DESCRIBING FUNCTION OF A SIMPLE MECHANICAL SYSTEM WITH NON-LINEAR FRICTION

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Abstract

This paper studies the describing function (DF) of systems constituted by a mass subjected to nonlinear friction. The friction force is decomposed in two components namely, the viscous and the Coulomb friction. The system dynamics is analyzed in the DF perspective revealing a fractional-order behaviour. The reliability of the DF method is evaluated through the signal harmonic contents.

Key words

Describing Function, Friction, Control, Modelling.

1 Introduction

The phenomenon of vibration due to friction is verified in many branches of technology where it plays a very useful role. On the other hand, its occurrence is often undesirable, because it causes additional dynamic loads, as well as faulty operation of machines and devices. Despite many investigations that have been carried out so far, this phenomenon is not yet fully understood, mainly due to the considerable randomness and diversity of reasons underlying the energy dissipation involving the dynamic effects (Armstrong *et al.*, 1994), (Armstrong and Amin, 1996), (Barbosa and Machado, 2002), (Barbosa *et al.*, 2003). In this paper we investigate the dynamics of systems that contain nonlinear friction, namely the Coulomb forces, in addition to the linear viscous component. Bearing these ideas in mind, the article is organized as follows. Section 2 introduces the fundamental aspects of the describing function method. Section 3 studies the describing function of mechanical systems with nonlinear friction. Finally, section 4 draws the main conclusions and addresses perspectives towards future developments.

2 Fundamental concepts

Let us consider the feedback system of Figure 1 with one nonlinear element N and a linear system with transfer function $G(s)$.

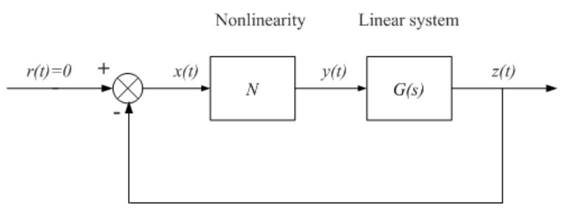

Figure 1. Nonlinear control system

Suppose that the input to a nonlinear element is sinusoidal $x(t) = X\sin(\omega t)$. In general the output of the nonlinear element $y(t)$ is not sinusoidal; nevertheless, the signal $y(t)$ is periodic, with the same period as the input, and containing higher harmonics in addition to the fundamental harmonic component.

If we assume that the nonlinearity is symmetrical with respect to the variation around zero, the Fourier series become:

$$y(t) = \sum_{k=1}^{\infty} Y_k \cos(k\,\omega\,t + \phi_k) \tag{1}$$

where Y_k and ϕ_k are the amplitude and the phase shift of the kth harmonic component of the output $y(t)$, respectively.

In the DF analysis, we assume that only the fundamental harmonic component of the output is significant. Such assumption is often valid since the higher harmonics in the output of a nonlinear element are usually of smaller amplitude than the fundamental component (Slotine and Li, 1991), (Vinagre and Monge, 2007), (Lanusse and Oustaloup, 2004). Moreover, most systems are “low-pass filters” with the result that the higher harmonics are further attenuated (Cox, 1987), (Atherton, 1975), (Dupont, 1992).

The DF, or sinusoidal DF, of a nonlinear element, $N(X,\omega)$, is defined as the complex ratio of the fundamental harmonic component of the output and the input, that is:

$$N(X,\omega) = \frac{Y_1}{X} e^{j\phi_1} \tag{2}$$

where the symbol N represents the DF, X is the amplitude of the input sinusoid, and Y_1 and ϕ_1 are the amplitude and the phase shift of the fundamental harmonic component of the output, respectively. Several analytical expressions of DFs of standard nonlinear elements can be found in the references (Haessig and Friedland, 1991), (Karnopp, 1985), (Azenha and Machado, 1998).

For nonlinear systems without involving energy storage the DF is merely amplitude-dependent, that is $N = N(X)$. However, when we have nonlinear elements that involve energy, the DF method is both amplitude and frequency dependent yielding N=$N(X,\omega)$. In this case, to determine the DF, usually we have to adopt a numerical approach because it is impossible to find a closed-form analytical solution. Once calculated, the DF can be used for the approximate stability analysis of the nonlinear control system.

