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FRACTIONAL-ORDER DESCRIBING FUNCTION OF SYSTEMS WITH BACKLASH

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This paper analyses the dynamical properties of systems with backlash and impact phenomena based on the describing function method. It is shown that this type of nonlinearity can be analysed in the perspective of the fractional calculus theory.

1 Introduction

Fractional calculus (FC) is a branch of mathematics that deals with the generalization of the operation of differentiation and integration to an arbitrary order. The theory of FC revealed a well adapted tool to the modeling of many physical phenomena, allowing the description to take into account some particularities that classical integer-order models simply neglect. Besides the intensive research carried out in the area of pure and applied mathematics, FC has found applications in various fields such as physics and chemistry, viscoelasticity, chaos, biology, signal processing, diffusion and wave propagation, electromagnetism and automatic control [1−10]. Nevertheless, in spite of the work that has been done in the area, many aspects remain to be investigated.

The phenomenon of vibration with impacts occurs in many branches of technology where plays a very useful role. On the other hand, its occurrence is often undesirable, because it causes additional dynamic loads, as well as faulty operation of machines and devices. Despite many investigations that have been carried out so far, this phenomenon is not fully understood yet, mainly due to the considerable randomness and diversity of reasons underlying the energy dissipation involving the dynamic effects.

In this paper we investigate the dynamics of systems with backlash and impacts and it is shown that FC is an appropriate tool for its analysis. Bearing these ideas in mind, the article is organized as follows. Section 2 studies the describing function of systems with backlash phenomena and section 3 draws the main conclusions.

2 Analysis of Systems with Backlash and Impact Phenomena

The standard approach to the backlash study is based on the adoption of a geometric model that neglects the dynamic phenomena involved during the impact process. Due to this reason often real results differ significantly from those predicted by that model. In this section we use the DF method to analyze systems with backlash and impact phenomena, usually called *dynamic backlash*.

The proposed mechanical model consists on two masses (M_1 and M_2) subjected to backlash and impact phenomenon as shown in Fig. 1.

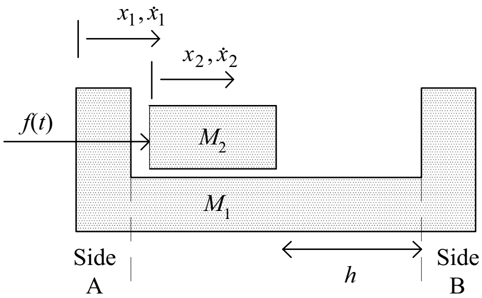

Fig. 1. System with two masses subjected to dynamic backlash.

A collision between the masses M_1 and M_2 occurs when $x_1 = x_2$ or $x_2 = x_1+h$. We can compute the velocities of masses M_1 and M_2 after the impact ($\dot{x}_1'$ and $\dot{x}_2'$) by relating them to the previous values ($\dot{x}_1$ and $\dot{x}_2$) through the Newton's law:

$$\left(\dot{x}_1' - \dot{x}_2'\right) = -\varepsilon\,\left(\dot{x}_1 - \dot{x}_2\right), \quad 0 \le \varepsilon \le 1 \tag{1}$$

where ε is the coefficient of restitution that represents the dynamic phenomenon occurring in the masses during the impact. In the case of a fully plastic (*inelastic*) collision ε = 0, while in the *ideally elastic* cases ε = 1.

The principle of conservation of momentum requires that the momentum, immediately before and immediately after the impact, must be equal:

$$M_1\dot{x}_1' + M_2\dot{x}_2' = M_1\dot{x}_1 + M_2\dot{x}_2 \tag{2}$$

From (8) –(9), we can find the velocities of both masses after an impact:

$$\dot{x}_1' = \frac{\dot{x}_1\left(M_1 - \varepsilon M_2\right) + \dot{x}_2\left(1+\varepsilon\right)M_2}{M_1 + M_2} \tag{3}$$

$$\dot{x}_2' = \frac{\dot{x}_1\left(1+\varepsilon\right)M_1 + \dot{x}_2\left(M_2 - \varepsilon M_1\right)}{M_1 + M_2} \tag{4}$$

The validity of the proposed model is restricted to frequencies of the exciting input force *f*(*t*) higher than a cut-off frequency ω_C. This frequency was determined numerically arriving to the approximate expression:

$$\omega_C \approx \left[\left(2\frac{F}{M_2 \cdot h}\right)^2 \cdot (1-\varepsilon)^5\right]^{1/4} \tag{5}$$

Fig. 2 illustrates the energy and power losses (*i.e.* W_L and P_L) *vs* the exciting frequency ω and the coefficient of restitution ε. As expected, the energy (and the power) loss decreases as ε increases. Moreover, as $\varepsilon \to 1$ it yields $\omega_C \to 0$, which is in accordance with (5).

