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A NEW METHOD FOR APPROXIMATING FRACTIONAL DERIVATIVES: APPLICATION IN NON-LINEAR CONTROL

J. A. Tenreiro Machado
Institute of Engineering of Porto
Dept. of Electrotechnical Engineering
Rua Dr. Antonio Bernardino de Almeida
4200-072 Porto, Portugal
jtm@isep.ipp.pt

Alexandra M. S. Galhano
Institute of Engineering of Porto
Dept. of Electrotechnical Engineering
Rua Dr. Antonio Bernardino de Almeida
4200-072 Porto, Portugal
amf@isep.ipp.pt

Abstract

The theory of fractional calculus goes back to the beginning of the theory of differential calculus, but its application received attention only recently. In the area of automatic control some work was developed but the proposed algorithms are still in a research stage. This paper discusses a novel method, with two degrees of freedom, for the design of fractional discrete-time derivatives. The performance of several approximations of fractional derivatives is investigated in the perspective of nonlinear system control.

Key words

Fractional calculus, control, discrete time.

1 Introduction

Fractional calculus (FC) is a natural extension of the classical mathematics. The fundamental aspects of the fractional calculus theory and the study of its properties can be addressed in references [Miller and Ross, 1993; Oldham and Spanier, 1974; Samko, *et al.* 1993]. In what concerns the application of FC concepts we can mention a large volume of research about viscoelasticity and damping, biology, signal processing, diffusion and wave propagation, modeling, identification and control [Anastasio, 1994; Bagley and Torvik, 1983; Machado, 1997; Mainardi, 1996; Méhauté, 1991; Miller and Ross, 1993; Nigmatullin, 1986; Oldham and Spanier, 1974; Oustaloup, 1991; Oustaloup, 1995; Podlubny. 1999a; Samko, *et al.*, 1993].

Several researchers on automatic control proposed algorithms based on the frequency [Oustaloup, 1991; Oustaloup, 1995] and the discrete-time [Machado, 1997; Podlubny. 1999a; Podlubny. 1999b] domains. This article introduces a novel method to implement fractional derivatives (FDs) in the discrete-time domain. The performance of the resulting algorithms is analyzed when adopted in the control of nonlinear systems. In this line of thought, the paper is organized as follows. Sections two and three develop the novel method of FD discrete-time approximation and investigate its performance in the control of a nonlinear system, respectively. Finally, section four draws the main conclusions.

2 On the Generalization of Fractional Discrete-Time Control Algorithms

The Grünwald-Letnikov definition of a FD of order α of the signal $x(t)$, $D^{\alpha}x(t)$, is given by the expression:

$$D^{\alpha}\left[x(t)\right]= = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}(-1)^{k}\frac{\Gamma(\alpha+1)x(t-kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \tag{1}$$

where Γ is the gamma function and h is the time increment. This formulation inspired the discrete-time FD calculation, by approximating the time increment h through the sampling period T, yielding the equation in the z domain:

$$Z\left\{D^{\alpha}\left[x(t)\right]\right\}\approx\frac{1}{T^{\alpha}}\sum_{k=0}^{\infty}\frac{(-1)^{k}\,\Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)}z^{-k}X(z) = \left(\frac{1-z^{-1}}{T}\right)^{\alpha}X(z) \tag{2}$$

where $X(z) = Z\{x(t)\}$.

The expression (2) represents the Euler (or first backward difference) approximation in the so-called $s\rightarrow z$ conversion schemes, being other possibilities the Tustin (or bilinear) and Simpson rules. The generalization to non-integer exponents of these conversion methods lead to the non-rational z-formulae:

$$s^{\alpha}\approx\left[\frac{1}{T}\left(1-z^{-1}\right)\right]^{\alpha}=\left[H_{0}\left(z^{-1}\right)\right]^{\alpha} \tag{3a}$$

$$s^{\alpha} \approx \left(\frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}} \right)^{\alpha} = \left[H_1\left(z^{-1}\right) \right]^{\alpha} \qquad (3b)$$

where H_0 and H_1 are often called generating approximants of zero and first order, respectively.
In order to get rational expressions the approximants need to be expanded into Taylor series and the final algorithm corresponds to a truncated series or to a rational Pade fraction.
We can obtain a family of fractional differentiators by the generating functions H_0 and H_1 weighted by the factors p and $1-p$, yielding:

$$H_{av}(p) = pH_0\left(z^{-1}\right) + (1-p)H_1\left(z^{-1}\right) \qquad (4)$$

For example, the Al-Alaoui operator corresponds to a weighted interpolation of the Euler and Tustin integration rules with $p = 3/4$ [Al-Alaoui, 1993; Al-Alaoui, 1997; Smith, 1987]. These approximation methods have been studied by several researchers [Barbosa, *et al.*, 2004; Barbosa, *et al.*, 2006; Chen, and Moore, 2002; Chen, *et al.*, 2004; Chen and Vinagre, 2003; Tseng, 2001; Vinagre, *et al.*, 2003] and motivated a novel averaging method based on the generalized formula of averages, or average of order $q \in \Re$:

$$H_{av}(q, p) = \left\{ p\left[H_0\left(z^{-1}\right)\right]^q + (1-p)\left[H_1\left(z^{-1}\right)\right]^q \right\}^{\frac{1}{q}} \qquad (5)$$

where (p,q) are two tuning degrees of freedom.
For example, when $q = \{-1,0,1\}$ we get the wel-known {harmonic, geometric, arithmetic} averages, respectively.

Bearing these ideas in mind we decided to examine the expression resulting from (5), for distinct values of (q,p). Tables 1 and 2 depict the coefficients of a second order Pade approximation $H_{av}(q, p) = 1/T^{1/2} \sum_{i=0}^{2} a_i z^{-i} / b_i z^{-i}$, $a_i, b_i \in \Re$, to D^{α} , $\alpha = 1/2$, for $q = \{-1, -1/2, 0, 1/2, 1, 3/2, 2\}$, $p = 3/4$, and $q = \{-1, 0, 1, 2\}$, $p = 1/2$, respectively.

3 Performance Evaluation in the Control of a Nonlinear System

In order to test the performance of the expressions the usual method is to examine either the frequency domain, by comparing the Bode plots, or the time domain, by comparing the step response. Nevertheless, often the differences are negligible and, furthermore, do not have a direct translation to control system performance. Therefore, in our study we decided to test the approximations by analyzing the step response when the FD represents the control algorithm. In this perspective, we consider the closed loop represented in Fig. 1. The inclusion of the on-off non-linearity in the forward loop leads to the simplification of the analysis because the controller gain is not relevant and is not necessary to tune, but, on the other hand, we have a stringent dynamic test that stimulates both the transient and steady-state behavior.

Figure 2 depicts the closed-loop step response for $q = \{-1, -1/2, 0, 1/2, 1, 3/2, 2\}$ and $p = 3/4$. Figure 2 presents the response for $q = \{-1,0,1,2\}$ and $p = 1/2$. In both cases, we consider three controller high sampling periods, namely T = {0.1, 0.2, 0.3} in order to test also the robustness for fast versus slow sampling controllers.

In all cases we verify that:

- In general the order $q = 1$ is the one that produces the best results;
- The sampling period T = 0.1 leads to a good performance, while the results degrade considerably for larger values of T;
- The difference between $p = 3/4$ and $p = 1/2$ seems to be negligible, particularly in the case of $q = 1$.

While these results seem clear the authors believe that further test, namely with other systems and other values of α, q and p are still required to establish a definitive conclusion.

4 Conclusions

In this paper a novel method for the discrete-time FD approximation was presented and evaluated. The new algorithm adopts the time domain and generates a family of possible approximations, having two distinct degrees of freedom, namely the order of the averaging and the weight of the generating functions. The properties of several expressions were studied for a simple non-linear system. The time response of the closed loop system was analyzed and the robustness for different sampling periods was tested. The conclusions are consistent and motivate an extensive test of all possibilities opened by the extra degrees of freedom.

