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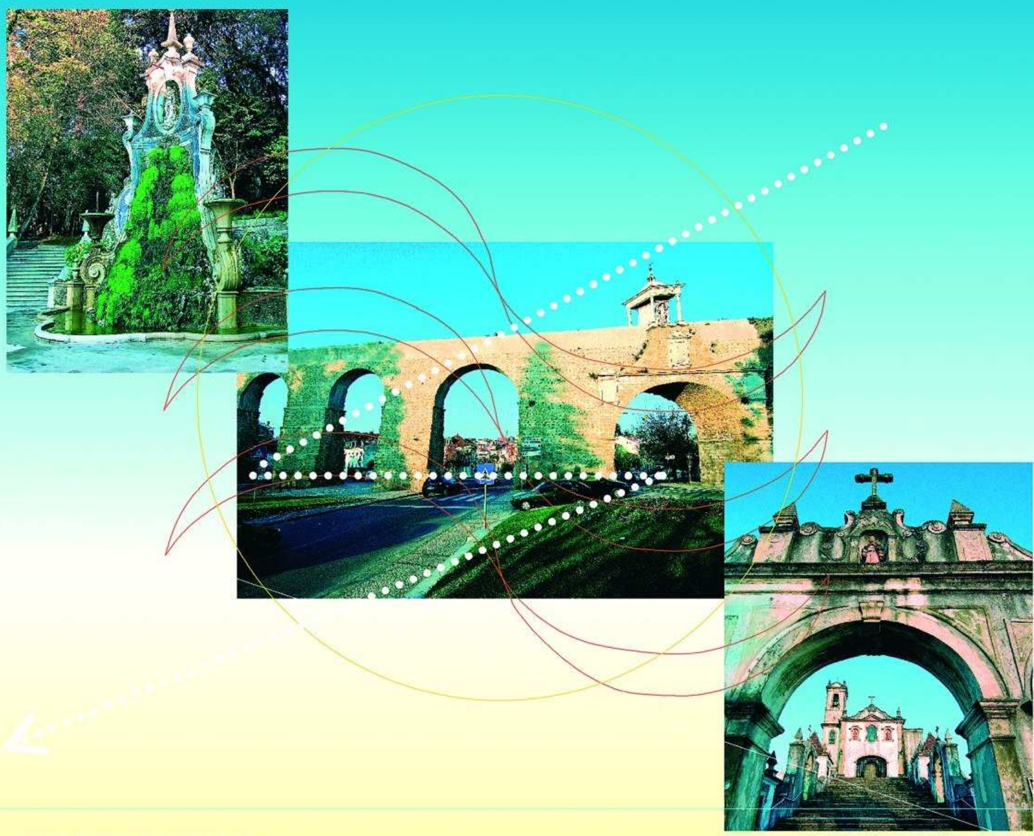

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DESCRIBING FUNCTION OF SYSTEMS WITH NONLINEAR FRICTION

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This paper studies the describing function (DF) of systems composed of a mass subjected to nonlinear friction. The friction force is decomposed in three components namely, the viscous, the Coulomb and the static forces. The system dynamics is analyzed in the DF perspective and the reliability of the DF method is evaluated through the signal harmonic content.

1. INTRODUCTION

The phenomenon of vibration due to friction occurs in many branches of technology where it plays a very useful role. On the other hand, its occurrence is often undesirable, because it causes additional dynamic loads, as well as faulty operation of machines and devices. Despite many investigations that have been carried out so far, this phenomenon is not yet fully understood, mainly due to the considerable randomness and diversity of reasons underlying the energy dissipation involving the dynamic effects [1], [6], [7]. These nonlinear dynamic phenomena have been an active area of research but well established conclusions are still lacking.

In this paper we investigate the dynamics of systems that contain nonlinear friction namely the Coulomb and the static forces in addition to the linear viscous, component. Bearing these ideas in mind, the article is organized as follows. Section 2 introduces the fundamental aspects of the describing function method. Section 3 studies the describing function of mechanical systems with nonlinear friction. Finally, section 4 draws the main conclusions and addresses perspectives towards future developments.

