An iterative approach for real roots of polynomials

Jean-Michel Billiot, Eric Fontenas

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Abstract

In the present study, we propose necessary and sufficient assumptions on the coefficients in order to only get distinct real roots of polynomials.

Keywords: Polynomials with only real roots; Polynomial sequences; Interlacing method, Sturm's theorem, Euclidean division.
AMS: 26A06, 26C10.

1 Introduction

It is well known that polynomials are very useful in order to approximate functions. The research of roots of polynomials is an old and famous problem. The theory of equations was studied by prestigious mathematicians such as d'Alembert, Cauchy, Gauss, Euler, Lagrange, Hermite, Galois among others. Polynomials with only real zeros arise often in different branches of mathematics. Nowadays, this is always the subject of an intense research, for example see [9] and references therein.

[9] proposed a unified approach to polynomial sequences with only real zeros. They give new sufficient conditions for a sequence of polynomials to have only real zeros based on the method of interlacing zeros. As applications, they derived the reality of zeros of orthogonal polynomials, matching polynomials, Narayana polynomials and Eulerian polynomials.

Recently [6], [7] studied cubic, quartic and quintic polynomials and proposed conditions on the coefficients derived from the Sturm sequence that will determine the real and complex root multiplicities together with the order of the real roots with respect to multiplicity.

Historically, equations of the first and second degree (where the coefficients are given numbers) are already solved with a general method by the Babylonians around 1700 bc. J.C and may be even earlier.

For the equations of degree three, it is necessary to wait until 1515 with the Italian Scipio del Ferro (1465-1526) whose papers are however lost. Then, his compatriots Nicolo Tartaglia and Gérolamo Cardano (1501-1576) continue his work. But it is Euler (1707-1783) who clarified the determination of the three roots in a Latin article of 1732.

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For the equations of the degree four, the answer comes from Jerome Cardan (1501-1576) and Lodovico Ferrari (1522-1565). Cardan gives a method in chapter 39 of the Ars Magna. He states that it was found by his pupil Lodovico Ferrari. In 1615, François Viète (1540-1603) clearly explains Ferrari's method. Descartes (1596-1650) also exposes another method of resolution.

For the equations of fifth order and above, the theorem, sometimes called Abel-Ruffini's theorem, indicates that: "For every polynomial with coefficients of degree greater than or equal to five, there is no expression by radicals of the roots of the polynomial, that is to say of expression using only the coefficients, the value one, the four operations and the extraction of the nth roots".

This result is expressed for the first time by Paolo Ruffini (1765-1822), then rigorously proved by Niels Henrik Abel (1802-1829).

However, it is the legendary French mathematician Evariste Galois (1811-1832) who gives a necessary and sufficient condition for a polynomial equation to be solvable by radicals. He introduces permutations groups of the roots, now called Galois groups. This more precise version makes it possible to exhibit equations of degree five, with integer coefficients, whose complex roots which exist according with D'Alembert-Gauss's theorem do not express themselves by radicals.

In this paper, we propose an iterative approach for real roots of polynomials based on the ideas of C Sturm [10]. First of all, we give the family of polynomials we are interested in. For simplicity reason (but our method works also in the general case), we are interested in polynomials having the coefficient of highest power equal one (if not we have just to divide the polynomial by this coefficient). Besides, we assume the coefficient of the second highest power is zero: we can always produce a translation that leads to this form. This means that the sum of the roots vanishes. We take as example the well known case of the order three. Then, we explain the general idea.
The goal is to give the motivation and convince the reader what kind of results we prove. Necessary and sufficient assumptions on the coefficients are given in order to obtain only distinct real roots. More precisely, we build a simple characterization of interlaced roots of the remainder of the polynomial and its derivative using extrema of the starting polynomial. We distinguish two cases according with the degree even or odd of the studied polynomial. In fact, the originality of our method is that we just need the same assumption for all degrees of the starting polynomial.

After recalling a consequence of the Sturm's theorem, we can identify the greatest common divisor (GCD) of a polynomial and its derivative with resultants and Sylvesters' matrices. But even if, as describe in [11] and [1], studying remainders of the Sturm sequence can be seen as minors of a single determinant, it remains in general a very difficult and laborious task.

We see in a new light the real roots of polynomials of orders three, four, five, six and seven. We use maple software to calculate different remainders of Euclidean divisions. We interpret our result because, for low degree, the expression of the roots are available since a long time (for example for orders three and four.) We think also interesting to compare our result with the assumptions arising from the Sturm's approach.
Besides, we concentrated on particular cases of multiple roots. This is because in such cases, the intervals we are given are reduced to a point. As well, the same limit cases are obtained using the Sturm's approach. On the other hand, it shows a way to establish upper and lower bounds for these intervals.
The case of the fifth order is studied in details and we explain how our assumptions can be easily satisfied. In some sense, our result can be considered as an extension of the Sturm's theorem and wonderful ideas he exhibited in [10]. We finish with some concluding remarks and perspectives.

2 Method

In this section, we introduce the family of studied polynomials. As example, we present the case of the order three. Next, we give the general idea. In fact, we will precise this later when we will present the cases of order five, six and seven. The idea here is to build a series of polynomials all having real roots introducing iteratively the same assumption on the last constant of the built polynomial. We consider the following polynomial sequence:

$$\left\{\begin{array}{l} P_2(x) = x^2 + c_0/3 \\ P_3(x) = x^3 + c_0 x + c_1 \\ P_4(x) = x^4 + \frac{4!}{2!3!} c_0 x^2 + \frac{4!}{1!3!} c_1 x + \frac{4!}{3!0!} c_2 \\ P_5(x) = x^5 + \frac{5!}{3!3!} c_0 x^3 + \frac{5!}{3!2!} c_1 x^2 + \frac{5!}{1!3!} c_2 x + \frac{5!}{0!3!} c_3 \\ \dots \\ P_n(x) = x^n + a_{n-2} x^{n-2} + a_{n-3} x^{n-3} + \dots + a_0 \end{array}\right.$$

where

$$\forall k \in \{0, \dots, n-2\}, \quad a_k = \frac{n!}{k!3!} c_{n-2-k}.$$

We note that

$$\forall n \geq 3,\ P_{n-1}(x) = \frac{1}{n} P_n^{'}(x).$$

As the notion of interlacing is crucial, a definition is welcome:

Definition 1 *Given both polynomials P and Q of order n and $n-1$ respectively and $\{\alpha_i\}_{1\leq i\leq n}$ and $\{\beta_j\}_{1\leq j\leq n-1}$ be all real roots of P and Q in nonincreasing order respectively. We say that the roots of Q are interlaced with the roots of P if*

$$\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \dots \leq \beta_{n-2} \leq \alpha_{n-1} \leq \beta_{n-1} \leq \alpha_n.$$

2.1 Examples of polynomials of degree two and three

1. Consider $P_2(x) = x^2 + p/3$. This polynomial has two distinct real roots as soon as $p < 0$.
2. For the polynomial $P_3(x) = x^3 + px + q$, by the Euclidean division by $P_3' = 3P_2$, it comes then

$$P_3(x) = \frac{x}{3} P_3^{'}(x) - R_1(x) = x P_2(x) - R_1(x)$$

with $R_1(x) = -(\frac{2}{3}px + q)$. The polynomial P_3 has three real roots if and only if P_2 has two real roots and if R_1 has a real root interlaced with those of P_2. Because of the previous remark, P_2 has two real roots if $p < 0$.
It then remains to fix the constant q so that the root of R_1 is interlaced with those of P_2. The root of R_1 is $\beta_1^{(1)} = -\frac{3q}{2p}$ and is interlaced with those of P_2 if

$$-\frac{3q}{2p} \in \left]\alpha_1^{(2)} = -\left(-\frac{p}{3}\right)^{1/2}; \alpha_2^{(2)} = \left(-\frac{p}{3}\right)^{1/2}\right[$$

or

$$q \in \left]-\frac{2}{3}p\alpha_1^{(2)}; -\frac{2}{3}p\alpha_2^{(2)}\right[.$$

If we call R_1^0 the function defined by $R_1^0(x) = R_1(x) + q = -\frac{2}{3}px$, we have that

$$q \in]R_1^0(\alpha_1^{(2)}); R_1^0(\alpha_2^{(2)})[.$$

This means that $R_1\left(\alpha_1^{(2)}\right) < 0$ and that $R_1\left(\alpha_2^{(2)}\right) > 0$. Note here that the result means that the discriminant of P_3 namely $\Delta(P_3) = -(4p^3 + 27q^2)$ is not negative. Moreover, if $p < 0$, it is well known that only cases of multiple root for P_3 correspond to $\beta_1^{(1)} = \alpha_1^{(2)}$ or $\beta_1^{(1)} = \alpha_2^{(2)}$ according with the sign of q. Remark that this real double root is also a root of P_2 and R_1.