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Table 1. Coefficients of the Pade fraction approximation for $p = 3/4$

q	a_0	a_1	a_2	b_0	b_1	b_2
−1	0.76597491	6.026479067	−7.984934523	1	1.054474431	−7.609965244
−1/2	1.324947376	3.621337338	−6.248676637	1	−0.000172433	−5.761497326
0	1.465218175	2.188544252	−4.788397277	1	−0.804204046	−4.788397277
1/2	1.734358402	1.669224885	−4.589703269	1	−1.164316112	−4.184478992
1	1.787304546	1.2339426	−4.155138823	1	−1.36761081	−3.765228028
3/2	1.779700183	0.975910656	−3.828827863	1	−1.464217163	−3.447457027
2	1.723916925	0.840452121	−3.56477439	1	−1.480178572	−3.188431144

Table 2. Coefficients of the Pade fraction approximation for $p = 1/2$

q	a_0	a_1	a_2	b_0	b_1	b_2
−1	0.218192312	2.932612925	−3.758174112	1	1.815189846	−8.845101094
0	1.645896792	2.146868296	−5.087608366	1	−1.403319874	−4.278151638
1	1.667798875	1.603885503	−4.470002812	1	−1.607438339	−3.703075415
2	1.488976016	1.498893033	−4.035502783	1	−1.52197687	−3.29498733

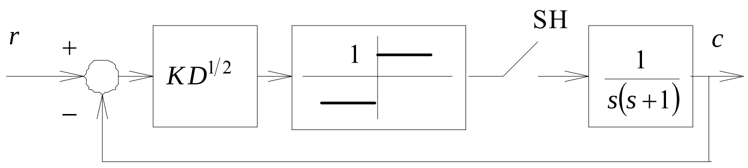

Figure 1. The $D^{1/2}$ controller for a system with a nonlinear actuator.

