Preprints

2nd IFAC Workshop on Fractional Differentiation and its Applications

19 - 21 July, 2006
Porto, Portugal

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Proceedings of the 2nd IFAC
Workshop on Fractional Differentiation and its Applications
Porto, Portugal, July 19-21, 2006

FRACTIONAL DYNAMICS IN THE DESCRIBING FUNCTION ANALYSIS OF NONLINEAR FRICTION

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Abstract: This paper studies the describing function (DF) of systems constituted by a mass subjected to nonlinear friction. The friction force is decomposed in three components namely, the viscous, the Coulomb and the static forces. The system dynamics is analyzed in the DF perspective revealing a fractional-order behaviour. The reliability of the DF method is evaluated through the signal harmonic content and the limit cycle prediction.

Keywords: Describing Function, Friction, Control, Modelling.

1. INTRODUCTION

The phenomenon of vibration due to friction is verified in many branches of technology where it plays a very useful role. On the other hand, its occurrence is often undesirable, because it causes additional dynamic loads, as well as faulty operation of machines and devices. Despite many investigations that have been carried out so far, this phenomenon is not yet fully understood, mainly due to the considerable randomness and diversity of reasons underlying the energy dissipation involving the dynamic effects (Armstrong and Amin, 1996), (Barbosa and Machado, 2002), (Barbosa *et al.*, 2003). In this paper we investigate the dynamics of systems that contain nonlinear friction namely the Coulomb and the static forces in addition to the linear viscous component. Bearing these ideas in mind, the article is organized as follows. Section 2 introduces the fundamental aspects of the describing function method. Section 3 studies the describing function of mechanical systems with nonlinear friction. Section 4 analyzes the prediction of cycle limit in the friction system under the action of a PID controller. Finally, section 5 draws the main conclusions and addresses perspectives towards future developments.

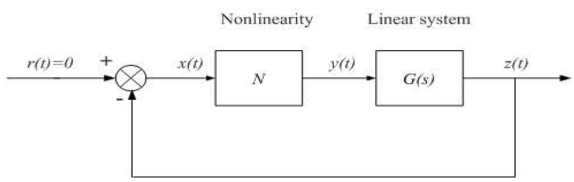

Fig. 1. Nonlinear control system

2. FUNDAMENTAL CONCEPTS

Let us consider the feedback system of Figure 1 with one nonlinear element N and a linear system with transfer function $G(s)$.

Suppose that the input to a nonlinear element is sinusoidal $x(t) = X \sin(\omega t)$. In general the output of the nonlinear element $y(t)$ is not sinusoidal; nevertheless $y(t)$ is periodic, with the same period as the input, and containing higher harmonics in addition to the fundamental harmonic component.

If we assume that the nonlinearity is symmetrical with respect to the variation around zero, the Fourier series become:

$$y\,(t) = \sum_{k=1}^{\infty} Y_k \cos\left(k\,\omega\,t + \phi_k\right) \qquad (1)$$

where Y_k and ϕ_k are the amplitude and the phase shift of the kth harmonic component of the output $y(t)$, respectively.

In the DF analysis, we assume that only the fundamental harmonic component of the output is significant. Such assumption is often valid since the higher harmonics in the output of a nonlinear element are usually of smaller amplitude than the fundamental component. Moreover, most control systems are "low-pass filters" with the result that the higher harmonics are further attenuated (Cox, 1987), (Atherton, 1975), (Dupont, 1992).

The DF, or sinusoidal DF, of a nonlinear element, $N\,(X,\omega)$, is defined as the complex ratio of the fundamental harmonic component of the output $y(t)$ and the input $x(t)$, that is:

$$N\,(X,\omega) = \frac{Y_1}{X} e^{j\phi_1} \qquad (2)$$

where the symbol N represents the DF, X is the amplitude of the input sinusoid and Y_1 and ϕ_1 are the amplitude and the phase shift of the fundamental harmonic component of the output, respectively. Several DFs of standard nonlinear system elements can be found in the references (Haessig and Friedland, 1991), (Karnopp, 1985), (Azenha and Machado, 1998).

For nonlinear systems not involving energy storage the DF is merely amplitude-dependent, that is $N = N(X)$. However, when we have nonlinear elements that involve energy, the DF method is both amplitude and frequency dependent yielding $N\,(X,\omega)$. In this case, to determine the DF, usually we have to adopt a numerical approach because it is impossible to find a closed-form analytical solution. Once calculated, the DF can be used for the approximate stability analysis of a nonlinear control system.

