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DISCRETIZATION OF COMPLEX-ORDER DIFFERINTEGRALS

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Abstract: This paper deals with the discretization of integrals and derivatives (*i.e.*, differintegrals) of complex order. Several methods for the discretization of the operator s^γ, where $\gamma = u+jv$ is a complex value, are proposed. The concept of conjugated-order differintegral is also presented. The conjugated-order operator allows the use of complex-order differintegrals while still resulting in real time responses and real transfer functions. The performance of the resulting approximations is evaluated both in the time and frequency domains.

Keywords: Fractional calculus, Fractional-order systems, Complex-order differintegrals, Conjugated-order differintegrals, IIR filters, Discretization, Rational approximations, CFE method, Fractional-order control.

1. INTRODUCTION

The fractional calculus (FC) theory refers to the study and application of the integrals and derivatives to an arbitrary order (real, rational, irrational or complex order). Nowadays, the FC theory is applied in almost all the areas of science and engineering, being recognized its ability to yield a superior modelling and control in many dynamical systems (Oldham and Spanier, 1974; Podlubny, 1999). However, the majority of the studies in this area deal with integrals and derivatives of real order. Furthermore, in these studies, the complex-order differintegrals are treated mainly from a mathematical point of view (Love, 1971; Samko, *et al.*, 1993).

Nevertheless, only in the last years we can find some works that deal with the applications of the integrals and derivatives of complex order (Oustaloup, *et al.*, 2000; Lanusse, *et al.*, 2005; Nigmatulin and Trujillo, 2005; Nigmatulin and Le Méhaute, 2005; Nigmatulin, 2005). The complex-order differintegrals has the “disadvantage” of yielding complex time responses (with real and imaginary parts) and are thus of an apparent limited application. To overcome this difficulty, Hartley, *et al*. (2005a) proposed the use of the concept of conjugated-order differintegrals, that is, fractional derivatives whose orders are complex conjugates. These conjugated-order differintegrals allow the use of complex-order differintegrals while still resulting in real time responses and real transfer functions.

This paper deals with the discretization of the two complex differintegrals: the complex-order and the conjugated-order differintegral. Their performance are evaluated both in the time and frequency domains. This work has also the objective to widespread the use of the complex operators, particularly in the areas of modelling, identification and control of dynamical systems, where we can foresee significant advantages. Although some work was already been made in this domain (Hartley, *et al*., 2005b; Silva, *et al.*, 2006), much more investigation is needed to allow an effective use and a deep understanding of the complex-order operators.

In this study, the approach for obtaining rational transfer functions approximations of complex differintegrals adopts the well-known continued fraction expansion (CFE) method. It must be mentioned that other techniques could be also employed as, for example, the least-squares based methods (Barbosa, *et al.*, 2005). It is well known that rational approximations (IIR filters) frequently converge faster than polynomial approximations (FIR filters) and have a wider domain of convergence in the complex domain. Therefore, here we only develop z-variable rational transfer functions approximations of the complex differintegrals operators. The determination process can be outlined by the following steps:

1. Discretize the complex differintegral using a suitable generating function $s = w(z^{-1})$;
2. Apply the CFE method in order to obtain the desired IIR-type approximation.

Bearing these ideas in mind, the paper is organized as follows. Section 2 studies the discretization of the complex-order differintegral and evaluates its performance in the time and frequency domains. Section 3 deals with the discretization of the conjugated-order differintegrals and presents some results that demonstrate its utility in control design. Finally, section 4 draws the main conclusions.

2. DISCRETIZATION OF COMPLEX-ORDER DIFFERINTEGRALS

In this section we present some fundamentals of the complex-order differintegrals, both in the frequency and time domains. Evaluation of the performance of the resulting IIR-type approximations is also performed.

2.1 *Notion of Complex-Order Differintegrals*

The complex-order differintegral operator of a function $f(t)$ is given by:

$$D^{u+jv} f(t) = y(t) \tag{1}$$

where u and v are the real and imaginary orders of the complex operator, respectively.