2.2 General idea

Consider the family of polynomials defined previously

$$P_n(x) = x^n + a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + \cdots + a_0$$

where

$$\forall k \in [0; n-2], \quad a_k = \frac{n!}{k!3!}c_{n-2-k}.$$

From a more general point of view, the result is based on the following remarks:

1. Set $P_n(x) = xQ_{n-1}(x) - R_{n-2}(x)$. If Q_{n-1} has $n-1$ real roots and if R_{n-2} has $n-2$ real roots interlaced with those of Q_{n-1}, then P_n has n real roots.
2. For a polynomial of degree n, to have n real roots, it is necessary that its derivative has $n-1$ distinct real roots $\alpha_1^{(n-1)} \dots \alpha_{n-1}^{(n-1)}$. So, $P_n(x) = \frac{x}{n}P_n'(x) - R_{n-2}(x)$ has n distinct real roots as soon as its derivative has $n-1$ distinct real roots and those of R_{n-2} denoted by $\beta_1^{(n-2)} \dots \beta_{n-2}^{(n-2)}$ are interlaced with those of P_n'.

The coefficient of highest degree of R_{n-2} is $-\frac{2}{n}a_{n-2}$. If $a_{n-2} < 0$ (as the sign of the discriminant of P_2) and if R_{n-2} has $n-2$ real distinct and interlaced roots with $P_n^{'}$ then P_n has n distinct real roots. If we write:

$$R_{n-2}(x) = \frac{1}{n-2}(x - \beta_1^{(1)})R_{n-2}^{'}(x) - T_{n-4}(x),$$

as $R_{n-2}^{'}(x) = (n-1)R_{n-3}(x)$, $R_{n-2}^{'}$ has $n-3$ distinct real roots. It is therefore necessary to ensure that T_{n-4} has $n-4$ distinct real roots denoted by $\gamma_1^{(n-4)} \dots \gamma_{n-4}^{(n-4)}$ and besides interlaced with those $\beta_1^{(n-3)} \dots \beta_{n-3}^{(n-3)}$ of $R_{n-2}^{'}$. The highest degree coefficient of T_{n-4} is at a positive factor proportional to Δ_2 (discriminant of R_2). Then

$$T_{n-4}(x) = \frac{1}{n-4}\left(x - \gamma_1^{(1)}\right)T_{n-4}^{'}(x) - U_{n-6}(x).$$

The roots of U_{n-6} noted $\delta_1^{(n-6)} \dots \delta_{n-6}^{(n-6)}$ will be interlaced with those $\gamma_1^{(n-5)} \dots \gamma_{n-5}^{(n-5)}$ of $T_{n-4}^{'}$. The coefficient of highest degree of U_{n-6} is a positive factor proportional to the discriminant of T_2. Then we have:

$$U_{n-6}(x) = \frac{1}{n-6}\left(x - \delta_1^{(1)}\right)U_{n-6}^{'}(x) - V_{n-8}(x)$$

and so on. We will specify this construction later when studying polynomials of degree five, six and seven, when we will explain how choosing iteratively the last coefficient a_0 of the polynomial P_n. This leads us now to the presentation of our main result.

3 Main result

First of all, we present a characteristic property of interlaced roots. Then, we explain how choosing the integration constant a_0 using the extrema of P_n and the remainder of Euclidean division of P_n by $P_n^{'}$. At last, we propose some necessary and sufficient assumptions on the coefficients in order to obtain n distinct real roots for P_n.

Proposition 1 *Let P_n be a polynomial and let*

$$P_n(x) = \frac{x}{n} P_n'(x) - R_{n-2}(x) = xP_{n-1}(x) - R_{n-2}(x).$$

Denote by $\alpha_1^{(n-1)} \dots \alpha_{n-1}^{(n-1)}$, the $n-1$ distinct real roots of P_{n-1} and $\beta_1^{(n-2)} \dots \beta_m^{(n-2)}$, the $n-2$ distinct real roots of R_{n-2}.
If n is even,

$$\alpha_1^{(n-1)} < \beta_1^{(n-2)} < \alpha_2^{(n-1)} < \beta_2^{(n-2)} < \dots < \alpha_{n-2}^{(n-1)} < \beta_{n-2}^{(n-2)} < \alpha_{n-1}^{(n-1)}$$

$$\Leftrightarrow \sup_{k\in\{1,\dots,\frac{n-2}{2}\}} R_{n-2}(\alpha_{2k}^{(n-1)}) < 0 < \inf_{k\in\{0,\dots,\frac{n-2}{2}\}} R_{n-2}(\alpha_{2k+1}^{(n-1)}).$$

If n is odd,

$$\alpha_1^{(n-1)} < \beta_1^{(n-2)} < \alpha_2^{(n-1)} < \beta_2^{(n-2)} < \dots < \alpha_{n-2}^{(n-1)} < \beta_{n-2}^{(n-2)} < \alpha_{n-1}^{(n-1)}$$

$$\Leftrightarrow \sup_{k\in\{0,\dots,\frac{n-3}{2}\}} R_{n-2}(\alpha_{2k+1}^{(n-1)}) < 0 < \inf_{k\in\{1,\dots,\frac{n-1}{2}\}} R_{n-2}(\alpha_{2k}^{(n-1)}).$$

Proof: We only present here the proof for n even. Suppose that R_{n-2} admits $n-2$ distinct roots $\beta_1^{(n-2)} \dots \beta_{n-2}^{(n-2)}$ interlaced with those of P_{n-1} :

$$\alpha_1^{(n-1)} < \beta_1^{(n-2)} < \alpha_2^{(n-1)} < \beta_2^{(n-2)} < \dots < \alpha_{n-2}^{(n-1)} < \beta_{n-2}^{(n-2)} < \alpha_{n-1}^{(n-1)}.$$

So,

$$\sup_{k\in\{0,\dots,\frac{n-2}{2}\}} P_n\left(\alpha_{2k+1}^{(n-1)}\right) < 0 < \inf_{k\in\{1,\dots,\frac{n-2}{2}\}} P_n\left(\alpha_{2k}^{(n-1)}\right).$$

which is also written

$$\sup_{k\in\{1,\dots,\frac{n-2}{2}\}} R_{n-2}\left(\alpha_{2k}^{(n-1)}\right) < 0 < \inf_{k\in\{0,\dots,\frac{n-2}{2}\}} R_{n-2}\left(\alpha_{2k+1}^{(n-1)}\right).$$

Conversely,

$$\sup_{k\in\{1,\dots,\frac{n-2}{2}\}} R_{n-2}\left(\alpha_{2k}^{(n-1)}\right) < 0 < \inf_{k\in\{0,\dots,\frac{n-2}{2}\}} R_{n-2}\left(\alpha_{2k+1}^{(n-1)}\right)$$

$$\Longleftrightarrow$$

$$\forall k \in \{1,..n-2\},\ R_{n-2}\left(\alpha_k^{(n-1)}\right) \times R_{n-2}\left(\alpha_{k+1}^{(n-1)}\right) < 0$$

This implies that R_{n-2} has $n-2$ interlaced roots with those of P_{n-1}.

