end{cases}$$

(B-4) Applying Lemma 1.4,

$$\begin{aligned}[\mathrm{Ind}(S_k(s_1,(1)) \otimes S_k(s_2,(1)) \otimes S_k(s_3,(1)))] &= [\mathrm{Ind}(D_k(s_1,(1)) \otimes D_k(s_2,(1)) \otimes D_k(s_3,(1)))] \\ &= [\mathrm{Ind}(D_k(s_1^\ell,(1)) \otimes D_k(s_2^\ell,(1)) \otimes D_k(s_3^\ell,(1)))].\end{aligned}$$

We have three further subcases:

(a) When s_i^ℓ are pairwise distinct in $\mathcal{C}$, then $\mathrm{Ind}(D_k(s_1^\ell,(1)) \otimes D_k(s_2^\ell,(1)) \otimes D_k(s_3^\ell,(1)))$ is irreducible and is in the classification.

(b) When $s_1^\ell \equiv s_2^\ell \equiv s_3^\ell$ in $\mathcal{C}$, then we apply Lemma 1.5 using the transitivity of induction first inducing to $\mathrm{GL}_2(\mathbb{F}_q) \times \mathrm{GL}_1(\mathbb{F}_q)$

$$\begin{aligned}&[\mathrm{Ind}(S_k(s_1^\ell,(1)) \otimes S_k(s_2^\ell,(1)) \otimes S_k(s_3^\ell,(1)))] \\ &= [\mathrm{Ind}(\mathrm{Ind}(S_k(s_1^\ell,(1)) \otimes S_k(s_1^\ell,(1))) \otimes S_k(s_1^\ell,(1)))] \\ &= [\mathrm{Ind}((S_k(s_1^\ell,(1^2)) + S_k(s_1^\ell,(2))) \otimes S_k(s_1^\ell,(1)))] \\ &= [\mathrm{Ind}((S_k(s_1^\ell,(1^2)) + S_k(s_1^\ell,(2))) \otimes D_k(s_1^\ell,(1)))]\end{aligned}$$

$$= \begin{cases} [\mathrm{Ind}((2D_k(s_1^\ell,(1^2)) + D_k(s_1^\ell,(2))) \otimes D_k(s_1^\ell,(1)))] & \text{if } e=2; \\ [\mathrm{Ind}((D_k(s_1^\ell,(1^2)) + D_k(s_1^\ell,(2))) \otimes D_k(s_1^\ell,(1)))] & \text{otherwise.} \end{cases}$$

$$= \begin{cases} [2\,\mathrm{Ind}((D_k(s_1^\ell,(1^2)) \otimes D_k(s_1^\ell,(1))) + \mathrm{Ind}(D_k(s_1^\ell,(2)) \otimes D_k(s_1^\ell,(1)))] & \text{if } e=2; \\ [\mathrm{Ind}((D_k(s_1^\ell,(1^2)) \otimes D_k(s_1^\ell,(1))) + \mathrm{Ind}(D_k(s_1^\ell,(2)) \otimes D_k(s_1^\ell,(1)))] & \text{otherwise.} \end{cases}$$

Now we write the D_k in terms of the S_k using the matrices $\Delta(1,w)_e$, and apply the relevent case from (2):

$$\begin{aligned}&= [\mathrm{Ind}((S_k(s_1^\ell,(1^2)) \otimes S_k(s_1^\ell,(1))) + \mathrm{Ind}(S_k(s_1^\ell,(2)) \otimes S_k(s_1^\ell,(1)))] \\ &= [S_k(s_1^\ell,(1^3)) + 2S_k(s_1^\ell,(21)) + S_k(s_1^\ell,(3))]\end{aligned}$$

$$= \begin{cases} 2D_k(s_1^\ell,(1^3)) + 2D_k(s_1^\ell,(21)) + D_k(s_1^\ell,(3)) & \text{if } e=2; \\ 3D_k(s_1^\ell,(1^3)) + 3D_k(s_1^\ell,(21)) + D_k(s_1^\ell,(3)) & \text{if } e=3; \\ D_k(s_1^\ell,(1^3)) + 2D_k(s_1^\ell,(21)) + D_k(s_1^\ell,(3)) & \text{otherwise.} \end{cases}$$

(c) When exactly two of s_i^ℓ are equal. We can assume that either $s_1^\ell = s_2^\ell$ or $s_2^\ell = s_3^\ell$ by the order imposed in Lemma 1.4 on the s_i. It is possible to rearrange the $S_k(s_i,(1))$ without changing the composition factors. Thus by symmetry we just need to consider one of the two cases. Assume $s_1^\ell \equiv s_2^\ell$ in $\mathcal{C}$. Following the calculations in the last subcase we find:

$$[\mathrm{Ind}(S_k(s_1^\ell,(1)) \otimes S_k(s_1^\ell,(1)) \otimes S_k(s_3^\ell,(1)))]$$

$$= \begin{cases} 2\,\mathrm{Ind}((D_k(s_1^\ell,(1^2)) \otimes D_k(s_3^\ell,(1))) + \mathrm{Ind}(D_k(s_1^\ell,(2)) \otimes D_k(s_3^\ell,(1))) & \text{if } e=2; \\ \mathrm{Ind}((D_k(s_1^\ell,(1^2)) \otimes D_k(s_3^\ell,(1))) + \mathrm{Ind}(D_k(s_1^\ell,(2)) \otimes D_k(s_3^\ell,(1))) & \text{otherwise.} \end{cases}$$

(B-5) Applying Lemmas 1.1 and 1.4

$$[S_k(s,(1))] = D_k(s,(1))$$

$$= D_k(s^\ell, (1)).$$

There are two possibilities (as the degree of the extension generated by s^ℓ has to divide 3):

(a) s^ℓ is of degree 3 over $\mathbb{F}_q$. Then $D_k(s^\ell, (1))$ is in the classification. These correspond to the cuspidal ordinary representations of $\mathrm{GL}_2(F)$ which on reduction are supercuspidal.

(b) s^ℓ is of degree 1 over $\mathbb{F}_q$. By Lemma 1.6

$$\begin{aligned}[S_k(s, (1))] &= D_k(s^\ell, (1)) \\ &= D_k(s^\ell, (3)).\end{aligned}$$

Assume that $\ell \neq 2, 3$. Following the same arguments as we gave for $\mathrm{GL}_2(\mathbb{F}_q)$ we find that if $\ell \mid q+1$ or $\ell \mid q-1$ there are no cuspidal subquotients and if $\ell \mid q^2-q+1$ there are $\frac{(\ell^a-1)(q-1)}{2}$ cuspidal subquotients.

Remark. For $\mathrm{GL}_n(\mathbb{F}_q)$ with n prime, the only cuspidal non-supercuspidal representations that appear are quotients of special representations (generalised Steinberg representations) by their maximal submodules.

APPENDIX B

Normal subgroups and decomposition numbers

In this appendix using Brauer characters and the decomposition matrices of $\mathrm{SU}_3(E/F)$, [**Gec90**], we find the decomposition numbers of $\mathrm{U}_3(E/F)$.

1. Clifford theory for Brauer characters

Let G be a finite group of Lie type, H a subgroup of G and $\theta \in \mathrm{IBr}(H)$. Denote by $\mathrm{IBr}(G|\theta)$ the subset of $\mathrm{IBr}(G)$ of Brauer characters which on restriction to H contain θ.

Theorem 1.1 ([**Nav98**, Theorems 8.9, 8.12, Corollaries 8.7, 8.20]). *Let N be a normal subgroup of G.*

(1) Let $\theta \in \mathrm{IBr}(N)$ and $\varphi \in \mathrm{IBr}(G)$. Then φ is an irreducible constituent of θ^G if and only if θ is an irreducible constituent of φ_N. Furthermore suppose θ is an irreducible constituent of φ_N and let $\theta_1, \theta_2, ..., \theta_r$ be the distinct conjugates of θ in G. Then

$$\varphi_N = e\sum_{i=1}^{r}\theta_i.$$

(2) (Clifford correspondence) Let $\theta \in \mathrm{IBr}(N)$. The map

$$\begin{aligned}\mathrm{IBr}(N_G(\theta)|\theta) &\to \mathrm{IBr}(G|\theta)\\ \psi &\mapsto \psi^G\end{aligned}$$

is a bijection.