The s-domain equation of time definition (1) is obtained by applying the Laplace transform:

$$L\left\{D^{u+jv} f(t)\right\} = s^{u+jv} F(s) = Y(s) \tag{2}$$

which gives the transfer function:

$$H(s) = \frac{Y(s)}{F(s)} = s^{u+jv} \tag{3}$$

Using the Euler identity, equation (3) can be rewritten as (Hartley, *et al.*, 2005a):

$$H(s) = s^{u}\left\{\cos\left[v\ln(s)\right] + j\sin\left[v\ln(s)\right]\right\} \tag{4}$$

Substituting $s = j\omega$ in (3), it yields the frequency response of the complex-order operator:

$$H(j\omega) = \omega^{u} e^{-v\frac{\pi}{2}} e^{j\left[u\frac{\pi}{2} + v\ln(\omega)\right]} \tag{5}$$

2.2 *Time-Domain Simulation of Complex-Order Differintegrals*

The discretization of the complex-order differintegral s^{γ} ($\gamma \in C$) will be expressed by using a generating function $s = w(z^{-1})$ (Machado, 2001; Chen, *et al.*, 2004; Barbosa, *et al.*, 2005). For that purpose, we use the Euler and Tustin operators, yielding, respectively, the discretization generating functions:

$$H_E(z^{-1}) = \left(\frac{1 - z^{-1}}{T}\right)^{u+jv} \tag{6}$$

$$H_T(z^{-1}) = \left(\frac{2}{T}\,\frac{1 - z^{-1}}{1 + z^{-1}}\right)^{u+jv} \tag{7}$$

By performing a power series expansion (PSE) over the irrational functions (6) or (7), we obtain an approximation of the complex-order operators in the form of a polynomial:

$$H\left(z^{-1}\right) = \bar{c}_0 + \bar{c}_1 z^{-1} + \bar{c}_2 z^{-2} + \ldots + \bar{c}_N z^{-N} \tag{8}$$

where the coefficients $\bar{c}_i$ ($i = 0, 1, \ldots, N$) $\in C$ are complex coefficients (with real and imaginary parts):

$$\bar{c}_l = c_{rl} + j c_{il}, \quad l = 0, 1, \ldots, N \tag{9}$$

and the indices r and i indicate the real and imaginary parts of the corresponding coefficients. The PSE scheme leads to approximations in the form of a complex FIR filter. Also, the coefficients of the FIR filter correspond to the impulse response of the discrete complex-order operator.

Taking a continuous fraction expansion (CFE), we get a rational transfer function of type:

$$H(z^{-1}) = \frac{\bar{a}_0 + \bar{a}_1 z^{-1} + \bar{a}_2 z^{-2} + \ldots + \bar{a}_m z^{-m}}{1 + \bar{b}_1 z^{-1} + \bar{b}_2 z^{-2} + \ldots + \bar{b}_n z^{-n}} \tag{10}$$

where $m \le n$ and the coefficients $(\bar{a}_l, \bar{b}_l) \in C$, that is:

$$\begin{aligned} \bar{a}_l &= a_{rl} + j a_{il}, \quad l = 0, 1, \ldots, m \\ \bar{b}_l &= b_{rl} + j b_{il}, \quad l = 1, 2, \ldots, n \end{aligned} \tag{11}$$

In this case, we obtain an IIR-type approximation which is a complex rational transfer function (*i.e.*, of complex coefficients). Such kind of complex transfer functions are not unusual and can be used, for example, to model induction motors (Aguiar and Cad, 2000).

Table 1 lists the CFE (4, 4)-order approximations to the complex-order differintegral obtained with the Euler and Tustin operators, for $u = -0.5$, $v = -0.5$ and $T = 1$ s. Figure 1 depicts the Bode diagrams of the resulting CFE approximations in comparison with the continuous complex-order differintegral $s^{-0.5-j0.5}$. The curves reveal that the approximations are well fitted into the ideal responses, particularly in the range of high frequencies, both in the magnitude and phase.

Figure 2 shows the pole-zero maps of the obtained approximations. As can be seen, the complex poles and complex zeros are distributed in an alternated fashion along the complex plane corresponding to the right semi-circle and to the entire circle, for the Euler and Tustin operators, respectively. Furthermore, all the complex poles and zeros lie inside the unit circle. Thus, the resulting approximations are simultaneously stable and minimum phase, as desired for a real time implementation.