Let us consider again the standard control system shown in Figure 1 where the block N denotes the DF of the nonlinear element. If the higher harmonics are sufficiently attenuated, N can be treated as a real or complex variable gain and the closed-loop frequency response becomes:

$$\frac{C\,(j\omega)}{R\,(j\omega)} = \frac{N(X,\omega)G(j\omega)}{1+N(X,\omega)G(j\omega)} \tag{3}$$

The characteristic equation is:

$$1 + N(X,\omega)G(j\omega) = 0 \tag{4}$$

If equation (4) can be satisfied for some values of X and ω, then a limit cycle is *predicted* for the nonlinear system. Moreover, since (4) is valid only if the nonlinear system is in a steady-state limit cycle, the DF analysis predicts only the presence or the absence of a limit cycle and cannot be applied to the analysis of other types of time responses.

3 Mechanical systems with nonlinear friction

In this section we analyze the DF of a dynamical system with nonlinear friction composed by a combination of the viscous and Coulomb components.

Let us consider a system (Figure 2) with a mass M, moving on a horizontal plane under the action of a input force $f(t)$, with a friction $F_f(t)$ effect composed of two components: a non-linear Coulomb K part and a linear viscous $B\dot{x}$ part (so-called CV model).

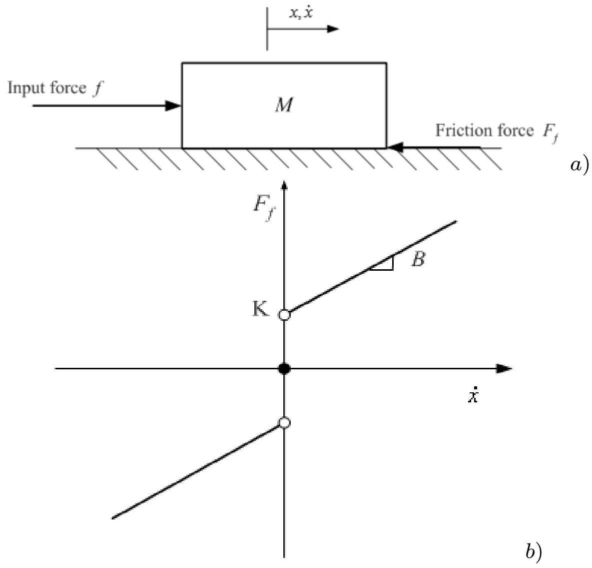

Figure 2. a) Elemental mass system subjected to nonlinear friction and b) Non-linear friction with Coulomb and viscous components (CV model).

The equation of motion in this system is as follows:

$$M\;\ddot{x}\,(t) + F_f\,(t) = f\,(t) \tag{5}$$

where x, $\dot{x}$ and $\ddot{x}$ are the displacement, velocity and acceleration, respectively.

For the system of Figure 2 we can calculate numerically $N(F,\omega)$ considering as input a sinusoidal force $f(t) = F\cos(\omega\,t)$ applied to mass M and as output the position $x(t)$.

Figure 3 shows the Nichols plot of $N(F,\omega)$ for $M = 1.0$ kg, $B = 0.5$ Nsm^{-1} and $K = 2.0$ N. Alternatively Figures 4 and 5 illustrate the log-log plots of $|Re\{N\}|$ and $|Im\{N\}|$ *vs* the exciting frequency ω, for different values of the input force $2.5 \leq F \leq 100.0$ N. We have different results according to the excitation force F and we get straight lines with slopes revealing clearly a fractional-order behaviour.

In Figure 6 it is depicted the harmonic content of the output signal $x(t)$ for an input force of $F = 10$ N. We verify that the output signal has a half - wave symmetry because the harmonics of even order are negligible. Moreover, the fundamental component of the output signal is the most important one, while the amplitude of the high order harmonics decay significantly. Therefore, we can conclude that, for the friction CV model, the DF method may lead to a good approximation.