Let us consider again the standard control system shown in Figure 1 where the block N denotes the DF of the nonlinear element. If the higher harmonics are sufficiently attenuated, N can be treated as a real or complex variable gain and the closed-loop frequency response becomes:

$$\frac{C\,(j\omega)}{R\,(j\omega)} = \frac{N\,(X,\omega)\,G\,(j\omega)}{1 + N\,(X,\omega)\,G\,(j\omega)} \qquad (3)$$

The characteristic equation is:

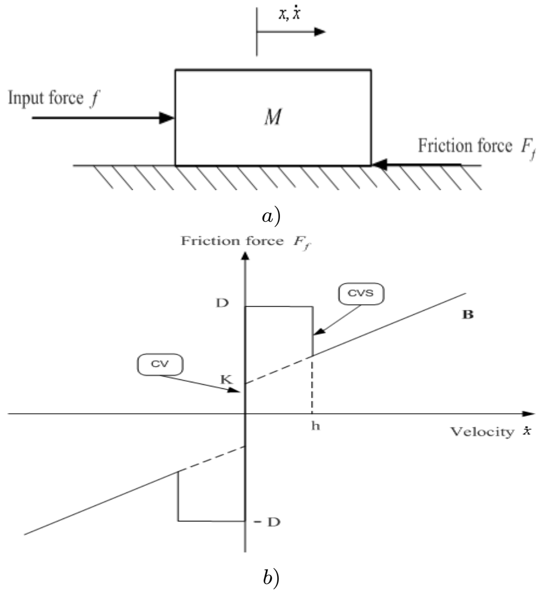

Fig. 2. a) Elemental mass system subjected to nonlinear friction and b) Non-linear friction with Coulomb, Viscous (CV model) and Static components (CVS model).

$$1 + N\,(X,\omega)\,G\,(j\omega) = 0 \qquad (4)$$

If (4) can be satisfied for some value of X and ω, a limit cycle is *predicted* for the nonlinear system. Moreover, since (4) is valid only if the nonlinear system is in a steady-state limit cycle, the DF analysis predicts only the presence or the absence of a limit cycle and cannot be applied to the analysis of other types of time responses.

3. SYSTEMS WITH NONLINEAR FRICTION

In this section we calculate the DF of a dynamical system with nonlinear friction and we study its properties. In sub-section 3.1 we start by a combination of the viscous and Coulomb components. In sub-section 3.2 we complement the study by including also the static friction.

3.1 Coulomb and viscous friction

Let us consider a system (Figure 2) with a mass M, moving on a horizontal plane under the action of a input force $f(t)$, with a friction $F_f\,(t)$ effect composed of two components: a non-linear Coulomb K part and a linear viscous B part (CV model).

The equation of motion in this system is as follows:

$$M\;\ddot{x}\,(t) + F_f\,(t) = f\,(t) \qquad (5)$$

For the simple system of Figure 2 we can calculate, numerically, the polar plot of $N(F,\omega)$ considering

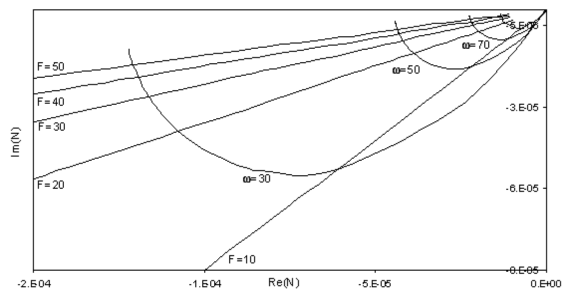

Fig. 3. Polar plot of $N(F,\omega)$ for the system subjected to nonlinear friction (CV model) with $F = \{10, 20, 30, 40, 50\}$ N and $\omega = \{30, 50, 70, 90\}$ rad s^{-1}.