(3) Suppose G/N is cyclic and $N_G(\theta) = G$. Then there exists $\varphi \in \mathrm{IBr}(G)$ such that $\varphi_N = \theta$.

(4) Let $\eta \in \mathrm{IBr}(G)$ and suppose $\eta_N = \theta$ for some $\theta \in \mathrm{IBr}(N)$. Then the characters $\beta\eta$ for $\beta \in \mathrm{IBr}(G/N)$ are irreducible, pairwise distinct and are all the irreducible constituents of θ^G.

Suppose we can extend $\theta \in \mathrm{IBr}(N)$ to $\varphi_\theta \in \mathrm{IBr}(N_G(\theta))$. Then by Theorem 1.1 part (4)

$$\mathrm{IBr}(N_G(\theta)|\theta) = \{\beta\varphi_\theta : \beta \in \mathrm{IBr}(N_G(\theta)/N)\}$$

and by Theorem 1.1 part (2)

$$\mathrm{IBr}(G|\theta) = \{(\beta\varphi_\theta)^G : \beta \in \mathrm{IBr}(N_G(\theta)/N)\}.$$

Furthermore

$$\mathrm{IBr}(G) = \bigcup_{\theta\in\mathrm{IBr}(N)} \mathrm{IBr}(G|\theta).$$

However this union is not necessarily disjoint: by (1) we see that $\mathrm{IBr}(G|\theta_1) = \mathrm{IBr}(G|\theta_2)$ if and only if there exists $g \in G$ such that $\theta_1^g = \theta_2$. If there is no such g then the sets are disjoint. Define an equivalence relation on $\mathrm{IBr}(N)$ by $\theta_1 \sim \theta_2$ if and only if there exists $g \in G$ such that $\theta_1^g = \theta_2$, then

$$\mathrm{IBr}(G) = \dot{\bigcup}_{\theta\in\mathrm{IBr}(N)/\sim} \mathrm{IBr}(G|\theta)$$

a disjoint union.

1.1. Direct products and decomposition numbers. Given a direct product of groups $G \times H$, by [**Nav98**, Theorem 8.21],

$$\mathrm{IBr}(G \times H) = \{\theta \times \varphi : \theta \in \mathrm{IBr}(G),\ \varphi \in \mathrm{IBr}(H)\},$$

where $\theta \times \varphi(g) = \theta(g)\varphi(g)$. The trick is to show that $\theta \times \varphi$ is irreducible in each case; it is easy to see that they are pairwise distinct then the equality follows by counting.

Therefore if we know the decomposition of the ordinary characters of G and H then we can work out the decomposition of the irreducible characters of $G \times H$. Suppose $d^1(\theta) = \sum \alpha_i\overline{\theta}_i$

with $\overline{\theta_i} \in \mathrm{IBr}(G)$ and $d^1(\varphi) = \sum \beta_j \overline{\varphi_j}$ with $\overline{\varphi_j} \in \mathrm{IBr}(H)$. Then

$$d^1(\theta \times \varphi) = \sum_{i,j} \alpha_i \beta_j (\overline{\theta}_i \times \overline{\varphi}_j).$$

1.2. The conjugacy classes of two and three dimensional finite unitary groups. Let $G = \mathrm{U}_3(E/F)$ and $N = \mathrm{SU}_3(E/F)$. Due to Theorem 1.1 we are interested in the action of G by conjugation on the classes of N. The conjugacy classes of G are given in [**Enn63**, Page 29] and the conjugacy classes of N in [**Gec90**, Table 1.1].

If $d = 1$ the N-conjugacy classes are the intersection of the G-conjugacy classes with N, i.e. the N-conjugacy classes are the G-conjugacy classes of determinant one.

If $d = 3$ there are three N-conjugacy classes, denoted by $C_3^{(k,0)}$,$C_3^{(k,1)}$, and $C_3^{(k,2)}$ in [**Gec90**], which are G-conjugate. The other N-conjugacy classes remain fixed under the action of G by conjugation. The group index $[G : N] = q + 1$ and by transitivity

$$[G : N] = [G : \mathrm{Z}(G)N][\mathrm{Z}(G)N : N].$$

By the second isomorphism theorem for groups

$$[\mathrm{Z}(G)N : N] = [\mathrm{Z}(G) : \mathrm{Z}(N)] = \frac{q+1}{d}.$$

Thus

$$[G : \mathrm{Z}(G)N] = d.$$

Therefore, if $d = 3$ there are three distinct cosets in the space $G/\,\mathrm{Z}(G)N$ which must permute the conjugacy classes $C_3^{(k,0)}$,$C_3^{(k,1)}$ and $C_3^{(k,2)}$ because the conjugacy classes remain fixed by $\mathrm{Z}(G)N$ yet form a single conjugacy class in G

Let $G_2 = \mathrm{U}_2(E/F)$ and $N_2 = \mathrm{SU}_2(E/F)$. The conjugacy classes of G_2 are found in [**Enn63**, Page 26], We assume q is odd. When intersected with N_2 the conjugacy classes of G_2 of determinant 1 are all N_2-conjugacy classes apart from two classes which are denoted by $C_2^{(0)}$ and $C_2^{(\frac{q+1}{2})}$ in [**Enn63**]. These have representatives

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}.$$

Explicit calculations show these G_2-conjugacy classes both split into two N_2-conjugacy classes.

2. Brauer characters of $\mathrm{SU}_3(E/F)$

2.1. The characters of $\mathrm{SU}_3(E/F)$ and $\mathrm{U}_3(E/F)$. The ordinary character table for $\mathrm{U}_3(E/F)$ is given in [**Enn63**, Pages 30-31]. The ordinary character of a representation ρ is denoted by

$$\chi^{(i)}_{\dim(\rho)},$$

where (i) is some list of parameters. For example, let ζ be an irreducible representation of $F^\times$ then $\mathrm{St}_G(\zeta)$ has character $\chi^{(u)}_{q^3}$ for some $u = 1, .., q - 1$. Relating the parameter u to ζ depends on a choice of $q - 1$-th root of unity.

We use the notation from the ordinary character table for $\mathrm{SU}_3(E/F)$, [**Gec90**, Table 3.1]. We follow the description of the decomposition numbers of $\mathrm{SU}_3(E/F)$ in non-defining characteristic given in [**Gec90**, Theorems 4.1-4.5]. In [**Gec90**] there are two parameters α and β missing from the decomposition matrices, these are found in [**OW02**, Lemma 2.2]. The order of $\mathrm{SU}_3(E/F)$ is $q^3(q-1)(q+1)^2(q^2-q+1)$. In non-describing characteristic, since $\gcd(q+1,q-1) \mid 2$, $\gcd(q+1,q^2-q+1) \mid 3$ and $\gcd(q+1,q^2+q+1)=1$ there are five different cases to consider:

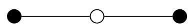
A-1 $\ell \neq 2$ and $\ell \mid q-1$,
A-2 $\ell \neq 2,3$ and $\ell \mid q+1$,
A-3 $\ell \neq 2,3$ and $\ell \mid q^2-q+1$,
A-4 $\ell = 3$ and $\ell \mid q+1$,
A-5 $\ell = 2$.

2.2. Decomposition matrices of $\mathrm{SU}_3(E/F)$ **if** $\ell \neq 2$ **and** $\ell \mid q-1$**,** [**Gec90**, Theorem 4.1]**.** Let $\ell^a \mid\mid q-1$.