The following theorem allows to choose a_0 so that the new polynomial P_n has n distinct real roots:

Theorem 1 *Let P_n be a polynomial and set*

$$P_n(x) = \frac{x}{n} P_n^{'}(x) - R_{n-2}(x) = xP_{n-1}(x) - R_{n-2}(x).$$

We call $R^0_{n-2} = R_{n-2} + a_0$.
Denote by $\alpha_1^{(n-1)} \ldots \alpha_{n-1}^{(n-1)}$ the $n-1$ distinct real roots of P_{n-1}.

For n even, if

$$\begin{cases} \sup\limits_{k\in\{1,\ldots,\frac{n-2}{2}\}} R_{n-2}(\alpha_{2k}^{(n-1)}) < 0 < \inf\limits_{k\in\{0,\ldots,\frac{n-2}{2}\}} R_{n-2}(\alpha_{2k+1}^{(n-1)}) \\ a_0 \in \left] \sup\limits_{k\in\{1,\ldots,\frac{n-2}{2}\}} R^0_{n-2}(\alpha_{2k}^{(n-1)}); \inf\limits_{k\in\{0,\ldots,\frac{n-2}{2}\}} R^0_{n-2}(\alpha_{2k+1}^{(n-1)}) \right[, \end{cases}$$

then P_n has n distinct real roots.

For n odd, if

$$\begin{cases} \sup\limits_{k\in\{0,\ldots,\frac{n-3}{2}\}} R_{n-2}(\alpha_{2k+1}^{(n-1)}) < 0 < \inf\limits_{k\in\{1,\ldots,\frac{n-1}{2}\}} R_{n-2}(\alpha_{2k}^{(n-1)}) \\ a_0 \in \left] \sup\limits_{k\in\{0,\ldots,\frac{n-3}{2}\}} R^0_{n-2}(\alpha_{2k+1}^{(n-1)}); \inf\limits_{k\in\{1,\ldots,\frac{n-1}{2}\}} R^0_{n-2}(\alpha_{2k}^{(n-1)}) \right[, \end{cases}$$

then P_n has n distinct real roots.

Remarks: 1. For example, in the case of n odd, if $a_0 = \sup\limits_{k\in\{0,\ldots,\frac{n-3}{2}\}} R^0_{n-2}(\alpha_{2k+1}^{(n-1)})$ or $a_0 = \inf\limits_{k\in\{1,\ldots,\frac{n-1}{2}\}} R^0_{n-2}(\alpha_{2k}^{(n-1)})$, that means that the polynomial P_n as a double root which is a root of P_{n-1} and R_{n-2}.
2. Either $a_0 = \sup\limits_{k\in\{0,\ldots,\frac{n-3}{2}\}} R^0_{n-2}(\alpha_{2k+1}^{(n-1)}) = \inf\limits_{k\in\{1,\ldots,\frac{n-1}{2}\}} R^0_{n-2}(\alpha_{2k}^{(n-1)})$ reach in two real distinct roots of P_{n-1}: $\alpha_i^{(n-1)}$ and $\alpha_j^{(n-1)}$. So these two real distinct roots are also roots of R_{n-2} and therefore of P_n . We deduce that these roots are double roots of P_n.
Or, either, $a_0 = \sup\limits_{k\in\{0,\ldots,\frac{n-3}{2}\}} R^0_{n-2}(\alpha_{2k+1}^{(n-1)}) = \inf\limits_{k\in\{1,\ldots,\frac{n-1}{2}\}} R^0_{n-2}(\alpha_{2k}^{(n-1)})$ reached in a real double roots of P_{n-1} then this is a triple root of P_n and a double root of R_{n-2}.

Applying recursively Theorem 1, the following theorem allows to choose all coefficients a_l, $0 \le l < n-2$,

Theorem 2 *Let $P_n(x) = x^n + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$ and define the sequence $[P_i, R_i, R_i^0]$, $i \in \{3, \ldots, n\}$, such that*

$$\begin{cases} P_{i-1}(x) = \frac{1}{i} P_i'(x) \\ P_i(x) = xP_{i-1}(x) - R_{i-2}(x) \end{cases}$$

and, $\forall i \in \{3, \ldots, n\}$, $R^0_{i-2}(x) = R_{i-2}(x) + a_{n-i}$.

If n is even, P_n has n distinct real roots if and only if, for all $l \in \{0, ..., \frac{n}{2} - 2\}$,

$$\left\{\begin{array}{l}
\bullet\, a_{n-2} < 0 \\
\bullet \sup_{k\in\{1,...,\frac{n-2}{2}-l\}} R^0_{n-2l-2}(\alpha^{(n-2l-1)}_{2k}) < \inf_{k\in\{0...\frac{n-2}{2}-1\}} R^0_{n-2l-2}(\alpha^{(n-2l-1)}_{2k+1}) \\
\bullet\, a_{2l} \in \left]\sup_{k\in\{1,...,\frac{n-2}{2}-l\}} R^0_{n-2l-2}(\alpha^{(n-2l-1)}_{2k})\,;\, \inf_{k\in\{0,...,\frac{n-2}{2}-1\}} R^0_{n-2l-2}(\alpha^{(n-2l-1)}_{2k+1})\right[\\
\bullet \sup_{k\in\{0,...,\frac{n-4}{2}-l\}} R^0_{n-2l-3}(\alpha^{(n-2l-2)}_{2k+1}) < \inf_{k\in\{1,...,\frac{n-2}{2}-l\}} R^0_{n-2l-3}(\alpha^{(n-2l-2)}_{2k}) \\
\bullet\, a_{2l+1} \in \left]\sup_{k\in\{0,...,\frac{n-4}{2}-l\}} R^0_{n-2l-3}(\alpha^{(n-2l-2)}_{2k+1})\,;\, \inf_{k\in\{1,...,\frac{n-2}{2}-1\}} R^0_{n-2l-3}(\alpha^{(n-2l-2)}_{2k})\right[.
\end{array}\right.$$

If n is odd, P_n has n distinct real roots if and only if,
• $a_{n-2} < 0$.

$$\bullet\, \forall\, l \in \{0, ..., \tfrac{n-3}{2}\},\ \left\{\begin{array}{l}
\bullet \sup_{k\in\{0,...,\frac{n-3}{2}-l\}} R^0_{n-2l-2}(\alpha^{(n-2l-1)}_{2k+1}) < \inf_{k\in\{0,...,\frac{n-1}{2}-l\}} R^0_{n-2l-2}(\alpha^{(n-2l-1)}_{2k}) \\
\bullet\, a_{2l} \in \left]\sup_{k\in\{0,...,\frac{n-3}{2}-l\}} R^0_{n-2l-2}(\alpha^{(n-2l-1)}_{2k+1})\,;\, \inf_{k\in\{0,...,\frac{n-1}{2}-l\}} R^0_{n-2l-2}(\alpha^{(n-2l-1)}_{2k})\right[
\end{array}\right.$$

$$\bullet\, \forall\, l \in \{0, ..., \tfrac{n-5}{2}\},\ \left\{\begin{array}{l}
\bullet \sup_{k\in\{1,...,\frac{n-3}{2}-l\}} R^0_{n-2l-3}(\alpha^{(n-2l-2)}_{2k}) < \inf_{k\in\{0,...,\frac{n-3}{2}-1\}} R^0_{n-2l-3}(\alpha^{(n-2l-2)}_{2k+1}) \\
\bullet\, a_{2l+1} \in \left]\sup_{k\in\{1,...,\frac{n-3}{2}-l\}} R^0_{n-2l-3}(\alpha^{(n-2l-2)}_{2k})\,;\, \inf_{k\in\{0,...,\frac{n-3}{2}-1\}} R^0_{n-2l-3}(\alpha^{(n-2l-2)}_{2k+1})\right[.
\end{array}\right.$$

4 Comparison with Sturm's approach

Of course, at this stage, it is difficult to see what our result means. An interpretation of our assumptions should be welcome. In particular, we may ask if we can compare our assumptions to those arising from Sturm's theorem. We denote S_n the first term of the Sturm sequence of the polynomial P_n defined as follows:

$$\left\{\begin{array}{l}
S_n = P_n \\
S_{n-1} = P_n' \\
S_n = Q_1 S_{n-1} - S_{n-2} \\
\dots \\
S_2 = Q_{n-1} S_1 - S_0.
\end{array}\right.$$