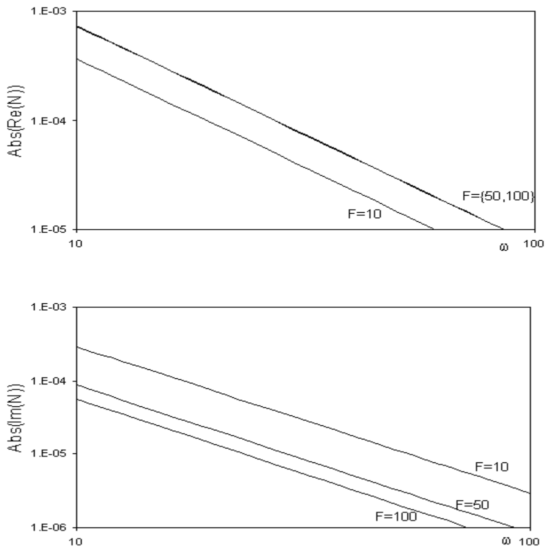

Fig. 4. Log-log plots of $|Re\{N\}|$ and $|Im\{N\}|$ *vs.* the exciting frequency ω for $F = \{10, 50, 100\}$ N, with de CV model.

as input a sinusoidal force $f(t) = F\cos(\omega t)$ applied to mass M and as output the position $x(t)$.

Figure 3 shows $N(F,\omega)$ when $M = 9$ kg, $B = 0.5$ Nsm^{-1}, $K = 5$ N. Figure 4 illustrates the log-log plots of $|Re\{N\}|$ and $|Im\{N\}|$ *vs* the exciting frequency ω, for different values of the input force $F = \{10, 50, 100\}$. The charts reveal that we have different results according to the excitation force F, being it more visible for the imaginary component.

In Figure 5 it is depicted the harmonic content of the output signal $x(t)$ for an input force of $F = 10$ N. From this chart we verify that the output signal has a half - wave symmetry because the harmonics of even order are negligible. Moreover, the fundamental component of the output signal is the most important one, while the amplitude of the high order harmonics decays significantly. Therefore,

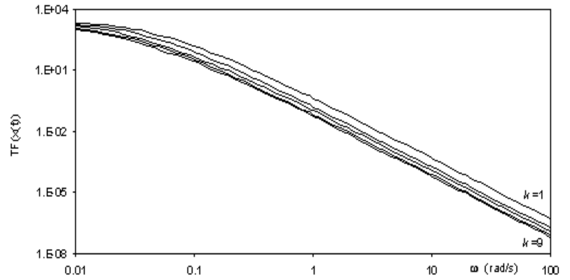

Fig. 5. Fourier transform of the output position $x(t)$, for the CV model, *vs.* the exciting frequency ω and the harmonic frequency index $k = \{1, 3, 5, 7, 9\}$ for an input force F= 10 N.

we can conclude that, for the friction CV model, the DF method leads to a good approximation.

In order to gain further insight into the system nature, we repeat the experiments for different mass values $M = \{0.1, 0.25, 0.5, 1, 2, 3, 5, 7, 9\}$ kg.

The results show that $Re\{N\}$ and $Im\{N\}$ vary with M. In order to study the relation between $Re\{N\}$ and $Im\{N\}$ *versus* F and M, we approximate the numerical results through power functions:

$$|Re\{N\}| = a\,\omega^{b},\ |Im\{N\}| = c\,\omega^{d},\ \{a,b,c,d\} \in \mathbb{R} \quad (6)$$

Figure 6 illustrates the variation of the parameters $\{a, b, c, d\}$ with F and M. Moreover, $Re\{N\}$ and $Im\{N\}$ reveal a distinct fractional order relationships with ω.

3.2 Coulomb, viscous and static friction

In this sub-section we incorporate the static friction (D, h) in the CV model (see Figure 2*b*) leading to the so-called CVS model. In this line of thought, we develop a study similar to the one adopted previously, with $M = 9$ kg, $B = 0.5$ Nsm^{-1}, $K = 5$ N, $D = 7$ N, $H = 0.5$ ms^{-1} .

Figures 7-10 depict the corresponding results. Comparing the results of the VC and CVS models we conclude that $Re\{N\}$ and $Im\{N\}$ are, in the two cases, of the same type, following a power law according with expression (6). Furthermore, once again we obtain fractional-order dynamics for $Re\{N\}$ and the $Im\{N\}$. Nevertheless, the CVS model is very sensitive to small input forces F (stimulation mainly the static component) leading to large values of N and to a higher harmonic content (Duarte and Machado, 2005).