2. FUNDAMENTAL CONCEPTS

In this section we present a summary of the DF method and its application on the prediction of limit cycles. The purpose is to analyse the controller performance in the presence of systems with nonlinear friction. Due to the nonlinear nature of the problem a possible approach would be the simulation of all possible systems which, obviously, is a time consuming and fastidious task. Therefore, the strategy taken here is to study the DF evolution in the Nyquist diagram of each controller and plant. By this way, we can study the stability and we can predict approximately the occurrence and the characteristics of limit cycles.

It is a well-known fact that many relationships among physical quantities are not linear, although they are often approximated by linear equations, mainly for mathematical simplicity. This simplification may be satisfactory as long as the resulting solutions are in agreement with experimental results. In fact, Cox [4] demonstrated that this is the case with the approximation of nonlinear systems by a DF where limit cycles can be predicted with reasonable accuracy. The DF method is not the only one tractable to limit cycle prediction; nevertheless, in the condition of limit cycle occurrence all of the methods are equivalent to the DF method [4].

Let us consider the feedback system of Figure 1 with one nonlinear element *N* and a linear system *G*(*s*).

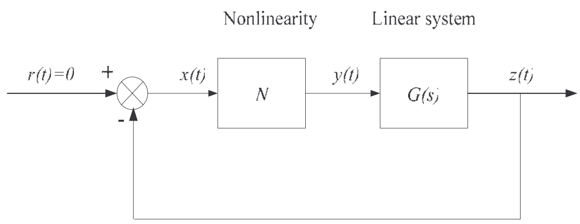

Figure 1. Nonlinear control system

Suppose that the input to a nonlinear element is sinusoidal $x(t) = X\sin(\omega t)$. In general the output of the nonlinear element is not sinusoidal, but it is periodic, with the same period as the input, containing higher harmonics in addition to the fundamental harmonic component.

If we assume that the nonlinearity is symmetric with respect to the variation around zero, the Fourier series become:

$$y(t) = \sum_{k=1}^{\infty} Y_k \cos(k\,\omega\, t + \phi_k) \qquad (1)$$

where Y_k and ϕ_k are the amplitude and the phase shift of the kth harmonic component of the output $y(t)$, respectively.

In the DF analysis, we assume that only the fundamental harmonic component of the output is significant. Such assumption is often valid since the higher harmonics in the output of a nonlinear element are usually of smaller amplitude than the fundamental component. Moreover, most control systems are "low-pass filters" with the result that the higher harmonics are further attenuated.

The DF, or sinusoidal DF, of a nonlinear element, $N(X,\omega)$, is defined as the complex ratio of the fundamental harmonic component of the output $y(t)$ and the input $x(t)$, that is:

$$N(X,\omega)=\frac{Y_1}{X}e^{j\phi_1} \tag{2}$$

where the symbol N represents the DF, X is the amplitude of the input sinusoid and Y_1 and ϕ_1 are the amplitude and the phase shift of the fundamental harmonic component of the output, respectively. Several DFs of standard nonlinear system elements can be found in the references [2], [3], [5].

For nonlinear systems that do not involve energy storage, the DF is merely amplitude-dependent, that is $N = N(X)$. When dealing with nonlinear elements that store energy, the DF method is both amplitude and frequency dependent, that is, $N = N(X, \omega)$. In this case, to determine the DF usually we have a numerical approach rather than a symbolic one because, in general, it is impossible to find a closed-form solution for the differential equations that model the nonlinear element. Nevertheless, it is possible to calculate the approximate analytical expressions for such DFs, namely with the aid of computer algebra packages. Once calculated, the DF can be used for the approximate stability analysis of a nonlinear control system.

