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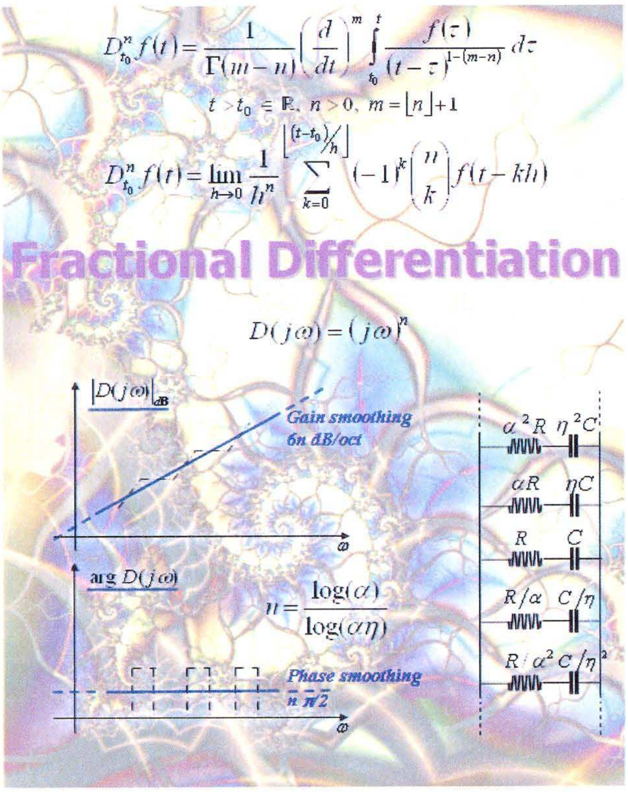

ENSEIRB, Bordeaux, France, July 19-21, 2004

WORKSHOP PREPRINTS/PROCEEDINGS N° 2004-1

LEAST-SQUARES DESIGN OF DIGITAL FRACTIONAL-ORDER OPERATORS

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Abstract: In this paper we develop a method for obtaining digital rational approximations to fractional-order operators of type s^γ, where $\gamma \in \Re$. The proposed method is based on the least-squares (LS) minimization between the impulse response of the fractional Euler/Tustin operators and the digital rational-fraction approximation. We make a comparison with other approaches and the results reveal that the LS method gives superior approximations. The effectiveness of the method is demonstrated both in the time and frequency domains through an illustrative example.

Keywords: Fractional-Order Operators, Least-Squares, Rational Approximations, IIR Filters, Filter Design, Padé Approximation, Fractional Calculus.

1. INTRODUCTION

In the literature we find several different definitions for the fractional-order operator D^γ, where the order γ can be an arbitrary non-integer value; see, for example, (Oldham and Spanier, 1974; Podlubny, 1999). In this study we admit only values of $\gamma \in \Re$. From a control and signal processing perspective, the Grünwald-Letnikov definition (Podlubny, 1999) seems to be the most appropriate, particularly for a digital realization (Machado, 2001). Furthermore, the definition poses fewer restrictions upon on the functions to which it is applied (Oldham and Spanier, 1974). It is given by the expression:

$$D^\gamma f(t) = \lim_{h \to 0} \left\{ \frac{1}{h^\gamma} \sum_{k=0}^{\infty} (-1)^k \binom{\gamma}{k} f(t - kh) \right\} \tag{1a}$$

$$\binom{\gamma}{k} = \frac{\Gamma(\gamma+1)}{\Gamma(k+1)\Gamma(\gamma-k+1)} \tag{1b}$$

where $f(t)$ is the applied function, Γ is the Gamma function and h the time increment. Note that formula (1a) is defined by an infinite series revealing that the fractional-order operators are *global* operators and that have, implicitly, a *memory* of all past function values.

One of the mathematical tools commonly used for the analysis and synthesis of automatic control systems is the Laplace transform. Fortunately, his generalization to a fractional-order is very straightforward. For instance, the Laplace transform of a fractional derivative/integral of order γ of the function $f(t)$, $D^\gamma[f(t)]$, under null initial conditions, is given by the simple expression:

$$L\left\{D^\gamma\left[f(t)\right]\right\} = s^\gamma F(s), \quad \gamma \in \Re \tag{2}$$

where $F(s) = L\{f(t)\}$. Note that (2) is a *direct* generalization of the classical integer-order scheme with the multiplication of the signal transform by the Laplace operator s. This means that frequency-based analysis methods have a straightforward adaptation to the fractional-order case.