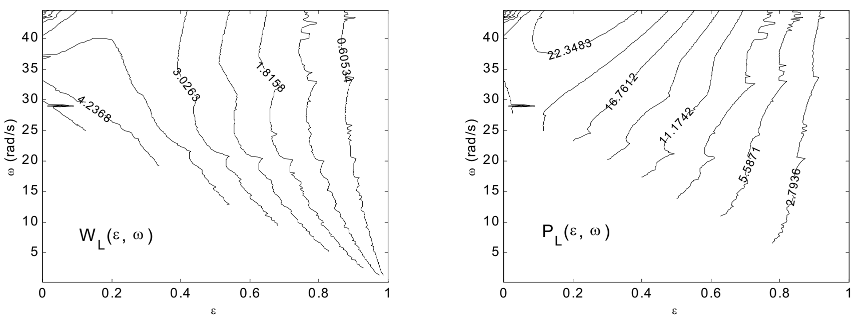

Fig. 2. Contour plot of energy and power losses (*i.e.* W_L and P_L) *vs* the exciting frequency ω and the coefficient of restitution ε, for an input force $F = 50$ N, $M_1 = M_2 = 1$ Kg and $h = 10^{-1}$ m.

On the other hand, there is also an limiting frequency ω_L determined by application of Newton's law to mass M_2, that is $f(t) = M_2\,\ddot{x}_2(t)$. Considering an input signal $f(t) = F\cos(\omega t)$ and solving for $x_2(t)$ we arrive to a expression for ω_L when the amplitude of the displacement is within the clearance $h/2$, yielding:

$$\omega_L = 2 \cdot \left(\frac{F}{h \cdot M_2}\right)^{1/2} \tag{6}$$

For the system model of Fig. 1 we can calculate numerically the Nyquist diagram of $-1/N(F,\omega)$ for a sinusoidal input force $f(t) = F\cos(\omega t)$ applied to mass M_2 while considering as output position $x_1(t)$ of mass M_1.

The values of the parameters adopted in the subsequent simulations are $M_1 = M_2 = 1$ Kg, and $h = 10^{-1}$ m. Fig. 3 shows the Nyquist plot for a constant input force $F = 50$ N and $\varepsilon = \{0.1,\ldots,0.9\}$. The Nyquist plot of Fig. 4 is depicted for $F = \{10, 20, 30, 40, 50\}$ N and a restitution coefficient of $\varepsilon = \{0.2, 0.8\}$.

The Nyquist charts of Fig. 3 and 4 reveal some interesting features. The most obvious is the occurrence of a jumping phenomena, which is a characteristic of nonlinear systems. This phenomenon is more visible around $\varepsilon \approx 0.5$, while for the limiting cases, $\varepsilon \to 0$ and $\varepsilon \to 1$ it disappears.

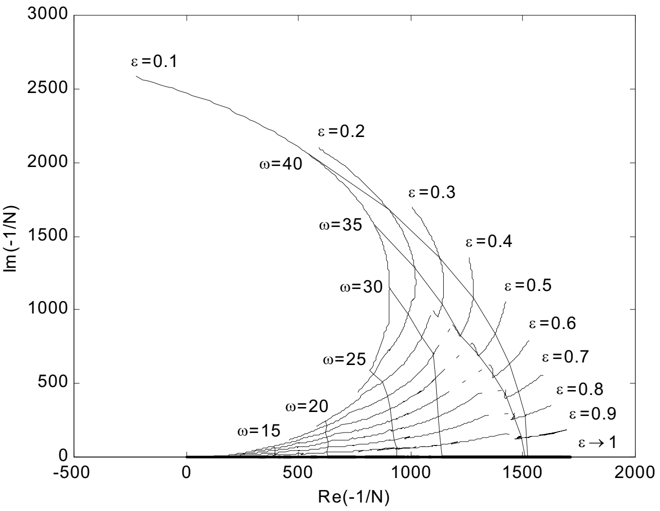

Fig. 3. Nyquist plot of $-1/N(F,\omega)$ for a system with dynamic backlash, F = 50 N and ε = {0.1,…,0.9}.

