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FRACTIONAL ORDER DYNAMICS IN CLASSICAL ELECTROMAGNETIC PHENOMENA

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Abstract

The Maxwell equations play a fundamental role in the well established formulation of the electromagnetic theory. These equations lead to the derivation of precise mathematical models useful in many applications in physics and engineering. The Maxwell equations involve only the integer-order calculus and, therefore, it is natural that the resulting classical models adopted in electrical engineering reflect this perspective.

Recently, a closer look of some phenomena present in electrical systems, such as motors, transformers and lines, and the motivation towards the development of comprehensive models, seem to point out the requirement for a fractional calculus approach.

Bearing these ideas in mind, in this study we shall address the well-known 'skin effect' and we re-evaluate the results demonstrating its fractional-order dynamics.

Key words

Skin effect, Eddy currents, Electromagnetism, Fractional calculus, Dynamical systems.

1 Introduction

Some experimentation with magnets was beginning in the late 19th century. By then reliable batteries had been developed and the electric current was recognized as a stream of charge particles. Maxwell developed a set of equations expressing the basic laws of electricity and magnetism, and he demonstrated that these two phenomena are complementary aspects of electromagnetism. He showed that electric and magnetic fields travel through space, in the form of waves, at a constant velocity. Maxwell is generally regarded as the nineteenth century scientist who had the greatest influence on twentieth century physics, making contributions to the fundamental models of nature.

The Skin Effect (*SE*) is one subject who can be explained by the Maxwell's equations. The *SE* is the tendency of a high-frequency electric current to distribute itself in a conductor so that the current density near the surface is greater than that at its core. This phenomenon increases the effective resistance of the conductor with the frequency of the current. The effect is most pronounced in radio-frequency systems, especially antennas and transmission lines, but it can also affect the performance of high-fidelity sound equipment, by causing attenuation in the treble range. The first study of *SE* was explained by Lord Kelvin in 1887, but many other scientists contributed to improve the comprehension of this theme, namely Nikola Tesla.

The *SE* can be reduced by using stranded rather than solid wire. This increases the effective surface area of the wire for a given wire gauge. It is simple to see that the spatial variation of the fields in vacuum is much smaller than the spatial variation in the metal. Therefore, in usual study, for the purposes of evaluating the fields in the conductor, the spatial variation from the wave length outside the conductor can be ignored. For the usual case the radii of curvature of the surface should be much larger than a skin depth, the solution is straightforward. To analyze this phenomenon, we apply the Maxwell's equations that relate the solutions for these fields. More often, however, some of the parameters that tend to be considered are the capacitance per length, inductance per length, and their relationship with the signals, the nominal propagation velocity and the characteristic impedance of the system.

In our study we apply the Bessel functions to compute values of cable impedance *Z*. For the sake of clarity we plot some values of the low and high frequency approximations of impedance. We verify the fractional order of these systems, namely the half-order nature of dynamic phenomenon.

Having these ideas in mind this paper is organized as follows. Section 2 summarizes the mathematical description of the *SE* and section 3 re-evaluates the results demonstrating its fractional-order dynamics. After clarifying the fundamental concepts it is addressed the case of Eddy (or Foucault) currents that occur in electrical machines. Finally, section 4 draws the main conclusions.

2 The Skin Effect

The internal impedance of a wire is function of the frequency. In a conductor, where the conductivity is sufficiently high, the displacement current density can be neglected. In this case, the conduction current density is given by the product of the electric field and the conductance. When we apply these simplifications, we get the Maxwell's equations.

One of the aspects of the high frequency effects is the *SE*. The fundamental problem with *SE* is the attenuation the higher frequency components of a signal.

In order to analyze the *SE* we start by recalling the classical model development for this electromagnetic phenomenon.

In the differential form the Maxwell equations are [Feynman, *et al.*, 1964]:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1a}$$

$$\nabla \times \mathbf{H} = \boldsymbol{\delta} + \frac{\partial \mathbf{D}}{\partial t} \tag{1b}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{1c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1d}$$

where **E**, **D**, **H**, **B**, δ are the vectors of electric field intensity, electric flux density (or electric displacement), magnetic field intensity, magnetic flux density and the current density, respectively, and ρ and *t* are the charge density and time. Moreover, for a homogeneous, linear and isotropic media, we can establish the relationships:

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{2a}$$

$$\mathbf{B} = \mu \mathbf{H} \tag{2b}$$

$$\boldsymbol{\delta} = \gamma \mathbf{E} \tag{2c}$$

where ε, μ and γ are the electrical permittivity, the magnetic permeability and the conductivity, respectively.

