Some observations and results concerning submeasures on Boolean algebras

A thesis submitted to the School of Mathematics of the University of East Anglia in partial fulfilment of the requirements for the degree of Doctor of Philosophy

Omar Selim

September 2012

Blank page.

For my father Fouad Abd-el-Aziz Selim, my mother Fatima Kassab and my sister Dina Selim, my singular good fortune.

Blank page.

Abstract

We investigate submeasures on Boolean algebras in the context of Maharam's problem and its solution. We generalise results that were originally proved for measures, to cases where additivity is not present. We investigate Talagrand's construction of a pathological exhaustive submeasure, attempting to give a more explicit description of this submeasure and we also consider some of its forcing properties. We consider the forcing consisting of submeasures that have as their domain a finite subalgebra of the countable atomless Boolean algebra. We find and investigate a linear association between the real vector space of all real-valued functionals on the countable atomless Boolean algebra, which includes the collection of all submeasures, and the space of all signed finitely additive measures on this Boolean algebra.

Contents

Acknowledgements

I would like to thank and acknowledge the help of my supervisor Mirna Džamonja with this dissertation and my studies in general. In fact, Mirna has been advising me from well before I arrived at UEA (a good three years before, by my calculations). It was also Mirna's suggestion that I read about Maharam submeasures and also to try to understand [33], and this is a topic that I have thoroughly enjoyed. I would like to thank Oren Kolman for his help. In particular, I would like to thank Oren for always finding the time to answer my questions, for the many interesting conversations we have had (both mathematical and otherwise) and for the many interesting seminars he gave to our logic group. I would like to thank Piotr Borodulin-Nadzieja, in particular for including me in his research, when it was very clear from the beginning that he was taking on a student rather than a collaborator. Because of this I learnt many new things, both mathematical and about mathematical research, all of which have had a very direct and positive impact on this dissertation. I wish to also thank Abeba Bahre, for providing me with employment during the writing up stage of this dissertation, and for always kindly accommodating every single one of my awkward "requests".

I of course must mention and thank the following people from UEA. Rob Kurinczuk, for all the absolutely absurd and ludicrous situations we found ourselves in during the last few year. Sharifa Al-Mahrouqi and Rima Al-Balushi, for teaching me about model theory, compactifications and Arabic. Ben Summers, who shared with me a particular interest in the finite and who never turned his nose up at a moderately good combinatorial lemma. Alan Tassin, for Keeping the Aspidistra Flying. Bander Al-Mutairi, for keeping me well nourished with his fabulous cooking. Alex Primavesi, who still owes me a solution to some problem about when some axiom fails, posed by some famous mathematician. I (of course) consider our weekly meetings at the Sainsbury centre an integral part of my training here at UEA. Nadir Matringe, for not looking for a mistake in this dissertation (yet). Rob Royals, for being my resident atheist and sceptic, and for keeping me in check, whenever I started babbling religious nonsense. Also Royals trained me to get sub 33 on int (30 3BV). Stuart Alder for `\emph{co}`-commissioning Book Club. Moritz Reinhard, with my sincerest apologies for never letting him work. Yousuff Lazar, for making sure *everyone* ate slowly. Francesco Piccoli and Carolina Macedo, for always allowing me to vent my frustrations, never with judgement and always with good advice. Alessia Freddo, for our regular coffee 'breaks'. Lydia Rickett, for showing the rest of us what hard graft really is. Matthias Krebs and Jeremie Guilhot, for making sure I stayed sharp on the golf [pitch and putt] course. Sawian Jaidee and Apisit Pakapongpun, for helping me settle in during my first year at UEA. David Maycock, for, well, just being Dave.

Finally, and not only to maximise the sentiment of this note (although this is an added bonus), I wish to also mention and thank the following people not from UEA. Everyone from Flat 52, and all my people from Chelsea - true pacemakers. Juncal Hernan, Sofiana Iliopoulou, George Kanellopoulos and George Neofytos and also Monika Rajca, for getting me through my first degree. My school mathematics teacher Mr Alex Capone, for not concentrating his efforts only on the so-called 'gifted and talented', a term which I still find deplorable today. If I may, just this once, use the word 'journey', well then, it is without a doubt that this particular journey of mine started with him.

Blank page.

1 Introduction

Given a non-empty set X, a collection $\mathcal{A}$ of subsets of X such that $X \in \mathcal{A}$ and $\mathcal{A}$ is closed under intersections and complementation (with respect to X) is called an *algebra of sets*.[1] A function $\mu : \mathcal{A} \to \mathbb{R}$ is called a *measure* if and only if $\mu(a) \geq 0$ and $a \cap b = \emptyset \to \mu(a \cup b) = \mu(a) + \mu(b)$, always. If $\mathcal{A}$ is closed under countable unions then $\mathcal{A}$ is called a *σ-algebra of sets*. If for every pairwise disjoint sequence $(a_i)_{i \in \mathbb{N}}$ from $\mathcal{A}$ we have $\mu(\bigcup_i a_i) = \sum_i \mu(a_i)$ then the measure μ is called *σ-additive*. A very familiar and central example is, of course, the Lebesgue measure on the collection of Borel sets of the unit interval $[0, 1]$. It is easy to check that every measure μ satisfies the following properties:

- $\mu(\emptyset) = 0$;
- $\mu(a) \leq \mu(b)$, if $a \subseteq b$;
- $\mu(a \cup b) \leq \mu(a) + \mu(b)$, always.

A function satisfying the above three properties is called a *submeasure*. If $\mathcal{A}$ is a σ-algebra then a submeasure μ is called *continuous* (or *Maharam*) if and only if for every sequence $a_1 \supseteq a_2 \supseteq \cdots$ from $\mathcal{A}$ with an empty intersection, we have $\inf_i \mu(a_i) = 0$. Every σ-additive measure is continuous.

These definitions extend in the natural way to the case when $\mathcal{A}$ is an arbitrary σ-complete Boolean algebra (that is not necessarily isomorphic to a σ-algebra of sets). A submeasure is called *strictly positive* if and only if $(\forall a)(a \neq \emptyset \to \mu(a) > 0)$. If we identify Borel subsets of $[0, 1]$ that differ by a set of measure 0 then the corresponding collection (of equivalence classes) defines a σ-complete Boolean algebra $\mathbb{M}$, the *random algebra*, on which the Lebesgue measure defines a strictly positive σ-additive measure.

According to [11, Page 880], the following problem was first posed by D. Maharam. It is also known as the *control measure problem*. It certainly appears in [28].

Problem A. (Maharam) *Does there exist a σ-complete Boolean algebra that carries a strictly positive continuous submeasure (a Maharam algebra) but does not carry a strictly positive σ-additive measure?*

Notice that it is not as difficult a task to find examples of strictly positive continuous submeasures that are not measures (see Example 4.4, below). Maharam's problem has many equivalent formulations (see [12]), but arguably the simplest and that carrying the "least structure" is as follows. Call an algebra of sets $\mathcal{A}$ *atomless* if and only if every non-empty member of $\mathcal{A}$ has a non-empty strict subset that is also a member of $\mathcal{A}$. There is (up to isomorphism) only one countably infinite atomless algebra of sets, which we denote by $\mathbb{A}$.

Problem B. *Does there exist a submeasure $\mu : \mathbb{A} \to \mathbb{R}$ such that:*

[1] We shall define everything properly in the next section.

- *For every sequence* $(a_i)_{i\in\mathbb{N}}$ *of pairwise disjoint members of* $\mathbb{A}$ *we have* $\lim_i \mu(a_i) = 0$*;*
- *There exists an* $\epsilon > 0$ *such that, for every* $n \in \mathbb{N}$*, there exists a sequence* $a_1, ..., a_n$ *of pairwise disjoint members of* $\mathbb{A}$ *such that* $\inf_i \mu(a_i) \geq \epsilon$*?*

Problem A has a positive solution if and only if Problem B does. Note that the transition from Problem A to Problem B (or rather from Problem B to Problem A) is not easy since it relies on the rather "deep" theorem of Kalton and Roberts from [19] (see Theorem 2.11, below).

A solution to Maharam's problem was announced in 2006 by M. Talagrand and was eventually published in [33]. In [33] a submeasure is constructed satisfying the properties listed in Problem B. Between the time of its formulation and the announcement of its solution, Maharam's problem obtained a considerable amount of attention and notoriety, and was generally considered an important problem of measure theory (for example, see [29]).

It is clear from the above two formulations that Maharam's problem is a very natural problem and, at the same time, a very simply stated problem. Moreover, the fact that a counter example has been provided means that there is now a theory of continuous submeasure open for investigation that is different from the classical theory of measures.

Despite the fact that Maharam's problem is no longer open, it is not clear how much more insight we have into the problem in light of its solution and, in particular, why it was so difficult. Talagrand's solution is a very intricate and difficult one, even after the dust has settled. Trying to master it poses a completely different set of problems altogether.

There are also many interesting questions surrounding Talagrand's solution. Most notably perhaps is the question of whether or not Talagrand's example of a complete Boolean algebra that does not carry a strictly positive σ-additive measure has a complete regular subalgebra that does. This is a very natural question to ask, but more than this, its solution is related to the well-known problem of Prikry (see [34]). Just as we can quotient the Borel sets of $[0, 1]$ by sets of Lebesgue measure 0, we can instead quotient by the collection of meagre subsets to obtain the so called *Cohen algebra* $\mathbb{C}$. Prikry's problem can then be stated as follows.

Problem. (Prikry) *Is it relatively consistent with ZFC that every ccc* σ*-complete Boolean algebra regularly embeds either* $\mathbb{M}$ *or* $\mathbb{C}$*?*

Recall that a Boolean algebra satisfies the *countable chain condition* (ccc) if and only if it contains no uncountable antichains. If it is the case that Talagrand's algebra does not contain a regular subalgebra carrying a strictly positive σ-additive measure then this algebra will be the first known example (in ZFC) of a ccc complete Boolean algebra that does not regularly embed $\mathbb{M}$ or $\mathbb{C}$, and of course Prikry's problem would have a negative solution.

In this dissertation we investigate submeasures on Boolean algebras in the context of Maharam's problem, its solution, and the many (still open) questions that have arisen as a result of this solution. Either directly or indirectly we have ultimately been motivated and influenced by these three themes. The techniques used in this dissertation are in the main combinatorial, although we do consider some forcing, but this is never more complicated than one step forcing (the point being that there is no iterated forcing).

We have organised this dissertation into sections. Each section will make use of the preliminaries (Section 2) but we have tried to make them as self contained as possible; each with their own definitions and motivating discussion. We have provided an index of symbols and definitions on page 96.

Let us now summarise the results of this dissertation. In Section 3 we generalise two results that were originally proved for measures to the case where additivity is not present. The first result (Theorem 3.2) states that under Todorcevic's Open Colouring Axiom, the Boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$ does not contain a Maharam algebra, as a subalgebra. The original result is from [5] which states that $\mathcal{P}(\omega)/\mathrm{Fin}$ does not contain $\mathbb{M}$ as a subalgebra. The second result of this section reads as as follows.

Theorem (3.6). *Let $\mathfrak{B}$ be a σ-complete Boolean algebra and $\mathfrak{A}$ a subalgebra carrying an exhaustive submeasure μ. Then there exists a continuous submeasure $\widehat{\mu}$ on $\sigma(\mathfrak{A})$ such that*

- $(\forall a \in \mathfrak{A})(\widehat{\mu}(a) \leq \mu(a))$.
- *If λ is another continuous submeasure on $\sigma(\mathfrak{A})$ such that $(\forall a \in \mathfrak{A})(\lambda(a) \leq \mu(a))$ then $(\forall a \in \sigma(\mathfrak{A}))(\lambda(a) \leq \widehat{\mu}(a))$.*
- *If μ is σ-subadditive then $(\forall a \in \mathfrak{A})(\mu(a) = \widehat{\mu}(a))$.*

Here $\sigma(\mathfrak{A})$ denotes the smallest σ-complete σ-regular subalgebra of $\mathfrak{B}$ generated by $\mathfrak{A}$. This is analogous to the corresponding classical result for measures from [37].

In Section 4 we show that if one forces with the collection of all normalised submeasures μ such that the domain of μ is a finite subalgebra of $\mathbb{A}$, then any generic for this forcing will define a submeasure that is not uniformly exhaustive but is exhaustive with respect to the antichains from the ground model (Theorem 4.1). We also discuss possible applications for this result.

In Section 5 we consider the forcing notion associated to the σ-ideal `path` of Borel sets that have ν-measure 0, where ν is Talagrand's pathological exhaustive submeasure. We show that the collection of random reals in any forcing extension due to this ideal is ν-null, once ν has been constructed in this extension. We also give a proof, following [24], that the collection of ground model reals will be ν-null and meagre in any such extension. We show, however, that the ideal `path` is analytic on G_δ, and therefore that this last result concerning the ground model reals actually follows from [8].

In Section 6 we attempt to give an explicit description of the values that Talagrand's submeasure ν takes. This is motivated by the fact that the values of the Lebesgue measure on 2^ω are easily calculable. However, we do not get particularly near to ν, but instead consider the very first pathological submeasure ψ constructed in [33]. We show for example that

$$\psi(\prod_{n\in\mathbb{N}}\{1,2,...,2^n\}) = 2^{\frac{2500}{216}}.$$

In the final section we find and investigate a linear map which sends each real-valued functional (and therefore each submeasure) g on $\mathbb{A}$, to a signed finitely additive measure $\mathfrak{f}(g)$ on $\mathbb{A}$ (Theorem 7.2). We define such a map explicitly and investigate the submeasure obtained as the preimage of the Lebesgue measure. We consider the corresponding forcing notion and show that it contains an antichain of length continuum. We show that the determining real added by this forcing cannot be a splitting real.

2 Preliminaries

In this section we present the background material needed for this dissertation. Our intention is not to give an introduction to these topics but only to consolidate the required information, provide references and also to establish notation. As a rule of thumb, we present only concepts that the author did not see as an undergraduate student. For example we do not define what a *topological space* is or what a *partial order* is, since these notions seem to be standard enough.

2.1 General notation and product spaces

We let $\mathbb{N} = \{1, 2, 3, ...\}$ and $\omega = \{0, 1, 2, 3, ...\}$. If $n \in \mathbb{N}$ then by $[n]$ we mean the set $\{1, 2, 3, ..., n\}$. If $n \in \omega$ then we will sometimes identify n with the collection $\{0, 1, 2, 3, ...n-1\}$ (by considering it as a von Neumann ordinal). In this case 0 is identified with $\emptyset$. The first infinite cardinal and the first uncountable cardinal are denoted by $\aleph_0$ and $\aleph_1$, respectively. The size of the continuum (the cardinality of $\mathbb{R}$) is denoted by $\mathfrak{c}$. The first uncountable ordinal is denoted by ω_1. Given two sets X and Y we let ${}^XY$ denote the collection of all Y valued functions with domain X. If $f \in {}^XY$, then we shall denote X by $\mathrm{dom}(f)$, and $\{f(x) : x \in X\}$ by $\mathrm{ran}(f)$. If $A \subseteq X$, then by $f[A]$ we shall mean the set $\{f(a) : a \in A\}$. The powerset of a set X is denoted by $\mathcal{P}(X)$. Given a set X we will write $[X]^{<\omega}$ to mean the collection of all finite subsets of X. The collection of countably infinite subsets of X will be denoted by $[X]^\omega$. The symbol **Id** will always represent an identity map. If s and t are sequences, then by $s^\frown t$ we shall mean the sequence that is formed by concatenating t to the right of s.

Given a sequence of non-empty sets $(X_i)_{i\in J}$ we will always equip

$$X := \prod_i X_i = \{f : \mathbb{N} \to \bigcup_i X_i : (\forall i)(f(i) \in X_i)\},$$

with the Tychanoff topology, where each X_i is equipped with the discrete topology. If $I \subseteq J$ and $s \in \prod_{i\in I} X_i$ then by $[s]$ we mean the collection

$$\{f \in X : (\forall i \in I)(s(i) = f(i))\}.$$

Sets of the form $[s]$, for $s \in \prod_{i\in I} X_i$ and I finite, form a base for the topology on X. These will be clopen (closed and open) and indeed the collection of clopen sets of X will be subsets of X that are a finite union of sets from this just described base. Thus X is 0-dimensional (has a base of clopen sets). We denote the collection of clopen sets of X by $\mathrm{Clopen}(X)$ and the collection of Borel sets of X by $\mathrm{Borel}(X)$. If the X_i are finite then the space X will be compact. This follows by Tychnoff's theorem, since each X_i is compact. To avoid Tychnoff's theorem see [13, Theorem 30]. It is straightforward to see that X is Hausdorff. In the case when each $X_i = \{0, 1\}$ and $I = \omega$, we will denote X by 2^ω.

Suppose that $J = \mathbb{N}$, then X is metrisable by

$$d(f,g) := 2^{-n}$$

where

$$n := \min\{k : f(k) \neq g(k)\}.$$

If for each i we have a group structure $(X_i, 0_i, +_i)$ then X may also be considered a group where addition is given by

$$(f +_X g)(i) = f(i) +_i g(i)$$

and the identity is given by

$$0_X(i) = 0_i.$$

2.2 Boolean algebras

Unless otherwise stated, everything in this subsection may be found in [12] or [22].[2] A **Boolean algebra** $\mathfrak{B} := (B, 0, 1, +, \cdot)$ is a commutative ring (with unity) such that multiplication is idempotent, that is, $(\forall b)(b^2 = b)$. We let $\mathfrak{B}^+$ be $\mathfrak{B} \setminus \{0\}$. There exists a natural partial order on $\mathfrak{B}^+$ defined by

$$a \leq_{\mathfrak{B}} b \leftrightarrow a \cdot b = a.$$

Notice that if $\mathfrak{B}$ is a Boolean algebra and $a \in \mathfrak{B}^+$ then the collection

$$\{b \in \mathfrak{B} : b \leq_{\mathfrak{B}} a\}$$

also forms a Boolean algebra with unit a. We denote this Boolean algebra by $\mathfrak{B}_a$.

A Boolean algebra $\mathfrak{B}$ is called **atomless** (or **non-atomic**) if

$$(\forall a \in \mathfrak{B}^+)(\exists b \in \mathfrak{B}^+)(b <_{\mathfrak{B}} a).$$

If $a \in \mathfrak{B}^+$ is such that $(\forall b \in \mathfrak{B}^+)(b \not< a)$ then a is called an **atom** of $\mathfrak{B}$. The collection of atoms of $\mathfrak{B}$ will be denoted by $\mathrm{atoms}(\mathfrak{B})$. Of course these definition apply to any partial order.

We can define a new binary operation on $\mathfrak{B}$ by

$$a \cup_{\mathfrak{B}} b = a + b + a \cdot b.$$

The use of the familiar '$\cup$' is not really an abuse of notation. Notice that if $\mathcal{A}$ is an algebra of subsets of a non-empty set X, then $(\mathcal{A}, \emptyset, X, \triangle, \cap)$ is a Boolean algebra and that $\cup_{\mathcal{A}} = \cup$.[3]

[2] The book [12] is available for free.

[3] For concreteness, recall that if X is a non-empty set then $\mathcal{A} \subseteq \mathcal{P}(X)$ is an **algebra of subsets of** X if and only if the following conditions hold:

- $\emptyset \in \mathcal{A}$;
- $a \in \mathcal{A} \to X \setminus a \in \mathcal{A}$;

Here $\triangle$ represents the symmetric difference of two sets

$$a \triangle b := a \setminus b \cup b \setminus a.$$

The converse of the above forms part of the well known *Stone representation theorem.*[4]

Theorem 2.1 (Marshall H. Stone). *Every Boolean algebra is (ring) isomorphic to an algebra of sets.*

In fact the algebra of sets in Stone's theorem will be the collection of clopen sets of a compact Hausdorff 0-dimensional topological space. This topological space is known as the **Stone space** of $\mathfrak{B}$ and it is unique up to homeomorphism.

The Stone space of a Boolean algebra has a simple enough description. Given a Boolean algebra $\mathfrak{B}$, a collection $u \subseteq \mathfrak{B}$ is called a **filter** if and only if the following conditions hold:

- $0 \notin u$;
- $(\forall a, b \in \mathfrak{B})(a \in u \wedge a \leq_{\mathfrak{B}} b \rightarrow b \in u)$;
- $(\forall a, b \in u)(a \cdot b \in u)$.

A filter u is called an **ultrafilter** if and only if it is maximal with respect to the above three conditions. In the case that $\mathfrak{B} = \mathcal{P}(\mathbb{N})$, an ultrafilter u is called **non-principal** if and only if it does not contain any finite sets. Let $\mathrm{Ult}(\mathfrak{B})$ denote the collection of all ultrafilters in $\mathfrak{B}$. On $\mathrm{Ult}(\mathfrak{B})$ we can define a topology in which the basic open sets are sets of the form

$$\{u \in \mathrm{Ult}(\mathfrak{B}) : a \in u\}, \tag{2.1}$$

for $a \in \mathfrak{B}$. This topological space is compact Hausdorff and 0-dimensional. The sets given by (2.1) are clopen and the map

$$F : a \mapsto \{u \in \mathrm{Ult}(\mathfrak{B}) : a \in u\}$$

defines an isomorphism between $\mathfrak{B}$ and the algebra of clopen subsets of $\mathrm{Ult}(\mathfrak{B})$.

Notice that for a Boolean algebra $\mathfrak{B}$ the operations $+$, $\cdot$ and $\cup_{\mathfrak{B}}$ correspond to $\triangle$, $\cap$ and $\cup$, respectively, in its Stone space. Notice also that

$$a \leq_{\mathfrak{B}} b \leftrightarrow F(a) \subseteq F(b).$$

As is usual, we shall often take advantage of this (notational) association and not distinguish between, for example, $+$ and $\triangle$. We will also drop the subscripts on the Boolean operations

- $a, b \in \mathcal{A} \rightarrow a \cap b \in \mathcal{A}$.

The algebra $\mathcal{A}$ is called a **σ-algebra** if for every sequence $(a_n)_{n \in \mathbb{N}}$ we have $\bigcup_n a_n \in \mathcal{A}$.

[4] The rest of Stone's theorem needs some category theory which does not concern us here.

since this should never cause any confusion.

Two members a and b of a Boolean algebra are **disjoint** if and only if $a \cdot b = 0$. A subset X of a Boolean algebra is called an **antichain** if and only if

$$(\forall a, b \in X)(a \neq b \to a \cdot b = 0).$$

If an antichain $X \subseteq \mathfrak{B}$ is maximal (as an antichain) then it is also called a **partition of** $\mathfrak{B}$. A Boolean algebra $\mathfrak{B}$ satisfies the **countable chain condition** (**ccc**) if and only if every antichain in $\mathfrak{B}$ is at most countable.

A subset $\mathcal{I} \subseteq \mathfrak{B}$ is called an **ideal** if and only if the following conditions hold:

- $1 \notin u$;
- $(\forall a, b \in \mathfrak{B})(a \in u \wedge a \geq b \to b \in u)$;
- $(\forall a, b \in u)(a \cup b \in u)$.

An ideal $\mathcal{I}$ on $\mathfrak{B}$ is called σ**-complete** if and only if for every countable $X \subseteq \mathcal{I}$ we can find $a \in \mathcal{I}$ such that

$$a = \sup X,$$

where the supremum here is taken with respect to $\leq$ on $\mathfrak{B}$.

Given an ideal $\mathcal{I}$ on a Boolean algebra $\mathfrak{B}$ we can construct a new Boolean algebra, the **quotient** of $\mathfrak{B}$ by $\mathcal{I}$, by identifying members of $\mathfrak{B}$ that differ by a member of $\mathcal{I}$. More precisely, we say that $a \sim b$ if and only if $a \triangle b \in \mathcal{I}$ and we form the collection of equivalence classes $\mathfrak{B}/\mathcal{I} = \{[b]_\sim : b \in \mathfrak{B}\}$. We define the operations of $+$ and $\cdot$ on $\mathfrak{B}/\mathcal{I}$ by $[a]_\sim + [b]_\sim = [a+b]_\sim$ and $[a]_\sim \cdot [b]_\sim = [a \cdot b]_\sim$. Under these operations $\mathfrak{B}$ becomes a Boolean algebra with constants $[0]_\sim$ and $[1]_\sim$.

A Boolean algebra $\mathfrak{B}$ is called σ**-complete** if and only if every countable subset $X \subseteq \mathfrak{B}$ has a least upper bound in $\mathfrak{B}$ (and therefore a greatest lower bound) with respect to $\leq$. A Boolean algebra is called **complete** if and only if *every* subset $X \subseteq \mathfrak{B}$ has a least upper bound in $\mathfrak{B}$. If $X \subseteq \mathfrak{B}$ and X has a least upper bound in $\mathfrak{B}$ then we shall denote this (unique) element by

$$\sum X.$$

Similarly we denote the greatest lower bound of a set (should it exist) by

$$\prod X.$$

If $\mathfrak{B}$ is a σ-complete ideal on a σ-complete Boolean algebra then $\mathfrak{B}/\mathcal{I}$ will be σ-complete also.

Fact 2.2. ([22, Lemma 10.2]) *If a σ-complete Boolean algebra is ccc then it is complete.*

Let us now discuss homomorphisms between Boolean algebras. Homomorphisms, epimorphisms, monomorphisms (or embeddings) and isomorphisms are defined as they are for rings (following [1], for example). A Boolean algebra $\mathfrak{A}$ is a **subalgebra** of $\mathfrak{B}$ if and only if the identity map on $\mathfrak{A}$ is a monomorphism from $\mathfrak{A}$ into $\mathfrak{B}$. If $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, then $\mathfrak{A}$ is called a **σ-regular subalgebra** of $\mathfrak{B}$ if and only if for every countable $X \subseteq \mathfrak{A}$ we have

$$\sum^{\mathfrak{A}} X \in \mathfrak{A} \to \sum^{\mathfrak{A}} X = \sum^{\mathfrak{B}} X.$$

We call $\mathfrak{A}$ a **regular subalgebra** of $\mathfrak{B}$ if and only if we can drop the restriction that X be countable in the definition of σ-regular. An embedding $f : \mathfrak{A} \to \mathfrak{B}$ is called **σ-regular embedding** if and only if $f[\mathfrak{A}]$ is a σ-regular subalgebra of $\mathfrak{B}$. Similarly we define a **regular embedding**.

Given a Boolean algebra $\mathfrak{B}$ and a subset $X \subseteq \mathfrak{B}$, we will denote by $\langle X \rangle$ the subalgebra of $\mathfrak{B}$ **generated** by X. More specifically we have

$$\langle X \rangle := \bigcap \{\mathfrak{A} : \mathfrak{A} \text{ is a subalgebra of } \mathfrak{B} \text{ and } X \subseteq \mathfrak{A}\}.$$

With regards to this last definition, we have the following.

Fact 2.3. ([22, Corollary 4.5]) *If $\mathfrak{B}$ is a Boolean algebra and $X \subseteq \mathfrak{B}$, then $|\langle X \rangle| = |X|$.*

If $\mathfrak{B}$ is σ-complete then we can consider

$$\sigma(X) := \bigcap \{\mathfrak{A} : \mathfrak{A} \text{ is a } \sigma\text{-complete } \sigma\text{-regular subalgebra of } \mathfrak{B} \text{ and } X \subseteq \mathfrak{A}\}.$$

We say that $\sigma(X)$ is **σ-generated** by X. There is of course the analogous definition without the 'σ', but since we will always be concerned with ccc Boolean algebras, these definitions will always coincide.

The following discussion follows [18]. A partial order $(P, \leq)$ is called **seperative** if and only if

$$(\forall a, b \in P)(a \not\leq b \to (\exists c \leq a)(\forall d \leq c)(d \not\leq b)). \tag{2.2}$$

Notice that every Boolean algebra gives rise to a separative partial order since we can always take $c := a \setminus b$ in (2.2). If $(P, \leq)$ is a separative partial order then there exists a complete Boolean algebra $\mathfrak{B}$ and an injective map $f : P \to \mathfrak{B}^+$ with the following properties:

- $(\forall a, b \in P)(a \leq b \leftrightarrow f(a) \leq f(b))$;
- $(\forall a \in \mathfrak{B}^+)(\exists b \in P)(f(b) \leq a)$.

The complete Boolean algebra here will be unique up to isomorphism, and is called the **completion of** P. It follows that if $(P, \leq)$ arrises from a Boolean algebra then $f(1) = 1$

and we can extend f so that $f(0) = 0$. Moreover, it will follow from the first item above that f is an embedding. The second item above says that the image of P under f is **dense** in $\mathfrak{B}$. Concisely then, we may say that every separative partial order embeds densely into a complete Boolean algebra. Given a partial order $(P, \leq)$, two of its members a and b are called **compatible** if and only if there exists $c \in P$ such that $c \leq a$ and $c \leq b$. Of course if P arises from a Boolean algebra then a and b are compatible if and only if they are not disjoint. In the case that a partial order $(P, \leq)$ is not separative we define a separative partial order $(Q, \preceq)$, called the **separative quotient of** P, and a map $g : P \to Q$ such that the following hold:

- $(\forall a, b \in P)(a \leq b \to g(a) \preceq g(b))$;
- $(\forall a, b \in P)(a$ and b are compatible in $P \leftrightarrow g(a)$ and $g(b)$ are compatible in $Q)$.

The partial order Q is defined as follows. Let $\sim$ be the equivalence relation on P defined by

$$a \sim b \leftrightarrow (\forall c \in P)(a \text{ is compatible with } c \leftrightarrow b \text{ is compatible with } c).$$

Then Q is the collection of equivalence classes of $\sim$ and $\preceq$ is defined by

$$[a] \preceq [b] \leftrightarrow (\forall c \leq a)(c \text{ and } b \text{ are compatible}).$$

Once again, for more details on separative partial orders and separative quotients see [18].

Recall that a subset of a topological space is **nowhere dense** if and only if the interior of its closure is empty. A set is **meagre** if and only if it is a countable union of nowhere dense sets. Two very central examples of Boolean algebras are as follows (the third, the *random algebra* is discussed in the next subsection).

Definition 2.4. *Fix once and for all the following Boolean algebras.*

- $\mathbb{A} = \mathrm{Clopen}(2^\omega)$ (**Cantor algebra**).
- $\mathbb{C} = \mathrm{Borel}(2^\omega)/\mathcal{I}$ *where* $\mathcal{I}$ *denotes the* σ*-ideal of meagre subsets of* 2^ω (**Cohen algebra**).

It is easy to check that the Cantor algebra is atomless and countable. The following is not so easy to check.

Fact 2.5 ([**22, Corollary 5.16**])**.** *The Cantor algebra is the unique countable atomless Boolean algebra.*

Of course one can replace the space 2^ω by $X := \prod_i X_i$ for finite and non-empty sets X_i and $\mathrm{Clopen}(X)$ will still be atomless and countable. The natural map from

$$\mathbb{A} \mapsto \mathbb{C} : a \mapsto [a]_{\mathcal{I}}$$

witnesses that the Cohen algebra has a countable dense subset, and therefore must be ccc. Since the ideal $\mathcal{I}$ is σ-complete and $\mathbb{C}$ is ccc, $\mathbb{C}$ must be complete (Fact 2.2). The same map witnesses that the Cohen algebra is the completion of the Cantor algebra. By uniqueness of such completions we obtain the following.

Fact 2.6. *The Cohen algebra is the unique complete atomless Boolean algebra with a countable dense subset.*

Finally let us recall here the concept of a *direct limit*. Everything is taken from [17, Pages 49-51]. Since we are only interested in Boolean algebras we present direct limits in terms of them only.

Definition 2.7. *A* **directed system** *is a triple $((I, \leq), (\mathfrak{B}_i)_{i\in I}, (f_{i,j})_{i,j\in I})$ where $(I, \leq)$ is a partial order, $(\mathfrak{B}_i)_{i\in I}$ is sequence of Boolean algebras and $(f_{i,j} : \mathfrak{B}_i \to \mathfrak{B}_j)_{i,j\in I}$ is a sequence of maps, such that the following hold:*

- *for every $i, j \in I$ there exists $k \geq i, j$;*
- *if $i = j$ then $f_{i,j} = \mathrm{Id}$;*
- *if $i \leq j$ then $f_{i,j}$ is a homomorphism;*
- *if $i \leq j \leq k$ then $f_{i,k} = f_{j,k} \circ f_{i,j}$.*

Given a directed system $((I, \leq), (\mathfrak{B}_i)_{i\in I}, (f_{i,j})_{i,j\in I})$ let $B = \bigcup_{i\in I}\{i\} \times \mathfrak{B}_i$ (the disjoint union of the $\mathfrak{B}_i$). For $(i, a), (j, b) \in B$ say that $(i, a) \sim (j, b)$ if and only if there exists $k \geq i, j$ such that $f_{i,k}(a) = f_{j,k}(b)$. Clearly $\sim$ is an equivalence relation. Let $g_i : \mathfrak{B}_i \to B/\sim$ be the map $a \mapsto [(i, a)]_\sim$ and call these the **limit maps**. Define a Boolean structure on $B/\sim$ as follows:

- $1 = g_i(1)$ and $0 = g_i(0)$, for some $i \in I$;
- $a + b = c$, if and only if, we can find some $i \in I$ and $a_i, b_i, c_i \in \mathfrak{B}_i$ such that $a_i \in a, b_i \in b, c_i \in c$ and $a_i + b_i = c_i$;
- $a \cdot b = c$, if and only if, we can find some $i \in I$ and $a_i, b_i, c_i \in \mathfrak{B}_i$ such that $a_i \in a, b_i \in b, c_i \in c$ and $a_i \cdot b_i = c_i$.

The structure $\mathfrak{B} := (B/\sim, +, \cdot, 1, 0)$ is called the **direct limit** of our directed system. The maps g_i will always be homomorphisms, and if $i \leq j$ then $g_i = g_j \circ f_{i,j}$. If the maps $f_{i,j}$ are injective (embeddings) then the g_i will be also. Moreover, if the $f_{i,j}$ are injective then the direct limit $\mathfrak{B}$ is unique up to isomorphism. More precisely, we have the following.

Fact 2.8. *Suppose that the $f_{i,j}$ are injective. If we have a Boolean algebra $\mathfrak{C}$ and homomorphisms $h_i : \mathfrak{B}_i \to \mathfrak{C}$ such that for each $i \leq j$ we have $h_i = h_j \circ f_{i,j}$ and $\mathfrak{C} = \bigcup_{i\in I} \mathrm{ran}(h_i)$, then there exists an isomorphism $F : \mathfrak{B} \to \mathfrak{C}$ such that $h_i = F \circ g_i$, always.*

2.3 Submeasures and Maharam's problem

Once again, unless otherwise stated everything in this subsection can be found in [12]. Recall from the introduction, the definition of a submeasure on a Boolean algebra.