(1) **The principal ℓ-block:**

			Conditions	Number
χ_1	1	0		1
χ_{q^3}	0	1		1
$\chi_{q^3+1}^{(u)}$	1	1	$\frac{q-1}{\ell^a} \mid u$	ℓ^a-1

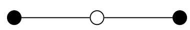
The non-exceptional characters are χ_1 and χ_{q^3} and we have the following Brauer tree:



(2) **ℓ-blocks $\mathbf{B}_1^{(u)}$:**

			Conditions	Number
$\chi_{q^2-q+1}^{(u)}$	1	0		1
$\chi_{q(q^2-q+1)}^{(u)}$	0	1		1
$\chi_{q^3+1}^{(v)}$	1	1	$\frac{q^2-1}{\ell^a} \mid v-(q-1)u$	ℓ^a-1

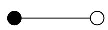
The non-exceptional characters are $\chi_{q^2-q+1}^{(u)}$ and $\chi_{q(q^2-q+1)}^{(u)}$ and we have the following Brauer tree:



(3) **ℓ-blocks $\mathbf{B}_2^{(u)}$:**

		Conditions	Number
$\chi_{q^3+1}^{(u)}$	1		1
$\chi_{q^3+1}^{(v)}$	1	$\frac{q^2-1}{\ell^a} \mid v-u$ or $v+uq$	ℓ^a-1

The non-exceptional character is $\chi_{q^3+1}^{(u)}$ and we have the following Brauer tree:



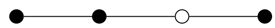
(4) **ℓ-blocks of defect zero.** All other Brauer characters are in ℓ-blocks of defect zero.

2.3. Decomposition matrices of $\mathrm{SU}_3(E/F)$ **if** $\ell \neq 2,3$ **and** $\ell \mid q^2-q+1$**,** [**Gec90**, Theorem 4.2]**.** Let $\ell^a \mid\mid q^2-q+1$.

(1) **The principal ℓ-block:**

				Conditions	Number
χ_1	1	0	0		1
χ_{q^3}	1	1	0		1
χ_{q^2-q}	0	0	1		1
$\chi^{(u)}_{(q+1)^2(q-1)}$	0	1	1	$\frac{q^2-q+1}{\ell^a} \mid u$	$\ell^a - 1$

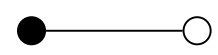
The non-exceptional characters are χ_1, χ_{q^3}, and χ_{q^2-q}, and we have the following Brauer tree:



(2) **ℓ-blocks $\mathbf{B}_1^{(u)}$:**

		Conditions	Number
$\chi^{(u)}_{(q+1)^2(q-1)}$	1		1
$\chi^{(u)}_{(q+1)^2(q-1)}$	1	$\frac{q^2-q+1}{\ell^a} \mid v-u$ or $v-uq^2$	$\ell^a - 1$

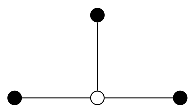
The non-exceptional character is $\chi^{(u)}_{(q+1)^2(q-1)}$ and we have the following Brauer tree:



(3) **ℓ-blocks $\mathbf{B}_2^{(v)}$, $v = 1, 2$:** If $d = 3$ we have the additional blocks:

				Conditions	Number
$\chi^{(0,v)}_{(q+1)^2(q-1)/3}$	1	0	0		1
$\chi^{(1,v)}_{(q+1)^2(q-1)/3}$	0	1	0		1
$\chi^{(2,v)}_{(q+1)^2(q-1)/3}$	0	0	1		1
$\chi^{(u)}_{(q+1)^2(q-1)}$	1	1	1	$\frac{q^2-q+1}{\ell^a} \mid u - \frac{q^2-q+1}{3}v$	$\ell^a - 1$

The non-exceptional characters are $\chi^{(0,v)}_{(q+1)^2(q-1)/3}$, $\chi^{(1,v)}_{(q+1)^2(q-1)/3}$ and $\chi^{(2,v)}_{(q+1)^2(q-1)/3}$. We have the following Brauer tree:



(4) **ℓ-blocks of defect zero.** All other Brauer characters are in ℓ-blocks of defect zero.

2.4. Decomposition matrices of $\mathrm{SU}_3(E/F)$ if $\ell \neq 2, 3$ and $\ell \mid q+1$, [**Gec90**, Theorem 4.3]**.** Let $\ell^a \mid\mid q+1$.

(1) **The principal ℓ-block:**

				Conditions	Number
χ_1	1	0	0		1
χ_{q^2-q}	0	1	0		1
χ_{q^3}	1	2	1		1
$\chi^{(u)}_{q^2-q+1}$	1	1	0	$\frac{q+1}{\ell^a} \mid u$	$\ell^a - 1$
$\chi^{(u)}_{q(q^2-q+1)}$	1	1	1	$\frac{q+1}{\ell^a} \mid u$	$\ell^a - 1$
$\chi^{(v,w)}_{(q-1)(q^2-q+1)}$	0	0	1	$\frac{q+1}{\ell^a} \mid v, w$	$\frac{1}{6}(\ell^a-1)(\ell^a-2)$

(2) **ℓ-blocks $\mathbf{B}_1^{(u)}$:**

		Conditions	Number
$\chi^{(u)}_{q^3+1}$	1		1
$\chi^{(v)}_{q^3+1}$	1	$\frac{q^2-1}{\ell^a} \mid v-u$ or $v+uq$	ℓ^a-1

The non-exceptional character is $\chi^{(u)}_{q^3+1}$ and we have the following Brauer tree:

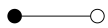
(3) **ℓ-blocks $\mathbf{B}_2^{(u)}$:**

			Conditions	Number
$\chi^{(u)}_{q^2-q+1}$	1	0		1
$\chi^{(u)}_{q(q^2-q+1)}$	1	1		1
$\chi^{(v)}_{q^2-q+1}$	1	0	$\frac{q+1}{\ell^a} \mid v-u$	ℓ^a-1
$\chi^{(v)}_{q(q^2-q+1)}$	1	1	$\frac{q+1}{\ell^a} \mid v-u$	ℓ^a-1
$\chi^{(v,w)}_{(q-1)(q^2-q+1)}$	0	1	$\frac{q+1}{\ell^a} \mid w, v-u$	$\frac{1}{2}\ell^a(\ell^a-1)$

(4) **ℓ-blocks $\mathbf{B}_3^{(u,v)}$, $\frac{q+1}{\ell^a} \nmid v$:**

		Conditions	Number
$\chi^{(u,v)}_{(q-1)(q^2-q+1)}$	1		1
$\chi^{(x,w)}_{(q-1)(q^2-q+1)}$	1	$\frac{q+1}{\ell^a} \mid x-u, w-v$	$\ell^{2a}-1$

The non-exceptional character is $\chi^{(u,v)}_{(q-1)(q^2-q+1)}$ and we have the following Brauer tree:



(5) **ℓ-blocks $\mathbf{B}_3$:** If $d=3$ we have in addition an ℓ-block:

				Conditions	Number
$\chi^{0}_{(q-1)(q^2-q+1)/3}$	1	0	0		1
$\chi^{1}_{(q-1)(q^2-q+1)/3}$	0	1	0		1
$\chi^{2}_{(q-1)(q^2-q+1)/3}$	0	0	1		1
$\chi^{(v,w)}_{(q-1)(q^2-q+1)}$	1	1	1	$\frac{q+1}{3\ell^a} \mid v,w$ $\frac{q+1}{\ell^a} \mid v+w,\ \frac{q+1}{\ell^a} \nmid v$	$\frac{\ell^{2a}-1}{3}$

(6) **ℓ-blocks of defect zero.** All other Brauer characters are in ℓ-blocks of defect zero.