The usual approach for obtaining discrete equivalents of the fractional-order operator s^γ, $\gamma \in \Re$, adopts a generating function (Vinagre, *et al.*, 2000; Chen and Moore, 2002). By other words, given a continuous transfer function, $G(s)$, a discrete equivalent, $G(z)$, can be found by the substitution:

$$G(z) = G(s)\Big|_{s^\gamma = H^\gamma(z)} \tag{3}$$

where $H^\gamma(z)$ denotes the fractional discrete equivalent of order γ of the fractional-order operator s^γ, expressed as a function of the complex variable z or the shift operator z^{-1}. In these $s \to z$ conversion schemes (also called analog to digital open-loop design methods) we usually adopt either the Euler (or first backward difference) or the Tustin (or bilinear) generating functions (Machado, 2001). Table 1 indicates the two mentioned conversion methods that will be used in this study.

Table 1 s → z conversion schemes

Method	$H^\gamma(z^{-1})$
Euler Grünwald-Letnikov	$\left(\frac{1}{T}(1-z^{-1})\right)^\gamma$
Tustin	$\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^\gamma$

In general, the irrational functions $H^\gamma(z)$ (Table 1) are approximated either through polynomials or through rational functions (*i.e.*, the ratio of two polynomials). It is well known that rational approximations frequently converge faster than polynomial approximations and have a wider domain of convergence in the complex domain. In the work that follows, we develop rational approximations of the z variable to fractional-order operators of type s^γ, which make them suited for Z-transform analysis and digital implementation.

Rational approximations $H_{m,n}(z^{-1})$ of m and n order to irrational transfer functions of type $H^\gamma(z^{-1})$ can be formally expressed as:

$$H^\gamma(z^{-1}) \approx \left[\frac{P_m(z^{-1})}{Q_n(z^{-1})}\right]_{m,n} = H_{m,n}(z^{-1}) \tag{4}$$

where P and Q are the polynomials of degree m and n, respectively.

In this paper we consider digital rational approximations of type (4) to fractional-order operators. In a first phase, we discretize the fractional-order operator s^γ, through the Euler/Tustin generating functions, yielding the irrational functions $H^\gamma(z^{-1})$, listed in Table 1, and we determine their impulse responses $h^\gamma(k)$. Then, in a second phase, we apply the least-squares (LS) minimization method to the impulse responses, $h^\gamma(k)$ and $h(k)$, of the digital fractional operator $H^\gamma(z^{-1})$ and of the digital rational approximation $H_{m,n}(z^{-1})$, respectively. We show that these new rational transfer functions of the z variable give better approximations, both in time and frequency domains, than other approaches, namely the Padé or continued fraction expansion (CFE) methods.

Bearing these ideas in mind, the paper is organized as follows. Section 2 derives the impulse responses of the fractional Euler/Tustin operators and section 3 gives an introduction to the problem. Based on the previous results, section 4 develops the Padé approximations to fractional-order operators and compares it with the CFE method. It is shown that the Padé approximations lead to the same rational functions as the CFE method. Section 5 develops the least-squares (LS) method for the identification of the rational approximation parameters. Section 6 presents an illustrative example showing the effectiveness of the proposed method both in the time and frequency domains. Finally, section 7 draws the main conclusions and addresses perspectives towards future developments.

2. IMPULSE RESPONSE OF DIGITAL FRACTIONAL-ORDER OPERATORS

In this section, we derive the impulse responses of the fractional Euler/Tustin operators, $h^\gamma(k)$. In obtaining the impulse responses we assume that $h^\gamma(k) = 0$ for $k < 0$ (*i.e.*, a causal system).

Expanding the fractional Euler function $H_E^\gamma(z^{-1})$ into a power series in z^{-1}, we have:

$$\begin{aligned} H_E^\gamma(z^{-1}) &= \left[\frac{1}{T}(1-z^{-1})\right]^\gamma \\ &= \left(\frac{1}{T}\right)^\gamma \sum_{k=0}^{\infty}(-1)^k \binom{\gamma}{k} z^{-k} = \sum_{k=0}^{\infty} h_E^\gamma(k) z^{-k} \end{aligned} \tag{5}$$

where the impulse sequence $h_L^\gamma(k)$ is given by:

$$h_E^\gamma(k) = \left(\frac{1}{T}\right)^\gamma (-1)^k \binom{\gamma}{k}, \quad k \ge 0 \tag{6}$$