Let us consider the standard control system shown in Figure 1 where the block N denotes the DF of the nonlinear element. If the higher harmonics are sufficiently attenuated, N can be treated as a real or complex variable gain and the closed-loop frequency response becomes:

$$\frac{C(j\omega)}{R(j\omega)}=\frac{NG(j\omega)}{1+NG(j\omega)} \tag{3}$$

The characteristic equation is:

$$1+NG(j\omega)=0 \Leftrightarrow G(j\omega)=-\frac{1}{N(X,\omega)} \tag{4}$$

If (4) can be satisfied for some value of X and ω, a limit cycle is *predicted* for the nonlinear system. Moreover, since (4) applies only if the nonlinear system is in a steady-state limit cycle, the DF analysis predicts only the presence or the absence of a limit cycle and cannot be applied to the analysis of other types of time responses.

3. SYSTEMS WITH NONLINEAR FRICTION

In this section we calculate the DF of a dynamical system with nonlinear friction and we study its properties. In sub-section 3.1 we start by a combination of the viscous and Coulomb components. In sub-section 3.2 we complement the study by including also the static friction.

3.1. Coulomb and Viscous Friction

Let us consider a system (Fig. 2) with a mass M, moving on a horizontal plane under the action of a force f, with a friction effect composed of two components: a non-linear Coulomb K part and a linear viscous B part (CV model).

The equation of motion in this system is as follows:

$$M\,\ddot{x}(t)+F_f(t)=f(t) \tag{5}$$

where M is the system mass, $F_f(t)$ is the friction force and $f(t)$ the applied input force.

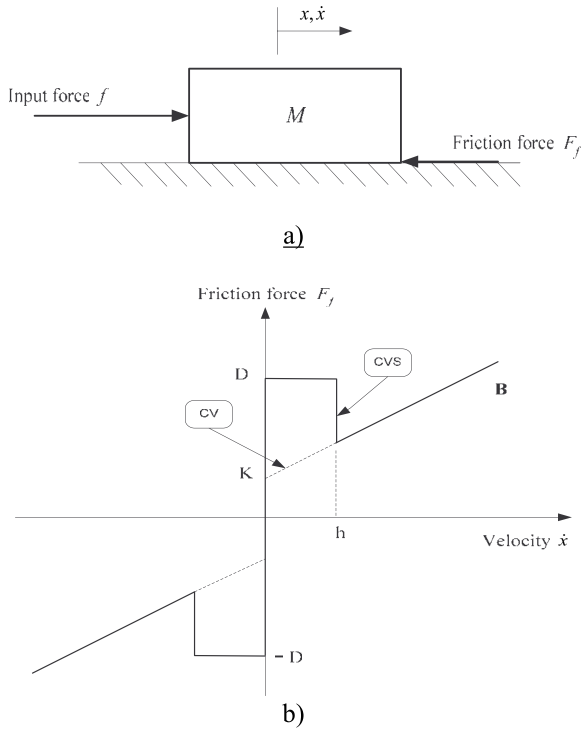

Figure 2. a) Elemental mass system subjected to nonlinear friction and b) Non-linear friction with Coulomb, Viscous (CV model) and Static components (CVS model).

For the simple system of Figure 2.a) we can calculate, numerically, the polar plot of $-1/N(F,\omega)$ considering as input a sinusoidal force $f(t)=F\cos(\omega t)$ applied to mass M and as output the position $x(t)$.

Figure 3 shows the function $-1/N(F,\omega)$ for several values of F when $M = 9$ Kg, $B = 0.5$ Ns/m, $K = 5$ N.

Figure 4 illustrates the log-log plots of Re{−1/N} and Im{−1/N} *vs*. the exciting frequency ω, for different values of the input force F = {10, 50, 100} N. The charts reveal that we have different results according to the excitation force F, being it more visible for the imaginary component.