In order to illustrate the effectiveness of the approximations in the time domain, we used them in the complex-order differintegration of the causal sine function $f(t) = \sin(\omega t)$ $(t > 0)$. Then, applying the complex-order operator, the analytical solution is obtained as (considering only the steady-state behaviour):

$$\begin{aligned} y(t) &= D^{u+jv}\left[\sin(\omega t)\right] \\ &= \omega^{u+jv} \sin\left(\omega t + u\frac{\pi}{2} + jv\frac{\pi}{2}\right) \end{aligned} \tag{12}$$

Table 1 Coefficients of the CFE (4, 4)-order approximations obtained with the Euler and Tustin operators, for $u = -0.5$, $v = -0.5$ and $T = 1$ s

Coef.	Euler	Tustin
$\bar{a}_0$	1.0000+j0.0000	0.6651–j0.2402
$\bar{a}_1$	–1.7500+j0.2500	0.4526+j0.2124
$\bar{a}_2$	0.9107–j0.3214	–0.5186+j0.3484
$\bar{a}_3$	–0.1339+j0.1042	–0.2472–j0.0897
$\bar{a}_4$	0.0007–j0.0060	0.0440–j0.0517
$\bar{b}_0$	1.0000+j0.0000	1.0000+j0.0000
$\bar{b}_1$	–2.2500–j0.2500	–0.5000–j0.5000
$\bar{b}_2$	1.6607+j0.4286	–0.8571+j0.2143
$\bar{b}_3$	–0.4375–j0.2113	0.2857+j0.2381
$\bar{b}_4$	0.0275+j0.0268	0.0833–j0.0476

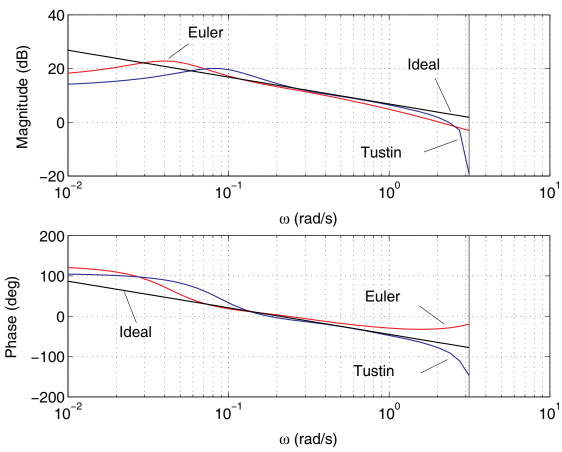

Fig. 1. Bode plots of the CFE (4, 4)-order approximations to complex-order differintegral with the Euler and Tustin operators.

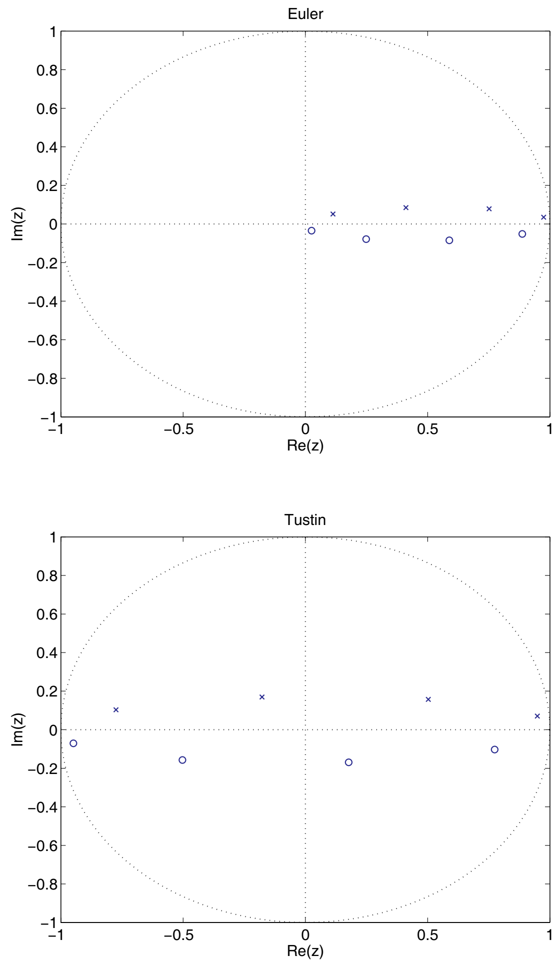

Fig. 2. Pole-zero maps of the CFE (4, 4)-order approximations to complex-order differintegral with the Euler and Tustin operators.