2.5. Decomposition matrix of the principal ℓ-block of $\mathrm{SU}_3(E/F)$ if $\ell=3$ and $\ell \mid q+1$, [**Gec90**, Theorem 4.5]. **The principal ℓ-block:**

						Conditions	Number
χ_1	1	0	0	0	0		1
χ_{q^2-q}	0	1	0	0	0		1
$\chi^{(0)}_{(q-1)(q^2-q+1)/3}$	0	0	1	0	0		1
$\chi^{(1)}_{(q-1)(q^2-q+1)/3}$	0	0	0	1	0		1
$\chi^{(2)}_{(q-1)(q^2-q+1)/3}$	0	0	0	0	1		1
χ_{q^3}	1	2	1	1	1		1
$\chi^{(u)}_{q^2-q+1}$	1	1	0	0	0	$\frac{q+1}{3^a} \mid u$	3^a-1
$\chi^{(u)}_{q(q^2-q+1)}$	1	1	1	1	1	$\frac{q+1}{3^a} \mid u$	3^a-1
$\chi^{(v,w)}_{(q-1)(q^2-q+1)}$	0	0	1	1	1	$\frac{q+1}{3^a} \mid v, w$	$\frac{3^a(3^{a-1}-1)}{2}$
$\chi^{(0,v)}_{(q+1)^2(q-1)/3}$	0	1	1	0	0	$v=1,2$	2
$\chi^{(1,v)}_{(q+1)^2(q-1)/3}$	0	1	0	1	0	$v=1,2$	2
$\chi^{(2,v)}_{(q+1)^2(q-1)/3}$	0	1	0	0	1	$v=1,2$	2

3. Brauer characters of $\mathrm{U}_3(E/F)$

In this section we find the decomposition matrices of $\mathrm{U}_3(E/F)$. We use Clifford theory for Brauer characters from $\mathrm{IBr}(\mathrm{SU}_3(E/F))$ to $\mathrm{IBr}(\mathrm{U}_3(E/F))$ and the decomposition matrices of $\mathrm{SU}_3(E/F)$. There are two cases, depending on whether 3 is prime to $q+1$ or not.

3.1. The Brauer Characters of $\mathrm{U}_3(E/F)$ when $d=1$, $\ell \neq 2,3,p$. Let $N := \mathrm{SU}_3(E/F)$ and $G := \mathrm{U}_3(E/F)$. Because $d=1$ we have a direct product decomposition

$$G \simeq N \times Z(G).$$

Thus the Brauer characters of G and decomposition matrices follow from Section 1.1.

3.2. Brauer characters of $\mathrm{U}(3)$ when $d=3$, $\ell \neq 2,3,p$.
In the first lemma we deal with certain complex characters whose normaliser is G and thus extend to G by Theorem 1.1 part (3).

Lemma 3.1. Let $\psi \in \mathrm{Irr}(N)$ such that $N_G(\psi) = G$ and $N_G(\varphi) = G$ for all irreducible constituents φ of $d^1(\psi)$. Suppose $d^1(\varphi) = \sum_{i=1}^n \varphi_i$ with φ_i and φ_j distinct when $i \neq j$. Let $\widetilde{\psi}$ be any extension of ψ to G. Then $d^1(\widetilde{\psi}) = \sum_{i=1}^n \widetilde{\varphi}_i$ where $\widetilde{\varphi}_i$ is an extension of φ_i to G.

Proof: Suppose $d^1(\widetilde{\psi}) = \sum_{i=1}^n e_j \eta_j$.

$$\begin{array}{ccccc} G & & \widetilde{\psi} & \xrightarrow{d^1} & d^1(\widetilde{\psi}) = \sum_{j=1}^m e_j \eta_j \\ \Big| & & \Big| & & \Big| \\ N & & \psi & \xrightarrow{d^1} & d^1(\psi) = \sum_{i=1}^n \varphi_i \end{array}$$

Restriction to the ℓ-regular elements commutes with restriction from G to N hence

$$\mathrm{Res}^G_N \left(d^1(\widetilde{\psi}) \right) = d^1(\psi).$$

By Theorem 1.1 part (3) for each i there exists an extension of φ_i to an irreducible Brauer character κ_i of G. Let $\theta \in \mathrm{IBr}(G)$, by Theorem 1.1 part (1), φ_i is an irreducible constituent of θ_N if and only if θ is an irreducible constituent of φ^G. The set $\{\beta\kappa_i : \beta \in \mathrm{IBr}(G/N)\}$ consists of all irreducible constituents of φ^G and these are pairwise distinct by Theorem 1.1 part (4). Hence the η_j must all be extensions of φ_i. Because the decomposition numbers are all ones we must have $m = n$ and $e_j = 1$ for all j. □

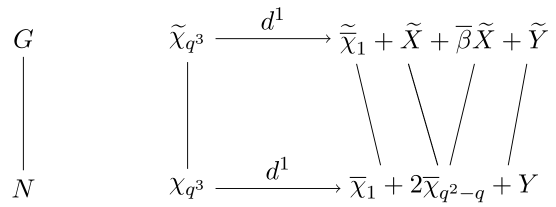
Comparing with the decomposition matrices of N in Section 2, Lemma 3.1 deals with extending all complex characters $\psi \in \mathrm{Irr}(G)$ whose normaliser is G and such that all irreducible constituents in $d^1(\psi)$ have normaliser G except one: the Steinberg character ψ_{q^3} when $\ell \mid q+1$

Lemma 3.2. Suppose $\ell \mid q+1$ and let $\widetilde{\psi}_{q^3}$ be any extension of ψ_{q^3} to G. Then

$$d^1(\widetilde{\psi}_{q^3}) = \widetilde{\chi}_1 + 2\widetilde{\chi}_{q^2-q} + \widetilde{Y}$$

where $\widetilde{\chi}_1$ is an extension of χ_1 to G, $\widetilde{\chi}_{q^2-q}$ is an extension of χ_{q^2-q} to G and $\widetilde{Y}$ is an extension of the irreducible Brauer character $d^1(\chi_{q^3}) - d^1(\chi_1) - 2d^1(\chi_{q^2-q})$ to G.

Proof: Let $\overline{\chi}_1 = d^1(\chi_1)$ and $\overline{\chi}_{q^2-q} = d^1(\chi_{q^2-q})$ which are irreducible Brauer characters of N. As in the proof of Lemma 3.1 we find $d^1(\widetilde{\psi}_{q^3}) = \widetilde{\overline{\chi}}_1 + \widetilde{X} + \overline{\beta}\widetilde{X} + \widetilde{Y}$ where $\widetilde{\overline{\chi}}_1$ extends $\overline{\chi}_1$ to G, $\widetilde{Y}$ extends Y to G, and $\widetilde{X}$ and $\overline{\beta}\widetilde{X}$ are two extensions of $\overline{\chi}_{q^2-q}$ to G which, by Theorem 1.1 part (4), are related by twisting by $\overline{\beta} \in \mathrm{IBr}(G/N)$. Claim: these two extensions coincide, i.e. β is trivial.



We can translate this into a problem using the notation of Chapter 2 Section 8: because d_ℓ commutes with Deligne-Lusztig induction, Chapter 3 Lemma 1.3, there is a cuspidal representation $\sigma_{T_1,\theta}$, by Chapter 2 Section 8, with the same decomposition modulo-ℓ as the virtual representation $\mathrm{St}_G(\chi) - 2\nu_\chi - 1_G(\chi)$. Thus the decomposition modulo-ℓ of the virtual representation $\mathrm{St}_G(\chi) - 2\nu_\chi - 1_G(\chi)$ is an ℓ-modular representation. Hence the decomposition modulo-ℓ of $\mathrm{St}_G(\chi)$ must contain the decomposition modulo-ℓ of ν_χ with multiplicity 2. □

Let $\psi \in \mathrm{Irr}(G)$ such that $N_G(\psi) = G$ then

$$G \supset N_G(d^1(\psi)) \supset N_G(\psi);$$

thus $N_G(d^1(\psi)) = G$. While not every Brauer character is equal to the restriction of an ordinary character to the ℓ-regular elements; the decomposition matrices of N can all be rearranged to be lower unitriangular. Starting at the top of these decomposition matrices and working top to bottom and left to right, if the first n ordinary characters have normaliser G, then so does the n-th Brauer character.