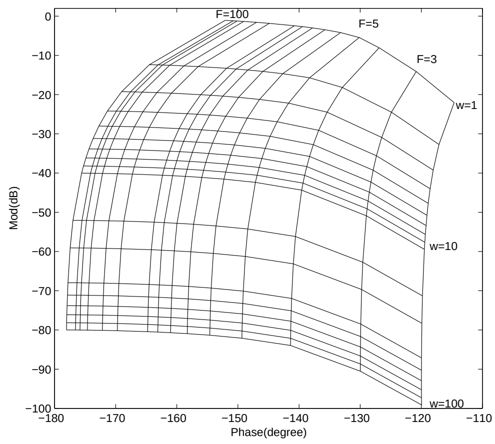

Figure 3. Nichols plot of $N(F,\omega)$ for the system subjected to nonlinear friction (CV model) with $M = 1.0$ kg, $2.5 \leq F \leq 100.0$ N, $1.0 \leq \omega \leq 100.0$ rad s^{-1} with $\{B, K\} = \{0.5\, Nsm^{-1}, 2.0\, N\}$.

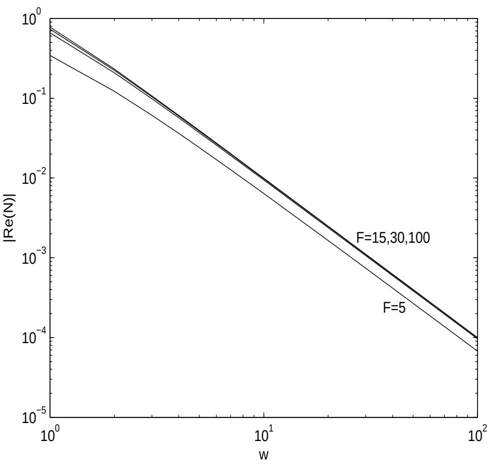

Figure 4. Log-log plots of $|Re\{N\}|$ *vs.* the exciting frequency $1.0 \leq \omega \leq 100$ rad s^{-1}, for the CV model with $\{B, K\} = \{0.5\ Nsm^{-1}, 2.0\ N\}$, $M = 1.0$ kg and $F = \{5, 15, 30, 100\}$ N.

In order to study $Re\{N(F,\omega)\}$ and $Im\{N(F,\omega)\}$, we approximate the numerical results through power functions:

$$Re\{N(F,\omega)\} = -a\,\omega^{-b},\ \{a,b\} \in \mathbb{R}^+ \qquad (6)$$
$$Im\{N(F,\omega)\} = -c\,\omega^{-d},\ \{c,d\} \in \mathbb{R}^+$$

Figure 7 illustrates the variation of the parameters $\{a,b\}$ and $\{c,d\}$ *versus* F for $K = \{1.0, 2.0, 3.0, 4.0, 5.0\}$. We verify that $Re\{N(F,\omega)\}$ and $Im\{N(F,\omega)\}$ reveal a distinct relationships with ω (Podlubny, 1999). In fact, we conclude that $Re\{N\}$ and $Im\{N\}$ are, in the two cases, of the same type,

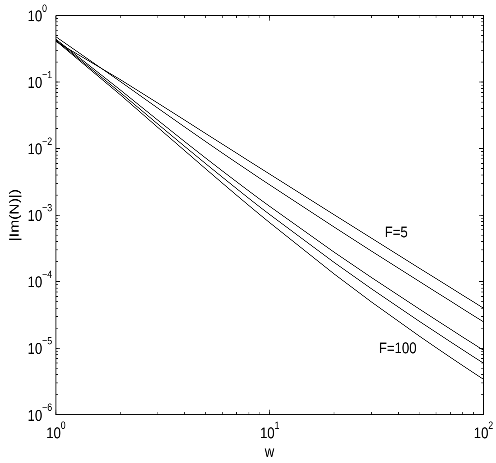

Figure 5. Log-log plots of $|Im\{N\}|$ *vs.* the exciting frequency $1.0 \leq \omega \leq 100$ rad s^{-1}, for the CV model with $\{B, K\} = \{0.5\ Nsm^{-1}, 2.0\ N\}$, $M = 1.0$ kg and $F = \{5, 15, 30, 100\}$ N

