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FRACTIONAL ORDER CAPACITORS

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ABSTRACT

The performance of electrical devices, depending on the processes of the electrolytes, have been fully described and the physical origin of each parameter is well established. However, the influence of the irregularity of the electrodes was not a subject of study and only recently, this problem became relevant in the viewpoint of fractional calculus. This paper describes an electrolytic process in the perspective of fractional order capacitors. In this line of thought, are developed several experiments for measuring the electrical impedance of the devices. The results are analyzed through the frequency response, revealing capacitances of fractional order that can constitute an alternative to the classical integer order elements.

KEY WORDS

Fractional calculus, Electrical Impedance, Capacitances, Electrolyze.

1. Introduction

Fractional calculus (FC) is a generalization of the integration and differentiation to a non-integer order. The fundamental operator is ${}_aD_t^{\alpha}$, where the order α is a real or, even, complex number, and the subscripts a and t represent the two limits of the operation, *cf*. [1], [2], [3].

Recent studies brought FC into attention revealing that many physical phenomena can be modeled by fractional differential equations [4], [5], [6], [7]. The importance of fractional order models is that they yield a more accurate description and give a deeper insight into the physical processes underlying a long range memory behavior.

Capacitors are one of the crucial elements in integrated circuits and are used extensively in many electronic systems [8]. However, Jonscher [9] demonstrated that the ideal capacitor cannot exist in nature, because an impedance of the form $1/(j\omega C)$ would violate causality [10], [11]. In fact, the dielectric materials exhibit a fractional behavior yielding electrical impedances of the form $1/[(j\omega C_F)^{\alpha}]$, with $\alpha \in \Re^+$.

Bearing these ideas in mind, this paper analyzes the fractional modelling of several electrical devices and is organized as follows. Section 2 introduces the fundamental concepts of electrical impedances. Section 3 describes the fractal geometries and fractional capacitors. Sections 4 and corresponding subsections present the experiments results and the impedance model for the study of the fractional order capacitors. Finally, section 5 draws the main conclusions.

2. On the Electrical Impedance

In an electrical circuit the voltage $u(t)$ and the current $i(t)$ can be expressed as a function of time t:

$$u(t) = U_0 \cos(\omega t), \quad (1)$$

$$i(t) = I_0 \cos(\omega t + \phi) \quad (2)$$

where U_0 and I_0 are the amplitudes of the signals, ω is the angular frequency and ϕ is the current phase shift. The voltage and current can be expressed in complex form as:

$$u(t) = \mathrm{Re}\left\{U_0 \mathrm{e}^{j(\omega t)}\right\}, \quad (3)$$

$$i(t) = \mathrm{Re}\left\{I_0 \mathrm{e}^{j(\omega t + \phi)}\right\} \quad (4)$$

where $\mathrm{Re}\{\ \}$ represents the real part and $j = \sqrt{-1}$.

Consequently, in complex form the electrical impedance $Z(j\omega)$ is given by the expression:

$$Z(j\omega) = \frac{U(j\omega)}{I(j\omega)} = Z_0 \mathrm{e}^{j\phi}. \quad (5)$$

Fractional order elements occur in several fields of engineering [8], [9], [11].

A brief reference about the Constant Phase Element (CPE) and the Warburg impedance is presented here due to their application in the work. In fact, to model an electrochemical phenomenon it is often used a CPE due to the fact that the surface is not homogeneous [12].

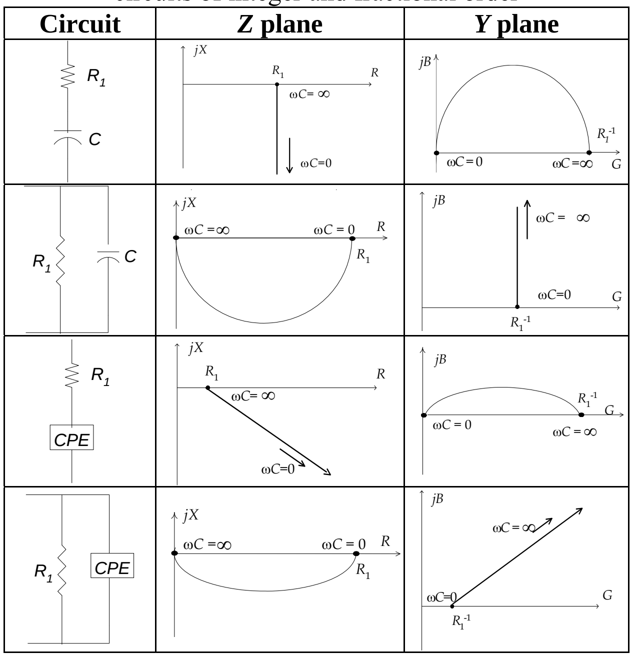
With a CPE we have the expression:

$$Z(j\omega) = \frac{1}{(j\omega C_F)^{\alpha}} \quad (6)$$

where C_F is a fractional order capacitance and the fractional order $0 < \alpha \le 1$ is a parameter, occurring the classical ideal capacitor when $\alpha = 1$.

