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Study of the Van der Pol Oscillator with Fractional Derivatives

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Abstract — *In this paper we propose a modified version of the classical Van der Pol oscillator by introducing fractional-order time derivatives into the state-space model. The resulting fractional-order Van der Pol oscillator is analyzed in the time and frequency domains, by using phase portraits, spectral analysis and bifurcation diagrams. The fractional-order dynamics is illustrated through numerical simulations of the proposed schemes by using approximations to fractional-order operators. Finally, the analysis is extended to the forced Van der Pol oscillator.*

1 Introduction

The study of nonlinear oscillators has been important in the development of the theory of dynamical systems. The Van der Pol oscillator (VPO), described by a second-order nonlinear differential equation, can be regarded as describing a mass-spring-damper system with a nonlinear position-dependent damping coefficient or, equivalently, an RLC electrical circuit with a negative-nonlinear resistor, and has been used for developing models in many applications, such as electronics, biology or acoustics. It represents a nonlinear system with an interesting behavior that arises naturally in several applications.

The VPO was used by Van der Pol in the twenties to study oscillations in vacuum tube circuits (part of the early radios). In the standard form, it is given by a second-order nonlinear differential equation of type:

$$\ddot{x} + \alpha\left(x^2 - 1\right)\dot{x} + x = 0 \tag{1}$$

where α is the control parameter that reflects the degree of nonlinearity of the system. The equation (1) possesses a periodic solution that attracts other solution except the trivial one at the unique equilibrium point $x = \dot{x} = 0$.

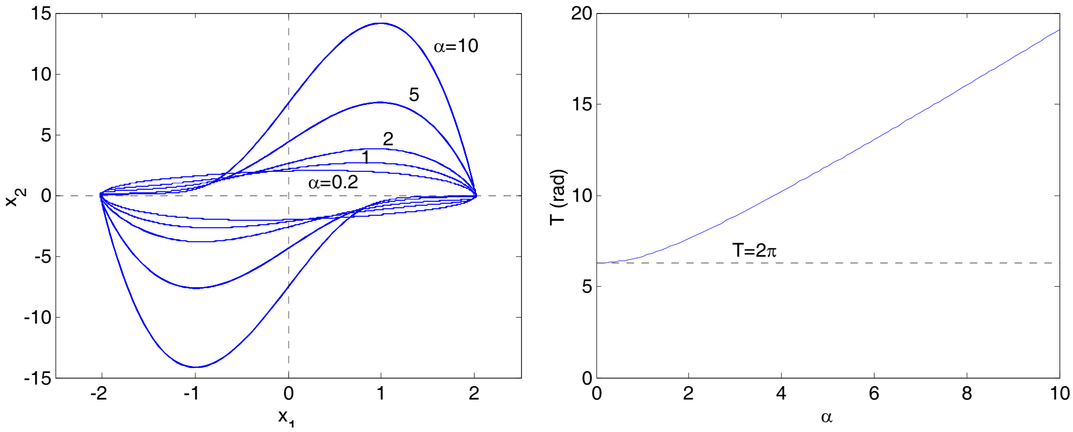

Figure 1: The Van der Pol oscillator: left) Phase portraits, right) Period of oscillation $T = 2\pi/\omega$ *versus* the parameter α.

The state-space model of the system, with $x_1 = x,\ \ x_2 = \dot{x}$ is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\alpha\left(x_1^2 - 1\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2}$$

Figure 1 (left) shows the phase portraits of the Van der Pol equation (2) for initial conditions $x_1(0) = 0$, $x_2(0) = 2$ as the control parameter α is varied. Clearly, the phase portraits are depending on α, namely: $\alpha = 0$, harmonic oscillator; $\alpha > 0$, stable limit cycle; α increasing, nonlinearity increasing. The amplitude of oscillations is nearly constant on the value $A = 2$, but the frequency of oscillation ω (period $T = 2\pi/\omega$) depends on α, as shown in Figure 1 (right). For lower values of α the frequency is approximately $\omega = 1$ ($T = 2\pi$).