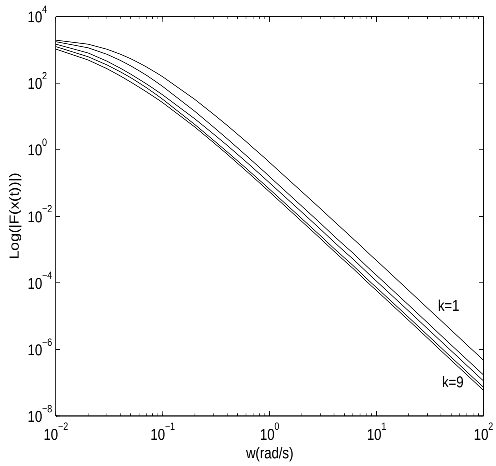

Figure 6. Fourier transform of the output position $x(t)$, for the CV model, *vs.* the exciting frequency $1.0 \leq \omega \leq 100.0$ rad s^{-1} and the harmonic frequency index $k = \{1, 3, 5, 7, 9\}$ for an input force F= 20 N, with $M = 1.0$ kg, $\{B, K\} = \{0.5\, Nsm^{-1}, 2.0\, N\}$.

following a power law according with expression (7). Furthermore, we obtain fractional-order dynamics as revealed by the Nichols chart in Figure 3. Nevertheless, $Re\{N(F,\omega)\}$ has an integer nature with $b \approx 2$, while $Im\{N(F,\omega)\}$ is clearly fractional with $2 < d < 2.7$ (Duarte and Machado, 2005), (Duarte and Machado, 2006).

To have a deeper insight into the effects of the different CV components several complementary experiments were performed varying separately the values of K and M while maintaining the rest constant. For example, Figure 8 present the values of the parameters $\{a,b\}$ and $\{c,d\}$ when approximating $Re\{N\}$ and $Im\{N\}$, for $2.5 \leq F \leq 100.0$

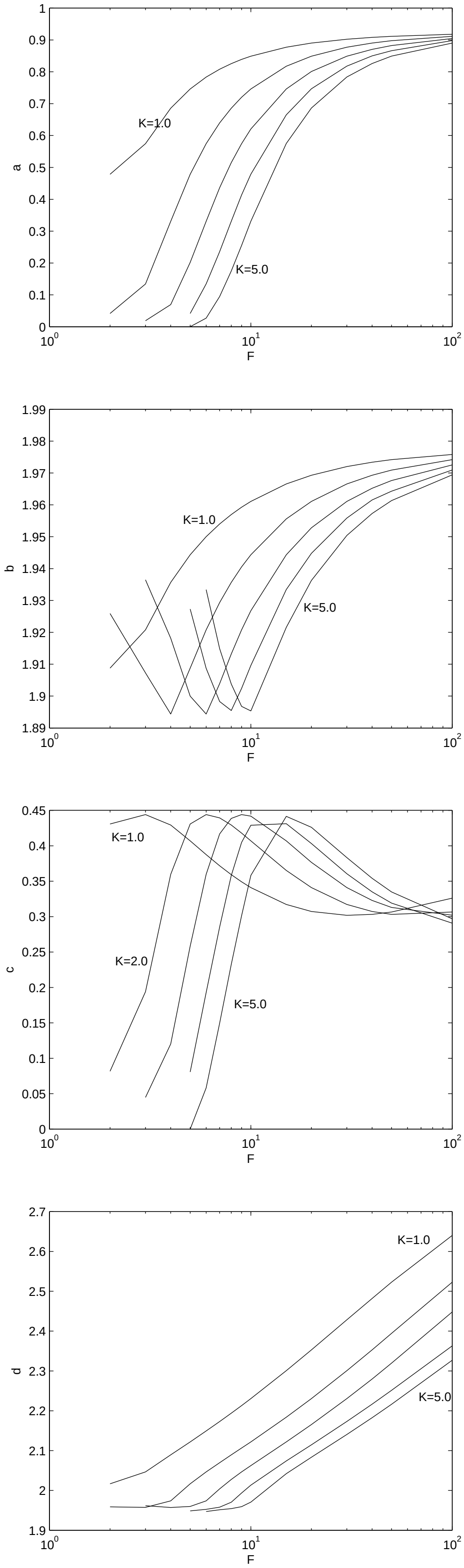

Figure 7. Variation of the parameters $\{a, b\}$ and $\{c, d\}$ *vs* $2.5 \leq F \leq 100.0$ N, in the CV model with $M = 1.0$ kg, $B = 0.5\,Nsm^{-1}$ and $K = \{1.0, 2.0, 3.0, 4.0, 5.0\}$ N.