Performing a power series expansion, over the fractional Tustin function $H_T^\gamma(z)$, we get:

$$\begin{aligned} H_T^\gamma(z^{-1}) &= \left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^\gamma \\ &= \left(\frac{2}{T}\right)^\gamma \sum_{k=0}^{\infty}\left[\sum_{i=0}^{k}(-1)^k \binom{\gamma}{i}\binom{-\gamma}{k-i}\right] z^{-k} = \sum_{k=0}^{\infty} h_T(k) z^{-k} \end{aligned} \tag{7}$$

where the impulse sequence $h_T^\gamma(k)$ is given by:

$$h_T^\gamma(k) = \left(\frac{2}{T}\right)^\gamma \sum_{i=0}^{k}(-1)^k \binom{\gamma}{i}\binom{-\gamma}{k-i}, \quad k \ge 0 \tag{8}$$

Notice that if $\gamma \in \aleph$ in (5), then we have the standard integer-order integrators ($\gamma \in \aleph^-$) and differentiators ($\gamma \in \aleph_0^+$). In this case, the impulse sequence is of finite duration and we obtain a *finite impulse response* (FIR) filter.

3. PROBLEM FORMULATION

Consider that the impulsional response $h^\gamma(k)$ of the fractional-order operator is specified for $k \ge 0$. The rational function $H_{m,n}(z^{-1})$ that approximates the irrational transfer function $H^\gamma(z^{-1})$ has the form:

$$H_{m,n}(z^{-1}) = \frac{\sum_{k=0}^{m} b_k z^{-k}}{1+\sum_{k=1}^{n} a_k z^{-k}} \approx \sum_{k=0}^{\infty} h(k) z^{-k} \qquad (9)$$

where $h(k)$ is its impulse response. The rational approximation has $L = m + n + 1$ parameters, namely, the coefficients $\{a_k\}_{k=1}^{n}$ and $\{b_k\}_{k=0}^{m}$, which are selected to minimize the sum of the squared errors:

$$J = \sum_{k=0}^{U} \left[h^{\gamma}(k) - h(k) \right]^2 \qquad (10)$$

where U is some preselected upper limit in the summation. In general, $h(k)$ is a nonlinear function of the rational model parameters and, consequently, the minimization of J involves the solution of a set of nonlinear equations.

4. PADÉ APPROXIMATION METHOD

In this section we present briefly the Padé approximation method in order to compare it with the proposed LS method.

If we select the upper limit in (10) as $U = L - 1$, then it is possible to match $h(k)$ perfectly in the fractional impulse response $h^{\gamma}(k)$ for $0 \le k \le m+n$. For that, we consider the impulse response $h(k)$ of the desired rational approximation $H_{m,n}(z^{-1})$, which is given by:

$$h(k) = \begin{cases} -\sum_{l=1}^{n} a_l h(k-l) + b_k, & 0 \le k \le m \quad (11a) \\ -\sum_{l=1}^{n} a_l h(k-l), & k > m \quad (11b) \end{cases}$$

This gives a set of $m + n + 1$ linear equations, which can be used to solve for the coefficients $\{a_k\}$ and $\{b_k\}$. First, we set $h(k) = h^{\gamma}(k)$ for $0 \le k \le m+n$ and we use the system of linear equations (11b) to calculate the coefficients $\{a_k\}$. Then we use the values of $\{a_k\}$ in equations (11a) and solve them for the coefficients $\{b_k\}$. Thus, we obtain a perfect match between $h(k)$ and the desired response $h^{\gamma}(k)$ for the first L values of the impulse response.

The success of this method depends strongly on the number of selected model coefficients. Since the design method matches $h^{\gamma}(k)$ only up to the number of model parameters, the more complex the model, the better the approximation to $h^{\gamma}(k)$ for $0 \le k \le m+n$. However, in practical applications, this introduces a major limitation of the Padé approximation method because the resulting approximation must contain a large number of poles and zeros.

It can be shown that rational approximations obtained by the CFE method are the same as those resulting by application of the Padé approximation to power series expansion (Lorentzen, 1992). Nevertheless, the CFE approach is computationally less expensive than the Padé technique.

5. LEAST SQUARES APPROXIMATION METHOD

In this section we adopt a new approach to the problem. The proposed method is based on the standard *least squares identification* algorithm (Franklin, *et al.*, 1990).