Definition 2.9. *Given a Boolean algebra $\mathfrak{B}$, a function $\mu : \mathfrak{B} \to \mathbb{R}$ is called a* **submeasure** *if and only if the following hold:*

- $\mu(0) = 0$*;*
- $(\forall a, b \in \mathfrak{B})(a \leq b \to \mu(a) \leq \mu(b))$ *(monotonicity);*
- $(\forall a, b \in \mathfrak{B})(\mu(a \cup b) \leq \mu(a) + \mu(b))$ *(subadditivity).*

We let $\mathrm{Null}(\mu)$ *denote the collection of all $a \in \mathfrak{B}$ such that $\mu(a) = 0$. The submeasure μ is called* **strictly positive** *if*

$$(\forall a)(a > 0 \to \mu(a) > 0).$$

A submeasure μ is **normalised** *if $\mu(1) = 1$.*

A submeasure on a Boolean algebra $\mathfrak{B}$ is **diffuse** if and only if for every $\epsilon > 0$ we can find a partition $a_1, ..., a_n$ of $\mathfrak{B}$ such that for every i, $\mu(a_i) \leq \epsilon$.

If a submeasure μ on a Boolean algebra $\mathfrak{B}$ satisfies

- $(\forall a, b \in \mathfrak{B})(a \cap b = 0 \to \mu(a) + \mu(b))$ (additivity),

then μ is called a **finitely additive measure**. Both additivity and subadditivity have their 'σ' analogues. A submeasure μ on a Boolean algebra $\mathfrak{B}$ is **σ-subadditive** if it satisfies the following condition:

- $(\forall X \in [\mathfrak{B}]^\omega)(\sum X \in \mathfrak{B} \to \mu(\sum X) \leq \sum_{x \in X} \mu(x))$ (σ-subadditivity).

The submeasure μ is called **σ-additive** if it satisfies the following condition:

- $(\forall X \in [\mathfrak{B}]^\omega)([(X \text{ is an antichain}) \wedge (\sum X \in \mathfrak{B})] \to \mu(\sum X) = \sum_{x \in X} \mu(x))$ (σ-additivity).

We will call a σ-additive submeasure a **measure**.

A **functional** on a Boolean algebra $\mathfrak{B}$ is a function $\mu : \mathfrak{B} \to \mathbb{R}$ such that $\mu(0) = 0$. Notice that any non-negative valued (finitely) additive functional is a submeasure and any non-negative valued σ-additive functional is a σ-subadditive submeasure. An additive functional that can take negative values will be called a **signed measure**.[5]

Crucial to Maharam's problem (and this dissertation) are the following properties.

Definition 2.10. *Let $\mathfrak{B}$ be a Boolean algebra carrying a submeasure μ. The submeasure μ is called*

- **exhaustive**, *if for every antichain $\{a_0, a_1, ...\} \subseteq \mathfrak{B}$ we have $\lim_n \mu(a_n) = 0$.*
- **uniformly exhaustive**, *if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for every pairwise disjoint $a_1, ...a_N \in \mathfrak{B}$ we have $\min_n \mu(a_n) < \epsilon$.*

[5] In [12] measures take values in $\mathbb{R} \cup \{\infty\}$. All measures we consider here are real-valued.

- **pathological**, *if the only finitely additive measure dominated by* μ *is the constant* 0 *function. A submeasure* λ *on* $\mathfrak{B}$ **dominates** *a submeasure* μ *if and only if* $(\forall a \in \mathfrak{B})(\mu(a) \leq \lambda(a))$, *and we write* $\mu \leq \lambda$.

- **continuous** *if for every sequence* $a_1 \geq a_2 \geq \cdots$ *from* $\mathfrak{B}$ *such that* $\prod_i a_i = 0$ *we have* $\mu(\prod_i a_i) = 0$.

It is useful to know that continuity of a submeasure μ on a complete Boolean algebra $\mathfrak{B}$, as stated above, implies that if $(a_i)_{i\in\mathbb{N}}$ is a sequence from $\mathfrak{B}$ such that

$$\prod_i \sum_{j\geq i} a_j = \sum_i \prod_{j\geq i} a_j \tag{2.3}$$

then we have

$$\lim_i \mu(a_i) = \mu(\prod_i \sum_{j\geq i} a_j).$$

This implies, for example, that every continuous submeasure is exhaustive. The property (2.3) gives rise to the *sequential topology* on $\mathfrak{B}$ which we do not consider here, but is given a thorough treatment in [2].

Given two submeasures μ and λ on a Boolean algebra $\mathfrak{B}$ we say that λ is **absolutely continuous** with respect to μ if for every sequence $(a_i)_{i\in\mathbb{N}}$ from $\mathfrak{B}$ we have

$$\lim_i \mu(a_i) = 0 \rightarrow \lim_i \lambda(a_i) = 0.$$

We write $\lambda \ll \mu$. We say that λ and μ are **equivalent** if and only if $\lambda \ll \mu$ and $\mu \ll \lambda$.

Notice that every finitely additive measure is uniformly exhaustive. A converse to this is the well known result due to N. J. Kalton and J. W. Roberts.

Theorem 2.11. ([19]) *A submeasure is uniformly exhaustive if and only if it is is equivalent to a finitely additive measure.*

Proof. Omitted, but see [9, Theorem 7H] for a concise proof. □

It is straightforward to see that if a submeasure μ is absolutely continuous with respect to a uniformly exhaustive submeasure then μ is also uniformly exhaustive. Thus the hard part of the Kalton-Roberts theorem is in the 'only if' direction.

Definition 2.12. *An atomless Boolean algebra* $\mathfrak{B}$ *is called a* **Maharam algebra** *if and only if it is* σ*-complete and carries a strictly positive continuous submeasure. An atomless Boolean algebra* $\mathfrak{B}$ *is called a* **measure algebra** *if and only if it is* σ*-complete and carries a strictly positive measure.*

If the domain of a measure is σ-complete then that measure will be continuous. This gives the following.

Fact 2.13. *Every measure algebra is a Maharam algebra.*

If $\mathfrak{B}$ is a Boolean algebra carrying a strictly positive exhaustive submeasure μ, then for each $\epsilon > 0$, every antichain contained in

$$\{a \in \mathfrak{B} : \mu(a) > \epsilon\}$$

must be finite. Since we have

$$\mathfrak{B}^+ = \bigcup_{n \in \mathbb{N}} \{a \in \mathfrak{B} : \mu(a) > 1/n\},$$

we obtain the following.

Fact 2.14. *Every Maharam algebra is ccc.*

We can define the 'Lebesgue measure' on 2^ω by defining $\lambda : \mathrm{Clopen}(2^\omega) \to \mathbb{R}$ by $\lambda([s]) = 2^{-|s|}$. This may be extended uniquely to $\mathrm{Borel}(X)$ (see Proposition 2.24, below). Another central example of a Boolean algebra, along with $\mathbb{C}$ and $\mathbb{A}$, is the following **random algebra**.

Definition 2.15. *Let* $\mathbb{M}$ *be the* σ*-complete Boolean algebra* $\mathrm{Borel}(2^\omega)/\mathrm{Null}(\lambda)$*, where* λ *is the Lebesgue measure on* 2^ω*.*

Notice that the Lebesgue measure λ on 2^ω defines a strictly positive measure on $\mathbb{M}$ by

$$\lambda([a]_{\mathrm{Null}(\lambda)}) = \lambda(a),$$

in particular $\mathbb{M}$ is ccc and therefore complete. Just as with $\mathbb{A}$ and $\mathbb{C}$, the algebra $\mathbb{M}$ has its own uniqueness property.

Fact 2.16. ([27]) *The random algebra is the unique measure algebra that is* σ*-generated by a countable set.*

Let us now discuss three formulations of Maharam's problem (two of which we have mentioned in the introduction). This will serve two purposes. The first is that it will nicely define the context in which we here investigated submeasures. The second purpose is that it will allow us to elaborate more on the properties of Definition 2.10 and their interactions with each other.

Recall Problem A from our introduction.

Is every Maharam algebra a measure algebra?

Recall also Problem B.

Is every exhaustive submeasure on $\mathbb{A}$ *uniformly exhaustive?*

We add to this one more formulation.

Problem C. *Does every exhaustive submeasure fail to be pathological?*

A positive answer to one of these questions yields a positive answer to the other, as summarised by the following theorem.

Theorem 2.17. *The following statements are equivalent.*

(A) Every Maharam algebra is a measure algebra.

(B) Every exhaustive submeasure on $\mathbb{A}$ is uniformly exhaustive.

(C) An exhaustive submeasure cannot be pathological.

Talagrand's result states that these statements have a negative answer.

Theorem 2.18. ([33]) *There exists a strictly positive pathological exhaustive submeasure on $\mathbb{A}$ that is not uniformly exhaustive.*

We present Talagrand's solution in Subsection 2.5. But for now let us prove Theorem 2.17, starting with $(A) \to (B)$. For this we will need Lemma 2.21, below. First let us extract two claims from the proof of Lemma 2.21, because we will need them again.

Lemma 2.19. ([12, Page 600]) *Let $\mathfrak{B}$ be a σ-complete Boolean algebra carrying a submeasure μ, and let $\mathfrak{A}$ be a subalgebra of $\mathfrak{B}$ such that the restriction of μ to $\mathfrak{A}$ is exhaustive. Let $b_0 \geq b_1 \geq \cdots \in \mathfrak{B}$ be such that, for each n and $\epsilon > 0$, there exists $a \in \mathfrak{A}$ satisfying $\mu(b_n \triangle a) < \epsilon$. Then for each $\epsilon > 0$, there exists $N \in \omega$ such that*

$$(\forall m, n \geq N)(\mu(b_m \triangle b_n) < \epsilon).$$

Proof. Omitted. □

Theorem 2.20. ([30, Theorems 10.9.1 and 10.12.5]) *Let (X, d) and (Y, d) be two metric spaces with Y complete, and let $S \subseteq X$ be a dense subset. Then every uniformly continuous function $f : S \to Y$, extends uniquely to a uniformly continuous function on the entirety of X. If (X', d) and (X'', d) are completions of X, and if $f' : X \to X''$ and $f'' : X \to X''$ are isometries onto dense sets, then there exists an isometry $f : X' \to X''$, that maps $f'[X]$ onto $f''[X]$.*

Proof. Omitted. □

Lemma 2.21. ([12, Lemma 393B]) *Let $\mathfrak{A}$ be a Boolean algebra carrying a strictly positive exhaustive submeasure μ. Then the metric completion $\widehat{\mathfrak{A}}$ of $\mathfrak{A}$ with respect to $d(a, b) = \mu(a \triangle b)$ is a complete Boolean algebra and μ extends to a strictly positive continuous submeasure $\widehat{\mu}$ on $\widehat{\mathfrak{A}}$.*

Proof (Sketch). The Boolean operations $+$ and $\cdot$ on $\mathfrak{A}$ are absolutely continuous with respect to the metric d. This follows from the identities

$$(a + b) + (c + d) \subseteq (a + c) \cup (b + d) \text{ and } (a \cdot b) + (c \cdot d) \subseteq (a + c) \cup (b + d),$$

for each $a, b, c, d \in \mathfrak{A}$. The function $\mu : \mathfrak{A} \to \mathbb{R}$ is also uniformly continuous since $a \subseteq b \cup (a+b)$, for each $a, b \in \mathfrak{A}$. Thus all these functions extend uniquely to uniformly continuous functions on the metric completion $\widehat{\mathfrak{A}}$ of $\mathfrak{A}$ and $\widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}}$ (Theorem 2.20). Ring identities are verified in $\widehat{\mathfrak{A}}$ by considering convergent sequences from $\mathfrak{A}$. For example, to verify the identity

$$(\forall a, b, c \in \widehat{\mathfrak{A}})(a \cdot (b + c) = a \cdot b + a \cdot c), \tag{2.4}$$

one can show that the set $C := \{(a, b, c) \in \widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}} : a \cdot (b + c) = a \cdot b + a \cdot c\}$, is a closed set (since we are working in a metric space, this can be done via convergent sequences). Since the identity expressed in (2.4) holds in $\mathfrak{A}$ and $\mathfrak{A} \times \mathfrak{A} \times \mathfrak{A}$ is dense in $\widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}}$, it follows that $C = \widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}} \times \widehat{\mathfrak{A}}$. In the same way, we see that μ extends to a strictly positive submeasure on $\widehat{\mathfrak{A}}$. The algebra $\widehat{\mathfrak{A}}$ is seen to be σ-complete by first showing that any sequence in $a_1 \geq a_2 \geq \cdots \in \widehat{\mathfrak{A}}$ is in fact Cauchy (Lemma 2.19). Then supremums and infimums in $\widehat{\mathfrak{A}}$ are obtained as limit points of such sequences, which exist by completeness of our metric space. Metric completeness is used in this way to also show that $\widehat{\mu}$ is Maharam. $\square$

We will also need the following.

Fact 2.22. **([28, Proof of Theorem 1])** *Any two strictly positive continuous submeasures μ and λ on a Maharam algebra $\mathfrak{B}$ are equivalent.*

Proof of (A) $\to$ (B). Let μ be an exhaustive submeasure on $\mathbb{A}$. Let $\mathfrak{A} = \mathbb{A}/\mathrm{Null}(\mu)$. By setting $\mu([a]_{\mathrm{Null}(\mu)}) = \mu(a)$ we define an exhaustive strictly positive measure on $\mathfrak{A}$. By Lemma 2.21 we can extend μ to a strictly positive continuous submeasure on a σ-complete Boolean algebra $\mathfrak{B}$. Without loss of generality we may assume that $\mathfrak{B}$ is atomless. By (A) the algebra $\mathfrak{B}$ carries a strictly positive measure λ. By Fact 2.22 we know that μ and λ are equivalent. But λ is a measure and is therefore uniformly exhaustive. $\square$

Proof of (B)$\to$(A). Let μ be a strictly positive continuous submeasure on a σ-complete atomless Boolean algebra $\mathfrak{B}$. Suppose that μ is not uniformly exhaustive. Then for some fixed $\epsilon > 0$ we can find, for each $n \in \mathbb{N}$, a partition of $\mathfrak{B}$ into pieces $a_1^n, ..., a_n^n$ such that $\mu(a_i^n) \geq \epsilon$, for each i. Let $\mathfrak{A}$ be any countable atomless subalgebra of $\mathfrak{B}$ containing each of the a_i^n. We can do this because $\mathfrak{B}$ is atomless. In particular μ will not be uniformly exhaustive on $\mathfrak{A}$ (which is isomorphic to $\mathbb{A}$), which contradicts (B). Thus μ is uniformly exhaustive on $\mathfrak{B}$ and so by Theorem 2.11 there exists a finitely additive measure λ equivalent to μ. It follows by definition that λ will be strictly positive and continuous and so σ-additive. Thus $\mathfrak{B}$ is a measure algebra. $\square$

To deal with (C) we quote the following result due to J. P. R. Christensen.

Theorem 2.23. **([4, Theorem 1])** *If λ is a non-trivial pathological submeasure on a Boolean algebra $\mathfrak{B}$ and μ is a non-trivial finitely additive measure, then $\lambda \not\ll \mu$ and $\mu \not\ll \lambda$.*

This allows us to complete the proof of Theorem 2.17.

Proof of (B)→(C). If μ is an exhaustive submeasure then it must be equivalent to a measure by (B) and Theorem 2.11. In particular it cannot be pathological by Theorem 2.23. □

Proof of (C)→(B). Let μ be an exhaustive submeasure on $\mathbb{A}$. By (C) we can find a maximal collection of non-trivial finitely additive measures $\{\mu_i : i \in I\}$ that are each dominated by μ and are such that

$$(\forall i, j)(i \neq j \to \inf\{\mu_i(A) + \mu_j(1 \setminus A) : A \in \mathfrak{A}\} = 0). \tag{2.5}$$

Suppose that I is uncountable. Then for some $\epsilon > 0$ there exists a countably infinite $J \subseteq I$ such that $\mu_j(1) > \epsilon$, for each $j \in J$. Using (2.5), find a countably infinite set $J' \subseteq J$ and a pairwise disjoint sequence $(a_j)_{j \in J'}$ such that $\mu_j(a_j) > \epsilon$, for each $j \in J'$. But this contradicts the exhaustivity of μ. Thus we can assume that $J = \mathbb{N}$. Let $c_j = \mu_j(1)/2^j$ and set

$$\lambda = \sum_j c_j \mu_j.$$

By maximality of $(\mu_j)_j$ the non-trivial finitely additive measure λ is equivalent to μ (see [19, Page 808]). □

2.4 Extension of submeasure

Lemma 2.21 says that one can extend a strictly positive exhaustive submeasure μ to a strictly positive continuous submeasure on a complete Boolean algebra that contains the domain of μ as a regular subalgebra. Here is another extension result that we will use.

Proposition 2.24. ([29, Proposition 7.1]) *Let K be a 0-dimensional compact topological space and let φ be an exhaustive submeasure on* $\mathrm{Clopen}(K)$*. Then φ extends uniquely to a continuous submeasure on* $\mathrm{Baire}(K)$*, the σ-algebra generated by* $\mathrm{Clopen}(K)$*.*

Proof. First extend φ to the collection $\{A \subseteq K : A \text{ is open or closed}\}$ by

$$\varphi(A) = \begin{cases} \sup\{\varphi(C) : C \subseteq A \wedge C \in \mathrm{Clopen}(K)\}, & \text{if } A \text{ is open;} \\ \inf\{\varphi(C) : A \subseteq C \wedge C \in \mathrm{Clopen}(K)\}, & \text{if } A \text{ is closed.} \end{cases}$$

Let $\mathfrak{M}$ be the algebra of subsets of K defined by, $A \in \mathfrak{M}$ if and only if for each $\epsilon > 0$ there exists an open set O and a closed set C such that $\varphi(O \setminus C) < \epsilon$ and $C \subseteq A \subseteq O$. Of course $\mathrm{Clopen}(K)$ is a subalgebra of $\mathfrak{M}$.

Since φ is exhaustive, for any open set O there exists a sequence of clopen sets $(O_n)_{n\in\mathbb{N}}$ such that each $O_n \subseteq O$ and $\lim_n \varphi(O \setminus O_n) = 0$. We use this to show that $\mathfrak{M}$ is a σ-algebra as follows. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of members of $\mathfrak{M}$ such that $A_n \subseteq A_{n+1}$ and fix $\epsilon > 0$. Let $(C_n)_{n\in\mathbb{N}}$ and $(O_n)_{n\in\mathbb{N}}$ be a sequence of closed sets and a sequence of open sets, respectively, such that for each n we have $\varphi(O_n \setminus C_n) < \epsilon 2^{-n}$ and $C_n \subseteq A_n \subseteq O_n$. Let $O = \bigcup_{n\in\mathbb{N}} O_n$ and find a clopen E such that $\varphi(O \setminus E) < \epsilon$. By compactness there exists some N such that

$E \subseteq \bigcup_{n\in[N]} O_n$. Let $C = \bigcup_{n\in[N]} C_n$. Then

$$\varphi(O \setminus C) \leq \varphi(O \setminus E) + \sum_{n\in[N]} \varphi(O_n \setminus C_n) < 2\epsilon.$$

Thus the sets C and O witness that $\bigcup_{n\in\mathbb{N}} A_n \in \mathfrak{M}$.

Now let us show that φ is continuous on Baire$(K) \subseteq \mathfrak{M}$. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence from Baire(K) such that $A_n \supseteq A_{n+1}$ and $\bigcap_n A_n = \emptyset$ and suppose for a contradiction that, for some $\epsilon > 0$, we have

$$\lim_n \varphi(A_n) = \inf_n \varphi(A_n) > \epsilon.$$

Find a sequence of closed sets $(C_n)_{n\in\mathbb{N}}$ such that $C_n \subseteq A_n$ and $\varphi(A_n \setminus C_n) < \epsilon 2^{-n-1}$. For each $k \in \mathbb{N}$ we have that

$$\varphi(A_k) \leq \varphi(\bigcap_{n\in[k]} C_n) + \sum_{n\in[k]} \varphi(A_n \setminus C_n),$$

so that $\varphi(\bigcap_{n\in[k]} C_n) \geq \epsilon/2$. By compactness once again, we have that

$$\bigcap_n A_n \supseteq \bigcap_n C_n \neq 0,$$

which is the desired contradiction. $\square$

2.5 Talagrand's construction

We present here the construction of the pathological exhaustive submeasure that is not uniformly exhaustive from [33] (see Theorem 2.18, above). For the rest of this dissertation, let

$$\mathcal{T} = \prod_{n\in\mathbb{N}} [2^n].$$

We also fix

$$\mathbb{T} = \text{Clopen}(\mathcal{T}).$$

For each $n \in \mathbb{N}$, let $\mathcal{A}_n = \{[f \restriction [n]] : f \in \mathcal{T}\}$ and $\mathcal{B}_n$ be the subalgebra of $\mathbb{T}$ generated by $\mathcal{A}_n$. Members of $\mathcal{B}_n$ will be finite unions of sets of the form $[s]$, for $s \in \prod_{k\in[n]}[2^k]$.

Let

$$\mathcal{M} = \mathbb{T} \times [\mathbb{N}]^{<\omega} \times \mathbb{R}_{\geq 0}.$$

For finite $X \subseteq \mathcal{M}$, where $X = \{(X_1, I_1, w_1), ..., (X_n, I_n, w_n)\}$, let

$$w(\emptyset) = 0, \quad w(X) = \sum_{i=1}^{n} w_i, \quad \bigcup X = \bigcup_{i=1}^{n} X_i.$$

The value $w(X)$ is called the **weight** of X.

We have the following general construction.

Definition 2.25. *If $Y \subseteq \mathcal{M}$ and is such that there exists a finite $Y' \subseteq Y$ such that $\mathcal{T} = \bigcup Y'$ then Y defines a submeasure ϕ_Y given by*

$$\phi_Y(B) = \inf\{w(Y') : Y' \subseteq Y \wedge Y' \textit{ is finite} \wedge B \subseteq \bigcup Y'\}.$$

For $k \in \mathbb{N}$ and $\tau \in [2^n]$ let

$$S_{n,\tau} = \{f \in \mathcal{T} : f(n) \neq \tau\}.$$

For $k \in \mathbb{N}$, let

$$\eta(k) = 2^{2k+10} 2^{(k+5)^4} (2^3 + 2^{k+5} 2^{(k+4)^4}),\ \alpha(k) = (k+5)^{-3}$$

and set

$$\mathcal{D}_k = \{(X, I, w) \in \mathcal{M} : |I| \in [\eta(k)] \wedge w = 2^{-k} \left(\frac{\eta(k)}{|I|}\right)^{\alpha(k)} \wedge (\exists \tau \in \prod_{n \in I} [2^n])(X = \bigcap_{n \in I} S_{n,\tau(n)})\}.$$

See Figure 1 on page 65, for the behaviour of the sequences $\alpha(k)$ and $\eta(k)$. Let $\mathcal{D} = \bigcup_{k \in \mathbb{N}} \mathcal{D}_k$ and

$$\psi = \phi_{\mathcal{D}}.$$

An important property of ψ is the following.

Proposition 2.26. ([31, Page 9]) *Any non-trivial submeasure μ such that $\mu \leq \psi$ must be pathological and cannot be uniformly exhaustive.*

Thus it is enough to now construct a non-trivial exhaustive submeasure that lies below ψ.

Definition 2.27. ([31, Page 11]) *Let $\mu : \mathbb{T} \to \mathbb{R}$ be a submeasure and let $m, n \in \mathbb{N}$.*

- *For each $s \in \prod_{i \in [m]} [2^i]$ we define the map*

  $$\pi_{[s]} : \mathcal{T} \to [s]$$

  by

  $$(\pi_{[s]}(x))(i) = \begin{cases} s(i), & \textit{if } i \in [m]; \\ x(i), & \textit{otherwise.} \end{cases}$$

- *For $m < n$ we say a set $X \subseteq \mathcal{T}$ is (m, n, μ)-**thin** if and only if*

  $$\forall A \in \mathcal{A}_m, \exists B \in \mathcal{B}_n \textit{ such that } B \subseteq A,\ B \cap X = \emptyset \textit{ and } \mu(\pi_A^{-1}[B]) \geq 1.$$

  *For $I \subseteq \mathbb{N}$, we say that X is (I, μ)-**thin** if it is (m, n, μ)-thin for each $m, n \in I$ with $m < n$.*

The rest of the construction proceeds by a downward induction. For $p \in \mathbb{N}$, let $\mathcal{E}_{p,p} = \mathcal{C}_{p,p} = \mathcal{D}$ and $\psi_{p,p} = \phi_{\mathcal{C}_{p,p}}$. Now for $k < p$, given $\mathcal{E}_{k+1,p}$, $\mathcal{C}_{k+1,p}$ and $\psi_{k+1,p}$ we let

$$\mathcal{E}_{k,p} = \{(X, I, w) \in \mathcal{M} : X \text{ is } (I, \psi_{k+1,p})\text{-thin, } |I| \in [\eta(k)] \text{ and } w = 2^{-k}\left(\tfrac{\eta(k)}{|I|}\right)^{\alpha(k)}\},$$

$\mathcal{C}_{k,p} = \mathcal{C}_{k+1,p} \cup \mathcal{E}_{k,p}$ and $\psi_{k,p} = \phi_{\mathcal{C}_{k,p}}$.

Next let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. For each $k \in \mathbb{N}$ let $\mathcal{E}_k$ and $\mathcal{C}_k$ be subsets of $\mathcal{M}$ defined by

$$x \in \mathcal{E}_k \leftrightarrow \{p : x \in \mathcal{C}_{k,p}\} \in \mathcal{U},$$

and $\mathcal{C}_k = \mathcal{D} \cup \bigcup_{l \geq k} \mathcal{E}_l$.

Finally, let $\nu_k = \phi_{\mathcal{C}_k}$. It is clear from Definition 2.25 that we have

$$\nu_1 \leq \nu_2 \leq \nu_3 \cdots \leq \psi.$$

Now the submeasure ν_1, which we shall denote by ν from here on, is the desired counter example to Maharam's problem. The fact that ν is non-trivial and exhaustive requires two separate arguments. Exhaustivity follows by showing that for each k and antichain $(a_n)_{n\in\mathbb{N}}$ from $\mathbb{T}$ we have

$$\limsup_n \nu_k(a_n) \leq 2^{-k}.$$

This last property is known as 2^{-k}-*exhaustivity*.

2.6 Set theory

We outline here the set theory that we shall need. Once again, our intention is not to give an introduction to these topics and as a starting point we take [23] (and in particular Chapter 7). Our forcing notation and technique (for example, via countable transitive models) is from [23] and we avoid forcing via Boolean valued models (for example, as described in [18]).

2.6.1 Forcing and Borel codes

Forcing notions (partial orders) $(\mathbb{P}, \leq)$ will have a *top element* which we will also denote by $\mathbb{P}$. By $a \leq b$ we will mean that *a is stronger than b*. We will not distinguish between the forcing relation $\Vdash$ and $\Vdash^*$ from [23]. When we do force over the universe V, we follow [23, Section 7.9]. We shall abbreviate *countable transitive model* by *c.t.m.*. Of course ZFC is used to denote the usual Zermelo-Fraenkel axioms of set theory with the Axiom of Choice. The statement "let M be a c.t.m. of ZFC" is to be interpreted as "let M be a c.t.m. of a finite fragment of ZFC that is enough to furnish the following definitions and argument" (see [23, Sections 7.1 and 7.9]). This is to avoid Gödel's second incompleteness theorem in the usual way. Given a c.t.m. of ZFC and a partial order $(\mathbb{P}, \leq)$, the letter 'G' will often denote a $\mathbb{P}$-generic filter over M. Canonical $\mathbb{P}$-names will be denoted by '$\check{r}$' and other names by '$\dot{r}$'.

The following describes the collection $\mathrm{BC} \subseteq {}^{\omega}\omega$ of *Borel codes*.

Lemma 2.28. ([3, Page 11] and[18, Page 504]) *There exists a set* $\mathrm{BC} \subseteq {}^{\omega}\omega$ *and a surjection* $\mathrm{BC} \to \mathrm{Borel}(2^{\omega}) : c \mapsto A_c$ *such that the following predicates of Borel codes are absolute for c.t.m.'s of* ZFC*:*

- $c \in \mathrm{BC}$*;*
- $A_c = \emptyset$*,* $A_c = A_d$*,* $A_c \cap A_d = A_e$*,* $A_c = A_d \triangle A_e$ *and* $A_c = \bigcup_{n\in\omega} A_{c_n}$*;*
- A_c *is meagre,* A_c *is Lebesgue null.*

The predicate '$x \in A_c$' is also absolute.

Borel codes provide a very convenient way to discuss the 'description' of a Borel set, regardless of what model of set theory we are considering. The idea is that the Borel code c contains all the information that is used in the construction of A_c from the open sets. Since there are continuum many Borel sets, there must be some that are not 'definable' like 2^{ω} or $[(1, 1, ..., 1)]$. We may, however, want to consider how some fixed (possibly undefinable) Borel set A behaves in two different models of set theory M and N, say. We can do this by considering the Borel code c such that $A_c = A$ and then considering the sets A_c^M and A_c^N. Just like we can construct 2^{ω} *in* M to get $(2^{\omega})^M$, we may also construct A_c in M to get A_c^M. We are assuming that $c \in M \cap N$. Of course by absoluteness of the predicate $x \in A_c$ we know that $A_c^M = A_c \cap M$.

2.6.2 Forcing with ideals

Given a σ-ideal $\mathcal{I}$ on $\mathrm{Borel}(2^{\omega})$ we can define a forcing notion $\mathbb{P}_{\mathcal{I}} = (\mathrm{Borel}(2^{\omega})/\mathcal{I})^+$ where the generic extensions due to $\mathbb{P}_{\mathcal{I}}$ are determined by (and determine) a single real $\dot{r} \in 2^{\omega}$. Examples of such reals are the Cohen real and the random real, which arise from the Cohen algebra and the random algebra, respectively. These are discussed in Subsection 2.6.3. Here we describe this *idealised forcing* and the determining generic real added.

Proposition 2.29. ([38, Page 15]) *Let $\mathcal{I}$ be a σ-complete ideal on* $\mathrm{Borel}(2^{\omega})$*. Then there exists a $\mathbb{P}_{\mathcal{I}}$-name $\dot{r}$ such that*

$$\mathbb{P}_{\mathcal{I}} \Vdash \dot{r} \in \dot{2^{\omega}} \wedge (\forall c \in \check{\mathrm{BC}})(\dot{r} \in A_c \leftrightarrow \check{A}_c \in \dot{G}). \tag{2.6}$$

Proof. By [23, Theorem 7.11] we may instead work with the forcing notion $\mathbb{P} = \{A \in \mathrm{Borel}(2^{\omega}) : A \notin \mathcal{I}\}$, ordered by inclusion. Let G be a $\mathbb{P}$-generic filter. For each $\epsilon > 0$ it is possible to find disjoint clopen $A_1, ..., A_n$ that cover 2^{ω} such that each A_i has diameter $\leq \epsilon$. By genericity, G must choose one of these. Now the collection of closed sets in G has the finite intersection property, so by compactness their intersection is non-empty. Since G contains closed sets of arbitrarily small diameter, this intersection must be a singleton. Thus let $\dot{r}$ be such that

$$\mathbb{P} \Vdash \dot{r} \in \bigcap \{A_c : c \in \check{\mathrm{BC}} \wedge A_c \text{ is closed} \wedge A_c \in \dot{G}\}.$$

Now let us show that the collection

$$\mathbb{B} = \{A_c : c \in \mathrm{BC} \wedge A_c \Vdash \dot{r} \in A_c\}$$

is closed under countable unions and countable intersections. Let $(C_n)_{n\in\mathbb{N}} \subseteq \mathbb{B}$ and $B = \bigcup_{n\in\mathbb{N}} C_n$. Let $D \in \mathbb{P}$ and $D \subseteq B$. By the σ-completeness of $\mathcal{I}$ we can find an n such that $D \cap C_n \notin \mathcal{I}$. But $D \cap C_n \leq C_n \Vdash \dot{r} \in C_n \subseteq B$. In particular, for any $D \leq B$, $D \Vdash \dot{r} \in B$ and so we must have $B \Vdash \dot{r} \in B$. Now let $B = \bigcap_{n\in\mathbb{N}} C_n$. Since $B \leq C_n$ for every n, $B \Vdash \dot{r} \in C_n$ for every n and so $B \Vdash \dot{r} \in B$. By definition $\mathbb{B}$ also contains all closed sets of $\mathbb{P}$ and so in fact $\mathbb{P} = \mathbb{B}$. That is,

$$\mathbb{P} \Vdash \dot{r} \in \dot{2^\omega} \wedge (\forall c \in \check{\mathrm{BC}})(\check{A}_c \in \dot{G} \to \dot{r} \in A_c).$$

For the other direction suppose that $A_c \Vdash \dot{r} \in A_{c'}$. If $\neg(A_c \leq A_{c'})$ then $B = A_c \setminus A_{c'} \notin \mathcal{I}$. But then $B \Vdash \dot{r} \in B \wedge B \cap A_{c'} = 0$. So $A_c \not\Vdash \dot{r} \in A_{c'}$, which is a contradiction. Now suppose that G is $\mathbb{P}_\mathcal{I}$-generic and that in $V[G]$, $\dot{r} \in A_{c'}$. Then for some $A_c \in G$, $A_c \Vdash \dot{r} \in A_{c'}$. By the previous argument, $A_c \leq A_{c'}$ and so in $V[G]$, $A_{c'} \cap V \in G$. $\square$

Let M be a c.t.m. of ZFC with $\mathcal{I} \in M$. Let $\mathbb{P}_\mathcal{I}$ denote $(\mathrm{Borel}(2^\omega)/\mathcal{I})^M$. Call a real $r \in 2^\omega$, **$\mathbb{P}_\mathcal{I}$-over** M, if and only if

$$(\forall c \in \mathrm{BC} \cap M)(A_c \cap M \in \mathcal{I} \to r \notin A_c). \tag{2.7}$$

The above is saying that a real is $\mathbb{P}_\mathcal{I}$ over M if and only if it misses every $\mathcal{I}$-positive Borel set from the ground model, after it has been computed in the universe.

Proposition 2.30. *If M is a countable transitive model of ZFC with $\mathcal{I} \in M$ and $\dot{r}$ is the $\mathbb{P}_\mathcal{I}$-name promised by Proposition 2.29 then for any $\mathbb{P}_\mathcal{I}$-generic G over M, $\dot{r}^G$ is $\mathbb{P}_\mathcal{I}$-over M.*

Proof. If for some $c \in B \cap M$ such that $A_c \cap M \in \mathcal{I}$ we have that $r \in A_c$ then $A_c \cap M \in G \cap \mathcal{I}$ (by (2.7)) which is a contradiction. $\square$

If one wants to avoid mentioning countable transitive models, then we can use the following definition.