After some trigonometric manipulation of equation (12), we get the final analytical expression as:

$$\begin{aligned} y(t) &= D^{u+jv}\left[\sin(\omega t)\right] \\ &= \omega^{u}\Big\{\Big[\sin\left(\omega t + u\frac{\pi}{2}\right)\cosh\left(v\frac{\pi}{2}\right)\cos\left(v\ln(\omega)\right) \\ &\quad - \cos\left(\omega t + u\frac{\pi}{2}\right)\sinh\left(v\frac{\pi}{2}\right)\sin\left(v\ln(\omega)\right)\Big] \\ &\quad + j\Big[\sin\left(\omega t + u\frac{\pi}{2}\right)\cosh\left(v\frac{\pi}{2}\right)\sin\left(v\ln(\omega)\right) \\ &\quad + \cos\left(\omega t + u\frac{\pi}{2}\right)\sinh\left(v\frac{\pi}{2}\right)\cos\left(v\ln(\omega)\right)\Big]\Big\} \end{aligned} \tag{13}$$

Note that the complex-order differintegral yields a complex time response (*i.e.*, with real and imaginary parts). Figure 3 shows the time responses of the approximations to a sinusoidal input (real and imaginary parts) with the Euler and Tustin operators for $u = -0.5$, $v = -0.5$ and $\omega = 0.2$ rad s^{-1}. The analytical solution (13) is also plotted. Clearly, the curves show a good accordance with the analytical solution (in stationary regime) demonstrating, once more, the effectiveness of the generated approximations.

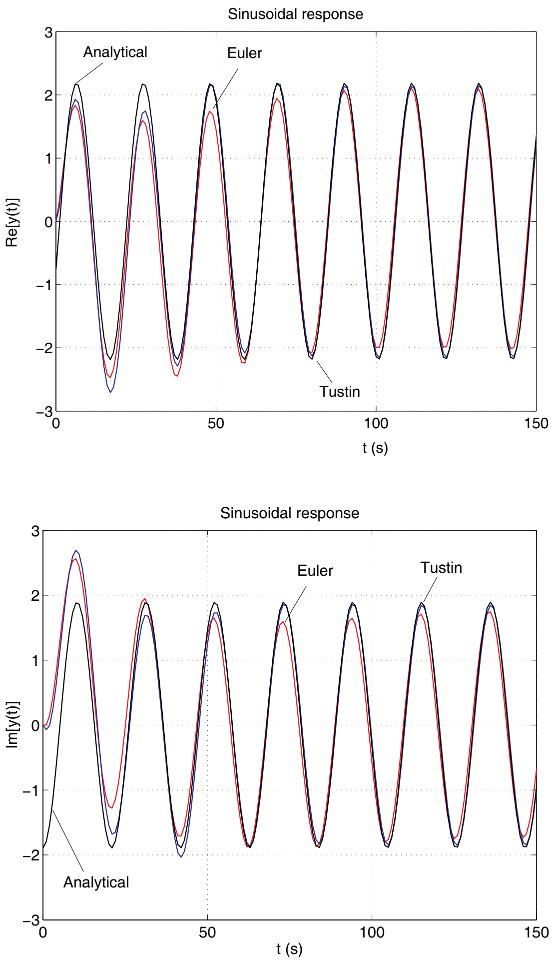

Fig. 3. Time responses to a sinusoidal input of the CFE (4, 4)-order approximations to complex-order differintegral with the Euler and Tustin operators. The analytical solution is also plotted.