In this paper we investigate the influence of a fractional-order time derivative introduced in the Van der Pol equation dynamics (2). The modified equation is called fractional Van der Pol oscillator (FrVPO). The system is analysed both in time and frequency domains and its dynamics illustrated through phase portraits, frequency spectra and bifurcation diagrams. The forced version of the system is also considered.

Bearing these ideas in mind, the article is organized as follows. Section 2 reviews the fundamentals of fractional calculus. Section 3 presents a frequency approximation method of fractional-order integrators. The approximations are used in the simulation of the FrVPO system. In section 4 we propose several versions of the VPO containing fractional derivatives. It is also presented numerical simulations of the fractional Van der Pol system under study. In section 5 we consider the forced version of the fractional Van der Pol system. Finally, section 6 draws the main conclusions.

2 Fundamentals of Fractional Calculus

The fractional calculus concerns the study and applications of integrals and derivatives of arbitrary order (real or complex order). There are different approaches to the fractional calculus, not being all equivalent. The two most commonly used definitions are the Riemann-Liouville and the Grünwald-Letnikov definitions [1–4]. The Riemann-Liouville definition of the fractional-order derivative is ($\lambda > 0$):

$$ {}_aD_t^{\lambda} f(t) = \frac{1}{\Gamma(n-\lambda)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\lambda-n+1}} d\tau, \quad n-1 < \lambda < n \tag{3} $$

where $\Gamma(x)$ is the well-known Gamma function of x.

By other hand, the Grünwald-Letnikov definition is formulated as ($\lambda \in \Re$):

$$ {}_aD_t^{\lambda} f(t) = \lim_{h \to 0} \frac{1}{\Gamma(\lambda)} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \binom{\lambda}{k} f(t-kh) \tag{4} $$

where h is the time increment and $[x]$ means the integer part of x.

For a wide class of functions, important for applications, both definitions are equivalent [2]. This allows one to use the Riemann-Liouville definition during problem formulation, and then turn to the Grünwald-Letnikov definition for obtaining the numerical solution.

An alternative definition, which reveals useful for the analysis and control design of dynamic systems, is given by the Laplace transform (L) method. Considering vanishing initial conditions, this definition is given by the expression ($\lambda \in \Re$):

$$ L\left\{ {}_aD_t^{\lambda} f(t) \right\} = s^{\lambda} F(s) \tag{5} $$

where $F(s) = L\{f(t)\}$. Expression (5) is a direct generalization of the integer-order scheme with the multiplication of the signal transform $F(s)$ by the Laplace s-variable raised to a fractional value λ. The frequency response of (5) is represented in the magnitude Bode diagram by a straight line of slope 20λ dB/dec and in the phase Bode diagram by a horizontal line positioned at $\lambda\pi/2$ rad.

3 Approximations of Fractional-Order Operators

From expressions (3)–(5) we note that the fractional-order operator has an unlimited memory, being the integer-order operators particular cases of this general case in which the memory is limited. These operators are characterized by having irrational continuous transfer functions in the Laplace domain or infinite dimensional discrete transfer functions in time domain. This fact poses evaluation problems when used in simulations. Then, the usual approach for analysing fractional-order systems is the development of continuous and discrete integer-order approximations of fractional-order operators [4–7].

In this paper we use the Charef's approximation frequency method [8] to obtain rational-type approximations of the fractional-order integrator $1/s^{\lambda}$. The basic idea is to aproximate the slope of the magnitude Bode diagram of the transfer function of a single-fractional power pole of the form:

$$ \frac{1}{s^{\lambda}} \approx \frac{1}{\left(1 + \frac{s}{p_T}\right)^{\lambda}} \tag{6} $$

with and succession of zeros and poles with slopes of 0 dB/dec and −20 dB/dec, respectively, over the required range of frequency. Thus, the obtained approximation is:

$$H(s)=\frac{\prod_{i=0}^{N-1}\left(1+\frac{s}{z_i}\right)}{\prod_{i=0}^{N}\left(1+\frac{s}{p_i}\right)} \tag{7}$$

where the coefficients are computed for obtaining a maximum deviation from the original magnitude response in the frequency domain of y dB. Defining:

$$a=10^{y/10(1-\lambda)},\quad b=10^{y/10\lambda},\quad ab=10^{y/10\lambda(1-\lambda)} \tag{8}$$

the poles and zeros of the approximation (7) are obtained by applying the following formulae:

$$p_0=p_T\sqrt{b},\quad p_i=p_0\left(ab\right)^i,\quad z_i=ap_0\left(ab\right)^i \tag{9}$$