And, with our notations,

$$\left\{\begin{array}{l}
P_n(x) = xP_{n-1}(x) - R_{n-2}(x) \\
R_{n-2}(x) = R^0_{n-2}(x) - a_0.
\end{array}\right.$$

Proposition 2

$$\begin{aligned}
S_0 &= K_1 \Delta(P_n) = K_2 \prod_{i=1}^{n-1} P_n(\alpha_i^{(n-1)}) \\
&= (-1)^{n-1} K_2 \prod_{i=1}^{n-1} R_{n-2}(\alpha_i^{(n-1)}) = (-1)^{n-1} K_2 \prod_{i=1}^{n-1} [R^0_{n-2}(\alpha_i^{(n-1)}) - a_0]
\end{aligned}$$

where $\alpha_i^{(n-1)}, i = 1, \ldots, n-1$, *distinct real roots of* $P_n^{'}$ *and* K_1, K_2 *strictly not negative constants.* $\Delta(P_n)$ *represents the discriminant of* P_n.

Remarks:
1. It is well known that, if the discriminant of the polynomial P_n is not negative, it is a necessary but not sufficient assumption to obtain only real roots. The previous proposition follows from well known results on resultants and Sylvesters' matrices see for example [5].
2. A direct consequence of Sturm's theorem is that P_n has n real roots if the terms of higher degree of S_j, $j \in \{0, \ldots, n\}$, are all not negative.

For illustrative purposes, let us describe what happen for polynomials of degree three, four, five, six and seven. Everytime, we study different cases of multiple roots up to order six.

4.1 Polynomial of order three

In our case, the polynomial P_3 has three roots if $p < 0$ and $4p^3 + 27q^2 < 0$. By Sturm's method,

$$\begin{cases} S_3(x) = P_3(x) = x^3 + px + q \\ S_2(x) = P_3^{'}(x) = 3x^2 + p \\ S_1(x) = R_1(x) = -(\frac{2}{3}px + q) \\ S_0 = K_2 \prod_{i=1}^{2} P_3(\alpha_i^{(2)}) = K_2 \prod_{i=1}^{2} R_1(\alpha_i^{(2)}) = K_2 \prod_{i=1}^{2} (R_1^0(\alpha_i^{(2)}) - q) = K_2 \left(-\frac{4p^3}{27} - q^2\right) = K_1 \Delta(P_3) \end{cases}$$

The form of S_0 can be deduced easily from proposition 2.

4.2 Polynomial of order four

The Sturm's sequence is given by

$$\begin{cases} S_4(x) = P_4(x) = x^4 + 2px^2 + 4qx + 4r \\ S_3(x) = P_4^{'}(x) = 4x^3 + 4px + 4q \\ S_2(x) = R_2(x) = -px^2 - 3qx - 4r \\ S_1(x) = \dfrac{-1}{p^2}[(-4pr + p^3 + 9q^2)x + q(12r + p^2)] \end{cases}$$

and

$$\begin{aligned} S_0 &= \frac{p^2}{(-4pr + p^3 + 9q^2)^2}(64r^3 - 32p^2r^2 + 4p^4r + 72prq^2 - 27q^4 - 2p^3q^2) \\ &= \frac{p^2\Delta(P_4)}{256(-4pr + p^3 + 9q^2)^2} \\ &= K_2 \prod_{i=1}^{3} P_4(\alpha_i^{(3)}) = -K_2 \prod_{i=1}^{3} R_2(\alpha_i^{(3)}) = -K_2 \prod_{i=1}^{3} (R_2^0(\alpha_i^{(3)}) - 4r) \end{aligned}$$

with $\alpha_i^{(3)}, i = 1..3$, the three distinct roots of P_3, $\Delta(P_4)$ the discriminant of P_4 and $R_2(x) = -px^2 - 3qx - 4r$, $R_2^0(x) = -px^2 - 3qx$.
Using Sturm's theorem, we have four distinct real roots if the coefficients of the term of highest degree of S_3, S_2, S_1 and S_0 are strictly positive. These assumptions are the following:

$$p < 0, \; -4pr + p^3 + 9q^2 < 0, \; \Delta(P_4) > 0.$$

Using our method, we have four distinct real roots for P_4 if we choose:

$$p < 0, \quad q \in \left]-2(-\frac{p}{3})^{3/2}; 2(-\frac{p}{3})^{3/2}\right[, \quad 4r \in]R_2^0(\alpha_2^{(3)}); \inf_{i\in\{1,3\}} R_2^0(\alpha_i^{(3)})[\subset]R_2^0(\beta_1^{(1)}); +\infty[$$

where $\beta_1^{(1)} = -\frac{3q}{2p}$ is the root of R_1.

1. Say $4r \in]R_2^0(\beta_1^{(1)}); +\infty[$ means R_2 has two distinct roots $\beta_1^{(2)}$ and $\beta_2^{(2)}$ (the discriminant of R_2 is given by $\Delta_2 = 9q^2 - 16pr > 0$). Clearly:
$$\Delta_2 > 0 \Leftrightarrow R_2(\beta_1^{(1)}) = R_2^0(\beta_1^{(1)}) - 4r < 0.$$
2. Say $4r \in]R_2^0(\alpha_2^{(3)}); \inf_{i\in\{1,3\}} R_2^0(\alpha_i^{(3)})[$ means the roots of R_2 are interlaced with $\alpha_1^{(3)}$, $\alpha_2^{(3)}$, $\alpha_3^{(3)}$.
3. If $q < 0$, we get: $\beta_1^{(2)}$ and $\beta_2^{(2)}$ interlaced with $\alpha_1^{(3)}, \alpha_2^{(3)}, \alpha_3^{(3)}$ is equivalent to $4r \in]R_2^0(\alpha_2^{(3)}); R_2^0(\alpha_1^{(3)})[$. Moreover we can show as $\alpha_3^{(3)} > 3q/p$ that $\alpha_2^{(3)}$ is closer to $-\frac{3q}{2p}$ than $\alpha_1^{(3)}$ and so that $R_2^0(\alpha_2^{(3)}) < R_2^0(\alpha_1^{(3)})$. This means $R_2(\alpha_2^{(3)}) < 0$ and $R_2(\alpha_1^{(3)}) > 0$. In that case, we remark that $\alpha_1^{(3)} < 0$, $\alpha_2^{(3)} < 0$ and $\alpha_3^{(3)} > 0$. So, $4r \in]R_2^0(\alpha_2^{(3)}); \inf_{i\in\{1,3\}} R_2^0(\alpha_i^{(3)})[=]R_2^0(\alpha_2^{(3)}); R_2^0(\alpha_1^{(3)})[$.
4. If $q > 0$, we have $4r \in]R_2^0(\alpha_2^{(3)}); \inf_{i\in\{1,3\}} R_2^0(\alpha_i^{(3)})[=]R_2^0(\alpha_2^{(3)}); R_2^0(\alpha_3^{(3)})[$.

Study of multiple roots for a polynomial of order four

1. If $r = -p^2/12$, we are in a limit case in the following sense

$$\Delta(P_4) = -\frac{256}{27}(4p^3 + 27q^2)^2 > 0$$
$$-4pr + p^3 + 9q^2 = \frac{1}{3}(4p^3 + 27q^2) < 0.$$

This implies $4p^3 + 27q^2 = 0$. We deduce that S_1 is identically zero. According with the sign of q, P_4 has a triple root $\pm\sqrt{-p/3}$ which is a double root of P_3 and of R_2.