The impulse response $h(k)$ of $H_{m,n}(z^{-1})$ to a unit sample input $\delta(k)$ corresponds to the expression:

$$h(k) + \sum_{l=1}^{n} a_l h(k-l) = \sum_{l=0}^{m} b_l \, \delta(k-l) \qquad (12)$$

where $\delta(k-l) = 1$ (for $k = l$) and $\delta(k-l) = 0$ (for $k \ne l$) and $k = 0, 1, \ldots, N-1$ corresponding to a collect of N values from the input and output sequences.

Expression (12) can be written in matrix form as:

$$h(k) = \boldsymbol{\theta}^T \boldsymbol{x}(k), \quad k \ge 0 \qquad (13)$$

where $\boldsymbol{x}(k)$ is the $(m + n + 1) \times 1$ *state vector* and $\boldsymbol{\theta}$ the $(m + n + 1) \times 1$ *parameter vector* defined as:

$$\boldsymbol{x}(k) = \left[-h(k-1), \ldots, -h(k-n), \delta(k), \ldots, \delta(k-m) \right]^T \qquad (14)$$

$$\boldsymbol{\theta} = \left[a_1, a_2, \ldots, a_n, b_0, b_1, \ldots, b_m \right]^T \qquad (15)$$

Let us introduce the matrix variables:

$$\boldsymbol{X} = \left[\boldsymbol{x}^T(0), \boldsymbol{x}^T(1), \ldots, \boldsymbol{x}^T(N-1) \right]^T \qquad (16)$$

$$\boldsymbol{h} = \left[h^{\gamma}(0), h^{\gamma}(1), \ldots, h^{\gamma}(N-1) \right]^T \qquad (17)$$

If the system can be represented by equation (13) for some $\boldsymbol{\theta} = \boldsymbol{\theta}^*$, then the vector of systems outputs becomes:

$$\boldsymbol{h} = \boldsymbol{X} \boldsymbol{\theta}^* \qquad (18)$$

where $\boldsymbol{X}$ is an $N \times (m + n + 1)$ matrix and $\boldsymbol{h}$ is an $N \times 1$ vector. For the construction of $\boldsymbol{X}$ we assume that the initial conditions of the system are zero, that is, $h(k) = 0$ for $k < 0$.

Usually, $N \ge m + n + 1$ and we define the *error vector* $e = \boldsymbol{h} - \boldsymbol{X\theta}$, where $\boldsymbol{\theta}$ is a general parameter vector. Hence, the objective is to find an estimate $\boldsymbol{\theta}$ that minimizes $J = \sum_{k=0}^{N-1} e^2(k)$:

$$J = e^T e = (\boldsymbol{h} - \boldsymbol{X\theta})^T (\boldsymbol{h} - \boldsymbol{X\theta}) \qquad (19)$$

Solving $\partial J / \partial \boldsymbol{\theta} = 0$ we obtain the following system of *normal equations*:

$$X^T X\theta = X^T h \tag{20}$$

If the matrix X^TX is nonsingular, a unique solution of (20) exists and the optimum θ is given by:

$$\theta = X^+ h = (X^T X)^{-1} X^T h \tag{21}$$

where $X^+ = (X^T X)^{-1} X^T$ is the pseudoinverse of X.

It is clear from (21) that if $N = m+n+1$, the system reduces to an $N \times N$ square matrix and, consequently, the parameter vector θ can be calculated simply by $\theta = X^{-1}h$. We verify that, in this case, we get the same rational approximation as those obtained by the application of the Padé or CFE methods.

6. ILLUSTRATIVE EXAMPLE

In this section we obtain rational approximation models $H_{m,n}(z^{-1})$ for the fractional-order operator s^γ, with $\gamma = -1/2$, using the LS method described in the previous section. We consider the fractional Euler/Tustin operators, sampled at $T = 0.01$ s, $m = n = \{1, 3, 5, 7\}$, and $N = 1000$. The approximations are given next.