The number of poles and zeros is related to the desired bandwidth and the error criteria used by the expression:

$$N=\left\lceil\frac{\log\left(\frac{\omega_{max}}{p_0}\right)}{\log\left(ab\right)}\right\rceil+1 \tag{10}$$

4 The Unforced Van der Pol Oscillator with Fractional Derivatives

The standard Van der Pol equation (1) is modelled by a differential equation for which the elastic restoring force is a linear function of the dependence variable. However, it may be of interest to consider modifications to this equation in which the dependent variable x and/or its derivatives occur to some fractional power [9–14]. Such nonlinear differential equations are called fractional Van der Pol equations.

Mickens (2002, 2003) have investigated the following two equations:

$$\ddot{x}+\alpha\left(x^2-1\right)\dot{x}+x^{1/3}=0 \tag{11}$$

$$\ddot{x}+\alpha\left(x^2-1\right)\left(\dot{x}\right)^{1/3}+x=0 \tag{12}$$

More recently, Pereira, *et al.* (2004) considered a fractional version of the Van der Pol equation given by:

$$x^{\lambda}+\alpha\left(x^2-1\right)\dot{x}+x=0,\quad 1<\lambda<2 \tag{13}$$

$$\begin{bmatrix}\dot{x}_1\\ x_2^{(\lambda)}\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & -\alpha\left(x_1^2-1\right)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{14}$$

which is obtained by substituting the capacitance by a fractance in the nonlinear RLC circuit model. Barbosa, *et al.* (2004) has also suggested the introduction of a fractional-order time derivative in the state-space equations (2) of the standard VPO in the form:

$$\begin{bmatrix} x_1^{(\lambda)} \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\alpha\left(x_1^2 - 1\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{15}$$

where $0 < \lambda < 1$ and $\alpha > 0$. A similar approach was performed for the Duffing [9] and Chua [10] systems. Note that the system (15) reduces to the classical VPO (2) when $\lambda = 1$ and that the total system order is changed to $\lambda+1 < 2$. The differential equation of system (15) is given by:

$$x^{(1+\lambda)} + \alpha\left(x^2 - 1\right)x^{(\lambda)} + x = 0, \quad 0 < \lambda < 1 \tag{16}$$

In this article we investigate the equation (16).

The block diagram representation of system (16) is illustrated in Figure 2. The fractional-order integrator $1/s^{\lambda}$ $(0 < \lambda < 1)$ was simulated by using approximations of type (7) with $p_T = 0.01$, $\omega_{max} = 100$ rad/s and $y = 2$ dB. Figure 3 shows the phase portraits for initial conditions $x_1(0) = 0$ and $x_2(0) = 1$ as the fractional-order λ (right plot) and the control parameter α (left plot) are varied, respectively. In both cases, we verify significant variations of the limit cycle, revealing a large impact of the λ-order derivative upon system dynamics. In order to clarify this point, Figure 4 illustrates the amplitude A and the period T of the output oscillation. It is clearly seen the large variation of the limit cycle, particularly in the period of the oscillation.

Figures 5 and 6 show the steady-state time responses and the Fourier spectra of the output $x_1(t)$ for several values of λ and for $\alpha = 1$ and $\alpha = 5$, respectively. The frequency spectrum was evaluated by using the FFT over $N = 2^{15}$ points after elapsing the initial transient up to $T_0 = 100$ s of the signal output $x_1(t)$. Once more, we observe the variation of the limit cycle as function of λ, noting that the amplitude gets smaller as λ is decreased. The system stops oscillating when $\lambda = 0.37$ $(\alpha = 1)$. On the other hand, analysing the Fourier spectra, we verify that the multiplicity of peaks and the amplitude of these peaks varies with α, which is in accordance with the time responses. Also note that the energy of the output signal, is not only concentrated in the peaks (fundamental and integer-odd harmonics), but distributed along all frequency domain, showing a long-term behaviour of type $C(\lambda)\omega^{-1}$ indicating different amplitude decays depending on λ [14].