To have a deeper insight into the effects of the different CVS components several experiences were performed varying $\{D,\ K,\ B\ ,\ H\}$. For example, Figures 11 and 12 present the

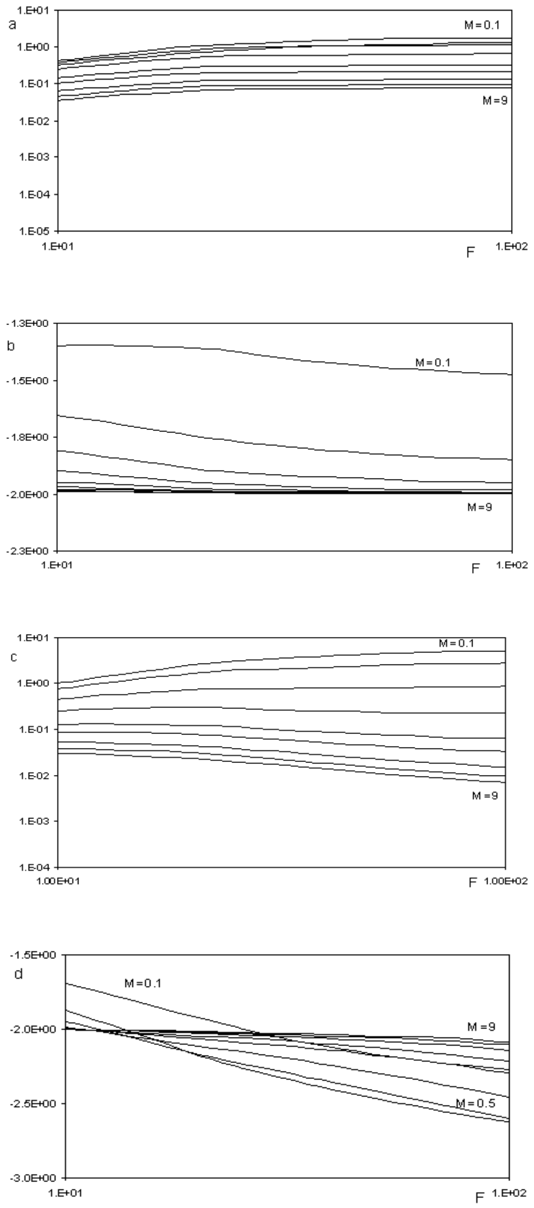

Fig. 6. Variation of the parameters $\{a, b, c, d\}$ versus $F = \{2,\ 3,\ 5,\ 10,\ 20,\ 40,\ 60,\ 80,\ 100\}$ N for $M = \{0.1, 0.25, 0.5, 1, 2, 3, 5, 7, 9\}$ kg, in the CV model with $\{K,\ B\}$ ={0.5 N, 0.5 Nsm^{-1}}.

values of the parameters $\{a, b, c, d\}$ when approximating $|Re\{N\}|$ and $|Im\{N\}|$, for $F = \{2,\ 3, 5,\ 10,\ 20,\ 40,\ 60,\ 80,\ 100\}$ N, and $M = \{0.1, 0.5, 1, 5, 10\}$ kg, with $\{D, K, B, H\}$ ={1.0 N, 0.5 N, 10.0 Nsm^{-1}, 0.05 ms^{-1}} and $\{D, K, B, H\}$ ={5.0 N, 1.0 N, 1.0 Nsm^{-1}, 0.1 ms^{-1}}, respectively.

4. LIMIT CYCLE PREDICTION

The characteristic equation (4) involves two nonlinear equations with the variables X and ω. It may be difficult to solve this equation by analytical methods and, therefore, a numerical and graphical approach is more adequate.

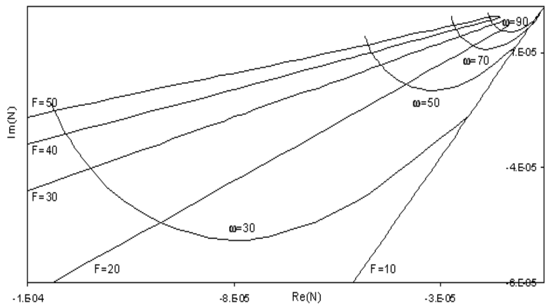

Fig. 7. Polar plot of $N(F, \omega)$ for the system subjected to nonlinear friction (CVS model) with $F = \{10, 20, 30, 40, 50\}$ N and $\omega = \{30, 50, 70, 90\}$ rad s^{-1}.