Table 1

Impedance $Z(j\omega)$ and admittance $Y(j\omega)$ *loci* of *RC* circuits of integer and fractional order



It should be noted that the SI base units of the C_F element are $[m^{-2/\alpha} kg^{-1/\alpha} s^{(\alpha+3)/\alpha} A^{2/\alpha}]$ [13], [14].

Table I shows the polar plots of the impedance $Z(j\omega)$ and the admittance $Y(j\omega) = Z^{-1}(j\omega)$ for simple series and parallel *RC* associations of integer and fractional order, where $G = \mathrm{Re}\{Y\}$ is the conductance and $B = \mathrm{Im}\{Y\}$ is the susceptance.

It is well known that, in electrochemical systems with diffusion, the impedance is modelled by the so-called Warburg element [12], [14]. The Warburg element arises from one-dimensional diffusion of an ionic species to the electrode. If the impedance is under an infinite diffusion layer, the Warburg impedance is given by:

$$Z(j\omega) = \frac{R}{\left(j\omega C_F\right)^{0.5}} \tag{7}$$

where R is the diffusion resistance. If the diffusion process has finite length, the Warburg element becomes:

$$Z(j\omega) = R\,\frac{\tanh\left(j\omega\tau\right)^{0.5}}{\left(\tau\right)^{0.5}} \tag{8}$$

with $\tau = \delta^2 / D$, where R is the diffusion resistance, τ is the diffusion time constant, δ is the diffusion layer thickness and D is the diffusion coefficient [14].

Based on these concepts, in the following sections some fractional order electric impedances are presented.

3. Fractals and Fractional Capacitors

Fractals can be found both in nature and abstract objects. The impact of the fractal structures and geometries, is presently recognized in engineering, physics, chemistry, economy, mathematics, art and medicine.

The concept of fractal is associated with Benoit Mandelbrot, that lead to a new perception of the geometry of the nature [15]. However, the concept was initially proposed by several well known mathematicians, such as George Cantor (1872), Giuseppe Peano (1890), David Hilbert (1891), Helge von Koch (1904), Waclaw Sierpinski (1916), Gaston Julia (1918) and Felix Hausdorff (1919) [15], [16].

An geometric important index consists in the fractal dimension (*FDim*) that represents the occupation degree in the space and that is related with its irregularity. The *FDim* is given by:

$$FDim = \lim_{z\to\infty} \frac{\ln N(z)}{\ln\left(1/\eta(z)\right)} \approx \frac{\ln(N)}{\ln(1/\eta)} \tag{9}$$

where N represents the number of boxes, with size $\eta(N)$ resulting from the z subdivisions of the original structure.

This is not the only description for the fractal geometry, but it is enough for the identification of groups with similar geometries.

Some of the classical fractals adopted in this work are the curve of Koch, triangle of Sierpinski, carpet of Sierpinski, curves of Hilbert and Peano, depicted in the Table II.

The dielectric absorption in the capacitors is difficult to characterize accurately, due to the high value of the involved time constant, and the necessity of high precision measuring equipment.

The simplest capacitors are constituted by two parallel electrodes separated by a layer of insulating dielectric. There are several factors susceptive of influencing the characteristics of a capacitor. However, three of them have a special importance, namely the surface of the electrodes, the distance among them and the material that constitutes the dielectric.