Definition 2.31. *Given $\mathbb{P}_\mathcal{I}$ and another forcing notion $\mathbb{Q}$, we will say that $\mathbb{Q}$* **adds a $\mathbb{P}_\mathcal{I}$-real** *if and only if there exists a $\mathbb{Q}$-name $\dot{r}$ such that*

$$\mathbb{Q} \Vdash \dot{r} \in \dot{2^\omega} \wedge (\forall c \in \check{\mathrm{BC}})(\check{A}_c \in \check{\mathcal{I}} \to \dot{r} \notin A_c) \tag{2.8}$$

In the case that $\mathbb{Q}$ arises from a complete Boolean algebra $\mathfrak{B}$ (if not then we may consider the Boolean completion of $\mathbb{Q}$, by [23, Theorem 7.11]), we have a combinatorial formulation of (2.8).

Fact 2.32. **([34])** *A complete Boolean algebra $\mathfrak{B}$ adds a $\mathbb{P}_\mathcal{I}$-real if and only if there exists a regular embedding from* $\mathrm{Borel}(2^\omega)/\mathcal{I}$ *to $\mathfrak{B}$.*

Proof. For the 'only if' direction if $\dot{G}$ is a $\mathbb{Q}$-name for the $\mathbb{P}_{\mathcal{I}}$-generic obtained from the $\mathbb{P}_{\mathcal{I}}$-real then we obtain the regular embedding:

$$\mathrm{Borel}(2^\omega)/\mathcal{I} \to \mathbb{Q} : p \mapsto \sum\{q : q \Vdash \check{p} \in \dot{G}\}.$$

For the other direction, if $f : \mathrm{Borel}(2^\omega)/\mathcal{I} \to \mathbb{Q}$ is a regular embedding then in any forcing extension due to a $\mathbb{Q}$-generic H, we can (as in [23, Corollary 7.6]) define a generic G for $\mathrm{Borel}(2^\omega)/\mathcal{I}$ (and therefore the determining real) by

$$G = \{f(p) : p \in H\}.$$

□

2.6.3 Generic reals

We saw in Subsection 2.6.1 how the generic extensions of $\mathbb{C}$ and $\mathbb{M}$, the Cohen and random algebras, are determined by certain members of 2^ω. Using Fact 2.32 we obtain the following.

Definition 2.33. *A complete Boolean algebra* $\mathfrak{B}$ **adds a Cohen real** *if and only if there exists a regular embedding from* $\mathbb{C}$ *to* $\mathfrak{B}$*. Similarly, a complete Boolean algebra* $\mathfrak{B}$ **adds a random real** *if and only if there exists a regular embedding from* $\mathbb{M}$ *to* $\mathfrak{B}$*.*

A forcing notion $\mathbb{P}$ is ω^ω**-bounding** if and only if every real $f \in 2^\omega$ in a given forcing extension due to $\mathbb{P}$ is dominated by a real $r \in 2^\omega$ from the ground model, that is to say,

$$(\forall m)(r(m) \geq f(m)).$$

The algebra $\mathbb{M}$ is ω^ω-bounding, while the algebra $\mathbb{C}$ is most certainly not! This is because the Cohen real added cannot be dominated by a real from the ground model. In fact the property of ω^ω-bounding, for ccc Boolean algebras, is equivalent to *weak distributivity*, which is the following combinatorial property. A complete ccc Boolean algebra $\mathfrak{B}$ is **weakly distributive** if and only if for every countable sequence $(A_n)_{n\in\omega}$, of partitions of $\mathfrak{B}$, there exists a dense subset A of $\mathfrak{B}$ such that

$$(\forall n)(\forall a \in A)(|\{b \in A_n : a \cap b \neq 0\}| < \infty).$$

There are many equivalent definitions of weak distributivity, but the above is perhaps the most transparent (see [2]).

Fact 2.34. ([2]) *Every Maharam algebra is weakly distributive, and therefore cannot add a Cohen real.*

The algebra $\mathbb{C}$ is actually **nowhere weakly distributive**, which is to say that for every $a \in \mathbb{C}^+$ the algebra $\mathbb{C}_a$ is not weakly distributive, and this is a simple consequence of Fact 2.6.

We will also consider *splitting reals.*

Definition 2.35 (**[35, Page 2]**). *A forcing notion $\mathbb{P}$ adds a* **splitting real** *if and only if there exists a $\mathbb{P}$-name $\dot{A}$ such that*

$$\mathbb{P} \Vdash \dot{A} \subseteq \check{\omega} \wedge (\forall B \in \check{\mathcal{P}(\omega)})(|B| = \infty \to (|\dot{A} \cap B| = \infty \wedge |B \setminus \dot{A}| = \infty)).$$

Informally, $\dot{A}$ is a splitting real if for every infinite set of natural numbers B from the ground model, $\dot{A}$ partitions (or splits*) B into two infinite pieces, one being a subset of $\dot{A}$ and the other being disjoint from $\dot{A}$.*

Given a function $f \in 2^\omega$ we can define a subset of ω by $\{n \in \omega : f(n) = 1\}$. In this way we may speak about members of 2^ω being splitting reals.

Fact 2.36. *Cohen reals and random reals are splitting reals.*

Proof. Let M be a c.t.m. of ZFC. Let $f \in 2^\omega$ and let $A = \{n \in \omega : f(n) = 1\}$. Suppose that (without loss of generality) there exists $B \in M \cap \omega$ such that $|B \cap A| < \omega$. Since $[\omega]^{<\omega} \subseteq M$, we know that

$$f \in C := \{g \in 2^\omega : (\forall n \in B \setminus A)(g(n) = 0)\} \in M$$

Since C is in the ground model and is both meagre and Lebesgue null, f cannot be a Cohen real nor can it be a random real. $\square$

The above fact, with respect to the random algebra, is actually a specific instance of the more general result that every Maharam algebra adds a splitting real ([35]).

3 Sometimes the same

In this section we present two results that were originally proved for additive submeasures but generalise to cases when additivity is not present. The first result says that under Todorcevic's Open Colouring Axiom (OCA) the Boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$, where Fin is the ideal of finite subsets of $\mathcal{P}(\omega)$, does not contain a Maharam algebra as a subalgebra. This generalises the result from [5] which states that under OCA, $\mathcal{P}(\omega)/\mathrm{Fin}$ does not contain a measure algebra as a subalgebra. In fact this was not so difficult to achieve. For this one needs to observe that the (sophisticated) proof from [5] goes through for any σ-complete Boolean algebra $\mathfrak{B}$ such that for some embedding $\mathbb{F} : \mathrm{Clopen}(\omega \times {}^{\omega}2) \to \mathfrak{B}$ we have

$$(\forall n \in \omega)(\forall f_1, f_2, ... \in {}^{\omega}2)(\exists N_1, N_2, ... \in \omega)(\sum_{i\in\omega} \mathbb{F}(\{n\} \times [f_i \restriction N_i]) < \mathbb{F}(\{n\} \times 2^{\omega})).$$

This we do in the next subsection. Once one substitutes the above promised embedding for the embedding used in [5] the rest follows the original proof identically. It is evident from the close proximity of the proof of the above statement to the original one, that really what we have observed is that it was not additivity but continuity of the Lebesgue measure on which the result from [5] relies on.

Our second result, which we prove in Subsection 3.2, states that if one has an exhaustive σ-subadditive submeasure μ on a subalgebra $\mathfrak{A}$ of a σ-complete Boolean algebra $\mathfrak{B}$ then μ extends to a continuous submeasure on the smallest σ-complete σ-regular subalgebra of $\mathfrak{B}$ containing $\mathfrak{A}$ (see Theorem 3.6). If we replace exhaustivity by additivity in the above then we get the original and classical result for measures. This second result seemed to require a little more effort, however, it does follow a standard procedure for constructing the collection of Lebesgue measurable sets.

Extension of submeasure has been observed before (see [9]). We discuss the extension theorems that we have already mentioned in Subsection 3.2, but we would like to point out here that we could not deduce the above from what we found in the literature.

3.1 OCA and Maharam algebras

Given a set X we let

$$[X]^2 = \{\{x, y\} : x, y \in X \wedge x \neq y\}.$$

If X is equipped with a topology then we can equip $[X]^2$ with the topology induced from the product space $X \times X$. Here is the statement of OCA.

OCA. *If X is a separable metric space and $[X]^2 = X_1 \cup X_2$ such that X_1 is open in $[X]^2$ then one of the following must hold:*

- *There exists an uncountable set $Y \subseteq X$ such that $[Y]^2 \subseteq X_1$;*

- *There exists a sequence $(Y_n)_{n\in\mathbb{N}}$ of subsets of X such that $X = \bigcup_n Y_n$ and, for each n, $[Y_n]^2 \subseteq X_2$.*

OCA is relatively consistent with ZFC, and one does not need any large cardinal assumptions for this (see [25], not [23]).

The main result of [5] is the following.

Theorem 3.1. *Assuming OCA, the random algebra is not a subalgebra of $\mathcal{P}(\omega)/\mathrm{Fin}$.*

Notice that this is in contrast to the situation in which one is working under the continuum hypothesis (CH). The Boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$ contains, as a subalgebra, every Boolean algebra of cardinality at most $\aleph_1$ (see [22, Section 5.5]). Thus under CH, which asserts that $\aleph_1 = \mathfrak{c}$ ($= |\mathbb{M}|$), the algebra $\mathcal{P}(\omega)/\mathrm{Fin}$ will indeed embed the random algebra.

In this section we observe that Theorem 3.1 remains true if we replace the random algebra by any Maharam algebra:

Theorem 3.2. *Assuming OCA, the Boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$ does not contain a Maharam algebra as a subalgebra.*

We work in the product space $\omega \times 2^\omega$ corresponding to the discrete topology on ω. It is straightforward to check that $\mathrm{Clopen}(\omega \times 2^\omega)$ is the collection of sets of the form

$$\bigcup_{n\in\omega} \{n\} \times B_n$$

for $B_n \in \mathrm{Clopen}(2^\omega)$. If λ is the Lebesgue measure on $\mathrm{Clopen}(2^\omega)$ then we can define a measure on $\mathrm{Clopen}(\omega \times 2^\omega)$ by

$$\lambda(\bigcup_{n\in\omega} \{n\} \times B_n) = \sum_{n\in\omega} 2^{-n}\lambda(B_n).$$

This will extend uniquely to $\mathrm{Borel}(\omega \times 2^\omega)$. The complete Boolean algebra $\mathrm{Borel}(\omega \times 2^\omega)/\mathrm{Null}(\lambda)$ will be σ-generated by the collection $\omega \times \mathrm{Clopen}(2^\omega)$ and will therefore be isomorphic to the random algebra (Fact 2.16).

Definition 3.3. *Let $\mathfrak{B}$ be a σ-complete Boolean algebra. Say that $\mathbb{C}(\mathfrak{B})$ holds if and only if there exists an embedding $\mathbb{F} : \mathrm{Clopen}(\omega \times {}^\omega 2) \to \mathfrak{B}$ such that*

$$(\forall n \in \omega)(\forall f_1, f_2, ... \in {}^\omega 2)(\exists N_1, N_2, ... \in \omega)(\sum_{i\in\omega} \mathbb{F}(\{n\} \times [f_i \restriction N_i]) < \mathbb{F}(\{n\} \times 2^\omega)).$$

We prove that Theorem 3.1 is true if we replace the random algebra with any σ-complete algebra $\mathfrak{B}$ such $\mathbb{C}(\mathfrak{B})$ holds. This involves nothing more than reproducing the arguments from [5] with the embedding $X \mapsto [X]_{\mathrm{Null}(\lambda)}$ replaced by the $\mathbb{F}$ that witnesses $\mathbb{C}(\mathfrak{B})$. Since these arguments are no different to those from [5] we will only present that part of [5] where $\mathbb{C}(\mathfrak{B})$ is used. This is done in Section A. For what remains we direct the reader to [5]. What is left to do here then is to show that $\mathbb{C}(\mathfrak{B})$ holds for any Maharam algebra $\mathfrak{B}$, and this is a straightforward consequence of continuity and the following fact.

Fact 3.4. **([12, 392Xg])** *If $\mathfrak{B}$ is an atomless σ-complete Boolean algebra then any strictly positive continuous submeasure on $\mathfrak{B}$ will be diffuse.*

Proposition 3.5. *$\mathbb{C}(\mathfrak{B})$ holds for every (atomless) Maharam algebra $\mathfrak{B}$.*

Proof. Let $\mathfrak{B}$ be a Maharam algebra carrying a strictly positive continuous submeasure μ. Let $(a_i)_{i\in\omega}$ be a partition of $\mathfrak{B}$. Since μ will be diffuse, for each $i, j \in \omega$ we can find a partition A_i^j of a_i into finitely many pieces each with μ-measure not greater than $\frac{1}{j+1}$. For each i, let $\mathfrak{B}_i$ be the (countable atomless) subalgebra of $\mathfrak{B}_{a_i}$ generated by $\bigcup_j A_i^j$ and let $f_i : \mathbb{A} \to \mathfrak{B}_i$ be any isomorphism. Now let

$$\mathbb{F}(\bigcup_{n\in\omega}\{n\} \times B_n) = \sum_{n\in\omega} f_n(B_n).$$

Let $f \in {}^\omega 2$ and $m \in \omega$. For each $\epsilon > 0$ there exists a finite partition $a_1, a_2, ..., a_n$ of $\mathrm{Clopen}(\{m\} \times 2^\omega)$ such that for each i, $\mu(\mathbb{F}(a_i)) \leq \epsilon$. But for k large enough there will be an i such that $\{m\}\times[f \restriction k] \subseteq a_i$, and so $\mu(\mathbb{F}(\{m\}\times[f \restriction k])) \to 0$ as $k \to \infty$. Thus given $f_0, f_1, ... \in {}^\omega 2$ for each i we can choose N_i such that $\mu(\mathbb{F}(\{m\} \times [f \restriction N_i])) < 2^{-i-2}\mu(\mathbb{F}(\{m\} \times 2^\omega))$. Then by σ-subadditivity of μ we get

$$\mu(\sum_{i\in\omega} \mathbb{F}(\{n\} \times [f_i \restriction N_i])) \leq \sum_{i\in\omega} \mu(\mathbb{F}(\{n\} \times [f_i \restriction N_i])) < \frac{1}{2}\mu(\mathbb{F}(\{n\} \times [\emptyset]),$$

and since μ is strictly positive we are done. □

3.2 Extension of submeasure (revisited)

We prove here the following.

Theorem 3.6. *Let $\mathfrak{B}$ be a σ-complete Boolean algebra and $\mathfrak{A}$ a subalgebra carrying an exhaustive submeasure μ. Then there exists a continuous submeasure $\widehat{\mu}$ on $\sigma(\mathfrak{A})$ such that*

- *$(\forall a \in \mathfrak{A})(\widehat{\mu}(a) \leq \mu(a))$.*
- *If λ is another continuous submeasure on $\sigma(\mathfrak{A})$ such that $(\forall a \in \mathfrak{A})(\lambda(a) \leq \mu(a))$ then $(\forall a \in \sigma(\mathfrak{A}))(\lambda(a) \leq \widehat{\mu}(a))$.*
- *If μ is σ-subadditive then $(\forall a \in \mathfrak{A})(\mu(a) = \widehat{\mu}(a))$.*

In fact $\widehat{\mu}$ will be of the form

$$\widehat{\mu}(a) = \sup\{\psi(a) : \psi \text{ is a continuous submeasure on } \sigma(\mathfrak{A}) \text{ and } (\forall b \in \mathfrak{A})(\psi(b) \leq \mu(b)\}$$

(see (4.4), below). This is analogous to the following classical result which describes the case when μ is finitely additive.

Theorem 3.7. **([37, Page 330])** *Let $\mathfrak{B}$ be a σ-complete Boolean algebra and $\mathfrak{A}$ a subalgebra carrying a finitely additive measure μ. Then there exists a σ-additive measure $\widehat{\mu}$ on $\sigma(\mathfrak{A})$ such that*

- $(\forall a \in \mathfrak{A})(\widehat{\mu}(a) \leq \mu(a))$.
- *If λ is another σ-additive measure on $\sigma(\mathfrak{A})$ such that $(\forall a \in \mathfrak{A})(\lambda(a) \leq \mu(a))$ then $(\forall a \in \sigma(\mathfrak{A}))(\lambda(a) \leq \widehat{\mu}(a))$.*
- *If μ is σ-additive then $(\forall a \in \mathfrak{A})(\mu(a) = \widehat{\mu}(a))$.*

Other extension results already exist (see [9, Theorems 1I, 10D, 10E, 10F]). Most relevant to us are Lemma 2.21 and Proposition 2.24. By compactness of the space K in the statement of Proposition 2.24, this result is implied by Theorem 3.6. Moreover, the proof of Proposition 2.24 relies on the compactness of K (on two occasions), while the techniques used to prove Theorem 3.6 are purely combinatorial. With regards to Lemma 2.21, at the end of this subsection we prove the following.

Proposition 3.8. *Let $\mathfrak{B}$ be a σ-complete Boolean algebra and $\mathfrak{A}$ a subalgebra carrying a strictly positive σ-subadditive exhaustive submeasure μ. Then $\sigma(\mathfrak{A})/\mathrm{Null}(\widehat{\mu})$ is isomorphic to the metric completion $\widehat{\mathfrak{A}}$ of $\mathfrak{A}$ with respect to the metric induced by μ, where $\widehat{\mu}$ is the submeasure promised by Theorem 3.6. Moreover, we can find an isomorphism $F : \sigma(\mathfrak{A})/\mathrm{Null}(\widehat{\mu}) \to \widehat{\mathfrak{A}}$ such that*

$$(\forall a \in \sigma(\mathfrak{A})/\mathrm{Null}(\widehat{\mu}))(\widehat{\mu}(a) = (\widetilde{\mu} \circ F)(a)).$$

where $\widetilde{\mu}$ is the extension of μ to $\widehat{\mathfrak{A}}$ (via Lemma 2.21).

Towards a proof of Theorem 3.6, fix a σ-complete Boolean algebra $\mathfrak{B}$, a subalgebra $\mathfrak{A}$ of $\mathfrak{B}$ and a submeasure $\mu : \mathfrak{A} \to \mathbb{R}$. Define the following *outer measure* $\mu^* : \mathfrak{B} \to \mathbb{R}$ by

$$\mu^*(a) = \inf\{\sum_{n\in\omega} \mu(a_n) : a_n \in \mathfrak{A} \wedge a \subseteq \sum_n a_n\}. \tag{3.1}$$

Lemma 3.9. *μ^* is a σ-subadditive submeasure on $\mathfrak{B}$ and $(\forall a \in \mathfrak{A})(\mu^*(a) \leq \mu(a))$. If μ is σ-subadditive (on $\mathfrak{A}$) then μ^* is an extension of μ.*

Proof. This is straightforward to see. For subadditivity consider a sequence $(a_i)_{i\in\omega}$ from $\mathfrak{B}$ and fix $\epsilon > 0$. For each $i \in \omega$ let $(a_{ij})_{j\in\omega} \subseteq \mathfrak{A}$ be such that $a_i \subseteq \sum_{j\in\omega} a_{ij}$ and

$$\sum_{j\in\omega} \mu^*(a_{ij}) \leq \mu^*(a_i) + \frac{\epsilon}{2^{i+1}}.$$

Then by definition we must have

$$\mu^*(\sum_{i\in\omega} a_i) \leq \sum_{i,j\in\omega} \mu^*(a_{ij}) \leq \sum_{i\in\omega}(\mu^*(a_i) + \frac{\epsilon}{2^{i+1}}) = \epsilon + \sum_{i\in\omega} \mu^*(a_i).$$

Since $\epsilon > 0$ was arbitrary, we are done. □

Definition 3.10. *Let $\overline{\mathfrak{A}}$ be the collection of all $a \in \mathfrak{B}$ such that there exists a sequence $(a_n)_{n\in\omega} \subseteq \mathfrak{A}$ with*

$$\mu^*(a_n \triangle a) \to 0.$$

We now work towards showing that $\overline{\mathfrak{A}}$ is a σ-complete σ-regular subalgebra of $\mathfrak{B}$.

Lemma 3.11. *The algebra $\mathfrak{A}$ is a subalgebra of $\overline{\mathfrak{A}}$ which is a subalgebra of $\mathfrak{B}$.*

Proof. Clearly $\mathfrak{A} \subseteq \overline{\mathfrak{A}}$. Let $a_1, a_2 \in \overline{\mathfrak{A}}$ and fix $\epsilon > 0$. Let $b_1, b_2 \in \mathfrak{A}$ be such that

$$\mu^*(a_1 \triangle b_1), \mu^*(a_2 \triangle b_2) < \epsilon.$$

Let $a = a_1 \cup a_2$, $b = b_1 \cup b_2 \in \mathfrak{A}$ and $c = b_1 \setminus b_2 \in \mathfrak{A}$. Since $a \triangle b, (a_1 \setminus a_2) \triangle c \subseteq (a_1 \triangle b_1) \cup (a_2 \triangle b_2)$ we have

$$\mu^*(a \triangle b), \mu^*(a_1 \setminus a_2 \triangle c) \leq \mu^*(a_1 \triangle b_1) + \mu^*(a_2 \triangle b_2) \leq 2\epsilon.$$

Thus $a_1 \cup a_2, a_1 \setminus a_2 \in \overline{\mathfrak{A}}$. □

Lemma 3.12. *If μ is exhaustive then μ^* is exhaustive on $\overline{\mathfrak{A}}$.*

Proof. Let $(a_i)_{i\in\omega}$ be an antichain $\mathfrak{A}$. Since the a_i are pairwise disjoint, for any sequence $(b_i)_{i\in\omega}$ and $n \in \omega$, we have

$$(b_n \setminus \sum_{i<n} b_i) \triangle a_n \leq b_n \triangle a_n \cup \sum_{i<n} b_i \setminus a_i. \tag{3.2}$$

Now fix $\epsilon > 0$ and find a sequence of positive reals $(\epsilon_n)_n$ such that

$$\sum_n \epsilon_n < \epsilon,$$

and for each n let $b_n \in \mathfrak{A}$ be such that $\mu^*(b_n \triangle a_n) < \epsilon_n$. Let $c_n = b_n \setminus \sum_{i<n} b_n$. By (3.2) we know that

$$\mu^*(c_n \triangle a_n) \leq \sum_{i\leq n} \epsilon_i < \epsilon.$$

But μ is exhaustive, so for n large enough we have $\mu^*(c_n) \leq \mu(c_n) < \epsilon$. In particular for n large enough we have

$$\mu^*(a_n) \leq \mu^*(a_n \setminus c_n) + \mu^*(c_n) < 2\epsilon.$$

□

Lemma 3.13. *Let $\epsilon > 0$ and $(a_n)_{n\in\omega}$ be a pairwise disjoint sequence from $\overline{\mathfrak{A}}$. If μ is exhaustive then we can find a sequence $(b_n)_{n\in\omega}$ from $\overline{\mathfrak{A}}$ such that for some $N \in \omega$ we have*

$$\sum_{n\geq N} a_n = \sum_{n\in\omega} b_n \wedge \sum_{n\in\omega} \mu^*(b_n) \leq \epsilon.$$

Proof. Let $c_k = \sum_{i\leq k} a_i$. This is a non-decreasing sequence in $\overline{\mathfrak{A}}$. Since μ^* is exhaustive (by Lemma 3.12) on $\overline{\mathfrak{A}}$, the sequence $(c_k)_{k\in\omega}$ is Cauchy with respect to the pseudometric

$d(a,b) = \mu^*(a \triangle b)$ (by Lemma 2.19). In particular, we can find a sequence $N_0 < N_1 < \cdots \in \omega$ such that for each $i \in \omega$ and $n, m \geq N_i$ we have

$$\mu^*(c_n \triangle c_m) \leq \frac{\epsilon}{2^{i+1}}.$$

Now take $N = N_0$ and $b_i = c_{N_i} \triangle c_{N_{i+1}}$. $\square$

Lemma 3.14. *If $(a_n)_{n\in\omega}$ is a pairwise disjoint sequence from $\overline{\mathfrak{A}}$ and if μ is exhaustive then $\sum_{i\in\omega} a_i \in \overline{\mathfrak{A}}$. Since $\overline{\mathfrak{A}}$ is an algebra, it follows that $\overline{\mathfrak{A}}$ is a σ-regular subalgebra of $\mathfrak{B}$.*

Proof. Let $(a_n)_{n\in\omega}$ be a pairwise disjoint sequence from $\overline{\mathfrak{A}}$ and lets show that $a = \sum_n a_n \in \overline{\mathfrak{A}}$. Fix $\epsilon > 0$ and let N and $(b_n)_{n\in\omega}$ be as promised by Lemma 3.13. Let $c = \sum_{n<N} a_n$ and $b \in \mathfrak{A}$ be such that

$$\mu^*(c \triangle b) < \epsilon.$$

Then

$$a \triangle b \subseteq (c \triangle b) \cup \sum_{n\geq N} a_n = (c \triangle b) \cup \sum_{n\in\omega} b_n$$

so that

$$\mu^*(a \triangle b) \leq \mu^*(c \triangle b) + \mu^*(\sum_{n\in\omega} b_n) \leq \epsilon + \sum_{n\in\omega} \mu^*(b_n) \leq 2\epsilon.$$

$\square$

Proof of Theorem 3.6. By Lemma 3.14 we know that on $\sigma(\mathfrak{A})$, a σ-regular subalgebra of $\overline{\mathfrak{A}}$, the submeasure $\widehat{\mu} = \mu^* \restriction \sigma(\mathfrak{A})$ is exhaustive and σ-subadditive and so it is continuous. The rest is easy to verify. $\square$

Finally, let us give a proof of Proposition 3.8.

Proof of Proposition 3.8. Let $\mathfrak{A}_1 = \sigma(\mathfrak{A})/\mathrm{Null}(\widehat{\mu})$ and $\mathfrak{A}_2 = \widehat{\mathfrak{A}}$. Then $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are both metric completions of $\mathfrak{A}$. Let $\psi_1 : \mathfrak{A} \to \mathfrak{A}_1$ and $\psi_2 : \mathfrak{A} \to \mathfrak{A}_2$ be the respective canonical embeddings. It is easy to check that ψ_1 and ψ_2 are isometries. Thus $F = \psi_2 \circ \psi_1^{-1} : \psi_1[\mathfrak{A}] \to \mathfrak{A}_2$ is an isometry onto a dense subspace of $\mathfrak{A}_2$. We can then uniquely extend F to an isometry $F' : \mathfrak{A}_1 \to \mathfrak{A}_2$ (Theorem 2.20). We claim that this is the desired isomorphism. Indeed, consider the map $f : \psi_1[\mathfrak{A}] \to \psi_1[\mathfrak{A}] : a \mapsto a^c$ and its uniformly continuous extension $f' : \mathfrak{A}_1 \to \mathfrak{A}_1 : a \mapsto a^c$. Consider also $g : \psi_2[\mathfrak{A}] \to \psi_2[\mathfrak{A}] : a \mapsto a^c$ and its uniformly continuous extension $g' : \mathfrak{A}_2 \to \mathfrak{A}_2 : a \to a^c$. Since for every $a \in \psi_1[\mathfrak{A}]$ we have

$$(F \circ f)(a) = (\psi_2 \circ \psi_1^{-1})(a^c) = ((\psi_2 \circ \psi_1^{-1})(a))^c = (F(a))^c = (g \circ F)(a)$$

We see that $F' \circ f'$ and $g' \circ F'$ are both uniformly continuous extensions of the same function. Thus $F' \circ f' = g' \circ F'$ and for every $a \in \psi_1[\mathfrak{A}]$ we have

$$F'(a^c) = (F' \circ f')(a) = (g' \circ F')(a) = (F'(a))^c.$$

Since the maps $(a, b) \mapsto a \cap b$, $a \mapsto \mu(a)$ and $a \mapsto a$ are also uniformly continuous we can use the same analysis to verify the remaining criteria for F' to be the desired isomorphism. $\square$

4 Forcing with submeasures of a finite domain

Let $\mathbb{P}$ be the collection of all normalised submeasures $\mu : \mathfrak{A} \to [0,1] \cap \mathbb{Q}$ where $\mathfrak{A}$ is a finite subalgebra of $\mathbb{A}$. Order $\mathbb{P}$ by reverse inclusion: $\mu \leq \lambda$ if and only if $\lambda \subseteq \mu$.

In this section we prove the following.

Theorem 4.1. *Let M be a c.t.m. of ZFC. If $G \subseteq \mathbb{P} \in M$ is $\mathbb{P}$-generic over M then $\lambda := \bigcup G$ is a normalised submeasure on $\mathbb{A}$ that is not uniformly exhaustive and is such that for any antichain $(a_i)_{i \in \mathbb{N}} \in M$ we have $\lim_i \lambda(a_i) = 0$.*

In fact $\mathbb{P}$ is a well known forcing notion.

Lemma 4.2. *The separative quotient of $\mathbb{P}$ is countably infinite and atomless and therefore (by Fact 2.6) its Boolean completion is the Cohen algebra.*

Proof. Let $\mathbb{P}'$ be the separative quotient of $\mathbb{P}$. Since the submeasures in $\mathbb{P}$ only take rational values and we have assumed $\mathfrak{B}$ is countable the partial order $\mathbb{P}$ is also countable and so $\mathbb{P}'$ is at most countable. Given $a \in \mathfrak{B}^+ \setminus \{1\}$ and $q \in \mathbb{Q} \cap [0,1]$ we can always find a submeasure $\lambda_q \in \mathbb{P}$ such that $\lambda_q(a) = q$. The λ_q correspond to countably many distinct equivalence classes of $\mathbb{P}'$, so $\mathbb{P}'$ is infinite. Now suppose that $\lambda \in \mathbb{P}$ and let a be an atom of $\mathrm{dom}(\lambda)$ such that $\lambda(a) > 0$. Let $c \in \mathbb{A}^+$ be such that $c < a$ and let $\mathfrak{A}$ be the subalgebra generated by $\mathrm{dom}(\lambda) \cup \{c\}$. Let λ_1 be the submeasure on $\mathfrak{A}$ defined by

$$\lambda_1(b) = \min\{\lambda(d) : d \in \mathfrak{A} \wedge b \subseteq d\}. \tag{4.1}$$

Then $\lambda_1 \leq \lambda$ and $\lambda_1(c) = \lambda(a)$. Now given $b \in \mathfrak{A}$ we can find $b' \in \mathrm{dom}(\lambda)$ and $b'' \in \{c, a \setminus c, 0\}$ such that $b = b' \sqcup b''$, so we can take λ_2 to be the submeasure on $\mathfrak{A}$ defined by

$$\lambda_2(b) = \begin{cases} \lambda(b' \cup a), & \text{if } b'' = a \setminus c; \\ \lambda(b'), & \text{otherwise.} \end{cases}$$

Then $\lambda_2 \leq \lambda$ and $\lambda_2(c) = 0 \neq \lambda_1(c)$. Thus λ_1 and λ_2 correspond to two different members of $\mathbb{P}'$ and so one of them must define a different equivalence class to λ, and we are done. $\square$

Thus, Theorem 4.1, together with Lemma 4.2, is saying that in any forcing extension adding a Cohen real there exists a submeasure, constructed from $\mathbb{P}$, that is not uniformly exhaustive but is exhaustive with respect to the antichains from the ground model.

Of course a priori this is uninteresting in light of Theorem 2.18. The motivation here is in the fact that it was known before [33] that the existence of an exhaustive submeasure that is not uniformly exhaustive is (equivalent to) a Π^1_2-*statement* and is therefore absolute for models of set theory (see [2]). It follows that if such a submeasure exists in some forcing extension then the existence of such a submeasure follows from ZFC. The aim then is to provide a new proof of the existence of such a submeasure using the theory of forcing. Of course Theorem 4.1 does not provide such a solution since in any forcing extension due to $\mathbb{P}$

new antichains might be added that have not been accounted for.

We prove Theorem 4.1 in the next subsection and in Subsection 4.2 we discuss possible directions for its development.

4.1 Proof of Theorem 4.1

We prove Theorem 4.1. The following is implicit in [16], and actually we have already used two instances of it in Lemma 4.2.

Lemma 4.3. *Let $\mathfrak{C}$ be an atomless Boolean algebra carrying a normalised submeasure μ and let $a_0, a_1, ..., a_n \in \mathfrak{C}^+$ be a finite partition of $\mathfrak{C}$. Let $\varphi_0, ..., \varphi_n$ be normalised submeasures on $\mathfrak{C}_{a_0}, ..., \mathfrak{C}_{a_n}$, respectively. Then $\mathfrak{C}$ carries a normalised submeasure φ such that for each $a \in \bigcup_i \mathfrak{C}_{a_i} \cup \langle a_0, ..., a_n\rangle$, we have*

$$\varphi(a) = \begin{cases} \mu(a_i)\varphi_i(a), & \textit{if } i \in n+1 \textit{ and } a \in \mathfrak{C}_{a_i}; \\ \mu(a), & \textit{otherwise.} \end{cases} \tag{4.2}$$

Moreover, if the μ and the φ_i take only rational values then so does φ.

Proof. Let $\mathfrak{A} = \langle a_0, a_1, ..., a_n\rangle$. Define the function $f : \bigcup_{i=0}^n \mathfrak{C}_{a_i} \cup \mathfrak{A} \to \mathbb{R}$ by

$$f(a) = \begin{cases} \mu(a_i)\varphi_i(a), & \text{if } i \in n+1 \text{ and } a \in \mathfrak{C}_{a_i}; \\ \mu(a), & \text{otherwise.} \end{cases}$$

Now define $\varphi : \mathfrak{C} \to \mathbb{R}$ by

$$\varphi(a) = \inf\{\sum_{c\in A} f(c) : A \in [\bigcup_{i=0}^n \mathfrak{C}_{a_i} \cup \mathfrak{A}]^{<\omega} \wedge c \leq \sum A\}. \tag{4.3}$$

It is straightforward to check that φ is a submeasure.

Finally let us observe that the value in (4.3) can always be achieved by a cover from $[\bigcup_{i=0}^n \mathfrak{C}_{a_i} \cup \mathfrak{A}]^{<\omega}$, that is to say, in (4.3) we are actually taking a minimum rather than an infimum. In particular the submeasure φ will be rational valued if μ and the φ_i are. To obtain the minimum, note that if $a \in \mathfrak{C}$ then a will be a disjoint union of sets $c_i \in \mathfrak{C}_{a_i}$ and so if $A \in [\bigcup_{i=0}^n \mathfrak{C}_{a_i} \cup \mathfrak{A}]^{<\omega}$ is a cover of a then we can find another cover $A' \in [\bigcup_{i=0}^n \mathfrak{C}_{a_i} \cup \mathfrak{A}]^{<\omega}$ such that the members of A' are pairwise disjoint, $(\forall i)(A' \cap \mathfrak{C}_{a_i} \in \{\emptyset, c_i\})$ and

$$\sum_{c\in A'} f(c) \leq \sum_{c\in A} f(c).$$

Since there are only finitely many such A' (given a) we are done. □

This previous lemma gives rise to the following example.