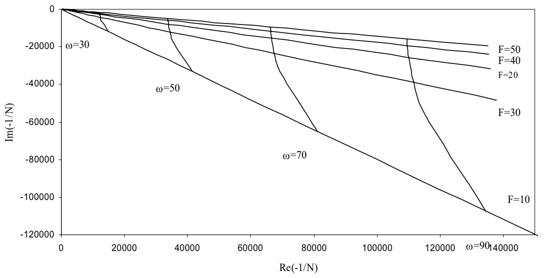

Figure 3. Polar plot of $-1/N(F,\omega)$ for the system subjected to nonlinear friction (CV model) and input forces F = {10, 20, 30, 40, 50} N.

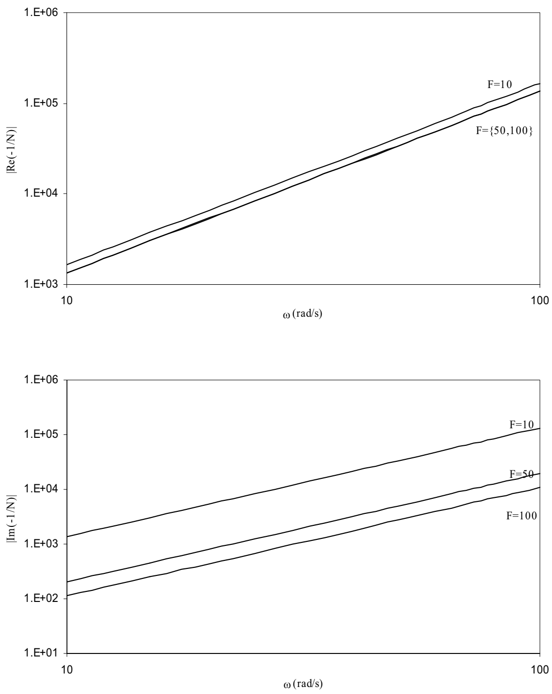

Figure 4. Log-log plots of Re{−1/N} and Im{−1/N} *vs*. the exciting frequency ω for F = {10, 50, 100} N, with de CV model.

In Figure 5 it is depicted the harmonic content of the output signal $x(t)$ for input forces of F = 10 N and F = 50 N. From this charts we conclude that the output signal has a half - wave symmetry, because the harmonics of even order are negligible. Moreover, the fundamental component of the output signal is the most important one, while the amplitude of the high order harmonics decays significantly. Therefore, we can conclude that, for the friction CV model, the DF method leads to a good approximation.

In order to gain further insight into the system nature, we repeat the experiment for different mass values M = {0.10, 0.25, 0.50, 1.0, 2.0, 3.0, 5.0, 7.0} Kg.

The results shows that the value of Re{−1/N} and Im{−1/N} fluctuate for different M values.

To study the relation between Re{−1/N} and Im{−1/N} *versus* F and M, we approximate the numerical results through power functions:

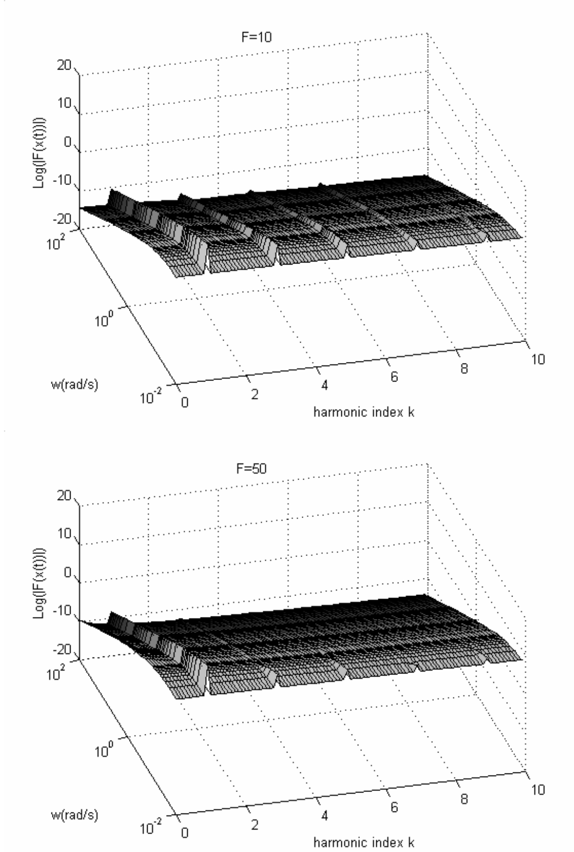

Figure 5. Fourier transform of the output position $x(t)$, over 50 cycles for the CV model, *vs*. the exciting frequency ω and the harmonic frequency index k for input forces F = 10 N and F = 50 N.