LS approach for the Euler operator:

$$H_{1,1}(z^{-1}) = \frac{0.1 - 0.04397z^{-1}}{1 - 0.9397z^{-1}}$$

$$H_{3,3}(z^{-1}) = \frac{\begin{array}{c} 0.1 - 0.161z^{-1} \\ +0.06789z^{-2} \quad 0.004722z^{\;3} \end{array}}{\begin{array}{c} 1 - 2.11z^{-1} \\ +1.359z^{-2} \quad 0.2479z^{\;3} \end{array}}$$

$$H_{5,5}(z^{-1}) = \frac{\begin{array}{c} 0.1 - 0.2789z^{-1} \quad 0.2792z^{-2} \quad 0.1184z^{\;3} \\ +0.01861z^{-4} \quad 0.0005166z^{\;5} \end{array}}{\begin{array}{c} 1 - 3.289z^{-1} \quad 4.062z^{\;2} \quad 2.294z^{\;3} \\ +0.5644z^{-4} \quad 0.04312z^{\;5} \end{array}}$$

$$H_{7,7}(z^{-1}) = \frac{\begin{array}{c} 0.1 - 0.3966z^{-1} \quad 0.6293z^{-2} \quad 0.5065z^{\;3} \\ +0.2155z^{-4} \quad 0.04542z^{\;5} \quad 0.003798z^{\;6} \\ -0.000056456z^{-7} \end{array}}{\begin{array}{c} 1 - 4.466z^{-1} \quad 8.151z^{\;2} \quad 7.778z^{\;3} \\ 4.109z^{-4} - 1.164z^{\;5} \quad 0.1544z^{\;6} \\ 0.006461z^{-7} \end{array}}$$

LS approach for the Tustin operator:

$$H_{1,1}(z^{-1}) = \frac{0.07071 + 0.008843z^{-1}}{1 - 0.8749z^{-1}}$$

$$H_{3,3}(z^{-1}) = \frac{\begin{array}{c} 0.07071 + 0.005302z^{-1} \\ -0.05281z^{-2} - 0.001747z^{-3} \end{array}}{\begin{array}{c} 1 - 0.925z^{-1} \\ -0.3218z^{-2} + 0.2596z^{-3} \end{array}}$$

$$H_{5,5}(z^{-1}) = \frac{\begin{array}{c} 0.07071 + 0.004728z^{-1} - 0.09396z^{-2} \\ -0.004356z^{-3} + 0.02656z^{-4} \\ +0.0005193z^{-5} \end{array}}{\begin{array}{c} 1 - 0.933z^{-1} - 0.8957z^{-2} + 0.8007z^{-3} \\ +0.1144z^{-4} - 0.08461z^{-5} \end{array}}$$

$$H_{7,7}(z^{-1}) = \frac{\begin{array}{c} 0.07071 + 0.004737z^{-1} - 0.1355z^{-2} \\ -0.007165z^{-3} + 0.0771z^{-4} + 0.00279z^{-5} \\ -0.0118z^{-6} - 0.0001729z^{-7} \end{array}}{\begin{array}{c} 1 - 0.933z^{-1} - 1.483z^{-2} + 1.348z^{-3} + \\ 0.5753z^{-4} - 0.4936z^{-5} - 0.04154z^{-6} + \\ 0.02785z^{-7} \end{array}}$$

For comparison purposes, we also plot the rational approximation obtained by the Padé method for $m = n = 5$, $G_{5,5}(z^{-1})$, for the Euler and Tustin operators, which are given next.

Padé approach for the Euler operator:

$$G_{5,5}(z^{-1}) = \frac{\begin{array}{c} 0.1 - 0.225z^{-1} + 0.175z^{-2} - 0.05469z^{-3} \\ +0.005859z^{-4} - 0.00009766z^{-5} \end{array}}{\begin{array}{c} 1 - 2.75z^{-1} + 2.75z^{-2} - 1.203z^{-3} \\ +0.2148z^{-4} - 0.01074z^{-5} \end{array}}$$

Padé approach for the Tustin operator:

$$G_{5,5}(z^{-1}) = \frac{\begin{array}{c} 0.07071 + 0.03536z^{-1} - 0.07071z^{-2} \\ -0.02652z^{-3} + 0.01326z^{-4} + 0.00221z^{-5} \end{array}}{\begin{array}{c} 1 - 0.5z^{-1} - z^{-2} + 0.375z^{-3} \\ +0.1875z^{-4} - 0.03125z^{-5} \end{array}}$$

We note that the sum of the coefficients of the numerator and denominator of the rational approximations $H_{m,n}(z^{-1})$ and $G_{5,5}(z)$ are approximately zero, that is $\sum_{k=0}^{m} b_k = \sum_{k=0}^{n} a_k \approx 0$.