3. DISCRETIZATION OF CONJUGATED-ORDER DIFFERINTEGRALS

In this section we develop complex-order differintegrals that yield purely real time responses. For that purpose it is adopted the concept of the conjugated differintegral, recently introduced by Hartley, *et al*. (2005a). The discretization of this type of differintegral operator is pursued and its performance is evaluated both in the time and frequency domains.

3.1 Notion of Conjugated-Order Differintegrals

The conjugated-order differintegral will be defined as (K is a real scale factor):

$$D^{\alpha(u,v)} f(t) = K\left[D^{\alpha} f(t) + D^{\bar{\alpha}} f(t)\right] = y(t)$$
$$= K\left[D^{u+jv} f(t) + D^{u-jv} f(t)\right] \quad (14)$$

Hartley, *et al*. (2005a) defines several other types of conjugated-order differintegrals, with real or complex weights. In the subsequent calculations we use only operator (14), but the procedure adopted will be identical if we use other alternative conjugated-order operators.

The corresponding transfer function of equation (14) is given by:

$$H^{\alpha(u,v)}(s) = \frac{Y(s)}{F(s)} = K\left(s^{u+jv} + s^{u-jv}\right) \quad (15)$$

Using the Euler identity, equation (15) can be rewritten as (Hartley, *et al*., 2005a):

$$H^{\alpha(u,v)}(s) = 2Ks^{u} \cos\left[v \ln(s)\right] \quad (16)$$

The frequency response of the conjugated-order operator (15) can be given in the form:

$$H^{\alpha(u,v)}(j\omega) = 2K\omega^{u} e^{ju\frac{\pi}{2}} \cosh\left[v\left(\frac{\pi}{2} - j\ln\omega\right)\right] \quad (17)$$

3.2 Time-Domain Simulation of Conjugated-Order Differintegrals

The discretization of the conjugated-order differintegral will be performed by using the Tustin operator (7) and an interpolation scheme of the Euler and Tustin operators called the Al-Alaoui generating function (Al-Alaoui, 1993):

$$H_A\left(z^{-1}\right) = \left(\frac{8}{7T}\frac{1-z^{-1}}{1+z^{-1}/7}\right) \quad (18)$$

For example, using the Al-Alaoui operator (18) we get the discretized conjugated operator as ($K = 1$):

$$D^{\alpha(u,v)}\left(z^{-1}\right) = \frac{Y(z)}{F(z)} = H^{(u,v)}\left(z^{-1}\right)$$
$$= \left(\frac{8}{7T}\frac{1-z^{-1}}{1+z^{-1}/7}\right)^{u+jv} + \left(\frac{8}{7T}\frac{1-z^{-1}}{1+z^{-1}/7}\right)^{u-jv} \quad (19)$$

Its impulse response sequence can be obtained by taking the PSE, yielding:

$$H^{\alpha(u,v)}\left(z^{-1}\right) = c_0 + c_1 z^{-1} + c_2 z^{-2} + \ldots + c_N z^{-N} \quad (20)$$

where the series coefficients c_i ($i = 0, 1, ..., N$) are now real values. The resulting approximation comes in the form of a real FIR filter. On the other hand, applying the CFE method, we obtain a rational transfer function of type:

$$H\left(z^{-1}\right) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_m z^{-m}}{1 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_n z^{-n}} \quad (21)$$

where $m \le n$ and the coefficients a_i ($i = 0, 1, ..., m$) and b_i ($i = 1, 2, \ldots, n$) are equally real values.

Table 2 lists the CFE (4, 4)-order approximations of the conjugated-order differintegral obtained with the Al-Alaoui and Tustin operators for $u = 0.5$, $v = 0.5$, $K = 1$ and $T = 1$ s. Figure 4 shows the Bode diagrams that exhibit a good agreement with the continuous conjugated-order differintegral. Note the better performance of the Al-Alaoui scheme in the high frequency range. In Figure 5 we plot the pole-zero maps of the approximations. First, we observe that there are no complex poles or zeros. We can further observe that the distribution of the poles and zeros are interlaced along the segment of the real axis. However, we note that this scheme may lead to zeros outside of the unit circle, that is, the resulting IIR-type approximation is stable and nonminimum phase.