In order to study the *SE* we start by considering a cylindrical conductor with radius r_0 conducting a current *I* along its longitudinal axis. In a conductor, even for high frequencies, the term $\partial \mathbf{D}/\partial t$ is negligible in comparison with the conduction term **δ** or, by other words, the displacement current is much lower than the conduction current. Therefore, for a radial distance $r < r_0$ the application of the Maxwell's equations with the simplification of (1b) leads to the expression [Küpfmüller and Einführung, 1939]-[Bessonov, 1968]:

$$\frac{\partial^2 E}{\partial r^2} + \frac{1}{r}\frac{\partial E}{\partial r} = \gamma\mu \frac{\partial E}{\partial t} \tag{3}$$

For a sinusoidal field we can adopt the complex notation $E = \sqrt{2}\tilde{E} e^{j\omega t}$, where $j = \sqrt{-1}$, yielding:

$$\frac{d^2 \tilde{E}}{dr^2} + \frac{1}{r}\frac{d\tilde{E}}{dr} + q^2 \tilde{E} = 0 \tag{4}$$

with $q^2 = -j\omega\gamma\mu$.

Equation (4) is a particular case of the Bessel equation that, for the case under study, has solution of the type:

$$\tilde{E} = \frac{q}{2\pi r_0 \gamma} \frac{J_0(qr)}{J_1(qr_0)} I,\ 0 \le r \le r_0 \tag{5}$$

where J_0 and J_1 are complex valued Bessel functions of the first kind of orders 0 and 1, respectively.

Equation (5) establishes the so-called *SE*, a phenomenon that yields a non-uniform current density along the conductor cross section. We have a low density near the conductor axis and an high density on the surface, being the larger the phenomenon the higher the frequency ω. Therefore, the total voltage drop is $\tilde{Z}\tilde{I} = \tilde{E}\tilde{I}$ that, for a conductor of length l_0, results:

$$\tilde{Z} = \tilde{E} = \frac{q l_0}{2\pi r_0 \gamma} \frac{J_0(q r_0)}{J_1(q r_0)} \tag{6}$$

where $\tilde{Z}$ is the equivalent electrical complex impedance.

Knowing [Abramowitz and Stegun, 1965] the Taylor series:

$$J_0(x) = 1 - \frac{x^2}{2^2} + \cdots,\ J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 4} + \cdots \tag{7}$$

and, for large values of *x,* the asymptotic expansion:

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - n\frac{\pi}{2} - \frac{\pi}{4}\right),\ n = 0,1,\cdots \tag{8}$$

we can obtain the low and high frequency approximations of $\tilde{Z}$:

$$\omega \to 0 \Rightarrow \tilde{Z} \approx \frac{l_0}{\pi r_0^{\,2} \gamma} \tag{9a}$$

$$\omega \to \infty \Rightarrow \tilde{Z} \approx \frac{l_0}{2\pi r_0} \sqrt{\frac{\omega\mu}{2\gamma}} (1 + j) \tag{9b}$$

3 A Fractional Calculus Approximation

In order to avoid the complexity of the transcendental equation (6) the standard approach in electrical engineering is to assign a resistance R and inductance L given by $R + j\omega L = \tilde{Z}$. Nevertheless, although widely used, this method is clearly inadequate because the model parameter values $\{L,R\}$ vary with the frequency. Moreover, (9b) points out the half-order nature of the dynamic phenomenon, at high frequencies (*i.e.*, $\tilde{Z} \sim \omega^{1/2}$), which is not captured by and integer-order approach. A possible approach that eliminates those problems is to adopt the fractional calculus [Aubourg and Mengue, 1998]-[Canat and Faucher, 2003]-[Machado and Jesus]-[Malpica, *et al.*, 2004], [Benchellal A, *et al.*, 2004]. Joining the two asymptotic expressions (9) we can establish several types of approximations, namely the zero and one parameter fractions:

$$\tilde{Z}_{app0} \approx \frac{l_0}{\pi r_0^{\,2} \gamma} \left[j\omega \left(\frac{r_0}{2} \right)^2 \mu\gamma + 1 \right]^{1/2} \tag{10a}$$

$$\tilde{Z}_{app1} \approx \frac{l_0}{\pi r_0^{\,2} \gamma} \frac{1 + j\omega\tau_1}{(1 + j\omega\tau_2)^{1/2}} \tag{10b}$$

where $\tau_i = a_i r_0^{\,2} \gamma\mu$, $i = 1,2$, and $a_1 = \sqrt{a_2/2}$ is a numerical value to be adjusted.