Following this algorithm all but two ℓ-blocks of N contain only ordinary and Brauer characters with normalisers equal to G. Using Lemmas 3.1 and 3.2 we can find the decomposition matrices for these blocks. We use the lower unitriangular shape of the decomposition matrices to choose the extensions of the ordinary characters to ensure compatibility with decomposition modulo-ℓ. We start with the ordinary character of the first row ψ_1 and fix any extension $\widetilde{\psi}_1$ of ψ_1 to G then

$$\mathrm{Irr}(G|\psi_1) = \{\gamma\widetilde{\psi}_1 : \gamma \in \mathrm{Irr}(G/N)\}$$

and $d^1(\gamma_1\widetilde{\psi}_1) = d^1(\gamma_2\widetilde{\psi}_2)$ if and only if $d^1(\gamma_1) = d^1(\gamma_2)$. We then move onto the next row with ordinary character ψ_2 in the decomposition matrix of N.

(1) If $d^1(\psi_2)$ contains $d^1(\psi_1)$ we fix an extension $\widetilde{\psi}_2$ of ψ_2 such that $d^1(\widetilde{\psi}_2)$ contains $d^1(\widetilde{\psi}_1)$. This is not necessarily unique, but exist: $d^1(\widetilde{\psi}_2)$ contains some extension of $d^1(\psi_1)$ thus the twists of this extension by $\beta \in \mathrm{IBr}(G/N)$, and hence all extensions of $d^1(\psi_1)$, appear in $d^1(\widetilde{\psi}_2)$ as $\widetilde{\psi}_2$ runs over all extensions of ψ_2. Let $\theta \in \mathrm{Irr}(G|\psi_2)$ then $\theta = \gamma\widetilde{\psi}_2$ with $\gamma \in \mathrm{Irr}(G/N)$ and θ is in the same ℓ-block as $\widetilde{\psi}_1$ if and only if $d^1(\gamma) = 1$.

(2) If $d^1(\psi_2)$ does not contain $d^1(\psi_1)$ we choose an ordinary character ζ in the ℓ-block which contains both $d^1(\psi_2)$ and $d^1(\psi_1)$, this is possible by Section 2. Fix an extension $\widetilde{\psi}_2$ of ψ_2 such that, for some extension $\widetilde{\zeta}$ of ζ, $d^1(\widetilde{\zeta})$ contains both $\widetilde{\psi}_1$ and $\widetilde{\psi}_2$. Then, as in the last case, if $\theta \in \mathrm{Irr}(G|\psi_2)$ then $\theta = \gamma\widetilde{\psi}_2$ with $\gamma \in \mathrm{Irr}(G/N)$ and θ is in the same ℓ-block as $\widetilde{\psi}_1$ if and only if $d^1(\gamma) = 1$.

It is easy to extrapolate from this and produce the decomposition matrices of G.

The two remaining blocks of N, one when $\ell \mid q+1$ and one when $\ell \mid q^2 - q + 1$, both have the following structure:

	$d^1(\chi_1)$	$d^1(\chi_2)$	$d^1(\chi_3)$	Conditions	Number
χ_1	1	0	0		1
χ_2	0	1	0		1
χ_3	0	0	1		1
χ_{ex}^i	1	1	1	$1 \leqslant i \leqslant k$	k

Furthermore $N_G(\chi_1) = N_G(\chi_2) = N_G(\chi_3) = \mathrm{Z}(G)N$ and $N_G(\chi_{ex}) = G$. Because $\ell \neq 3$ the conjugacy classes $C_3^{(k,0)}$, $C_3^{(k,1)}$ and $C_3^{(k,2)}$ are ℓ-regular and as the restriction of the χ_i to the ℓ-regular elements is irreducible $N_G(d^1(\chi_1)) = N_G(d^1(\chi_2)) = N_G(d^1(\chi_3)) = \mathrm{Z}(G)N$.

Using Theorem 1.1, as explained in Section 1, we have $\mathrm{Irr}(G|\chi_1) = \mathrm{Irr}(G|\chi_2) = \mathrm{Irr}(G|\chi_3)$; $\mathrm{IBr}(G|d^1(\chi_1)) = \mathrm{IBr}(G|d^1(\chi_2)) = \mathrm{IBr}(G|d^1(\chi_3))$ and it remains to describe d^1 restricting the domain to $\mathrm{Irr}(G|\chi_1) \cup (\cup_{i=1}^k \mathrm{Irr}(G|\chi_{ex}^i))$.

Lemma 3.3. For all $\rho \in \mathrm{Irr}(G|\chi_1)$ the Brauer character $d^1(\rho)$ is irreducible and contained in $\mathrm{IBr}(G|d^1(\chi_1))$.

Proof: Fix an extension $\widetilde{\chi}_1$ of χ_1 to $\mathrm{Z}(G)N$ which exists by Theorem 1.1 part (3). Alternatively, as $\mathrm{Z}(G) \cap N$ is equal to $\mathrm{Z}(N)$ we can extend by choosing an extension of the central character of χ_1 to $\mathrm{Z}(G)$.

$$
\begin{array}{ccccc}
G & \qquad & \widetilde{\chi}_1^G & \xrightarrow{d^1} & d^1(\widetilde{\chi}_1^G) \\
\big| & & \big| & & \big| \\
\mathrm{Z}(G)N & & \widetilde{\chi}_1 & \xrightarrow{d^1} & d^1(\widetilde{\chi}_1) \\
\big| & & \big| & & \big| \\
N & & \chi_1 & \xrightarrow{d^1} & d^1(\chi_1)
\end{array}
$$

By the second isomorphism theorem for groups $\mathrm{Z}(G)N/N \simeq \mathrm{Z}(G)/\mathrm{Z}(N)$. Hence by Theorem 1.1 part (4)

$$\mathrm{Irr}\left(\mathrm{Z}(G)N|\chi_1\right) = \left\{\gamma\widetilde{\chi}_1 : \gamma \in \mathrm{Irr}(\mathrm{Z}(G)/\mathrm{Z}(N))\right\}.$$

The Brauer character $d^1(\widetilde{\chi}_1)$ extends $d^1(\chi_1)$ to $\mathrm{Z}(G)N$ and by Theorem 1.1 (4)

$$\mathrm{IBr}\left(\mathrm{Z}(G)N|d^1(\chi_1)\right) = \left\{\beta d^1(\widetilde{\chi}_1) : \beta \in \mathrm{IBr}(\mathrm{Z}(G)/\mathrm{Z}(N))\right\}.$$

Furthermore $d^1(\gamma\widetilde{\chi}_1) = d^1(\gamma)d^1(\widetilde{\chi}_1)$. By Theorem 1.1 part (2),

$$\mathrm{Irr}\left(G|\chi_1\right) = \left\{\left(\gamma\widetilde{\chi}_1\right)^G : \gamma \in \mathrm{Irr}(\mathrm{Z}(G)/\mathrm{Z}(N))\right\};$$
$$\mathrm{IBr}\left(G|d^1(\chi_1)\right) = \left\{\left(\beta d^1(\widetilde{\chi}_1)\right)^G : \beta \in \mathrm{IBr}(\mathrm{Z}(G)/\mathrm{Z}(N))\right\}.$$

Because induction commutes with d^1

$$d^1\left(\left(\gamma\widetilde{\chi}_1\right)^G\right) = \left(d^1(\gamma)d^1(\widetilde{\chi}_1)\right)^G \in \mathrm{IBr}(G|d^1(\chi_1)).$$

□

The last sets of ordinary characters to consider are $\mathrm{Irr}(G|\chi_{ex}^i)$ and we let $\mathrm{Irr}(G|\chi_{ex})$ denote any one of these sets.

LEMMA 3.4. For all $\rho \in \mathrm{Irr}(G|\chi_{ex})$ the Brauer character $d^1(\rho)$ is irreducible and contained in $\mathrm{IBr}(G|d^1(\chi_1))$.