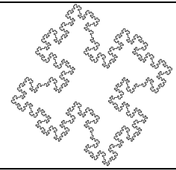
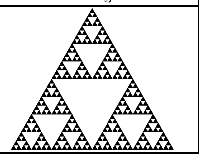
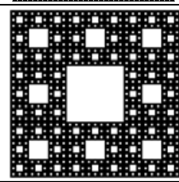
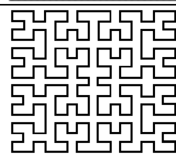
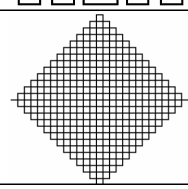
In this work it is studied another aspect that can also influence the capacity of a capacitor, namely the wrinkling of theirs electrodes, and a non-homogenous dielectric structure. The electrodes are implemented through one-side cooper-based printed circuit boards with the fractal geometries presented in Table II. The choice of these fractals is due to the value of *FDim* that it is intended to evaluate cases with dimension from 1 up to 2.

4. Experimental Results

In this section, we consider four types of electrolytes, five different fractal structures and a fractional order model of an electrical circuit. In the first subsection, we analyze the system of Figure 1 by adopting two electrodes with the carpet of Sierpinski fractal and four different electrolytes.

In the second subsection we adopt the approximation electrical model for analyzing several fractal electrodes and dielectric structures.

Table 2
Fractals structures

Fractal Name	Fractal Dimension	Structure
Curve of Koch	1.262	
Triangle of Sierpinski	1.585	
Carpet of Sierpinski	1.893	
Curve of Hilbert	2.000	
Curve of Peano	2.000	

4.1 Experiments with the Carpet of Sierpinski

In the experiments (Figure 1) we apply sinusoidal excitation signals $v(t)$ to the apparatus, for several distinct frequencies ω, and the impedance $Z(j\omega)$ between the electrodes is measured based on the resulting voltage $u(t)$ and current $i(t)$.

We study the influence of several factors such as *FDim*, different chloride of sodium (NaCl) solution concentrations (Ψ) and the introduction of fractal materials in the solution, namely gravel and sand. Moreover, we test also the linearity and the variation of the impedance $Z(j\omega)$ with the amplitude V_0 of the input signal.

In each experiment we use two identical single face electrodes. The voltage, the adaptation resistance R_a and the distance between electrodes d_{elec} are kept identical during the different experiments namely, V_0 = 10 V, R_a = 1.2 kΩ and d_{elec} = 0.13 m.

This methodology help us to understand the influence of the relevant factors in the impedance $Z(j\omega)$ and, consequently, the behavior of the fractal capacitor.

In a first experiment, the electrolyte process consists in a aqueous solution of NaCl with $\Psi = 5$ gl^{-1} (AS5) and two copper electrodes with the carpet of Sierpinski printout with area $S = 0.423$ m^2.

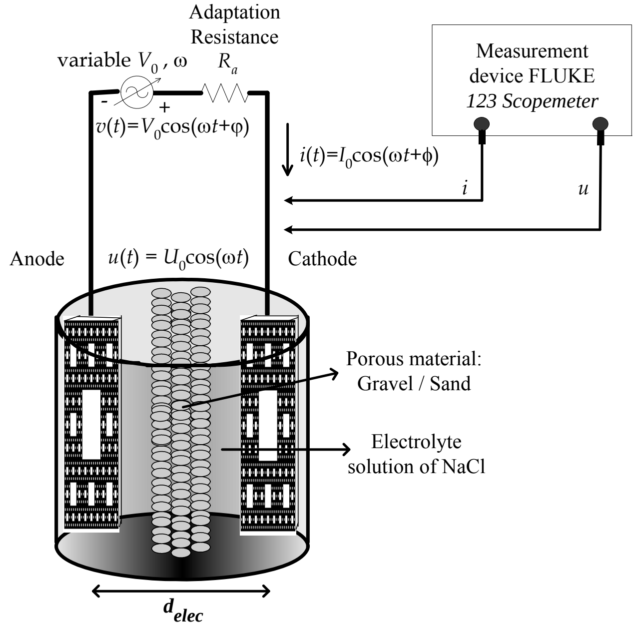


Figure 1 Electrolyte process

We apply the *R* and *CPE* (6) series circuit of Table I, to model the fractional electrical impedance yielding:

$$Z_{\text{app}}(j\omega) = R + \frac{1}{\left(j\omega C_F\right)^{\alpha}}. \tag{10}$$

Figure 2 presents the polar diagrams of $Z(j\omega)$ and $Y(j\omega)$, and the corresponding model approximation, Z_{app}, Y_{app}. The electric element reveals fractional order impedance leading to the parameters $(R, C_F, \alpha) = (19.10, 1.02\times10^{-4}, 0.589)$.