In order to analyze the feasibility of (10a) and (10b) with define the errors in the amplitude and phase as:

$$\varepsilon_k = Max\left\{ \left| Mod(\tilde{Z}) - Mod\,\tilde{Z}_{appk} \right|, \omega \right\} \tag{11a}$$

$$\phi_k = Max\left\{ \left| Phase(\tilde{Z}) - Phase\,\tilde{Z}_{appk} \right|, \omega \right\} \tag{11b}$$

for the $k = 0,1$ parameter approximations.

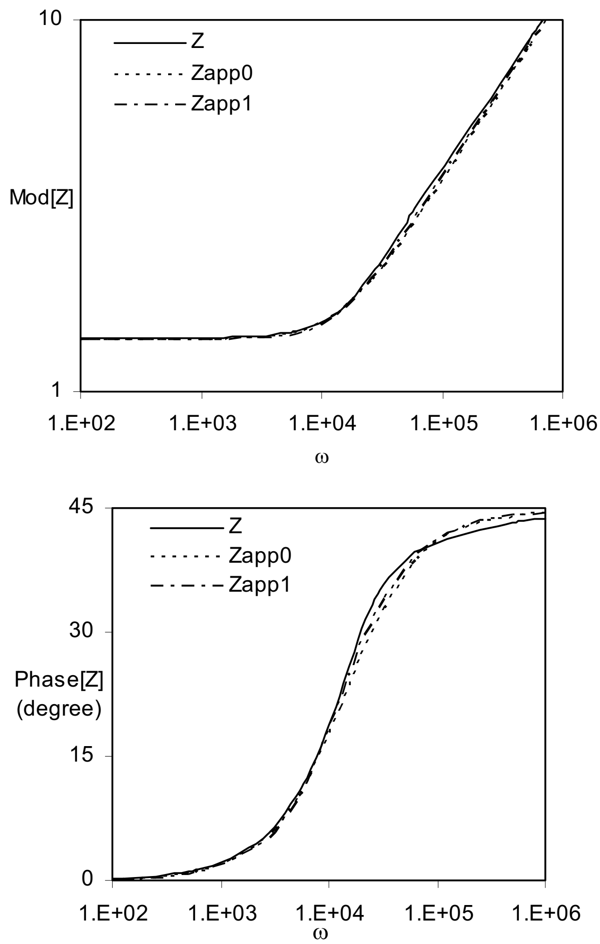

Figure 1 Amplitude and phase Bode diagrams of $\tilde{Z}$ for the theoretical and the approximate expressions with $\gamma = 5.7\;10^7\,\Omega^{-1}\text{m}$, $l_0 = 10^3\,\text{m}$, $r_0 = 2.0\;10^{-3}\,\text{m}$, $\mu = 1.257\;10^{-6}\,\text{Hm}^{-1}$.

Figure 1 compares the Bode diagrams of amplitude and phase for expressions (6) and (10) (eq. 10b with a numerical non-linear regression lead $a_1 = 0.3647$) revealing a very good fit. On the other hand, Fig. 2 depicts the approximation errors that lead to $\{\varepsilon_0, \phi_0\} \le \{6.5\%, 8.3\%\}$ and $\{\varepsilon_1, \phi_1\} \le \{4.2\%, 9.6\%\}$ in the frequency charts of amplitude and phases, respectively.

These physical concepts and mathematical tools can be adopted in more complex systems. In fact, the 'Eddy Currents' phenomenon, usual in electrical machines such as transformers and motors, can be modeled using identical guidelines.

For example, let us consider the magnetic circuit of an electrical machine constituted by a laminated iron core.