PROOF: Let $\widetilde{\chi}_{ex}$ be an extension of χ_{ex} to G which exists by Theorem 1.1 part (3). By Theorem 1.1 part (4)

$$\mathrm{Irr}\left(G|\chi_{ex}\right) = \left\{\gamma\widetilde{\chi}_{ex} : \gamma \in \mathrm{Irr}(G/N)\right\}.$$

The restriction of $d^1(\widetilde{\chi}_{ex})$ to N is $d^1(\chi_{ex})$.

$$
\begin{array}{ccccc}
G & \qquad & \widetilde{\chi}_{ex} & \xrightarrow{d^1} & d^1(\widetilde{\chi}_{ex}) \\
\big| & & \big| & & \big| \\
N & & \chi_{ex} & \xrightarrow{d^1} & d^1(\chi_{ex}) = \sum_{i=1}^3 d^1(\chi_i)
\end{array}
$$

Let $\zeta \in \mathrm{IBr}(G)$ be an irreducible constituent of $d^1(\widetilde{\chi}_{ex})$ By Theorem 1.1 part (1):

$$\mathrm{Res}_N^G(\zeta) = e\sum_{i=1}^{3} d^1(\chi_i).$$

Therefore $e = 1$ and $\zeta = d^1(\widetilde{\chi}_{ex})$. Thus $d^1(\gamma\widetilde{\chi}_{ex}) = d^1(\gamma)d^1(\widetilde{\chi}_{ex})$ is irreducible as a Brauer character and is contained in $\mathrm{IBr}(G|d^1(\chi_1))$. □

Using Lemmas 3.3 and 3.4 we can find the decomposition matrices of the remaining ℓ-blocks of G. If $\theta_1, \theta_2 \in \mathrm{Irr}(G|\chi_{ex})$ then $\theta_1 = \gamma_1\widetilde{\chi}_{ex}$, $\theta_2 = \gamma_2\widetilde{\chi}_{ex}$ with $\gamma_1, \gamma_2 \in \mathrm{Irr}(G/N)$ and $d^1(\theta_1) = d^1(\theta_2)$ if and only if $d^1(\gamma_1) = d^1(\gamma_2)$.

Let $\theta_1, \theta_2 \in \mathrm{Irr}(G|\chi_1)$ then $\theta_1 = (\gamma_1\widetilde{\chi}_1)^G$, $\theta_2 = (\gamma_2\widetilde{\chi}_1)^G$ with $\gamma_1, \gamma_2 \in \mathrm{Irr}(Z(G)N/N)$ and $d^1(\theta_1) = d^1(\theta_2)$ if and only if $d^1(\gamma_1) = d^1(\gamma_2)$ because d^1 commutes with induction.

Let $\theta_1 \in \mathrm{Irr}(G|\chi_1)$, $\theta_2 \in \mathrm{Irr}(G|\chi_{ex})$ then $\theta_1 = (\gamma_1\widetilde{\chi}_1)^G$ with $\gamma_1 \in \mathrm{Irr}(Z(G)N/N)$ and $\theta_2 = \gamma_2\widetilde{\chi}_{ex}$ with $\gamma_2 \in \mathrm{Irr}(G/N)$. Fix the extension $\widetilde{\chi}_{ex}$ so that $d^1(\widetilde{\chi}_{ex}) = d^1(\widetilde{\chi}_1)^G$. By restriction induction $\mathrm{Res}^G_{Z(G)N}(\widetilde{\chi}_{ex}) \simeq \widetilde{\chi}_1 \oplus \widetilde{\chi}_1^g \oplus \widetilde{\chi_1}^h$ where $\{1, g, h\}$ is a set of coset representatives of $G/Z(G)N$. By restriction to N, we see that $\widetilde{\chi}_1, \widetilde{\chi}_1^g, \widetilde{\chi_1}^h$ are distinct. Therefore $d^1(\theta_1) = d^1(\theta_2)$ if and only if $d^1(\gamma_1) = \mathrm{Res}^G_{Z(G)N}(d^1(\gamma_2))$.

Similar arguments apply to the sets $\mathrm{Irr}(G|\chi_2)$ and $\mathrm{Irr}(G|\chi_3)$. To relate $\theta_1 \in \mathrm{Irr}(G|\chi^1_{ex})$ and $\theta_2 \in \mathrm{Irr}(G|\chi^2_{ex})$ we can go via $\mathrm{Irr}(G|\chi_1)$.

APPENDIX C

The building of parahoric subgroups

In this appendix we briefly describe a model for the reduced building of $\mathrm{U}(2,1)(E/F)$.

1. The reduced building of $\mathrm{U}(2,1)(E/F)$

The geometry of parahoric subgroups of a p-adic reductive group can be described using the reduced building $\mathcal{B}(G)$ of G. We give a description of the reduced building of a unitary group in terms of lattice functions.

An $\mathcal{O}_E$-lattice function on V is a map

$$\Lambda : \mathbb{R} \to \mathrm{Lat}_{\mathcal{O}_E}(V)$$

such that

(1) Λ is decreasing and periodic,
(2) Λ is left continuous.

Denote the set of $\mathcal{O}_E$-lattice functions on V by $\mathrm{Latt}^1_{\mathcal{O}_E}(V)$. The image of a lattice function defines a lattice sequence. Define an equivalence relation on $\mathrm{Latt}^1_{\mathcal{O}_E}(V)$ by $\Lambda_1 \sim \Lambda_2$ if there exists $r \in \mathbb{R}$ such that for all $x \in \mathbb{R}$

$$\Lambda_1(x) = \Lambda_2(x + r).$$

The set of equivalence classes is denoted $\mathcal{B}(\mathrm{GL}(V))$ and called the reduced building of $\mathrm{GL}(V)$. We have a transitive action of $\mathrm{GL}(V)$ on $\mathrm{Latt}^1_{\mathcal{O}_E}(V)$ by

$$g \cdot \Lambda(x) = g(\Lambda(x)),$$

and this action stabilises equivalence classes, hence we get an action of $\mathrm{GL}(V)$ on $\mathcal{B}(\mathrm{GL}(V))$.

A decomposition of V

$$V = \bigoplus_{i=1}^{n} V_i,$$

splits $\Lambda \in \mathrm{Latt}^1_{\mathcal{O}_E}(V)$ if for all $x \in \mathbb{R}$

$$\Lambda(x) = \bigoplus_{i=1}^{n} V_i \cap \Lambda(x),$$

and splits $[\Lambda] \in \mathcal{B}(\mathrm{GL}(V))$ if it splits some representative of $[\Lambda]$.

Theorem 1.1. For any two lattice functions Λ_1 and Λ_2 in V, there exists a basis $(a_i)_{i=1}^n$ of V such that $\bigoplus_{i=1}^n E a_i$ splits Λ_1 and Λ_2.

The subset of $[\Lambda] \in \mathcal{B}(\mathrm{GL}(V))$ which are split by the decomposition of V given by a chosen basis $(v_i)_{i=1}^n$ is called an apartment $\mathcal{A}$ of $\mathcal{B}(\mathrm{GL}(V))$. The equivalence class $[\Lambda] \in \mathrm{Latt}_{\mathcal{O}_E}(V)$ is in the apartment given by a basis $(v_i)_{i=1}^n$ if and only if, there exists a representative Λ of $[\Lambda]$ and $(c_i)_{i=1}^n \in \mathbb{R}^n$ such that for all $r \in \mathbb{R}$

$$\Lambda(r) = \varpi^{\lceil r+c_1 \rceil}\mathcal{O}_E v_1 \oplus \varpi^{\lceil r+c_2 \rceil}\mathcal{O}_E v_1 \oplus \cdots \oplus \varpi^{\lceil r+c_n \rceil}\mathcal{O}_E v_n.$$

An equivalence class $[\Lambda]$ is called hyperspecial if the period of the image of a representative of $[\Lambda]$ as a lattice sequence is 1, i.e. $(c_i)_{i=1}^n \in \mathbb{Z}^n$. Thus the hyperspecial points can be indexed by the lattices in V.