2. If $q = 0$, the bounds become : $\Delta(P_4) = 64r(r - p^2/4)^2$ and the other is$-4p(r - p^2/4)$. Taking $r = p^2/4$, S_1 is identically zero. This case corresponds with two double roots $\pm\sqrt{-p}$ for P_4. We can establish that the quotient of the Euclidean division of P_4 by $(x - a)^2$ is $3a^2 + 2xa + 2p + x^2$. This one has two real roots if its discriminant $-8p - 8a^2$ is not negative. This requires that

$$a \in] - \sqrt{-p}; \sqrt{-p}[$$

which are precisely the double roots previously obtained for $q = 0$.

Both previous points make it possible to find bounds for r:

$$4r \in]R_2^0(\alpha_2^{(3)}); \inf_{k\in\{1,3\}} R_2^0(\alpha_k^{(3)})[\subset \left]-\frac{p^2}{3}; p^2\right[.$$

4.3 Polynomial of order five

In this section, we concentrate on the order five. After giving the Sturm's assumptions, we express three particular cases. Then, we describe the assumptions of our theorem. In particular, the choice of s is discussed using the general idea. We explain how our assumption can be satisfied. Next, different cases of multiple roots are specified. We can remark once more that our assumptions and Sturms' assumptions lead to the same result.

4.3.1 The Sturm's assumptions

The Sturm's sequence is given by

$$\left\{\begin{array}{l} S_5(x) = P_5(x) = x^5 + \frac{10p}{3}x^3 + 10qx^2 + 20rx + 20s \\ S_4(x) = P_5^{'}(x) = 5x^4 + 10px^2 + 20qx + 20r \\ S_3(x) = R_3(x) = -(\frac{4}{3}px^3 + 6qx^2 + 16rx + 20s) \\ S_2(x) = -5\left[\frac{(8p^3+81q^2-48pr)x^2}{4p^2} + \frac{(-15ps+4p^2q+54qr)x}{p^2} + 4r + \frac{135qs}{2p^2}\right] \\ S_1(x) = -a_{S_1}\,x - b_{S_1} \end{array}\right.$$

with

$$\left\{\begin{array}{rcl} a_{S_1} & = & -80p^4r - 2106q^2pr + 1056p^2r^2 - 3456r^3 + 240p^2qs + 3240qsr \\ & & +40p^3q^2 + 729q^4 - 450ps^2 \\ b_{S_1} & = & -120p^4s - 1755spq^2 + 1560p^2rs - 4320r^2s + 40p^3qr + 729q^3r \\ & & -864qpr^2 + 2025s^2q. \end{array}\right.$$

$$\begin{array}{rcl} S_0 & = & 1800s^2p^5 - 3600p^4qsr + 1600r^3p^4 - 27000p^3s^2r - 600q^2r^2p^3 + 1200p^3q^3s \\ & & +37125p^2s^2q^2 - 23040r^4p^2 + 50400p^2r^2qs - 85050prq^3s + 38880r^3q^2p \\ & & +108000pr^2s^2 - 101250ps^3q + 182250rq^2s^2 - 10935r^2q^4 + 21870q^5s \\ & & -259200r^3qs + 82944r^5 + 50625s^4 \\ & = & K_1\Delta(P_5) = K_2\prod_{i=1}^{4}P_5(\alpha_i^{(4)}) = K_2\prod_{i=1}^{4}R_3(\alpha_i^{(4)}) = K_2\prod_{i=1}^{4}(R_3^0(\alpha_i^{(4)}) - 20s). \end{array}$$

The result of Sturm gives five distinct real roots for P_5 under the following assumptions:

$$p < 0 \quad (1)$$

$$8p^3 + 81q^2 - 48pr < 0 \quad (2)$$

$$a_{S_1} < 0 \quad (3)$$

$$\prod_{i=1}^{4}(R_3^0(\alpha_i^{(4)}) - 20s) > 0. \quad (4)$$

The polynomial a_{S_1} in s in the inequality (3) having $-450p > 0$ as the coefficient before s^2 has to be not positive: we must have that its discriminant

$$144p^4q^2 - 2484q^2p^2r + 11664q^2r^2 - 160p^5r + 2112p^3r^2 - 6912pr^3 + 1458pq^4 > 0.$$

Its roots in r are:

$$r_1 = \frac{p^2}{6} + \frac{27q^2}{16p},\ r_2 = \frac{5p^2 - \sqrt{25p^4 - 648q^2p}}{72},\ r_3 = \frac{5p^2 + \sqrt{25p^4 - 648q^2p}}{72}.$$

In fact, three particular cases are interesting to explain.

1. The particular case $q = 0$ deserves to be detailed. It comes $r_1 = p^2/6$, $r_2 = 5p^2/36$ and $r_3 = 0$. If we take $r = \frac{5p^2}{36}$, it gives two double roots $-(-\frac{5p}{3})^{1/2}$ and $(-\frac{5p}{3})^{1/2}$ and $s = 0$.

2. If $r_1 = r_2$ or $r_1 = r_3$, then $4p^3 + 27q^2 = 0$ or $8p^3 + 27^2q^2 = 0$. When $4p^3 + 27q^2 = 0$, the particular case $r = r_1 = \frac{-p^2}{12}$ is interesting: the polynomial P_4 has one triple root. The first three Sturm's assumptions become

$$\left\{\begin{array}{l} p < 0 \\ 8p^3 + 81q^2 - 48pr = 3(4p^3 + 27q^2) < 0 \\ a_{S_1} = (4p^3 + 27q^2)^2 - 450p(s + \frac{pq}{30})^2 < 0. \end{array}\right.$$

s is necessary equal to $\frac{-pq}{30}$: according with the sign of q, $\pm\sqrt{-p/3}$ is the quadruple root of P_5. If $8p^3 + 27^2q^2 = 0$, the particular case $r = r_1 = \frac{4p^2}{27}$ is interesting: the first three Sturms' assumptions become

$$\begin{cases} p < 0 \\ 8p^3 + 81q^2 - 48pr = \frac{1}{9}(8p^3 + 729q^2) = 0 \\ a_{S_1} = -450p(s - \frac{4pq}{5})^2 + \frac{1}{729}(8p^3 + 729q^2)^2. \end{cases}$$

This situation shows a limit case where, if $q < 0$, taking $q = -\frac{\sqrt{-8p^3}}{27}$, s is necessary equal to $\frac{4pq}{5}$. Then, P_5 have a real triple root $\alpha_1^{(5)} = \alpha_1^{(4)} = \alpha_1^{(3)} = -\frac{2\sqrt{-2p}}{3}$ and a real double root $\alpha_2^{(5)} = \alpha_2^{(4)} = \sqrt{-2p}$. Notice that we have

$$4r = -R_2^0(\alpha_1^{(3)}) = \frac{16p^2}{27}.$$

Using Sturms' polynomials, if $r = \frac{4p^2}{27}$ and $729q^2 + 8p^3 = 0$ and $s = \frac{4pq}{5}$ or, if $r = \frac{-p^2}{12}$ and $27q^2 + 4p^3 = 0$ and $s = \frac{2p^4}{405q} = \frac{-pq}{30}$, then S_2 is identically zero. Both cases correspond respectively with a real triple root with a double real root and a quadruple real root and a simple real root for the polynomial P_5: this result is consistent with [7].

4.3.2 Our assumptions

For our method, recall the assumptions of order four:

$$p < 0, \quad q \in \left]-2\left(-\frac{p}{3}\right)^{3/2}; 2\left(-\frac{p}{3}\right)^{3/2}\right[, \quad 4r \in \left]R_2^0(\alpha_2^{(3)}); \inf_{k\in\{1,3\}} R_2^0(\alpha_k^{(3)})\right[.$$

Applying the theorem 1, we explain how choosing the parameter s.

