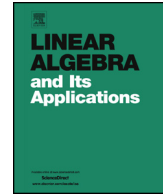




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www.elsevier.com/locate/laaThe number of P -vertices in a matrix with maximum nullityRosário Fernandes ^{a,*}, Henrique F. da Cruz ^{b,2}^a *CMA and Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, 2829-516 Caparica, Portugal*^b *Universidade da Beira Interior, Centro de Matemática e Aplicações (CMA-UBI), Rua Marquês D'Avila e Bolama, 6201-001 Covilhã, Portugal*

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ABSTRACT

Let T be a tree with $n \geq 2$ vertices. Set $\mathcal{S}(T)$ for the set of all real symmetric matrices whose graph is T . Let $A \in \mathcal{S}(T)$ and $i \in \{1, \dots, n\}$. We denote by $A(i)$ the principal submatrix of A obtained after deleting the row and column i . We set $m_A(0)$ for the multiplicity of the eigenvalue zero in A (the nullity of A). When $m_{A(i)}(0) = m_A(0) + 1$, we say that i is a P -vertex of A . As usual, $M(T)$ denotes the maximum nullity occurring of $B \in \mathcal{S}(T)$. In this paper we determine an upper bound and a lower bound for the number of P -vertices in a matrix $A \in \mathcal{S}(T)$ with nullity $M(T)$. We also prove that if the integer b is between these two bounds, then there is a matrix $E \in \mathcal{S}(T)$ with b P -vertices and maximum nullity.

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1. Introduction

Let $T = (X, \mathcal{E})$ be a tree with $n \geq 2$ vertices, where $X = \{1, 2, \dots, n\}$, and let $\mathcal{S}(T)$ be the set of all n -by- n real symmetric matrices $A = [a_{ij}]$ whose graph is T ($a_{ij} \neq 0$, with $i \neq j$, if and only if there is an edge between i and j). No restriction is placed on the main diagonal. If $\alpha \subseteq \{1, 2, \dots, n\}$, then we denote the principal submatrix of $A \in \mathcal{S}(T)$ resulting from deletion (respectively, retention) of rows and columns in α by $A(\alpha)$ (respectively, $A[\alpha]$). When α consists of a single element i , instead of $A(\{i\})$ we simply write $A(i)$. Note that, if we denote by $T - \alpha$ the subgraph of T after removing vertices in α and edges incident in at least one vertex of α , then $A(\alpha)$ is a matrix whose graph is $T - \alpha$. Using a similar notation, when $\alpha = \{i\}$, we write $T - i$ instead of $T - \{i\}$. When H is a subgraph of T , $A[H]$ denotes the principal submatrix of A resulting from deletion of rows and columns that correspond to vertices not in H .

The multiplicity of $\lambda \in \mathbb{R}$ as an eigenvalue of $A \in \mathcal{S}(T)$ is denoted by $m_A(\lambda)$, and the spectrum of A is denoted by $\sigma(A)$. As a consequence of the Cauchy interlacing theorem [7], we have

$$m_A(\lambda) - 1 \leq m_{A(i)}(\lambda) \leq m_A(\lambda) + 1, \tag{1}$$

for all $1 \leq i \leq n$.

Consequently, we have three possibilities for a vertex removed in T :

- In the case of equality in the right-hand side of (1), the element i is known as a *Parter vertex* of A .
If $\lambda = 0$, then the element i (vertex i of T) is known as a *P-vertex* of A .
- If $m_{A(i)}(\lambda) = m_A(\lambda)$, then i is known as a *neutral vertex* of A .
- If $m_{A(i)}(\lambda) = m_A(\lambda) - 1$, then i is known as a *downer vertex* of A .

When a vertex removed in T verifies $m_{A(i)}(\lambda) \geq m_A(\lambda)$, then i is a Parter vertex or a neutral vertex of A , and it is known as an *F-vertex* of A .

Note that if $\lambda \neq 0$ and λ is an eigenvalue of A , then zero is an eigenvalue of $B = A - \lambda I$ and $m_B(0) = m_A(\lambda)$. So, we can focus our study in the eigenvalue zero.

One may also derive from the right-hand of (1) that $m_{A(\alpha)}(0) \leq m_A(0) + |\alpha|$, for any nonempty subset α of $\{1, 2, \dots, n\}$. When we have $m_{A(\alpha)}(0) = m_A(0) + |\alpha|$, we call α a *P-set* of A , [10]. The existence of the *P-vertices* was stated in the next theorem, a generalization of the Parter–Wiener Theorem [15,16]. We denote by $d_T(v)$ the degree of a vertex v of T .