In a second case, with the purpose of studying the effect of the dielectric, we introduce gravel into the aqueous solution of $\Psi = 5$ gl^{-1} (AS5G). We use the same electrodes and the gravel covers completely the electrodes. In this case, we obtain a dielectric having also fractal characteristics. The values of the voltage and of the adaptation resistance are identical to the previous experiment (*i. e.*, V_0 = 10 V, R_a = 1.2 kΩ) leading to the approximation parameters $(R, C_F, \alpha) = (58.00, 1.40\times10^{-5}, 0.500)$.

In a third experiment the gravel is replaced by sand (AS5S) leading to $(R, C_F, \alpha) = (90.90, 3.9\times10^{-5}, 0.540)$.

In all three experiments was varied the amplitude V_0 and it was verified that the device has linear characteristics. Figure 3 illustrates the polar diagrams of $Z(j\omega)$, $Y(j\omega)$, and the corresponding approximations. The results reveal a good fit between the experimental data and the approximation model.

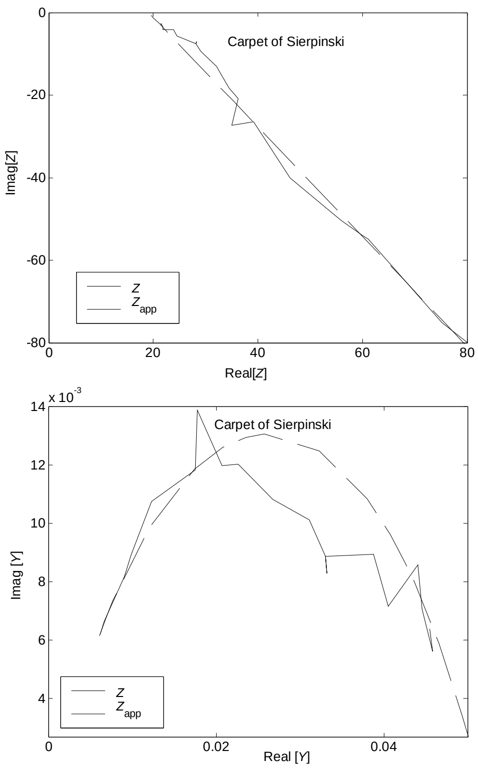


Figure 2 Polar diagrams of the impedance and admittance for electrodes with the carpet of Sierpinski fractal

The fourth experiment studies the influence of the fractal surface by using two electrodes printed with the carpet of Sierpinski having an area of $S / 3$. In this case the values of the voltage, the resistance of adaptation and the solution remain unchanged, namely $V_0 = 10$ V, $R_a = 1.2$ kΩ and $\Psi = 5$ gl^{-1}, without introducing neither gravel or sand in the dielectric. The approximation model leads to (R, C_F, α) = (27.10, 5.30×10^{-17}, 0.175).

In a fifth case, $Z(j\omega)$ is evaluated for electrodes with the carpet of Sierpinski and the initial electrode (S = 0. 423 m^2) but with a aqueous solution concentration of the $\Psi = 10$ gl^{-1} (AS10). The voltage and the resistance of adaptation remain the same. The approximation leads to the parameters (R, C_F, α) = (13.00, 2.66×10^{-4}, 0.690).

Table III presents the values of the parameters (R, C_F, α) for the five experiments described previously. Comparing, experiments 1, 2 and 3, we conclude that the introduction of the gravel or sand in the solution increases R, and diminishes C_F while α remains almost invariant.

The comparison of the experiments 1 and 4 reveals that R increases and that C_F and α decrease. Moreover, we verify that the fractional order α decreases approximately by a factor of 1/3.

Finally, comparing experiments 1 and 5, we verify that R decreases and that C_F and α increase.

Based on these initial results, in the next subsection, we organize similar experiments for the other fractals presented in Table II, in order to analyze their influence in the electrical impedance.