Example 4.4. *Let $\mathfrak{B}$ be a measure algebra, $a \in \mathfrak{B} \setminus \{0,1\}$ and μ be the submeasure on $\mathfrak{B}$ defined by $(\forall a \in \mathfrak{A}^+)(\mu(a) = 1)$. By Lemma 4.3 we can find a submeasure φ on $\mathfrak{B}$ such that $\varphi \restriction \mathfrak{B}_a$ and $\varphi \restriction \mathfrak{B}_{a^c}$ are σ-additive but such that $\varphi(a) = \varphi(a^c) = 1$. It is straightforward to verify that φ is continuous (and not additive).*

Now we deal with the density arguments needed for $\mathbb{P}$.

Lemma 4.5. *If $\mathfrak{A}$ is a finite subalgebra of $\mathbb{A}$ and $\mu : \mathfrak{A} \to [0,1] \cap \mathbb{Q}$ is a normalised submeasure then there exists an exhaustive submeasure $\lambda : \mathbb{A} \to [0,1] \cap \mathbb{Q}$ extending μ.*

Proof. Let $a_0, ..., a_n$ be the atoms of $\mathfrak{A}$ and for each $i \in n+1$ let $\varphi_i : \mathbb{A}_{a_i} \to \mathbb{Q}$ be a normalised finitely additive measure (take the Lebesgue measure for example). Let φ be the submeasure promised by Lemma 4.3. To see that φ is exhaustive let $(b_i)_{i\in\omega}$ be a disjoint sequence in $\mathbb{A}$ and fix $\epsilon > 0$. For each $i \in n+1$ let $D_i = \{b_j \cap a_i : j \in \omega\}$. Then each D_i is a disjoint sequence in $\mathbb{A}_{a_i}$ and so, since each φ_i is exhaustive, we can find an N such that

$$(\forall m \geq N)(\varphi(b_m) = \varphi(\bigsqcup_{i\in n+1} b_m \cap a_i) \leq \sum_{i\in n+1} \varphi(b_m \cap a_i) = \sum_{i \in n+1} \varphi_i(b_m \cap a_i) \leq (n+1)\epsilon).$$

Since ϵ was arbitrary (and n was fixed) we are done. □

Lemma 4.6. *Let $\epsilon \in (0,1]$ and $(a_i)_{i\in\omega} \subseteq \mathbb{A}$ be a disjoint sequence. Then for any finite subalgebra $\mathfrak{A}$ of $\mathbb{A}$ and submeasure $\mu : \mathfrak{A} \to [0,1] \cap \mathbb{Q}$, we can find a finite subalgebra $\mathfrak{C}$ of $\mathbb{A}$ and a submeasure $\mu' : \mathfrak{C} \to [0,1] \cap \mathbb{Q}$ extending μ, such that for some $n \in \omega$ we have, $a_n \in \mathfrak{C}$ and $\mu'(a_n) < \epsilon$.*

Proof. By Lemma 4.5 we can find an exhaustive submeasure $\lambda : \mathbb{A} \to [0,1] \cap \mathbb{Q}$ extending μ. Then for some n, $\lambda(a_n) < \epsilon$. Let $\mathfrak{C}$ be the algebra generated by $\mathfrak{A} \cup \{a_n\}$ and take $\mu' = \lambda \restriction \mathfrak{C}$. □

If $\mathfrak{C}$ is a Boolean algebra carrying a normalised submeasure $\mu : \mathfrak{C} \to [0,1]$ and $n \in \mathbb{N}$ then we will say that μ is **n-pathological** if and only if we can find disjoint $a_1, ..., a_n \in \mathfrak{C}$ such that

$$(\forall i)(\mu(a_i) = 1).$$

Lemma 4.7. *Let $\mathfrak{A}$ be a finite subalgebra of $\mathbb{A}$ and $\mu : \mathfrak{A} \to [0,1] \cap \mathbb{Q}$ a normalised submeasure. Then for any $n \in \mathbb{N}$ we can find a finite subalgebra $\mathfrak{C}$ containing $\mathfrak{A}$ and an n-pathological submeasure $\lambda : \mathfrak{C} \to [0,1] \cap \mathbb{Q}$ extending μ.*

Proof. Let $b_0, ..., b_k$ be the atoms of $\mathfrak{A}$. For each $i \in k+1$ let $b^i_1, ..., b^i_n$ be a partition of b_i into non-zero pieces. Let $\mathfrak{C}$ be the subalgebra of $\mathbb{A}$ generated by $\{b^i_j : i \in k+1, j \in [n]\}$). For $a \in \mathfrak{C}$ let

$$\lambda(a) = \mu(\bigcap\{b \in \mathfrak{C} : a \leq b\}). \tag{4.4}$$

Then $\lambda : \mathfrak{A} \to [0,1]$ is a submeasure extending μ. Also if, for $l \in [n]$, we let $a_l = \bigcup_{i \in k+1} b^i_l$ then (since $a_l \notin \mathfrak{A}$ and a_l intersects each atom of $\mathfrak{C}$) $\lambda(a_l) = 1$ and of course the a_l are pairwise disjoint. □

Proof of Theorem 4.1. The fact that $\lambda \in {}^{\mathfrak{B}}\mathbb{R}$ follows by the genericity of G and the fact that for any $p \in \mathbb{P}$ and $a \notin \mathrm{dom}(p)$, we can find $q \leq p$ such that $a \in \mathrm{dom}(q)$ (for example, see (4.4)). It is a normalised submeasure because its restriction to any finite subalgebra of $\mathfrak{B}$ is. By Lemma 4.7, for each $n \in \mathbb{N}$, the set $\{p \in \mathbb{P} : p \text{ is } n\text{-pathological}\}$ is dense in $\mathbb{P}$ and so for each n, we can find an n-pathological $p \in G$. The disjoint sequence that witnesses this, $a_1, ..., a_n$, is such that $(\forall i)(\lambda(a_i) = p(a_i) = 1)$. Thus λ cannot be uniformly exhaustive. Suppose for a contradiction that for some antichain $(a_i)_{i\in\omega}$ in M and $\epsilon \in (0, 1]$ we have $(\forall i)(\lambda(a_i) \geq \epsilon)$. By Lemma 4.6 the set $D = \{p \in \mathbb{P} : (\exists i \in \mathbb{N})(a_i \in p \wedge p(a_i) < \epsilon)\}$ is dense. Thus we can find a $p \in G \cap D$ and an $i \in \omega$ such that $a_i \in \mathrm{dom}(p) \subseteq \mathrm{dom}(\lambda)$ and $\lambda(a_i) = p(a_i) < \epsilon$, which is a contradiction. □

4.2 Remarks

It is more than likely that a more sophisticated forcing is required to achieve what we want. However, if we are to stick with the $\mathbb{P}$ described in this section there are (at least) two possible directions one could pursue, which we try to describe here.

The first is to assume that Theorem 4.1 is enough. Notice that since one can always find a filter that intersects countably many sets (see [23]), Theorem 4.1 is essentially saying that given countably many antichains of $\mathbb{A}$ one can find a submeasure that is not uniformly exhaustive but is exhaustive with respect to these antichains. Now then perhaps it is possible that there exists a countable collection of antichains, such that exhaustivity on this collection ensures exhaustivity proper. Here is a non-example to illustrate. Given two sequence $(a_i)_{i\in\omega}$ and $(b_i)_{i\in\omega}$ from $\mathbb{A}$, say that $(a_i)_{i\in\omega} \to (b_i)_{i\in\omega}$ if and only if there exists subsequences $(a_{i_k})_k$ and $(b_{j_k})_k$ such that for each k we have $a_{i_k} \leq b_{j_k}$. Call a collection $\mathcal{A}$ of antichains from $\mathbb{A}$ **exhaustive** if and only if for any antichain a from $\mathbb{A}$ we can find $b \in \mathcal{A}$ such that $a \to b$. Clearly then if there exists a countable exhaustive collection of antichains then we are done. This, as would be expected, is not the case.

Lemma 4.8. *If $\mathcal{A}$ is an exhaustive family of antichains then $|\mathcal{A}| \geq \aleph_1$.*

Proof. In what follows, for each antichain a in $\mathbb{A}$, fix an enumeration $a = \{a(0), a(1), ...\}$. We may without loss of generality assume that each antichain in $\mathcal{A}$ is maximal. Let $\mathcal{A} = \{a_0, a_1, ...\}$. By maximality of the members of $\mathcal{A}$, for any $b \in \mathbb{A}^+$ and $a \in \mathcal{A}$ we can find $i \in \omega$ such that $b \cap a(i) \neq 0$. In this way we can find an $f \in {}^{\omega}\omega$ such that for each $n \in \omega$ we have $c_n := \bigcap_{i=0}^{n} a_i(f(i)) \neq 0$. If, for some n, we have that $m \geq n \to c_n = c_m$, then every member of every antichain a in $\mathbb{A}_{c_n}$, will be a subset $a_i(f(i))$, for every i. That is

$$(\forall i)(\forall j)(a(j) \leq a_i(f(i))). \tag{4.5}$$

This contradicts the exhaustivity of $\mathcal{A}$. Otherwise, we can find a subsequence $(c_{n_k})_{k\in\omega}$ such that $c_{n_k} > c_{n_{k+1}}$. We can then take $a(k) = c_{n_k} \setminus c_{n_{k+1}}$ and reach the same contradiction as before. □

The second direction involves assuming that Theorem 4.1 is *not* enough! In this case one could try to employ an iteration of some sort. At each stage of the iteration a submeasure is generated that 'kills' the previous antichains, with the aim that at the end of the iteration all antichains will have been taken care of (much like the proof of the consistency of Martin's Axiom, as described in [23]). The problem here is one of taking limits of submeasures where as of yet we have no control over the coherence of these submeasures. This is not a new observation, since it is evident from Talagrand's construction that the limit process of submeasures is indeed an important one with regards to Maharam's problem. Indeed, it was a limit argument that motivated the definition of uniform exhaustivity in the first place (see [32, Page 102]). To illustrate consider the following straightforward lemma.

Lemma 4.9. *If $\mathfrak{B}$ is a Boolean algebra carrying a collection of normalised submeasures $(\varphi_i)_{i\in\omega}$ such that for every $a \in \mathfrak{B}$ the limit $\varphi(a) = \lim_i \varphi_i(a)$ exists then φ defines a normalised submeasure.*[6]

In particular, suppose that at each (successor) stage of the above described iteration the submeasure added is dominated by the previous one. At limit stage one could just take the pointwise infimum of the previous submeasures to obtain a submeasure that is exhaustive with respect to all the antichains that have already appeared (since it is dominated by all the submeasures that have already been constructed). Unfortunately, there seems to be no guarantee that this submeasure will not become uniformly exhaustive. Notice that although we are in fact dealing with the Cohen forcing, it is perfectly plausible that a new submeasure may be dominated by an old one. This is of course in contrast to members of ${}^{\omega}2$ where the Cohen real added cannot be dominated by a ground model real. Indeed *all* submeasures will be dominated by the submeasure that takes the value 1 everywhere but on the empty set, and this is of course in the ground model.

If we are to attempt the above procedure, we must make sure that the submeasures added are not *superpathological*. A submeasure is **superpathological** if it does not dominate a non-zero exhaustive submeasure. Superpathological submeasures have been constructed in [32, Example 3]. One possible way to ensure that the added submeasures are not superpathological is to fix a finitely additive measure in the ground model and to see to it that any newly constructed submeasures lie above it. Notice that a finitely additive measure will stay finitely additive (and therefore exhaustive) in any forcing extension. One could take the Lebesgue measure which has the added benefit of being rational valued. Thus we are lead to the following two problems.

Question 4.10. *Given a sequence $\varphi_i \geq \varphi_{i+1}$ of normalised submeasures under what conditions can we ensure that $\lim_i \varphi_i$ is not uniformly exhaustive?*

Question 4.11. *Suppose we have a finite subalgebra $\mathfrak{A}$ of $\mathbb{A}$, a submeasure φ on $\mathfrak{A}$, a finitely additive submeasure λ on $\mathbb{A}$ and a submeasure μ on $\mathbb{A}$ that is not uniformly exhaustive.*

[6] Notice that this can be seen as a 'weak* convergence' for submeasures. We are not aware of any integration theory for submeasures to really call it weak* convergence.

Suppose also that $\lambda \leq \varphi \leq \mu$ and $\lambda \leq \mu$. Can we, for each $n \in \mathbb{N}$, extend φ to an exhaustive n-pathological submeasure φ_n on $\mathbb{A}$ such that $\lambda \leq \varphi_n \leq \mu$?

A positive answer to Question 4.11 would allow us to obtain the required decreasing sequence of submeasures. The obvious way to tackle Question 4.10 is to fix from the outset a collection of partitions $a_1, a_2, \ldots$ of $\mathfrak{B}$ such that $|a_i| \geq n$ and to see to it that each φ_i is n-pathological witnessed by a_n. The methods employed in the proof of Theorem 4.1 do not allow us to do this.

5 Talagrand's ideal

The main result of this section is that in any forcing extension corresponding to Talagrand's submeasure ν, the collection of random reals will be ν-null, once ν has been computed in this extension. This is proved in Subsection 5.1 where the reader will also find a more precise statement of the above (Theorem 5.2). This is related to the problem of whether or not this forcing actually adds a random real, which we have already mentioned in the introduction, but has also been raised in the literature a number of times (see [7, Question 12], [10, Problem 3A], [36, Question 3]).

In Subsection 5.2 we give a proof that in any forcing extension due to ν, the set of ground model reals will be both Lebesgue null and meagre (Theorem 5.7). This result actually follows from the results of [8], where it is shown that the σ-ideal of Lebesgue null sets, is the only *analytic on* G_δ ideal that does not force the ground model reals to be Lebesgue null. Similarly, the σ-ideal of meagre sets is the only analytic on G_δ ideal that does not force the ground model reals to be meagre. We show that the σ-ideal corresponding to ν is indeed analytic on G_δ (see Proposition 5.10). The main ingredient of the proof given here is the result proved by Fremlin, that ν is invariant under the action of the isometry group of $\mathcal{T}$ (Proposition 5.11). The rest then follows [24].

5.1 Random reals are ν-null

We work in the context of Subsection 2.5, in particular $\mathcal{T}$ is the product space $\prod_{i\in\mathbb{N}}[2^i]$, $\mathbb{T} = \text{Clopen}(\mathcal{T})$ and $\nu : \mathbb{T} \to \mathbb{R}$ is Talagrand's submeasure. We may extend ν to a σ-subadditive submeasure on $\mathcal{P}(\mathcal{T})$ by

$$\nu(A) = \inf\{\sum_{i\in\mathbb{N}} \nu(A_i) : A_i \in \mathbb{T} \wedge A \subseteq \bigcup_{i\in\mathbb{N}} A_i\} \tag{5.1}$$

where the restriction of ν to Borel($\mathcal{T}$) is a continuous submeasure (by Lemma 3.9 and Proposition 2.24). This extension remains pathological, since any non-trivial finitely additive measure dominated by this extension, will restrict to one dominated by ν on $\mathbb{T}$. Let

$$\texttt{path} = \{A \in \mathcal{P}(\mathcal{T}) : \nu(A) = 0\}.$$

For the rest of this subsection fix a countable transitive model M of ZFC and $\mathcal{U} \in M$ such that, in M, $\mathcal{U}$ is a non-principal ultrafilter. By ν^M we mean Talagrand's submeasure as defined in M and with respect to $\mathcal{U}$. By $\texttt{path}_M$ we mean the collection (in M) of ν^M-null sets. We will also denote the complete Boolean algebra Borel($\mathcal{T}$)/$\texttt{path}$, as computed in M, by $\texttt{path}_M$. By N we mean either a countable transitive model of ZFC such that $M \subseteq N$ or V itself. By ν^N we mean ν as defined in N with respect to *any* non-principal ultrafilter $\mathcal{V}$ (in N) such that $\mathcal{U} \subseteq \mathcal{V}$. Such an ultrafilter exists since $\mathcal{U}$ will always have the finite intersection property and will not contain any finite sets, and so any non-principal extension will do. We do not know if different ultrafilters produce different ideals, nevertheless, the choice of the ultrafilter here will not matter. We let $\mathtt{path}_N$ denote the collection of ν^N-null sets. If $\mathcal{V}$ is any non-principal ultrafilter over $\mathbb{N}$ we let $\nu_{\mathcal{V}}$ be Talagrand's submeasure defined with respect to the ultrafilter $\mathcal{V}$.

Given a subset A of BC let $R(A) = \{f \in \mathcal{T} : (\forall c \in A)(A_c \in \mathtt{null} \to f \notin A_c)\}$ and $C(A) = \{f \in \mathcal{T} : (\forall c \in A)(A_c \in \mathtt{meagre} \to f \notin A_c)\}$. If H is a countable transitive model of ZFC then $R(\mathrm{BC} \cap H)$ is just the collection of random reals over H and similarly $C(\mathrm{BC} \cap H)$ is just the collection of Cohen reals over H. We prove the following.

Theorem 5.1. *Let G be a $\mathtt{path}_M$-generic filter over M. Then in $M[G]$ we have*

$(\forall \mathcal{V})((\mathcal{V}$ *is a non-principal ultrafilter on* $\mathbb{N} \wedge \mathcal{U} \subseteq \mathcal{V}) \to \nu_{\mathcal{V}}(R(\mathrm{BC} \cap M) \cup C(\mathrm{BC} \cap M)) = 0)$.

We state the following related question which is asking if the ground model reals become ν-null.

Question 5.2. $\mathtt{path} \Vdash (\forall \mathcal{V})((\mathcal{V}$ *is a non-principal ultrafilter on* $\mathbb{N} \wedge \mathcal{U} \subseteq \mathcal{V}) \to \nu_{\mathcal{V}}(\check{\mathcal{T}}) = 0)$*?*

Notice that the corresponding result for random and Cohen reals requires a Fubini property for these ideals (see [3, Theorem 3.2.39] and [24, Theorem 3.22]), which we do not have for $\mathtt{path}$ (see [7, Theorem 7]).

Towards a proof of Theorem 5.1 we first state the following result due to Christensen.

Theorem 5.3. ([4, Theorem 2]) *If $\mathfrak{B}$ is a Boolean algebra, $\mu : \mathfrak{B} \to [0,1]$ is a pathological submeasure and $\lambda : \mathfrak{B} \to [0,1]$ is finitely additive then*

$$\inf\{\mu(A) + \lambda(A^c) : A \in \mathfrak{B}\} = 0$$

In [24] two σ-ideals $\mathcal{I}$ and $\mathcal{J}$ on $\mathcal{P}(\mathcal{T})$ are called **dual** if and only if there exists a Borel set A such that $A \in \mathcal{I}$ and $\mathcal{T} \setminus A \in \mathcal{J}$.

Corollary 5.4. *If $\mathfrak{B}$ is a σ-complete Boolean algebra, $\mu : \mathfrak{B} \to [0,1]$ is a continuous pathological submeasure and $\lambda : \mathfrak{B} \to [0,1]$ is a σ-additive measure then there exists $A \in \mathfrak{B}$ such that*

$$\mu(A) = 0 = \lambda(A^c).$$

In particular, the ideals $\mathtt{null}$ *and* $\mathtt{path}$ *are dual.*

Proof. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence such that $\mu(A_n), \lambda(A_n^c) \leq 2^{-n}$. By continuity if we set $A = \bigcap_{k\in\mathbb{N}} \bigcup_{n\geq k} A_n$ we have that $\mu(A) = 0$. But

$$\lambda(A^c) = \lambda(\bigcup_{k\in\mathbb{N}} \bigcap_{n\geq k} A_n^c).$$

Now for each k, $\lambda(\bigcap_{n\geq k} A_n^c) \leq \lambda(A_n^c) \leq 2^{-n} \to 0, \quad n \to \infty$. By continuity of λ then $\lambda(A^c) = 0$. $\square$

To gain some control on the ν-null sets in a forcing extension of $\texttt{path}_M$, we have the following.

Proposition 5.5. *If $c \in \mathrm{BC} \cap M$ then $\nu^M(A_c^M) \geq \nu^N(A_c^N)$. In particular for any $c \in \mathrm{BC} \cap M$, if $A_c \cap M \in \texttt{path}_M$ then $A_c \cap N \in \texttt{path}_N$.*

Proof. Let $\mathcal{T}^*$ be the collection $\bigcup_{I \in [\mathbb{N}]^{<\omega}} \prod_{n \in I} [2^n]$. Let $\phi_1(f, \tau)$ be the formula

$$\tau \in \mathcal{T}^* \wedge f \in \mathcal{T} \wedge (\forall n \in \mathrm{dom}(\tau))(f(n) \neq \tau(n)))$$

Of course

$$f \in \bigcap_{n \in \mathrm{dom}(\tau)} S_{n,\tau(n)} \leftrightarrow \phi_1(f, \tau).$$

Since $\mathcal{T}^{*M} = \mathcal{T}^*$ and $\mathcal{T}^M = \mathcal{T} \cap M$, we have

$$(\forall \tau)(\forall f \in M)(\phi_1(f, \tau) \leftrightarrow \phi_1^M(f, \tau)).$$

So if $\tau \in \mathcal{T}^*$

$$(\bigcap_{n \in I} S_{n,\tau(n)})^M = \{f : \phi_1(f, \tau)\}^M = \{f \in M : \phi_1^M(f, \tau)\} = \bigcap_{n \in I} S_{n,\tau(n)} \cap M. \tag{5.2}$$

Let $\phi_2(x)$ be the formula

$$\begin{aligned} x \text{ is a function} \wedge \mathrm{dom}(x) = 3 \quad \wedge \quad & (\exists \tau \in \mathcal{T}^*)(\exists k)(x(0) = \bigcap_{n \in \mathrm{dom}(\tau)} S_{n,\tau(n)} \\ \wedge \quad & x(1) = |\tau| \wedge x(2) = \left(\frac{\eta(k)}{|\tau|}\right)^{\alpha(k)}). \end{aligned}$$

Of course

$$\phi_2(X) \leftrightarrow X \in \mathcal{D}.$$

By this and (5.2) we see that

$$\mathcal{D}^M = \{(A \cap M, I, w) : (A, I, w) \in \mathcal{D}\}. \tag{5.3}$$

Note that the sequences $(\eta(k))_{k \in \mathbb{N}}$ and $(\alpha(k))_{k \in \mathbb{N}}$ are in M.

Now we follow the proof of Proposition 5.11 (below) and proceed by downwards induction. Let $[k, p]$ be the statement that

$$(\mathcal{C}_{k,p}^M = \{(A^M, I, w) : (A^M, I, w) \in \mathcal{C}_{k,p}\}) \wedge (\forall A \in \mathbb{T})(\psi_{\mathcal{C}_{k,p}}^M(A^M) = \psi_{\mathcal{C}_{k,p}}^N(A^N)).$$

We show that for each $k \leq p$ the statement $[k, p]$ holds. First we show that $\psi_{\mathcal{D}}^M(A^M) = \psi_{\mathcal{D}}^N(A^N)$, this along with (5.3) will prove $[p, p]$. Suppose $\psi_{\mathcal{D}}^M(A^M) < \eta$, for some $\eta \in \mathbb{Q}_{>0}$. Then we can find $\{(X_i \cap M, I_i, w_i) : i \in [N]\} \subseteq \mathcal{D}^M$ such that $A^M = A \cap M \subseteq \bigcup_{i \in [N]} X_i \cap M = (\bigcup_{i \in [N]} X_i)^M$ and $\sum_{i \in [N]} w_i < \eta$. Thus $\{(X_i \cap N, I_i, w_i) : i \in I\} \subseteq \mathcal{D}^N$ witnesses $\psi_{\mathcal{D}}^N(A) < \eta$. The other direction is the same but just using the fact that if $\{(X_i \cap N, I_i, w_i) : i \in I\} \subseteq \mathcal{D}^N$

then $\{(X_i \cap M, I_i, w_i) : i \in I\} \subseteq \mathcal{D}^M$.

Suppose now for some $[k+1,p]$ holds. By $[k+1,p]$, for every $s \in \mathcal{T}^*$ and $B \in \mathbb{T}$, we have

$$\psi^M_{k+1,p}((\pi^{-1}_{[s]}(B))^M) = \psi^N_{k+1,p}((\pi^{-1}_{[s]}(B))^N),$$

from which it follows that

$$(\forall X \in \mathbb{T})(\ X \cap M \text{ is } (I, \psi^M_{k+1,p})\text{-thin if and only if } X \cap N \text{ is } (I, \psi^N_{k+1,p})\text{-thin}).$$

From this, arguing as in the case for $[p,p]$, we obtain $[k,p]$.

Finally, since $\mathcal{U} \subseteq \mathcal{V}$, we have for each $k \in \mathbb{N}$:

($\star$) *If* $(X \cap M, I, w) \in \mathcal{E}^M_k$ *then* $(X \cap N, I, w) \in \mathcal{E}^N_k$,

where of course $\mathcal{E}^M_k = \{(X, I, w) : \{p : (X, I, p) \in \mathcal{C}^M_{k,p}\} \in \mathcal{U}\}$ and $\mathcal{E}^N_k = \{(X, I, w) : \{p : (X, I, p) \in \mathcal{C}^N_{k,p}\} \in \mathcal{V}\}$. This completes the proof. □

Proof of Theorem 5.1. Since, in M, the ideals `path` and `null` are dual, we can find $c, d \in \mathrm{BC} \cap M$ such that that $A_c \cap M \in \mathtt{null}_M$ and $A_d \cap M = \mathcal{T} \setminus A_c \cap M \in \mathtt{path}_M$. Let G be a `path`-generic filter over M. In $M[G]$, if $f \in R(\mathrm{BC} \cap M)$ then $f \notin A_c$ so that $R(\mathrm{BC} \cap M) \subseteq A_d$. But by Proposition 5.5 we know that, since $A_d \cap M \in \mathtt{path}_M$, for any appropriate $\mathcal{V}$ we have $\nu^{M[G]}_{\mathcal{V}}(A_d \cap M[G]) = 0$. The same proof works for `meagre` (using Lemma 5.14, below), but of course we can use the fact that Borel$(\mathcal{T})$/`path` is a Maharam algebra and therefore cannot add any Cohen reals (Fact 2.34). □

Finally, let us comment on the absoluteness of the above ideals.

Definition 5.6 (**[24]**)**.** *An ideal* $\mathcal{I} \subseteq \mathcal{P}(\mathcal{T})$ *is called* **absolute** *if and only if there exists a formula, in the language of set theory,* $\phi(x)$ *such that* $\mathcal{I} = \{x \in \mathcal{P}(\mathcal{T}) : \phi(x)\}$ *and* $(\forall c \in \mathrm{BC} \cap M)(\phi(A_c) \leftrightarrow (\phi(A_c))^M)$*, for every countable transitive model* M *of ZFC).*

Both `meagre` and `null` are absolute (see [24]). With regards to the absoluteness of `path`, the problem is that the ideal `path` is defined in terms of a non-principal ultrafilter on $\mathbb{N}$, which of course is not absolute. Thus we can find an (absolute) formula $\phi(x, y)$ such that for some non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$

$$\mathtt{path} = \{A \subseteq \mathcal{P}(\mathcal{T}) : \phi(A, \mathcal{U})\}.$$

We remark that by [35], if N is a forcing extension of M due to $\mathtt{path}_M$ then $\mathcal{V}$, being maximal, contains a splitting real over M. In particular $\mathcal{V}$ will contain a subset of $\mathbb{N}$ that does not contain (as a subset) any member of $\mathcal{U}$. Thus with regards to ($\star$), in the proof of Proposition 5.5, we see that $\mathcal{E}^M_k$ will be strictly smaller than $\mathcal{E}^N_k$, this suggests that we do not have $\mathtt{null}_N = \mathtt{null}_M$. On the other hand, Fremlin has asked whether or not these ultrafilters could be replaced by actual converging sequences ([10, Problem 3B]). If this is indeed the case then it would follow that `path` is absolute.

5.2 The ground model reals are Lebesgue null and meagre

We prove the following.

Theorem 5.7. $\mathtt{path} \Vdash \check{\mathcal{T}} \in \dot{\mathtt{meagre}} \cap \dot{\mathtt{null}}$.

Recall that a set is **analytic** if and only if it is the continuous image of a Borel set (see [18, Chapter 11]). Given a set $A \subseteq \mathcal{T} \times \mathcal{T}$, for $x \in \mathcal{T}$, we define

$$A_x = \{y \in \mathcal{T} : (x, y) \in A\}.$$

An ideal $\mathcal{I}$ on Borel($\mathcal{T}$) is called **analytic on** G_δ if and only if for every G_δ set $A \subseteq \mathcal{T} \times \mathcal{T}$, the set $\{x : A_x \in \mathcal{I}\}$ is analytic. As we have already remarked, Theorem 5.7 actually follows from [8], since path is analytic on G_δ. To see this we have Proposition 5.10, below. But first we recall the following.

Definition 5.8. ([6, Page 116]) *Given two sets X and Y and two σ-algebras $\mathfrak{A} \subseteq \mathcal{P}(X)$ and $\mathfrak{B} \subseteq \mathcal{P}(Y)$, a function $f : X \to Y$ is called* $(\mathfrak{A}, \mathfrak{B})$**-measurable** *if and only if*

$$(\forall B \in \mathfrak{B})(f^{-1}(B) \in \mathfrak{A}).$$

Theorem 5.9. ([6, Pages 123 and 125]) *Let X be a set and $\mathfrak{A} \subseteq \mathcal{P}(X)$ be a σ-algebra. Suppose that for each $n \in \mathbb{N}$ we have an $(\mathfrak{A}, \mathrm{Borel}(\mathbb{R}))$-measurable map $f_n : X \to \mathbb{R}$, and in addition, for each $x \in X$ the limit $\lim_n f_n(x)$ exists. Then the map*

$$X \mapsto \mathbb{R} : x \mapsto \lim_n f_n(x)$$

is $(\mathfrak{A}, \mathrm{Borel}(\mathbb{R}))$-measurable.

In what follows call a map $f : \mathcal{T} \to \mathbb{R}$ **measurable** if and only if it is $(\mathrm{Borel}(\mathcal{T}), \mathrm{Borel}(\mathbb{R}))$-measurable. The following should be compared to [20, Theorem 17.25].

Proposition 5.10. *Let $\mu : \mathrm{Borel}(\mathcal{T}) \to \mathbb{R}$ be a Maharam submeasure. Then for each Borel set $A \subseteq \mathcal{T} \times \mathcal{T}$, the map*

$$x \mapsto \mu(A_x)$$

is measurable. In particular, by considering the preimage of $\{0\}$, the ideal Null(μ) *is analytic on G_δ (in fact* Borel on Borel*).*

Proof. For concreteness first recall the Borel hierarchy on $\mathcal{T} \times \mathcal{T}$ (see [18, Page 140]). Let Σ^0_1 be the collection of all open sets in $\mathcal{T} \times \mathcal{T}$, and let Π^0_1 be the collection of all closed sets in $\mathcal{T} \times \mathcal{T}$. If $\alpha < \omega_1$ and Σ^0_β and Π^0_β have been defined for each $\beta < \alpha$, then we let Σ^0_α be the collection of all countable unions of sets from $\bigcup_{\beta<\alpha} \Pi^0_\beta$. We let Π^0_α be the collection of all countable intersections of sets from $\bigcup_{\beta<\alpha} \Sigma^0_\beta$. Then we have

$$\mathrm{Borel}(\mathcal{T} \times \mathcal{T}) = \bigcup_{\alpha<\omega_1} \Sigma^0_\alpha = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha.$$

Each Σ^0_α and Π^0_α is closed under finite intersections and finite unions, and we also have $\Pi^0_\beta \subseteq \Pi^0_\alpha$ and $\Sigma^0_\beta \subseteq \Sigma^0_\alpha$, for each $\beta \leq \alpha < \omega_1$

Now fix a Maharam submeasure μ on Borel($\mathcal{T}$). Given $A \in$ Borel($\mathcal{T} \times \mathcal{T}$), let $[A]$ be the statement:

$$\text{The map } \mathcal{T} \to \mathcal{T} : x \mapsto \mu(A_x) \text{ is measurable.}$$

We claim that, if $\mathcal{R} \subseteq$ Borel($\mathcal{T} \times \mathcal{T}$) is such that $(\forall A \in \mathcal{R})([A])$ and $\mathcal{R}$ is closed under finite intersections, then

$$(\forall (A_i)_{i\in\mathbb{N}} \subseteq \mathcal{R})([\bigcap_i A_i]).$$

Indeed, let $(A_i)_{i\in\mathbb{N}}$ be a sequence from $\mathcal{R}$ and let $A = \bigcap_i A_i$. Since $\mathcal{R}$ is closed under finite intersections, we may assume that $A_i \supseteq A_{i+1}$, for each i. Let $f : \mathcal{T} \to \mathbb{R}$ be the map $x \mapsto \mu(A_x)$ and, for each $n \in \mathbb{N}$, let $f_n : \mathcal{T} \to \mathbb{R}$ be the map

$$x \mapsto \mu((A_n)_x).$$

By the monotonicity of μ, we have that $f_1(x) \geq f_2(x) \geq \cdots$, and since μ is Maharam we have

$$f(x) = \lim_n f_n(x).$$

By Theorem 5.9, since $(\forall i)([A_i])$ holds and therefore each f_i is measurable, we must have $[\bigcap_i A_i]$. The same argument shows that, if $\mathcal{R} \subseteq$ Borel($\mathcal{T} \times \mathcal{T}$) is such that $(\forall A \in \mathcal{R})([A])$ and $\mathcal{R}$ is closed under finite unions, then

$$(\forall (A_i)_{i\in\mathbb{N}} \subseteq \mathcal{R})([\bigcup_i A_i]).$$

Let us now show that $[A]$ holds for each open set A of $\mathcal{T} \times \mathcal{T}$. If $A = \bigcup_{i\in[n]} [s_i] \times [t_i] \subseteq \mathcal{T} \times \mathcal{T}$, for some finite sequences s_i and t_i, then for each $x \in \mathcal{T}$ and function $\mu :$ Borel($\mathcal{T}$) $\to \mathbb{R}$ we have

$$\mu(A_x) = \mu(\bigcup\{[t_i] : i \in [n] \wedge x \in [s_i]\}).$$

From this it is straightforward to see that the map $x \mapsto \mu(A_x)$ is continuous (and so measurable). Now suppose A is an open set in $\mathcal{T} \times \mathcal{T}$. Then we can find finite sequences $(s_i)_{i\in\mathbb{N}}$ and $(t_i)_{i\in\mathbb{N}}$ such that $A = \bigcup_{i\in\mathbb{N}} [s_i] \times [t_i]$. For each n, let $A_n = \bigcup_{i\in[n]} [s_i] \times [t_i]$. Then, by the above applied to $\mathcal{R} := \{A_n : n \in \mathbb{N}\}$, we see that $[A]$ holds.