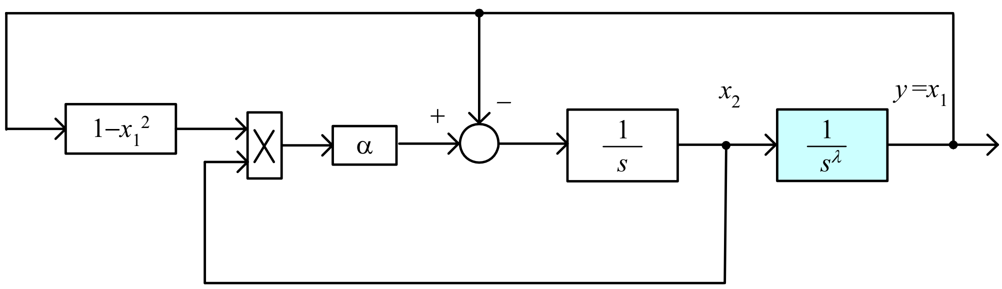

Figure 2: Block diagram of the unforced fractional Van der Pol system under study.

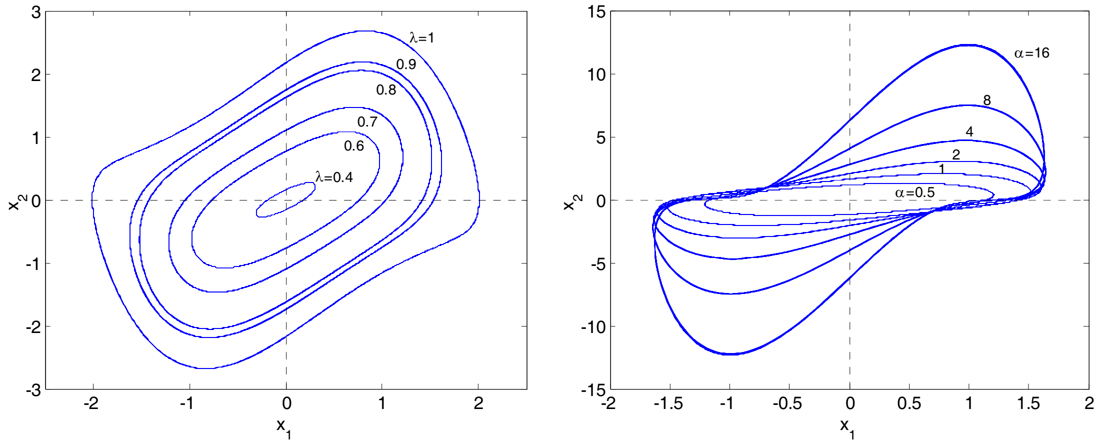

Figure 3: Phase portraits: left) λ = {0.4, 0.6, 0.7, 0.8, 0.9, 1.0} and α = 1, right) λ = 0.8 and α = {0.5, 1, 2, 4, 8, 16}.

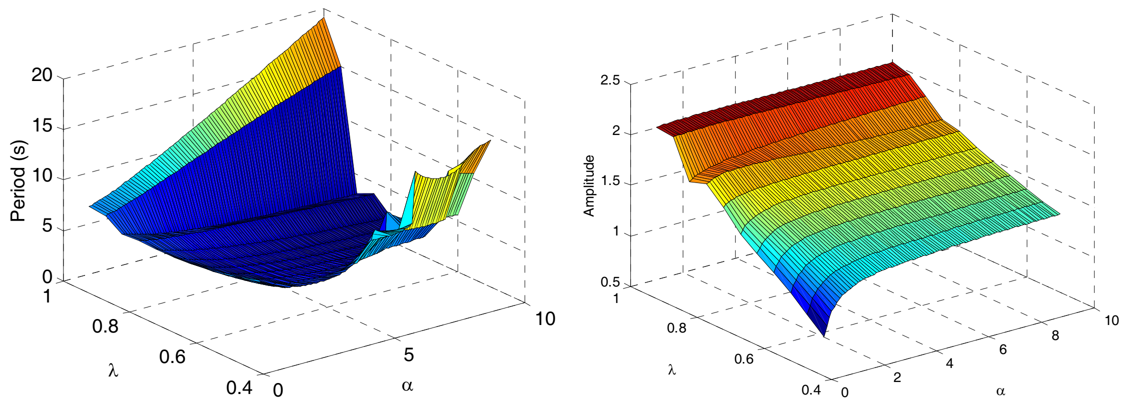

Figure 4: Limit cycle: period and amplitude of the output oscillation for 1 ≤ α ≤ 10 and 0.5 ≤ λ ≤ 1.