$$\mathrm{Re}\{-1/N\} = a\,\omega^{b},\ \mathrm{Im}\{-1/N\} = c\,\omega^{d} \qquad (6)$$

Figure 6 illustrates the variation of the {a, b, c, d} parameters with F and M.

The {a, b, c, d} parameters can also be approximated by heuristic analytical expressions, namely:

$$\begin{aligned} a &= \alpha F/(F-\beta) \\ b &\approx 2.0 \\ c &= \chi F/(F-\delta)^{\varepsilon} \\ d &= -\rho \ln(F) + \eta \end{aligned} \qquad (7)$$

where F is the input force and $\{\alpha, \beta, \chi, \delta, \varepsilon, \rho, \eta\}$ are parameters that depend on the mass M. We conclude that the parameters β and δ seems similar to K. Moreover, Re{−1/N} and Im{−1/N} have distinct relationships with ω, namely integer and fractional order dependences. The second case is of utmost importance because it establishes a link towards the area of fractional calculus [8] and it properties of dynamical memory.

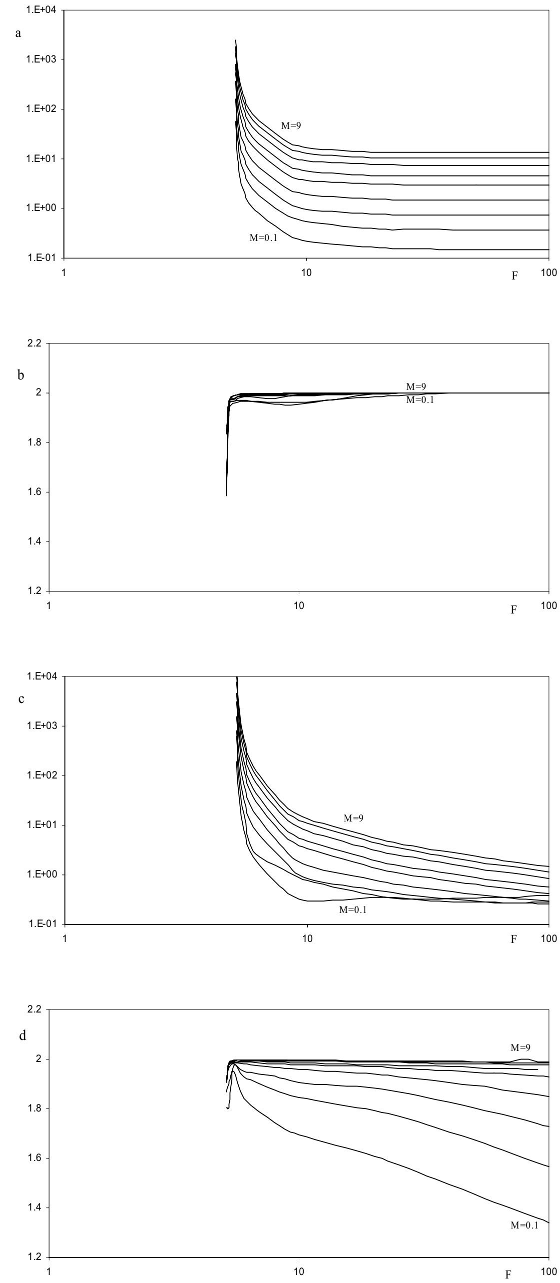

Figure 6. Variation of the {a, b, c, d} parameters versus F and M = {0.1, 0.25, 0.5, 1, 2, 3, 5, 7, 9} Kg, in the CV model

3.2. Coulomb, Viscous and Static Friction

In this sub-section we incorporate the static friction (D, h) in the CV model leading to the so-called CVS model. In this line of thought, we develop a study similar to the one adopted previously, with M = 9 Kg, B = 0.5 Ns/m, K = 5 N, D=7 N, h=0.5 ms^{-1} (Figures 7-10).