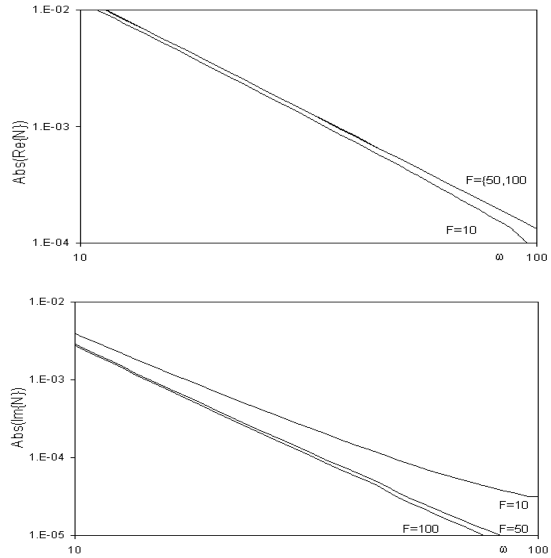

Fig. 8. Log-log plots of $|Re\{N\}|$ and $|Im\{N\}|$ *vs.* the exciting frequency ω for $F = \{10, 50, 100\}$ N, with de CVS model.

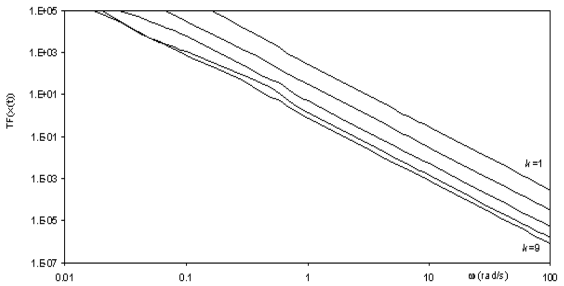

Fig. 9. Fourier transform of the output position $x(t)$, for the CVS model, *vs.* the exciting frequency ω and the harmonic frequency index $k = \{1, 3, 5, 7, 9\}$ for an input force F= 10 N.

Adopting a classical PID controller $f(t) = K_p\, e(t) + K_d\, e(t) + K_i \int e(t)\, dt$, with gains $K_p = 1500$, $K_d = 44250$, and $K_i = 130$ in the closed-loop system, the FD *predicts* (Figure 13) a limit cycle with