Figures 1 and 2 depict the Bode diagrams and the step responses of the approximations $H_{m,n}(z)$, with $m = n = \{1, 3, 5, 7\}$ and $N = 1000$, for the Euler and the Tustin operators, respectively. Figures 4 and 5 show the results when we vary the length of the impulsional sequence $N = \{11, 100, 200, 500, 1000\}$ for a fixed order of the approximations, namely for $m = n = 5$.

It is clear that the higher the order $m = n$ (or the impulse sequence N) of the approximations the better the fitting, in a least-squares sense, both in the frequency and the step responses, of the fractional-order integrator $s^{-0.5}$. Furthermore, with the LS method we can tune the approximations for achieving better accuracy on a prescribed range of time t (or frequency ω) in contrast with other approximations that matches only the initial-time transient corresponding to the high frequency range.

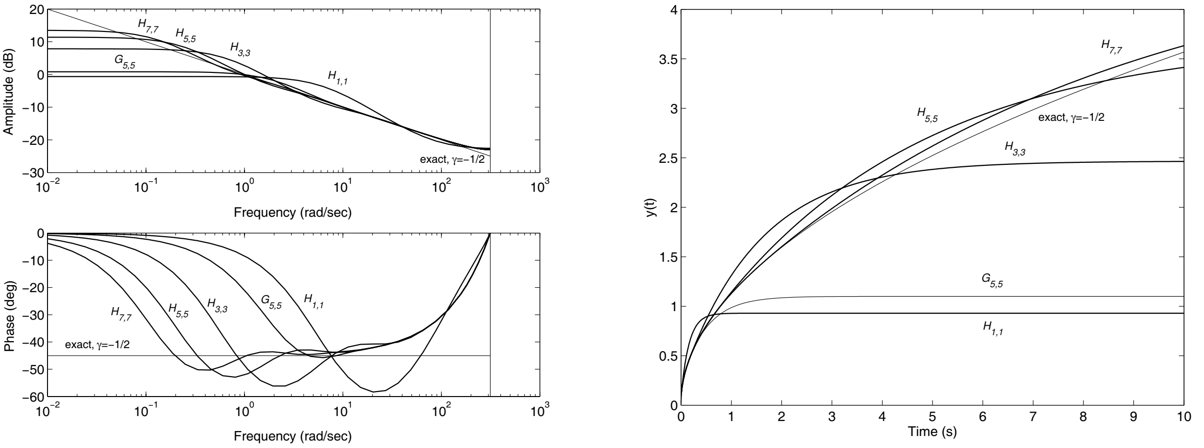

Fig. 1. Bode diagrams (left) and step responses (right) of the LS approximation $H_{m,n}(z)$, $m = n = \{1, 3, 5, 7\}$, *vs.* the Padé approximation $G_{5,5}(z^{-1})$ for the Euler operator with $\gamma = -1/2$ and $N = 1000$.

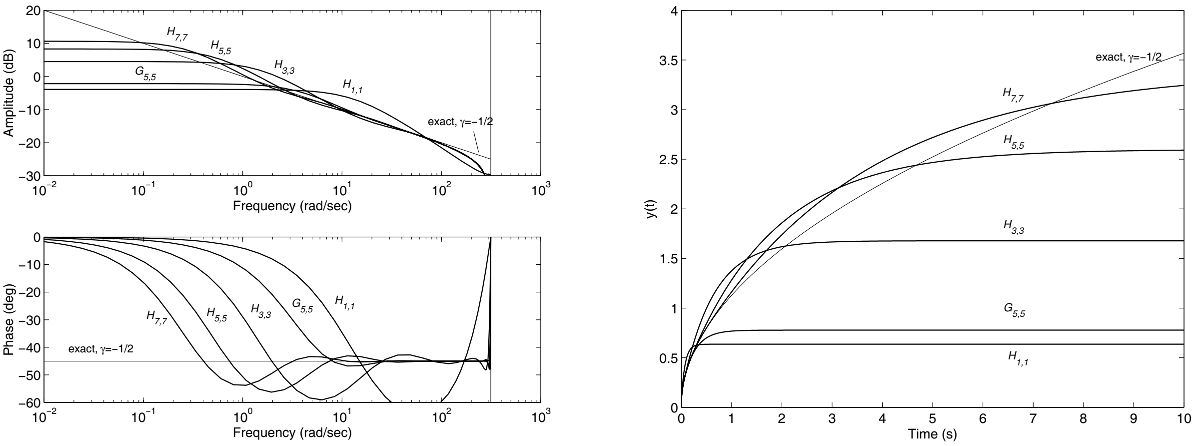

Fig. 2. Bode diagrams (left) and step responses (right) of the LS approximation $H_{m,n}(z^{-1})$, $m = n = \{1, 3, 5, 7\}$, *vs.* the Padé approximation $G_{5,5}(z^{-1})$ for the Tustin operator with $\gamma = -1/2$ and $N = 1000$.