Theorem 1.1. [10] *Let A be a Hermitian matrix whose graph is a tree T , and suppose that there exists a vertex v of T and a real number λ such that $\lambda \in \sigma(A) \cap \sigma(A(v))$. Then*

1. *there is a vertex v' of T such that $m_{A(v')}(\lambda) = m_A(\lambda) + 1$;*

2. if $m_A(\lambda) \geq 2$, then v' may be chosen so that $d_T(v') \geq 3$, and there are at least three components T_1, T_2 , and T_3 of $T - v'$ such that $m_{A[T_i]}(\lambda) \geq 1, i = 1, 2, 3$;
3. if $m_A(\lambda) = 1$, then v' may be chosen so that $d_T(v') \geq 2$ and there are two components T_1 and T_2 of $T - v'$ such that $m_{A[T_i]}(\lambda) = 1, i = 1, 2$.

A characterization of F -vertices was stated in [14]:

Theorem 1.2. [14] *Let A be an n -by- n singular, symmetric matrix, and $i \in \{1, \dots, n\}$. The following statements are equivalent:*

1. *The i th coordinate of every nullvector of A is equal to zero;*
2. *The vertex i is an F -vertex of A .*

Many researchers ([2,3,10,11,13]) have studied P -vertices and P -sets of matrices whose graph is a tree. In [2], the authors found an upper bound for the size of any P -set of any symmetric matrix of order n . They also characterized the trees where this upper bound is attained.

As usual, [9], $M(T)$ is the maximum multiplicity occurring for an eigenvalue of an $A \in \mathcal{S}(T)$.

The thrust of this paper is to provide an upper bound and a lower bound on the number of P -vertices of a matrix $A \in \mathcal{S}(T)$ with nullity $M(T)$. We also prove that if the integer b is a number between these two bounds then there is a matrix $C \in \mathcal{S}(T)$ with b P -vertices and maximum nullity. The structure of matrices with maximum nullity was studied in [11]. The following result is a combination of results that were stated there.

Theorem 1.3. [11] *Suppose that T is a tree, and $A \in \mathcal{S}(T)$ is such that $m_A(0) = M(T)$. Then*

1. *no vertex of T is a neutral vertex of A ,*
2. *the removal of a P -vertex of A in T does not change the status of any other vertex,*
3. *the each set of P -vertices of A is a P -set of A .*

The paper is organized as follows. In Section 2 we introduce some additional notation, study the problem when T is a path and present some results useful in the rest of the paper. In Section 3 we define a family of subsets of vertices of the tree, T , necessary to find bounds of the number of P -vertices in a matrix $A \in \mathcal{S}(T)$ with maximum nullity. In Section 4 we prove results about the sets defined in Section 3, with smallest cardinality. The study about the sets with largest cardinality appears in Section 5. Section 6 is dedicated to establish a relation between some of the sets defined in Section 3, in a tree T , and a particular subtree of T . Finally, in Section 7 we present the bounds for the number of P -vertices of a matrix $A \in \mathcal{S}(T)$ with maximum nullity.

2. The number of P -vertices in a path

In this section we studied the particular case of paths. We do this for motivation, and for the readers who may only be interested in paths. Many of the known results presented in this work can be found in a good book of spectral graph theory, besides the references. The path cover number, $P(T)$, of a tree T is defined as the minimum number of vertex disjoint paths that cover all vertices of T . We will use the following fundamental theorem (see [9] or [6]):

Theorem 2.1. *If T is a tree then*

$$M(T) = P(T).$$

When T is a path, using last theorem, we conclude that

Corollary 2.2. *Let T be a path, $A \in \mathcal{S}(T)$ and λ be an eigenvalue of A . Then $m_A(\lambda) = 1$.*

As noted in [10], if T is a path with n vertices, and $A \in \mathcal{S}(T)$, then A is permutationally similar to an n -by- n irreducible tridiagonal symmetric matrix. Consequently, by Corollary 2.2, any eigenvalue of A has multiplicity one. Moreover, we have the classical fact.

Proposition 2.3. *If A is an n -by- n irreducible tridiagonal symmetric matrix, then the eigenvalues of $A(1)$ and $A(n)$ each strictly interlace those of A .*

When T is a tree, as usual, we denote by $L(T)$ the Laplacian matrix, i.e., the matrix $L(T) = A(T) - D(T)$ where $A(T)$ is the adjacency matrix of T and $D(T)$ is a diagonal matrix with the degree of vertices of T . It is well known that zero is an eigenvalue of $L(T)$. Moreover, if λ is an eigenvalue of $L(T)$ then $\lambda \geq 0$. The next result gives the multiplicity of the eigenvalue 0 of the Laplacian matrix of a path. It follows from general results on for graphs (see theorem 7.1.2 of [1]), and from results in [5]. We include a simple proof here for convenience.