Comparing the results of the VC and VCS models we conclude that Re{−1/N} and Im{−1/N} are, in the two cases, of the same type, following power law according with (6).

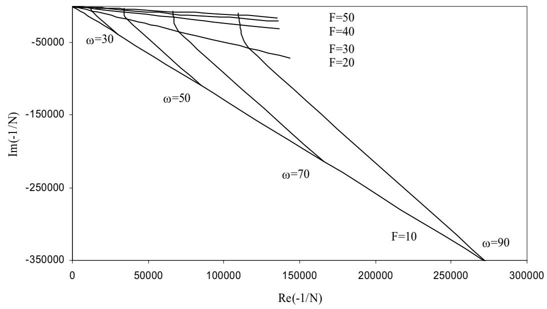

Figure 7. Polar plot of $-1/N(F,\omega)$ for the system subjected to nonlinear friction (CVS model) and input forces F = {10, 20, 30, 40, 50} N.

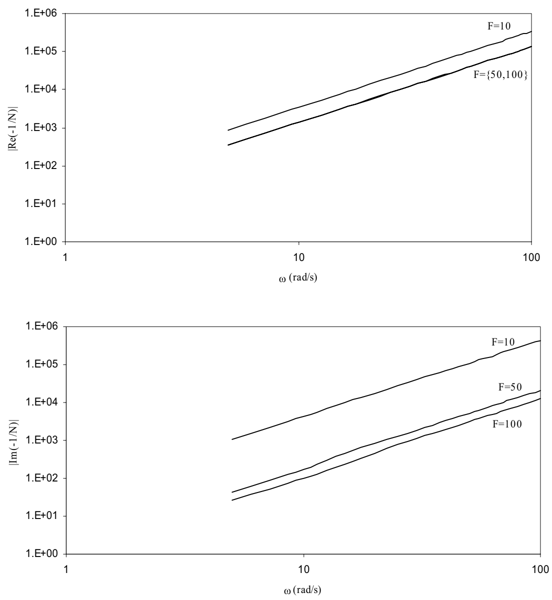

Figure 8. Log-log plots of Re{−1/N} and Im{−1/N} *vs*. the exciting frequency ω, for F = {10, 50, 100} N, with the CVS model.

Furthermore, once again we obtain integer-order and fractional-order dynamics for Re{−1/N} and the Im{−1/N}, respectively.

On the other hand, the CVS model is very sensitive to small input forces F (stimulation mainly the static component) leading to large values of $-1/N$ and to a higher harmonic content.

4. CONCLUSIONS

This paper addressed several aspects of the phenomena involved in systems with nonlinear friction. The dynamics of elemental mechanical system was analysed through the describing function method and compared with standard models. The results encourage further studies of nonlinear systems in a similar perspective and the analysis of limit cycle prediction. The conclusion may lead to the development of compensation schemes capable of improving control system performance.