5 The Forced Van der Pol Oscillator with Fractional Derivatives

Let us now consider the forced FrVPO defined in state-space form as:

$$\begin{aligned} x_1^{(\lambda)} &= x_2 \\ \dot{x}_2 &= -x_1 - \alpha\left(x_1^2 - 1\right)x_2 + f\cos\left(\omega_f t\right) \end{aligned} \tag{17}$$

where f and ω_f are the amplitude and the frequency of the forcing sinusoidal input, respectively. The block diagram representation of equations (17) is depicted in Figure 7.

It is well-known that for the parameters $\alpha = 5$, $\omega_f = 2.46$ rad/s and $\lambda = 1$ the classical forced VPO oscillator exhibits chaos. For the forced FrVPO, by modifying the order λ, the system will now reveals a different behavior. For example, Figure 8 shows the bifurcation diagram of the sampled output position $x_1(nT)$ as function of the forcing amplitude f for a fractional-order of $\lambda = 0.85$. This graph was obtained by applying the method of Poincaré sections.

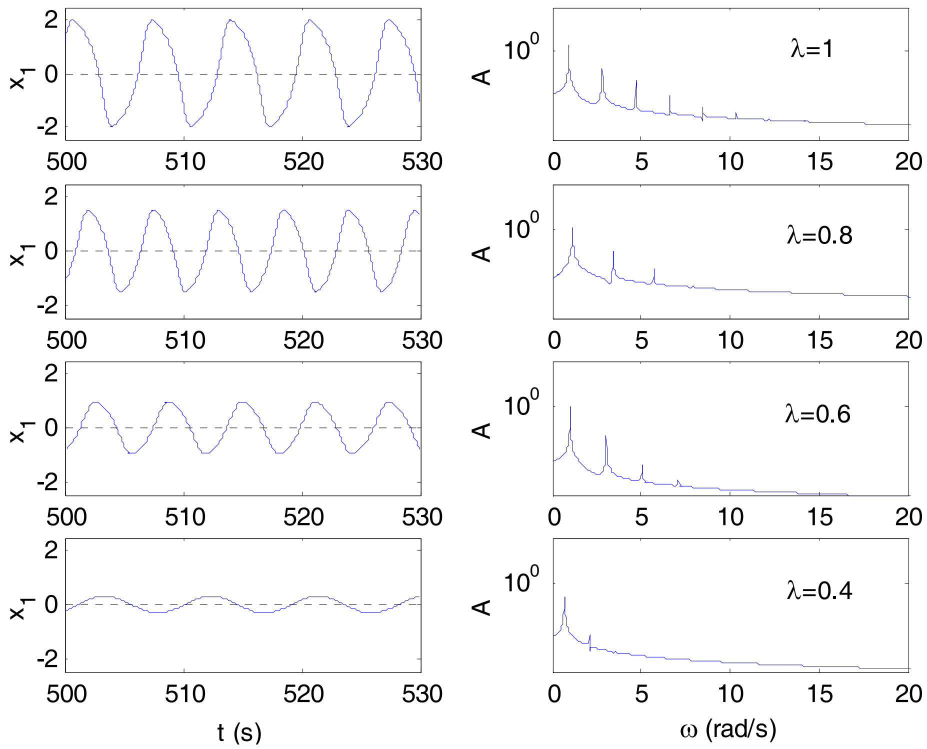

Figure 5: Time responses and Fourier spectra for $\lambda = \{1, 0.8, 0.6, 0.4\}$ and $\alpha = 1$.