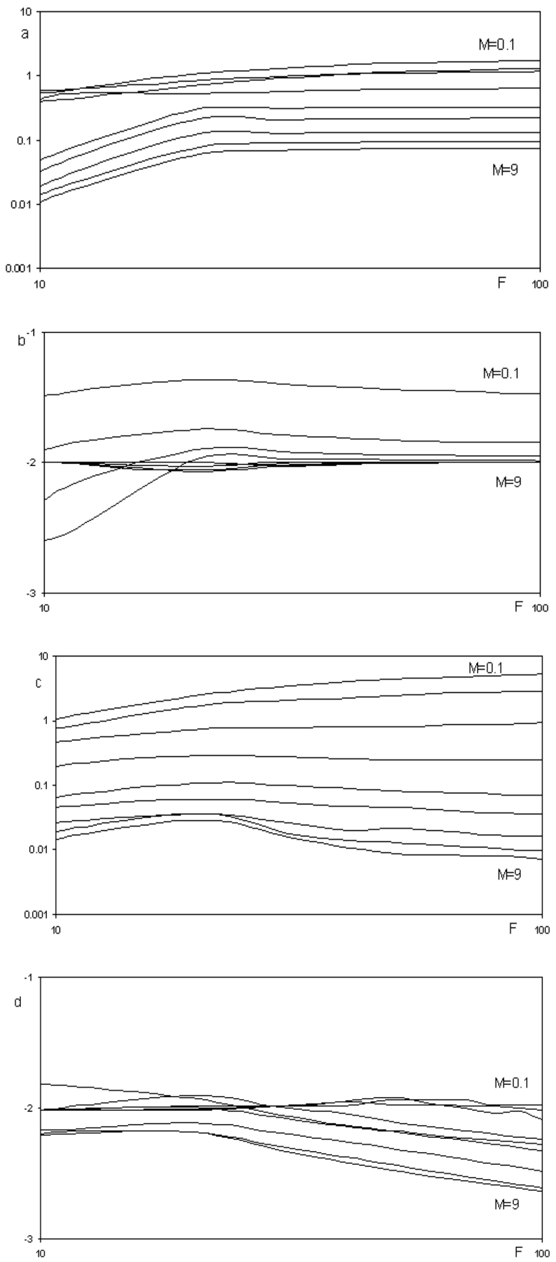

Fig. 10. Variation of the parameters $\{a, b, c, d\}$ *vs* $F = \{2,\ 3, 5,\ 10,\ 20,\ 40,\ 60,\ 80,\ 100\}$ N in the CVS model, with $M = 9$ kg, and $\{D, K, B, H\}$ ={7.0 N, 5.0 N, 0.5 Nsm^{-1}, 0.5 ms^{-1}}.

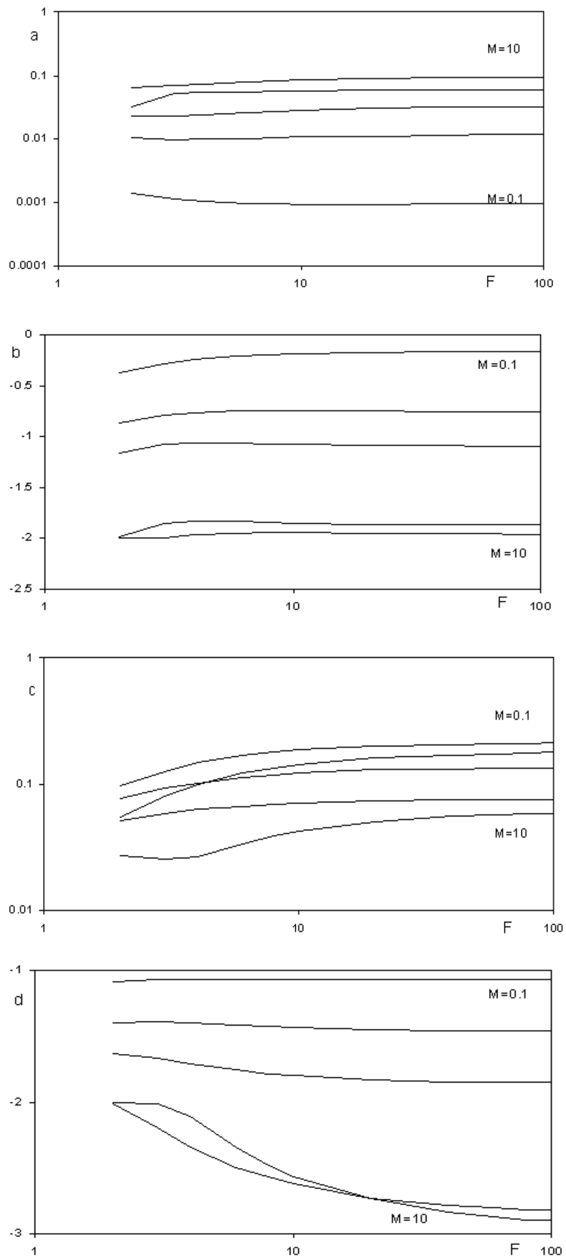

Fig. 11. The parameters $\{a, b, c, d\}$ *vs* $F = \{2,\ 3, 5,\ 10,\ 20,\ 40,\ 60,\ 80,\ 100\}$ N and $M = \{0.1, 0.5, 1, 5, 10\}$ kg, in the CVS model, with $\{D, K, B, H\}$ ={1.0 N, 0.5 N, 10.0 Nsm^{-1}, 0.05 ms^{-1}}.

$\omega = 2.2$ rad s^{-1} and $F = 0.01$ N which is very close to the real response $f(t)$ depicted in Figure 14. This result confirms the reliability of the DF method.

5. CONCLUSIONS

This paper addressed the limit cycle prediction of systems with nonlinear friction. The dynamics of elemental mechanical system was analyzed through the describing function method and compared with standard models. The results encourage further studies of nonlinear systems in a similar perspective and the adoption of the tools of fractional calculus. The conclusions may lead to the development of compensation schemes capable of improving the control system performance.