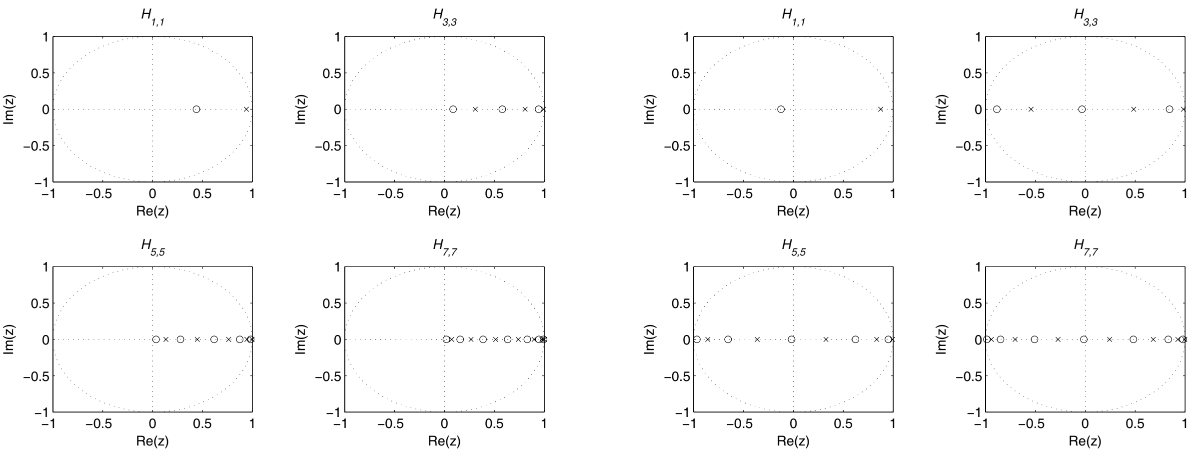

Fig. 3. Pole-zero map of the LS approximation $H_{m,n}(z^{-1})$, $m = n = \{1 , 3, 5, 7\}$ for the Euler (left) and Tustin (right) operators with $\gamma = -1/2$ and $N = 1000$.

Figure 3 shows the pole-zero map of the approximations $H_{m,n}(z)$, with $m = n = \{1, 3, 5, 7\}$, for the Euler and Tustin operators. We observe that the distribution of the zeros and poles satisfies two desired properties: (*i*) all the poles and zeros lie inside the unit circle and (*ii*) they are interlaced along the segment of the real axis, corresponding to $z \in\ ]0, 1[$ and $z \in\ ]-1, 1[$ for the Euler and Tustin operators, respectively.

In conclusion, the proposed LS method provides causal, stable and minimum-phase rational approximations as imposed for a digital realization. Its superior nature, in comparison with the Padé and the CFE approximation methods, is illustrated in the case of typical paradigms. The results presented here seem to indicate that the LS method is a suitable technique for obtaining discrete approximations of the fractional-order operators.