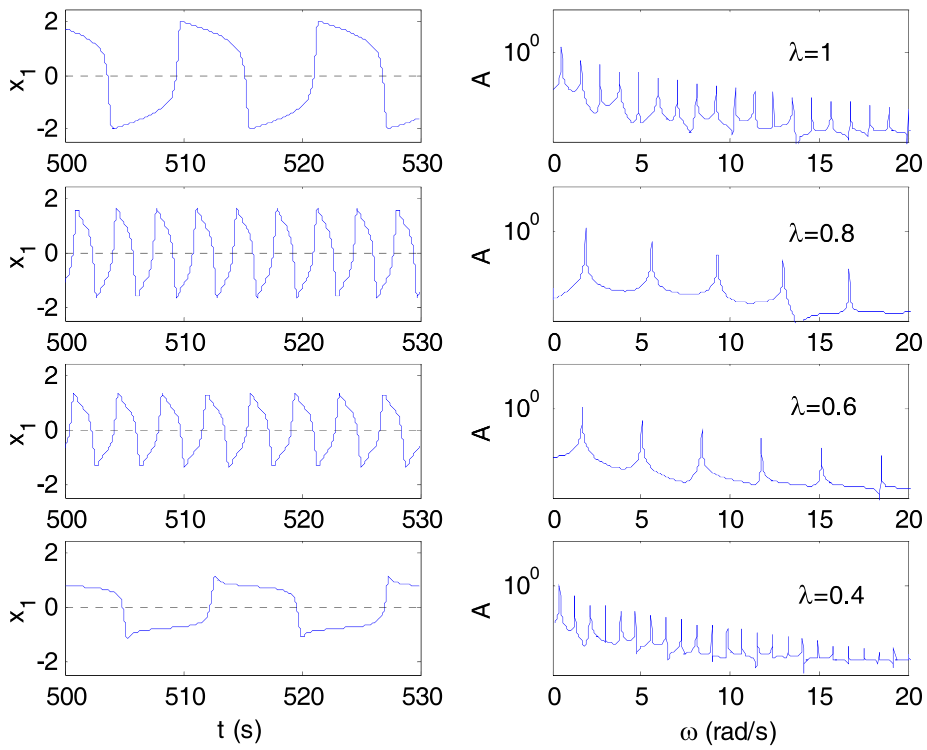

Figure 6: Time responses and Fourier spectra for $\lambda = \{1, 0.8, 0.6, 0.4\}$ and $\alpha = 5$.

From the bifurcation plot we can distinguish different modes of the forced Van der Pol system, namely: periodic motion, quasiperiodic motion and period locked motion. These types of motion are illustrated through Figures 9-11 by phase portraits and Fourier spectra. In the periodic motion the phase-plane exhibits period doubling, as shown in Figure 9. Figure 10 depicts the quasiperiodic motion in which the system is oscillating at multiple or sub-multiple periods of the forcing frequency. In this case the frequency and amplitude varies with time. Finally, Figure 11 illustrates the period locked motion in which the system is oscillating at the forcing frequency. Note that all these modes correspond to a periodic behaviour of the system. The non-periodic behaviour is characterized by the chaos (or sensitivity to initial conditions). It is well-kwon that the classical forced Van der Pol equation can display chaos for specific set of parameters, even not always easy to find. The same difficulty can be expected for the case of the forced fractional Van der Pol equation. This is a subject that will be investigated in future research.

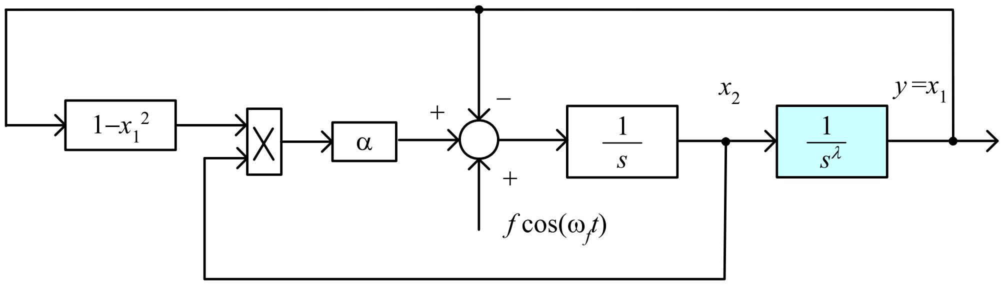

Figure 7: Block diagram of the forced fractional Van der Pol system under study.