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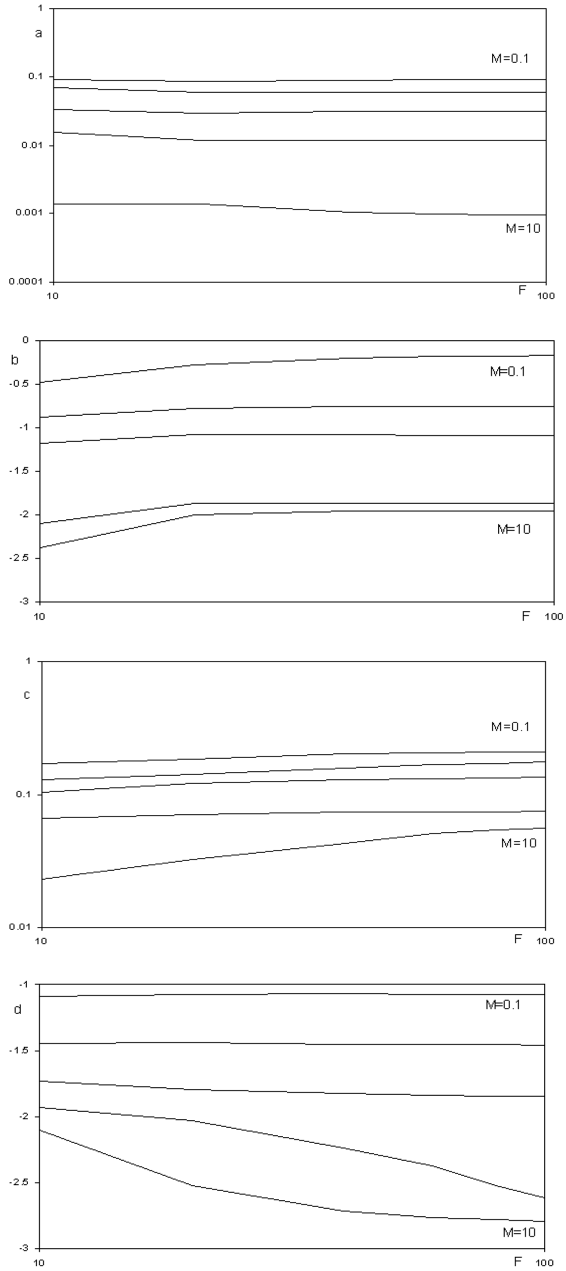

Fig. 12. The parameters $\{a, b, c, d\}$ *vs* $F = \{2,\ 3, 5,\ 10,\ 20,\ 40,\ 60,\ 80,\ 100\}$ N and $M = \{0.1, 0.5, 1, 5, 10\}$ kg, in the CVS model, with $\{D, K, B, H\}$ ={5.0 N, 1.0 N, 1.0 Nsm^{-1}, 0.1 ms^{-1}}.

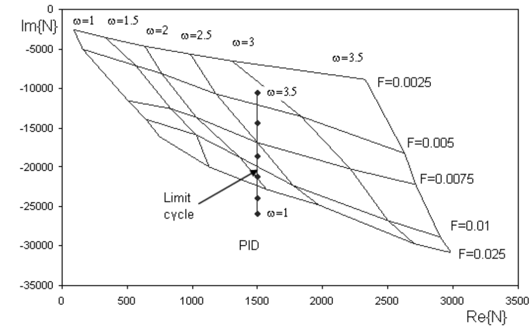

Fig. 13. Limit cycle for the CVS friction model with $M = 5$ kg, $B = 1$ Nsm^{-1}, $K = 0.5$ N, $D = 0.5$ N, $H = 0.01$ ms^{-1} .

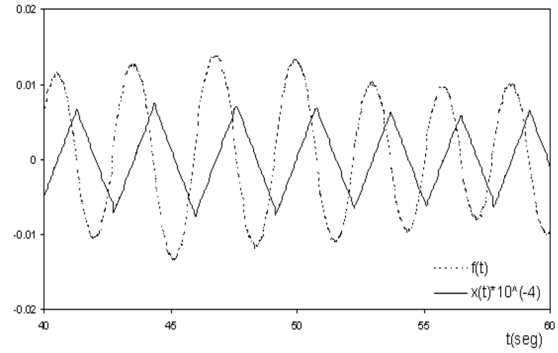

Fig. 14. Time response of system with CVS friction with $M = 5$ kg, $B = 1$ Nsm^{-1}, $K = 0.5$ N, $D = 0.5$ N, $H = 0.01$ ms^{-1} : Actuation force ($f(t)$), Output displacement ($x(t)$).

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