Choice of s

First, R_3 must have three distinct real roots and then they have to be interlaced with $\alpha_i^{(4)}, i = 1\ldots 4$. According with the result of order three, we find

$$20s \in]R_3^0(\beta_2^{(2)}), R_3^0(\beta_1^{(2)})[.$$

The Euclidean division of R_3 by $R_3^{'}$ give as a remainder whose sign we change

$$T_1(x) = \frac{2(16pr - 9q^2)}{3p}x + 20s - \frac{8qr}{p}$$

and this remainder vanishes at

$$\gamma_1^{(1)} = \frac{3(-10sp + 4qr)}{16pr - 9q^2}$$

It is enough now that $\gamma_1^{(1)}$ is interlaced with $\beta_1^{(2)}$ and $\beta_2^{(2)}$:

$$\beta_1^{(2)} < \gamma_1^{(1)} < \beta_2^{(2)}.$$

We obtain :

$$20s \in]R_3^0(\beta_2^{(2)}); R_3^0(\beta_1^{(2)})[= \left]\frac{-9q^3}{p^2} + \frac{24qr}{p} - \frac{(\Delta_2)^{3/2}}{3p^2}; \frac{-9q^3}{p^2} + \frac{24qr}{p} + \frac{(\Delta_2)^{3/2}}{3p^2}\right[.$$

With the assumption $\Delta_2 = 9q^2 - 16pr > 0$ (the discriminant of R_2), we deduce that R_3 will have three distinct real roots.
Or, as

$$-T_1(\beta_1^{(2)}) = R_3(\beta_1^{(2)}) = R_3^0(\beta_1^{(2)}) - 20s$$

and

$$-T_1(\beta_2^{(2)}) = R_3(\beta_2^{(2)}) = R_3^0(\beta_2^{(2)}) - 20s.$$

The condition can be written as $R_3(\beta_1^{(2)}) > 0$ and $R_3(\beta_2^{(2)}) < 0$ or $T_1(\beta_1^{(2)}) < 0$ and $T_1(\beta_2^{(2)}) > 0$. Now, if the three roots of R_3 are interlaced with those of P_4, then

$$20s \in \left] \sup_{k\in\{1,3\}} R_3^0(\alpha_k^{(4)}); \inf_{k\in\{2,4\}} R_3^0(\alpha_k^{(4)}) \right[\subset \left] R_3^0(\beta_2^{(2)}); R_3^0(\beta_1^{(2)}) \right[$$

under the assumption

$$\sup_{k\in\{1,3\}} R_3^0(\alpha_k^{(4)}) < \inf_{k\in\{2,4\}} R_3^0(\alpha_k^{(4)}).$$

4.3.3 Discussion : our interval is reduced to a point

Now for a better understanding of our assumption, we need to precise when

$$20s = \sup_{k\in\{1,3\}} R_3^0(\alpha_k^{(4)}) = \inf_{k\in\{2,4\}} R_3^0(\alpha_k^{(4)}).$$

This corresponds with two cases of multiple roots for P_5: one special case of two real double roots and another one of a real triple root. This is the subject of the two following paragraphs.

Case of two real double roots The polynomial P_5 has two double roots a and b which are roots of P_4 and R_3. The Euclidean division of P_5 by $(x-a)^2(x-b)^2$ gives a remainder that is identically zero if:

$$\begin{cases} 3(a^2+b^2) + 4ab + \frac{10p}{3} = 0 \\ -2(a^3+b^3) - 8ab(a+b) + 10q = 0 \\ 7a^2b^2 + 4ab(a^2+b^2) + 20r = 0 \\ 20s - 2a^2b^2(a+b) = 0. \end{cases}$$

We deduce

$$\begin{cases} (ab)^2 - \frac{8p}{3}ab + 12r = 0 \\ (a+b)^3 + \frac{2p}{3}(a+b) - 2q = 0 \\ a+b = 3q/[ab - \frac{2p}{3}] \\ -2(ab-\frac{2p}{3})^3 + 2p(ab-\frac{2p}{3})^2 + 27q^2 = 0. \end{cases}$$

1. The discriminant of the first equation in (ab) is not negative if $r \le \frac{4p^2}{27}$.
2. The discriminant of the second equation in $(a+b)$ must be not negative : $-\frac{4}{27}(8p^3 + 729q^2) \ge 0$.
3. If $q \ne 0$, we conclude having two double real roots if $8p^3 + 729q^2 \le 0$ and $r \le \frac{4p^2}{27}$.
4. The case $q = 0$ gives $r = \frac{5p^2}{36}$, $a = -b = \sqrt{\frac{-5p}{3}}$ and $s = 0$ or $r = \frac{p^2}{9}$ and $ab = \frac{2p}{3} = -2\sqrt{r}$ and $s = \frac{2\sqrt{-2p}p^2}{45\sqrt{3}}$.
5. The third equation gives the particular case $ab = \frac{2p}{3}$: this implies that $q = 0$, $r = p^2/9$ and $a + b = \pm\sqrt{-2p/3}$.

Case of a real triple root It also matches a double root of P_4 and R_3, a root of P_3 and R_2. We can look for the order five under which conditions P_5 has a triple root. If we divide P_5 by $(x-a)^3$, we have, if the remainder is zero, that:

$$\begin{cases} a^3 + ap + q = 0 \\ 4r - 3a^4 - 2a^2 p = 0 \\ 6a^5 + 20s + \frac{10a^3 p}{3} = 0. \end{cases}$$

The conditions on a are those for a double root at the order four or quadruple at the order six (see later). The roots of $4r - 3a^4 - 2a^2 p = 0$ are

$$\pm\frac{1}{3}\sqrt{-3p \pm 3\sqrt{p^2+12r}}.$$

When $r = -\frac{p^2}{12}$ then $4p^3 + 27q^2 = 0$ and $s = \frac{-pq}{30}$. One of these values (according with the sign of q)

$$\sqrt{-p/3}, -\sqrt{-p/3}$$

should be a double root of P_3, triple root of P_4 and quadruple of P_5.
The quotient of P_5 by $(x-a)^3$ gives

$$6a^2 + 3ax + x^2 + \frac{10p}{3}.$$

So, if, in addition to the triple root, we also want two real roots: the discriminant of the previous polynomial is not negative if

$$a \in \left]-\frac{2\sqrt{-2p}}{3}; \frac{2\sqrt{-2p}}{3}\right[.$$

If $a = \pm 2\sqrt{-2p}/3$, P_5 has a triple root a and a double root.

4.3.4 Discussion : our interval is empty or not empty

Under the assumptions

$$p < 0, \quad q \in \left]-2(-\frac{p}{3})^{3/2}; 2(-\frac{p}{3})^{3/2}\right[, \quad 4r \in]R_2^0(\alpha_2^{(3)}); \inf_{k\in\{1,3\}} R_2^0(\alpha_k^{(3)})[,$$

the assumption

$$\sup_{k\in\{1,3\}} R_3^0(\alpha_k^{(4)}) < \inf_{k\in\{2,4\}} R_3^0(\alpha_k^{(4)}) \tag{5}$$

may not be satisfied if $R_3^0(\alpha_1^{(4)}) > R_3^0(\alpha_4^{(4)})$ with $\alpha_1^{(4)}$ the smallest root and $\alpha_4^{(4)}$ the biggest root. So, as the sum of the roots is zero, $\alpha_1^{(4)} < 0$ and $\alpha_4^{(4)} > 0$. Taking $X = \alpha_1^{(4)}\alpha_4^{(4)}$ (note, as the product of the four roots is equal to $4r$, $X \leq -2\sqrt{|r|}$) , the inequality (5) can be written as

$$(\frac{6r}{p} - p)X(X^2 - 4r) + (\frac{9q^2}{p} + 2r)X^2 - 8r^2 < 0. \tag{6}$$

1. $r < 0$
   (a) This inequality is satisfied if $r < 0$ or better if $r < \frac{-9q^2}{2p}$. P_5 has five distinct real roots.

(b) If, for example, $\alpha_1^{(3)}$ or $\alpha_3^{(3)}$ according with the sign of q is lower in absolute value than $|\frac{-3q}{p}|$ ($\inf_{k\in\{1,3\}} R_2^0(\alpha_k^{(3)}) < 0$), the proposed interval at order four for r is included in $]-\infty;0[$: P_5 has five distinct real roots.

2. $r > 0$

(a) As $X \leq -2\sqrt{r}$, the inequality (6) is satisfied for $r \leq \frac{p^2}{9}$: P_5 has five distinct real roots.