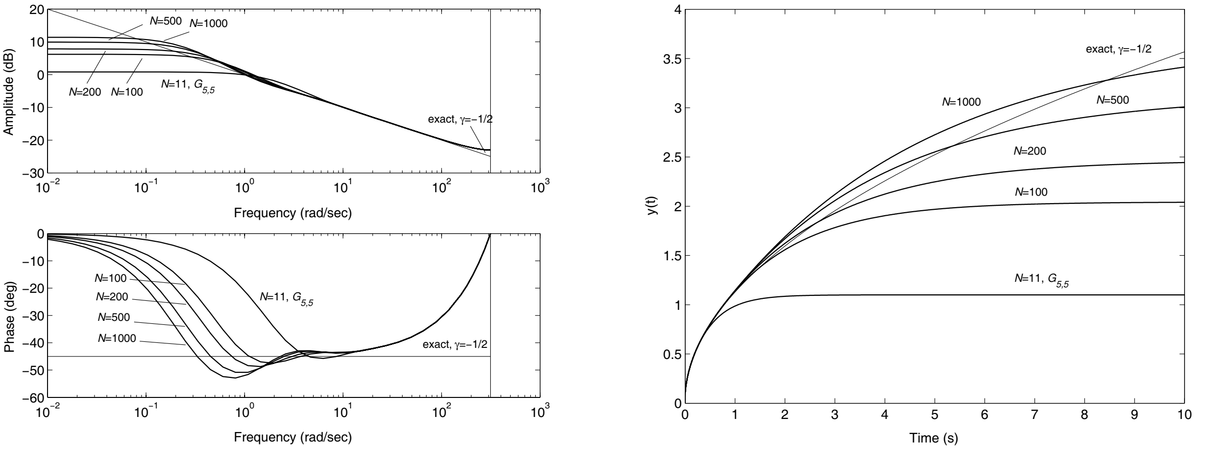

Fig. 4. Bode diagrams (left) and step responses (right) of the LS approximation $H_{5,5}(z)$ *vs*. the Padé approximation $G_{5,5}(z^{-1})$ for the Euler operator with $\gamma = -1/2$ and $N = \{11, 100, 200, 500, 1000\}$.

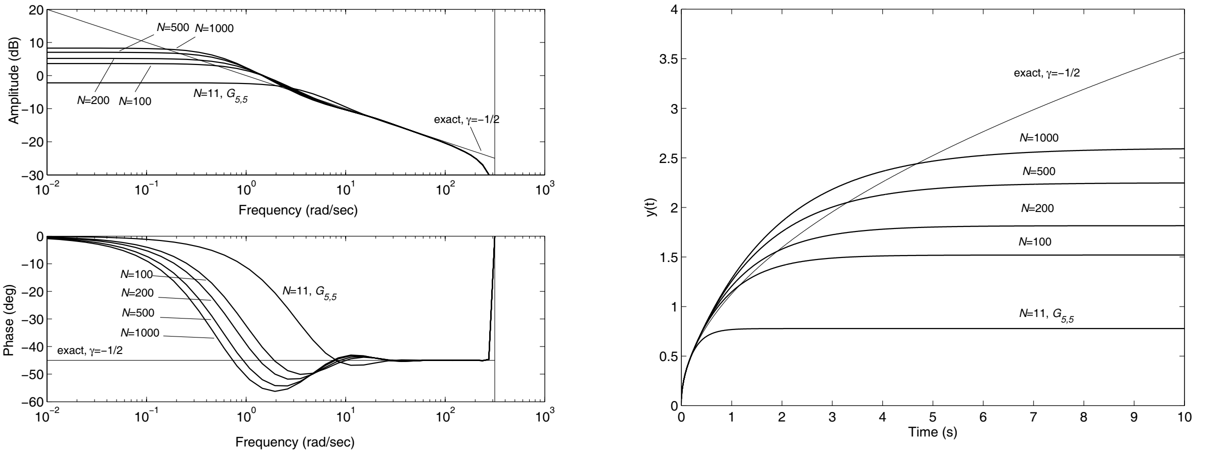

Fig. 5. Bode diagrams (left) and step responses (right) of the LS approximation $H_{5,5}(z)$ *vs*. the Padé approximation $G_{5,5}(z^{-1})$ for the Tustin operator with $\gamma = -1/2$ and $N = \{11, 100, 200, 500, 1000\}$.

7. CONCLUSIONS

We have described the adoption of the LS method in the design of digital rational transfer functions that approximates fractional-order operators of type s^γ, $\gamma \in \Re$. The method was illustrated for a fractional integrator of order $\gamma = -1/2$, but it can be generalized to others real noninteger values. It was shown that the new discrete rational functions give better results, both in time and frequency domains, than other approaches used for the same purpose, namely the Padé or the CFE approximations. Furthemore, the LS method yields causal, stable and minimum-phase rational transfer functions suitable for real-time implementation.

However, further research on this topic is needed. One may point out several lines of investigation on this subject. For instance, the use of interpolation techniques between discretization schemes based on the Euler, Tustin or Simpson operators and to apply the proposed LS method in the resulting new schemes. This matter is under study and will be the subject of future publications. In this line of thought, this paper represents a step towards the implementation of practical digital fractional-order differentiators and integrators.

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