Theorem 2.4. *Let T be a path. Let $L(T)$ be the Laplacian matrix of T . Then*

$$m_{L(T)}(0) = 1 = P(T)$$

and there are no P -vertices of $L(T)$.

Proof. If T is a path with n vertices, then $P(T) = 1$. Since $P(T) = 1$ then $m_{L(T)}(0) = 1 = P(T)$.

Let $\{1, 2, \dots, n\}$ be the set of vertices of T with i adjacent to $i - 1$ and $i + 1$, for $2 \leq i \leq n - 1$.

If $i = 1$ or $i = n$, then by Proposition 2.3, i is not a P -vertex of $L(T)$.

If $i \in \{2, \dots, n-1\}$, consider the paths T_1 and T_2 , where T_1 is the subpath of T indexed by vertices $1, \dots, i-1, i$ and T_2 is the subpath of T indexed by vertices $i, i+1, \dots, n$. Since

$$m_{L(T)(i)}(0) = m_{L(T_1)(i)}(0) + m_{L(T_2)(i)}(0),$$

by Proposition 2.3, i is not a P -vertex of $L(T)$.

So, we conclude that no vertex of T is a P -vertex of $L(T)$. \square

The next result follows from general results in [12], concerning how the multiplicity of an eigenvalue can change when two vertices are deleted (see proposition 4.7). We give a self contained proof here.

Lemma 2.5. *Let T be a path and let $A \in \mathcal{S}(T)$ such that $m_A(0) = 1$. If x and y are P -vertices of A then x is not adjacent to y .*

Proof. Suppose that x is adjacent to y . By Proposition 2.3 neither x nor y are terminal vertices of T . Let T_1 and T_2 be the two paths of $T - x$. If A_1 is the submatrix of $A(x)$ associated with T_1 and A_2 is the submatrix of $A(x)$ associated with T_2 then $A(x) = A_1 \oplus A_2$, $m_{A(x)}(0) = 2$ and $m_{A_1}(0) = m_{A_2}(0) = 1$. Suppose that y is a vertex of T_1 . Using y instead of x , we conclude that $A(y) = B_1 \oplus B_2$, $m_{A(y)}(0) = 2$ and $m_{B_1}(0) = m_{B_2}(0) = 1$, where B_1 and B_2 are the submatrices of $A(y)$ associated with the paths of $T - y$. Suppose that x is a vertex of the path associated with B_1 . By Proposition 2.3, $m_{B_1(x)}(0) = 0$. This is impossible because $A_2 = B_1(x)$. \square

As in [13], the number of P -vertices of $A \in \mathcal{S}(T)$ is denoted by $P_v(A)$.

Proposition 2.6. *Let T be a path with n vertices and let $A \in \mathcal{S}(T)$ such that $m_A(0) = 1$. Then*

$$0 \leq P_v(A) \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Proof. Using Proposition 2.3 we know that a P -vertex is not a terminal vertex of T . By Lemma 2.5 we know that there are not adjacent P -vertices. So, $P_v(A) \leq \lfloor \frac{n-1}{2} \rfloor$. \square

Theorem 2.7. *Let T be a path with n vertices and let b be an integer such that*

$$0 \leq b \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Then, there is a matrix $A \in \mathcal{S}(T)$ such that $m_A(0) = 1$ and $P_v(A) = b$.

Proof. Let T be a path. If $b = 0$, then by Theorem 2.4, $L(G)$ is the sought matrix. Assume now that $b \geq 1$. Remove b vertices i_1, \dots, i_b from T such that none of the removed vertices is a terminal vertex of T , and no two of them are adjacent.

Let P_1, \dots, P_{b+1} be the paths of $T - \{i_1, \dots, i_b\}$ such that P_1 , and P_{b+1} contain the terminal vertices of T . We denote the forest of the paths P_1, \dots, P_{b+1} by G . Let $A = [a_{i,j}] \in \mathcal{S}(T)$ be a matrix such that $A(\{i_1, \dots, i_b\}) = L(G)$ (the Laplacian matrix of G), and assuming that vertex i_t , $1 \leq t \leq b$ is adjacent to vertices j_t and j'_t , $a_{i_t, j_t} = a_{j_t, i_t} = c$, for some $c \neq 0$, and $a_{i_t, j'_t} = a_{j'_t, i_t} = -c$.

As $m_{L(G)}(0) = b + 1$, by the Cauchy interlacing Theorem and using Corollary 2.2, $m_A(0) = 1$, and i_1, \dots, i_b are P -vertices of A .