Thus we can now work our way up the Borel hierarchy.[7] Since for each $A \in \Sigma^0_1$ we have $[A]$, and Σ^0_1 is closed under finite intersections, we know (by the above arguments) that $[A]$ holds, for each $A \in \Pi^0_1$. Proceeding in this way along ω_1, we see that $(\forall A \in$ Borel$(\mathcal{T} \times \mathcal{T}))([A])$, and we are done. $\square$

[7] Professor G. Plebanek has pointed out to me that at this point we may apply the Monotone Class Theorem (see [15, Theorem 6B]), and by doing so avoid any commentary on the Borel hierarchy.

The proof of Theorem 5.7 here uses the following.

Proposition 5.11. **([10, Proposition N])** *If g is a bijective isometry of $\mathcal{T}$ then*

$$(\forall E \in \mathbb{T})(\nu(g[E]) = \nu(E)).$$

We remark that in [10], Fremlin has made some modifications to the definitions found in [33]. It is not clear to us that Proposition 5.11 is valid for Talagrand's construction as it appears in [33]. It is clear that the result holds true if one considers only isometries that are pointwise defined by permutations of the sets $[2^n]$.[8] It is not difficult to find other bijective isometries (see, for example, Remark 7.33). This is enough for our purposes.

Clearly, from (5.1), this invariance lifts to $\mathcal{P}(\mathcal{T})$ and in particular we have

$$(\forall A \in \mathrm{Borel}(\mathcal{T}))(\forall f \in \mathcal{T})(\nu(A + f) = \nu(A)).$$

This is actually saying that `path` is **0-1-invariant** according to [24]. The same is true for `meagre` and `null` (again see [24]).

We will also need the following, which is just the observation that Theorem 3.20 from [24] goes through without the absoluteness for the ideal that one is forcing with.

Theorem 5.12. *Let $\mathcal{J}$ be a* 0-1*-invariant absolute ideal on $\mathcal{T}$. Let M be a countable transitive model of ZFC and consider $\mathtt{path}_M$ as in the previous subsection. Let G be $\mathtt{path}_M$-generic over M. If ($\mathtt{path}_M$ and $\mathcal{J}$ are dual)^M then $(\mathcal{T} \cap M \in \mathcal{J})^{M[G]}$.*

We prove Theorem 5.12 at the end of this subsection, but from this and Corollary 5.4, we may already conclude that

$$\mathtt{path} \Vdash \check{\mathcal{T}} \in \dot{\mathtt{null}}. \tag{5.4}$$

Let us now show that the ideals `meagre` and `path` are also dual.

Lemma 5.13. *For every $A \in \mathrm{Borel}(\mathcal{T}) \setminus \mathtt{path}$ there exists $B \in (\mathrm{Borel}(\mathcal{T}) \cap \mathtt{meagre}) \setminus \mathtt{path}$ such that $B \subseteq A$.*

Proof. Suppose that for some $A \in \mathrm{Borel}(\mathcal{T}) \setminus \mathtt{path}$ we have $\mathrm{Borel}(\mathcal{T}) \cap \mathtt{meagre} \cap \mathcal{P}(A) \subseteq \mathtt{path}$. Let $\dot{r}$ be a name, via Propositions 2.29 and 2.30, such that

$$\mathtt{path} \Vdash (\forall c \in \check{\mathrm{BC}})(\dot{A}_c \cap \check{\mathcal{T}} \in \check{\mathtt{path}} \to \dot{r} \notin A_c) \tag{5.5}$$

(i.e. $\dot{r}$ is the name determined by `path`). We claim that

$$A \Vdash \dot{r} \text{ is a Cohen real.} \tag{5.6}$$

[8] That is, isometries $F : \mathcal{T} \to \mathcal{T}$, such that for some sequence of permutations $h_n : [2^n] \to [2^n]$ we have $F(f)(n) = h_n(f(n))$.

If not then for some $B \subseteq A$ and some $c \in \mathrm{BC}$ with $A_c \in \mathtt{meagre}$ we have $B \Vdash \dot{r} \in A_c$. If $d \in \mathrm{BC}$ is such that $B = A_d$ then $B \Vdash \dot{r} \in A_d \cap A_c$. Let $e \in \mathrm{BC}$ be such that $A_e = A_c \cap A_d$. But then

$$A_c \cap A_d \in \mathrm{Borel}(\mathcal{T}) \cap \mathtt{meagre} \cap \mathcal{P}(A) \subseteq \mathtt{path}.$$

In particular $B \Vdash \check{A}_e \in \mathtt{path} \wedge \dot{r} \in A_e$, which contradicts (5.5). Thus (5.6) holds which contradicts the fact that $\mathtt{path} \Vdash$ 'there are no Cohen reals over M' (Fact 2.34). $\square$

Lemma 5.14. *The ideals* $\mathtt{path}$ *and* $\mathtt{meagre}$ *are dual.*

Proof. Use Lemma 5.13 to find for each $A \in \mathrm{Borel}(\mathcal{T}) \setminus \mathtt{path}$ a meagre Borel set $\Gamma(A) \notin \mathtt{path}$ such that $\Gamma(A) \subseteq A$. Let $B_1 = \Gamma(\mathcal{T})$. If B_β for $\beta < \alpha < \omega_1$ has been constructed let

$$B_\alpha = \begin{cases} \Gamma(\mathcal{T} \setminus (\bigcup_{\beta<\alpha} B_\beta)), & \text{if } \mathcal{T} \setminus (\bigcup_{\beta<\alpha} B_\beta) \notin \mathtt{path} \\ \emptyset, & \text{otherwise} \end{cases}$$

Since $\mathrm{Borel}(\mathcal{T})/\mathtt{path}$ is ccc (Fact 2.14), we know that $B := \{B_\alpha : \alpha < \omega_1 \wedge B_\alpha \notin \mathtt{path}\}$ is countable. Thus $\mathcal{T} \setminus \bigcup B \in \mathtt{path}$ and $\bigcup B \in \mathtt{meagre}$, since each $B_\alpha \in \mathtt{meagre}$. $\square$

By Lemma 5.14 and Theorem 5.12 we obtain

$$\mathtt{path} \Vdash \check{\mathcal{T}} \in \check{\mathtt{meagre}}, \tag{5.7}$$

which, along with (5.4), proves Theorem 5.7.

Proof of Theorem 5.12. Let $c \in \mathrm{BC} \cap M$ be such that $A_c \cap M \in \mathcal{J}^M$ and $(\mathcal{T} \setminus A_c) \cap M \in \mathtt{path}_M$. Let $F = \dot{r}^G \in M[G]$ be the generic real determined $\mathtt{path}_M$ (as outlined in Subsection 2.6.2). Let

$$B = A_c \cap M[G] - F := \{f - F : f \in A_c \cap M[G]\}.$$

Since $\mathcal{J}$ is 0-1 invariant and absolute, $B \in \mathcal{J}^{M[G]}$. Let $d \in \mathrm{BC} \cap M[G]$ be such that, in $M[G]$, $A_d = B$. We claim that we have $\mathcal{T} \cap M \subseteq A_d \cap M$. Indeed, let $h \in \mathcal{T} \cap M$. In M, we can find a Borel code e such that $A_e = (\mathcal{T} \setminus A_c) - h = \mathcal{T} \setminus (A_c - h)$. Since $\mathtt{path}_M$ is 0-1 invariant we know that $A_e \cap M \in \mathtt{path}_M$. Thus in $M[G]$, $F \notin \mathcal{T} \setminus (A_c - h)$ and so $F \in A_c - h$. By definition it follows that $h \in B$. $\square$

6 Talagrand's ψ

This section is motivated by the fact that the values the Lebesgue measure λ takes on Clopen(2^ω) are easily calculable. Indeed, if $A \in$ Clopen(2^ω) then we know that for some $n \in \omega$ we have

$$\lambda(A) = |\{s \in {}^n2 : [s] \subseteq A\}| \cdot 2^{-n}.$$

Many results about Cohen and random reals rely on the fact that Cohen forcing has a very transparent tree representation (see for example [7]) and that the Lebesgue measure is easily calculable (see for example [3, Section 2.5.A]). With the forcing associated to Talagrand's construction one does not, yet, have either; a nice tree representation or any control over the defining submeasure.

We would like to know if it is possible to find an explicit description of Talagrand's submeasure analogous to the one we have for the Lebesgue measure. Actually we do not consider Talagrand's submeasure at all since we did not get so far. Instead, and as a start, we attempt to calculate explicit values for the first (pathological) submeasure constructed in [33], this is the submeasure denoted by ψ in Subsection 2.5 (and in [33]). We remark that in [33] the value $\eta(k)$ was set to $2^{2k+10}2^{(k+5)^4}(2^3 + 2^{k+5}2^{(k+4)^4})$. As pointed out by Talagrand anything larger will do (see [31, Page 8]), so for simplicity we take the value

$$\eta(k) = 2^{2500k^4}$$

(see Inequality 1, on page 62). We start by trying to measure the entire space and in Subsection 6.1 we show that

$$\psi(\mathcal{T}) = \eta(1)^{\alpha(1)} = 2^{\frac{2500}{216}}$$

(see (6.6)). In Subsection 6.2 we try to measure sets of the form $[s]$, for $s \in \prod_{n\in I}[2^n]$ and $I \in [\mathbb{N}]^{<\omega}$. We show that

$$\psi([s]) = \min\{2^{-\delta(|I|)+1}, 2^{-\delta(|I|)}\left(\frac{\eta(\delta(|I|))}{|I|}\right)^{\alpha(\delta(|I|))}\},$$

where $\delta(m) = \min\{n \in \mathbb{N} : \eta(n) \geq m\}$ (see (6.8)). We were unable to measure more complicated sets.

Subsection 6.3 consolidates certain inequalities that we will call upon throughout.

Finally let us introduce a diagram which might be helpful to the reader, and which was certainly how we arrived at most (if not all) of what follows in this section. We will not mention them again and so the uninterested reader need only pay attention to the next definition before moving on.

Definition 6.1. *For $X \in \mathbb{T}$ we say that X is a* **$\mathcal{D}$-set** *if and only if for some (non-empty)* *finite set $I \subseteq \mathbb{N}$ and some $\tau \in \prod_{n\in I}[2^n]$ we have*

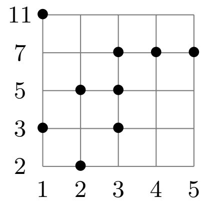
$$X = \bigcap_{n\in I}\{y \in \mathcal{T} : (\forall n \in I)(y(n) \neq \tau(n)\}) = \bigcap_{n\in I} S_{n,\tau(n)}.$$

Since we can recover I and τ from X we allow ourselves to denote I by X^{Ind} and $\tau(n)$ by $X(n)$.

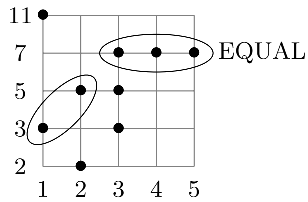
If $X = \{X_1, ..., X_n\}$ is a collection of $\mathcal{D}$-sets then we may represent X by a diagram as follows. We first consider the grid $I \times \bigcup_{i=1}^n X_i^{\mathrm{Ind}}$ and then place a mark on each point (i, m) of our grid such that $m \in X_i^{\mathrm{Ind}}$. As an example suppose, in the above, that $n = 5$ and

$$X_1^{\mathrm{Ind}} = \{3, 11\}, X_2^{\mathrm{Ind}} = \{2, 5\}, X_3^{\mathrm{Ind}} = \{3, 5, 7\}, X_4^{\mathrm{Ind}} = \{7\}, X_5^{\mathrm{Ind}} = \{7\}.$$

Then X can be represented by:



Of course these diagrams do not tell us anything about the particular values $X_i(l)$, but these particular values won't matter. What will matter is whether these values are constant or all distinct across each row of our diagram. For example suppose in the above diagram we know that $X_3(7) = X_4(7) = X_5(7)$ (so X is constant across row 7). Then we can very quickly spot from this:



that $\bigcup_i X_i$ is *not* a cover of $\mathcal{T}$. This is because for any $s \in \mathcal{T}$ such that $s(3) = X_1(3)$, $s(5) = X_2(5)$ and $s(7) = X_3(7) = X_4(7) = X_5(7)$, the sequence s will not be a member of $\bigcup_i X_i$.

6.1 Measuring the entire space

We begin with with the following natural definition.

Definition 6.2. *Let $A \subseteq \mathcal{T}$, X a collection of $\mathcal{D}$-sets and $Y \in [\mathcal{D}]^{<\omega}$. We say that X (resp. Y) is a* **cover of** *A if and only if $A \subseteq \bigcup X$ (resp. $A \subseteq \bigcup Y$). We say that X (resp. Y) is a* **proper cover of** *A if and only if it is a cover of A and for any $X' \subsetneq X$ (resp. $Y' \subsetneq Y$)*

$$A \not\subseteq \bigcup X' (\text{resp. } A \not\subseteq \bigcup Y').$$

Clearly then given $A \subseteq \mathcal{T}$ we have

$$\psi(A) = \inf\{w(X) : X \subseteq \mathcal{D} \text{ and } X \text{ properly covers } A\}. \tag{6.1}$$

The idea of this subsection (and indeed the rest of this section) is as follows. For each proper cover X of $\mathcal{T}$ we find another cover Y of $\mathcal{T}$ of lower weight, where the Y here will have a very regular structure (it will be a *rectangle*) and so will have an easily calculable weight. Of course it will be sufficient to consider the infimum over all such regular structures.

Definition 6.3. *For any $n \in \mathbb{N}$ let*

$$\delta(n) = \min\{k \in \mathbb{N} : \eta(k) \geq n\}$$

and

$$w(n) = 2^{-\delta(n)} \left(\frac{\eta(\delta(n))}{n}\right)^{\alpha(\delta(n))}.$$

If X is a finite collection of $\mathcal{D}$-sets then we will denote the **weight** *of X by*

$$w(X) = \sum_{Y \in X} w(|Y^{\mathrm{Ind}}|).^{9}$$

By Inequality 2 (page 63), we see that if X is a $\mathcal{D}$-set then $w(|I(X)|)$ will be the least weight that we can possibly attach to it. Specifically, we will always have $(X, I(X), w(|I(X)|)) \in \mathcal{D}$ and, if $(X, I(X), w) \in \mathcal{D}$ then $w \geq w(|I(X)|)$.

Here is the regular structure we mentioned above.

Definition 6.4. *Let $X = \{X_i : i \in I\}$ be a collection of $\mathcal{D}$-sets. We call X an N-***rectangle** *for some integer $N \geq 2$ if and only if the following hold:*

- *$|I| = N$;*
- *$X_i^{\mathrm{Ind}} = X_j^{\mathrm{Ind}}$ for all $i, j \in I$;*
- *$X_i(m) \neq X_j(m)$, whenever $i \neq j$ and $m \in X_i^{\mathrm{Ind}}$;*
- *$|X_i^{\mathrm{Ind}}| = N - 1$ for all (any) $i \in I$.*

Notice that the weight of an N-rectangle is given by

$$N \cdot w(N-1). \tag{6.2}$$

Rectangles give rise to proper covers of $\mathcal{T}$:

Lemma 6.5. *If $X := \{X_i : i \in I\}$ is an N-rectangle then X is a proper cover of $\mathcal{T}$.*

[9] We now are using the term *weight* for $\mathcal{D}$-sets and members of $\mathcal{D}$, but with two different meanings.

Proof. Assume that $x \in \mathcal{T} \setminus \bigcup_i X_i$. Then for each i we can find an $m_i \in X_i^{\text{Ind}}$ such that $x(m_i) = X_i(m_i)$. These m_i must be distinct for if $i \neq j$ and $m := m_i = m_j$, then $X_i(m) = x(m) = X_j(m)$, contradicting the third item from Definition 6.4. But then $\{m_1, ..., m_N\} \subseteq X_i^{\text{Ind}}$, for some i, a (cardinality) contradiction. To see that this cover is proper let J be a non-empty strict subset of $\{1, 2, ..., N\}$. Then $|J| \leq N - 1 = |X_i^{\text{Ind}}|$, for each $i \in J$. Enumerate

$$J = \{a_1, a_2, ..., a_k\}.$$

Inductively, choose $b_1 \in X_{a_1}^{\text{Ind}}$, $b_2 \in X_{a_2}^{\text{Ind}} \backslash \{b_2\}$, $b_3 \in X_{a_3}^{\text{Ind}} \backslash \{b_1, b_2\}, ..., b_k \in X_{a_k}^{\text{Ind}} \backslash \{b_1, ..., b_{k-1}\}$. Now define $y \in \prod_{i \in J} [2^{b_i}]$ by

$$y_i = \begin{cases} X_{a_i}(b_i), & \text{if } i \in \{b_1, ..., b_k\}; \\ 1, & \text{if } i \notin J. \end{cases}$$

and note that $y \in \mathcal{T} \setminus \bigcup_{i \in J} X_i$. □

Given a proper cover of $\mathcal{T}$ we claim that we can find an N-rectangle of lower weight. Before we can demonstrate this we need one more claim.

Lemma 6.6. *Let $X = \{X_i : i \in I\}$ be a collection of $\mathcal{D}$-sets that properly covers $\mathcal{T}$. Then*

$$|\bigcup_{i \in I} X_i^{\text{Ind}}| \leq |I| - 1.$$

Proof. For each $i \in I$ let $I_i = X_i^{\text{Ind}}$. Recall that a *complete system of distinct representatives* for $\{I_i : i \in I\}$ (a CDR) is an injective function $F : I \to \bigcup_{i \in I} I_i$ such that $(\forall i \in I)(F(i) \in I_i)$, and that by Hall's marriage theorem a CDR exists if and only if

$$(\forall J \subseteq I)(|J| \leq |\bigcup_{i \in J} I_i|),$$

see [14]. Clearly if a CDR existed for $\{I_i : i \in I\}$ then $\bigcup_{i \in I} X_i$ would not cover $\mathcal{T}$ (just argue as in the proof of Lemma 6.5). So for some $J \subseteq I$ we have $|\bigcup_{i \in J} I_i| \leq |J| - 1$. Assume that $|J|$ is as large as possible so that

$$(J' \subseteq I \wedge |J'| > |J|) \to (|J'| \leq |\bigcup_{i \in J'} I_i|). \tag{6.3}$$

If $J = I$ then we are done. So we may assume that $J \subsetneq I$. Since X is a proper cover of $\mathcal{T}$ there exists $t \in \mathcal{T}$ such that $t \notin \bigcup_{i \in J} X_i$. For $i \in I \setminus J$ let $I_i' = I_i \setminus \bigcup_{j \in J} I_i$. Suppose that $\{I_i' : i \in I \setminus J\}$ has a CDR $F : I \setminus J \to \bigcup_{i \in I \setminus J} I_i'$. Let $s \in \prod_{k \in \text{ran}(F)} [2^k]$ be defined by $s(k) = X_{F^{-1}(k)}(k)$. Then the function $(t \setminus \{(k, t(k)) : k \in \text{ran}(F)\}) \cup s \notin \bigcup_{i \in I} X_i$, which is a contradiction. Thus no such CDR can exist and so by Hall's theorem again, we may find a $J' \subseteq I \setminus J$ such that $|\bigcup_{i \in J'} I_i'| \leq |J'| - 1$. But then

$$|\bigcup_{i \in J \cup J'} I_i| = |\bigcup_{i \in J} I_i \cup \bigcup_{i \in J'} I_i'| \leq |J| - 1 + |J'| - 1 \leq |J| + |J'| - 1 = |J \cup J'| - 1.$$

But $|J \cup J'| > |J|$, contradicting (6.3). $\square$

Proposition 6.7. *For every proper cover of $\mathcal{T}$ there exists an N-rectangle of lower weight.*

Proof. Let $X = \{(X, I_i, w_i) : i \in [M]\}$ is a proper cover of $\mathcal{T}$ and assume that I_1 is such that $(\forall i \in [M])(w(|I_1|) \leq w(|I_i|))$. By Lemma 6.6 we have

$$(\forall i)(|I_i| + 1 \leq |\bigcup_{i \in [N]} I_i| + 1 \leq M). \tag{6.4}$$

So if Y is an $|I_1| + 1$-rectangle we get:

$$w(X) \geq \sum_{i \in [M]} w(|I_i|) \geq Mw(|I_1|) \geq (|I_1| + 1)w(|I_1|) \overset{(6.2)}{=} w(Y).$$

$\square$

Thus we have

$$\psi(\mathcal{T}) = \inf\{w(X) : X \text{ is an } N\text{-rectangle, for some } N\}. \tag{6.5}$$

But by Inequality 6 (page 64) we see that $\psi(\mathcal{T})$ is just the weight of a 2-rectangle, that is to say,

$$\psi(\mathcal{T}) = \eta(1)^{\alpha(1)}. \tag{6.6}$$

6.2 Measuring an atom

Throughout this subsection fix a non-empty finite subset $\mathcal{I}$ of $\mathbb{N}$ and an $\tau \in \prod_{i \in \mathcal{I}}[2^i]$. In this subsection we try to measure $A := [\tau]$.

Note that as in the previous subsection

$$\psi(A) = \inf\{w(X) : X \subseteq \mathcal{D} \text{ is a proper cover of } A\}.$$

The idea here is the same as before but instead of rectangles we use the following analogue of Definition 6.4 and also Definition 6.11, below.

Definition 6.8. *Let $X := \{X_i : i \in I\}$ be a collection of $\mathcal{D}$-sets. We call X a (J, S, N)-***rectangle** *for some non-empty finite subset J of $\mathbb{N}$, $S \subseteq \prod_{j \in J}[2^j]$ and integer $N \geq 2$ if and only if the following hold:*

- *$J \subsetneq X_i^{\mathrm{Ind}}$, always;*
- *$\{\bigcap_{l \in X_i^{\mathrm{Ind}} \setminus J} S_{l, X_i(l)} : i \in I\}$ is an N-rectangle;*
- *$(\forall s \in S)(\forall i \in I)(\forall j \in J)(X_i(j) \neq s(j))$.*

*In the case that $S = \{s\}$, we shall call X a (J, s, N)-***rectangle***.*

For example, in the case that $J = [m]$, for some $m \in \mathbb{N}$, this new type of rectangle is just an old rectangle with m rows attached to the bottom (most likely with a gap) where the values of the determining sequences along these rows miss the corresponding values of s.

Of course the weight of a (J, s, N)-rectangle is given by

$$N \cdot w(|J| + N - 1)$$

Lemma 6.9. *Every $(\mathcal{I}, \tau, N)$-rectangle covers A.*

Proof. Let $X = \{X_i : i \in I\}$ be an $(\mathcal{I}, \tau, N)$-rectangle, as in the above statement. Assume that we can find a $y \in A \setminus \bigcup X$. The assumption that $y \notin X$ cannot be witnessed by $y(i)$ for some $i \in \mathcal{I}$ since for each such i, we have $y(i) = \tau(i) \neq X_j(i)$, for each $j \in I$. In particular y witnesses that $Y = \{\bigcap_{l \in X_i^{\mathrm{Ind}} \setminus \mathcal{I}} S_{l, X_i(l)} : i \in I\}$ does not cover $\mathcal{T}$, which contradicts Lemma 6.5 and the fact that Y is an N-rectangle. $\square$

Next we see how to use Lemma 6.6 in this new situation and adapt what we have already done with $\psi(\mathcal{T})$ to $\psi(A)$ (compare (6.4) above, and (6.10) below).

Lemma 6.10. *If $X = \{X_i : i \in I\}$ is a proper cover of A such that $(\forall i \in \mathcal{I})(X_i^{\mathrm{Ind}} \setminus \mathcal{I} \neq 0)$ then $\{\bigcap_{l \in X_i^{\mathrm{Ind}} \setminus \mathcal{I}} S_{l, X_i(l)} : i \in I\}$ is a proper cover of $\mathcal{T}$*

Proof. For each $i \in I$, let $I_i = X_i^{\mathrm{Ind}}$. Let $X_i' = \bigcap_{l \in I_i \setminus \mathcal{I}} S_{l, X_i(l)}$ and lets show that $Y := \{X_i' : i \in I\}$ is a cover of $\mathcal{T}$. Suppose not and let $x \in \mathcal{T} \setminus \bigcup Y$. Thus for every $i \in I$ there exists an $m_i \in X_i^{\mathrm{Ind}} \setminus \mathcal{I}$ such that $x(m_i) = X_i(m_i)$. Let $y \in A$ be such that $y(j) = x(j)$, for each $j \in \{m_i : i \in I\}$. Then it is straightforward to see that $y \notin \bigcup X$, which contradicts the assumption that X is a cover of A. Suppose now that Y is not proper. Then there exists $I' \subsetneq I$ such that $\{X_i' : i \in I'\}$ is a cover of $\mathcal{T}$. But then $\{X_i : i \in I'\}$ is a cover of A, contradicting the properness of X. $\square$

Definition 6.11. *A $\mathcal{D}$-set X is a (I, S, J)-**spike** for some non-empty finite subset I of $\mathbb{N}$, $S \subseteq \prod_{j \in I} [2^j]$ and $J \subseteq I$ if and only if X is of the form*

$$X = \bigcap_{j \in J} S_{j, t(j)} \tag{6.7}$$

*such that $t \in \prod_{j \in J} [2^j]$ and $(\forall s \in S)(\forall j \in J)(t(j) \neq s(j))$. In the case that $S = \{s\}$, we shall call X an (I, s, J)-**spike**.*

Of course, every $(\mathcal{I}, \tau, J)$-spike covers A.

Proposition 6.12. *For every proper cover of A there exists an $(\mathcal{I}, \tau, J)$-spike of lower weight.*

Assuming this for now we obtain

$$\begin{aligned} \psi(A) &= \min\{w(X) : X \text{ is an } (\mathcal{I}, \tau, J)\text{-spike for some } J \subseteq \mathcal{I}\} \\ &= \min\{2^{-\delta(|\mathcal{I}|)+1}, w(|\mathcal{I}|)\}. \end{aligned} \tag{6.8}$$

Proof of Lemma 6.12. Let $X = \{(X_i, I_i, w_i) : i \in [N]\}$ be a proper cover of A and let $m = |\mathcal{I}|$. If there exists $i \in [N]$ such that $|I_i| \le m$ then any $(\mathcal{I}, \tau, J)$-spike such that $|J| = |I_i|$ will have a lower weight than X and will cover A and we will be done. So we may assume that

$$(\forall i \in [N])(|I_i| > m). \tag{6.9}$$

By Lemma 6.10 and Lemma 6.6 we get

$$(\forall i \in [N])(|I_i| \le N + m - 1). \tag{6.10}$$

We now divide the proof into the following cases.

- $\delta(N + m - 1) = 1$. Then

$$w(X) \ge \sum_{i \in [N]} 2^{-1} \left(\frac{\eta(1)}{|I_i|} \right)^{\alpha(1)} \overset{(6.10)}{\ge} N 2^{-1} \left(\frac{\eta(1)}{N + m - 1} \right)^{\alpha(1)},$$

  and this lower bound can be achieved by any $(\mathcal{I}, \tau, N)$-rectangle.

- $\delta(N + m - 1) > 1$. Let $\delta_1 = \delta(N + m - 1) - 1$, $\delta_2 = \delta(N + m - 1)$, $J_1 = \{i \in [N] : \delta(|I_i|) \le \delta_1\}$ and $J_2 = [N] \setminus J_1$. Of course

$$\eta(\delta_1) < N + m - 1 \le \eta(\delta_2). \tag{6.11}$$

  Notice that if $2 > \eta(\delta_1) - m + 1$ then

$$(\forall i \in [N])(\eta(\delta_1) \overset{(6.11)}{\le} m \overset{(6.9)}{<} |I_i| \le N + m - 1 \le \eta(\delta_2)),$$

  and so

$$w(X) \ge \sum_{i \in [N]} 2^{-\delta_2} \left(\frac{\eta(\delta_2)}{|I_i|} \right)^{\alpha(\delta_2)} \overset{(6.10)}{\ge} N 2^{-\delta_2} \left(\frac{\eta(\delta_2)}{N + m - 1} \right)^{\alpha(\delta_2)},$$

  which can be achieved by any $(\mathcal{I}, \tau, N)$-rectangle. So we may assume that

$$2 \le \eta(\delta_1) - m + 1.$$

  By Inequality 3 we have

$$w(X) = \sum_{i \in J_1} w_{|I_i|} + \sum_{i \in J_2} w_{|I_i|} \ge |J_1| 2^{-\delta_1} + |J_2| 2^{-\delta_2} \left(\frac{\eta(\delta_2)}{N + m - 1} \right)^{\alpha(\delta_2)}.$$

  – $2^{-\delta_2} \left(\frac{\eta(\delta_2)}{N+m-1} \right)^{\alpha(\delta_2)} \le 2^{-\delta_1}$. Then

$$w(X) \ge N 2^{-\delta_2} \left(\frac{\eta(\delta_2)}{N + m - 1} \right)^{\alpha(\delta_2)},$$

which can be achieved by any $(\mathcal{I}, \tau, N)$-rectangle..

– $2^{-\delta_2}\left(\frac{\eta(\delta_2)}{N+m-1}\right)^{\alpha(\delta_2)} > 2^{-\delta_1}$. Then

$$w(X) \geq N2^{-\delta_1} \overset{(6.11)}{>} (\eta(\delta_1) - m + 1)2^{-\delta_1}.$$

But this can be achieved by any $(\mathcal{I}, \tau, \eta(\delta_1) - m + 1)$-rectangle since

$$w(\eta(\delta_1){-}m{+}1) = 2^{-\delta(\eta(\delta_1))}\left(\frac{\eta(\delta(\eta(\delta_1)))}{\eta(\delta_1)}\right)^{-\alpha(\delta(\eta(\delta_1)))} = 2^{-\delta_1}\left(\frac{\eta(\delta_1)}{\eta(\delta_1)}\right)^{-\alpha(\delta_1)} = 2^{-\delta_1}.$$

Now, by Inequality 7, any $(\mathcal{I}, \tau, \mathcal{I})$-spike has a lower weight than any $(\mathcal{I}, \tau, k)$-rectangle, and this completes the proof. $\square$

We may generalise Proposition 6.12 slightly as follows.

Definition 6.13. *A collection $\mathcal{A} \subseteq \prod_{n\in[m]}[2^n]$ is called l-**empty**, for $l \in [m]$, if and only if $[2^l] \setminus \{s(l) : s \in \mathcal{A}\} \neq 0$. If $I \in \mathcal{P}([m])^+$, then $\mathcal{A}$ is I-**empty** if and only if $I = \{l : \mathcal{A}$ is l-empty$\}$. If no such I exists then call $\mathcal{A}$ **full**.*

Definition 6.14. *Let $\mathcal{A} \subseteq \prod_{n\in[m]}[2^m]$ and X a finite collection of $\mathcal{D}$ -sets properly covering $\bigcup\{[s] : s \in \mathcal{A}\}$. Call X a **hereditary cover** of $\mathcal{A}$ if and only if X properly covers each $[s]$ for $s \in \mathcal{A}$.*

Now the proof of the following is exactly the same as that of Proposition 6.12.

Proposition 6.15. *If $\mathcal{A} \subseteq \prod_{n\in[m]}[2^n]$ is is I-empty then for any hereditary cover of $\mathcal{A}$ there exists an $(I, \mathcal{A}, J)$-spike, of lower weight.*

6.3 Inequalities

Here we provide the various inequalities that are needed for the previous subsections. These identities were motivated by Figure 1, below, which was computed using the numerical computing package MATLAB. We used the symbolic manipulation package Maple to carry out the computations in the proofs of Inequalities 1, 2 and 5.