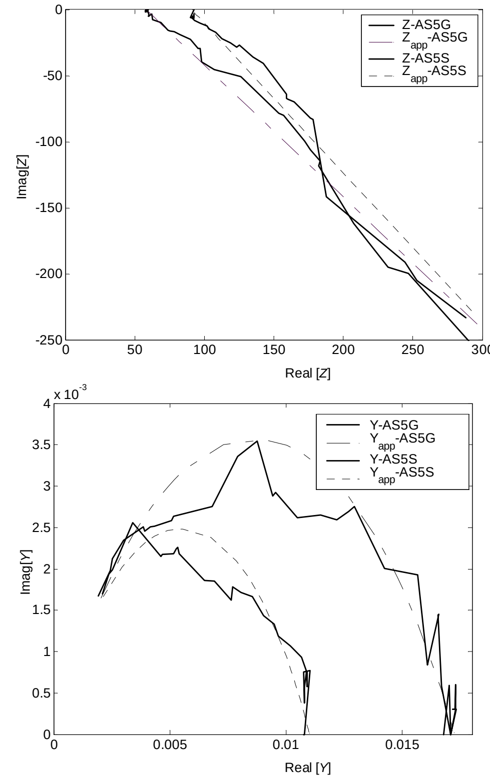


Figure 3 Polar diagrams of the impedance and admittance for electrodes with the carpet of Sierpinski fractal, and the AS5G and AS5S dielectrics

Table 3
Numerical values of the parameters for the circuit with R and CPE series association, with the carpet of Sierpinski

Case	**Sur-face**	Ψ	R [Ω]	C_F [$m^{-2/\alpha}kg^{-1/\alpha}s^{(\alpha+3)/\alpha}A^{2/\alpha}$]	α
1	S	AS5	19.10	1.02×10^{-4}	0.589
2	S	AS5G	58.00	1.40×10^{-5}	0.500
3	S	AS5S	90.90	3.90×10^{-5}	0.540
4	$S/3$	AS5	27.10	5.30×10^{-17}	0.175
5	S	AS10	13.00	2.66×10^{-4}	0.690

4.2 Experiments with Other Fractals

In this subsection are analyzed the curve of Koch, triangle of Sierpinski, curve of Hilbert and curve of Peano.

The values of the voltage amplitude V_0, the resistance of adaptation R_a and the dielectric solution (AS5) are identical to those used in the previous experiments.

The size of the fractals was adjusted so that their surface yields identical values, namely S = 0.423 m^2.

Table IV shows the values of the approximation parameters (R, C_F, α) and figures 4 and 5 depict the polar diagrams of $Z(j\omega)$ and $Y(j\omega)$. The results reveal again a good fit between the experimental data and the electrical model.

These figures reveal also similarities with the results presented in the Table I. Moreover, it is clear that adopting more complex circuit models we can have better approximations. Nevertheless, models with a larger number of components make difficult to compare different cases and to assign a physical meaning to each parameter.

Table 4
Numerical values of the parameters of the circuit with *R* and CPE series association for several fractal electrodes

Fractal	**Sur-face**	Ψ	$\boldsymbol{R}$ **[Ω]**	$\boldsymbol{C_F}$ $[\mathbf{m^{-2/\alpha}kg^{-1/\alpha}s^{(\alpha+3)/\alpha}A^{2/\alpha}}]$	$\boldsymbol{\alpha}$
Curve of Koch	*S*	AS5	20.87	7.03×10^{-5}	0.602
Triangle of Sierpinski	*S*	AS5	8.00	4.00×10^{-6}	0.480
Curve of Hilbert	*S*	AS5	19.30	1.40×10^{-4}	0.640
Curve of Peano	*S*	AS5	19.50	3.00×10^{-5}	0.550

5. Conclusion

During several centuries the FC was developed mainly in a mathematical viewpoint, but presently it addresses a considerable range of applications.

In this paper the FC concepts were applied to the electrical fractional impedances. Therefore, fractal structures were adopted in an electrolyte process. This system is a possible prototype for the development of fractional electrical devices, and an alternative to the classical integer order capacitors.

It was verified that is possible to get fractional order elements by adopting non classical electrodes and dielectrics.