Let v be the n -by-1 vector whose j th coordinates, for $j \in \{i_1, \dots, i_b\}$ are zero, and all the others coordinates are 1. Then v is a nullvector of A . We have proved that A has a nullvector whose j th coordinates, for all $j \in \{1, \dots, n\} - \{i_1, \dots, i_b\}$, are not zero. By Theorem 1.2, none of the vertices of $\{1, \dots, n\} - \{i_1, \dots, i_b\}$ is a P -vertex of A . Hence, $P_v(A) = b$. \square

3. The optimal P -sets

Let T be a tree. We say that a set P_T of $P(T)$ paths realizes T , if P_T is a set of $P(T)$ vertex disjoint paths, occurring as induced subgraphs of T , that cover all vertices of T . In [4], the number $P(T)$ was studied. In particular the next two propositions appeared in that paper.

Proposition 3.1. (Proposition 2.2 in [4]) *Let T be a tree and i be a vertex of T . Then*

$$P(T) - 1 \leq P(T - i) \leq P(T) + 1.$$

Proposition 3.2. (Proposition 3.3 in [4]) *Let T be a tree and i be a vertex of T . Then $P(T - i) = P(T) + 1$ if and only if i is a vertex of degree two in every set of $P(T)$ paths that realizes T .*

Remark 3.3. If G is a forest (a union of trees) with components T_1, \dots, T_l , then

$$P(G) = \sum_{j=1}^l P(T_j).$$

First we define a family of subsets of vertices of T , \mathcal{A}_T , that contains two kinds of sets necessary to determine a lower and an upper bound of the number of P -vertices in a matrix $A \in \mathcal{S}(T)$ with maximum nullity.

Definition 3.4. Let $T = (X, \mathcal{E})$ be a tree and $Y \subseteq X$. Let $c_T(Y)$ be the number of components of $T - Y$ and let

$$\mathcal{A}_T = \{R : R \subseteq X, c_T(R) = P(T) + |R|\}.$$

A set $R \in \mathcal{A}_T$ is called an optimal P -set of T .

Let s_T be the number

$$s_T = \min\{|R| : R \in \mathcal{A}_T\},$$

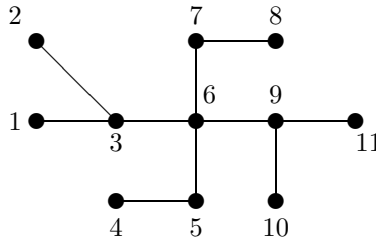
and let l_T be the number

$$l_T = \max\{|R| : R \in \mathcal{A}_T\}.$$

A set $R \in \mathcal{A}_T$ of smallest cardinality is called an s -optimal P -set of T .

A set $R \in \mathcal{A}_T$ of largest cardinality is called an l -optimal P -set of T .

Example 3.5. Consider the following tree T with $n = 11$ vertices.



Note that $P(T) = 3$ and no pendant vertex can be a P -vertex. The set \mathcal{A}_T is the set

$$\{\{6\}, \{3, 9\}, \{3, 6\}, \{6, 9\}, \{3, 6, 9\}, \{3, 5, 9\}, \{3, 7, 9\}, \{3, 5, 7, 9\}\}.$$

So,

- $\{6\}$ is an s -optimal P -set and $s_T = 1$, and
- $\{3, 5, 7, 9\}$ is an l -optimal P -set and $l_T = 4$.

The next theorem shows another characterization of the elements of \mathcal{A}_T that justifies the name of optimal P -sets.

Theorem 3.6. Let $T = (X, \mathcal{E})$ be a tree and R be a subset of X . Then

$$c_T(R) \leq P(T) + |R|.$$

Moreover, $c_T(R) = P(T) + |R|$ if and only if

1. there exists $A \in \mathcal{S}(T)$ with $m_A(0) = M(T)$ and whose set of P -vertices is R , and
2. each component of $T - R$ is a path.

Proof. By Proposition 3.1,

$$P(T - R) \leq P(T) + |R|.$$

Since

$$c_T(R) \leq P(T - R)$$

we conclude that

$$c_T(R) \leq P(T) + |R|.$$

Now assume that $c_T(R) = P(T) + |R|$. Then $P(T - R) = c_T(R)$ and consequently, each component of $T - R$ is a path.

Let A be a matrix in $\mathcal{S}(T)$ with $A(R)$ equal to the Laplacian matrix of $T - R$. Then by Theorem 2.4 we have $m_{A(R)}(0) = M(T - R) = P(T - R) = c_T(R)$, and

$$M(T) \geq m_A(0) \geq m_{A(R)}(0) - |R| = c_T(R) - |R| = P(T) = M(T).$$

Therefore, $M(T) = m_A(0) = m_{A(R)}(0) - |R|$ and R is a P -set of A .

Using Theorems 1.3 and 2.4 we know that the elements of $X - R$ are downer vertices of A . So, R is the set of P -vertices of A .