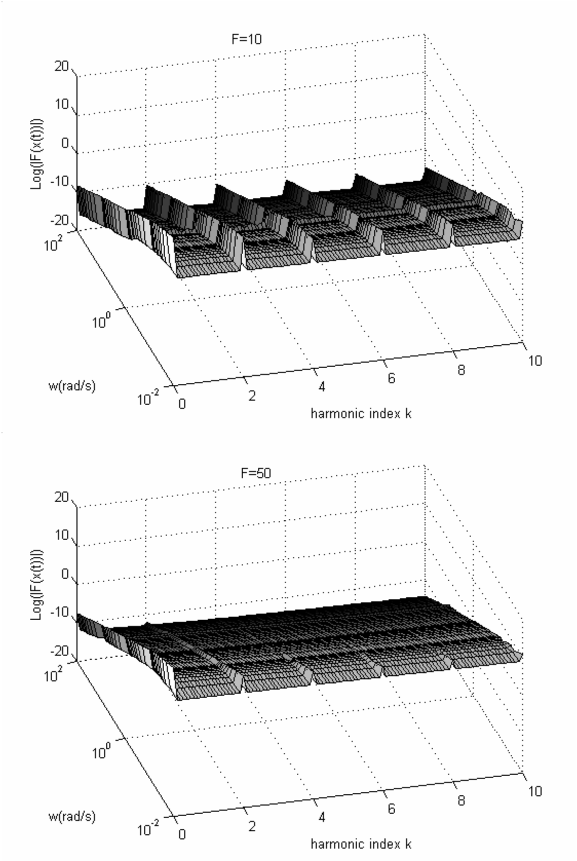

Figure 9. Fourier transform of the output position $x(t)$, over 50 cycles for the CV model, *vs*. the exciting frequency ω and the harmonic frequency index k for input forces $F = 10$ N and $F = 50$ N.

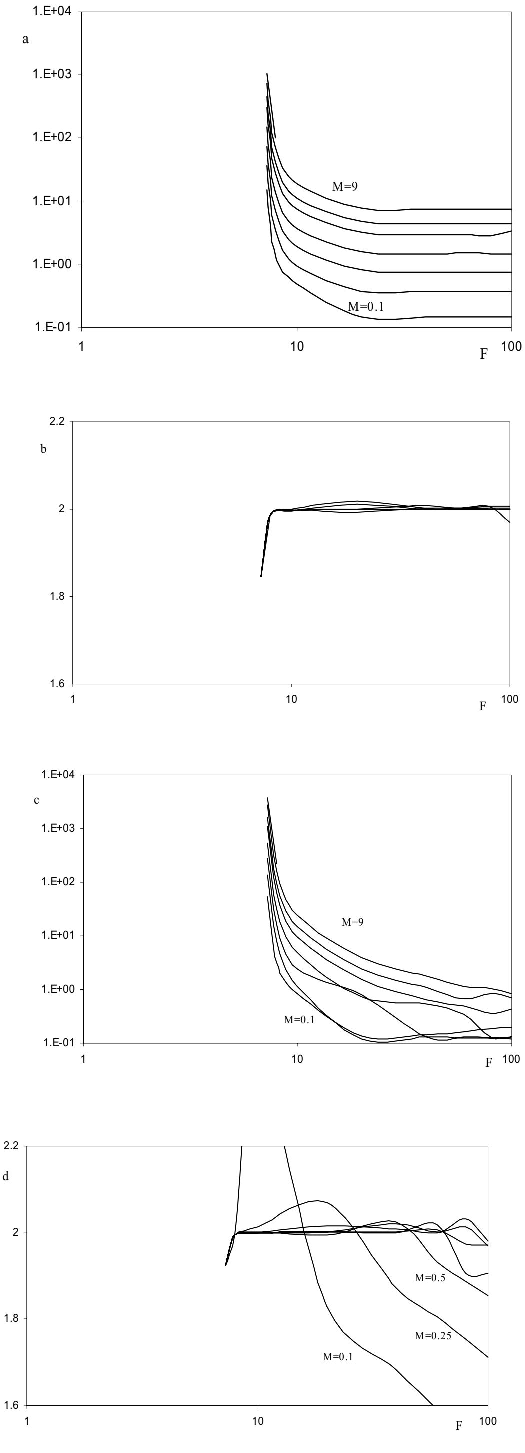

Figure 10. Variation of the $\{a, b, c, d\}$ parameters versus F and $M = \{0.1, 0.25, 0.5, 1, 2, 3, 5, 7, 9\}$ Kg, in the CVS model

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