Inequality 1. *For each $k \in \mathbb{N}$, we have*

$$2^{2k+10}2^{(k+5)^4}(2^3 + 2^{k+5}2^{(k+4)^4}) \leq 2^{2500k^4}.$$

Proof. We have

$$\begin{aligned} 2^{2k+10}2^{(k+5)^4}(2^3 + 2^{k+5}2^{(k+4)^4}) &\leq 2^{2k+10}2^{(k+5)^4}(2^{(k+4)^4} + 2^{k+5}2^{(k+4)^4}) \\ &\leq 2^{2k+10}2^{(k+5)^4}2^{(k+4)^4}(1 + 2^{k+5}) \\ &\leq 2^{2k+10}2^{(k+5)^4}2^{(k+4)^4}2^{k+6} \\ &\leq 2^{2k+10}2^{(k+5)^4}2^{(k+4)^4}2^{k+6} \end{aligned}$$

$$
\begin{aligned}
&\leq 2^{4(k+5)^4} \\
&\leq 2^{4k^4+80k^3+600k^2+2000k+2500} \leq 2^{2500k^4}.
\end{aligned}
$$

□

Inequality 2. *For each $k \in \mathbb{N}$ and $n \in [\eta(k)]$*

$$2^{-k}\left(\frac{\eta(k)}{n}\right)^{\alpha(k)} < 2^{-(k+1)}\left(\frac{\eta(k+1)}{n}\right)^{\alpha(k+1)}.$$

Proof. Since $n^{\alpha(k+1)-\alpha(k)} < 1$ for each n it is enough to show that $1 \leq \frac{1}{2}\frac{\eta(k+1)^{\alpha(k+1)}}{\eta(k)^{\alpha(k)}}$. Thus it is enough to show that $2^{(k+5)^3(k+6)^3+2500k^4(k+6)^3} \leq 2^{2500(k+1)^4(k+5)^3}$. But

$$
\begin{aligned}
& (k+5)^3(k+6)^3 + 2500k^4(k+6)^3 - 2500(k+1)^4(k+5)^3 \\
= \; & -285500 - 1407800k - 2648910k^2 - 2524189k^3 \\
- \; & 757047k^4 - 82467k^5 - 2499k^6 < 0.
\end{aligned}
$$

□

Inequality 3. *For $\delta_1, \delta_2, k \in \mathbb{N}$ such that $\delta_1 \leq \delta_2$ and $k \in [\eta(\delta_1)]$ we have*

$$2^{-\delta_1}\left(\frac{\eta(\delta_1)}{k}\right)^{\alpha(\delta_1)} \geq 2^{-\delta_2}.$$

Proof. We have

$$2^{(\delta_2-\delta_1)(\delta_1+5)^3}\eta(\delta_1) \geq \eta(\delta_1) \geq k,$$

which implies the desired inequality. □

Inequality 4. *Let $N, M, \delta_1, \delta_2 \in \mathbb{N}$ be such that $2 \leq N \leq M$ and $\delta_1 \leq \delta_2$. Then*

$$\frac{M}{N}\frac{(N-1)^{\alpha(\delta_1)}}{(M-1)^{\alpha(\delta_2)}} \geq 1.$$

Proof. If $N = M$ then we are done, so assume $N < M$. Since $\alpha(\delta_1) \geq \alpha(\delta_2)$

$$\frac{M}{N}\frac{(N-1)^{\alpha(\delta_1)}}{(M-1)^{\alpha(\delta_2)}} \geq \frac{M}{N}\frac{(N-1)^{\alpha(\delta_1)}}{(M-1)^{\alpha(\delta_1)}}.$$

So it is enough to show that

$$\left(\frac{M}{N}\right)^{(\delta_1+5)^3} \geq \frac{M-1}{N-1}.$$

Let $n = M - N > 0$ so that

$$\left(\frac{M}{N}\right)^{(\delta_1+5)^3} = \left(\frac{N+n}{N}\right)^{(\delta_1+5)^3} > \frac{1}{N^{(\delta_1+5)^3}}(N^{(\delta_1+5)^3} + (\delta_1+5)^3 N^{(\delta_1+5)^3-1}n) = 1 + (\delta_1+5)^3 N^{-1} n.$$

Thus it is enough to show that $(N-1)(1+(\delta_1+5)^3N^{-1}n) \geq N+n-1$. Now

$$\begin{aligned}
(N-1)(1+(\delta_1+5)^3N^{-1}n) &= N+(\delta_1+5)^3n-1-(\delta_1+5)^3N^{-1}n \geq N+n-1 \\
&\leftrightarrow (\delta_1+5)^3n-(\delta_1+5)^3N^{-1}n \geq n \\
&\leftrightarrow (\delta_1+5)^3(1-N^{-1}) \geq 1,
\end{aligned}$$

so we are done. □

Inequality 5. *Let $\delta_1, \delta_2 \in \mathbb{N}$ be such that $\delta_1 \leq \delta_2$. Then*

$$2^{\delta_1-\delta_2}\frac{\eta(\delta_2)^{\alpha(\delta_2)}}{\eta(\delta_1)^{\alpha(\delta_1)}} \geq 1.$$

Proof. If $\delta_1 = \delta_2$ then we are done, so assume $\delta_1 < \delta_2$. It is enough to show that

$$(\delta_1-\delta_2)(\delta_1+5)^3(\delta_2+5)^3 + 2500\delta_2^4(\delta_1+5)^3 - 2500\delta_1^4(\delta_2+5)^3 \geq 0.$$

But if we let $n = \delta_2 - \delta_1$ then we have

$$(\delta_1-\delta_2)(\delta_1+5)^3(\delta_2+5)^3 + 2500\delta_2^4(\delta_1+5)^3 - 2500\delta_1^4(\delta_2+5)^3 = \sum_{i=1}^{7} c_i\delta_1^i,$$

where we have the following values for the c_i:

i	c_i
1	$312375n^4 - 1875n^3 - 9375n^2 - 15625n$
2	$187425n^4 + 1248500n^3 - 9375n^2 - 18750n$
3	$37485n^4 + 749550n^3 + 1871250n^2 - 9375n$
4	$2499n^4 + 149940n^3 + 1124250n^2 + 1247500n$
5	$7497n^3 + 187425n^2 + 562125n$
6	$7497n^2 + 74970n$
7	$2449n$

Each of these coefficients is strictly positive. □

Inequality 6. *Let $N, M, \delta_1, \delta_2 \in \mathbb{N}$ be such that $2 \leq N \leq M$ and $\delta_1 \leq \delta_2$. Then*

$$N2^{-\delta_1}\left(\frac{\eta(\delta_1)}{N-1}\right)^{\alpha(\delta_1)} \leq M2^{-\delta_2}\left(\frac{\eta(\delta_2)}{M-1}\right)^{\alpha(\delta_2)}.$$

Proof. We need to show that

$$1 \leq \left[\frac{M}{N}\frac{(N-1)^{\alpha(\delta_1)}}{(M-1)^{\alpha(\delta_2)}}\right]\left[2^{\delta_1-\delta_2}\frac{\eta(\delta_2)^{\alpha(\delta_2)}}{\eta(\delta_1)^{\alpha(\delta_1)}}\right].$$

Which follows from Inequalities 4 and 5. □

Inequality 7. *Let $k, N, \delta_1, \delta_2 \in \mathbb{N}$ be such that $\delta_1 \leq \delta_2$. Then*

$$2^{-\delta_1}\left(\frac{\eta(\delta_1)}{k}\right)^{\alpha(\delta_1)} \leq N2^{-\delta_2}\left(\frac{\eta(\delta_2)}{N+k-1}\right)^{\alpha(\delta_2)}. \tag{6.12}$$

Proof. We need to show that

$$1 \leq \left[N\frac{k^{\alpha(\delta_1)}}{(N+k-1)^{\alpha(\delta_2)}}\right]\left[2^{\delta_1-\delta_2}\frac{\eta(\delta_2)^{\alpha(\delta_2)}}{\eta(\delta_1)^{\alpha(\delta_1)}}\right].$$

The second term in this product is Inequality 5. So let us show that $1 \leq \left[Nk^{\alpha(\delta_1)}(N+k-1)^{-\alpha(\delta_2)}\right]$. Since $\delta_1 \leq \delta_2$ we know that

$$N\frac{k^{\alpha(\delta_1)}}{(N+k-1)^{\alpha(\delta_2)}} \geq N\left(\frac{k}{N+k-1}\right)^{\alpha(\delta_2)},$$

and we are done since $1 \leq N^{(\delta_2+5)^3}k(N+k-1)^{-1}$. $\square$

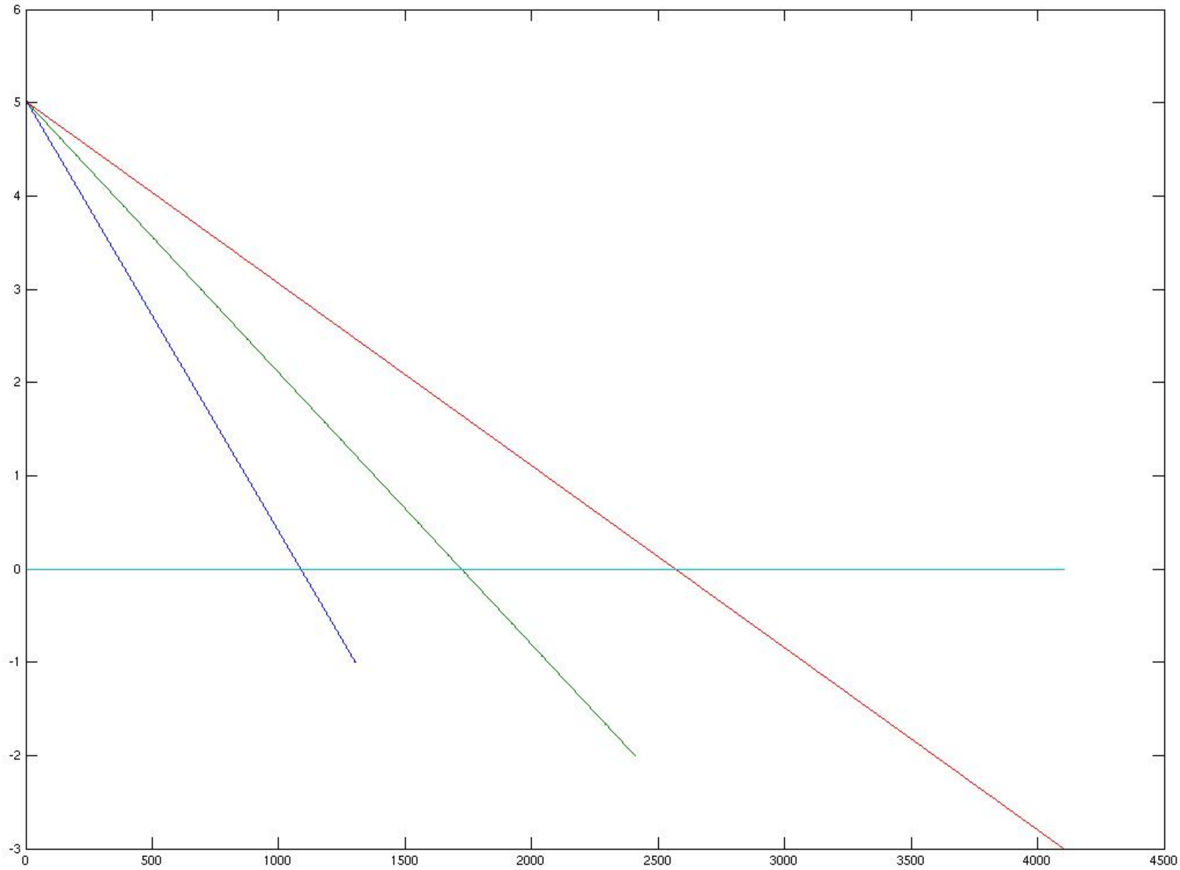


Figure 1: Behaviour of the function $x \mapsto \log_2(2^{-k}\left(\frac{\eta(k)}{x}\right)^{\alpha(k)})$, for fixed k, and for $x \in [1, \eta(k)]$. Here we take the $\eta(k)$ as defined in Subsection 2.5. The blue, green and red plots (from left to right) are for $k = 1, 2, 3$, respectively. The horizontal line is just to indicate the function $x \mapsto 0$.

7 Submeasures and signed measures

We begin with the following definition.

Definition 7.1. *If $\mathfrak{B}$ is a Boolean algebra, call a collection $\{a_i : i \in [n]\} \subseteq \mathfrak{B}$,* **$*$-free** *if and only if for every non-empty $J \subseteq [n]$ we have*

$$\left(\bigcap_{j\in J} a_j\right) \cap \left(\bigcap_{j\notin J} a_j^c\right) \neq 0 \wedge \bigcup_{i\in[n]} a_i = 1.$$

In this section we prove and investigate the following result.

Theorem 7.2. *For every countable Boolean algebra $\mathfrak{A}$ there exists a Boolean algebra $\mathfrak{B}$ and an injective map $\mathfrak{f} : \mathfrak{A} \to \mathfrak{B}$ with the following properties:*

(T.1) $\mathfrak{B} = \langle \mathfrak{f}[\mathfrak{A}] \rangle$, in particular $\mathfrak{B}$ will also be countable (Fact 2.3);

(T.2) if $\mathfrak{A}' \subseteq \mathfrak{A}$ is a finite subalgebra, then the collection $\mathfrak{f}[\mathrm{atoms}(\mathfrak{A}')]$ is $*$-free in $\mathfrak{B}$;

(T.3) $(\forall a, b \in \mathfrak{A})(\mathfrak{f}(a \cup b) = \mathfrak{f}(a) \cup \mathfrak{f}(b))$.

Moreover, if $\mathfrak{D}$ is a Boolean algebra and $\mathfrak{g} : \mathfrak{A} \to \mathfrak{D}$ satisfies the above, then for any functional μ on $\mathfrak{A}$, there exists a unique signed finitely additive measure λ on $\mathfrak{D}$ such that $\mu(a) = \lambda(\mathfrak{g}(a))$, for each $a \in \mathfrak{A}$.

Thus to each functional on a given countable Boolean algebra we associate a signed measure. In fact this association will be a linear map from the real vector space of all functionals on $\mathfrak{A}$ to the real vector space of all signed measures on $\mathfrak{B}$. We are of course interested in the case when μ is a submeasure. Unfortunately even for very simple submeasures, the corresponding measure may be unbounded. We prove Theorem 7.2 in Subsection 7.1 and in Subsection 7.2 we attempt to generalise the above to the case when $\mathfrak{A}$ a Maharam algebra.

In subsection 7.3, given a sequence $(X_i)_{i\in\mathbb{N}}$ of finite non-empty sets, we construct another sequence $(Y_i)_{i\in\mathbb{N}}$, consisting also of finite non-empty sets, and an injective map

$$\mathfrak{f} : \mathrm{Clopen}(\prod_{i\in\mathbb{N}} X_i) \to \mathrm{Clopen}(\prod_{i\in\mathbb{N}} Y_i)$$

that satisfies properties (T.2) and (T.3) of Theorem 7.2. To obtain the rest of Theorem 7.2, we can take $\mathfrak{B} = \langle \mathfrak{f}[\mathfrak{A}] \rangle$ where $\mathfrak{A} := \mathrm{Clopen}(\prod_{i\in\mathbb{N}} X_i)$. In the above context, if λ is a (non-negative) measure on $\mathrm{Clopen}(\prod_{i\in\mathbb{N}} Y_i)$, then $\mu : \mathfrak{A} \to \mathbb{R}$ defined by $\mu(a) = \lambda(\mathfrak{f}(a))$, will be a submeasure (see Remark 7.9). This raises the following question: If λ is the Lebesgue measure on $\mathrm{Clopen}(\prod_{i\in\mathbb{N}} Y_i)$ then what submeasure do we get on $\mathfrak{A}$? Understanding this submeasure reduces to counting. For arbitrary X_i this becomes difficult, however, when we restrict to $|X_i| = 2$, the counting becomes manageable.

It will be straightforward to see, for arbitrary X_i, that sets of the form $\{f \in \prod_{i\in\mathbb{N}} X_i : f(n) =$

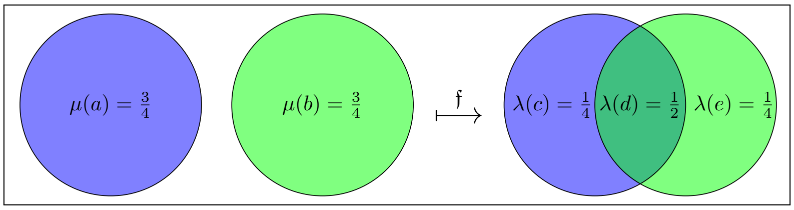


Figure 2: Finite version of Theorem 7.2.

$m\}$ have (sub)measure bounded way from 0 (Lemma 7.23). In particular, if $\sup_i |X_i| = \infty$ then the submeasure we obtain from the Lebesgue measure will not be uniformly exhaustive. One might hope then that this submeasure might be exhaustive. We show that this is not the case when we restrict to $|X_i| = 2$, and that this is witnessed by an antichain of length continuum (Theorem 7.31). As a consequence of this, we show that the determining real added by the corresponding idealised forcing cannot be a splitting real (Corollary 7.34). We cannot prove in general that this submeasure will not be exhaustive for arbitrary X_i.

In section 7.4 we collect some miscellaneous counting arguments concerning the above.

7.1 Proof of Theorem 7.2

We prove here Theorem 7.2. To illustrate the motivating idea of this construction consider the submeasure μ defined on the finite Boolean algebra $\mathfrak{A}$ of two atoms, a and b, given by

$$\mu(a) = \mu(b) = \frac{3}{4}, \quad \mu(a \cup b) = 1.$$

This is clearly not additive. If we supposed for a moment that μ was additive then a and b would have to intersect. Thus we view the atoms a and b as not having enough *space* for the submeasure μ. We try to insert this space by allowing a and b to intersect, and by doing so turning μ into a measure. To this end we consider the algebra $\mathfrak{B}$ of three atoms c, d and e and the map $\mathfrak{f} : \mathfrak{A} \to \mathfrak{B}$ define by

$$a \mapsto c \cup d, b \mapsto d \cup e, a \cup b \mapsto c \cup d \cup e.$$

The atom d then becomes the inserted space, and on $\mathfrak{B}$ we can take the measure

$$\lambda(c) = \lambda(e) = \frac{1}{4}, \quad \lambda(d) = \frac{1}{2}$$

(see Figure 2). Notice that no matter what values we had for μ, we would still be able to solve (uniquely) for λ and so we have a finite version of Theorem 7.2. Indeed, one need only solve the following system of linear equations:

$$\lambda(c) + \lambda(d) = \mu(a), \quad \lambda(d) + \lambda(e) = \mu(b), \quad \lambda(c) + \lambda(d) + \lambda(e) = 1.$$

The final $\mathfrak{f}$ and $\mathfrak{B}$ will be obtained as a direct limit of these finite constructions.

In this way we are led to the definition of $*$-free from Definition 7.1, and the following.

Definition 7.3. *For $n \in \mathbb{N}$ let $\mathrm{Fr}^* n$ be the Boolean algebra $\mathcal{P}(\mathcal{P}([n])^+)$. Call the sets $\{y \in \mathcal{P}([n])^+ : i \in y\}$, for $i \in [n]$, the* **$*$-free generators** *of $\mathrm{Fr}^* n$.*

Remark 7.4. *Clearly the $*$-free generators of $\mathrm{Fr}^* n$ are $*$-free and generate $\mathrm{Fr}^* n$. If $\mathrm{Fr}\, n$ is the freely generated Boolean algebra over n elements with free generators $a_1, ..., a_n$ then $\mathrm{Fr}^* n$ may be viewed as the Boolean algebra*

$$\mathrm{Fr}\, n_{\bigcup_{i=1}^n a_i} (= \{a \in \mathrm{Fr}\, n : a \subseteq \bigcup_{i \in [n]} a_i\}).$$

In the motivating example we see that the algebra $\mathfrak{B}$ with three atoms c, d and e is given by $\mathrm{Fr}^* 2$ where we can take

$$c = \{\{1\}\}, d = \{\{1, 2\}\}, e = \{\{2\}\}.$$

Notice that the atoms of $\mathfrak{A}$ are mapped to the $*$-free generators of $\mathfrak{B}$ $(= \mathrm{Fr}^* 2)$.

The fact that we can always solve for λ (as in the motivating example) is given by the following two lemmas.

Lemma 7.5. *For each $n \in \mathbb{N}$ enumerate $\mathcal{P}([n])^+ = \{y_i : i \in [2^n - 1]\}$. Then the matrix $(a_{ij})_{i,j \in [2^n - 1]}$ defined by*

$$a_{ij} = \begin{cases} 1, & \text{if } y_i \cap y_j \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

is invertible.

Since we could not find a particularly enlightening proof of Lemma 7.5 we leave it to the end of this subsection.

Lemma 7.6. *Let $a_1, ..., a_n$ be the $*$-free generators of $\mathrm{Fr}^* n$ and $\mu : \{\bigcup_{i \in I} a_i : I \in \mathcal{P}([n])^+\} \to \mathbb{R}$ any functional. Then there exists a unique signed measure $\lambda : \mathrm{Fr}^* n \to \mathbb{R}$ such that $(\forall I \in \mathcal{P}([n])^+)(\lambda(\bigcup_{i \in I} a_i) = \mu(\bigcup_{i \in I} a_i))$.*

Proof. Since we only need to decide the values that λ should take on the atoms of $\mathrm{Fr}^* n$ we need only find a solution to the following set of linear equations:

$$\sum_{y \in \mathcal{P}([n])^+ \wedge q \cap y \neq 0} X_y = \mu(\bigcup_{i \in q} a_i) : q \in \mathcal{P}([n])^+.$$

These equations have a unique solution by Lemma 7.5. Now set $\lambda(\{y\}) = X_y$. $\square$

We give an explicit expression for λ in Section 7.4 (Lemma 7.36).

Definition 7.7. *Let $\mathfrak{A}$ be a finite Boolean algebra with n atoms. A map $f : \mathfrak{A} \to \mathrm{Fr}^* n$ is called* **$\mathfrak{A}$-good** *if and only if the following hold:*

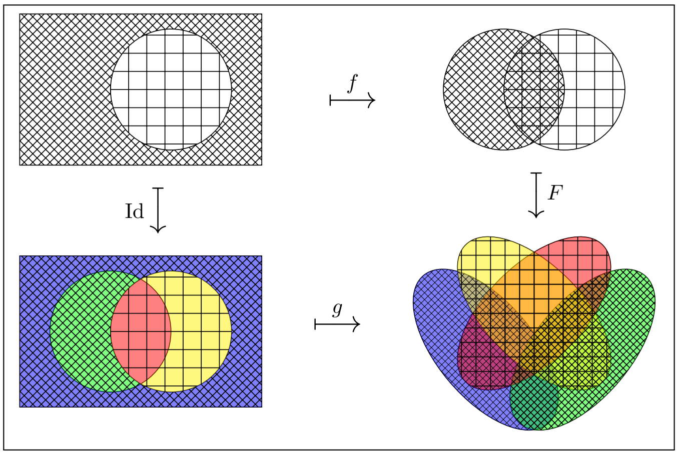


Figure 3: Commutative maps of Lemma 7.11 with $m = 2$ and $n = 4$.

- *f injectively maps the atoms of $\mathfrak{A}$ onto the $*$-free generators of* Fr^*n*;*
- *for each $a \in \mathfrak{A}$ we have $f(a) = \bigcup\{f(b) : b \in \mathrm{atoms}(\mathfrak{A}) \wedge b \leq a\}$.*

Of course in the context of the above definition any map sending the atoms of $\mathfrak{A}$ onto the $*$-free generators of Fr^*n, induces an $\mathfrak{A}$-good map (by just taking unions).

Lemma 7.8. *Let $\mathfrak{A}$ be a finite Boolean algebra with n atoms and let f be an $\mathfrak{A}$-good map. Then f is injective and satisfies the following properties:*

- *$f(0) = 0_{\mathrm{Fr}^*n}$ and $f(1) = 1_{\mathrm{Fr}^*n}$,*
- *$(\forall a, b \in \mathfrak{A})(f(a \cup b) = f(a) \cup f(b))$.*

Moreover, for any functional μ on $\mathfrak{A}$, we can find a unique signed measure λ on Fr^*n *such that $(\forall a \in \mathfrak{A})(\mu(a) = \lambda(f(a)))$.*

Proof. The properties of f follow by definition. The last part is just Lemma 7.6. □

Remark 7.9. *If $f : \mathfrak{C} \to \mathfrak{C}'$ is an injective map such that we always have $f(c \cup d) = f(c) \cup f(d)$ then for any measure λ on $\mathfrak{C}'$ one can define a submeasure μ on $\mathfrak{C}$ by $\mu(c) = \lambda(f(d))$.*

The fact that we can coherently put together the maps from Lemma 7.8 to build the map $\mathfrak{f}$ from Theorem 7.2 is justified by following two lemmas (see Figure 3).

Lemma 7.10. *Let $n \in \mathbb{N}$ and for each $i \in [n]$, let $a_i = \{y \in \mathcal{P}([n])^+ : i \in y\}$, so that $a_1, ..., a_n$ are the $*$-free generators of* Fr^*n*. Let $\mathfrak{B}$ be a finite Boolean algebra and let $b_1, ..., b_n$ be $*$-free members of $\mathfrak{B}$. Then the map $a_i \mapsto b_i$ extends uniquely to a monomorphism from* Fr^*n *to $\mathfrak{B}$.*

Proof. We need only define the embedding on the atoms of Fr^*n and this is given by

$$\{y\} = \left(\bigcap_{j \in y} a_j\right) \cap \left(\bigcap_{j \notin y} a_j^c\right) \mapsto \left(\bigcap_{j \in y} b_j\right) \cap \left(\bigcap_{j \notin y} b_j^c\right).$$

□

Lemma 7.11. *Let $\mathfrak{A}$ be a subalgebra of a finite Boolean algebra $\mathfrak{B}$. Let f be $\mathfrak{A}$-good and g be $\mathfrak{B}$-good. Let m be the number of atoms of $\mathfrak{A}$ and n the number of atoms of $\mathfrak{B}$. Then there exists an embedding $F : \mathrm{Fr}^*m \to \mathrm{Fr}^*n$ such that*

$$g \restriction \mathfrak{A} = F \circ f. \tag{7.1}$$

Proof. Let $F' : f[\mathfrak{A}] \to g[\mathfrak{B}]$ be the map $g \circ f^{-1}$. By Lemma 7.8 we see that

$$\bigcup_{a \in f[\mathfrak{A}]} F'(a) = \bigcup_{a \in f[\mathfrak{A}]} g \circ f^{-1}(a) = g \circ f^{-1}(\bigcup_{a \in f[\mathfrak{A}]} a) = g(1_{\mathfrak{A}}) = g(1_{\mathfrak{B}}) = 1_{\mathrm{Fr}^*n},$$

and so the map F' sends the $*$-free generators of Fr^*m to $*$-free members of Fr^*n. By Lemma 7.10 we can find an embedding $F : \mathrm{Fr}^*m \to \mathrm{Fr}^*n$ which agrees with F' on $f[\mathfrak{A}]$. $\square$

Proof of Theorem 7.2. See Subsection 2.2 for the definitions relating to direct limits. Fix a countable Boolean algebra $\mathfrak{A}$ let $(\mathfrak{A}_i)_{i \in \mathbb{N}}$ be a sequence of finite subalgebras of $\mathfrak{A}$ such that $\mathfrak{A}_i \subseteq \mathfrak{A}_{i+1} \subseteq \mathfrak{A}$. For each i, let $n_i = |\mathrm{atoms}(\mathfrak{A}_i)|$ and, by choosing the $\mathfrak{A}_i$ appropriately, see to it that $n_i < n_{i+1}$. For each i, let $\mathfrak{C}_i = \mathrm{Fr}^*n_i$ and let f_i be an $\mathfrak{A}_i$-good map. For $i < j$, let $f_{i,j} : \mathfrak{C}_i \to \mathfrak{C}_j$ be the embeddings promised by Lemma 7.11, with respect to the good maps f_i. If $i = j$ then we let $f_{i,j} = \mathrm{Id}$ in $\mathfrak{A}_i$. Now suppose that $i \leq j \leq k$ and let $a_1, ..., a_l$ be the $*$-free generators of $\mathfrak{C}_i$. By applying (7.1) appropriately, it is straightforward to compute that both $f_{i,k}$ and $f_{j,k} \circ f_{i,j}$ map a_m to $f_k(a_m)$, for each $m \in [l]$. Thus both these embeddings map the $*$-free generators of $\mathfrak{C}_i$ to the same $*$-free members of $\mathfrak{C}_k$ and so, by the uniqueness part of Lemma 7.10, we see that

$$f_{i,k} = f_{j,k} \circ f_{i,j}.$$

This shows that $((\mathbb{N}, \leq), (\mathfrak{C}_i)_{i \in \mathbb{N}}, (f_{i,j})_{i,j \in \mathbb{N}})$ is a directed system. Let $\mathfrak{B}$ be the corresponding direct limit and let $g_i : \mathfrak{C}_i \to \mathfrak{B}$ be the corresponding limit maps. We have the following commutative diagram for $i \leq j$:

$$\begin{tikzcd}
& \mathfrak{B} & \\
\mathfrak{C}_i \arrow[ur, "g_i"] \arrow[rr, "f_{i,j}"'] & & \mathfrak{C}_j \arrow[ul, "g_j"'] \\
\mathfrak{A}_i \arrow[u, "f_i"] \arrow[rr, "\mathrm{Id}"] & & \mathfrak{A}_j \arrow[u, "f_j"']
\end{tikzcd}$$

Set $\mathfrak{f}(a) = (g_i \circ f_i)(a)$ for any i such that $a \in \mathfrak{A}_i$. Let us now check that $\mathfrak{f}$ satisfies the desired properties. The fact that $\mathfrak{f}$ is injective follows since each g_i is an embedding (and in particular injective), and each f_i is an $\mathfrak{A}_i$-good map (and in particular injective). Properties (T.2) and (T.3) follow by the properties of good maps. Property (T.1) follows since for every $b \in \mathfrak{B}$, we can find a finite subalgebra $\mathfrak{A}' \subseteq \mathfrak{A}$, such that $b \in \langle \mathfrak{f}[\mathfrak{A}] \rangle$.

Let $\mu : \mathfrak{A} \to \mathbb{R}$ be any functional. By the final part of Lemma 7.8 for each i we can find a unique measure $\lambda_i : \mathfrak{C}_i \to \mathbb{R}$ such that $(\forall a \in \mathfrak{A}_i)(\mu(a) = \lambda_i(f_i(a)))$. We now define the measure $\lambda : \mathfrak{B} \to \mathbb{R}$ by

$$\lambda(b) = \lambda_i(g_i^{-1}(b))$$

for any i such that $b \in \operatorname{ran}(g_i)$. To see that this is well defined we just notice that for $i \leq j$, the uniqueness of λ_i implies that $\lambda_j \circ f_{i,j} = \lambda_i$.

Suppose now that $\mathfrak{D}$ is a Boolean algebra and $\mathfrak{g} : \mathfrak{A} \to \mathfrak{D}$ is an injective map satisfying (T.1), (T.2) and (T.3). Let $(\mathfrak{A}_i)_{i\in\mathbb{N}}$ and $(n_i)_{i\in\mathbb{N}}$ be as above. For each $i \in \mathbb{N}$, let $\mathfrak{D}_i = \mathfrak{g}[\mathfrak{A}_i]$, $\mathfrak{g}_i = \mathfrak{g} \restriction \mathfrak{A}_i$ and $p_i : \mathfrak{D}_i \to \mathrm{Fr}^* n_i$ be any isomorphism which injectively maps the $*$-free generators of $\mathfrak{D}_i$ to the $*$-free generators of $\mathrm{Fr}^* n_i$. For each i, let $f_{i,i+1} = p_{i+1} \circ p_i^{-1}$ and $f_i = p_i \circ \mathfrak{g}_i$. For $i < j$ let, $f_{i,j} = f_{j-1,j} \circ \cdots f_{i+1,i+2} \circ f_{i,i+1}$ and $f_{i,i} = \mathrm{Id}$. The system $((\mathbb{N}, \leq), (f_{i,j})_{i,j\in\mathbb{N}}, (\mathrm{Fr}^* n_i)_{i\in\mathbb{N}})$ is a directed system. Let $\mathfrak{B}$ be its direct limit and g_i be the corresponding limit maps. Define $\mathfrak{f} : \mathfrak{A} \to \mathfrak{B}$ by $\mathfrak{f}(a) = (g_i \circ f_i)(a)$, for any i such that $a \in \mathfrak{A}_i$. As above, we see that $\mathfrak{B}$ and $\mathfrak{f}$ satisfy the properties in the statement of Theorem 7.2, with respect to $\mathfrak{A}$. Finally, for each $i \in \mathbb{N}$, let $h_i = p_i^{-1}$ and notice that by construction, for $i \leq j$ we have $h_i = h_j \circ f_{i,j}$. By Fact 2.8, we can find an isomorphism $F : \mathfrak{B} \to \mathfrak{D}$, such that $h_i = F \circ g_i$, for each i. In particular, given a functional μ on $\mathfrak{A}$, if we let λ be the signed measure on $\mathfrak{B}$ defined by $(\forall a)(\mu(a) = \lambda(\mathfrak{f}(a)))$, then we can define a signed measure on $\mathfrak{D}$ by $\varphi(a) = \lambda(F^{-1}(a))$, and for each $a \in \mathfrak{A}$ we have,

$$\begin{aligned}
\mu(a) = \lambda(\mathfrak{f}(a)) = \lambda((g_i \circ f_i)(a)) = \lambda((F^{-1} \circ h_i \circ f_i)(a)) &= \lambda((F^{-1} \circ p_i^{-1} \circ f_i)(a)) \\
&= \lambda((F^{-1} \circ \mathfrak{g}_i)(a)) \\
&= \varphi(\mathfrak{g}(a)).
\end{aligned}$$

□

As we have already mentioned, even for very simple submeasures the corresponding measure obtained from Theorem 7.2 may be signed and, even worse, unbounded, as the following example shows.

Example 7.12. *For each n let $\mu : \mathcal{P}([n]) \to \mathbb{R}$ be the submeasure $\mu([n]) = 1$, $\mu(0) = 0$ and $\mu(a) = \frac{1}{2}$, otherwise. If $\lambda : \mathrm{Fr}^* n \to \mathbb{R}$ is the corresponding measure from Lemma 7.8 then*

$$(\exists a \in \mathrm{Fr}^* n)(\lambda(a) = -\frac{1}{2}\binom{n}{2})$$

In particular, in the context of Theorem 7.2 and its proof, if we take $\mu : \mathfrak{A} \to \mathbb{R}$ to be $(\forall a \in \mathfrak{A} \setminus \{0, 1\})(\mu(a) = \frac{1}{2})$ and $\mu(1) = 1$ then for each i, the algebra $\langle \mathfrak{f}[\mathfrak{A}_i] \rangle$ will contain an element of λ-measure $-\frac{1}{2}\binom{n_i}{2}$. Thus $\inf_{b\in\mathfrak{B}} \lambda(b) = -\infty$.

Proof. Notice that $I \in \mathcal{P}([n])^+$ we have $\{I\} = (\bigcap_{i\in I} a_i) \setminus (\bigcup_{i\notin I} a_i)$. Notice also that $\lambda(\{\{i\}\}) = \lambda(\bigcup_{l\in[n]} a_l) - \lambda(\bigcup_{l\in[n]\setminus\{i\}} a_l)$. We also have that for $i \neq j$ we have $\lambda(\{\{i, j\}\}) =$

$\lambda(\bigcup_{l\in[n]} a_l) - (\lambda(\bigcup_{l\in[n]\setminus\{i,j\}} a_l) + \lambda(\{\{i\}\}) + \lambda(\{\{j\}\}))$. This shows that if $i \neq j$ then

$$\lambda(\{\{i,j\}\}) = -\lambda(\bigcup_{l\in[n]} a_l) - \lambda(\bigcup_{l\in[n]\setminus\{i,j\}} a_l) + \lambda(\bigcup_{l\in[n]\setminus\{i\}} a_l) + \lambda(\bigcup_{l\in[n]\setminus\{j\}} a_l)).$$

Thus we can take $a = \{y : y \in [[n]]^2\}$. $\square$

The following table defines the measure of Example 7.12 for $n = 3$.

I	$\mu(\bigcup_{i\in I} a_i)$	$\lambda(\bigcap_{i\in I} a_i \cap \bigcap_{i\notin I} a_i^c)$
$\{1\}$	0.5	0.5
$\{2\}$	0.5	0.5
$\{3\}$	0.5	0.5
$\{1,2\}$	0.5	-0.5
$\{1,3\}$	0.5	-0.5
$\{2,3\}$	0.5	-0.5
$\{1,2,3\}$	1	1

It is not clear to us how to predict when a submeasure will generate a non-negative measure, or even just a bounded signed measure (in the case that the measure is bounded but signed, one could hope to employ, for example, the Jordan decomposition theorem (see [37, Page 25])).

Before we move on, we record the following.