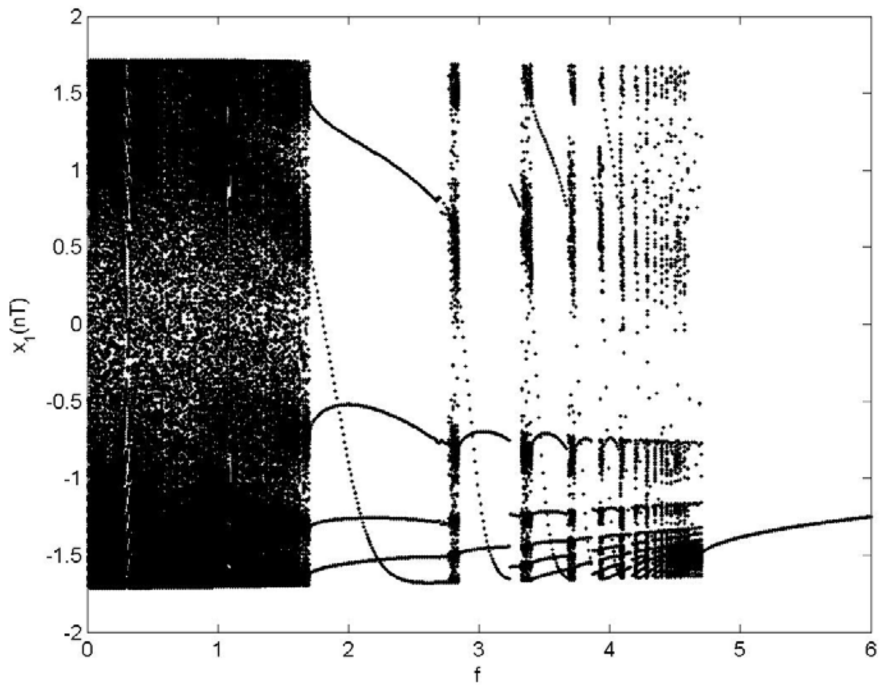

Figure 8: Bifurcation diagram for $\alpha = 5$, $\omega_f = 2.46$ rad/s and fractional-order $\lambda = 0.85$ *versus* the forcing amplitude f.

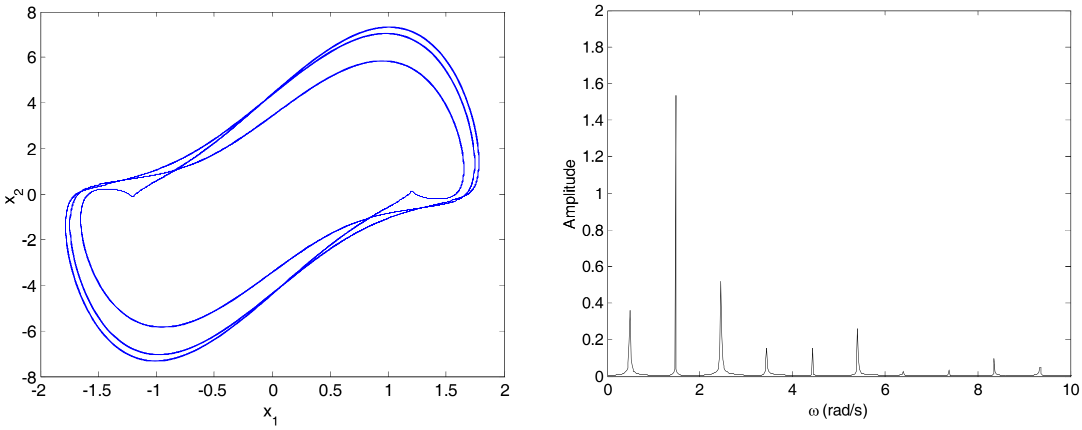

Figure 9: Phase plane (left) and Fourier spectrum (right) for $\alpha = 5$, $\omega_f = 2.46$ rad/s, $f = 2.0$ and fractional-order $\lambda = 0.85$: periodic motion.