(b) If $r > p^2/9$, we must look for negative roots of the polynomial (6) which are less than $-2\sqrt{r}$.

(c) If $r = \frac{4p^2}{27}$ and $729q^2 + 8p^3 = 0$, the polynomial with variable X in (6) has a not positive double root $\frac{4p}{3}$. By taking $X = 4p/3$, P_5 has a real triple root and a real double (here, $s = 4pq/5$).

4.4 Polynomial of order six

In this section, another time we compare our assumptions with the Sturms' assumptions. Then, we explain how choosing t and we describe some cases of multiple roots. Take

$$S_6(x) = P_6(x) = x^6 + 5px^4 + 20qx^3 + 60rx^2 + 120sx + 120t$$

and

$$S_5(x) = P_6^{'}(x) = 6P_5(x) = 6x^5 + 20px^3 + 60qx^2 + 120rx + 120s.$$

We call $\alpha_1^{(5)}, \alpha_2^{(5)}, \alpha_3^{(5)}, \alpha_4^{(5)}, \alpha_5^{(5)}$ the five distinct roots of P_5. We find:

$$S_4(x) = R_4(x) = -\left(\frac{5}{3}px^4 + 10qx^3 + 40rx^2 + 100sx + 120t\right)$$

with $P_6(x) = \frac{x}{6}P_6^{'}(x) - S_4(x)$. We multiply $P_6^{'}$ by $\frac{p^2}{4}$ and divide by S_4. It provides S_3:

$$\begin{array}{rcl} S_3(x) & = & -[(5p^3 + 54q^2 - 36pr)x^3 + (216qr + 15p^2q - 90ps)]x^2 \\ & & -[(540qs + 30p^2r - 108pt)x + 648qt + 30p^2s]. \end{array}$$

Now, we multiply S_4 by

$$\frac{(5p^3 + 54q^2 - 36pr)^2}{15p^2}.$$

We divide this polynomial by S_3 and we obtain as the opposite of the remainder:

$$S_2(x) = -a_{S_2}x^2 - b_{S_2}x - c_{S_2}$$

with

$$\left\{ \begin{array}{rcl} a_{S_2} & = & -12(5p^3 + 54q^2 - 36pr)t - 50p^4r - 1620q^2pr + 840p^2r^2 + 300p^2qs \\ & & -3456r^3 + 4320rqs + 25p^3q^2 + 540q^4 - 900ps^2 \\ b_{S_2} & = & -150p^4s - 2520psq^2 + 2580p^2sr + 180p^2qt - 8640r^2s + 2592rqt \\ & & +50qp^3r + 1080q^3r - 1440qpr^2 + 5400s^2q - 1080pst \\ c_{S_2} & = & -200tp^4 - 3240tpq^2 + 2880tp^2r - 10368tr^2 + 50qp^3s + 1080q^3s \\ & & -1440qprs + 300p^2s^2 + 6480sqt \end{array} \right.$$

Likewise:

$$S_1(x) = -a_{S_1}x - b_{S_1}$$

with

$$\begin{cases} a_{S_1} &= -135000s^4 + 250q^2p^3r^2 + 48600q^3spr + 16200tp^2rq^2 - 16200p^2s^2t \\ &+194400s^2rt - 38880qt^2s - 27900p^2s^2q^2 + 23100p^3s^2r + 60480tr^3p \\ &-250tp^4q^2 + 500tp^5r + 900sqtp^3 - 45360sqtpr - 3240p^2rt^2 - 69120r^5 \\ &+3888pt^3 - 750p^4r^3 + 600p^4t^2 - 32400qp^2r^2s - 194400rq^2s^2 - 129600ps^2r^2 \\ &-21600pq^2r^3 - 38880tr^2q^2 - 500q^3sp^3 - 11100tr^2p^3 - 31104r^2t^2 \\ &+45360sq^3t - 5400tpq^4 + 3240q^2t^2p + 241920r^3qs + 144000ps^3q \\ &+1750qp^4rs + 5400q^4r^2 - 10800q^5s - 1125p^5s^2 + 14400r^4p^2 \\ b_{S_1} &= -194400rq^2s^2 - 135000s^4 + 250q^2p^3r^2 + 48600q^3spr + 16200tp^2rq^2 \\ &-16200p^2s^2t + 194400s^2rt - 38880qt^2s - 27900p^2s^2q^2 + 23100p^3s^2r \\ &+60480tr^3p - 250tp^4q^2 + 500tp^5r + 900sqtp^3 - 45360sqtpr - 3240p^2rt^2 \\ &-69120r^5 + 3888pt^3 - 750p^4r^3 + 600p^4t^2 - 32400qp^2r^2s - 194400rq^2s^2 \\ &-129600ps^2r^2 - 21600pq^2r^3 - 38880tr^2q^2 - 500q^3sp^3 - 11100tr^2p^3 - 31104r^2t^2 \\ &+45360sq^3t - 5400tpq^4 + 3240q^2t^2p + 241920r^3qs + 144000ps^3q + 1750qp^4rs \\ &+5400q^4r^2 - 10800q^5s - 1125p^5s^2 + 14400r^4p^2 \end{cases}$$

Finally, S_0 is

$$S_0 = K_1\Delta(P_6) = K_2\prod_{i=1}^{5} P_6(\alpha_i^{(5)}) = -K_2\prod_{i=1}^{5} R_4(\alpha_i^{(5)}) = -K_2\prod_{i=1}^{5}(R_4^0(\alpha_i^{(5)}) - 120t)$$

According with Sturm's theorem, the assumptions for having six real distinct roots for the order six are:

$$\begin{cases} p < 0 \\ 5p^3 + 54q^2 - 36pr < 0 \\ a_{S_2} < 0 \\ a_{S_1} < 0 \\ S_0 > 0. \end{cases}$$

In these cases, a_{S_2} is a polynomial of order one in t and a_{S_1} a polynomial of order three in t. The study of the sign will remove the unnecessary intervals of $\Delta(P_6) > 0$.
By our method, we must assume the conditions described in order five on p, q, r, s.

Choice of t: R_4 must have distinct roots. For this, we need three distinct roots $\beta_1^{(3)}, \beta_2^{(3)}, \beta_3^{(3)}$ of $R_3 = R_4'/5$,

$$R_3(x) = -\left(\frac{4}{3}px^3 + 6qx^2 + 16rx + 20s\right).$$

We note once more

$$R_4^0(x) = -\left(\frac{5}{3}px^4 + 10qx^3 + 40rx^2 + 100sx\right).$$

It is necessary that $R_4(\beta_2^{(3)}) > 0$ and that $R_4(\beta_1^{(3)}) < 0$ and $R_4(\beta_3^{(3)}) < 0$ or else $R_4^0(\beta_2^{(3)}) - 120t > 0$ and $R_4^0(\beta_1^{(3)}) - 120t < 0$ and $R_4^0(\beta_3^{(3)}) - 120t < 0$. Then we have to show that the four roots of R_4 obtained are interlaced with $\alpha_1^{(5)}, \alpha_2^{(5)}, \alpha_3^{(5)}, \alpha_4^{(5)}, \alpha_5^{(5)}$ which will imply that $120t$ belongs to the interval we want, that is to say,

$$120t \in \left]\sup_{k\in\{2,4\}} R_4^0(\alpha_k^{(5)}); \inf_{k\in\{1,3,5\}} R_4^0(\alpha_k^{(5)})\right[\subset \left]\sup_{k\in\{1,3\}} R_4^0(\beta_k^{(3)}); R_4^0(\beta_2^{(3)})\right[$$

under the assumption

$$\sup_{k\in\{2,4\}} R_4^0(\alpha_k^{(5)}) < \inf_{k\in\{1,3,5\}} R_4^0(\alpha_k^{(5)}).$$

To ensure that R_4 had four real roots, we divide R_4 by $R_4^{'}$. The remainder will give a polynomial T_2 of degree two whose we change the sign

$$T_2(x) = \frac{5}{4}\frac{(-9q^2+16pr)x^2}{p} + \frac{15(5ps-2qr)x}{p} + 120t - \frac{75qs}{2p}.$$