To illustrate the effectiveness of the approximations in the time domain, we use them to calculate the conjugated differintegral of the causal sine function $f(t) = \sin(\omega t)$ $(t > 0)$. The analytical solution $y(t)$ is obtained as (considering only the steady-state behaviour):

$$\begin{aligned} y(t) &= K\left(D^{u+jv}\left[\sin(\omega t)\right] + D^{u-jv}\left[\sin(\omega t)\right]\right) \\ &= K\left\{\omega^{u+jv}\sin\left(\omega t + u\frac{\pi}{2} + jv\frac{\pi}{2}\right)\right. \\ &\quad \left. + \omega^{u-jv}\sin\left(\omega t + u\frac{\pi}{2} - jv\frac{\pi}{2}\right)\right\} \end{aligned} \quad (22)$$

After some trigonometric manipulation of equation (22), we obtain the final analytical solution as:

$$\begin{aligned} y(t) &= 2K\omega^{u}\left\{\sin\left(\omega t + u\frac{\pi}{2}\right)\cosh\left(v\frac{\pi}{2}\right)\cos\left[v\ln(\omega)\right]\right. \\ &\quad \left. - \cos\left(\omega t + u\frac{\pi}{2}\right)\sinh\left(v\frac{\pi}{2}\right)\sin\left[v\ln(\omega)\right]\right\} \end{aligned} \quad (23)$$

It should be noted that the conjugated-order differintegral yields a purely real time response. Figure 6 shows the time responses of the approximations to a sinusoidal input with the Al-Alaoui and Tustin operators for $u = 0.5$, $v = 0.5$, $K = 1$ and $\omega = 0.2\ \mathrm{rad\ s^{-1}}$. The analytical solution (23) is also plotted. As can be seen, the curves are almost coincident (in stationary regime) demonstrating its effectiveness.

The impulse response $h(t)$ of the conjugated operator can be easily obtained as (Hartley, *et al.*, 2005a):

$$\begin{aligned} h(t) &= L^{-1}\left\{2Ks^{u}\cos\left[v\ln(s)\right]\right\} \\ &= 2Kt^{-u-1}\left\{\mathrm{Re}\left[\frac{1}{\Gamma(-u-jv)}\right]\cos\left[v\ln(t)\right]\right. \\ &\quad \left. + \mathrm{Im}\left[\frac{1}{\Gamma(-u-jv)}\right]\sin\left[v\ln(t)\right]\right\} \end{aligned} \quad (24)$$

Figure 7 shows the impulse responses of the approximations with the Al-Alaoui and Tustin operators for $u = -0.5$, $v = -0.5$, $K = 1$ and $T = 1$ s. As can be seen, the curves show a good accordance with the ideal impulse response, particularly the Al-Alaoui operator (note that this operator is better at higher frequencies) while the Tustin scheme presents a limited oscillatory behaviour.

Table 2 Coefficients of the CFE (4, 4)-order approximations with the Al-Alaoui and Tustin operators, for $u = 0.5$, $v = 0.5$, $K = 1$ and $T = 1$ s

Coef.	Al-Alaoui	Tustin
a_0	2.1333	2.6603
a_1	–3.8238	–0.3190
a_2	1.6366	–3.7027
a_3	0.1136	0.8480
a_4	–0.0872	0.3655
b_0	1.0000	1.0000
b_1	–1.2592	0.5189
b_2	0.3461	–0.6992
b_3	0.0309	–0.1813
b_4	–0.0083	0.0516

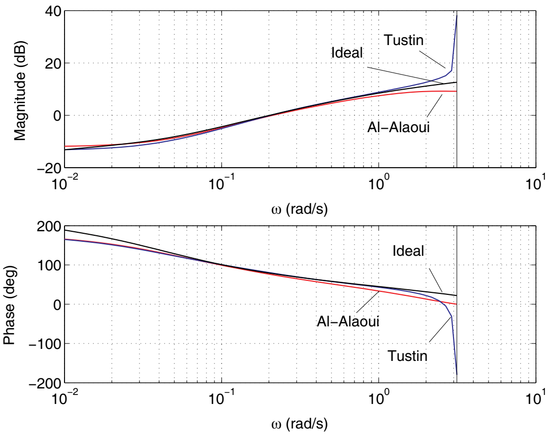

Fig. 4. Bode plots of the CFE (4, 4)-order approximations to conjugated-order differintegral with the Al-Alaoui and Tustin operators. The continuous solution is also plotted.