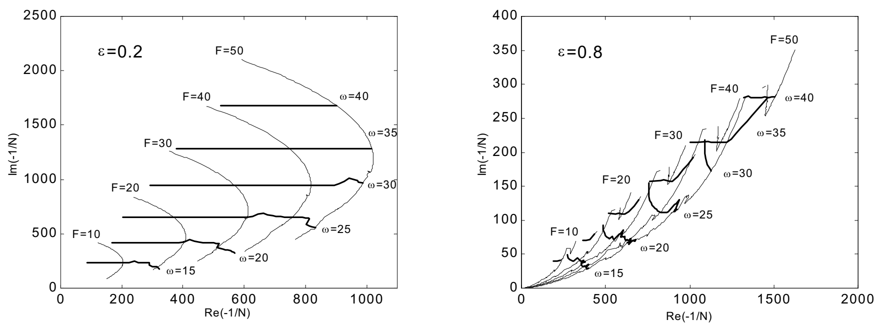

Fig. 4. Nyquist plot of $-1/N(F,\omega)$ for a system with dynamic backlash, F = {10, 20, 30, 40, 50} N and ε = {0.2, 0.8}.

The frequency for which the jumping phenomena occurs (ω_J) has the relation:

$$\omega_J \propto \left(\frac{F}{h \cdot M_2}\right)^{1/2} \tag{7}$$

Moreover, Fig. 4 shows also that for a fixed value of ε the charts are proportional to the input amplitude F.

Fig. 5 presents the harmonic content of $x_1(t)$ for an input force $f(t) = 50\cos(\omega t)$, $\omega_C < \omega < \omega_L$, and ε = {0.2, 0.8}.

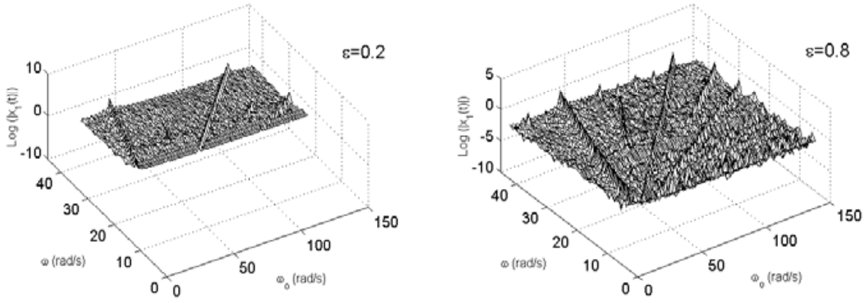

Fig. 5. Fourier transform of the output displacement $x_1(t)$, over 500 cycles, *vs* the exciting frequency ω and the harmonic frequency ω_o, for ε = {0.2, 0.8}.

The charts demonstrate that the fundamental harmonic of the output has a much higher magnitude than the other higher-harmonic components. This fact supports the application of the DF method in the prediction of limit cycles for this system.

Fig. 6 illustrates the variation of the slope α of the tangent to the curve of the Nyquist plots of $-1/N(F,\omega)$ as function of the exciting frequency ω, for a constant input force F = 50 N and ε = {0.1, 0.3, 0.5, 0.7, 0.9}. The curves reveal a fractional slope α (unlike classical systems where we have integer values) which illustrates the fractional-order dynamics of this system.

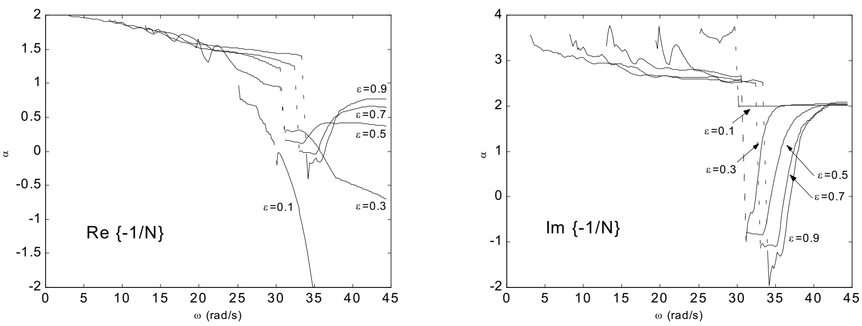

Fig. 6. Plots of slope α *vs* the exciting frequency ω, for F = 50 N and ε = {0.1, 0.3, 0.5, 0.7, 0.9} for Re{−1/N}, Im{−1/N}.

3 Conclusions

This paper addressed the modeling of systems with backlash and impact phenomena through the describing function method. It was shown that these systems may lead to a fractional-order dynamics. This results encourages further studies of nonlinear systems in the perspective of the fractional calculus.

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