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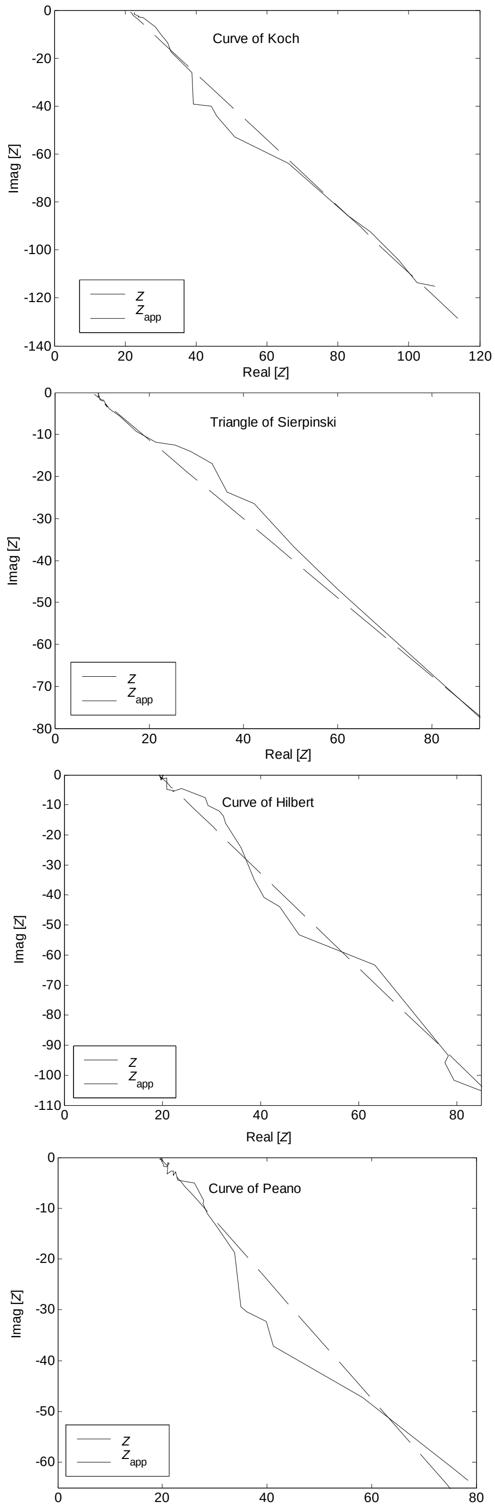

Figure 4 Polar diagrams of the impedance and for the fractal electrodes {Koch, Triangle Sierpinski, Hilbert and Peano}

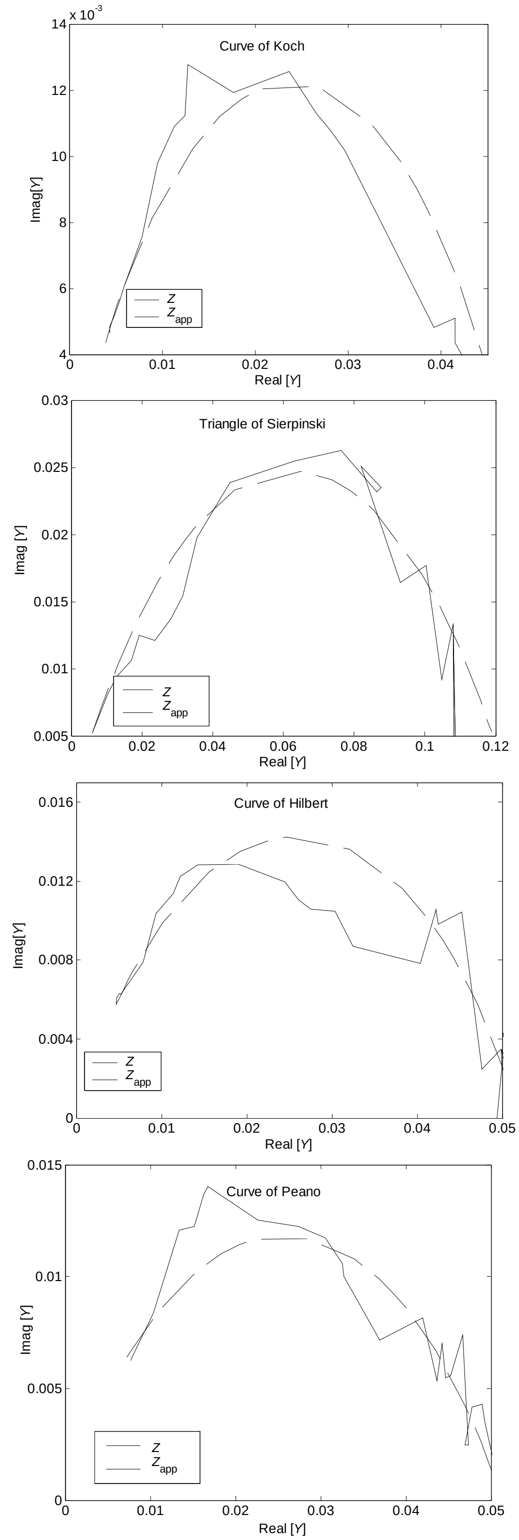

Figure 5 Polar diagrams of the admittance for the fractal electrodes {Koch, Triangle Sierpinski, Hilbert and Peano}