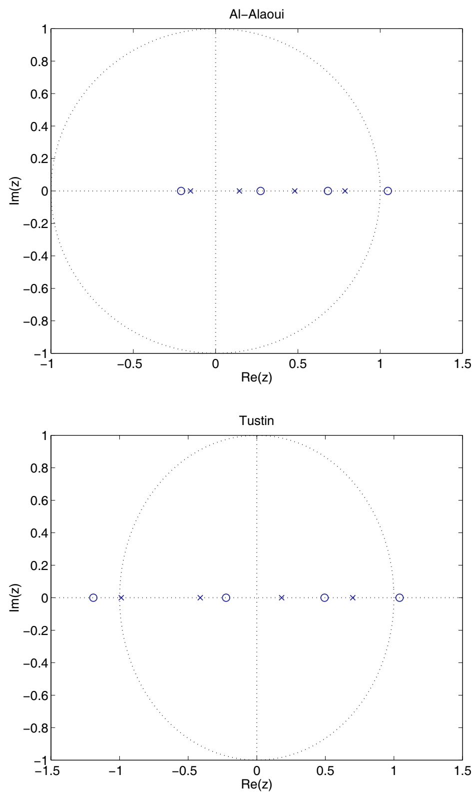

Fig. 5. Pole-zero maps of the CFE (4, 4)-order approximations to conjugated-order differintegral with the Al-Alaoui and Tustin operators.

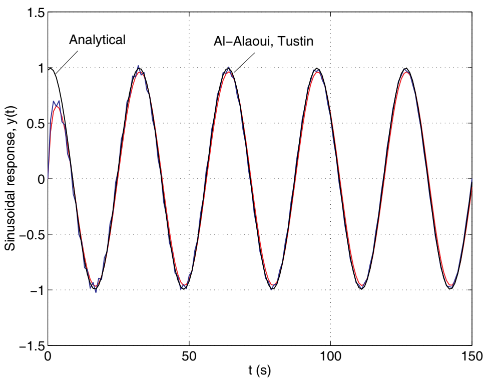

Fig. 6. Time responses to a sinusoidal input of the CFE (4, 4)-order approximations to conjugated-order differintegral with the Al-Alaoui and Tustin operators. The analytical solution is also plotted.

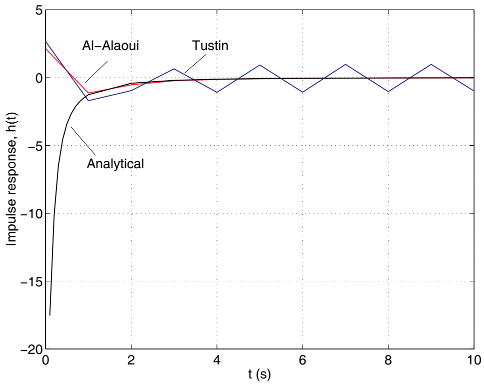

Fig. 7. Impulse responses of the CFE (4, 4)-order approximations to conjugated-order differintegral with the Al-Alaoui and Tustin operators. The analytical solution is also plotted.

4. CONCLUSIONS

In this paper we have introduced the discretization of complex-order operators. Also, we obtained discrete approximations of the complex-order differintegral. This complex operator generates complex time responses which, consequently, are of limited application. In order to ensure real time responses and real transfer functions we used the concept of conjugated-order differintegral. In both cases, the linear invariant time (LTI) transfer functions (CFE-type IIR filters) produce good approximations both in the frequency and time domains. If such a LTI representation of these complex differential operators, in time domain, is the most effective, is yet an open question. In conclusion, it will be necessary more exhaustive research of these operators in order to clarify all its implications and, particularly, when used for the modelling, control and identification of dynamical systems.

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