Let $[\Lambda_i] \in \mathcal{B}(\mathrm{GL}(V))$, $i = 1, 2$ and Λ_i, $i = 1, 2$ be representatives of $[\Lambda_i]$. We define a metric d on $\mathcal{B}(\mathrm{GL}(V))$ via choosing a common splitting basis for Λ_i

$$\Lambda_1(r) = \varpi^{\lceil r+c_1 \rceil}\mathcal{O}_E v_1 \oplus \varpi^{\lceil r+c_2 \rceil}\mathcal{O}_E v_1 \oplus \cdots \oplus \varpi^{\lceil r+c_n \rceil}\mathcal{O}_E v_n,$$

$$\Lambda_2(r) = \varpi^{\lceil r+d_1 \rceil}\mathcal{O}_E v_1 \oplus \varpi^{\lceil r+d_2 \rceil}\mathcal{O}_E v_1 \oplus \cdots \oplus \varpi^{\lceil r+d_n \rceil}\mathcal{O}_E v_n,$$

then

$$d([\Lambda_1],[\Lambda_2]) = \min\{\max\{|c_j - d_j| : 1 \leqslant j \leqslant n\} : \Lambda_i \in [\Lambda_i]\}.$$

The apartment has a $(n-1)$-simplicial structure. The vertices are the hyperspecial points in $\mathcal{B}(\mathrm{GL}(V))$. A sequence of k hyperspecial points $[\Lambda_i]$, $i = 1, \ldots, k$ form a k-simplex if and only if there exist lattices L_i in the image of representatives Λ_i of $[\Lambda_i]$ such that

$$\varpi L_1 \subsetneq \cdots \subsetneq L_k \subsetneq L_1.$$

The maximal simplices are called chambers.

Let $\Lambda \in \mathrm{Latt}^1_{\mathcal{O}_E}(V)$, define

$$\Lambda(r+) = \bigcup_{s>r} \Lambda(s),$$

the dual lattice function $\Lambda^\sharp \in \mathcal{B}(\mathrm{GL}(V))$ of Λ is defined by

$$\Lambda^\sharp(r) = (\Lambda((-r)+))^\sharp$$

for all $r \in \mathbb{R}$. The lattice function Λ is called self dual if $\Lambda = \Lambda^\sharp$. The equivalence class $[\Lambda]$ is called self dual if $\Lambda \in [\Lambda]$ implies that $\Lambda^\sharp \in [\Lambda]$. If $[\Lambda]$ is self dual, there exists a unique self dual lattice function $\Lambda \in [\Lambda]$.

The involution attached to the hermitian form h acts on the (reduced) building of $\mathrm{GL}(V)$ by taking the equivalence class of lattice function Λ to the equivalence class of its dual lattice function $\Lambda^\sharp$. The building $\mathcal{B}(\mathrm{U}(V,h))$ of $\mathrm{U}(V,h)$ is the space of self dual lattice functions on V. Thus

$$\mathcal{B}(\mathrm{U}(V,h)) = \mathcal{B}(\mathrm{GL}(V))^h,$$

and we can embed $\mathcal{B}(\mathrm{U}(V,h))$ in the building of $\mathrm{GL}(V)$. The group $\mathrm{U}(V,h)$ acts on the building.

The subset of $[\Lambda] \in \mathcal{B}(\mathrm{U}(V,h))$ which are split by a chosen basis $(v_i)_{i=1}^n$ of V which is stable under h is called an apartment of $\mathcal{B}(\mathrm{U}(V,h))$. The apartment has a simplicial structure. An $\mathcal{O}_E$-lattice sequence L is called almost self dual if

$$\mathcal{P}_E L \subseteq L^\sharp \subseteq L.$$

The equivalence class of Λ is a vertex if and only if Λ is constant on the interval $[0, \frac{1}{2})$. If $[\Lambda]$ is a vertex then

$$\Lambda(x) = \begin{cases} L^\sharp & \text{if } x \in [0, \frac{1}{2}); \\ L & \text{if } x \in [\frac{1}{2}, 1). \end{cases}$$

Hence a vertex corresponds to an almost self dual lattice sequence. A sequence of r vertices $[\Lambda_i]$, $i = 1, \ldots, k$ form an $r-1$-simplex if and only if there exist lattices lattices L_i in the image of representatives Λ_i of $[\Lambda_i]$ such that

$$\mathcal{P}_E L_r^\sharp \subseteq L_r \subsetneq L_{r-1} \subsetneq L_{r-2} \subsetneq \cdots \subsetneq L_0 \subseteq L_0^\sharp \subsetneq \cdots \subsetneq L_{r-1}^\sharp \subsetneq L_r^\sharp.$$

The maximal simplices are called chambers. Then h induces a nondegenerate form $\overline{h}$ on

$$L_r^\sharp/L_{r-1}^\sharp \oplus L_{r-1}^\sharp/L_{r-2}^\sharp \oplus \cdots \oplus L_0^\sharp/L_0 \oplus L_0/L_1 \oplus \cdots \oplus L_r/\mathcal{P}_E L_r^\sharp$$

via $\overline{h} = \overline{h}_r \oplus \overline{h}_{r-1} \oplus \cdots \oplus \overline{h}_0 \oplus \overline{h}_{r+1}$. Where if $1 \leqslant i \leqslant r$

$$\overline{h}_i : L_{i-1}/L_i \times L_i^\sharp/L_{i-1}^\sharp \to k_E,$$

$$\overline{h}_i : (x + L_i, y + L^\sharp_{i-1}) \mapsto h(x, y) + \mathcal{P}_E,$$

and

$$\overline{h}_0 : L^\sharp_0/L_0 \times L^\sharp_0/L_0 \to k_E,$$
$$\overline{h}_0 : (x + L_0, y + L_0) \mapsto h(x, y) + \mathcal{P}_E,$$

and

$$\overline{h}_{r+1} : L_r/\mathcal{P}_E L^\sharp_r \times L_r/\mathcal{P}_E L^\sharp_r \to k_E,$$
$$\overline{h}_{r+1} : (x + \mathcal{P}_E L^\sharp_r, y + \mathcal{P}_E L^\sharp_r) \mapsto \varpi_E^{-1} h(x, y) + \mathcal{P}_E.$$

A facet $\mathcal{F}$ is a simplex in the building, a vertex is a 0-simplex. The stabiliser in G of $\mathcal{F}$ under the action of G on $\mathcal{B}(G)$ is a compact open subgroup of G, we denote by $G^+_\mathcal{F}$. Letting x be the barycentre of $\mathcal{F}$ we have $G^+_x = G^+_\mathcal{F}$. Let $\mathcal{F}$ be an $r-1$-simplex, as above, which corresponds to a flag of lattices

$$\mathcal{P}_E L^\sharp_r \subseteq L_r \subsetneq L_{r-1} \subsetneq L_{r-2} \subsetneq \cdots \subsetneq L_0 \subseteq L^\sharp_0 \subsetneq \cdots \subsetneq L^\sharp_{r-1} \subsetneq L^\sharp_r.$$

We have a map

$$\pi : G^+_\mathcal{F} \to \prod_{i=0}^{r} \mathrm{Aut}_{k_E}(L_i/L_{i+1})$$

whose image consists of elements which preserve the form $\overline{h}$, i.e.

$$\mathrm{im}(\pi) = \prod_{i=0}^{r-1} \mathrm{Aut}_{k_E}(L_i/L_{i-1}) \times \mathrm{U}(L^\sharp_0/L_0) \times \mathrm{U}(L_r/\mathcal{P}_E L^\sharp_r).$$

The kernel of π is the pro-unipotent radical of $G^+_\mathcal{F}$ and is denoted $G^1_\mathcal{F}$. In general, the finite reductive group $M^+_\mathcal{F} = \mathrm{im}(\pi)$ need not be connected. A parahoric subgroup of G, associated to the facet $\mathcal{F}$, is the preimage $G_\mathcal{F}$ of the connected component $M_\mathcal{F}$ of $M^+_\mathcal{F}$ in $G^+_\mathcal{F}$. We then have a short exact sequence