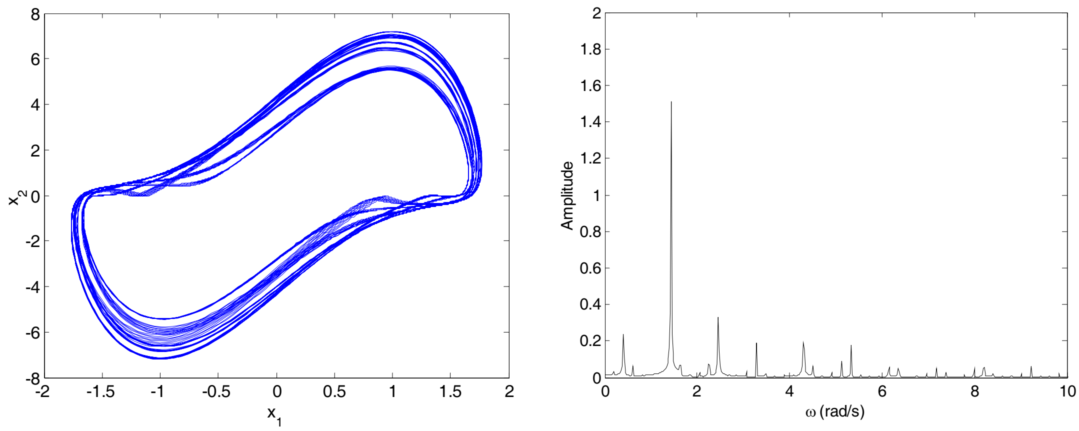

Figure 10: Phase plane (left) and Fourier spectrum (right) for $\alpha = 5$, $\omega_f = 2.46$ rad/s, $f = 1.5$ and fractional-order $\lambda = 0.85$: quasiperiodic motion.

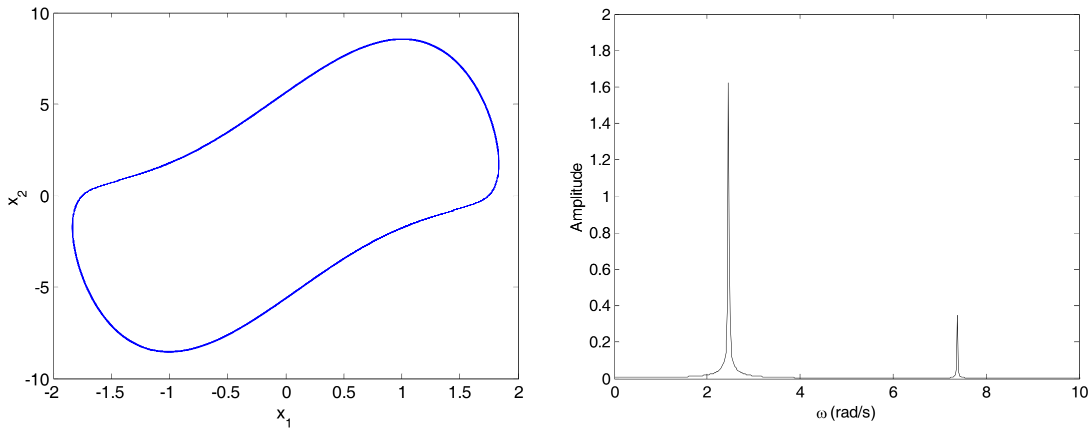

Figure 11: Phase plane (left) and Fourier spectrum (right) for $\alpha = 5$, $\omega_f = 2.46$ rad/s, $f = 5.5$ and fractional-order $\lambda = 0.85$: period locked motion.

6 Conclusions

In this paper we have proposed several versions of the modified Van der Pol equation. Such modifications consisted on the introduction of a fractional-order time derivative in the sate-space equations of the standard Van der Pol oscillator. The unforced and forced versions of the resulting fractional-order Van der Pol oscillators were studied in the time and frequency domains. The results reveal that the fractional-order systems can exhibit different behaviour from those obtained with the standard Van der Pol oscillator depending on order's derivative (or system's order). The fractional-order can act as a modulation parameter that may be useful for a better understanding and control of such systems.

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