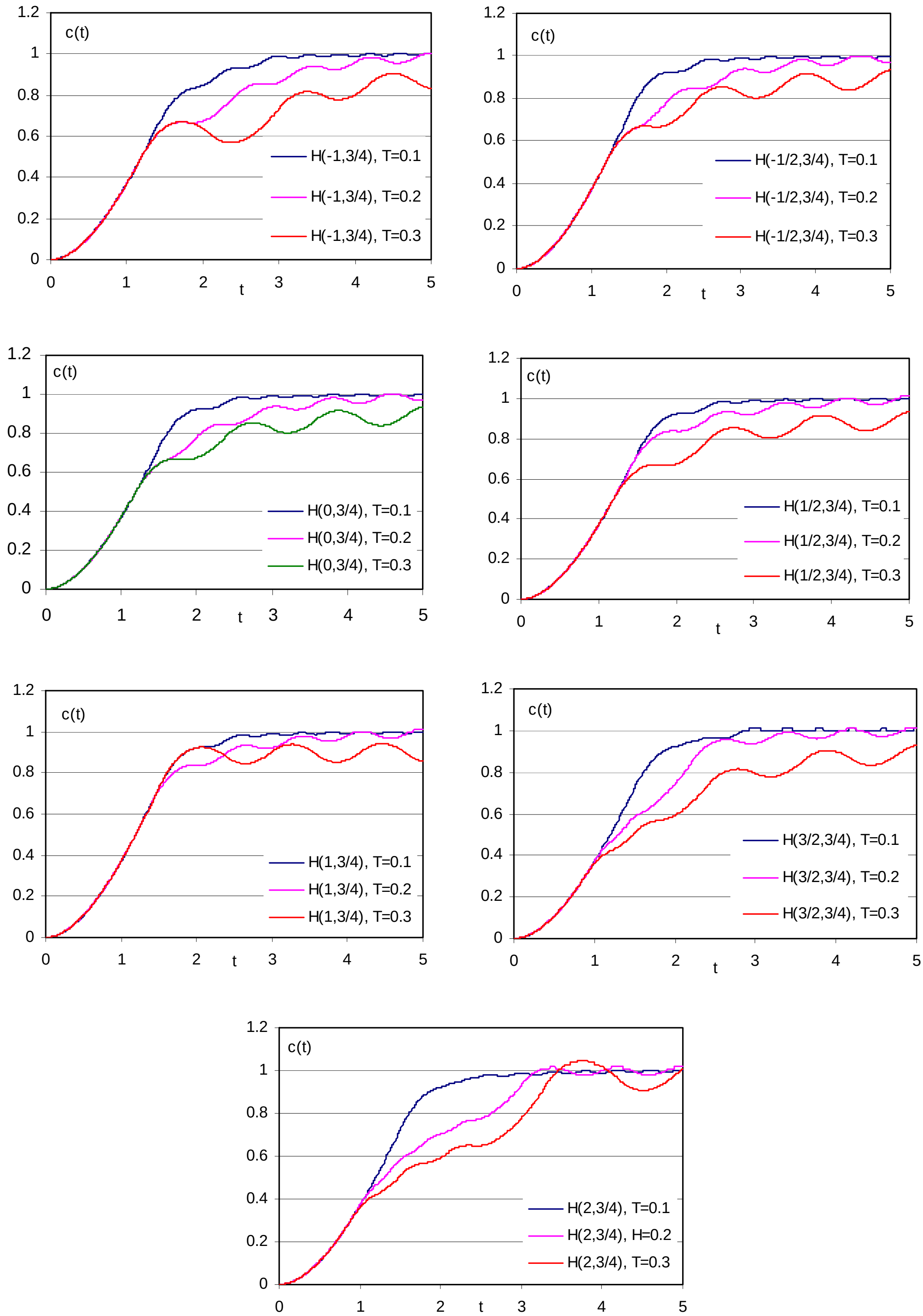

Figure 2. Closed loop step response for a $D^{1/2}$ controller with $q = \{-1, -1/2, 0, 1/2, 1, 3/2, 2\}$ and $p = 3/4$.

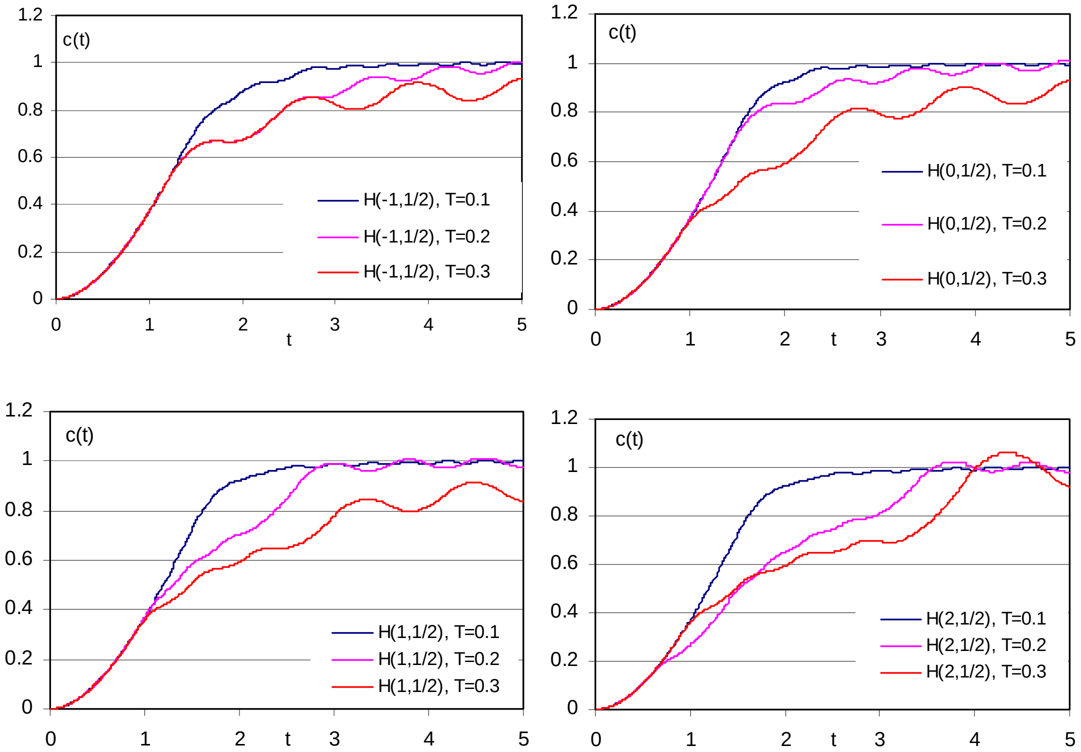

Figure 3. Closed loop step response for a $D^{1/2}$ controller with $q = \{-1, 0, 1, 2\}$ and $p = 1/2$.