Conversely, suppose that R is a subset of X and 1. and 2. hold. Using Theorem 1.3, $X - R$ is the set of downer vertices of A . Let S be a component of $T - R$ and x be a vertex of S . By Theorem 1.3, x is a downer vertex of $A[S]$. This implies that $m_{A[S]}(0) \geq 1$. Since S is a path then $m_{A[S]}(0) = 1$.

So,

$$c_T(R) = m_{A(R)}(0) = m_A(0) + |R| = P(T) + |R|. \quad \square$$

Using the proof of last theorem we have the following corollary.

Corollary 3.7. *Let T be a tree and R be an optimal P -set of T . Let $A \in \mathcal{S}(T)$ such that $A(R)$ is equal to the Laplacian matrix of $T - R$. Then $m_A(0) = M(T)$ and R is the P -set of A .*

4. The optimal P -sets with smallest cardinality

The purpose of this section is to establish proprieties of the optimal P -sets with smallest cardinality of T , where T is a tree different from a path.

Theorem 4.1. *Let T be a tree. Let R be an optimal P -set and let $x \in R$. Then*

1. x is a vertex of degree two in every set of $P(T)$ paths that realizes T ,
2. $d_T(x) \geq 2$,
3. if $d_T(x) = 2$, then $R - x$ is an optimal P -set of T .

Proof. Since R is a set of \mathcal{A}_T ,

$$c_T(R) = P(T) + |R|.$$

By Theorem 3.6, there exists $A \in \mathcal{S}(T)$ with $m_A(0) = M(T)$ and whose set of P -vertices is R , and each component of $T - R$ is a path. So, $P(T - R) = c_T(R)$. Consequently,

$$P(T - R) \leq P(T - (R - x)) + 1 \leq P(T - x) + |R - x| \leq P(T) + |R| = P(T - R),$$

and we conclude that $P(T - x) + |R| - 1 = P(T) + |R|$ and $P(T - R) = P(T - (R - x)) + 1$. By Proposition 3.2, x is a vertex of degree two in every set of $P(T)$ paths that realizes T , and by Remark 3.3 and Proposition 3.2, x is a vertex of degree two in every set of $P(T - (R - x))$ paths that realizes $T - (R - x)$. Thus, $d_T(x) \geq 2$ and $d_{T - (R - x)}(x) \geq 2$.

If $d_T(x) = 2$ then $d_{T - (R - x)}(x) = 2$ and $c_T(R) = c_T(R - x) + 1$. Therefore, $c_T(R - x) = c_T(R) - 1 = P(T) + |R| - 1 = P(T) + |R - x|$ and $R - x$ is an optimal P -set of T . \square

Using the last theorem we conclude the following corollary.

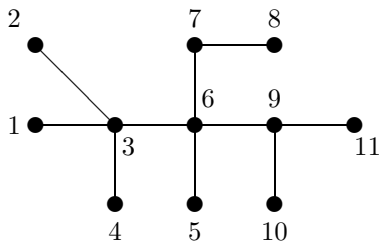
Corollary 4.2. *Let T be a tree that is not a path. Let R be an s -optimal P -set of T . Then*

$$d_T(x) > 2, \forall x \in R.$$

In [8] the authors defined a generalized star. Let T be a tree that is not a path. We say that S is a pendant generalized star of T with central vertex x if

- S is a generalized star with central vertex x ,
- $d_T(x) > 2$,
- when we remove the vertices of S from T , we obtain a nonempty tree, denoted by $T \setminus S$, and
- x is adjacent to a vertex of $T \setminus S$.

Example 4.3. Consider the following tree T with $n = 11$ vertices.



The subgraph of T induced by vertices $1, 2, 3, 4$ is a pendant generalized star of T with central vertex 3 .

Proposition 4.4. Let T be a tree. Then $s_T \leq P(T) - 1$.

Proof. The proof is by induction on the number l of vertices with degree greater than 2. If $l = 0$, then T is a path, and so $s_T = 0$ and $P(T) = 1$. Therefore, $s_T \leq P(T) - 1$.

Let $l \geq 0$, and assume that the result is true for trees with at most l vertices of degree greater than two. Let T_1 be a tree with $l + 1$ vertices of degree greater than two, and let P_{T_1} be a set of $P(T_1)$ paths that realizes T_1 . Let S be a pendant generalized star of T_1 with central vertex y . Let P be the path in P_{T_1} where y belongs. Then $d_P(y) = 2$. In fact suppose that $d_P(y) \leq 1$. Since there are at least 2 components of $T_1 - y$ that are paths, if C_1 is one of these paths that has no common vertices with P , then the concatenation of C_1 and P results in a set, of $P(T_1) - 1$ paths, that realizes T_1 , which is impossible. By Proposition 3.2, $P(T_1 - y) = P(T_1) + 1$. By Remark 3.3, $P(T_1 - y) = P(T_1 \setminus S) + d_{T_1}(y) - 1$. As $T_1 \setminus S$ is a tree with at most l vertices of degree greater than two, by induction, $s_{T_1 \setminus S} \leq P(T_1 \setminus S) - 1$.