Proposition 7.13. *In the context of Theorem 7.2, if $\mathfrak{A}$ is atomless then so is $\mathfrak{B}$.*

Proof. Assume the notation from the proof of Theorem 7.2. Suppose for a contradiction that $b \in \mathfrak{B}$ is an atom. Let $i \in \mathbb{N}$ be such that $b \in \mathrm{ran}(g_i)$ and let $c = g_i^{-1}(b)$. Of course c is still an atom of $\mathfrak{C}_i$ so we can find $a \in \mathrm{atoms}(\mathfrak{A}_i)$ such that $c \leq f_i(a)$. Enumerate $\mathrm{atoms}(\mathfrak{A}_i) = \{a, a_1, ..., a_k\}$. Let $d \in \mathfrak{A}_a^+$ and j be such that $\langle \mathfrak{A}_i \cup \{d\}\rangle = \langle d, a\setminus d, a_1, ..., a_k\rangle \subseteq \mathfrak{A}_j$. Now

$$c = (f_i(a) \cap \bigcap_{l\in L} f_i(a_l)) \cap (\bigcap_{l\notin L} f_i(a_l))^c,$$

for some $L \subseteq [k]$. Let $q = f_j(a \setminus d)$, $p = f_j(d)$ and $r = \bigcap_{l\in L} f_j(a_l) \cap (\bigcap_{l\notin L} f_j(a_l))^c$. We have

$$f_{i,j}(c) \quad = \quad f_j(a) \cap r = (q \cup p) \cap r = (q \setminus p \sqcup p) \cap r = ((q \setminus p) \cap r) \sqcup (p \cap r).$$

Since the sets $f_j(d), f_j(a \setminus d), f_j(a_1), ..., f_j(a_k)$ are $*$-free, we have that $(q \setminus p) \cap r \neq \emptyset \neq p \cap r$, and so $0 < f_j(p \cap r) < b$, which is a contradiction. $\square$

Proof of Lemma 7.5. By induction on n. For the case $n = 1$, the matrix in question is the identity, so let us show that this is true for $n+1$ assuming it is true for n. Let $m = 2^n - 1$ and $m' = 2^{n+1} - 1$. Enumerate $\mathcal{P}([n+1])^+ = \{y_i : i \in [m']\}$ so that $\{y_i : i \in [m]\}$ is an enumeration of $\mathcal{P}([n])^+$, $y_{m+1} = \{n\}$ and $y_{i+m+1} = y_i \cup \{n\}$ for $i \in [m]$. Let A_n be $m \times m$

matrix where, for $i, j \in [m]$, we let

$$A_n(i,j) = \begin{cases} 1, & \text{if } y_i \cap y_j \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Let A_{n+1} be the $m' \times m'$ matrix where, for $i, j \in [m']$, we set

$$A_{n+1}(i,j) = \begin{cases} 1, & \text{if } y_i \cap y_j \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

We want to show that A_{n+1} is invertible. By induction the rows of A_n are linearly independent. Let v_i denote the ith row of A_{n+1} and u_i the ith row of A_n. Then

$$v_i = \begin{cases} u_i{}^\frown 0^\frown u_i, & \text{if } i \in [m]; \\ 0^m{}^\frown 1^{m'-m}, & \text{if } i = m+1; \\ u_{i-m-1}{}^\frown 1^{m'-m}, & \text{otherwise.} \end{cases}$$

That is

$$A_{n+1} = \begin{pmatrix} A_n & (0^m)^T & A_n \\ 0^m & 1 & 1 \\ A_n & 1 & 1 \end{pmatrix}.$$

Here, $(0^m)^T$ denotes the column vector of length m containing only 0's. Now let $\lambda_i \in \mathbb{R}$ be such that $\sum_{i \in [m']} \lambda_i v_i = 0^{m'}$. Since

$$\sum_{i \in [m]} \lambda_i u_i + \sum_{i=m+2}^{m'} \lambda_i u_{i-m-1} = \sum_{i \in [m]} (\lambda_i + \lambda_{i+m+1}) u_i = 0^m,$$

by the linear independence of the u_i, we must have

$$(\forall i \in [m])(\lambda_{i+m+1} = -\lambda_i). \tag{7.2}$$

Considering the $(m+1)$th column of A_{n+1}, by (7.2), we see that $\lambda_{m+1} - \sum_{i \in [m]} \lambda_i = 0$. We now have

$$\begin{aligned} 0 = \sum_{i \in [m']} \lambda_i v_i &= \sum_{i=1}^{m} \lambda_i v_i + \lambda_{m+1} v_{m+1} + \sum_{i=m+2}^{m'} \lambda_i v_i \\ &= \sum_{i=1}^{m} \lambda_i v_i + \sum_{i=1}^{m} \lambda_i 0^m{}^\frown 1^{m'-m} + \sum_{i=1}^{m} \lambda_{i+m+1} v_{i+m+1} \\ &= \sum_{i=1}^{m} \lambda_i u_i{}^\frown 0^\frown u_i + \sum_{i=1}^{m} \lambda_i 0^m{}^\frown 1^{m'-m} - \sum_{i=1}^{m} \lambda_i u_i{}^\frown 1^{m'-m} \\ &= \sum_{i=1}^{m} \lambda_i 0^{m+1}{}^\frown u_i + \sum_{i=1}^{m} \lambda_i 0^m{}^\frown 1^{m'-m} - \sum_{i=1}^{m} \lambda_i 0^m{}^\frown 1^{m'-m} \end{aligned}$$

$$= \sum_{i\in[m]} \lambda_i 0^{m+1} {}^\frown u_i.$$

By the linear independence of the u_i and (7.2), we may conclude that $\lambda_i = 0$ for each $i \neq m+1$. But then

$$\lambda_{m+1} 0^m {}^\frown 1^{m'-m} = 0^{m'}$$

and so we must have $\lambda_{m+1} = 0$, also. Thus the rows $\{v_i : i \in [m']\}$ are linearly independent and A_{n+1} is invertible. $\square$

7.2 On Maharam algebras

In the context of Theorem 7.2, call a submeasure μ on $\mathfrak{A}$ **true** if and only if the corresponding λ , with respect to some $\mathfrak{f}$, is non-negative. In this section we prove the following, which can be seen as an analogue of Theorem 7.2 in the case when $\mathfrak{A}$ is a (true) countably generated Maharam algebra.

Proposition 7.14. *Let μ be a continuous submeasure on* $\mathrm{Borel}(2^\omega)$ *such that* $\mu \restriction \mathrm{Clopen}(2^\omega)$ *is strictly positive and true, with respect to some $\mathfrak{f}$ from Theorem 7.2. Then there exists a strictly positive σ-additive measure λ on the random algebra* $\mathbb{M}$ *and a uniformly continuous function* $f : \mathrm{Borel}(2^\omega)/\mathrm{Null}(\mu) \to \mathbb{M}$ *such that the following hold:*

- $f(0) = 0$ *and* $f(1) = 1$*;*
- $f(a \cup b) = f(a) \cup f(b)$*;*
- $\mu(a) = \lambda(f(a))$*.*

Before we prove Proposition 7.14 we will need the following two lemmas.

Lemma 7.15. *Let $\mathfrak{C}$ and $\mathfrak{C}'$ be two Boolean algebras and let $f : \mathfrak{C} \to \mathfrak{C}'$ be a map such* $(\forall a, b \in \mathfrak{C})(f(a \cup b) = f(a) \cup f(b))$*. Suppose that μ and λ are submeasures on $\mathfrak{C}$ and $\mathfrak{C}'$, respectively, such that* $(\forall a \in \mathfrak{C})(\mu(a) = \lambda(f(a))$*. Then for every $\epsilon > 0$ we have*

$$(\forall a, b \in \mathfrak{A})(\mu(a \triangle b) < \epsilon \to \lambda(f(a) \triangle f(b)) < \epsilon).$$

In particular, f is uniformly continuous with respect to the pseudometrics induced by μ and λ.

Proof. This is because for any $a, b \in \mathfrak{C}$ we have $f(a) \triangle f(b) \leq f(a \setminus b) \cup f(b \setminus a)$. Indeed, let $a, b \in \mathfrak{A}$, then

$$f(a)\triangle f(b) = f(a\setminus b\cup a\cap b)\triangle f(b\setminus a\cup a\cap b) = (f(a\setminus b)\cup f(a\cap b))\triangle(f(b\setminus a)\cup f(a\cap b)) \subseteq f(a\setminus b)\cup f(b\setminus a).$$

In particular $\lambda(f(a)\triangle f(b)) \leq \lambda(f(a \setminus b) \cup f(b \setminus a)) = \mu(a \setminus b \cup b \setminus a) = \mu(a\triangle b)$. $\square$

Lemma 7.16. *In the context of Theorem 7.2, if μ is a diffuse true submeasure then λ is diffuse and* $\mathfrak{B}/\mathrm{Null}(\lambda)$ *is atomless.*

Proof. Since a is diffuse, for every $\epsilon > 0$, we can find a partition of $\mathfrak{A}$, $a_1, ...a_k$, such that $\mu(a_i) \leq \epsilon$ for each i. Now the atoms of the algebra generated by the $\mathfrak{f}(a_i)$ will be a partition of $\mathfrak{B}$ and will have λ-measure less than ϵ also (this uses the monotonicity of λ which we might not have if λ were not true). It follows that $\text{atoms}(\mathfrak{B}) \subseteq \text{Null}(\lambda)$ and we are done. $\square$

Proof of Proposition 7.14. Let $\mathfrak{f} : \text{Clopen}(2^\omega) \to \text{Clopen}(2^\omega)$ and λ be that promised by Theorem 7.2. By Lemma 7.16 we know that $\mathfrak{C} := \text{Clopen}(2^\omega)/\text{Null}(\lambda)$ is still atomless (and countable). Consider the strictly positive finitely additive measure on $\mathfrak{C}$, which we also denote by λ, defined by

$$\lambda([a]_{\text{Null}(\lambda)}) = \lambda(a).$$

Let $\mathfrak{D}$ be the metric completion of $\text{Clopen}(2^\omega)$ with respect to the metric induced by μ (Lemma 2.21). The metric completion of $\mathfrak{C}$ will be a measure algebra σ-generated by $\text{Clopen}(2^\omega)$, and so, by Fact 2.16, will be isomorphic to $\mathbb{M}$. Denote the extension of λ to $\mathbb{M}$ by λ also. Let $f : \text{Clopen}(2^\omega) \to \mathbb{M}$ be the map $a \mapsto [\mathfrak{f}(a)]_{\text{Null}(\lambda)}$. By Lemma 7.15 and Theorem 2.20, we may extend f to the entirety of $\mathfrak{D}$, so that we have a uniformly continuous function $f : \mathfrak{D} \to \mathbb{M}$ such that

$$(\forall a, b \in \text{Clopen}(2^\omega))(f(a \cup b) = f(a) \cup f(b) \wedge \mu(a) = \lambda(f(a)).$$

Since $f, \cup, \mu$ and λ are all uniformly continuous (see the proof of Lemma 2.21), it is straightforward to see (by taking convergent sequences from $\text{Clopen}(2^\omega)$, for example), that we actually have

$$(\forall a, b \in \mathfrak{D})(f(a \cup b) = f(a) \cup f(b) \wedge \mu(a) = \lambda(f(a)). \tag{7.3}$$

But by Proposition 3.8, we can replace $\mathfrak{D}$ by $\text{Borel}(2^\omega)/\text{Null}(\mu)$ in (7.3), and we are done. $\square$

Unfortunately Proposition 7.14 does not apply to Talagrand's submeasure. A submeasure μ on a Boolean algebra $\mathfrak{A}$ is called **submodular** if and only if, for each $a, b \in \mathfrak{A}$, we have

$$\mu(a \cup b) + \mu(a \cap b) \leq \mu(a) + \mu(b). \tag{7.4}$$

This terminology is taken from [9]. In [38] submodular submeasures are called *strongly subadditive*. By [21, Theorem 14], submodular submeasures always dominate a non-trivial finitely additive measure (i.e. they cannot be pathological). True submeasures are always submodular, and so in particular Talagrand's submeasure cannot be true:

Lemma 7.17. *Let μ be a submeasure on a countable Boolean algebra $\mathfrak{A}$ and let $\lambda, \mathfrak{B}, \mathfrak{f}$ be as in Theorem 7.2. Then μ is submodular if and only if, for each $a, b \in \mathfrak{A}$, we have*

$$\lambda((\mathfrak{f}(a \setminus b) \cap \mathfrak{f}(b \setminus a)) \setminus \mathfrak{f}(a \cap b)) \geq 0. \tag{7.5}$$

Proof. Given $a, b \in \mathfrak{A}$ we show that (7.4) holds if and only if (7.5) holds. If $a \subseteq b$ or $b \subseteq a$ then both identities always hold. If $a \cap b = 0$ then (7.4) always holds, by the subadditivity of μ. In this case (7.5) also holds, again by the subadditivity of μ. Indeed, since $\mu(a \cup b) \leq \mu(a) + \mu(b)$

we have

$$\lambda(\mathfrak{f}(a) \setminus \mathfrak{f}(b)) + \lambda(\mathfrak{f}(a) \cap \mathfrak{f}(b)) + \lambda(\mathfrak{f}(b) \setminus \mathfrak{f}(a)) \leq \lambda(\mathfrak{f}(a) \setminus \mathfrak{f}(b)) + 2\lambda(\mathfrak{f}(a) \cap \mathfrak{f}(b)) + \lambda(\mathfrak{f}(b) \setminus \mathfrak{f}(a))$$

from which it follows that

$$\lambda(\mathfrak{f}(a) \cap \mathfrak{f}(b)) \geq 0$$

as required. Thus we can assume that $c_1 := a \setminus b, c_2 := b \setminus a$ and $c_3 := a \cap b$ are all non-empty. For brevity, in the following calculation, let $y \in \mathcal{P}([3])^+$ represent the set $(\bigcap_{i \in y} \mathfrak{f}(c_i)) \setminus (\bigcup_{i \in [3] \setminus y} \mathfrak{f}(c_i))$. Now we have that $\mu(a \cup b) + \mu(a \cap b) \leq \mu(a) + \mu(b)$ if and only if we have

$$\sum_{y \in \mathcal{P}([3])^+} \lambda(y) + \sum_{y \in \mathcal{P}([3])^+ : 3 \in y} \lambda(y) \leq \sum_{y \in \mathcal{P}([3])^+ : 3 \in y \vee 1 \in y} \lambda(y) + \sum_{y \in \mathcal{P}([3])^+ : 3 \in y \vee 2 \in y} \lambda(y).$$

But this last inequality is equivalent to asserting that

$$\lambda(\mathfrak{f}(a \setminus b) \cap \mathfrak{f}(b \setminus a) \setminus \mathfrak{f}(a \cap b)) = \lambda(\{1, 2\}) \geq 0.$$

as required. □

7.3 The preimage of the Lebesgue measure

In this section we find an explicit instance of Theorem 7.2. The main result is that if $\mathfrak{A} = \mathrm{Clopen}({}^{\mathbb{N}}[2])$ then the submeasure obtained (via Remark 7.9) as the preimage of the Lebesgue measure is not exhaustive (see Theorem 7.31). As a consequence we can show that the real determining the corresponding idealised forcing of this submeasure, cannot be a splitting real (see Corollary 7.34).

Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of finite non-empty sets. Let $X^{(n)} = \prod_{i \in [n]} X_i$ and $X = \prod_{i \in \mathbb{N}} X_i$. For convenience we assume that for each i, we have

$$|X_i| > 1, \tag{7.6}$$

and we also set $X^{(0)} = \{\emptyset\}$. Let $T_1 = \mathcal{P}(X_1)^+$ and

$$T_{i+1} = \{A \subseteq X^{(i+1)} : (\forall t \in X^{(i)})(\exists s \in A)(s \restriction [i] = t)\}. \tag{7.7}$$

Let $T^{(n)} = \prod_{i \in [n]} T_i$ and $T = \prod_{i \in \mathbb{N}} T_i$. Let $\mathfrak{A} = \mathrm{Clopen}(X)$ and $\mathfrak{B} = \mathrm{Clopen}(T)$.

Definition 7.18. *Let $\mathfrak{f} : \mathfrak{A} \to \mathfrak{B}$ be defined as follows. For every $n \in \mathbb{N}$ and $t \in X^{(n)}$ we set*

$$\begin{aligned} \mathfrak{f}([t]) &= \bigcup \{[f] : f \in T^{(n)} \wedge (\forall i \in [n])(t \restriction [i] \in f(i))\} \\ &= \{f \in T : (\forall i \in [n])(t \restriction [i] \in f(i))\} \end{aligned}$$

We then extend $\mathfrak{f}$ to all members of $\mathfrak{A}$ by taking unions.

To see that $\mathfrak{f}$ is well defined we need to check that for any $a \in \mathfrak{A}$, $B \subseteq X^{(m)}$ and $C \subseteq X^{(n)}$

such that $a = \bigcup_{t\in B}[t] = \bigcup_{t\in C}[t]$, we have $\bigcup_{t\in B}\mathfrak{f}([t]) = \bigcup_{t\in C}\mathfrak{f}([t])$. We may assume that $m < n$. Let $f \in \mathfrak{f}([t])$ for some $t \in B$. In particular $(\forall i \in [m])(t \restriction [i] \in f(i))$. By induction (via (7.7)) we can find $s \in \prod_{i\in(m,n]} X_i$ such that $(\forall i \in (m,n])(t^\frown s \restriction [i] \in f(i))$. But since $t^\frown s \in [t] \subseteq a$ we have $t^\frown s \in C$. Thus $f \in \bigcup_{t\in C}\mathfrak{f}([t])$. The other inclusion is immediate.

Proposition 7.19. *The function $\mathfrak{f}$ is injective and satisfies (T.2) and (T.3) of Theorem 7.2.*

Before we prove Proposition 7.19, it will be helpful to record the following.

Definition 7.20. *For $f \in T^{(n)}$ say that $t \in X^{(n)}$* **generates** *f, if and only if,*

$$(\forall i \in [n])(t \restriction [i] \in f(i)).$$

Lemma 7.21. *For every $n \in \mathbb{N}$ and $f \in T^{(n)}$, there exists $t \in X^{(n)}$ that generates f. Conversely, for every n and $A \subseteq X^{(n)}$ there exists an $f \in T^{(n)}$ that is generated by precisely the members of A.*

Proof. The first claim may be seen by induction on n using (7.7). Indeed, for the case $n = 1$, any member of $f(1)$ generates f. Suppose it is true for n and let $f \in T^{(n+1)}$. By induction, find $t \in X^{(n)}$ that generates $f \restriction [n]$. By (7.7), there exists $s \in f(n+1)$ such that $s \restriction [n] = t$ and so, since t generates $f \restriction [n]$, it must be the case that s generates f. The second claim also proceeds by induction on n. For the case $n = 1$, if $A \subseteq X^{(1)}$ then the function $\{(1, A)\} \in T^{(1)}$ and is generated precisely by A. Now suppose it is true for n, and let $A \subseteq X^{(n+1)}$. Let $g \in T^{(n)}$ be generated by precisely the members of $B := \{t \restriction [n] : t \in A\}$. Now fix $x \in X_{n+1}$ and let $f = g^\frown(A \cup \{t^\frown x : t \in X^{(n)} \setminus B\})$. It is clear that $f \in T^{(n+1)}$. Suppose that $t \notin A$ and t generates f. Then $t = s^\frown x$, for some $s \notin B$. On the other hand $s = t \restriction [n]$ generates g, so that $s \in B$, which is a contradiction. If $t \in A$, then by definition $t \in f(n+1)$, and since $t \restriction [n]$ generates g, we must have that t generates f. □

Proof of Proposition 7.19. This all follows from Lemma 7.21. For injectivity, Let $n \in \mathbb{N}$ and suppose that $C, B \subseteq X^{(n)}$, such that $C \neq B$. Without loss of generality, we can find $t \in C\setminus B$. Now let $f \in T^{(n)}$ be generated only by t. Then $f \in \bigcup_{s\in C}\mathfrak{f}([s]) \setminus \bigcup_{s\in B}\mathfrak{f}([s])$. For property (T.2), it is enough to check that for each $n \in \mathbb{N}$, the collection $\{\mathfrak{f}([t]) : t \in X^{(n)}\}$ forms a $*$-free collection in $\mathfrak{B}$. For this just observe that if A is a non-empty subset of $X^{(n)}$ and $f \in T^{(n)}$ is generated by precisely the members of A, then

$$f \in \left(\bigcap_{t\in A}\mathfrak{f}[t]\right) \cap \left(\bigcap_{t\in A}\mathfrak{f}[t]\right)^c \neq \emptyset.$$

Property (T.3) follows from how we constructed $\mathfrak{f}$ (by taking unions). □

Now let $\lambda : \mathfrak{B} \to \mathbb{R}$ be the Lebesgue measure and define the submeasure $\mu : \mathfrak{A} \to \mathbb{R}$ by

$$\mu(a) = \lambda(\mathfrak{f}(a)).$$

Recall that for each $a \in \mathfrak{B}$ we can find $A \subseteq T^{(n)}$ such that $a = \{[t] : t \in A\}$ and that

$$\lambda(a) = |A||T^{(n)}|^{-1}.$$

Thus to understand the submeasure μ we must count functions in T. We will apply the following lemma (usually without reference) several times from now on.

Lemma 7.22. *Let $A_1, ..., A_n$ be disjoint non-empty sets. Then*

$$|\{A \subseteq \bigcup_{i\in[n]} A_i : (\forall i)(A \cap A_i \neq \emptyset)\}| = |\prod_{i\in[n]} \mathcal{P}(A_i)^+| = \prod_{i\in[n]} (2^{|A_i|} - 1)$$

Proof. The map $A \mapsto (A \cap A_1, A \cap A_2, ..., A \cap A_n)$ is a bijection. $\square$

From this we can calculate the cardinality of T_n. Indeed, if for each $t \in X^{(n-1)}$ we take $A_t^{(n)} = \{t^\frown i : i \in X_n\}$ then, by Lemma 7.22, we get

$$|T_n| = |\{A \subseteq \bigcup_{t\in X^{(n-1)}} A_t^{(n)} : (\forall t)(A \cap A_t^{(n)} \neq \emptyset)\}| = \prod_{t\in X^{(n-1)}} (2^{|X_n|} - 1). \tag{7.8}$$

Now we can already obtain some bounds for μ. For the rest of this section, given $i \in \mathbb{N}$ and $j \in X_i$, let

$$C_{i,j} = \{f \in X : f(i) = j\}.$$

Lemma 7.23. *We have the following.*

- *For $t \in X^{(n)}$ we have $\mu([t]) \leq (\frac{2}{3})^n$.*
- *For every i, j we have $\mu(C_{i,j}) \geq \frac{1}{2}$.*

Proof. For the first claim, we want to count the number of $f \in T^{(n)}$ generated by t (and possibly by other things), and then divide by $|T^{(n)}|$. That is, $\mu([t])$ is given by

$$|\{f \in T^{(n)} : (\forall i \in [n])(t \restriction [i] \in f(i))\}|/|T^{(n)}| = \prod_{i\in[n]} |\{A \in T_i : t \restriction [i] \in A\}|/(\prod_{i\in[n]} \prod_{s\in X^{(i-1)}} (2^{|X_i|}-1)).$$

But since

$$|\{A \in T_i : t \restriction [i] \in A\}| = |\mathcal{P}(X_i \setminus t(i))| \prod_{s\in X^{(i-1)}\setminus\{t\restriction[i-1]\}} |\mathcal{P}(X_i)^+| = 2^{|X_i|-1}(2^{|X_i|} - 1)^{|X^{(i-1)}|-1}.$$

we get

$$\mu([t]) = \prod_{i\in[n]} \frac{2^{|X_i|-1}}{2^{|X_i|} - 1} \leq \left(\frac{2}{3}\right)^n,$$

since for $k \geq 2$ we have $\frac{2^{k-1}}{2^k-1} \leq \frac{2}{3}$ (see (7.6)).

For the second claim, we have $\mathfrak{f}(C_{n+1,i}) = \bigcup_{t\in X^{(n)}} \mathfrak{f}([t^\frown i])$. For every $f \in T^{(n)}$, let $t_f \in X^{(n)}$

be a sequence that generates f. For each such f let

$$C_f = \{A \in T_{n+1} : t_f^\frown i \in A\}.$$

Then

$$|C_f| = |\mathcal{P}(X_{n+1} \setminus \{i\})||\mathcal{P}(X_{n+1})^+|^{|X^{(n)}|-1}.$$

But

$$\bigsqcup_{f \in T^{(n)}} \{f^\frown A : A \in C_f\} \subseteq C_{n+1,i}.$$

So that

$$\lambda(\bigsqcup_{f \in T^{(n)}} \{f^\frown A : A \in C_f\}) = \frac{|C_f| \cdot |T^{(n)}|}{|T^{(n+1)}|} = \frac{|C_f|}{|T_{n+1}|} = \frac{2^{|X_{n+1}|-1}}{2^{|X_{n+1}|} - 1} \geq \frac{1}{2}$$

□

Remark 7.24. *Lemma 7.23 says that relative atoms in $\mathfrak{A}$ will have arbitrarily small μ-measure, in particular it follows that the submeasure μ is diffuse (and that singletons in* Borel(X) *will have μ-measure* 0*, see below). Moreover, if we take X_i to be such that $|X_i| = n$ then the coordinate sets $(C_{i,j})_{i,j}$ witness that μ will not be uniformly exhaustive.*

For the remainder of this section assume that $X_i = [2]$. Notice that in this case, by (7.8), we get

$$|X^{(n)}| = 2^n,\ |T_n| = 3^{2^{n-1}} \text{ and } |T^{(n)}| = 3^{2^n - 1}. \tag{7.9}$$

We will need the following function.

Definition 7.25. *For $k, n \in \mathbb{N}$, let $\delta(k, n)$ be the number of f in $T^{(n)}$ that are generated by precisely k members of $X^{(n)}$. When the context is clear we will write '$f \in \delta(k, n)$' to mean '$f \in T^{(n)}$ and f is generated by precisely k members of $X^{(n)}$'.*

We could not get an explicit expression for $\delta(k, n)$ (but see Lemma 7.37). However, for our immediate purposes we can make do with the following.

Lemma 7.26. *We have $\delta(1, 1) = 2$ and $\delta(2, 1) = 1$. For each $n \in \mathbb{N}$ and $k \in [2^{n+1}]$ we have,*

$$\delta(k, n+1) = \sum_{l \in [k/2, k] \cap \mathbb{N}} \delta(l, n) \binom{l}{k-l} 2^{2l-k} 3^{2^n - l}. \tag{7.10}$$

Proof. Any member of $T^{(n+1)}$ will be an extension of a member of $T^{(n)}$. If $f \in T^{(n)}$ is generated by precisely l sequences $t_1, ..., t_l$, then any extension of f to $g \in T^{(n+1)}$ will be generated by the extensions of the t_i that appear in $g(n+1)$. In particular any extension of f will be generated by at least l sequences. If $l < k/2$ then any extension of f will be generated by at most $k - 1$ sequences. This gives

$$\delta(k, n+1) = \sum_{l \in [k/2, k] \cap \mathbb{N}} \sum_{f \in \delta(l, n)} |\{A \in T_{n+1} : A \text{ contains precisely } k \text{ extensions of generators of } f\}|.$$

It is clear that for each l the number

$$|\{A \in T_{n+1} : A \text{ contains precisely } k \text{ extensions of generators of } f\}|$$

is independent of which $f \in \delta(l,n)$ we consider. Thus if we enumerate $X^{(n)} = \{t_1, ..., t_{2^n}\}$ we get

$$\delta(k, n+1) = \sum_{l \in [k/2,k] \cap \mathbb{N}} \delta(l,n) |\{A \in T_{n+1} : A \text{ contains precisely } k \text{ extensions of } t_1, ...t_l\}|. \tag{7.11}$$

Now fix $l \in [k/2, k] \cap \mathbb{N}$ and suppose that A contains precisely k extensions of $t_1, ..., t_l$. Then we can find some $I \subseteq [l]$ and, for each $i \in [l] \setminus I$, a natural number $n_i \in [2]$ such that

$$A = \{t_i^\frown 1 : i \in I\} \sqcup \{t_i^\frown 2 : i \in I\} \sqcup \{t_i^\frown n_i : i \in [l] \setminus I\} \sqcup A'$$

where $A' \subseteq \bigcup \{A_s^{(n+1)} : s \in X^{(n)} \setminus \{t_1, ..., t_l\}\}$ such that $A' \cap A_s^{(n+1)} \neq \emptyset$ for each $s \in X^{(n)} \setminus \{t_1, ..., t_l\}$ (recall the notation from (7.8)). There are $\binom{l}{k-l}$ such I, 2^{2l-k} such n_i (given I) and $3^{2^n - l}$ such A'. This together with (7.11) gives (7.10). □

The expression corresponding to (7.10) for general X_i is given in Lemma 7.40 on page 89.

Now we give an upper bound the $\delta(k,n)$, which is essentially the central calculation of this subsection.

Lemma 7.27. *For every $n \in \mathbb{N}$ and $k \in [2^n]$ we have*

$$\frac{\delta(k,n)}{|T^{(n)}|} \leq \left(\frac{9}{10}\right)^n. \tag{7.12}$$

Proof. We proceed by induction on n. The fact that

$$\delta(k,1) = \begin{cases} 2, & \text{if } k = 1, \\ 1, & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases}$$

deals with the base step.

For the induction we first recall the following well known fact. Let $F(0), F(1), F(2), ...$ enumerate the Fibonacci sequence $0, 1, 1, 2, 3, 5, \cdots$. Then

$$\sum_{i=0}^{\text{Floor}(n/2)} \binom{n-i}{i} = F(n+1). \tag{7.13}$$

This is proved by an easy induction using the fact that $F(n+1) = F(n) + F(n-1)$ and

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

We imitate this result to obtain the following.

Claim. *Let $G(0) = 0$ and $G(1) = 1$ and set $G(n+1) = G(n) + \frac{3}{4}G(n-1)$. Then*

$$G(n) = \frac{1}{2}\left(\frac{3}{2}\right)^n - \frac{1}{2}\left(-\frac{1}{2}\right)^n$$

and

$$\sum_{i=0}^{\mathrm{Floor}(n/2)} \binom{n-i}{i}\left(\frac{3}{4}\right)^i = G(n+1). \tag{7.14}$$

Proof. The sequence G is an example of a linear homogeneous recurrence relation with constant coefficients and obtaining a closed solution for these is standard. The identity (7.14) is obtained in precisely the same way as (7.13). □

Now by a change of variable $l = k - i$ in (7.10) we get

$$\begin{aligned}
\frac{\delta(k,n)}{|T^{(n)}|} &= \frac{1}{3^{2^n-1}} \sum_{i=0}^{\mathrm{Floor}(k/2)} \delta(k-i, n-1)\binom{k-i}{i} 2^{k-2i} 3^{2^{n-1}-k+i} \\
&= \left(\frac{2}{3}\right)^k 3^{1-2^{n-1}} \sum_{i=0}^{\mathrm{Floor}(k/2)} \delta(k-i, n-1)\binom{k-i}{i}\left(\frac{3}{4}\right)^i \\
&\leq \left(\frac{2}{3}\right)^k \sum_{i=0}^{\mathrm{Floor}(k/2)} \delta(k-i, n-1)\binom{k-i}{i}\left(\frac{3}{4}\right)^i
\end{aligned}$$

By the above claim and induction we obtain

$$\begin{aligned}
\frac{\delta(k,n)}{|T^{(n)}|} &\leq \left(\frac{9}{10}\right)^{n-1}\left(\frac{2}{3}\right)^k \sum_{i=0}^{\mathrm{Floor}(k/2)} \binom{k-i}{i}\left(\frac{3}{4}\right)^i \\
&= \left(\frac{9}{10}\right)^{n-1}\left(\frac{2}{3}\right)^k G(k+1) \\
&= \left(\frac{9}{10}\right)^{n-1}\left(\frac{2}{3}\right)^k \left(\frac{1}{2}\left(\frac{3}{2}\right)^{k+1} - \frac{1}{2}\left(-\frac{1}{2}\right)^{k+1}\right) \\
&= \left(\frac{9}{10}\right)^{n-1} \frac{10}{12} \leq \left(\frac{9}{10}\right)^n.
\end{aligned}$$

□

Definition 7.28. *Given $f \in T^{(n)}$ let $G(f) = \{t \in X^{(n)} : t \text{ generates } f\}$. Given $A \subseteq X^{(n)}$ let*

$$\delta(k, A) = |\{f \in T^{(n)} : |G(f) \cap A| = k\}|.$$

As before we write '$f \in \delta(k, A)$' to mean '$f \in T^{(n)}$ and $|G(f) \cap A| = k$'.

Lemma 7.29. *Suppose $A \subseteq X^{(n)}$ and $a = \bigcup_{t\in A}[t]$. Then for each $j \in X_{n+1}$ we have*

$$|\{f \in T^{(n+1)} : [f] \subseteq \mathfrak{f}(a \cap C_{n+1,j})\}| = \sum_{k=1}^{2^n} 3^{2^n-k}\delta(k,A)(3^k-1). \tag{7.15}$$

Thus we have

$$\mu(a \cap C_{n+1,j}) = \mu(a) - \frac{1}{|T^{(n)}|}\sum_{k\in[2^n]}\frac{\delta(k,A)}{3^k}. \tag{7.16}$$

Proof. The idea here is precisely the same as that used to obtain (7.10). Without loss of generality we consider the case $j = 1$. It is easily seen that

$$\{f \in T^{(n+1)} : [f] \subseteq \mathfrak{f}(a \cap C_{n+1,1})\} = \bigsqcup_{k=1}^{2^n}\bigsqcup_{f\in\delta(k,A)}\{f^\frown C : t^\frown 1 \in C \text{ for some } t \in A \text{ that generates } f\}.$$

So say that f is generated by precisely $t_1, ..., t_k$ members of A and suppose that $[f^\frown C] \subseteq \mathfrak{f}(a \cap C_{n+1,1})$ for some $C \in T_{n+1}$. Then we can find some $I \in \mathcal{P}([k])^+$ and $J \subseteq I$ such that

$$C = \{t_i^\frown 1 : i \in I\} \sqcup \{t_i^\frown 2 : i \in J\} \sqcup \{t_i^\frown 2 : i \in I \setminus [k]\} \sqcup A'.$$

where $A' \subseteq \bigcup\{A_s^{(n+1)} : s \in X^{(n)} \setminus \{t_1, ..., t_k\}\}$ such that $A' \cap A_s^{(n+1)} \neq \emptyset$, for each $s \in X^{(n)} \setminus \{t_1, ..., t_k\}$. From this we obtain

$$|\{f \in T^{(n+1)} : [f] \subseteq \mathfrak{f}(a \cap C_{n+1,1})\}| = \sum_{k=1}^{2^n}\delta(k,A)\sum_{i=1}^{k}\binom{k}{i}2^i 3^{2^n-k},$$

from which we easily obtain (7.15). Now using the fact that

$$\sum_{k\in[2^n]}\delta(k,A) = |\{f \in T^{(n)} : [f] \subseteq \bigcup_{t\in A}\mathfrak{f}([t])\}|,$$

we get that $\mu(a) = \frac{1}{|T^{(n)}|}\sum_{k\in[2^n]}\delta(k,A)$, and from this we obtain (7.16). □

Proposition 7.30. *For every $a \in \mathfrak{A}$ we have*

$$\mu(a \cap C_{n,2}) = \mu(a \cap C_{n,1}) \to \mu(a), \quad n \to \infty.$$

It follows that for every clopen a and $\epsilon > 0$ there exists $b, b' \subseteq a$ such that $b \sqcup b' = a$, $\mu(b) = \mu(b')$ and $\mu(a) - \mu(b) < \epsilon$. In particular, μ cannot be exhaustive.