The fact that the roots of this polynomial $\gamma_1^{(2)}$ and $\gamma_2^{(2)}$ which are a function of t are interlaced with the three distinct roots $\beta_1^{(3)}, \beta_2^{(3)}, \beta_3^{(3)}$ of $R_4^{'}$ give the condition for t

$$\beta_1^{(3)} < \gamma_1^{(2)} < \beta_2^{(3)} < \gamma_2^{(2)} < \beta_3^{(3)}.$$

More precisely, we consider T_2 whose coefficient of highest degree is positive: if $p < 0$ and $\Delta_2 > 0$, so the conditions described in the order four give $T_2(\beta_2^{(3)}) < 0$, $T_2(\beta_1^{(3)}) > 0$ and $T_2(\beta_3^{(3)}) > 0$. This is equivalent to $R_4(\beta_2^{(3)}) > 0$, $R_4(\beta_1^{(3)}) < 0$ and $R_4(\beta_3^{(3)}) < 0$; this interval for t is included in that which ensures that T_2 has two distinct real roots: $T_2(\gamma_1^{(1)}) < 0$ if $\gamma_1^{(1)}$ is the root of $T_2^{'}$ that we defined at the order five.

Some bounds for these intervals can be obtain by studying different cases of multiple roots for the order six. In that cases, several polynomials of the Sturm's sequence vanishes identically. Let us now describe three particular cases.

Multiple Roots We can focus for the order six under which conditions P_6 has a root of multiplicity five. We already meet the case at the order three, four and five. If we divide P_6 by $(x-a)^5$, we have, if the remainder is zero, that:

$$a^2 = -\frac{p}{3}, \quad q^2 = 4a^6 = -\frac{4p^3}{27}, \quad 4r = -3a^4 = -\frac{p^2}{3},\ s = \pm\frac{p^2\sqrt{-p}}{45\sqrt{3}}, \quad t = -\frac{a^6}{24} = \frac{p^3}{648}.$$

In that case, we also have a quadruple root for P_5 and for R_4. Otherwise, if we want a real root of multiplicity four and a real double root, then $b = -2a$ with a such that

$$a^2 = -\frac{5p}{6}, \quad q^2 = \frac{a^6}{25} = -\frac{5p^3}{216}, \quad 4r = \frac{3a^4}{5} = \frac{5p^2}{12},\ s = \pm\frac{p\sqrt{-5p}}{72\sqrt{6}}, \quad t = \frac{a^6}{30} = -\frac{25p^3}{1296}.$$

This case corresponds also with a triple root for P_5 and for R_4.
At last, the case of two real triple roots for which we find two double roots for P_5 and for R_4.
It comes:

$$a^2 = -\frac{5p}{3}, \quad q = 0, \quad 4r = \frac{a^4}{5} = \frac{5p^2}{9},\ s = 0, \quad t = -\frac{a^6}{120} = \frac{25p^3}{648}.$$

In these three cases, we remark that S_3 vanishes identically.

4.5 Polynomial of order seven

In this section, according with our theorem and our family of polynomials, we only explain the choice of the last constant. In that case,we have

$$P_7(x) = x^7 + 7px^5 + 35qx^4 + 140rx^3 + 420sx^2 + 840tx + 840u$$
$$P_7^{'}(x) = 7(x^6 + 5px^4 + 20qx^3 + 60rx^2 + 120sx + 120t).$$

We call $\alpha_1^{(6)}, \alpha_2^{(6)}, \alpha_3^{(6)}, \alpha_4^{(6)}, \alpha_5^{(6)}, \alpha_6^{(6)}$ the six distinct real roots of $P_7^{'}$. We define :

$$R_5(x) = -(2px^5 + 15qx^4 + 80rx^3 + 300sx^2 + 720tx + 840u)$$

with

$$P_7(x) = \frac{x}{7} P_7^{'}(x) - R_5(x).$$

We put once again

$$R_5^0(x) = -(2px^5 + 15qx^4 + 80rx^3 + 300sx^2 + 720tx).$$

We assume the same assumptions for p, q, r, s, t of the order six.

Choice of u: As $R_5^{'} = 6R_4$, we call $\beta_1^{(4)}, \beta_2^{(4)}, \beta_3^{(4)}, \beta_4^{(4)}$ their four distinct real roots. Then, the Euclidean division of R_5 by $R_5^{'}$ gives changing the sign of the remainder:

$$T_3(x) = \frac{2}{p}(-9q^2 + 16pr)x^3 + \frac{36(5ps - 2qr)}{p}x^2$$
$$+\frac{36s(16pt - 5qs)}{p}x + 840u - \frac{216qt}{p}.$$

The Euclidean division of T_3 by $T_3^{'}$ gives changing the sign of the remainder a polynomial U_1 of degree one whose numerator of the term in x is none other than the discriminant of the polynomial T_2 of the order six which is positive to have two real roots ($T_2(\gamma_1^{(1)}) < 0$). We deduce the following expression for U_1:

$$U_1(x) = -\left[\frac{24(-144ptq^2 + 256p^2tr + 45q^3s + 40qspr - 150p^2s^2 - 24q^2r^2)}{p(-9q^2 + 16pr)}x\right.$$
$$\left.+\frac{24(81q^3t - 48qtpr - 315puq^2 + 560p^2ur - 240p^2st + 75ps^2q - 30q^2rs)}{p(-9q^2 + 16pr)}\right]$$

whose the root $\delta_1^{(1)}$ is a polynomial of degree one in u:

$$\delta_1^{(1)} = \frac{-[81q^3t - 48qtpr - 35pu(9q^2 - 16pr) - 240p^2st + 75ps^2q - 30q^2rs]}{-16pt(9q^2 - 16pr) + 45q^3s + 40qspr - 150p^2s^2 - 24q^2r^2}.$$

If this root is interlaced with those $\gamma_1^{(2)}$ and $\gamma_2^{(2)}$ of T_2 - as established for the degree three-, T_3 has three real distinct roots. If these roots are interlaced with $\beta_1^{(4)}, \beta_2^{(4)}, \beta_3^{(4)}, \beta_4^{(4)}$, this gives an interval for u included in the previous one. So according with the result for the order five, R_5 has five real distinct roots. Finally, if these five real roots are interlaced with $\alpha_1^{(6)}, \alpha_2^{(6)}, \alpha_3^{(6)}, \alpha_4^{(6)}, \alpha_5^{(6)}, \alpha_6^{(6)}$, we conclude that P_7 has seven real distinct roots. We find:

$$840u \in \left]\sup_{k\in\{1,3,5\}} R_5^0(\alpha_k^{(6)}), \inf_{k\in\{2,4,6\}} R_5^0(\alpha_k^{(6)})\right[,$$

if we have

$$\sup_{k\in\{1,3,5\}} R_5^0(\alpha_k^{(6)}) < \inf_{k\in\{2,4,6\}} R_5^0(\alpha_k^{(6)}).$$

5 Concluding Remarks

Our assumptions are explicit and depend on the roots of the previous order. That is why after the order five, things become harder. Indeed, as we recall in the introduction, exact expressions of the roots are unknown. Obviously, for cases of order three and four, Cardano's, Ferrari's, Descartes's or Euler's formula of the roots are available and can be used for the order five. Notice that, for example in different cases of multiple roots, the resolvant cubic takes a nice and simple form.
Of course, recall that there exists methods, for example, Brings-Jerrard efficient for solving polynomials of degree five. Different transformations are needed in order to obtain expressions of the roots.
Otherwise, Cayley [3] and more recently [4] and [8] proposed different methods for some kind of polynomials of degree six those roots as functions of the roots for solvable quintics. On the other side, Ramanujan [2] solve some polynomials of degree three, four, five, six and seven. His approach is very original. It seems that he often started with roots having product one. In this paper, we rather assume that the sum of the roots is zero even if our result is yet true in the general case, but it takes a more complex form. Naturally, this is particularly true for the Sturm's sequence. Some connections with the theory of elliptic functions would be surely promising and successful.

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