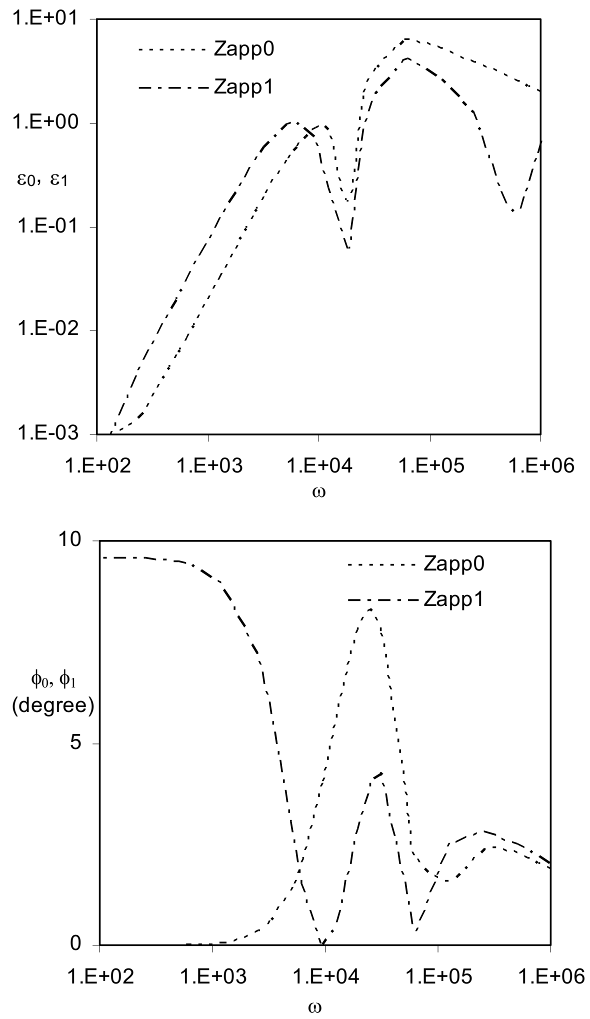

Figure 2 Amplitude and phase errors $\{\varepsilon_i, \phi_i\}$, $i = 0,1$, for the two approximate expressions.

Each ferromagnetic metal sheet with permeability μ has thickness *d* and width *b* (*b* >> *d*) making a closed magnetic circuit with an average length l_0. The total pack of ferromagnetic metal sheet make a height *a* while embracing a coil having *n* turns with current *I*.

The contribution of the magnetic core to the coil impedance is (for details see [Küpfmüller, 1939]):

$$\tilde{Z} = \frac{2\mu ab\, j\omega\, n^2}{(1+j)\,\beta L d}\tanh\left[(1+j)\,\beta\frac{d}{2}\right] \qquad (12)$$

where $\beta = \sqrt{\omega\gamma\mu/2}$.

Alternatively, expression (12) can be re-written as:

$$\tilde{Z} = \frac{\mu ab\, n^2}{l_0}\omega \cdot \frac{[\sinh(\beta d) - \sin\beta d] + j[\sinh(\beta d) + \sin\beta d]}{(\beta d)[\cosh\beta d + \cos\beta d]} \qquad (13)$$

We can obtain the low and high frequency approximations of $\tilde{Z}$:

$$\omega \to 0 \Rightarrow \tilde{Z} \approx j\omega\frac{\mu ab\, n^2}{l_0} \qquad (14a)$$

$$\omega \to \infty \Rightarrow \tilde{Z} \approx \frac{\mu ab\, n^2}{l_0}\frac{1}{d}\sqrt{\frac{2\omega}{\gamma\mu}}(1+j) \qquad (14b)$$

Once more we have a clear half-order dependence of $\tilde{Z}$ (*i.e.*, $\tilde{Z} \sim \omega^{1/2}$) while the standard approach is to assign frequency-dependent 'equivalent' resistance *R* and inductance *L* given by $R + j\omega L = \tilde{Z}$.

A possible approach that eliminates those problems is to joint the two asymptotic expressions (14).