N with $M = \{0.5, 1.0, 2.0, 3.0\}$ kg and $\{B, K\} = \{0.5\,Nsm^{-1}, 2.0\,N\}$.

As we should expect $Re\{N(F,\omega)\}$ and $Im\{N(F,\omega)\}$ vary with the system parameters, but we conclude that the integer *vs.* order behaviour remain identical, respectively. Furthermore, the fractional characteristics of $Im\{N\}$ are a direct consequence of the nonlinear action of the Coulomb friction, since the viscous friction leads simply to a linear integer order result.

4 Conclusions

This paper addressed the study of systems with nonlinear friction. The dynamics of elemental mechanical system was analyzed through the describing function method and compared with standard models. The polar plot reveals a fractional order behaviour which was further analyzed in the real and imaginary components. The results encourage further studies of nonlinear systems in a similar perspective and the adoption of the tools of fractional calculus.

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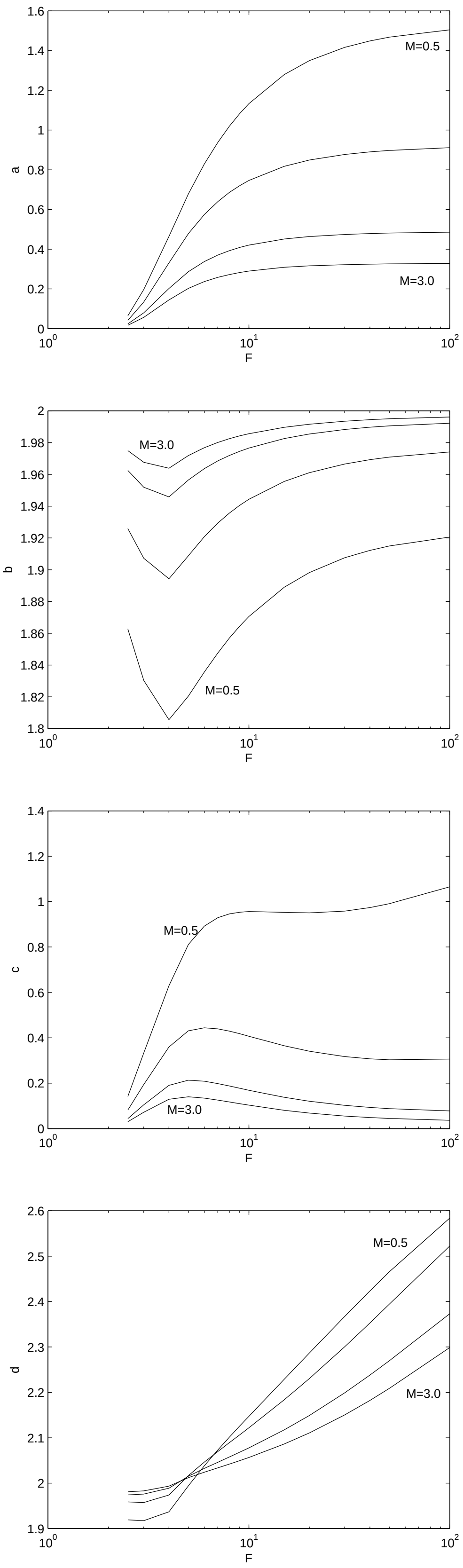

Figure 8. Variation of the parameters $\{a, b\}$ and $\{c, d\}$ *vs* $2.5 \leq F \leq 100.0$ N, in the CV model with $M = \{0.5, 1.0, 2.0, 3.0\}$ kg, $\{B, K\} = \{0.5\, Nsm^{-1}, 2.0\, N\}$.

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