Let R be an s -optimal P -set of $T_1 \setminus S$. So,

$$c_{T_1}(R \cup \{y\}) = c_{T_1 \setminus S}(R) + d_{T_1}(y) - 1 = P(T_1 \setminus S) + |R| + d_{T_1}(y) - 1 =$$

$$P(T_1 - y) - d_{T_1}(y) + 1 + |R| + d_{T_1}(y) - 1 = P(T_1) + 1 + |R| = P(T_1) + |R \cup \{y\}|.$$

Therefore, $R \cup \{y\}$ is an optimal P -set of T_1 and

$$\begin{aligned} s_{T_1} &\leq |R| + 1 = s_{T_1 \setminus S} + 1 \leq P(T_1 \setminus S) = P(T_1 - y) - d_{T_1}(y) + 1 = \\ &= P(T_1) - d_{T_1}(y) + 2 \leq P(T_1) - 1. \end{aligned}$$

So, we have the result. \square

5. The optimal P -sets with largest cardinality

As expected, the purpose of this section is to establish properties of the optimal P -sets with largest cardinality of T , where T is a tree different from a path.

Theorem 5.1. *Let $T = (X, \mathcal{E})$ be a tree. Let R be an optimal P -set and let $x \in X - R$ such that $d_{T-R}(x) = 2$, then $R \cup \{x\}$ is an optimal P -set of T .*

Proof. Since R is an optimal P -set then $c_T(R) = P(T) + |R|$. Because $d_{T-R}(x) = 2$ then $c_T(R \cup \{x\}) = c_T(R) + 1$. Consequently, $c_T(R \cup \{x\}) = P(T) + |R| + 1 = P(T) + |R \cup \{x\}|$ and $R \cup \{x\}$ is an optimal P -set of T . \square

The following corollary is the characterization of the components of $T - R$ when R is an l -optimal P -sets of a tree and it is an easy consequence of last theorem.

Corollary 5.2. *Let $T = (X, \mathcal{E})$ be a tree. Let R be an l -optimal P -set. Then each component of $T - R$ is isomorphic to K_1 or K_2 .*

6. Pendant generalized stars and optimal P -sets

Let T be a tree and S be a pendant generalized star of T . In this section, we describe relations between some optimal P -sets of $T \setminus S$ and of T .

Lemma 6.1. *Let $T = (X, \mathcal{E})$ be a tree. Let S be a pendant generalized star of T with central vertex x . Let $R \in \mathcal{A}_T$ such that $x \notin R$ and $d_T(z) > 2, \forall z \in R$. Then $d_T(x) = 3$.*

Proof. Suppose $d_T(x) = s \geq 4$. Since, by Theorem 3.6, $T - R$ is a union of paths, $s \geq 4$ and $x \notin R$, then at least two neighbors of x belong to R and have degree 3 or more. We obtain a contradiction because one of these two neighbors of x is a vertex of S and has degree at most 2. \square

Lemma 6.2. *Let $T = (X, \mathcal{E})$ be a tree. Let S be a pendant generalized star of T with central vertex x . Let w be the vertex of $T \setminus S$ adjacent to x . Let $R \in \mathcal{A}_T$ such that $x \notin R$ and $d_T(z) > 2, \forall z \in R$. Then*

1. $w \in R$ and $d_T(w) \geq 3$.
2. w and x are not in the same path for any set of $P(T)$ paths that realizes T .
3. $P(T) = P(T \setminus S) + 1$.
4. R is an optimal P -set of $T \setminus S$.

Proof. 1. Assume that $w \notin R$. Since by Theorem 3.6, $T - R$ is a union of paths, and $x \notin R$ and $d_T(x) \geq 3$, we conclude that there is a vertex a_2 , adjacent to x , such that

a_2 is not a vertex of $T \setminus S$ and $a_2 \in R$. Since $d_T(a_2) \leq 2$ we obtain a contradiction. Then $w \in R$ and by hypothesis, $d_T(w) \geq 3$.