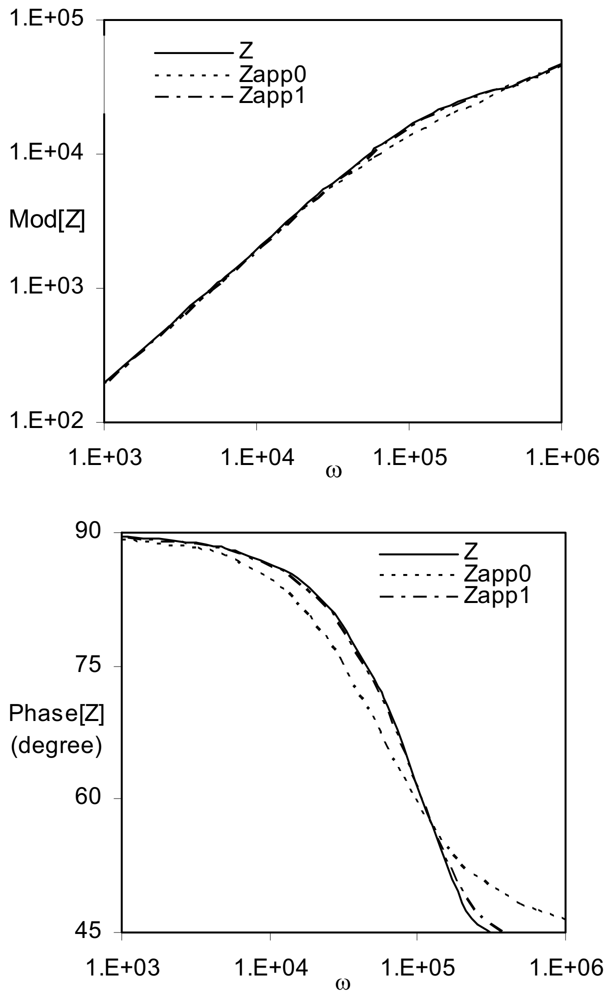

Figure 3 Amplitude and phase Bode diagrams of $\tilde{Z}$ for the theoretical and the approximate expressions with: $l_0 = 1.0$m, $a = 0.28$m, $b = 0.28$m, $d = 2.0\,10^{-3}$m, $n = 100$, $\gamma = 7.0\,10^{4}\,\Omega^{-1}$m, $\mu = 200 \cdot 1.257\,10^{-6}\,\mathrm{Hm}^{-1}$

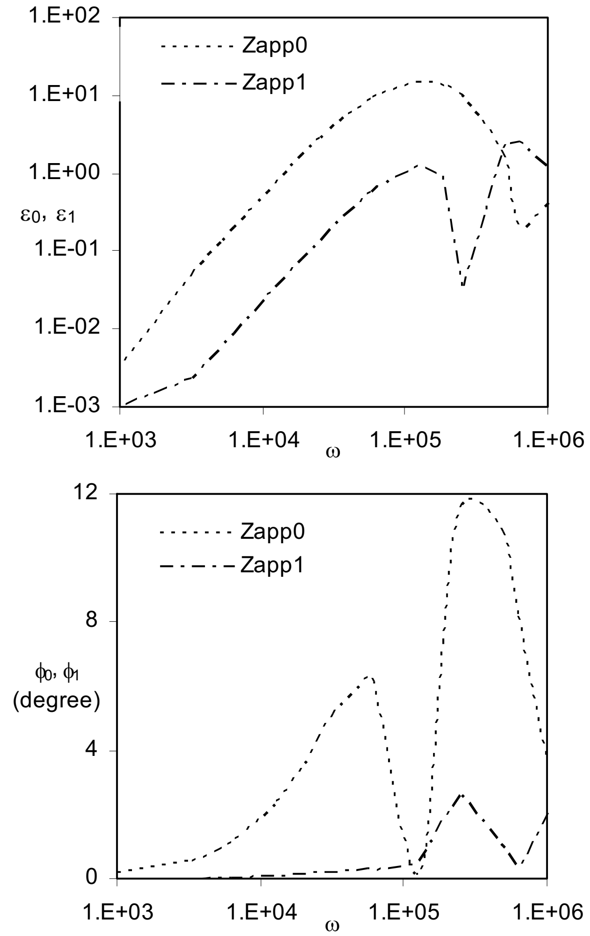

Figure 4 Amplitude and phase errors $\{\varepsilon_i, \phi_i\}$, $i = 0,1$, for the two approximate expressions.

Therefore, we can establish several types of approximations, namely the zero and one parameter fractions:

$$\tilde{Z}_{app0} \approx \frac{j\omega\,\mu ab\,n^2}{l_0}\left[j\omega\left(\frac{d}{2}\right)^2 \mu\gamma + 1\right]^{-1/2} \tag{15a}$$

$$\tilde{Z}_{app1} \approx \frac{j\omega\,\mu ab\,n^2}{l_0}\left(\frac{1 + j\omega\tau_1}{1 + j\omega\tau_2}\right)^{1/2} \tag{15b}$$

where $\tau_i = a_i \gamma\mu (d/2)^2$, $i = 1,2$, and $\sqrt{a_1} = a_2$ is a numerical value to be adjusted.

Figure 3 compares the Bode diagrams of amplitude and phase for expressions (12) and (15) (eq. 15b with a numerical non-linear regression lead $a_1 = 0.1860$) revealing a very good fit. Figure 4 depicts the errors $\{\varepsilon_0, \phi_0\} \le \{15.6\%, 11.7\%\}$, $\{\varepsilon_1, \phi_1\} \le \{2.6\%, 2.8\%\}$ in the frequency charts of amplitude and phases, respectively.

4 Conclusions

In conclusion, we have that the classical electromagnetism and the Maxwell equation, with integer order derivatives, lead to models requiring a fractional calculus perspective to be fully interpreted. Another aspect of interest is that in all cases we get only half-order models. Therefore, the meaning of the D^{α} for the particular case of $\alpha = 1/2$ and its relationship with integer-order calculus remains to be investigated.

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