Proof. Let $A \subseteq X^{(n)}$ be such that $a = \bigcup_{t\in A}[t]$. Let $M = |A|$ and let $G(f) = \{t \in X^{(n)} : t \text{ generates } f\}$ (for $f \in T^{(n)}$). Now for any $k \in [M]$ and $f \in \delta(k,A)$ we have $f \in \delta(k+|G(f)\setminus A|, n)$. In particular

$$\delta(k,A) \leq \sum_{l=0}^{2^n-M}\delta(k+l,n).$$

But then by Lemma 7.27
$$\frac{\delta(k, A)}{|T(n)|} \leq (2^n - M + 1)\left(\frac{9}{10}\right)^n.$$
Given $b \in \mathfrak{A}$ and $n \in \mathbb{N}$ let $\alpha(b, n) = 2^n - |B| + 1$, for the (unique) $B \subseteq X^{(n)}$ such that $b = \bigcup_{t \in B}[t]$. Then $\alpha(b, n)$ is constant for fixed b. Thus we have
$$\frac{1}{|T^{(n)}|}\sum_{k\in[M]}\frac{\delta(k, A)}{3^k} \leq \alpha(a, n)\left(\frac{9}{10}\right)^n \sum_{k\in[M]}\frac{1}{3^k} \leq \frac{1}{2}\alpha(a, n)\left(\frac{9}{10}\right)^n \to 0.$$

The result now follows by (7.16). □

Consider now the extension of μ to Borel($\mathfrak{A}$). That is we set
$$\mu(A) = \inf\{\sum_{i\in\mathbb{N}} \mu(a_i) : a_i \in \mathfrak{A} \wedge A \subseteq \bigcup_i a_i\}.$$

We have the following.

Theorem 7.31. *For any $\epsilon \in [0, 1)$ and $I \in [\mathbb{N}]^\omega$ there exists $J \in [I]^\omega$ such that, for each $g \in \prod_{j\in J} X_j$, the sets $\{f \in X : f(j) = g(j)\}$ have μ-measure at least ϵ.*

For this we will first show that μ is invariant with respect to the isometry group of X.

Proposition 7.32. *If g is a bijective isometry of X then for every $a \in \mathfrak{A}$ we have $\mu(g[a]) = \mu(a)$.*

Proof. Let g be a bijective isometry of X. Let $g_0 : X_1 \to X_1$ be the permutation defined by $g_0(i) = g(i^\frown s)(1)$, for any $s \in \prod_{n>1} X_n$. For each $n \in \mathbb{N}$ and $t \in X^{(n)}$ we define the permutation $g_t : X_{n+1} \to X_{n+1}$ by $g_t(i) = g(t^\frown i^\frown s)(n + 1)$ where s is any member of $\prod_{j>n+1} X_i$. For $n \in \mathbb{N}$ let $G_n : X^{(n)} \mapsto X^{(n)}$ be the map
$$t \mapsto (g_0(t(1)), g_{t\restriction[1]}(t(2)), ..., g_{t\restriction[n-1]}(t(n))). \tag{7.17}$$

Claim. *The map G_n is a permutation of $X^{(n)}$.*

Proof. Clearly this is true for G_1. For $n \in \mathbb{N}$, let $t, s \in X^{(n+1)}$, and suppose that $s \neq t$. Then we can find an $i \in [n + 1]$ such that $s(i) \neq t(i)$ but $s \restriction [i - 1] = t \restriction [i - 1]$. But then
$$g_{t\restriction[i-1]}(t(i)) = g_{s\restriction[i-1]}(t(i)) \neq g_{s\restriction[i-1]}(s(i)),$$
since $g_{t\restriction[i-1]}$ and $g_{s\restriction[i-1]}$ are themselves injective. Here by $g_{t\restriction[0]}$ we mean g_0. This shows that G_{n+1} is injective. To see that G_{n+1} is surjective let $t \in X^{(n+1)}$ and define $s \in X^{(n+1)}$ as follows. Let $s(1) = g_0^{-1}(t(1))$. If $s_i = (s(1), ...s(i))$ has been defined, let $s(i+1) = g_{s_i}^{-1}(t(i+1))$. Then $G_{n+1}(s) = t$.[10] □

[10] In fact, from this last argument we obtain the following (recursive) definition for G_{n+1}^{-1}:
$$G_{n+1}^{-1}(t) = (g_0^{-1}(t(1)), g^{-1}_{G^{-1}_{n+1}(t)\restriction[1]}(t(2)), g^{-1}_{G^{-1}_{n+1}(t)\restriction[1]}(t(2)), ..., g^{-1}_{G^{-1}_{n+1}(t)\restriction[n]}(t(n + 1))).$$

Now for each $i \in \mathbb{N}$, the map $T_i \to T_i : A \mapsto G_i[A]$, defines a permutation of T_i. It needs to be checked that if $A \in T_i$ then $G_i[A] \subseteq X^{(i)}$ remains in T_i, but this follows from (7.17). Thus the map

$$F : T \to T : f \mapsto (G_1[f(1)], G_2[f(2)], ...)$$

defines a bijective isometry of T.

Claim. *If $t \in X^{(n)}$ then $F[\mathfrak{f}([t])] = \mathfrak{f}([G_n(t)])$.*

Proof. Just note that $f \in T^{(n)}$ is generated by t if and only if $(G_1[f(1)], G_2[f(2)], ..., G_n[f(n)])$ is generated by $G_n(t)$. □

Finally let $a \in \mathfrak{A}$ and $A \subseteq X^{(n)}$ be such that $a = \bigcup_{t\in A}[t]$. Since the Lebesgue measure is invariant under the isometry group of T, for any $A \subseteq X^{(n)}$, we have

$$\begin{aligned} \mu(a) = \lambda(\bigcup_{t\in A} \mathfrak{f}([t])) = \lambda(F[\bigcup_{t\in A} \mathfrak{f}([t])]) &= \lambda(\bigcup_{t\in A} F[\mathfrak{f}([t])]) \\ &= \lambda(\bigcup_{t\in A} \mathfrak{f}([G_n(t)])) = \lambda(\bigcup_{t\in G_n[A]} \mathfrak{f}([t])) = \mu(g[a]). \end{aligned}$$

□

Remark 7.33. *The maps g_t in the proof of Proposition 7.32 actually characterise bijective isometries of X. Indeed, it is straightforward to check that given a sequence $(g_x)_{x\in\bigcup_{i\in\omega} X^{(i)}}$ such that each g_x is a permutation of $X_{|x|+1}$, then the map $g : X \mapsto X$ defined by*

$$g(x)(i) = g_{x\restriction[i-1]}(x(i))$$

is a bijective isometry.

Proof of Theorem 7.31. Fix $\epsilon > 0$. By a repeated application of Proposition 7.30 we can find an increasing sequence of integers $n_1 < n_2 < \cdots \in I$ such that $\mu(\bigcap_{i\in[k]} C_{n_i,1}) \geq \epsilon$, always. Now we want to measure $c = \bigcap_{i\in\mathbb{N}} C_{n_i,1}$. This is of course a closed set so that, by compactness of X, we have

$$\mu(c) = \inf\{\mu(a) : a \in \mathfrak{A} \wedge c \subseteq a\}.$$

Now suppose that $a \in \mathfrak{A}$ is such that $c \subseteq a$. Let $A \subseteq X^{(n)}$ be such that $\bigcup_{t\in A}[t] = a$. Let $n_k > n$. Then it is straightforward to check that $\bigcap_{i\in[k]} C_{n_i,1} \subseteq a$. This shows that any cover of c by a clopen set must have μ-measure at least ϵ. Thus $\mu(c) \geq \epsilon$. Now let $J = \{n_i : i \in \mathbb{N}\}$ and notice that for any $g \in \prod_{i\in I} X_i$ the map $x \mapsto x + g$ is a bijective isometry of X and so by Proposition 7.32, we will also have $\mu(g[c]) \geq \epsilon$. □

Consider now the forcing $\mathbb{P} := (\mathrm{Borel}(X)/\mathrm{Null}(\mu))^+$. We note that the generic real determining the extension of $\mathbb{P}$ cannot be a Cohen real or a random real beneath any condition. This follows from the fact that the maximal antichain obtained from Theorem 7.31 consists of meagre Lebesgue null sets. In fact we can say slightly more. Recall that both Cohen and random reals are splitting reals (Fact 2.36).

Corollary 7.34. *If $\dot{a}$ is a name for the subset of natural numbers corresponding to the determining real of $\mathbb{P}$, then for each $p \in \mathbb{P}$ we can find $q \leq p$ and a set $B_q \in [\mathbb{N}]^\omega$ such that*

$$q \Vdash \check{B}_q \subseteq \dot{a} \vee \check{B}_q \cap \dot{a} = 0.$$

In particular, $\dot{a}$ cannot be a splitting real beneath any condition.

Proof. Let $\dot{r}$ be the determining real of $\mathbb{P}$. In particular for all Borel sets of positive μ-measure c, we have $c \Vdash \dot{r} \in c$. Let $\dot{a}$ be such that $\mathbb{P} \Vdash (\forall n \in \mathbb{N})(n \in \dot{a} \leftrightarrow \dot{r}(n) = 1)$. Fix $p \in \mathbb{P}$. Let $J \in [\mathbb{N}]^\omega$ be that promised by Theorem 7.31 (with $I = \mathbb{N}$). For each $g \in \prod_{i \in J} X_i$ let

$$c_g = \{f \in X : (\forall j \in J)(f(j) = g(j))\}.$$

Then the c_g form a maximal antichain in $\mathbb{P}$, so we can find some c_g and a $q' \leq c_g \wedge p$. Now let $B_1 = \{i \in J : g(i) = 1\}$ and $B_2 = \{i \in J : g(i) = 2\}$. Since $c_g \Vdash \dot{r} \restriction J = g$ we must have $q' \Vdash \check{B}_1 \subseteq \dot{a} \wedge \check{B}_2 \cap \dot{a} = 0$. Since one of B_1 or B_2 is infinite we have that $q' \Vdash (\exists B \in [\check{I}]^\omega)(B \subseteq \dot{a} \vee B \cap \dot{a} = 0)$. In particular, for some name $\check{B} \in \mathrm{dom}([\check{I}]^\omega)$ and $q \leq q'$ we have $q \Vdash \check{B} \subseteq \dot{a} \vee \check{B} \cap \dot{a} = 0$ (see [23, 3.7 Corollary (d)]). Now let $B_q = B$. □

Since μ is an example of a *pavement submeasure* we state the following taken from [38, Question 7.3.7].

Question 7.35. *Does $\mathbb{P}$ add a Cohen real?*

7.4 Miscellaneous countings

In this section we gather some miscellaneous counting arguments from the previous sections. We begin by generalising the calculations from Example 7.12 on page 71. In fact the following may be viewed as a replacement for Lemmas 7.5 and 7.6.

Lemma 7.36. *Let $n \in \mathbb{N}$ and and $a_1, ..., a_n$ be the $*$-free generators of $\mathrm{Fr}^* n$. Let $\lambda : \{\bigcup_{i \in I} a_i : I \in \mathcal{P}([n])^+\} \to \mathbb{R}$ be any map. Then λ extends uniquely to a measure on $\mathrm{Fr}^* n$ and is defined by:*

$$\lambda(I) = (-1)^{|I|} \sum_{l=0}^{|I|} (-1)^{l+1} \sum_{y \in [I]^l} \lambda([n] \setminus y). \tag{7.18}$$

where in the above, if $z \subseteq [n]$, by $\lambda(z)$ we mean $\lambda(\bigcup_{i \in z} a_i)$.

Before we prove this we note the following. If $\sigma(l, m, k) = |\{a \in [k]^m : [l] \subseteq a\}|$, then clearly $\sigma(l, m, k) = \binom{k-m}{m-l}$ and so

$$\sum_{i=0}^{k-l} \sigma(l, l+i, k+1) x^i = \sum_{i=0}^{k-l} \binom{k+1-l}{i} x^i = (1+x)^{k+1-l} - x^{k+1-l}. \tag{7.19}$$

Proof of Lemma 7.36. By Lemma 7.6 such an extension exists (or just proceed to show that the above defines the required signed measure). Clearly this extension must satisfy the above for $|I| = 1$ (since it is additive). Suppose it is true for all $|I| \leq k$. Let $I \in [n]^{k+1}$ and notice that

$$1_{\mathrm{Fr}^* n} = \{I\} \sqcup \bigsqcup_{J \in [I]^{\leq k}} \{J\} \sqcup \left(\bigcup_{i \notin I} a_i \right). \tag{7.20}$$

Now for each $p \in [k]$ we have

$$\begin{aligned} \sum_{J \in [I]^p} \lambda(J) &= (-1)^p \sum_{J \in [I]^p} \sum_{l=0}^{p} (-1)^{l+1} \sum_{y \in [J]^l} \lambda([n] \setminus y) \\ &= (-1)^p \sum_{l=0}^{p} (-1)^{l+1} \sigma(l, p, k+1) \sum_{y \in [I]^l} \lambda([n] \setminus y). \end{aligned}$$

To see this just observe that for each $l \in [k]$ and $y \in [I]^l$, the number of times that the term $\lambda([n] \setminus y)$ will appear in the summand will be equal to the number of subsets of I, of size p, that contain y. This gives

$$\begin{aligned} \sum_{p \in [k]} \sum_{J \in [I]^p} \lambda(J) &= \sum_{p=1}^{k} \sum_{l=0}^{p} (-1)^{l+p+1} \sigma(l, p, k+1) \sum_{y \in [I]^l} \lambda([n] \setminus y) \\ &= \sum_{l=1}^{k} (-1)^{l+1} \sigma(0, l, k+1) \lambda([n]) \\ &- \sum_{l=1}^{k} \sum_{i=0}^{k-l} (-1)^i \sigma(l, l+i, k+1) \sum_{y \in [I]^l} \lambda([n] \setminus y) \\ &\overset{(7.19)}{=} (1 - (-1)^k) \lambda([n]) + \sum_{l=1}^{k} (-1)^{k+1-l} \sum_{y \in [I]^l} \lambda([n] \setminus y) \end{aligned}$$

From (7.20) we then get

$$\begin{aligned} \lambda(I) &= \lambda([n]) - \left((1 - (-1)^k) \lambda([n]) + \sum_{l=1}^{k} (-1)^{k+1-l} \sum_{y \in [I]^l} \lambda([n] \setminus y) \right) - \lambda([n] \setminus I) \\ &= (-1)^k \lambda([n]) - \lambda([n] \setminus I) + \sum_{l=1}^{k} (-1)^{k-l} \sum_{y \in [I]^l} \lambda([n] \setminus y), \end{aligned}$$

which is easily seen to give (7.18). □

Now we describe a procedure that allows us to obtain explicit values for the $\delta(k, n)$ via (7.10) on page 79. From this we can obtain reasonably nice expressions for the first few values of

$\delta(k,n)$, but it is not clear to us how to proceed. Let

$$\gamma_{k,n+1} = \sum_{l\in[k/2,k)\cap\mathbb{N}} \delta(l,n)\binom{l}{k-l}2^{2l-k}3^{2^n-l},$$

(the sum of all but the last expression in (7.10)). Then of course we have

$$\delta(k,n+1) = \gamma_{k,n+1} + \delta(k,n)2^k3^{2^n-k}. \tag{7.21}$$

An(other tedious) induction on n using (7.21) gives

$$\delta(k,n+1) = 2^{(n+1)k}3^{2^{n+1}-(n+1)k}\sum_{i=2}^{n+1}\gamma_{k,i}2^{-ik}3^{-2^i+ik} + \delta(k,1)2^{nk}3^{2^{n+1}-2-nk}. \tag{7.22}$$

Lemma 7.37. *We have* $\delta(2,n+1) = \delta(1,n)|T_{n+1}| - 2\delta(1,n)^2$ *and*

$$\delta(3,n+1) = \frac{3^{n+1}-2^{n+1}}{3^n}(\frac{3^{n+1}+2^{n+1}}{53^n}\delta(1,1+n) - 2\delta(1,n)^2).$$

Proof. We prove the first expression but omit the second. A straightforward induction on n using (7.10) gives

$$\delta(1,n) = 2^n3^{2^n-1-n}.$$

Then we have

$$\gamma_{2,n+1} = \delta(1,n)3^{2^n-1} = 2^n3^{2^{n+1}-2-n}.$$

By (7.22) we get

$$\begin{aligned}
\delta(2,n+1) &= 2^{(n+1)2}3^{2^{n+1}-(n+1)2}\sum_{i=2}^{n+1}2^{i-1}3^{2^i-i-1}2^{-2i}3^{-2^i+2i} + \delta(2,1)2^{2n}3^{2^{n+1}-2-2n}\\
&= 2^{2n+1}3^{2^{n+1}-2n-3}\sum_{i=2}^{n+1}\left(\frac{3}{2}\right)^i + 2^{2n}3^{2^{n+1}-2-2n}\\
&= 2^{2n+1}3^{2^{n+1}-2n-3}\sum_{i=1}^{n+1}\left(\frac{3}{2}\right)^i\\
&= \frac{2}{3}\delta(1,n)^2\sum_{i=1}^{n+1}\left(\frac{3}{2}\right)^i = 2\delta(1,n)^2\left(\frac{3}{2}\right)^{n+1} - 2\delta(1,n)^2,
\end{aligned}$$

and we are done since $|T_n| = 3^{2^n}$ (see (7.9) on page 79). □

Next we give an alternative expression for (7.10). This expression is not inductively defined but it relies on understanding certain partition numbers of integers. This makes it complicated.

For $n\in\mathbb{N}$ let $b(n) = \max\{k\in\omega : 2^k\le n\}$ and let $B(n) = \{(a_0,...,a_{b(n)}) : \sum_{i=0}^{b(n)} a_i2^{b(n)-i} =$

$n \wedge a_i \in \omega\}$. Of course each a_i need only range over $\{0, 1, ...i+1\}$ (not ω). Notice that $n \mapsto |B(n)|$ is well known as the *binary partition function* (for example, see [26]). Let

$$B'(n) = \{a \in B(n) : (\forall i \in [b(n)])(a(i) \leq 2^i - \sum_{j=0}^{i-1} 2^{i-j-1} a(j))\}.$$

Given $A \subseteq X^{(m)}$ and $n \geq m$, let $(A)_n = \{t \in X^{(n)} : t \restriction [m] \in A\}$. Now suppose $n \in \mathbb{N}$ and $l \in 2^n$. Let $A_0 = X^{(n-b(l))} \setminus f(n-b(l))$. And set

$$A_{i+1} = [X^{(n-b(l)+i)} \setminus f(n-b(l)+i)] \setminus \bigcup_{j \leq i} (A_j)_{n-b(l)+i}.$$

The motivation for considering the sets $B'(n)$ comes from the following lemma.

Lemma 7.38. *If $f \in \delta(2^n - l, n)$ then $(|A_0|, |A_1|, ..., |A_{b(l)}|) \in B'(l)$.*

Proof. We observe that $t \in X^{(n)}$ generates f precisely when $(\forall i)(t \notin (A_i)_n)$. So we have

$$l = |\bigsqcup_i (A_i)_n| = \sum_{i=0}^{b(l)} |A_i| 2^{b(l)-i}.$$

□

We can go the other way too! Let $n \in \mathbb{N}$ and $l \in \{0, 1, ..., 2^n - 1\}$. Fix $a \in B'(l)$ and define a member $f \in \delta(2^n - l, n)$ as follows. Let $C_0 \in [X^{(n-b(l)-1)}]^{a(0)}$ and let $A_0 = \{t^\frown i_t : t \in C_0\}$, for some $i_t \in X_{(n-b(l))}$. Now let

$$C_{i+1} \in [X^{n-b(l)+i} \setminus \bigcup_{j \leq i} (A_j)_{n-b(l)+i}]^{a(i+1)},$$

and set $A_{i+1} = \{t^\frown i_t : t \in C_{i+1}\}$, for some $i_t \in X_{n-b(l)+i+1}$. Now let $f \in T^{(n)}$ be defined by $f \restriction [n-b(l)-1] = (X^{(1)}, X^{(2)}, ..., X^{(n-b(l)-1)})$ and for $i \in [0, b(l)]$ we let $f(n-b(l)+i)$ be any member of

$$Z_i := \{A \in T_{n-b(l)+i} : X^{n-b(l)+i} \setminus \bigcup_{j \leq i} (A_j)_{n-b(l)+i} \subseteq A \wedge A \cap A_i = 0\}. \tag{7.23}$$

It can now be checked that $t \in X^{(n)}$ generates f precisely when $(\forall i)(t \notin (A_i)_n)$. In particular the A_i constructed in the discussion preceding Lemma 7.38 will coincide with the A_i here. From this we may conclude that $\delta(2^n - l, n)$ is equal to the number of f's one can construct in the above manner. This gives the following.

Lemma 7.39. *For $n \in \mathbb{N}$ and $l \in 2^n$ then we have*

$$\delta(2^n - l, n) = \sum_{a \in B'(l)} \prod_{i=0}^{b(l)} \binom{2^{(n-b(l)-1)+i} - \sum_{j=0}^{i-1} 2^{i-j-1} a(j)}{a(i)} 2^{a(i)} \prod_{j=0}^{i-1} 3^{2^{i-j-1} a(j)}. \tag{7.24}$$

Proof. For each $a \in B'(l)$ the binomial coefficient counts the number of choices for the C_i. The term $2^{a(i)}$ counts the number of choices for A_i, given C_i. The term $\prod_{j=0}^{i-1} 3^{2^{i-j-1}a(j)}$ is the cardinality of Z_i from (7.23). $\square$

Finally for this subsection we give an expression for $\delta(k,n)$ for general X_i. This again seems to require integer partitions. For natural numbers $l \leq k \leq |X^{(n+1)}|$ let

$$P(n,k,l) = \{(a_1,...,a_s) : a_i \in \omega \wedge \sum_{i\in[s]} ia_i = k \wedge \sum_i a_i = l \wedge s \in [|X_{n+1}|]\}$$

That is $P(n,k,l)$ represents a subcollection of integer partitions of k into no more than $|X_{n+1}|$. To see why we are interested in such a collection let $(a_1,...,a_s) \in P(n,k,l)$ and $f \in T^{(n)}$ be generated by $t_1,...,t_l$. Let $(C_j)_{j\in[s]}$ be a partition of $[l]$ such that $|C_j| = a_j$. For each $i \in C_j$ let $B_{i,j} \subseteq X_{n+1}$ such that $|B_{i,j}| = j$. Now notice that if $A \in T_{n+1}$ is such that

$$\{t_i^\frown x : x \in B_{i,j}\} \subseteq A \tag{7.25}$$

then $f^\frown A \in \delta(k, n+1)$.

Conversely, if $f \in \delta(k,n+1)$ let $\{t_1,...,t_l\} = \{t \restriction [n] : t \text{ generates } f\}$. Then the t_i will be the generators of $f \restriction [n]$. If, for $j \in \omega$, we let $C_j = \{i \in [l] : t_i \text{ has precisely } i \text{ extensions in } f(n+1)\}$ and set $s = \max\{j : C_j \neq \emptyset\}$ then we see that $(|C_1|,...,|C_s|) \in P(n,k,l)$. If, for $i \in C_j$, we set $B_{i,j} = \{x : t_i^\frown x \in f(n+1)\}$ then of course

$$\{t_i^\frown x : x \in B_{i,j}\} \subseteq f(n+1).$$

The number $\delta(k,n+1)$ is given by

$$\sum_{l\in[k]} \delta(l,n)|\{A \in T_{n+1} : (\exists t_1,...,t_l \in X^{(n)})(A \text{ contains precisely } k \text{ extensions of } t_1,...,t_l)\}|.$$

So for each $l \in [k]$ and $f \in \delta(l,n)$ we need to count the number of $A \in T_{n+1}$ such that $f^\frown A \in \delta(k,n+1)$. But from the above discussion this is just the number of A's satisfying (7.25), given $(a_1,...,a_s) \in P(n,k,l)$. Thus we have the following.

Lemma 7.40. *For general X_i we have*

$$\delta(k,n+1) = \sum_{l\in[k]} \sum_{a\in P(n,k,l)} \prod_{j\in \mathrm{dom}(a)\wedge a(j)\neq 0} \delta(l,n) \binom{l}{a} \binom{|X_{n+1}|}{j}^{a(j)} (2^{|X_{n+1}|}-1)^{|X^{(n)}|-l}$$

where by $\binom{k}{a}$ we mean the multinomial coefficient

$$\binom{k,}{a(1),a(2),...,a(k)} = \frac{k!}{\prod_{i\in[k]} a_i!}.$$

Proof. The multinomial coefficient counts the number of choices for the C_j. The binomial coefficient counts the number of $B_{i,j}$, given the C_j. The last term counts the number of A that will satisfy (7.25). $\square$

A Dow and Hart

We prove the following Proposition A.1 (below), which is Proposition 2.1 from [5], but under our more general conditions. This is the only part in which one needs to consider the property $\mathbb{C}(\mathfrak{B})$ in [5]. Everything is taken from [5] so we excuse ourselves from any further referencing. Throughout this section fix a σ-complete Boolean algebra $\mathfrak{B}$ such that $\mathbb{C}(\mathfrak{B})$ holds, witnessed by $\mathbb{F}$. If $\varphi : \mathfrak{B} \to \mathcal{P}(\omega)/\mathrm{Fin}$ is an embedding then a map $\Phi : \mathfrak{B} \to \mathcal{P}(\omega)$ is a **lifting** for φ if and only if

$$(\forall x \in \mathfrak{B})(\varphi(x) = [\Phi(x)]_{\mathrm{fin}}).$$

If Φ is a lifting then for brevity we will write $\Phi(x)$ instead of $\Phi(\mathbb{F}(x))$, for $x \in \mathrm{Clopen}(\omega \times {}^{\omega}2)$. We will also write (n, s) instead of $\{n\} \times [s]$.

If $A \subseteq \omega$ then a lifting Φ is **exact** on A if and only if for every $n, m \in A$ and $s \in \bigcup_{n\in\omega} {}^{n}2$ we have

$$n \neq m \to \Phi(n, \emptyset) \cap \Phi(m, \emptyset) = \emptyset \text{ and } \Phi(n, s) = \Phi(n, s^\frown 0) \sqcup \Phi(n, s^\frown 1).$$

If $f \in {}^{\omega}\omega$ and $A \subseteq \omega$ then set

$$B_{f,A} = \{(n, s) : n \in A \wedge s \in 2^{f(n)}\}.$$

If $A \subseteq \omega$ then a lifting Φ is **complete on** A if and only if for every $f \in {}^{\omega}\omega$ and $O \subseteq B_{f,A}$ we have

$$\Phi[O] := \bigcup_{(n,s)\in O} \Phi(n, s) =^* \Phi(\sum O),$$

where two subsets $M =^* N$ if and only if $M \triangle N \in \mathrm{Fin}$ (that is, they are identified in $\mathcal{P}(\omega)/\mathrm{Fin}$). Theorem 3.1 for $\mathfrak{B}$ follows from the following two results.

Proposition A.1. *If $\varphi : \mathfrak{B} \to \mathcal{P}(\omega)/\mathrm{fin}$ is an embedding, $A \subseteq \omega$ is infinite and Φ is a lifting for φ that is exact on A then Φ is not complete on A.*

Theorem A.2. (OCA) *If there exists an embedding $\varphi : \mathfrak{B} \to \mathcal{P}(\omega)/\mathrm{fin}$ then there exists a lifting Φ for φ and an infinite $A \subseteq \omega$ such that Φ is both exact and complete on A.*

Proof of Proposition A.1. Let $n \in A$. For each $i \in \Phi(n, \emptyset)$ let $f_i^n \in 2^\omega$ be the unique function such that

$$(\forall m)(i \in \Phi(n, f_i^n \restriction m)). \tag{A.1}$$

By $\mathbb{C}(\mathfrak{B})$ we can find $N_1^n, N_2^n, \ldots$ such that

$$U_n := \sum_{i\in\Phi(n,\emptyset)} \mathbb{F}(n, f_i^n \restriction N_i^n) < \mathbb{F}(n, \emptyset).$$

Let $C_n = \mathbb{F}(n, \emptyset) \setminus U_n$ and $F = \sum_{n\in A} C_n$. Now for each $n \in A$ we have

$$[\Phi(F)] \cdot [\Phi(n, \emptyset)] = \varphi(F) \cdot \varphi(n, \emptyset) = \varphi(F \cdot \mathbb{F}(n, \emptyset)) = \varphi(\sum_{k\in A} C_k \cdot (n, \emptyset)) = \varphi(C_n) \neq 0.$$

Thus

$$(\forall n \in A)(|\Phi(F) \cap \Phi(n, \emptyset)| = \omega).$$

For each $n \in A$ let

$$k_n = \min \Phi(F) \cap \Phi(n, \emptyset). \tag{A.2}$$

Set $I = \{k_n : n \in A\}$ and $O = \{\mathbb{F}(n, f^n_{k_n} \restriction N^n_{k_n}) : n \in A\}$. Since $\sum O \cdot F = 0$ we must have

$$|\Phi(\sum O) \cap \Phi(F)| < \omega. \tag{A.3}$$

By (A.1) we have $I \subseteq \Phi[O]$ and by (A.2) we have $I \subseteq \Phi(F)$. Thus by (A.3)

$$\Phi[O] \neq^* \Phi(\sum O).$$

□

References

[1] R. B. J. T. Allenby. *Rings, fields and groups.* Edward Arnold, London, second edition, 1991. An introduction to abstract algebra.

[2] B. Balcar and T. Jech. Weak distributivity, a problem of von Neumann and the mystery of measurability. *Bull. Symbolic Logic*, 12(2):241–266, 2006.

[3] T. Bartoszyński and H. Judah. *Set theory.* A K Peters Ltd., Wellesley, MA, 1995. On the structure of the real line.

[4] J. P. R. Christensen. Some results with relation to the control measure problem. In *Vector space measures and applications (Proc. Conf., Univ. Dublin, Dublin, 1977), II*, volume 77 of *Lecture Notes in Phys.*, pages 27–34. Springer, Berlin, 1978.

[5] A. Dow and K. P. Hart. The measure algebra does not always embed. *Fund. Math.*, 163(2):163–176, 2000.

[6] R. M. Dudley. *Real analysis and probability*, volume 74 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002. Second edition.

[7] I. Farah and B. Veličković. Maharam algebras and Cohen reals. *Proc. Amer. Math. Soc.*, 135(7):2283–2290 (electronic), 2007.

[8] I. Farah and J. Zapletal. Between Maharam's and von Neumann's problems. *Math. Res. Lett.*, 11(5-6):673–684, 2004.

[9] D. H. Fremlin. *Maharam Algebras.* Version 22/10/06. Available from: `http://www.essex.ac.uk/maths/staff/fremlin/preprints.htm`.

[10] D. H. Fremlin. *Talagrand's Example.* Version 04/08/08. Available from: `http://www.essex.ac.uk/maths/staff/fremlin/preprints.htm`.

[11] D. H. Fremlin. Measure algebras. In *Handbook of Boolean algebras, Vol. 3*, pages 877–980. North-Holland, Amsterdam, 1989.

[12] D. H. Fremlin. *Measure theory. Vol. 3.* Torres Fremlin, Colchester, 2004. Measure algebras, Corrected second printing of the 2002 original.

[13] S. Givant and P. Halmos. *Introduction to Boolean algebras.* Undergraduate Texts in Mathematics. Springer, New York, 2009.

[14] P. Hall. On representatives of subsets. *J. London Math. Soc*, (10):26–30, 1935.

[15] P. Halmos. *Measure Theory.* D. Van Nostrand Company, Inc., New York, N. Y., 1950.

[16] W. Herer and J. P. R. Christensen. On the existence of pathological submeasures and the construction of exotic topological groups. *Math. Ann.*, 213:203–210, 1975.

[17] W. Hodges. *Model theory*, volume 42 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

[18] T. Jech. *Set theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[19] N. J. Kalton and J. W. Roberts. Uniformly exhaustive submeasures and nearly additive set functions. *Trans. Amer. Math. Soc.*, 278(2):803–816, 1983.

[20] A. S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[21] J. L. Kelley. Measures on Boolean algebras. *Pacific J. Math.*, 9:1165–1177, 1959.

[22] S. Koppelberg. *Handbook of Boolean algebras. Vol. 1*. North-Holland Publishing Co., Amsterdam, 1989. Edited by J. Donald Monk and Robert Bonnet.

[23] K. Kunen. *Set theory*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1980. An introduction to independence proofs.

[24] K. Kunen. Random and Cohen reals. In *Handbook of set-theoretic topology*, pages 887–911. North-Holland, Amsterdam, 1984.

[25] K. Kunen. *Set theory*, volume 34 of *Studies in Logic (London)*. College Publications, London, 2011.

[26] M. Latapy. Partitions of an integer into powers. In *Discrete models: combinatorics, computation, and geometry (Paris, 2001)*, Discrete Math. Theor. Comput. Sci. Proc., AA, pages 215–227 (electronic). Maison Inform. Math. Discrèt. (MIMD), Paris, 2001.

[27] D. Maharam. On homogeneous measure algebras. *Proc. Nat. Acad. Sci. U. S. A.*, 28:108–111, 1942.

[28] D. Maharam. An algebraic characterization of measure algebras. *Ann. of Math. (2)*, 48:154–167, 1947.

[29] J. W. Roberts. Maharam's problem. *Proceedings of the Orlicz Memorial Conference*, Kranz and Labuda editors, 1991.

[30] M. Searcóid. *Metric spaces*. Springer Undergraduate Mathematics Series. Springer-Verlag London Ltd., London, 2007.

[31] M. Talagrand. Maharam's problem. Preprint. arXiv:math/0601689v1.

[32] M. Talagrand. A simple example of pathological submeasure. *Math. Ann.*, 252(2):97–102, 1979/80.

[33] M. Talagrand. Maharam's problem. *Ann. of Math. (2)*, 168(3):981–1009, 2008.

[34] B. Veličković. The basis problem for CCC posets. In *Set theory (Piscataway, NJ, 1999)*, volume 58 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 149–160. Amer. Math. Soc., Providence, RI, 2002.

[35] B. Veličković. CCC forcing and splitting reals. *Israel J. Math.*, 147:209–220, 2005.

[36] B. Veličković. Maharam algebras. *Ann. Pure Appl. Logic*, 158(3):190–202, 2009.

[37] D. A. Vladimirov. *Boolean algebras in analysis*, volume 540 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2002. Translated from the Russian manuscript, Foreword and appendix by S. S. Kutateladze.

[38] J. Zapletal. *Forcing idealized*, volume 174 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2008.

Index

The End