2. Let $S_1, \dots, S_{P(T)}$ be a set of $P(T)$ paths that realizes T , and assume that $x, w \in S_1$. Since $x \notin R$, by Lemma 6.1, $d_T(x) = 3$. Let a_1 and a_2 be the vertices adjacent to x , different from w . Suppose that $d_{S_1}(x) = 1$. Let S_2 be the path where a_1 belongs. So, $d_{S_2}(a_1) = 1$ and $S_1 \cup x \cup S_2, S_3, \dots, S_{P(T)}$ is a set of $P(T) - 1$ paths that realizes T . This is impossible, therefore $d_{S_1}(x) = 2$. Let a_2 be the vertex of S_1 . Assume that a_1 is a vertex of S_2 . Let S'_1 and S''_1 be the subpaths of $S_1 - x$, and assume that a_2 is a vertex of S'_1 . Then $S''_1, S_2 \cup x \cup S'_1, S_3, \dots, S_{P(T)}$ is a set of $P(T)$ paths that realizes T , and w is a terminal vertex of S''_1 . This is impossible by Theorem 4.1, because $w \in R$.
3. By Theorem 4.1, w is a vertex of degree two in every set of $P(T)$ paths that realizes T . Using 2. and the fact that $d_T(x) = 3$ we conclude that $P(T) = P(T \setminus S) + 1$.
4. Using 1., 3., then

$$c_{T \setminus S}(R) = c_T(R) - 1 = P(T) + |R| - 1 = P(T \setminus S) + |R|.$$

So, R is an optimal P -set of $T \setminus S$. \square

Lemma 6.3. *Let $T = (X, \mathcal{E})$ be a tree. Let S be a pendant generalized star of T with central vertex x . Let $R \in \mathcal{A}_T$ such that $x \in R$ and $d_T(z) > 2, \forall z \in R$. Then $R - x$ is an optimal P -set of $T \setminus S$.*

Proof. By Theorem 4.1, x is a vertex of degree two in every set of $P(T)$ paths that realizes T . By Proposition 3.2, $P(T-x) = P(T)+1$. Since $P(T \setminus S)+d_T(x)-1 = P(T-x)$, we conclude that

$$\begin{aligned} c_{T \setminus S}(R - x) &= c_T(R) - d_T(x) + 1 = P(T) + |R| - d_T(x) + 1 = \\ &= P(T \setminus S) + |R| - 1 = P(T \setminus S) + |R - x|. \end{aligned}$$

So, $R - x$ is an optimal P -set of $T \setminus S$. \square

The next theorem will be used in Section 7 to show that there are no gaps in the cardinality of \mathcal{A}_T , when T is a tree.

Theorem 6.4. *Let $T = (X, \mathcal{E})$ be a tree that is not a path, R^* be an s -optimal P -set of T , and U^* be an l -optimal P -set of T . Let U be the set of all elements of U^* that are vertices of degree greater than two of T and let Y be a subset of $X - R^*$ of largest cardinality that*

- the subgraph spanned by vertices in Y is the null graph, and
- $d_{T-R^*}(v) = 2$, for all $v \in Y$.

Then,

$$|U| \leq s_T + |Y|.$$

Proof. Let S be a pendant generalized star of T with central vertex x . By Theorem 4.1, U is an optimal P -set of T . We will divide the proof in four cases:

Case 1: If $x \notin R^*$ and $x \notin U$ then, by Lemma 6.2, R^* and U are optimal P -sets of $T \setminus S$. Let w be the vertex of $T \setminus S$ adjacent to x , by Lemma 6.2, then $w \in R^* \cap U$.

We repeat the process with $T \setminus S$ and

- if $d_{T \setminus S}(w) > 2$, we consider the sets R^* , U and $Y \cap (R \setminus S)$,
- if $d_{T \setminus S}(w) = 2$, we consider the sets $R^* - w$, $U - w$ and $Y \cap (T \setminus S)$.

Case 2: If $x \notin R^*$ and $x \in U$.

Because $x \notin R^*$, by Lemma 6.1, $d_T(x) = 3$. Let w be the vertex of $T \setminus S$ adjacent to x , by Lemma 6.2, then $w \in R^*$. Let P_1 be the path of $T - R^*$ where x belongs. Then $d_{P_1}(x) = 2$. So $d_{T - R^*}(x) = 2$. If $x \notin Y$, then at least one of the vertices of P_1 that are adjacent to x is in Y . So, we associate the vertex x to a vertex of P_1 that is in Y .

By Lemma 6.2 we have that R^* is an optimal P -set of $T \setminus S$ and by Lemma 6.3, $U - x$ is an optimal P -set of $T \setminus S$.

We repeat the process with $T \setminus S$ and

- if $d_{T \setminus S}(w) > 2$, we consider the sets R^* , $U - x$ and $Y \cap (T \setminus S)$,
 - if $d_{T \setminus S}(w) = 2$, we consider the sets $R^* - w$, $U - x$ and $(Y \cup \{w\}) \cap (T \setminus S)$.
- (Note that in this case, $d_T(w) = 3$ and since $w \in R^*$ and $x \in U$, by Theorem 4.1 and Lemma 6.2, w and x are vertices of degree two in every set of $P(T)$ paths that realizes T and they do not belong at the same of these paths. So, the vertices adjacent to w , different from x , are not in Y .)