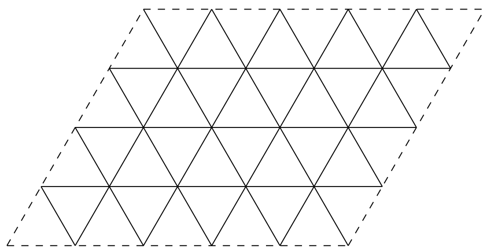
$$1 \to G^1_\mathcal{F} \to G_\mathcal{F} \to M_\mathcal{F} \to 1.$$

When the facet $\mathcal{F}$ is a chamber in the building, a parahoric subgroup is called an Iwahori subgroup. All Iwahori subgroups are conjugate in G. We fix a choice of Iwahori subgroup $\mathfrak{I}$, this determines a chamber C in the building $B(G)$. If x is any point in the closure $\mathcal{C}$ the Iwahori subgroup $\mathfrak{I}$ is equal to the inverse image of a Borel subgroup $B_{x,\mathcal{C}}$ of M_x in G_x. In fact, for any point $x \in \mathcal{B}(G)$, G^+_x is equal to $\mathcal{G}(\mathcal{O}_F)$ for some smooth affine $\mathcal{O}_F$-group scheme $\mathcal{G}$ whose generic fibre is G, [**Tit79**, 3.4.1].

A parahoric subgroup G_x corresponding to a point in the closure of the chamber $\mathcal{C}$ is called maximal if it is maximal under inclusion. There can be multiple non-conjugate maximal parahoric subgroups in G. Fix a maximal parahoric subgroup G_x. The non-maximal parahoric subgroups G_z contained in G_x and which correspond to a point z in the closure of C are equal to the preimage of parabolic subgroups $P_{z,\mathcal{C}}$ of M_x which contain $B_{x,\mathcal{C}}$ in G_x.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G_x^1 & \longrightarrow & G_x & \longrightarrow & M_x & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & G_x^1 & \longrightarrow & G_z & \longrightarrow & P_{z,\mathcal{C}} & \longrightarrow & 1 \\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & G_x^1 & \longrightarrow & \mathfrak{I} & \longrightarrow & B_{x,\mathcal{C}} & \longrightarrow & 1
\end{array}$$

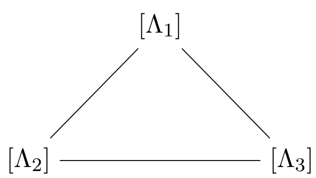
EXAMPLE 1.2. Let V be a three dimensional E-vector space. The (reduced) building $\mathcal{B}(\mathrm{GL}(V))$ is a union of apartments that are isometric to the plane. The simplicial structure on an apartment is a tesselation of the plane by equalateral triangles.



Choosing the standard basis $(e_i)_{i=1}^3$ for V, and identifying $\mathrm{GL}(V)$ with $\mathrm{GL}_3(E)$, let

$$\Lambda_1(r) = \varpi^{\lceil r \rceil}\mathcal{O}_E e_1 \oplus \varpi^{\lceil r \rceil}\mathcal{O}_E e_2 \oplus \varpi^{\lceil r \rceil}\mathcal{O}_E e_3,$$
$$\Lambda_2(r) = \varpi^{\lceil r \rceil}\mathcal{O}_E e_1 \oplus \varpi^{\lceil r \rceil}\mathcal{O}_E e_2 \oplus \varpi^{\lceil r+1 \rceil}\mathcal{O}_E e_3,$$
$$\Lambda_3(r) = \varpi^{\lceil r \rceil}\mathcal{O}_E e_1 \oplus \varpi^{\lceil r+1 \rceil}\mathcal{O}_E e_2 \oplus \varpi^{\lceil r+1 \rceil}\mathcal{O}_E e_3,$$

and let $[\Lambda_i]$, $i = 1, 2, 3$ be the equivalence class of Λ_i. The standard chamber of $B(\mathrm{GL}_3(E))$ is



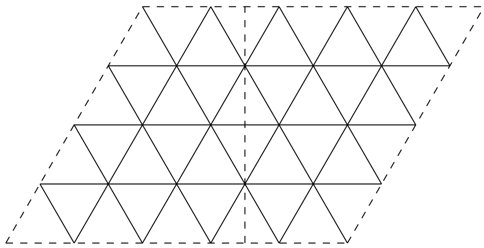
We can describe representatives of the equivalence classes of other lattice functions in this chamber in terms of linear combinations of Λ_i, $i = 1, 2, 3$. For example the midpoint of the line from $[\Lambda_2]$ to $[\Lambda_3]$ has representative

$$\Lambda_{23} = \varpi^{\lceil r \rceil}\mathcal{O}_E e_1 \oplus \varpi^{\lceil r+\frac{1}{2} \rceil}\mathcal{O}_E e_2 \oplus \varpi^{\lceil r+1 \rceil}\mathcal{O}_E e_3.$$

Let h be the form on V given by

$$(h((e_i, e_j)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

see Example 2.3. Then the duality acts as a reflection in the vertical line on the standard apartment of $\mathrm{GL}_3(E)$



The points $[\Lambda_1]$ and $[\Lambda_{23}]$ in $\mathcal{B}(\mathrm{GL}_3(E))$ are self dual, whereas the dual of $[\Lambda_2]$ is $[\Lambda_3]$ and vice-versa. Thus the standard chamber of $\mathcal{B}(\mathrm{U}(2,1)(E/F))$ is the line with vertices $[\Lambda_1]$ and $[\Lambda_{23}]$. The standard parahoric subgroups of $\mathrm{U}(2,1)(E)$ are contained in the pointwise stabilisers of the simplices in the chamber. We have two maximal standard parahoric subgroups: one contained in

$$\mathrm{Stab}_{\mathrm{U}(2,1)(E/F)}[\Lambda_1] = \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \end{pmatrix}^{\times} \cap \mathrm{U}(2,1)(E/F);$$

the other contained in

$$\mathrm{Stab}_{\mathrm{U}(2,1)(E/F)}[\Lambda_{23}] = \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \varpi^{-1}\mathcal{O}_E \\ \varpi\mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \varpi\mathcal{O}_E & \varpi\mathcal{O}_E & \mathcal{O}_E \end{pmatrix}^{\times} \cap \mathrm{U}(2,1)(E/F).$$

The stabilizer of the chamber is the standard Iwahori subgroup $\mathfrak{I}$ of $\mathrm{U}(2,1)(E/F)$

$$\mathfrak{I} = \begin{pmatrix} \mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \varpi\mathcal{O}_E & \mathcal{O}_E & \mathcal{O}_E \\ \varpi\mathcal{O}_E & \varpi\mathcal{O}_E & \mathcal{O}_E \end{pmatrix}^{\times} \cap \mathrm{U}(2,1)(E/F).$$

This is equal to the stabiliser of the central point in the chamber $[\Lambda_{123}]$ which is the equivalence class of

$$\Lambda_{123}(r) = \varpi^{\lceil r \rceil}\mathcal{O}_E e_1 \oplus \varpi^{\lceil r+\frac{1}{4} \rceil}\mathcal{O}_E e_2 \oplus \varpi^{\lceil r+\frac{1}{2} \rceil}\mathcal{O}_E e_3.$$

If E/F is unramified then the form $\overline{h}$ is hermitian and the parahoric subgroup associated to a point is equal to the pointwise stabiliser of that point. If E/F is ramified then the form $\overline{h}$ is orthogonal and the parahoric subgroup associated to a point has index 2 in the pointwise stabiliser of that point.

The building of $\mathrm{U}(2,1)(E/F)$ is a tree. If E/F is unramified then the vertices conjugate to $[\Lambda_1]$ have q^3+1 neighbours and the vertices conjugate to $[\Lambda_{23}]$ have $q+1$ neighbours. If E/F is ramified then the tree of $\mathrm{U}(2,1)(E/F)$ is $q+1$-regular.

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