Case 3: If $x \in R^*$ and $x \notin U$.

By Lemma 6.3 we have that $R^* - x$ is an optimal P -set of $T \setminus S$ and by Lemma 6.2, U is an optimal P -set of $T \setminus S$ and $w \in U$.

We repeat the process with $T \setminus S$ and

- if $d_{T \setminus S}(w) > 2$, we consider the sets $R^* - x$, U and $Y \cap (T \setminus S)$,
- if $d_{T \setminus S}(w) = 2$ and $w \in R^*$, we consider the sets $R^* \setminus \{x, w\}$, $U - w$ and $Y \cap (T \setminus S)$,
- if $d_{T \setminus S}(w) = 2$, $w \notin R^*$ we associate the vertex w with the vertex x , and we consider the sets $R^* - x$, $U - w$ and $(Y - w) \cap (T \setminus S)$.

Case 4: $x \in R^*$ and $x \in U$.

By Lemma 6.3, $R^* - x$ and $U - x$ are optimal P -sets of $T \setminus S$. We repeat the process with $T \setminus S$ and we consider the sets $R^* - x$, $U - x$ and $Y \cap (T \setminus S)$.

Because each time we repeat the process and we remove a vertex of U we also remove a vertex of R^* or of Y , we conclude that $|U| \leq s_T + |Y|$. \square

7. The number of P -vertices in a tree

In this section, using the last sections, we determine an upper and a lower bound for the number of P -vertices of a matrix $A \in \mathcal{S}(T)$ with nullity $M(T)$, where T is a tree. We also prove that if b is an integer between these two bounds, then there is a matrix $A \in \mathcal{S}(T)$ such that $m_A(0) = M(T)$ and A has b P -vertices.

Proposition 7.1. *Let T be a tree and let $A \in \mathcal{S}(T)$ with $m_A(0) = M(T)$. Then*

$$s_T \leq P_v(A) \leq l_T.$$

Proof. Let B be the set of all P -vertices of A . If T_1 is a component of $T - B$ that is not a path, using the fact that B is the set of P -vertices of A , by Theorem 1.1,

$$m_{A[T_1]}(0) \leq 1.$$

Since we are assuming that T_1 is not a path there is a vertex x of T_1 such that $d_{T_1}(x) > 2$. Let $A'_1 \in \mathcal{S}(T_1)$ such that $m_{A'_1}(0) = d_{T_1}(x) - 1$ (for instance we can consider the Laplacian matrix associated to each component of $T_1 - x$). Denote by $A' \in \mathcal{S}(T)$ a matrix whose summands of $A'(B)$ are the same of $A(B)$ except $A[T_1]$ that is replaced by A'_1 . Therefore, by the Cauchy interlacing Theorem

$$m_{A'}(0) \geq m_{A'(B)}(0) - |B| > m_{A(B)}(0) - |B| = m_A(0) + |B| - |B| = M(T).$$

This is impossible. Then $T - B$ is a union of paths. By Theorem 3.6, B is an optimal P -set of T . So, $s_T \leq |B| = P_v(A) \leq l_T$. \square

Theorem 7.2. *Let $T = (X, \mathcal{E})$ be a tree and let b be an integer such that*

$$s_T \leq b \leq l_T.$$

Then, there is a matrix $A \in \mathcal{S}(T)$ such that $m_A(0) = M(T)$ and $P_v(A) = b$.

Proof. Let B be an s -optimal P -set of T . If all vertices of $T - B$ have degree at most one, let $Y = \emptyset$. If $T - B$ has at least one vertex of degree 2, let Y be a subset of $X - B$ of largest cardinality that

- the subgraph spanned by vertices in Y is the null graph, and
- $d_{T-B}(v) = 2$, for all $v \in Y$.

By Theorem 5.1, $B \cup Y$ is an optimal P -set of T . Let U^* be an l -optimal P -set of T and let U be the elements of U^* that have degree greater than two in T . By Theorem 4.1, U is an optimal P -set of T . By Theorem 6.4, $|U| \leq |B \cup Y|$.

- If $|B| \leq b \leq |B \cup Y|$, since by Theorem 4.1, there is an optimal P -set, F , such that $B \subseteq F \subseteq (B \cup Y)$ with cardinality b , by Theorem 3.6 we have the result.
- If $|U| \leq b \leq |U^*|$, since by Theorem 5.1, there is an optimal P -set, H , such that $U \subseteq H \subseteq U^*$ with cardinality b , by Theorem 3.6 we have the result. \square

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